CEMC at Home

Grade 11/12 - Monday, May 11, 2020
Contest Day 2

Today's resource features two questions from the 2020 CEMC Mathematics Contests.

2020 Canadian Team Mathematics Contest, Team Problem #7

What is the smallest four-digit positive integer that is divisible by both 5 and 9 and has only even digits?

2020 Euclid Contest, #4(b)

A geometric sequence has first term 10 and common ratio $\frac{1}{2}$.
An arithmetic sequence has first term 10 and common difference d.
The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence.
Determine all possible values of d.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 3, 6, 12 is a geometric sequence with three terms.)

More Info:

Check out the CEMC at Home webpage on Thursday, May 21 for solutions to the Contest Day 2 problems.

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Grade 11/12 - Monday, May 11, 2020
Contest Day 2 - Solution

Solutions to the two contest problems are provided below, including a video for the first problem.

2020 Canadian Team Mathematics Contest, Team Problem #7

What is the smallest four-digit positive integer that is divisible by both 5 and 9 and has only even digits?

Solution 1:

Suppose the number is $abcd$ where a, b, c, and d are digits.
Since the number is divisible by 5, we must have that $d = 0$ or $d = 5$.
The digits are all even, which means $d \neq 5$, so $d = 0$.

The smallest that a can be is 2 since it must be even and greater than 0 (a four-digit number cannot have $a = 0$). So we will try to find such a number with $a = 2$.
In order to be divisible by 9, we must have $a+b+c+d$ divisible by 9. Substituting $a = 2$ and $d = 0$, this means $2+b+c+0 = 2+b+c$ is divisible by 9.

Since b and c are even, $2+b+c$ is even, which means it cannot equal 9. Thus, we will try to find b and c so that $2+b+c = 18$, which is the smallest multiple of 9 that is greater than 9.
This equation rearranges to $b+c = 16$. Since b and c are even and satisfy $0 \leq b \leq 9$ and $0 \leq c \leq 9$, the only possibility is $b = c = 8$.

Solution 2:

An integer is divisible by both 5 and 9 exactly when it is divisible by 45.
Since we seek a number having only even digits, its last digit is one of $0, 2, 4, 6$, or 8, so the number itself is even.
Thus, we are looking for an even multiple of 45.

An even number is a multiple of 45 exactly when it is a multiple of 90.
This means we are looking for the smallest 4-digit multiple of 90 that has only even digits.
The smallest 4-digit multiple of 90 is 1080, but the first digit of this number is 1, which is odd.
Each of the next 10 multiples of 90 has a first digit equal to 1, so the number we seek must be larger than 2000.

The four-digit multiples of 90 that have a thousands digit of 2 are

$$2070, 2160, 2250, 2340, 2430, 2520, 2610, 2700, 2790, 2880, 2970$$

and the only number in this list that has all even digits is 2880.
Therefore, 2880 is the smallest 4-digit number that is a multiple of 5, a multiple of 9, and has only even digits.

Video

Visit the following link for a discussion of two different approaches to solving the first contest problem: https://youtu.be/hnksZR1etAg.

2020 Euclid Contest, #4(b)

A geometric sequence has first term 10 and common ratio $\frac{1}{2}$.
An arithmetic sequence has first term 10 and common difference d.
The ratio of the 6th term in the geometric sequence to the 4th term in the geometric sequence equals the ratio of the 6th term in the arithmetic sequence to the 4th term in the arithmetic sequence.
Determine all possible values of d.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant, called the common ratio. For example, 3, 6, 12 is a geometric sequence with three terms.)

Solution:

The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are 10, $10 + d$, $10 + 2d$, $10 + 3d$, $10 + 4d$, $10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

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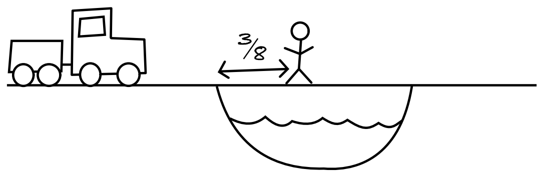
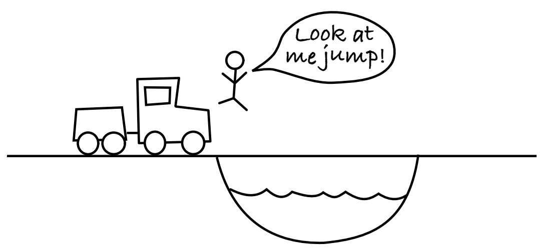
Grade 11/12 - Tuesday, May 12, 2020

Crocodiles!

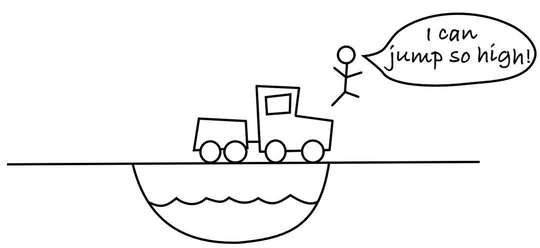
As part of an astounding race, you foolishly try to take a shortcut across a bridge over a river full of hungry crocodiles. When you are $\frac{3}{8}$ of the way across the bridge you notice something terrifying. This bridge is a railway bridge and there is a train travelling towards you!



You definitely aren't going to jump in the river with all those crocodiles. You quickly do some calculations and you realize that if you run towards the train, you will reach the end of the bridge just as the train reaches the bridge and you will have just enough time to jump off and be safe.



You also realize that if you run away from the train, you will reach the other end of the bridge just as the train catches up to you and again you will have just enough time to jump off and be safe.



The train is travelling at 40 km/h. How fast can you run?

Here are the answers to some questions you may want to ask. Yes, you do have enough information to solve the problem. No, we don't know how far the train is from the bridge at the beginning. Yes, we will assume our speed is constant (we can instantaneously run at our top speed!).

Next week we will present two different solutions to this problem. One solution will use algebra to solve the problem and the other will use a different type of reasoning. Can you figure out how to solve this problem in two very different ways?

More Info:

Check out the CEMC at Home webpage on Tuesday, May 19 for two different solutions to this problem. We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Tuesday, May 12, 2020
Crocodiles! - Solution

Solution 1:

Let d be initial distance from the train to the bridge, let x be the length of the bridge, and let s be your speed. The two different scenarios given in the problem give us two equations in terms of these variables. We will make use of the fact that

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

In the first scenario, we are told that the time it takes the train to reach the start of the bridge is equal to the time it takes you to run back to the start of the bridge. Therefore,

$$\frac{d}{40} = \frac{\frac{3}{8}x}{s},$$

which simplifies to

$$ds = 15x. \qquad (1)$$

In the second scenario, we are told that the time it takes the train to reach the start of the bridge and then cross it is equal to the time it takes you to run to the end of the bridge. Therefore,

$$\frac{d+x}{40} = \frac{\frac{5}{8}x}{s},$$

which simplifies to

$$ds + xs = 25x. \qquad (2)$$

Substituting equation (1) into equation (2), we get $15x + xs = 25x$ or $xs = 10x$. We know that $x \neq 0$ and so we divide both sides of the equation by x to obtain $s = 10$. Therefore, your speed is 10 km/h.

Solution 2:

Consider the scenario where you run away from the train. When the train reaches the start of the bridge, where are you? We know that if you run towards the train, then you will be at the start of the bridge when the train reaches the bridge. In other words, you run $\frac{3}{8}$ of the bridge in the time it takes for the train to reach the bridge. So if you run away from the train, when the train reaches the bridge you will be $\frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$ of the way across the bridge. You have $\frac{1}{4}$ of the bridge left to run. In the time it takes you to run what's left, the train will go the length of the bridge. Therefore, your speed is $\frac{1}{4}$ the speed of the train and so your speed is 10 km/h.

Discussion: Isn't the second solution here amazing? Many students are really excited when they see it! We often solve problems by introducing variables, creating equations and solving for unknowns like we did in the first solution. In many situations this process is necessary to solve the problem. However, in this problem, if we use the information in a different way, we can find a simple solution. As you continue to study mathematics you will add more and more tools to your problem solving toolbox. These tools are very powerful and very useful, but always be on the lookout for alternative ways to tackle problems.