CEMC at Home

Grade 4/5/6 - Monday, May 25, 2020
Contest Day 4

Today's resource features a question from one of the recently released 2020 CEMC Mathematics Contests, along with a question from one of our past contests.

2020 Gauss Contest, #11

Each of 7 boxes contains exactly 10 cookies. If the cookies are shared equally among 5 people, how many cookies does each person receive?

(A) 14 **(B)** 12 **(C)** 9 **(D)** 11 **(E)** 13

2011 Gauss Contest, #16

A 51 cm rod is built from 5 cm rods and 2 cm rods. All of the 5 cm rods must come first, and are followed by the 2 cm rods. For example, the rod could be made from seven 5 cm rods followed by eight 2 cm rods. How many ways are there to build the 51 cm rod?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

More Info:

Check out the CEMC at Home webpage on Monday, June 1 for solutions to the Contest Day 4 problems.

CEMC at Home

Grade 4/5/6 - Monday, May 25, 2020
Contest Day 4 - Solution

Solutions to the two contest problems are provided below, including a video for the second problem.

2020 Gauss Contest, #11

Each of 7 boxes contains exactly 10 cookies. If the cookies are shared equally among 5 people, how many cookies does each person receive?

(A) 14 **(B)** 12 **(C)** 9 **(D)** 11 **(E)** 13

Solution:

Each of 7 boxes contains exactly 10 cookies, and so the total number of cookies is $7 \times 10 = 70$.
If the cookies are shared equally among 5 people, then each person receives $70 \div 5 = 14$ cookies.

Answer: (A)

2011 Gauss Contest, #16

A 51 cm rod is built from 5 cm rods and 2 cm rods. All of the 5 cm rods must come first, and are followed by the 2 cm rods. For example, the rod could be made from seven 5 cm rods followed by eight 2 cm rods. How many ways are there to build the 51 cm rod?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Solution 1:

Using the method of trial and error, we begin by trying one rod with length 5 cm.
If there is one 5 cm rod, then there is $51 - 5 = 46$ cm remaining to be made up of 2 cm rods.
Since $46 \div 2 = 23$, then using one 5 cm rod and twenty-three 2 cm rods gives a possible combination.
Next, let's try two 5 cm rods.
Since $2 \times 5 = 10$ cm, then there is $51 - 10 = 41$ cm remaining to be made up of 2 cm rods.
However, 2 does not divide into 41 evenly, so it is not possible to have two 5 cm rods.
Next, we try three rods with length 5 cm.
If there are three 5 cm rods, then there is $51 - (3 \times 5) = 51 - 15 = 36$ cm remaining to be made up of 2 cm rods.
Since $36 \div 2 = 18$, then using three 5 cm rods and eighteen 2 cm rods gives a possible combination.
Next, let's try four 5 cm rods.
Since $4 \times 5 = 20$ cm, then there is $51 - 20 = 31$ cm remaining to be made up of 2 cm rods.
However, 2 does not divide into 31 evenly, so it is not possible to have four 5 cm rods.
We can see that using an odd number of 5 cm rods allow us to find a solution that works, but using an even number of 5 cm rods does not give a possible combination that works.
(Can you see why this is true? Below, Solution 2 explains this idea further.)
Five 5 cm rods will give a combination that works.
If there are five 5 cm rods, then there is $51 - (5 \times 5) = 51 - 25 = 26$ cm remaining to be made up of 2 cm rods.

See the next page for the rest of Solution 1 and for Solution 2.

Since $26 \div 2 = 13$, then using five 5 cm rods and thirteen 2 cm rods gives a possible combination.
If there are seven 5 cm rods, then there is $51 - (7 \times 5) = 51 - 35 = 16$ cm remaining to be made up of 2 cm rods.
Since $16 \div 2 = 8$, then using seven 5 cm rods and eight 2 cm rods gives a possible combination.
If there are nine 5 cm rods, then there is $51 - (9 \times 5) = 51 - 45 = 6$ cm remaining to be made up of 2 cm rods.
Since $6 \div 2 = 3$, then using nine 5 cm rods and three 2 cm rods gives a possible combination.
However, using eleven 5 cm rods does not work because $11 \times 5 = 55$ cm is greater than 51 cm.
Using any number of 5 cm rods that is greater than 11 will similarly have a total length that is more than 51 cm.
Therefore, there are exactly 5 possible combinations that add to 51 cm using 5 cm rods first followed by 2 cm rods.

Solution 2:

Any number of 2 cm rods add to give a rod having an even length.
Since we need an odd length, 51 cm, then we must combine an odd length from the 5 cm rods with the even length from the 2 cm rods to achieve this.
An odd length using 5 cm rods can only be obtained by taking an odd number of them.
All possible combinations are shown in the table below.

Number of 5 cm rods	Length in 5 cm rods	Length in 2 cm rods	Number of 2 cm rods
1	5	$51 - 5 = 46$	$46 \div 2 = 23$
3	15	$51 - 15 = 36$	$36 \div 2 = 18$
5	25	$51 - 25 = 26$	$26 \div 2 = 13$
7	35	$51 - 35 = 16$	$16 \div 2 = 8$
9	45	$51 - 45 = 6$	$6 \div 2 = 3$

Note that attempting to use 11 (or more) 5 cm rods gives more than the 51 cm length required.
Thus, there are exactly 5 possible combinations that add to 51 cm using 5 cm rods first followed by 2 cm rods.

ANSWER: (A)

Video

Visit the following link to view a discussion of a solution to the second contest problem, and some extensions of the problem: https://youtu.be/Hy-qFPnNatQ

CEMC at Home

Grade 4/5/6 - Tuesday, May 26, 2020
Catching a Code

Try out the following two-player game involving codebreaking!

You Will Need: Two players, paper and a pencil

How to Play:

- The two players in this game alternate between being the "Code Maker" and the "Code Breaker" in each round.
- The players start by choosing an *even* number of rounds to play in the game.
- At the start of each round, the Code Maker chooses a three-digit code, using only the digits 1 through 6 (for example, 146 or 222), and writes the code on a piece of paper, keeping it secret from the Code Breaker.
- The Code Breaker then gets up to ten attempts to guess the Code Maker's code. After each guess, the Code Maker provides two pieces of information about the guess:
  - the number of digits in the guess that appear in the code and are in the correct place, and
  - the number of digits in the guess that appear in the code but are in the wrong place.

  For example, suppose that the Code Maker's code is 263. If the Code Breaker guesses 361, then the Code Maker would give the following information: one of the digits in the guess is in the correct place, and one of the digits in the guess appears in the code but is in the wrong place. If the Code Breaker guesses 336, then the Code Maker would give the following information: no digits are in the correct place, and two digits appear in the code but are in the wrong place. *Note that the two digits here are the 6 and one of the 3s. The Code Maker does not say that all three digits appear in the code because there is only one 3 in the code.*
- The round ends when either the Code Breaker correctly guesses the code or the Code Breaker has made ten incorrect guesses. If the Code Breaker guesses the code in ten or fewer attempts, then the Code Breaker's score for the round is equal to the total number of attempts needed. If the Code Breaker does not guess the code in ten or fewer attempts, then the Code Breaker's score for the round is 11. (The Code Maker does not score.)
- The game ends when the chosen even number of rounds are completed. The winner is the player with the *lowest* total score.

Example: Here is a sample game of Catching a Code, organized in a table.

Code Maker's Code: 234			
Guess #	Guess	Correct Digit – Correct Place	Correct Digit – Wrong Place
1	566	0	0
2	113	0	1
3	423	0	3
4	243	1	2
5	234	3	0

Here, the Code Breaker found the correct code after five attempts. This means the Code Breaker scores 5 points for this round.

Play this game a number of times.

Below, you will find some questions to think about that relate to the sample game provided.

A blank table is also given below for you to use while you play.

Questions:

1. Based on the first guess in the sample game provided, the Code Breaker decides that the correct secret code only contains the digits 1 through 4. Why is this true?
2. After the third guess in the sample game provided, how many possibilities remain for the Code Maker's code? What are they?
3. After the fourth guess in the sample game provided, how many possibilities remain for the Code Maker's code? What are they?

Sample Table:

Guess #	Guess	Correct Digit – Correct Place	Correct Digit – Wrong Place
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

More Info:

Check out the CEMC at Home webpage on Tuesday, June 2 for answers to the questions above.

This game is a pencil-and-paper version of a popular board game. The advantage to the pencil-and-paper version is that you can easily change the rules as you wish. For example, the length of the secret code can be made longer, consisting of four, five, or even more digits. Or, you can increase or decrease the number of possible digits allowed in the code (for example, you can use only the digits 1 through 4, or you can use all of the digits 0 through 9).

CEMC at Home

Grade 4/5/6 - Tuesday, May 26, 2020
Catching a Code – Solution

Code Maker's Code: 234			
Guess #	Guess	Correct Digit – Correct Place	Correct Digit – Wrong Place
1	566	0	0
2	113	0	1
3	423	0	3
4	243	1	2
5	234	3	0

1. Based on the first guess in the sample game provided, the Code Breaker decides that the correct secret code only contains the digits 1 through 4. Why is this true?

   Solution: The feedback given by the Code Maker says that none of the digits in the guess 566 are an exact match with the code, nor are they correct digits in the wrong place. Therefore, the Code Maker's code cannot contain the digits 5 or 6, leaving only 1 through 4 as valid possibilities. Notice that this already cuts down on the number of possible guesses!

2. After the third guess in the sample game provided, how many possibilities remain for the Code Maker's code? What are they?

   Solution: Given the feedback on the third guess above, we know that the Code Maker's code must contain the digits $2, 3, and 4$ in some order, but not the order 423. There are six arrangements of the digits $2, 3, and 4$, and therefore only five arrangements different from 423. These are: 234, 243, 324, 342, and 432. Thus, at first, it looks like there are five remaining possibilities.

   However, we also know that *none* of the digits in the guess 423 are in the same place as the Code Maker's code. This rules out 243, 324, and 432 as well, leaving only two valid possibilities: 234 and 342. In other words, we are looking for a rearrangement of 423 in which *none* of the digits stay in the same place.

3. After the fourth guess in the sample game provided, how many possibilities remain for the Code Maker's code? What are they?

   Solution: Notice we already eliminated 243 as a possible choice for the Code Maker's code at the previous step. In particular, we are already down to two possible codes: 234 and 342. The information we get after the fourth guess is that 243 matches the Code Maker's code in exactly one digit. But 234 matches 243 in only the first digit, and 342 matches 243 in only the second digit. No matter which of the two possibilities is true, the Code Maker will give us the same feedback on the guess 243.

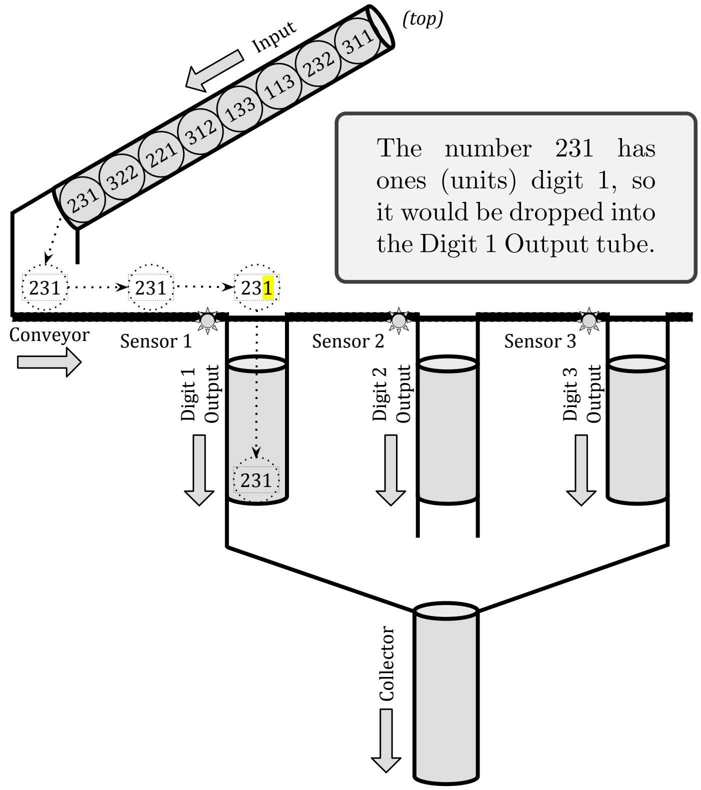
   In other words, the fourth guess is not a useful guess because it adds no new information! It would have been better to guess one of the two remaining possibilities. This would provide a 50% chance of the Code Breaker winning on Guess 4, and would guarantee that the Code Breaker wins by Guess 5.

Note: The famous mathematician and computer scientist Donald Knuth analyzed a version of this game with codes that are four digits long, and formed using six possible digits. Knuth showed that this version of the game can always be won by the Code Breaker in a maximum of five guesses, if they play with the right strategy!

CEMC at Home

Grade 4/5/6 - Wednesday, May 27, 2020
Let's Sort All This Out

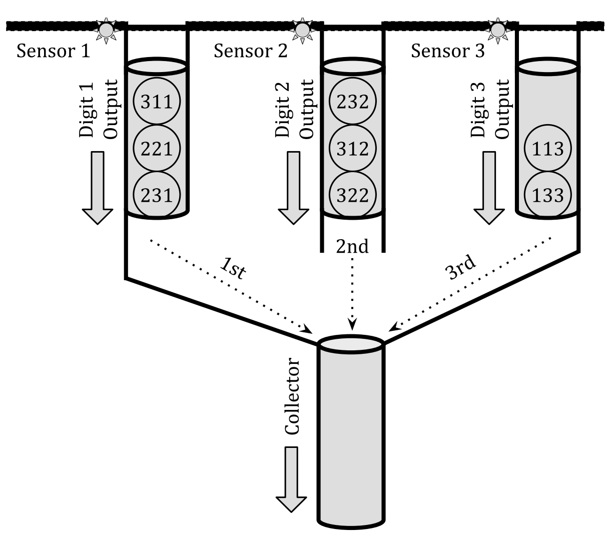
The CEMC has developed a number sorting machine as shown to the right. It sorts three-digit numbers that only contain the digits 1, 2, or 3. Here's how the machine works:



The number 231 has ones (units) digit 1, so it would be dropped into the Digit 1 Output tube.

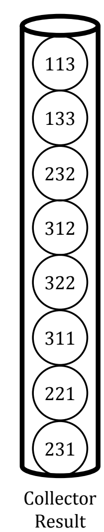
1. The numbers are loaded in the input tube. They drop, one at a time, from the bottom of the input tube and move along a conveyor belt that moves from left to right.
2. There are three sensors along the conveyor belt that can identify the ones, tens, or hundreds digit of a passing number as one of 1, 2, or 3. The sorting machine starts with the sensors set to scan the **ones** (units) digit of each passing number.
3. There are three output tubes, one attached to each sensor. The sensors determine whether a passing number will drop into the tube or pass by the tube.

   If a number with ones digit 1 passes by sensor 1, then a trap door above the Digit 1 Output tube will open and the number drops into the tube. If a number with ones digit 2 or 3 passes by sensor 1, then the trap door stays shut and the number moves on. The other sensors work similarly for the other two digits. The numbers are stored temporarily in the output tubes.
4. Once all of the numbers have been processed from the input tube, they are released from each output tube, one at a time from left to right, and funnelled to the collector tube at the bottom. In other words, all *Digit 1 Output* numbers drop first, followed by the *Digit 2 Output* numbers, and finally the *Digit 3 Output* numbers.



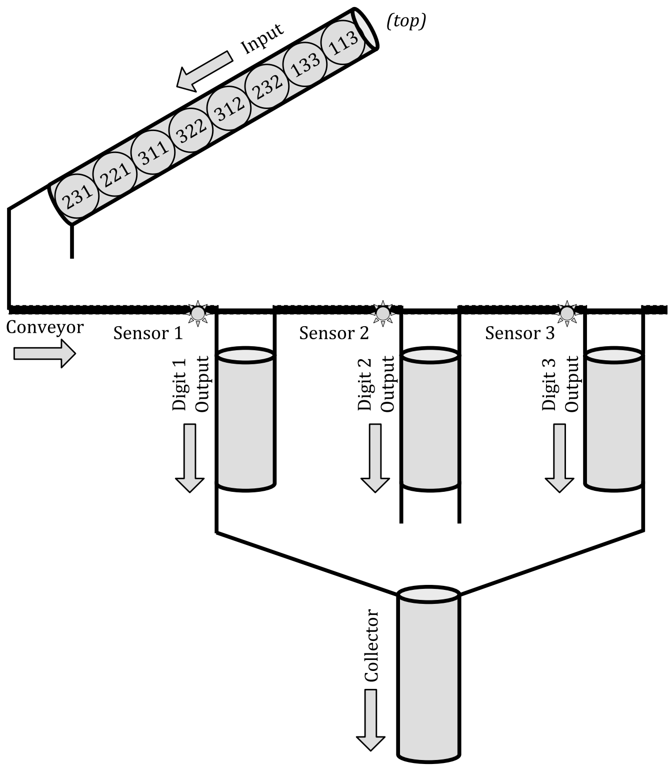
← The diagram on the left shows the result of processing all of the numbers in the input tube when all sensors are set to scan **ones** digits.

The diagram on the right shows the result after the numbers are released from the output tubes one at a time from left to right. →



5. Next, we take the numbers from the collector tube (the result showing on the first page) and feed them back into the input tube of the machine. We must keep the numbers in the *same order* from bottom to top as shown in the input tube below. We now run the machine again, but this time we set all sensors to scan the **tens** digit of each passing number.

   In the table below, list the numbers as they would appear in the collector tube after the numbers pass through the machine for the second time.



(top)

(bottom)

Collector result when sensing the **tens** digits

6. Finally, we take the numbers from the collector tube after Step 5 is completed and feed them back into the input tube of the machine. Again we must keep the numbers in the *same order* from bottom to top. We now run the machine for a third time, but this time we set all sensors to scan the **hundreds** digit of each passing number.

   In the table to the right, list the numbers as they would appear in the collector tube after the numbers pass through the machine for the third time.

   What do you notice about the order of these numbers?

(top)

(bottom)

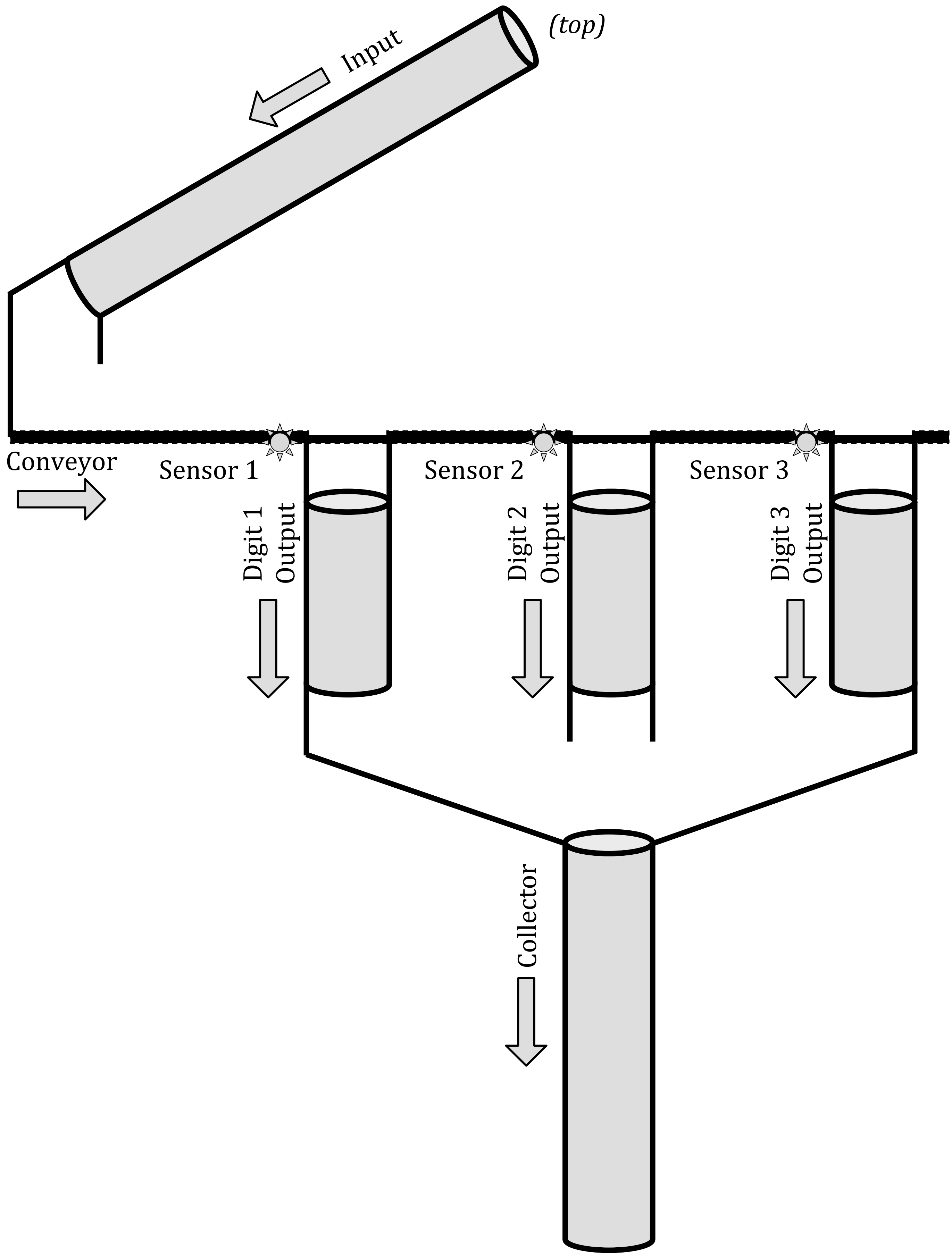
Collector result when sensing the **hundreds** digits

A diagram of the machine without any numbers is provided on the next page. You may find it helpful to keep track of the numbers as they pass through the machine each time.

More Info:

Check the CEMC at Home webpage on Wednesday, June 3 for a solution to Let's Sort All This Out.

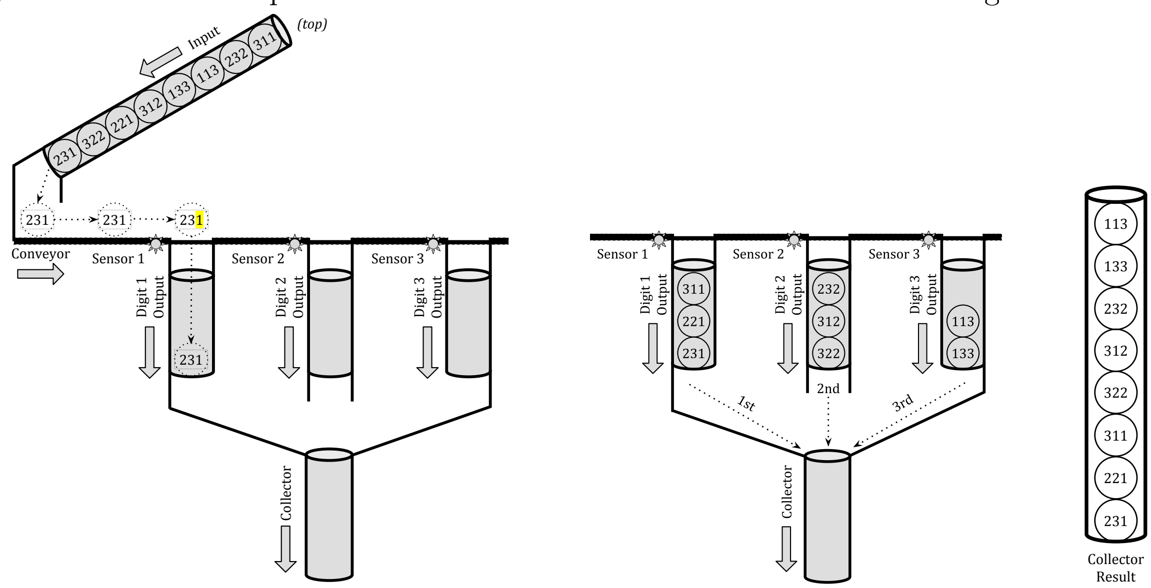
Blank Machine



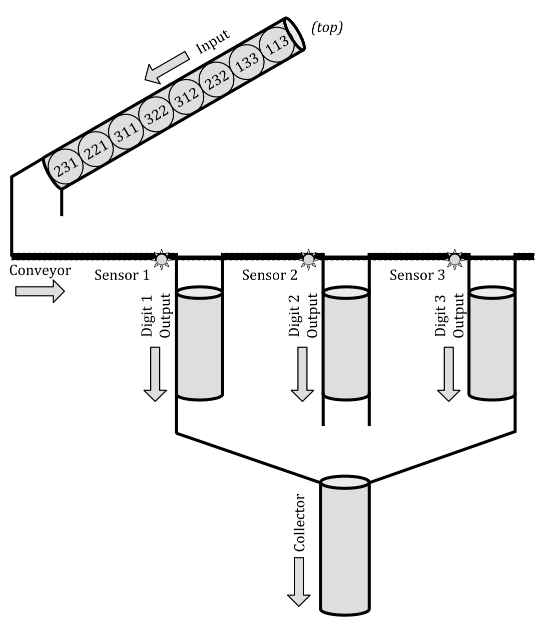
CEMC at Home

Grade 4/5/6 - Wednesday, May 27, 2020
Let's Sort All This Out - Solution

After the numbers pass through the machine the first time (with the sensors set to scan **ones** digits), the numbers end up in the order shown in the collector below on the right.



The numbers are now put back into the input tube of the machine in the order shown.
After the numbers pass through the machine the second time (with the sensors set this time to scan **tens** digits), the numbers end up in the order shown in the table.



(top)

133
232
231
322
221
113
312
311

(bottom)

After the numbers pass through the machine a third time (with the sensors set this time to scan **hundreds** digits), the numbers end up in the order shown in the collector below.

(top)

322
312
311
232
231
221
133
113

(bottom)

Notice that after the third pass through the machine, the numbers are sorted! The numbers in the collector tube are in order from smallest to largest, if you read them from bottom to top.

Computer Science Connections:

The CEMC Sorting Machine mimics a sorting technique known as *bucket sort*. This sorting technique works with any integers. To do a true bucket sort, you would need output tubes for all ten possible digits (0, 1, 2, 3, 4, 5, 6, 7, 8, and 9) rather than just for the digits 1, 2, and 3.

A big part of why the machine works is the way that numbers enter and exit the tubes. The first number that enters a tube (for example, the input tube) will be the first number to eventually leave that same tube. This idea is known as FIFO (First In, First Out). In Computer Science, we use a FIFO data structure called a *queue* which will hold a collection of values. A queue will manage that collection in a way that is similar to the way the tubes in this machine work.

CEMC at Home

Grade 4/5/6 - Thursday, May 28, 2020

Flower Powers

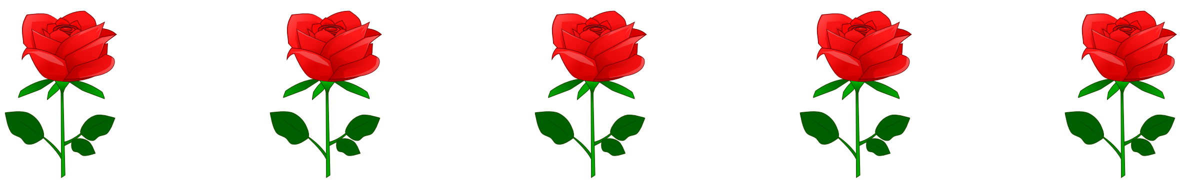
Mr. Digme planted five rose plants in a row along one side of his property.

He then planted one tulip plant in each of the spaces between the roses already in the row.

Next, he planted one daffodil plant in each of the spaces between the plants already in the row.

He then repeated this procedure with daisies, then marigolds, and finally with lilies.

Determine the total number of plants in the row.



To help you get started, think about how many spaces there are between the roses. After planting the tulips, how many roses and tulips would Mr. Digme have planted in total? How many spaces are there between the plants in the row now?
You may not be able to draw out all the plants in the end, but drawing the first few steps and looking for a pattern may be helpful.

More Info:

Check out the CEMC at Home webpage on Friday, May 29 for a solution to Flower Powers.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 4/5/6 - Thursday, May 28, 2020
Flower Powers - Solution

Problem:

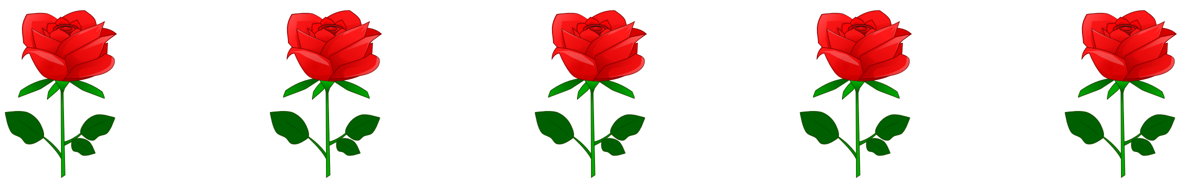
Mr. Digme planted five rose plants in a row along one side of his property.

He then planted one tulip plant in each of the spaces between the roses already in the row.

Next, he planted one daffodil plant in each of the spaces between the plants already in the row.

He then repeated this procedure with daisies, then marigolds, and finally with lilies.

Determine the total number of plants in the row.



Solution:

After planting 5 roses, there were four spaces between the plants. So, Mr. Digme then plants 4 tulips. At this point he has planted $5 + 4 = 9$ plants.

Since there are now 9 plants, there are 8 spaces between plants. So, Mr. Digme plants 8 daffodil plants. At this point he has planted $9 + 8 = 17$ plants.

Since there are now 17 plants, there are 16 spaces between plants. So, Mr. Digme plants 16 daisies. At this point he has planted $17 + 16 = 33$ plants.

Since there are now 33 plants, there are 32 spaces between plants. So, Mr. Digme plants 32 marigold plants. At this point he has planted $33 + 32 = 65$ plants.

Finally, since there are now 65 plants, there are 64 spaces between plants. So, Mr. Digme plants 64 lily plants. At this point he has planted $65 + 64 = 129$ plants.

Therefore, Mr. Digme planted a total of 129 plants in the row.

CEMC at Home

Grade 4/5/6 - Friday, May 29, 2020
The Leaky Tap

An experiment was performed in various households to determine the amount of water that is wasted by a single leaky tap. If you would like to perform this experiment on your own then the instructions are provided below. We have also provided the results of our own experiment for you to work with if you would prefer.

To Perform the Experiment You Will Need:

- A sink with a tap
- A watch or clock that measures seconds
- A clear metric measuring cup (to see the water level)
- A copy of Table 1 below

Instructions for the Experiment:

Turn on the tap, just a little, so that the water drips at a slow but steady rate.

The water should drip slowly enough that you can accurately count the drips, but not so slow that the drips are too irregular.

While the tap drips, complete the following two steps.

1. Count the number of drips in 20 seconds, and record this number in item 1. of Table 1.

   During our experiment, we observed 18 drips in 20 seconds. You can enter the value "18" or the value from your own experiment in the empty cell in the first row of the table.

2. Then place the measuring cup under the tap, and catch all the water that drips out during a 5 minute interval. Record this volume in item 3. of Table 1.

   During our experiment, we accumulated 65 mL of water in 5 minutes.

Table 1: Measurements

	Quantity	Measure	Units
1.	Drips in 20 seconds		drips
2.	Drips in one minute		drips/min
3.	Volume leaked in 5 minutes		mL
4.	Volume that would leak in 1 hour		mL/hr
5.	Volume that would leak in 1 day		L/day
6.	Volume that would leak in 1 year		L/year

After the Experiment:

Complete the table above by calculating the remaining items; use a calculator where needed.

For item 2., think about how many intervals of 20 seconds there are in one minute. For item 4., think about how many intervals of 5 minutes there are in 1 hour. Recall that 1 L is equal to 1000 mL.

See the next page for some questions to think about and an extra activity.

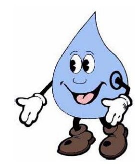
Questions:

1. On average, taking a bath uses about 160 L of water. If a tap leaked at the same rate as yours dripped (or the one from our experiment), about how many days would it take for the tap to leak this amount of water (160 L)?

2. Canada had about 10.2 million households as of 2019. If 1% of those households (about 102 000) had a tap that leaked at the same rate as yours dripped (or the one from our experiment), how many litres of water in total would be leaked in one year? How many inground swimming pools, with capacity about 76 000 L each, could be filled once per year with the water wasted by the leaking taps?

Activity:

Discover your family's weekly water consumption by completing the table below with the help of the members of your household.



My Family's Weekly Water Consumption

Activity	Average* Amount of Water Used (L)	Number per Week	Water Used (L)
Shower (10 minutes)	200 L (standard) 100 L (low flow)		
Tub Bath	160 L		
Washing Machine Load	110 L (top loading) 55 L (front loading)		
Dishwasher Load	20 L		
Cooking and Food Preparation	20 L per day		
Hygiene (teeth, hand washing, etc)	10 L/person/day		
Drinking water	2 L/person/day		

Total Weekly Average Water Used ______

* These averages were compiled by examining various web sources to determine suitable values.

Comment: While this may give an idea of your family's direct water use, the products and services we all enjoy (such as clothing, food, heating, etc.) involve many indirect uses of water which contribute greatly to the depletion of fresh water. You can learn more about your "water footprint" online.

More info:

Check out the CEMC at Home webpage on Friday, June 5 for a sample solution to The Leaky Tap.

CEMC at Home

Grade 4/5/6 - Friday, May 29, 2020
The Leaky Tap - Solution

Here are the results of one leaky tap experiment, and the follow up calculations.

	Quantity	Measure (with units)
1.	Drips in 20 seconds	18 drips
2.	Drips in 1 minute	54 drips/min
3.	Volume leaked in 5 minutes	65 mL
4.	Volume that would leak in 1 hour	780 mL/hr
5.	Volume that would leak in 1 day	18.72 L/day
6.	Volume that would leak in 1 year	6 832.8 L/year

1. In the experiment, 18 drips were observed in 20 seconds.
2. Since there were 18 drips in 20 seconds and one minute is $60 = 5 \times 20$ seconds, we would expect $5 \times 18 = 54$ drips in 1 minute.
3. In the experiment, 65 mL of water leaked in 5 minutes.
4. Since 65 mL of water leaked in 5 minutes, and 1 hour is $60 = 12 \times 5$ minutes, we would expect $12 \times 65 = 780$ mL of water to leak in 1 hour.
5. Since 780 mL of water would leak in 1 hour, and 1 day is 24 hours, we would expect $24 \times 780 = 18\,720$ mL of water to leak in 1 day. Since 1 L is 1000 mL, this means a rate of 18.72 L in 1 day.
6. Since 18.72 L of water would leak in 1 day, and 1 year has 365 days (as long as it is not a leap year), we would expect $365 \times 18.72 = 6\,832.8$ L of water to leak in 1 year.

Questions:

1. On average, taking a bath uses about 160 L of water. If a tap leaked at the same rate as the tap in the experiment, about how many days would it take for the tap to leak this amount of water?

   Solution: Since 18.72 L of water leak each day, the number of days it would take to leak 160 L is equal to $160 \div 18.72$. Since $160 \div 18.72$ is around 8.55, it would take about $8\frac{1}{2}$ days.

2. Canada had about 10.2 million households as of 2019. If 1% of those households (about 102 000) had a tap that leaked at the same rate as the tap in the experiment, how many litres of water in total would be leaked in one year?

   Solution: Since 6 832.8 L would be leaked from each of the 102 000 households, the total water leaked would be $6\,832.8 \times 102\,000 = 696\,945\,600$ L.

   Note: This wasted water could fill approximately $696\,945\,600 \div 76\,000 \approx 9\,170$ *inground pools (each with capacity about 76 000 L) once per year.*