CEMC at Home

Grade 11/12 - Monday, May 25, 2020
Contest Day 4

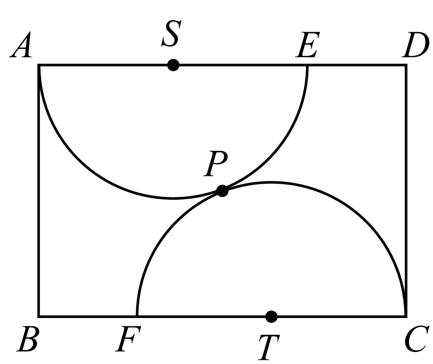
Today's resource features two questions from the recently released 2020 CEMC Mathematics Contests.

2020 Euclid Contest, #2(c)

Suppose that n is a positive integer and that the value of $\dfrac{n^2+n+15}{n}$ is an integer. Determine all possible values of n.

2020 Euclid Contest, #6(a)

Rectangle $ABCD$ has $AB = 4$ and $BC = 6$. The semi-circles with diameters AE and FC each have radius r, have centres S and T, and touch at a single point P, as shown. What is the value of r?



More Info:

Check out the CEMC at Home webpage on Monday, June 1 for solutions to the Contest Day 4 problems.

CEMC at Home

Grade 11/12 - Monday, May 25, 2020
Contest Day 4 - Solution

Solutions to the two contest problems are provided below, including a video for the first problem.

2020 Euclid Contest, #2(c)

Suppose that n is a positive integer and that the value of $\dfrac{n^2+n+15}{n}$ is an integer. Determine all possible values of n.

Solution:

First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.

This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n+1+\dfrac{15}{n}$ is an integer.

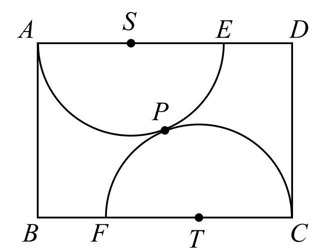
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.

The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.

Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

Video
Visit the following link for a discussion of a solution to the first contest problem:
https://youtu.be/MDV_HBu3-v4.

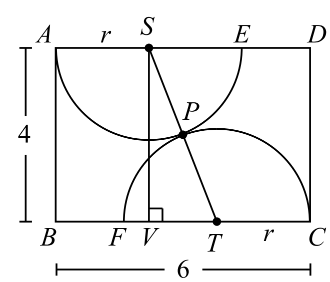
2020 Euclid Contest, #6(a)



Rectangle $ABCD$ has $AB = 4$ and $BC = 6$. The semi-circles with diameters AE and FC each have radius r, have centres S and T, and touch at a single point P, as shown. What is the value of r?

Solution:

Draw a perpendicular from S to V on BC. Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle. Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.



Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.

Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$. Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$. By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

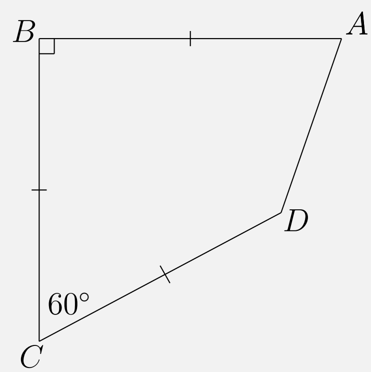
CEMC at Home

Grade 11/12 - Tuesday, May 26, 2020
Geometric Constructions

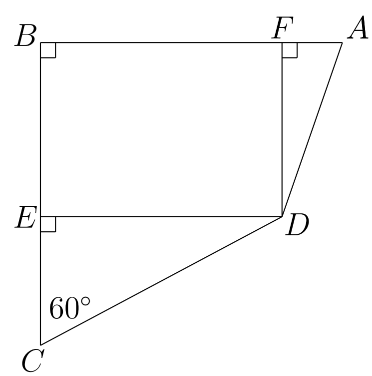
Sometimes when solving problems involving geometry, you can find a solution more easily by drawing something (eg. a point, a line, or an arc) that was not given in the original diagram. This is known as a *construction*. For example, consider the following problem.

Example

In the diagram, $ABCD$ is a quadrilateral with $AB = BC = CD = 6$, $\angle ABC = 90^\circ$, and $\angle BCD = 60^\circ$. Determine the length of AD.



One way to solve this problem is to start by dropping a perpendicular from D to E on BC and a perpendicular from D to F on BA. This gives us the diagram shown.



We start with $\triangle EDC$. We have that $ED = 6\sin 60^\circ = 3\sqrt{3}$ and $EC = 6\cos 60^\circ = 3$.

Consider rectangle $BFDE$. We have that $BE = BC - EC = 3$. Since $BFDE$ is a rectangle, we have that $BF = ED = 3\sqrt{3}$ and $FD = BE = 3$. Also, $FA = BA - BF = 6 - 3\sqrt{3}$.

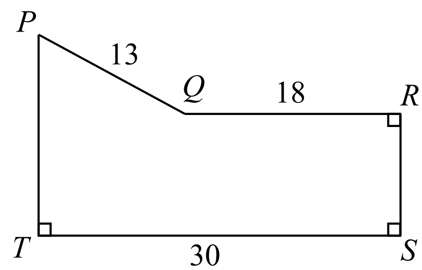
Finally, consider $\triangle FAD$. By the Pythagorean Theorem, we have that $AD^2 = FA^2 + FD^2 = (6 - 3\sqrt{3})^2 + 3^2 = 36(2 - \sqrt{3})$. Since $AD > 0$ we have $AD = 6\sqrt{2 - \sqrt{3}}$.

Notice how drawing in ED and FD made the problem more manageable by breaking it into smaller parts that were easier to work with.

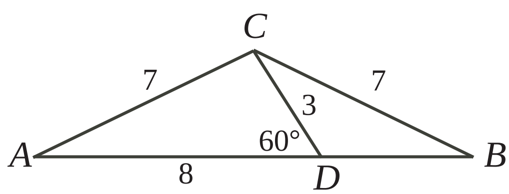
Each of the problems on the following page can be solved using a construction.

Problems

1. In the diagram, pentagon $PQRST$ has $PQ = 13$, $QR = 18$, $ST = 30$, and a perimeter of 82. Also, $\angle QRS = \angle RST = \angle STP = 90°$. Determine the area of pentagon $PQRST$.

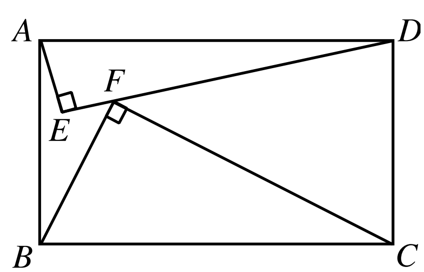


2. In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60°$, $AD = 8$, and $CD = 3$. Determine the length of BD.



Note: This problem can be solved without a construction (e.g. using the cosine law) but the solution becomes much simpler if you can find the right line to draw in the diagram!

3. In the diagram, right-angled triangles $\triangle AED$ and $\triangle BFC$ are constructed inside rectangle $ABCD$ so that F lies on DE. If $AE = 21$, $ED = 72$, and $BF = 45$, what is the length of AB?



More Info:

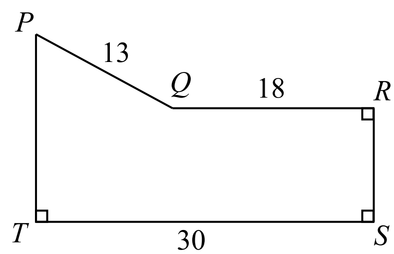
Check out the CEMC at Home webpage on Tuesday, June 2 for a solution to Geometric Constructions.

For more geometry problems check out this geometry unit in the CEMC Courseware.

CEMC at Home

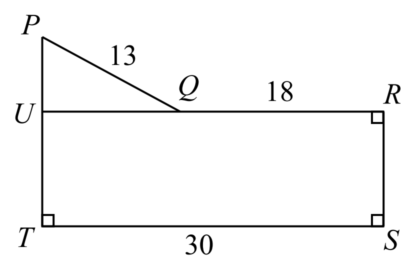
Grade 11/12 - Tuesday May 26, 2020
Geometric Constructions - Solution

1. In the diagram, pentagon $PQRST$ has $PQ = 13$, $QR = 18$, $ST = 30$, and a perimeter of 82. Also, $\angle QRS = \angle RST = \angle STP = 90^\circ$. Determine the area of pentagon $PQRST$.



Solution:

We extend RQ to the left until it meets PT at point U, as shown. Because quadrilateral $URST$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $UT = RS$ and $UR = TS = 30$.
Since $UR = 30$, then $UQ = UR - QR = 30 - 18 = 12$.
Now $\triangle PQU$ is right-angled at U.
By the Pythagorean Theorem, since $PU > 0$, we have

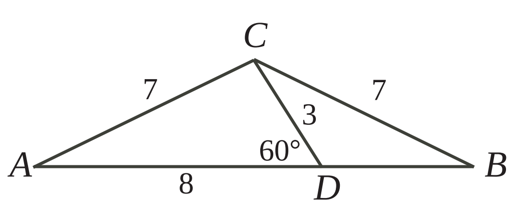


$$PU = \sqrt{PQ^2 - UQ^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$$

Since the perimeter of $PQRST$ is 82, then $13 + 18 + RS + 30 + (UT + 5) = 82$.
Since $RS = UT$, then $2 \times RS = 82 - 13 - 18 - 30 - 5 = 16$ and so $RS = 8$.
Finally, we can calculate the area of $PQRST$ by splitting it into $\triangle PQU$ and rectangle $URST$.
The area of $\triangle PQU$ is $\frac{1}{2} \times UQ \times PU = \frac{1}{2} \times 12 \times 5 = 30$.

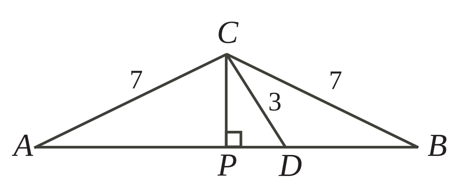
The area of rectangle $URST$ is $RS \times TS = 8 \times 30 = 240$.
Therefore, the area of pentagon $PQRST$ is $30 + 240 = 270$.

2. In the diagram, $\triangle ABC$ is isosceles with $AC = BC = 7$. Point D is on AB with $\angle CDA = 60^\circ$, $AD = 8$, and $CD = 3$. Determine the length of BD.



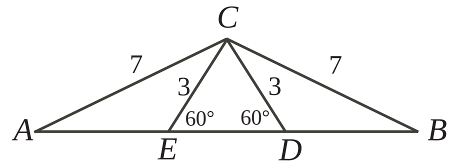
Solution 1:

Drop a perpendicular from C to P on AD. Since $\triangle ACB$ is isosceles, then $AP = PB$.
Since $\triangle CDP$ is a 30°-60°-90° triangle, then $PD = \frac{1}{2}(CD) = \frac{3}{2}$.
Thus, $AP = AD - PD = 8 - \frac{3}{2} = \frac{13}{2}$.
This tells us that $DB = PB - PD = AP - PD = \frac{13}{2} - \frac{3}{2} = 5$.



Solution 2:

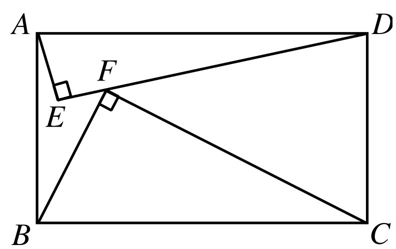
Since $\triangle ACB$ is symmetric about the vertical line through C, we can reflect CD in this vertical line, finding point E on AD with $CE = 3$ and $\angle CED = 60^\circ$.



Then $\triangle CDE$ has two 60° angles, so must have a third, and so is equilateral.

Therefore, $ED = CD = CE = 3$ and so $DB = AE = AD - ED = 8 - 3 = 5$.

3. In the diagram, right-angled triangles $\triangle AED$ and $\triangle BFC$ are constructed inside rectangle $ABCD$ so that F lies on DE. If $AE = 21$, $ED = 72$, and $BF = 45$, what is the length of AB?

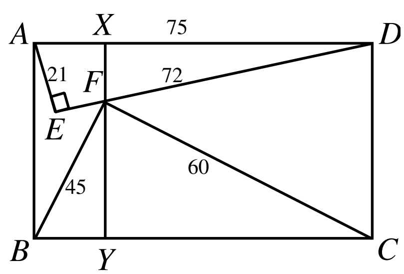


Solution:

By the Pythagorean Theorem in $\triangle AED$, $AD^2 = AE^2 + ED^2 = 21^2 + 72^2 = 5625$, so $AD = 75$. Since $ABCD$ is a rectangle, $BC = AD = 75$. Also, by the Pythagorean Theorem in $\triangle BFC$, $FC^2 = BC^2 - BF^2 = 75^2 - 45^2 = 3600$, so $FC = 60$.

Draw a line through F parallel to AB, meeting AD at X and BC at Y.

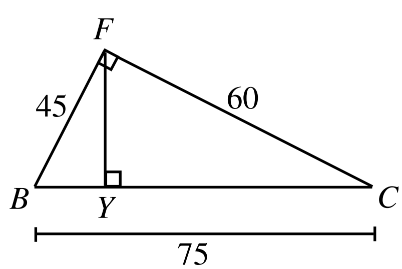
To determine the length of AB, we can find the lengths of FY and FX.



Step 1: Calculate the length of FY

One way to do this is to calculate the area of $\triangle BFC$ in two different ways.

We know that $\triangle BFC$ is right-angled at F, so its area is equal to $\frac{1}{2}(BF)(FC)$ or $\frac{1}{2}(45)(60) = 1350$. Also, we can think of FY as the height of $\triangle BFC$, so its area is equal to $\frac{1}{2}(FY)(BC)$ or $\frac{1}{2}(FY)(75)$.



Therefore, $\frac{1}{2}(FY)(75) = 1350$, so $FY = 36$.

Step 2: Calculate the length of FX

Since $FY = 36$, then by the Pythagorean Theorem,

$$BY^2 = BF^2 - FY^2 = 45^2 - 36^2 = 729$$

so $BY = 27$.
Thus, $YC = BC - BY = 48$.
Since $\triangle AED$ and $\triangle FXD$ are right-angled at E and X respectively and share a common angle D, then they are similar.
Since $YC = 48$, then $XD = 48$.
Since $\triangle AED$ and $\triangle FXD$ are similar, then $\dfrac{FX}{XD} = \dfrac{AE}{ED}$ or $\dfrac{FX}{48} = \dfrac{21}{72}$ so $FX = 14$.

Therefore, $AB = XY = FX + FY = 36 + 14 = 50$.

CEMC at Home

Grade 11/12 - Wednesday, May 27, 2020
Triangle of Integers

Here we revisit the idea of using computer science to solve previous CEMC math contest problems.

First, read Question 21 from the 2011 Fermat Contest and do your best to determine the correct answer. Read the solution to this question, and then answer the following related questions.

Problem Set 1

Suppose integers are arranged in a triangle as described in Question 21 of the 2011 Fermat Contest. Further, we will number the rows from top to bottom starting from 1 at the top.

1. What is the number of the row that contains the number 2020?
2. What is the sum of the numbers in the row immediately below the row that contains the number 500?

We will now look at a Python computer program that has been built to help you test your solutions to the questions above, and also to solve further related questions.

Here are instructions for using the program:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open the free CS Circles console in another tab.
3. Copy the code and paste it into the console of the interpreter.
4. Hit *Run program*.
5. You should see that the program outputs the correct answer to Question 21 of the 2011 Fermat Contest.

Try to understand how the program computes and displays the correct answer to the question. It is okay if you are new to computer programming or the language Python! The code is also included below for convenience and the notes below outline some of the details.

Python program

```
num = 400

# set n to be the number of the row containing num
n = 1
while n*(n+1) <= 2*num:
    n = n + 1

# compute the last numbers in rows n-1 and n
a = ((n-1)*n) // 2
b = (n*(n+1)) // 2

# add up the numbers in row n and display this sum
total = 0
for i in range(a+1,b+1):
    total = total + i
print(total)
```

When a Python program is run, anything beginning with a `#` character through to the end of the line is a *comment* which is ignored when the program is run.

When the `while` *loop* is reached, Python tests if the *condition* `n*(n+1) <= 2*num` is true. If it is true, the indented line `n = n + 1` is executed. This process is repeated until a test determines that the condition is false in which case the program continues after the indented line.

The line `n = n + 1` increases the value of the variable `n` by 1. For example, if the value of `n` is 4 when this line is reached, then after this line is executed, the value of `n` becomes $4 + 1 = 5$.

The two characters `//` form an *operator* used to divide two integers.

Do you see where the formulas for a and b above come from? See the solution to the contest problem.

Revisiting Problem Set 1

Let's revisit the questions on the previous page, and try to answer them using elements of our Python program. Suppose integers are arranged in a triangle as described in Question 21 of the 2011 Fermat Contest. Further, we will number the rows from top to bottom starting from 1 at the top.

1. What is the number of the row that contains the number 2020?

   To answer this question using the Python program, change the line `num = 400` to `num = 2020`, remove the lines after the while loop and add the line `print(n)` at the end. Your code should now look like the code in Program 1 below. Run the program. Does the answer given by this program agree with the answer you had calculated? Can you see why this program produces the desired output?

2. What is the sum of the numbers in the row immediately below the row that contains the number 500?

   Use Program 2 below to answer this question. Can you see what changes have been made to the original Python program to obtain Program 2? Run the program. Does the answer given by this program agree with the answer you had calculated? Can you see why this program produces the desired output?

Program 1

```
num = 2020
n = 1
while n*(n+1) <= 2*num:
    n = n + 1
print(n)
```

Program 2

```
num = 500
n = 1
while n*(n+1) <= 2*num:
    n = n + 1
n = n + 1
a = ((n-1)*n) // 2
b = (n*(n+1)) // 2
total = 0
for i in range(a+1,b+1):
    total = total + i
print(total)
```

Problem Set 2

Now, it's your turn! A solution to each of the following problems can be found that uses elements of the given Python program. Modify the given program, or use parts of it, to answer the questions below about the integers arranged in a triangle as described in the Fermat Contest question. Of course, you should feel free to use any other features that you know or learn about. The correct answers are provided so you can test your programs.

Of course, you can try to solve these problems by hand as well!

1. The sum of the numbers of a row is 34481. What is the number of this row?
   Answer: 41

2. For how many rows is the sum of the numbers in the row between 50000 and 90000?
   Answer: 10

More Info:

Check out the CEMC at Home webpage on Wednesday, June 3 for a solution to Triangle of Integers.

CEMC at Home

Grade 11/12 - Wednesday, May 27, 2020
Triangle of Integers - Solution

Recall that the questions in this resource are based around Question 21 from the 2011 Fermat Contest. Please remind yourself of the question and the solution.

Problem Set 1

1. What is the number of the row that contains the number 2020? (*Answer: 64*)

   Solution: After n rows, the total number of integers appearing in the pattern is

   $$1 + 2 + \cdots + +(n-1) + n = \tfrac{1}{2}n(n+1)$$

   In other words, the largest number in the nth row is $\frac{1}{2}n(n+1)$. To determine the row the number 2020 is in, we want to determine the smallest value of n such that $\frac{1}{2}n(n+1) \geq 2020$. If $n = 63$ then $\frac{1}{2}n(n+1) = 2016$ and if $n = 64$ then $\frac{1}{2}n(n+1) = 2080$. Therefore, 2020 appears in the 64th row.

2. What is the sum of the numbers in the row immediately below the row that contains the number 500? (*Answer: 17 985*)

   Solution: When $n = 31$ we have $\frac{1}{2}n(n+1) = 496$ and when $n = 32$ we have $\frac{1}{2}n(n+1) = 528$. This means that the number 500 must be in the 32nd row, and so we want the sum of the numbers in 33rd row. By the computation above, the largest number in the 32nd row is 528, and so the smallest number in the 33rd row is 529. Also, when $n = 33$ we have $\frac{1}{2}n(n+1) = 561$, which means that the largest number in the 33rd row is 561. Therefore, the 33rd row consists of the integers from 529 to 561. We calculate the sum of these numbers as follows:

   $$\begin{aligned}
   &529 + 530 + \cdots + 560 + 561 \\
   &= (1 + 2 + \cdots + 528 + 529 + 530 + \cdots + 560 + 561) - (1 + 2 + \cdots + 528) \\
   &= \tfrac{1}{2}(561)(562) - \tfrac{1}{2}(528)(529) \\
   &= 157\,641 - 139\,656 \\
   &= 17\,985
   \end{aligned}$$

Problem Set 2

On the next page you will find programs that compute and display the correct answers for the remaining two problems from the resource. Note that there are many different ways of using Python (or any other programming language) in each case. We have selected programs that are very similar to the program shown on the right that answers Question 21 from the 2011 Fermat Contest and was provided with the resource.

Original Python Program

```
num = 400
n = 1
while n*(n+1) <= 2*num:
    n = n + 1
a = ((n-1)*n) // 2
b = (n*(n+1)) // 2
total = 0
for i in range(a+1,b+1):
    total = total + i
print(total)
```

1. The sum of the numbers of a row is 34481. What is the number of this row?

```
n = 1
target = 34481
total = 0
while total < target:
    a = ((n-1)*n) // 2
    b = (n*(n+1)) // 2
    total = 0
    for i in range(a+1,b+1):
        total = total + i
    n = n + 1
print(n-1)
```

Answer: 41

2. For how many rows is the sum of the numbers in the row between 50000 and 90000?

```
n = 1
low = 50000
high = 90000
total = 0
while total < low:
    a = ((n-1)*n) // 2
    b = (n*(n+1)) // 2
    total = 0
    for i in range(a+1,b+1):
        total = total + i
    n = n + 1
numrows = 1
while total < high:
    a = ((n-1)*n) // 2
    b = (n*(n+1)) // 2
    total = 0
    for i in range(a+1,b+1):
        total = total + i
    n = n + 1
    numrows = numrows + 1
print(numrows-1)
```

Answer: 10

CEMC at Home

Grade 11/12 - Thursday, May 28, 2020

So Many Unknowns

In a sequence of twelve numbers, each number after the first three is equal to the sum of the previous three numbers.

The 3$^{\text{rd}}$ number in the sequence is 6, the 6$^{\text{th}}$ number in the sequence is 11, and the 11$^{\text{th}}$ number in the sequence is 14.

Determine all twelve numbers in the sequence.

?	?	6	?	?	11	?	?	?	?	14	?

More Info:

Check out the CEMC at Home webpage on Friday, May 29 for a solution to So Many Unknowns.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Thursday, May 28, 2020
So Many Unknowns - Solution

Problem:

In a sequence of twelve numbers, each number after the first three is equal to the sum of the previous three numbers.

The 3rd number in the sequence is 6, the 6th number in the sequence is 11, and the 11th number in the sequence is 14.

Determine all twelve numbers in the sequence.

?	?	6	?	?	11	?	?	?	?	14	?

Solution:

Let a_1 be the first number in the sequence, a_2 be the second, a_4 be the fourth, and so on, until a_{12} which is the 12th number in the sequence. The twelve boxes are labelled in the following diagram.

a_1	a_2	6	a_4	a_5	11	a_7	a_8	a_9	a_{10}	14	a_{12}

Each number after the third number is equal to the sum of the previous three numbers. Therefore, looking at the 6th term, we have $11 = 6 + a_4 + a_5$, or $a_4 + a_5 = 5$.

Looking at the 7th term, $a_7 = a_4 + a_5 + 11 = 5 + 11 = 16$, since $a_4 + a_5 = 5$.

a_1	a_2	6	a_4	a_5	11	16	a_8	a_9	a_{10}	14	a_{12}

Looking at the 9th term, $a_9 = 11 + 16 + a_8 = 27 + a_8$.
Looking at the 10th term, $a_{10} = 16 + a_8 + a_9 = 16 + (a_8) + (a_8 + 27) = 2a_8 + 43$.
Looking at the 11th term, $a_{11} = a_8 + a_9 + a_{10} = (a_8) + (a_8 + 27) + (2a_8 + 43) = 4a_8 + 70$.

We are given that the 11th term is $a_{11} = 14$.
Therefore, $4a_8 + 70 = 14$, or $4a_8 = -56$, or $a_8 = -14$.

Therefore, $a_9 = a_8 + 27 = -14 + 27 = 13$ and $a_{10} = 2a_8 + 43 = 2(-14) + 43 = 15$.

Also, $a_{12} = a_9 + a_{10} + a_{11} = 13 + 15 + 14 = 42$.

So far, we know the sequence is

a_1	a_2	6	a_4	a_5	11	16	-14	13	15	14	42

Working backwards, $-14 = a_5 + 11 + 16$, so $a_5 = -41$.

From earlier, $a_4 + a_5 = 5$ and $a_5 = -41$, so $a_4 = 46$.

Continuing backwards, $a_5 = a_2 + 6 + a_4$, so $-41 = a_2 + 6 + 46$, or $a_2 = -93$.

And finally, $a_4 = a_1 + a_2 + 6$, so $46 = a_1 + (-93) + 6$, or $a_1 = 133$.

Therefore, the sequence of twelve numbers is

133	-93	6	46	-41	11	16	-14	13	15	14	42

We can indeed check that in this sequence each number after the first three numbers is equal to the sum of the previous three numbers.

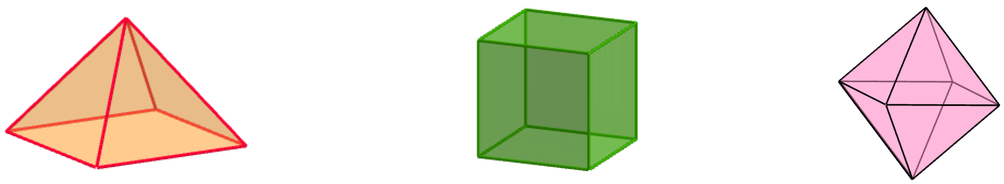
CEMC at Home

Grade 11/12 - Friday, May 29, 2020
Euler Characteristic - Part 2

Last Friday we introduced the Euler Characteristic of a polyhedron in Part 1. Given a polyhedron, we let V, E, and F denote the number of vertices, edges, and faces, respectively, of the polyhedron.

The **Euler characteristic** of a polyhedron, denoted χ, is defined to be the value of $V - E + F$. It turns out that $\chi = 2$ for every convex polyhedron.

The following are examples of *convex polyhedra*. We verified that each of these polyhedra have $\chi = 2$ directly.



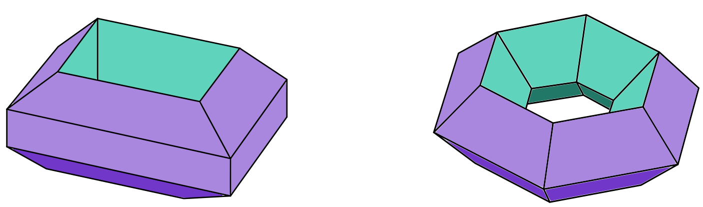
What makes a polyhedron convex?

A polyhedron is said to be **convex** if for any two points we choose on the polyhedron, the line segment joining these two points is always on or inside polyhedron.

Looking at the images of the pyramid, the cube, and the octahedron, convince yourself that they are indeed convex polyhedra.

Examples of Non-Convex Polyhedra

The following two figures show three-dimensional objects with polygons for their faces. These objects are polyhedra by definition; however, they are not convex polyhedra. One way to form a non-convex polyhedron is to include a "hole". Notice that if you pick two points that are on opposite sides of the hole, then at least some portion of the line segment joining the two points must lie outside of the polyhedron.



Polyhedron 1 **Polyhedron 2**

Assume that the "bottom" of each of the polyhedra is identical to the "top", and that the "back" is identical to the "front".

Example

Note that Polyhedron 1 has 4 "inner faces" and 12 "outer faces". We can verify that Polyhedron 1 has $V = 16$, $E = 32$, and $F = 16$. Therefore, we can calculate that $\chi = V - E + F = 16 - 32 + 16 = 0$.

Notice that $\chi = 0$ for this polyhedron, rather than $\chi = 2$ as we observed for the convex polyhedra.

Problem 1
Determine the value of χ for Polyhedron 2.

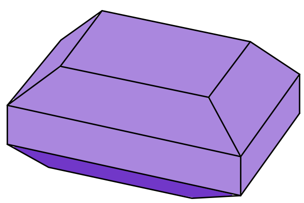
Solve this problem before moving on to the next page.

If you did the calculation correctly in Problem 1, you should have gotten an answer of $\chi = 0$, again! Is there something about these two polyhedra that make them have the same Euler characteristic, similar to how all convex polyhedra have $\chi = 2$?

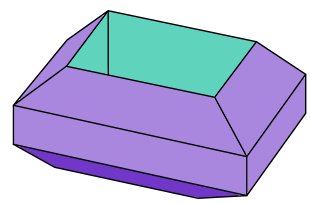
It turns out that a polyhedron with "one hole" in its interior always has $\chi = 0$. The number of holes a polyhedron has will affect its value of χ. (Note that if a polyhedron has a hole, then it cannot be convex, but there are non-convex polyhedra without holes. Try to visualize one and think about its value of χ.)

Investigation: *What happens when we take a polyhedron and "cut a hole"?*

Consider the convex polyhedron shown below on the left. We obtain the non-convex polyhedron shown below on the right by removing a vertical chunk of the convex polyhedron. (Notice that this is Polyhedron 1 from the first page.) Doing so may change the number of vertices, edges, and/or faces. Can we use this construction to help us understand why the Euler characteristic changes from $\chi_{old} = 2$ to $\chi_{new} = 0$ when a hole is introduced?



$\Longrightarrow$



Problem 2

Let $V_{old}, E_{old}, F_{old}$, and χ_{old} denote the vertices, edges, faces, and Euler characteristic for the polyhedron on the left above, and $V_{new}, E_{new}, F_{new}$, and χ_{new} denote these values for the polyhedron on the right above. Use the images above to determine the following values:

- $V_{new} - V_{old}$
- $E_{new} - E_{old}$
- $F_{new} - F_{old}$

Use this information to determine the relationship between χ_{new} and χ_{old}.

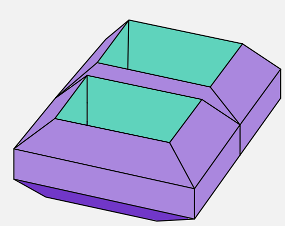
To get started, explain why $V_{new} - V_{old} = 0$. Next, do we "gain" or "lose" edges when we cut the hole?

Your work in Problem 2 should lead you to the fact that $\chi_{new} = \chi_{old} - 2$. This suggests that "adding a hole" may reduce the value of χ by 2. This would mean that if we started with a convex polyhedron, which has $\chi_{old} = 2$, and cut a hole in the interior, then we will be left with a non-convex polyhedron with $\chi_{new} = \chi_{old} - 2 = 2 - 2 = 0$. Of course, we already observed that $\chi = 0$ for Polyhedron 1 in the example on the first page.

What about polyhedra with more than one hole?

Problem 3

Consider the polyhedron shown to the right that has two holes. What is the value of χ for this polyhedron?



Assume that the "bottom" of this polyhedron is identical to the "top", and that the "back" is identical to the "front". If you imagine slicing this polyhedron down the centre, then you would essentially get two copies of Polyhedron 1 from the first page.

It turns out that if you know the number of holes that a polyhedron has, then you can easily calculate its Euler characteristic. In particular, if two polyhedra have the same number of holes, then they will have the same value of χ, regardless of what they look like. Proving this fact is challenging, but the formula for calculating χ based on the number of holes is simple!

We will make an attempt to outline the reasoning behind the fact above in the solutions for this resource, but the math involved in a formal proof is beyond the scope of this activity. Assuming the fact is true, try the following:

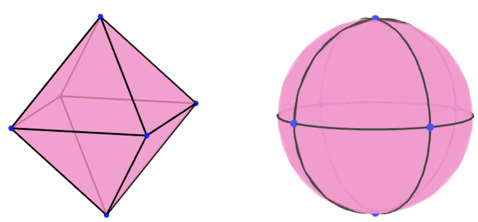
Extension: Suppose that a polyhedron has g holes, where g is a non-negative integer. Guess a formula for χ in terms of g. Your formula should give an answer of 2 when $g = 0$ and 0 when $g = 1$, and the value you calculated in Problem 3 when $g = 2$.

More Info: Check out the CEMC at Home webpage on Friday, June 5 for a solution to Part 2.

Further Discussion

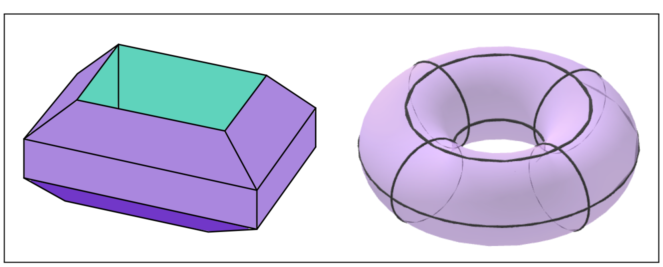
The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a convex polyhedron is always 2, and understand the formula for the Euler characteristic of a non-convex polyhedron with g holes.

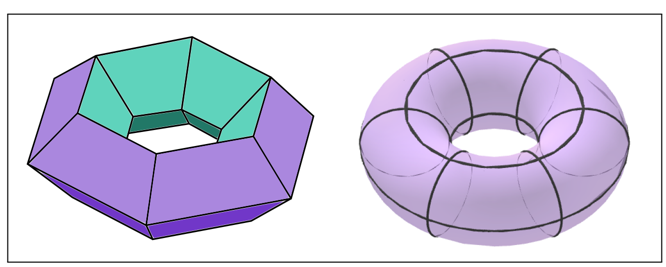
For example, if we "inflate" the octahedron, then we obtain a sphere as shown below. We visualize the vertices, edges, and faces of the octahedron on the spherical surface.



We call this result a *polygonization of a sphere*. Every convex polyhedron, when inflated, will produce such a polygonization. What happens when we "inflate" polyhedra with a hole?

We can imagine "inflating" Polyhedron 1 and Polyhedron 2 from the first page in a similar way to how we "inflated" the octahedron. The result is again a polygonization of a surface, but not of a sphere; these "inflate" to become a *polygonization of a torus*, which is a surface shaped like a donut as shown.





If you "inflated" the polyhedron with two holes from Problem 3, then you would get a tiling of a different surface. What would it look like? Can you describe it? If you have been to a water park that has inner tubes for more than one person, then you may already be familiar with the surface!

It turns out that every polyhedron that corresponds to a polygonization of a sphere (when "inflated") has Euler characteristic $\chi = 2$, and every polyhedron that corresponds to a polygonization of a torus (when "inflated") has Euler characteristic $\chi = 0$. There is a similar situation for g holes with $g \geq 2$.

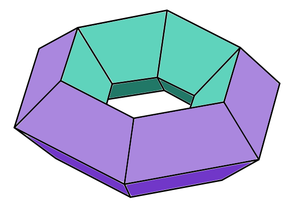
CEMC at Home

Grade 11/12 - Friday, May 29, 2020
Euler Characteristic - Part 2 - Solution

Problem 1

Determine the value of χ for Polyhedron 2.

Solution:



Polyhedron 2

The polyhedron has a total of 24 vertices: 6 vertices at the "top level", 12 vertices at the "middle level" (6 on the outer surface and 6 on the inner surface) and 6 vertices at the "bottom level".

The polyhedron has a total of 48 edges: 6 edges running horizontally at the "top level", 12 edges running horizontally at the "middle level" (6 on the outer surface and 6 on the inner surface), 6 edges running horizontally at the "bottom level", and 24 slanted edges (12 on the outer surface and 12 on the inner surface).

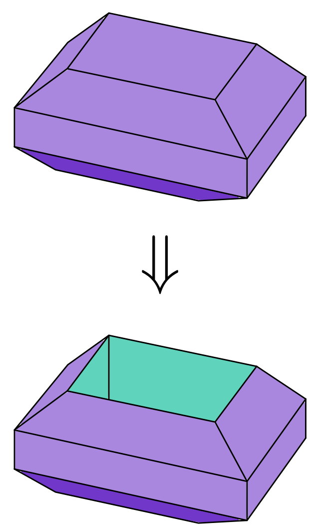
The polyhedron has a total of 24 faces: 6 top inner faces, 6 top outer faces, 6 bottom inner faces, and 6 bottom outer faces.

Therefore, $\chi = V - E + F = 24 - 48 + 24 = 0$.

Problem 2

Let $V_{old}, E_{old}, F_{old}$, and χ_{old} denote the vertices, edges, faces, and Euler characteristic for the convex polyhedron (top image), and $V_{new}, E_{new}, F_{new}$, and χ_{new} denote these values for the polyhedron after the vertical "hole" is introduced (bottom image). Use the images to determine the following values:

- $V_{new} - V_{old}$
- $E_{new} - E_{old}$
- $F_{new} - F_{old}$



Polyhedron 1

Use this information to determine the relationship between χ_{new} and χ_{old}.

Solution:

We could count all of the vertices, edges, and faces of each of the polyhedra, but instead we focus on how introducing the hole "changes" the values.

First, we observe that no vertices are gained or lost. Therefore, $V_{new} - V_{old} = 0$.

Second, we observe that 4 edges are gained: the 4 "vertical edges" of the inner surface of the object. Therefore, $E_{new} - E_{old} = 4$.

Finally, we observe that 2 faces are removed (the top and bottom rectangular faces) but 4 new faces are created (the 4 interior faces). This means a net gain of 2 faces and therefore, $F_{new} - F_{old} = 2$.

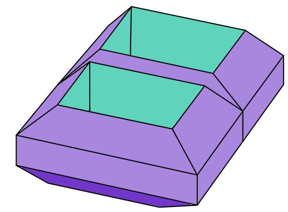
Using these four values, we determine the relationship between χ_{new} and χ_{old}:

$$\begin{aligned}
\chi_{new} - \chi_{old} &= (V_{new} - E_{new} + F_{new}) - (V_{old} - E_{old} + F_{old}) \\
&= (V_{new} - V_{old}) - (E_{new} - E_{old}) + (F_{new} - F_{old}) \\
&= 0 - 4 + 2 \\
&= -2
\end{aligned}$$

Therefore, introducing this hole results in $\chi_{new} = \chi_{old} - 2$.

Problem 3

Consider the polyhedron shown to the right that has two holes. What is the value of χ for this polyhedron?



Assume that the "bottom" of this polyhedron is identical to the "top", and that the "back" is identical to the "front". If you imagine slicing this polyhedron down the centre, then you would essentially get two copies of Polyhedron 1 from the first page.

Solution:

We could directly count up vertices, edges, and faces for this polyhedron, but we will instead use what we already know about Polyhedron 1.

Let V_{old}, E_{old}, and F_{old} denote the number of vertices, edges, and faces of Polyhedron 1. We can think of the polyhedron in this question as two copies of Polyhedron 1 "glued together" at rectangular faces on the side as shown in the image above.

Let V_{new}, E_{new}, and F_{new} denote the number of vertices, edges, and faces of the given polyhedron.

When we "glue" the two copies of Polyhedron 1 together, we lose two faces: the two rectangular faces on the sides where the polyhedra are "glued". Therefore, $F_{new} = 2F_{old} - 2$.

We also lose 4 edges. The two rectangular faces which are "glued" start with 8 edges in total, which is reduced to 4 once they are "glued" together. Therefore, $E_{new} = 2E_{old} - 4$.

Similarly, we have $V_{new} = 2V_{old} - 4$.

Note that we already know from earlier that $\chi_{old} = V_{old} - E_{old} + F_{old} = 0$. Therefore,

$$\begin{aligned}
\chi_{new} &= V_{new} - E_{new} + F_{new} \\
&= (2V_{old} - 4) - (2E_{old} - 4) + (2F_{old} - 2) \\
&= 2(V_{old} - E_{old} + F_{old}) - 4 + 4 - 2 \\
&= 2(0) - 4 + 4 - 2 \\
&= -2
\end{aligned}$$

Extension: Suppose that a polyhedron has g holes, where g is a non-negative integer. Guess a formula for χ in terms of g. Your formula should give an answer of 2 when $g = 0$ and 0 when $g = 1$, and the value you calculated in Problem 3 when $g = 2$.

Solution:

All polyhedra we have seen with 0 holes have $\chi = 2$, the two polyhedra we have seen with 1 hole have $\chi = 0$, and the polyhedron we just saw with 2 holes has $\chi = -2$. These findings suggest a possible pattern: every hole added reduces the Euler characteristic by 2. Note that our work in Problem 2 (where we showed $\chi_{new} = \chi_{old} - 2$) also provides evidence of this pattern.

If we guess that this pattern continues, then we would arrive at the following formula:

If a polyhedron has g holes, then $\chi = 2 - 2g$.

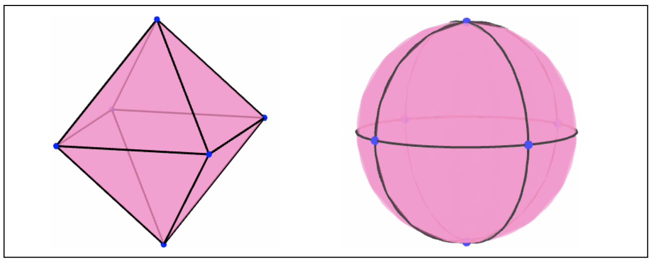
Fun fact: This formula is indeed true! Mathematicians often use the value of χ of a polyhedron to define the number of holes the polyhedron has. That is, we define a polyhedron to have $\frac{2-\chi}{2}$ holes.

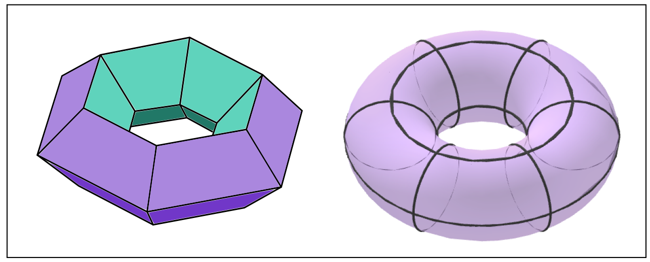
See the next page for an outline of an argument for why $\chi = 2$ for all polyhedra with 0 holes. (Since every convex polyhedron has 0 holes, in particular this argues that $\chi = 2$ for all convex polyhedra.)

Further Discussion

The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a polyhedron with 0 holes is always 2, and understand the formula for the Euler characteristic of a polyhedron with g holes.

Every polyhedron with 0 holes will result in a polygonization of a sphere when "inflated", and every polyhedron with 1 hole will result in a polygonization of a torus when "inflated".





What type of surface would you get if you "inflated" a polyhedron with g holes for some $g \geq 2$?

We can use the idea of a polygonization to relate the Euler characteristic of all polyehedra with the same number of holes. We outline this for the polyhedra with 0 holes below.

Altering a polygonization

We can make a small alteration to a polygonization in many ways. One way to do so is to "divide one face into two faces". This is an example of a *refinement* of a polygonization.

Start with a polygonization of a sphere as shown to the right.

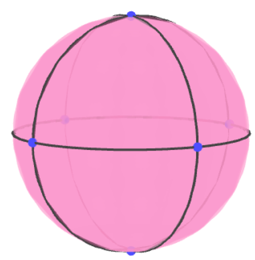
Recall that this polygonization arose from the octahedron.

To refine the polygonization, we can choose a "polygonal face", add 2 new vertices on 2 different edges of this face, and join the new vertices with a new edge.

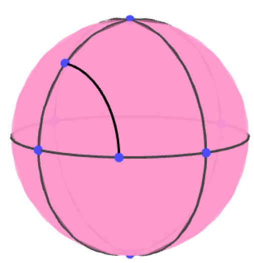
An example of such a refinement is shown below on the right.

What are the new values of V, E, and F for the polyhedron corresponding to this new polygonization? We can determine these values by looking at the vertices, edges, and faces of the polygonal faces on the surface of the sphere.

We can verify that this polygonization corresponds to a polyhedron with 8 vertices, 15 edges, and 9 faces and so we have $\chi = V - E + F = 8 - 15 + 9 = 2$ as expected! However, the most important thing to note here is how the values of V, E, and F *changed*.



$$V_{old} = 6$$
$$E_{old} = 12$$
$$F_{old} = 8$$



$$V_{new} = 6 + 2$$
$$E_{new} = 12 + 3$$
$$F_{new} = 8 + 1$$

When we change the polygonization as we did above, it is not hard to see that we gain 2 vertices, and gain 1 face during the process. What happens to the number of edges? We actually gain 3 edges overall. Can you see why? We "drew" 1 new edge, but in adding the 2 vertices, we "split" 2 edges and so we gained another 2 edges over all.

We could have added these two vertices in different places on the edges of our chosen face above, or added the vertices on a different face altogether, but the result will always be the same. How the values of V, E, and F change is summarized below. Notice that these changes have no effect on χ.

$$\begin{aligned} V_{new} &= V_{old} + 2 \quad \text{(gain 2 vertices)} \\ E_{new} &= E_{old} + 3 \quad \text{(gain 3 edges)} \\ F_{new} &= F_{old} + 1 \quad \text{(gain 1 face)} \end{aligned}$$

$$\begin{aligned} \chi_{new} &= V_{new} - E_{new} + F_{new} \\ &= (V_{old} + 2) - (E_{old} + 3) + (F_{old} + 1) \\ &= V_{old} - E_{old} + F_{old} \\ &= \chi_{old} \end{aligned}$$

Exploration:

See what happens if instead of adding two vertices to the polygonization as we did above, you make one of the following alterations.

- Choose a polygonal face, add 1 new vertex in the interior of this face, and draw new edges from this vertex so that you get another polygonization.
- Choose a vertex, remove this vertex and all edges that are attached to it.

Determine how performing each of these alterations will change the values of V, E, and F. Confirm that in each case you have $\chi_{new} = \chi_{old}$.

All polyhedra we have seen with 0 holes have $\chi = 2$, and changing their polygonizations in the ways described above does not affect the value of χ. Using this idea, it can be shown that every polyhedron with 0 holes has $\chi = 2$. Explaining exactly how this is done is beyond the scope of this activity, but we encourage you to think about this and research this topic further on your own if you are interested. The rough idea is that given any two polyhedra that both "inflate" into polygonizations of a sphere, you can always get from one polygonization to the other by doing the operations described above, none of which change the value of χ. This means they must have the same Euler characteristic, and we know this must be $\chi = 2$.

The argument is similar for the polyhedra that correspond to polygonizations of the torus. If we already know χ on one of them is 0 (for example Polyhedron 1) then we can deduce that any polyhedron that is inflated onto a torus must have $\chi = 0$ as well. If you would like, you can complete the "altering a polygonization" activity again for the given polygonization of the torus, and convince yourself that making each of these alterations leaves the value of χ unchanged.