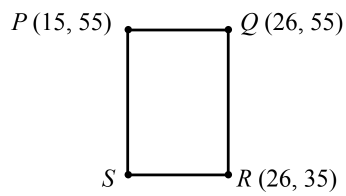
CEMC at Home

Grade 7/8 - Monday, May 11, 2020
Contest Day 2

Today's resource features two questions from the recently released 2020 CEMC Mathematics Contests.

2020 Gauss Contest, #13

Points $P(15, 55)$, $Q(26, 55)$ and $R(26, 35)$ are three vertices of rectangle $PQRS$. The area of this rectangle is



(A) 360 **(B)** 800 **(C)** 220

(D) 580 **(E)** 330

2020 Gauss Contest, #23

The list $11, 20, 31, 51, 82$ is an example of an increasing list of five positive integers in which the first and second integers add to the third, the second and third add to the fourth, and the third and fourth add to the fifth. How many such lists of five positive integers have 124 as the fifth integer?

(A) 10 **(B)** 7 **(C)** 9 **(D)** 6 **(E)** 8

More Info:

Check out the CEMC at Home webpage on Thursday, May 21 for solutions to the Contest Day 2 problems.

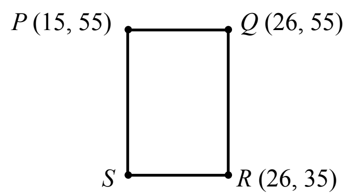
CEMC at Home

Grade 7/8 - Monday, May 11, 2020
Contest Day 2 - Solution

Solutions to the two contest problems are provided below, including a video for the second problem.

2020 Gauss Contest, #13

Points $P(15, 55)$, $Q(26, 55)$ and $R(26, 35)$ are three vertices of rectangle $PQRS$. The area of this rectangle is



(A) 360 **(B)** 800 **(C)** 220 **(D)** 580 **(E)** 330

Solution:

The y-coordinates of points $P(15, 55)$ and $Q(26, 55)$ are equal.
Therefore, the distance between P and Q is equal to the positive difference between their x-coordinates, or $26 - 15 = 11$.
Similary, the x-coordinates of points $R(26, 35)$ and $Q(26, 55)$ are equal.
Therefore, the distance between R and Q is equal to the positive difference between their y-coordinates, or $55 - 35 = 20$.
Since $PQ = 11$ and $RQ = 20$, the area of rectangle $PQRS$ is $11 \times 20 = 220$.

ANSWER: (C)

2020 Gauss Contest, #23

The list $11, 20, 31, 51, 82$ is an example of an increasing list of five positive integers in which the first and second integers add to the third, the second and third add to the fourth, and the third and fourth add to the fifth. How many such lists of five positive integers have 124 as the fifth integer?

(A) 10 **(B)** 7 **(C)** 9 **(D)** 6 **(E)** 8

Solution:

If the first positive integer in the list is a and the second is b, then the third integer is $a + b$, the fourth is $b + (a + b)$ or $a + 2b$, and the fifth is $(a + b) + (a + 2b)$ or $2a + 3b$.
Thus, we are asked to find the number of pairs of positive integers a and b, where a is less than b (since the list is increasing), and for which $2a + 3b = 124$.
What is the largest possible value for b?
If $b = 42$, then $3b = 3 \times 42 = 126$ which is too large since $2a + 3b = 124$. (Note that a larger value of b makes $3b$ even larger.)
If $b = 41$, then $3b = 3 \times 41 = 123$.
However in this case, we get that $2a = 124 - 123 = 1$, which is not possible since a is a positive integer.

Solution continued on the next page.

If $b = 40$, then $3b = 3 \times 40 = 120$ and so $2a = 4$ or $a = 2$.
Thus, the largest possible value for b is 40.
What is the smallest value for b?
If $b = 26$, then $3b = 3 \times 26 = 78$ and so $2a = 124 - 78 = 46$ or $a = 23$.
If $b = 25$, then $3b = 3 \times 25 = 75$.
However in this case, we get that $2a = 124 - 75 = 49$, which is not possible since a is a positive integer.
If $b = 24$, then $3b = 3 \times 24 = 72$ and so $2a = 124 - 72 = 52$ or $a = 26$.
However, if the first integer in the list is 26, then the second integer can not equal 24 since the list is increasing.
Smaller values of b will give larger values of a, and so the smallest possible value of b is 26.
From the values of b attempted thus far, we notice that when b is an odd integer, $3b$ is also odd (since the product of two odd integers is odd), and $124 - 3b$ is odd (since the difference between an even integer and an odd integer is odd).
So when b is odd, $124 - 3b$ is odd, and so $2a$ is odd (since $2a = 124 - 3b$).
However, $2a$ is even for every choice of the integer a and so b cannot be odd.
Conversely, when b is even, $124 - 3b$ is even (as required), and so all even integer values of b from 26 to 40 inclusive will satisfy the requirements.
These values of b are $26, 28, 30, 32, 34, 36, 38, 40$, and so there are 8 such lists of five integers that have 124 as the fifth integer.
Here are the 8 lists:

$2, 40, 42, 82, 124$; $5, 38, 43, 81, 124$; $8, 36, 44, 80, 124$; $11, 34, 45, 79, 124$;
$14, 32, 46, 78, 124$; $17, 30, 47, 77, 124$; $20, 28, 48, 76, 124$; $23, 26, 49, 75, 124$.

ANSWER: (E)

Video

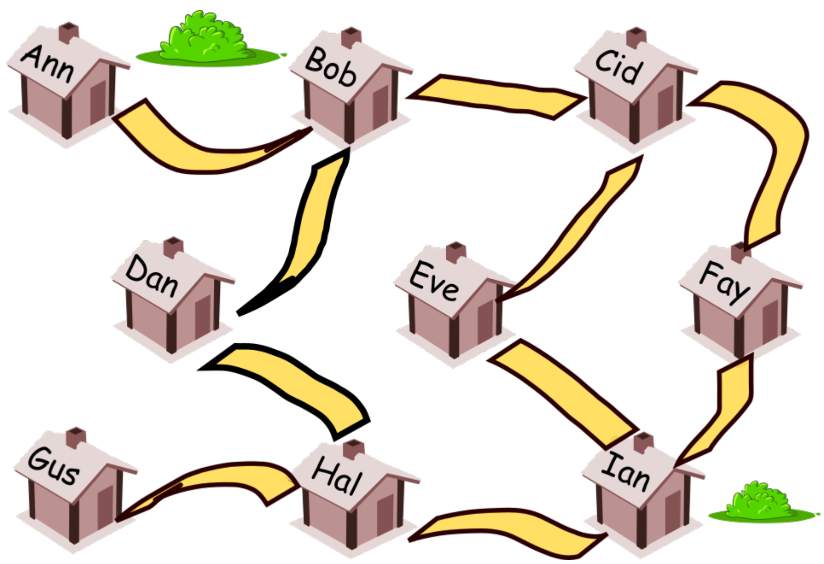
Visit the following link to view another solution to the second contest problem which makes use of a spreadsheet to solve the problem: https://youtu.be/RBM5Q88WvNk.

Don't worry if you have never used spreadsheets before. The video will walk you through the parts that you need to know.

CEMC at Home

Grade 7/8 - Tuesday, May 12, 2020
Volunteer Firefighters

The mayor of a small town is looking for volunteer firefighters. A map showing the possible volunteers' homes and how they are connected by roads is shown below. The mayor's goal is to ensure that every home in the town is either the home of a volunteer or is connected by a single road to the home of a volunteer.



Problem 1: The mayor's goal outlined above can be achieved with only 3 volunteer firefighters. Find a group of 3 volunteer firefighters that will achieve the mayor's goal.

Use the online exploration to test different combinations of firefighters.

Problem 2: Find all possible groups of 3 volunteer firefighters that will achieve the mayor's goal. Explain how you know you have found them all.

Problem 3: Explain why it is not possible for the mayor's goal to be achieved if there are only 2 volunteer firefighters.

Extension: A new road is built between two of the homes in the town and the mayor's goal can now be achieved with only 2 volunteer firefighters. Between which homes in the town could this road have been added?

This new road may be longer and take a less direct route than the other roads on the map.

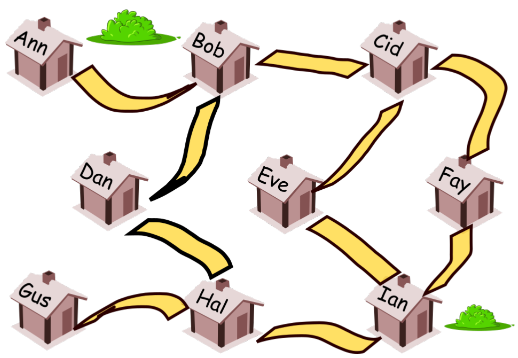
More Info:

Check the CEMC at Home webpage on Wednesday, May 13 for a solution to Volunteer Firefighters.

CEMC at Home

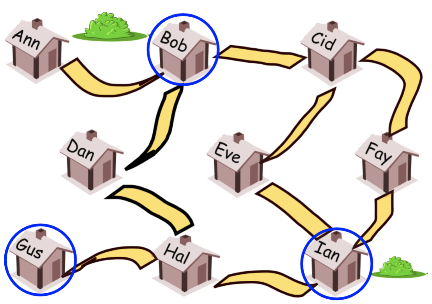
Grade 7/8 - Tuesday, May 12, 2020
Volunteer Firefighters - Solution

Set Up: The mayor's goal is to ensure that every home in the town is either the home of a volunteer or is connected by a single road to the home of a volunteer.



Problem 1: The mayor's goal outlined above can be achieved with only 3 volunteer firefighters. Find a group of 3 volunteer firefighters that will achieve the mayor's goal.

Solution: Here is one possible choice of the group of 3 firefighters.



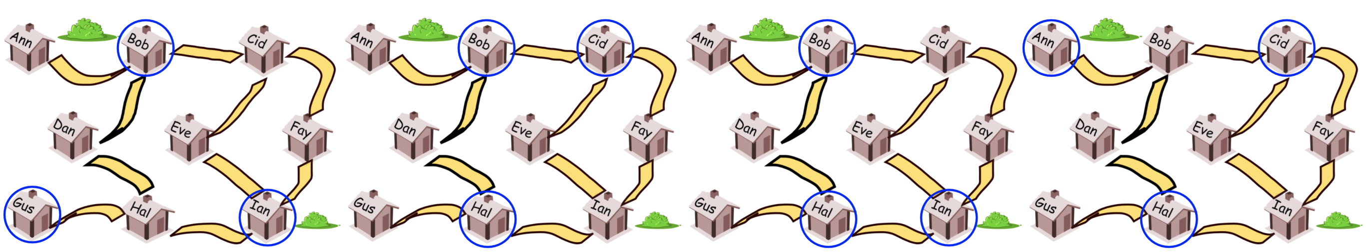
The mayor's goal is achieved with this group because

- Bob, Gus and Ian are firefighters,
- Ann, Dan and Cid are one road from Bob, and
- Eve, Fay and Hal are one road from Ian.

Problem 2: Find all possible groups of 3 volunteer firefighters that will achieve the mayor's goal. Explain how you know you have found them all.

Problem 3: Explain why it is not possible for the mayor's goal to be achieved if there are only 2 volunteer firefighters.

Solution: We will give solutions to Problem 2 and Problem 3 at the same time. First, there are four possible groups of 3 firefighters that will achieve the mayor's goal. They are shown below:



You can check for yourself that each of these four groups achieve the mayor's goal. Let's explain why there cannot be any other groups. (On our way to this, we will also explain why the goal cannot be achieved with only 2 firefighters.)

Ann and Gus have houses that have only one road leading from them. (Ann's road leads to Bob's home and Gus's road leads to Hal's home.) Since each of Ann and Gus needs to either be a firefighter or be next to a firefighter, we need to take one of Ann or Bob as well as one of Gus or Hal in our group of 3 firefighters. (This shows us we need at least 2 firefighters to achieve the goal.)

Notice that no matter what combination of the above is used (Ann and Gus or Ann and Hal or Bob and Gus or Bob and Hal), the 2 firefighters chosen cannot achieve the mayor's goal on their own. For example, Fay will not be a firefighter and will not be one road away from one. This tells us that there is no way to achieve the Mayor's goal with only 2 firefighters (Problem 3).

Now we need to consider all four combinations of Ann, Gus, Bob, and Hal outlined above, and see in how many ways these groups of 2 can be extended to groups of 3 achieving the goal.

Suppose we start with Bob and Hal as firefighters. We see that in this case, the only homes that are not yet connected to either Bob or Hal via a single road are Eve's and Fay's. To fix this using one more firefighter we can add either Cid or Ian. (Notice that adding Ann, Dan, Gus, Eve, or Fay will still leave us with at least one of Eve's and Fay's homes too far from a firefighter.)

This give us two possibilities for the group of 3:

Bob, Hal, and Cid
Bob, Hal, and Ian

Now suppose we start with Bob and Gus. We see that in this case, the only homes that are not yet connected to either Bob or Gus via a single road are Eve's, Fay's, and Ian's. To fix this by adding one more firefighter we *must* add Ian. (Notice that adding Ann, Dan, Hal, Eve, Cid, or Fay will still leave us with at least one home too far from a firefighter.)

This gives us one more possibility for the group of 3:

Bob, Gus, and Ian

If we start with Ann and Hal, then Cid, Eve, and Fay are not yet connected to the home of a volunteer firefighter. The only way to fix this by adding one more firefighter is to add Cid.

This gives us one more possibility for the group of 3:

Ann, Hal, and Cid

Finally, if we start with Ann and Gus, then Dan, Cid, Eve, Fay, and Ian are not connected to the home of a volunteer firefighter. We cannot fix this by adding only one more firefighter. If we add Bob, Dan, or Hal, then Eve and Fay stay disconnected; if we add Cid, Eve, Fay, or Ian, then Dan stays disconnected.

We have now covered all possibilities. Using this reasoning we can be sure that the four groups we have found must be the only groups of 3 that achieve the goal.

Extension: A new road is built between two of the homes in the town and the mayor's goal can now be achieved with only 2 volunteer firefighters. Between which homes in the town could this road have been added?

Solution:

- Having Bob and Ian as the 2 firefighters would leave out only Gus. This could be fixed by adding a road between Gus and Bob or Gus and Ian.
- Having Cid and Hal as the 2 firefighters would leave out only Ann. This could be fixed by adding a road between Ann and Cid or Ann and Hal.