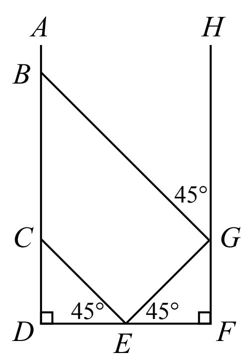
CEMC at Home

Grade 11/12 - Thursday, May 21, 2020
Contest Day 3

Today's resource features two questions from the recently released 2020 CEMC Mathematics Contests.

2020 Euclid Contest, #3(a)

Donna has a laser at C. She points the laser beam at the point E. The beam reflects off of DF at E and then off of FH at G, as shown, arriving at point B on AD. If $DE = EF = 1$ m, what is the length of BD, in metres?



2020 Euclid Contest, #5(b)

Determine all triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned} (x-1)(y-2) &= 0 \\ (x-3)(z+2) &= 0 \\ x + yz &= 9 \end{aligned}$$

More Info:

Check out the CEMC at Home webpage on Monday, May 25 for solutions to the Contest Day 3 problems.

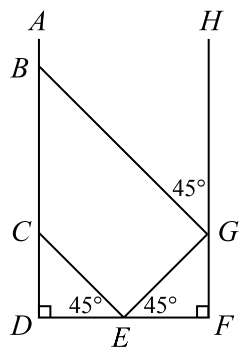
CEMC at Home

Grade 11/12 - Thursday, May 21, 2020
Contest Day 3 - Solution

Solutions to the two contest problems are provided below.

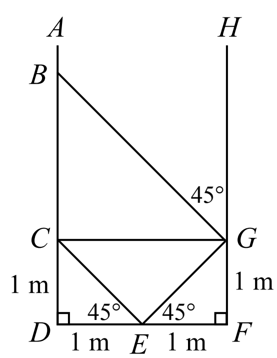
2020 Euclid Contest, #3(a)

Donna has a laser at C. She points the laser beam at the point E. The beam reflects off of DF at E and then off of FH at G, as shown, arriving at point B on AD. If $DE = EF = 1$ m, what is the length of BD, in metres?



Solution:

First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

See the next page for a solution to the second problem.

2020 Euclid Contest, #5(b)

Determine all triples (x, y, z) of real numbers that satisfy the following system of equations:

$$\begin{aligned}(x-1)(y-2) &= 0\\(x-3)(z+2) &= 0\\x+yz &= 9\end{aligned}$$

Solution:

Since $(x-1)(y-2)=0$, then $x=1$ or $y=2$.
Suppose that $x=1$. In this case, the remaining equations become:

$$\begin{aligned}(1-3)(z+2) &= 0\\1+yz &= 9\end{aligned}$$

or

$$\begin{aligned}-2(z+2) &= 0\\yz &= 8\end{aligned}$$

From the first of these equations, $z=-2$.
From the second of these equations, $y(-2)=8$ and so $y=-4$.
Therefore, if $x=1$, the only solution is $(x,y,z)=(1,-4,-2)$.
Suppose that $y=2$. In this case, the remaining equations become:

$$\begin{aligned}(x-3)(z+2) &= 0\\x+2z &= 9\end{aligned}$$

From the first equation $x=3$ or $z=-2$.
If $x=3$, then $3+2z=9$ and so $z=3$.
If $z=-2$, then $x+2(-2)=9$ and so $x=13$.
Therefore, if $y=2$, the solutions are $(x,y,z)=(3,2,3)$ and $(x,y,z)=(13,2,-2)$.
In summary, the solutions to the system of equations are

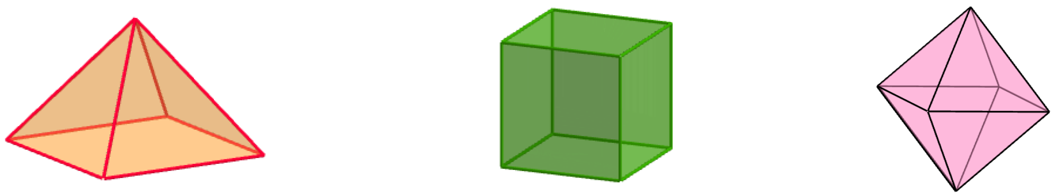
$$(x,y,z)=(1,-4,-2),(3,2,3),(13,2,-2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

CEMC at Home

Grade 11/12 - Friday, May 22, 2020
Euler Characteristic - Part 1

A *polyhedron* (plural: polyhedra or polyhedrons) is a three-dimensional solid with polygons for its faces. Below are three examples of polyhedra: a square-based pyramid, a cube, and an octahedron.



A polyhedron has vertices, edges, and faces. Given a polyhedron, we let V, E, and F denote the number of vertices, edges, and faces, respectively, of the polyhedron.

In this activity, we will explore the relationship between the values of V, E, and F.

Example
Verify that the square-based pyramid has 5 vertices, 8 edges, and 5 faces. If we calculate the value of $V - E + F$ then we get $5 - 8 + 5 = 2$.

Example
Verify that the cube has 8 vertices, 12 edges, and 6 faces. If we calculate the value of $V - E + F$ then we get $8 - 12 + 6 = 2$.

Question
What is the value of $V - E + F$ for the octahedron?
You should get an answer of 2, again! Confirm this for yourself.

It seems unlikely that it is just a coincidence that all three of these polyhedra produce the same value of $V - E + F$. Is there some reason to believe that this will always happen?

> The **Euler characteristic** of a polyhedron, denoted χ, is defined to be the value of $V - E + F$. It turns out that $\chi = 2$ for every (convex) polyhedron.

Explaining why the Euler characteristic is always 2 is challenging, and we will explore this idea a bit further in next week's activity. You can use this fact, when needed, to solve the following problems.

Problem 1
Verify directly that $\chi = 2$ for a tetrahedron, a dodecahedron, and an icosahedron.
You may need to look up one or two of these platonic solids first!

Problem 2
A particular polyhedron has 26 faces and has twice as many edges as vertices. How many edges must the polyhedron have?

Problem 3
An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

Problem 4

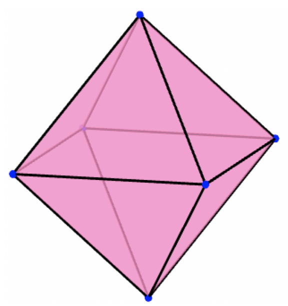
A polyhedron is formed with exactly P pentagons, exactly H hexagons, and no other polygons as its faces and has the property that three polygonal faces meet at each vertex.

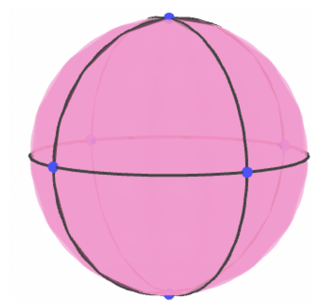
1. Explain why it is true that $E = \dfrac{5P + 6H}{2}$.
2. Explain why it is true that $V = \dfrac{5P + 6H}{3}$.
3. Using the fact that $\chi = 2$ for this polyhedron, show that it must be the case that $P = 12$.

A standard soccer ball can be thought of as an "inflation" of such a polyhedron.

Further Discussion

The idea of "inflating" a polyhedron can be used to help us understand why the Euler characteristic of a convex polyhedron is always 2. Imagine "inflating" a polyhedron, as if its surface is elastic like a balloon. For example, if we "inflate" the octahedron, then we obtain a sphere as shown below. We visualize the vertices, edges, and faces of the octahedron on the spherical surface.





Can you visualize what would happen if we "inflated" the pyramid or the cube in a similar way?

It can be helpful to have a common way in which to view all possible polyhedra; we can identify each polyhedron with how it looks after it is "inflated" to form a sphere. Notice that the resulting figure looks like a "tiling" of a spherical surface that uses polygon-like shapes as the tiles. (Of course, these shapes are not flat as they tile a curved surface.) For this reason, we call the result of "inflating" a polyhedron a *polygonization of a sphere*. This type of model gives us a nice way to compare polyhedra and their values of χ. We will revisit this idea in next Friday's activity.

Extra Problems to Think About

- Choose one face in the polygonization of a sphere shown above. Add two new vertices on two different edges of this face and join the new vertices with a new edge. How does this alteration affect the values of V, E, and F? What is the value of χ for this new polygonization?
- Choose one vertex in the polygonization of a sphere shown above. Remove this vertex and all of the edges that are directly connected to it. How does this alteration affect the values of V, E, and F? What is the value of χ for this new polygonization? (What is the polyhedron corresponding to this new polygonization? Think about "deflating the sphere".)

More Info: Check out the CEMC at Home webpage on Friday, May 29 for solutions to the problems in this activity and further discussion of polygonizations of surfaces and the Euler characteristic.

CEMC at Home

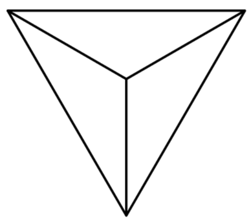
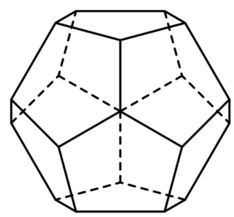
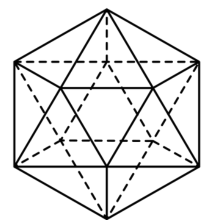
Grade 11/12 - Friday, May 22, 2020
Euler Characteristic - Part 1 - Solution

Problem 1

Verify directly that $\chi = 2$ for a tetrahedron, a dodecahedron, and an icosahedron.

Solution:

Here are diagrams outlining the shape of a tetrahedron, dodecahedron, and icosahedron. You can find many good images of these solids (three of the platonic solids) by searching online.

A tetrahedron has 4 triangular faces. We can count directly that $V = 4$, $F = 4$, and $E = 6$, and so $\chi = V - E + F = 4 - 6 + 4 = 2$.

A dodecahedron has 12 pentagonal faces and so we have $F = 12$. It is a bit harder to keep track of the vertices and edges as you count, and so instead we reason the numbers as follows:

There are 12 pentagons and each pentagon has 5 vertices. This means there are $12(5) = 60$ vertices among the faces. But 3 pentagons meet at every vertex and so the number 60 is triple counting the vertices. Therefore, $V = \frac{12(5)}{3} = 20$.

There are 12 pentagons and each pentagon has 5 sides. This means there are $12(5) = 60$ sides among the faces. But 2 pentagons meet at every edge of the polyhedron and so the number 60 is double counting the edges of the polyhedron. Therefore, $E = \frac{12(5)}{2} = 30$.

Putting it all together, we get $\chi = V - E + F = 20 - 30 + 12 = 2$, as desired.

Similar reasoning works for the icosahedron which has 20 triangular faces. We have $F = 20$, $V = \frac{20(3)}{5} = 12$ (since 5 faces meet at each vertex), and $E = \frac{20(3)}{2} = 30$ (since 2 faces meet at each edge). Therefore, $\chi = V - E + F = 12 - 30 + 20 = 2$.

One thing you may want to take away from this is that even though we had (2-dimensional) images of these tetrahedra in front of us, it was still easier to use combinatorics to determine E and V for the dodecahedron and icosahedron, rather than directly counting!

Problem 2

A particular polyhedron has 26 faces and has twice as many edges as vertices. How many edges must the polyhedron have?

Solution:

We are told that $F = 26$ and $E = 2V$. Using the fact that $\chi = 2$, we have that

$$2 = V - E + F = V - 2V + 26$$

which can be rearranged to give $V = 24$.

Problem 3

An Elongated Pentagonal Orthocupolarotunda is a polyhedron with exactly 37 faces, 15 of which are squares, 7 of which are regular pentagons, and 15 of which are triangles. How many vertices does it have?

Solution:

The given polyhedron has 37 faces, so $F = 37$.
The 15 square faces have $4(15) = 60$ sides in total, the 7 pentagonal faces have $5(7) = 35$ sides in total, and the 15 triangular faces have $3(15) = 45$ sides in total. Thus, the faces have a total of 140 sides.
Since these sides are paired up to form the edges of the polyhedron, we have $E = \frac{1}{2}(140) = 70$.
Since $V - E + F = 2$, we have $V = E - F + 2 = 70 - 37 + 2 = 35$, so the polyhedron has 35 vertices.

We have not included an image of this polyhedron here, but you can find an image on the internet if you are interested in seeing what it looks like.

Problem 4

A polyhedron is formed with exactly P pentagons, exactly H hexagons, and no other polygons as its faces and has the property that three polygonal faces meet at each vertex.

1. Explain why it is true that $E = \dfrac{5P + 6H}{2}$.

   Solution: There are P pentagons and H hexagons. Each pentagon has 5 sides and each hexagon has 6 sides. However, each edge is "shared" between two polygons. Therefore the total number of edges is $E = \dfrac{5P + 6H}{2}$.

2. Explain why it is true that $V = \dfrac{5P + 6H}{3}$.

   Solution: There are P pentagons and H hexagons. Each pentagon has 5 vertices and each hexagon has 6 vertices. However, each vertex is "shared" between three polygons (the statement of the question tells us this). Therefore the total number of vertices is $V = \dfrac{5P + 6H}{3}$.

3. Using the fact that $\chi = 2$ for this polyhedron, show that it must be the case that $P = 12$.

   Solution: We already have an expression for V and E. We also want an expression for the number of faces. Since P and H are, respectively, the number of pentagons and the number of hexagons, we have that $F = P + H$. Let's now calculate the Euler Characteristic of this polyhedron.

$$\begin{aligned} V - E + F &= \left(\frac{5P + 6H}{3}\right) - \left(\frac{5P + 6H}{2}\right) + (P + H) \\ &= \frac{(10P + 12H) - (15P + 18H) + (6P + 6H)}{6} \\ &= \frac{(10P - 15P + 6P) + (12H - 18H + 6H)}{6} \\ &= \frac{P}{6} \end{aligned}$$

   Since $\chi = V - E + F = 2$ we must have $\dfrac{P}{6} = 2$ and hence $P = 12$.