CEMC at Home

Grade 9/10 - Monday, May 11, 2020
Contest Day 2

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Team Problem #9

How many times does the digit 0 appear in the integer equal to 20^{10}?

2020 Canadian Team Mathematics Contest, Individual Problem #7

Twenty-seven unit cubes are each coloured completely black or completely red. The unit cubes are assembled into a larger cube. If $\frac{1}{3}$ of the surface area of the larger cube is red, what is the smallest number of unit cubes that could have been coloured red?

More Info:

Check out the CEMC at Home webpage on Thursday, May 21 for solutions to the Contest Day 2 problems.

CEMC at Home

Grade 9/10 - Monday, May 11, 2020
Contest Day 2 - Solution

Solutions to the two contest problems are provided below, including a video for the second problem.

2020 Canadian Team Mathematics Contest, Team Problem #9

How many times does the digit 0 appear in the integer equal to 20^{10}?

Solution:

By factoring and using exponent rules, we have $20^{10} = (2 \times 10)^{10} = 2^{10} \times 10^{10}$.
Therefore, $20^{10} = 1024 \times 10^{10}$, which is the integer 1024 followed by ten zeros.
Thus, 20^{10} has eleven digits that are 0. That is, 10 zeros at the end and one coming from the 1024 at the beginning.

2020 Canadian Team Mathematics Contest, Individual Problem #7

Twenty-seven unit cubes are each coloured completely black or completely red. The unit cubes are assembled into a larger cube. If $\frac{1}{3}$ of the surface area of the larger cube is red, what is the smallest number of unit cubes that could have been coloured red?

Solution:

Since $\sqrt[3]{27} = 3$, the dimensions of the larger cube must be $3 \times 3 \times 3$.
Therefore, each side of the larger cube has area $3 \times 3 = 9$.
A cube has 6 faces, so the total surface of the cube is made up of $9 \times 6 = 54$ of the 1 by 1 squares from the faces of the unit cubes.
Since $\frac{1}{3}$ of the surface area is red, this means $\frac{54}{3} = 18$ of these unit squares must be red.

The unit cube at the centre of the larger cube has none of its faces showing, the 6 unit cubes in the centres of the outer faces have exactly 1 face showing, the 12 unit cubes on the edge but not at a corner have 2 faces showing, one of each of two adjacent sides, and the 8 unit cubes at the corners each have 3 faces showing.
For any unit cube, there are either 0, 1, 2, or 3 of its faces showing on the surface of the larger cube. This means at most three faces of any unit cube are on the surface of the larger cube. Thus, there must be at least 6 cubes painted red in order to have 18 red unit squares on the surface of the larger cube.

There are 8 unit cubes on the corners, so if we colour exactly 6 unit cubes red and the other 21 black, then arrange the cubes into a $3 \times 3 \times 3$ cube so that the 6 red unit cubes are at the corners, there will be exactly 18 of the unit squares on the surface coloured red.
Therefore, the answer is 6.

Video

Visit the following link for an explanation of the solution to the second contest problem:
https://youtu.be/K9ax9uQESME

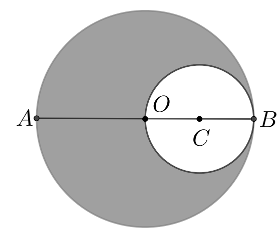
CEMC at Home

Grade 9/10 - Tuesday, May 12, 2020
Shady Circles

For each problem, use the information and diagram given to **find the area of the shaded region**. Express your answers as simplified exact numbers. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

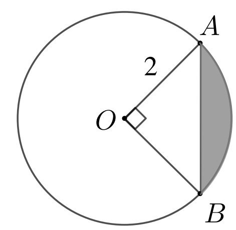
Problem 1

Two circles are centred at C and O as shown. AB is a diameter of the larger circle. OB is a diameter of the smaller circle. The larger circle has a diameter of 20.



Problem 2

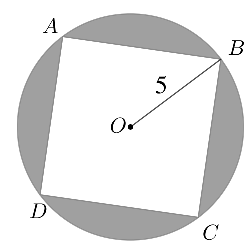
The circle with centre O has a radius of 2. Points A and B are on the circle and $\angle AOB = 90^\circ$ as shown.



Problem 3

Square $ABCD$ is inscribed in the circle with centre O and radius 5 as shown.

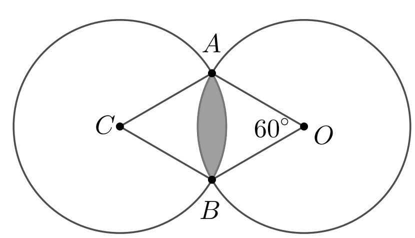
A square is inscribed in a circle if all four vertices of the square lie on the circle.



Problem 4

Two circles, each with a radius of 10, are centred at C and O as shown. The circles intersect at points A and B with $\angle AOB = 60^\circ$.

Can you visualize what this diagram would look like if you pulled the circles apart?



More Info:

Check out the CEMC at Home webpage on Tuesday, May 19 for a solution to Shady Circles.

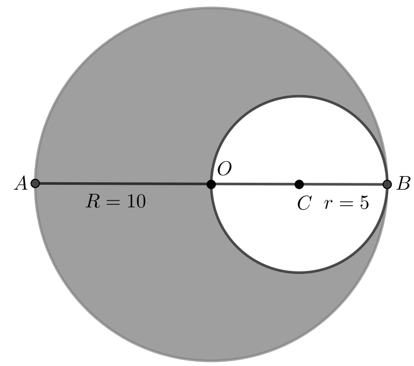
To review area calculations involving circles and triangles, visit the following videos in the CEMC courseware: area of triangles and area of circles.

CEMC at Home

Grade 9/10 - Tuesday, May 12, 2020
Shady Circles - Solution

Problem 1 Solution

We will find the area of the shaded region by subtracting the area of the smaller circle from the area of the larger circle.
Let R be the radius of the larger circle and r be the radius of the smaller circle.



Since the diameter of the smaller circle is the radius of the larger circle, we have $R = \frac{20}{2} = 10$ and $r = \frac{10}{2} = 5$.

$A_{\text{larger}} = \pi R^2 = \pi(10)^2 = 100\pi$
$A_{\text{smaller}} = \pi r^2 = \pi(5)^2 = 25\pi$

Therefore, the area of the shaded region is $100\pi - 25\pi = 75\pi$.

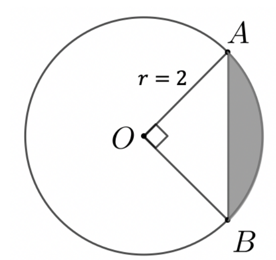
Problem 2 Solution

To find the area of the shaded region, we will find the area of the sector of the circle with arc AB and subtract the area of $\triangle AOB$. Note that the triangle is a right isosceles triangle and therefore, the base and height are both equal to the radius which is 2.

$A_{wholeCircle} = \pi r^2 = \pi(2)^2 = 4\pi$

$A_{sector} = \left(\dfrac{90}{360}\right) 4\pi = \pi$

$A_{triangle} = \dfrac{bh}{2} = \dfrac{(2)(2)}{2} = 2$



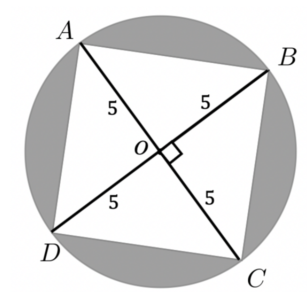
Therefore, the area of the shaded region is $\pi - 2$.

Problem 3 Solution

There are a few different ways to approach this problem. We will outline two approaches. Each of these approaches relies upon the following facts that we will not prove:

1) The two diagonals of the inscribed square intersect at the centre, O, of the circle.

2) The two diagonals of the inscribed square bisect each other and meet at right angles.



Approach 1: Recognize that the shaded region in this problem consists of four identical shaded regions, each having an area that can be calculated by subtracting the area of a triangle from the area of a sector of a the circle (as in Problem 2).

The final calculation is as follows: Area $= 4\left(\frac{\pi(5)^2}{4} - \frac{5^2}{2}\right) = 25\pi - 50$.

Approach 2: Recognize that the area of the shaded region is the area of the circle minus the area of the inscribed square.

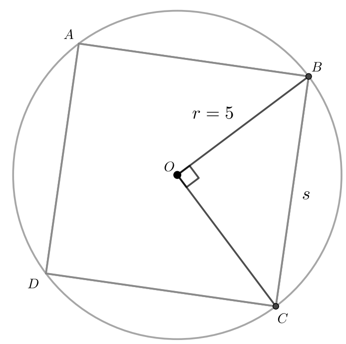
The area of the circle $\pi r^2 = \pi(5)^2 = 25\pi$.

Let s be the side length of the square as shown in the figures below.

Note that $\triangle BOC$ is a right isosceles triangle. Therefore, its base and height are both equal to the radius which is $r = 5$. We can calculate the value of s^2 as follows:

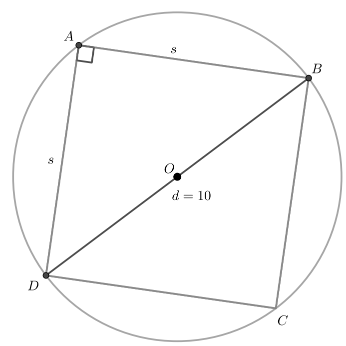
Using the Pythagorean Theorem on $\triangle BOC$, we get

$$\begin{aligned} s^2 &= r^2 + r^2 \\ s^2 &= (5)^2 + (5)^2 \\ s^2 &= 25 + 25 \\ s^2 &= 50 \end{aligned}$$



Alternatively, using the Pythagorean Theorem on $\triangle BAD$, with diameter $d = 10$, we get

$$\begin{aligned} d^2 &= s^2 + s^2 \\ 10^2 &= 2s^2 \\ 100 &= 2s^2 \\ 50 &= s^2 \end{aligned}$$

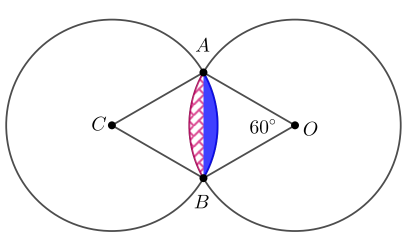


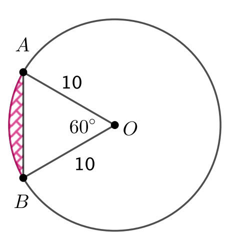
Each of these calculations tells us that the area of the square is $s^2 = 50$.

Therefore, the area of the shaded region is $25\pi - 50$.

Problem 4 Discussion

One way to find the area of the shaded region is to observe that it is made up of two identical regions as shown below. You can find the area of each of the regions using a similar method to that in the solution to Problem 2, although the area of the triangle will not be as easy to calculate in this case. We leave the details to you, but give the key values in the calculations here.





Area of triangle AOB is $\frac{1}{2}(10)(\sqrt{75})$

Area of sector AOB is $\frac{60}{360}(\pi(10)^2)$