CEMC at Home

Grade 9/10 - Monday, March 30, 2020
Careful Clipping

You Will Need:

- Two players
- 10 paper clips
  (or other small objects)

How to Play:

1. Start with a pile of 10 paper clips.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On your turn, you can remove 1, 2 or 3 paper clips from the pile.
4. The player who removes the last paper clip, *loses*.

Play this game a number of times. Alternate which player goes first.
Can you determine a winning strategy* for this game?
Does the winning strategy depend on whether you are Player 1 or Player 2?

> * A *strategy* is a pre-determined set of rules that a player will use to play the game. The strategy dictates what the player will do for every possible situation in the game. It's a *winning strategy* if the strategy allows the player to always win, regardless of what the other player does.

Is there a connection between this game and the game we played on March 23 (Addition Magician)?

Variations:

A. Which player has a winning strategy if the game is won (instead of lost) by the player who removes the last paper clip? Describe this winning strategy.

B. Which player has a winning strategy if, in addition to variation A, players are instructed to instead take 1, 3 or 4 paper clips from the pile? Describe this winning strategy.

C. Which player has a winning strategy if, in addition to variations A and B, the pile starts with 14 paper clips? Describe this winning strategy.

More Info:

Check out the CEMC at Home webpage on Monday, April 6 for a solution to Careful Clipping.
We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 9/10 - Monday, March 30, 2020
Careful Clipping - Solution

The Strategy

Let the two players be Player 1 and Player 2.

You likely noticed that the player that brings the number of paper clips in the pile to 1 is guaranteed to win the game, and the player that brings the number of paper clips in the pile to 2, 3, or 4 generally loses the game. This is because the next player can bring the pile to 1 paper clip by removing 1, 2 or 3 paper clips, respectively.

Using similar reasoning, we can show that the player that brings the number of paper clips to 5 is guaranteed to be able to bring the number to 1 on their next turn, and the player that brings the number of paper clips to 9 is guaranteed to be able to bring the number to 5 on their next turn. This means that Player 1 has a winning strategy for this game and it goes as follows:

Start by removing 1 paper clip, reducing the total number of paper clips to 9. On your next turn, remove whatever number of paper clips are needed to bring the total to 5. On your turn after that, remove whatever number of paper clips are needed to bring the total to 1. (Our analysis above explains why each of these moves will be possible within the rules of the game.)

Notice that the "target numbers" (9, 5, and 1) all differ by 4. We can instead describe the strategy as follows: Go first and start by removing 1 paper clip. For all turns that follow, if the other player just removed n paper clips, then you remove $4 - n$ paper clips. (These two turns, combined, will reduce the number of paper clips by 4.)

The Variations

Variation A

The player that reduces the pile to 1, 2, or 3 paper clips will lose the game since the next player can remove all of the remaining paper clips. Therefore, you want to be the player that reduces the pile to 4 paper clips as you are guaranteed to be able to win the game on your next turn. Player 1 now has the following winning strategy: Start by removing 2 paper clips, reducing the pile to 8 paper clips. On your next turn, remove whatever number of paper clips are needed to bring the total to 4. On your turn after that, remove all remaining paper clips.

Variations B and C

In each of these variations, players can remove 1, 3, or 4 paper clips on their turn, and you *win* by removing the last paper clip from the pile. Player 1 has a winning strategy starting from 10 paper clips (Variation B) and Player 2 has a winning strategy starting from 14 paper clips (Variation C). We outline these strategies in the table on the next page by analyzing how to win the game starting with each of 1 through 14 paper clips, in turn. We give the first move(s) in each strategy and then give guidance on how to use earlier rows in the table to fill in the rest of the strategy.

Let the two players be Ally and Bri. In each game, Ally will go first.

Starting Pile	Winner	Reasoning
1	Player 1	Ally takes the clip and wins.
2	Player 2	Ally must take 1 clip. Bri takes the remaining clip and wins.
3	Player 1	Ally takes all 3 clips and wins.
4	Player 1	Ally takes all 4 clips and wins.
5	Player 1	Ally takes 3 clips, leaving 2 clips for Bri's turn. As above, if a pile has 2 clips, the second player will win. Since it is Bri's turn, the second player (starting from 2 clips) is Ally (Player 1).
6	Player 1	Ally takes 4 clips, leaving 2 clips for Bri's turn. As above, if a pile has 2 clips, the second player will win. Since it is Bri's turn, the second player (starting from 2 clips) is Ally (Player 1).
7	Player 2	Ally must take 1, 3, or 4 clips, leaving 6, 4, or 3 clips for Bri's turn. As above, if a pile has 6, 4, or 3 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).
8	Player 1	Ally takes 1 clip, leaving 7 clips for Bri's turn. As above, if a pile has 7 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
9	Player 2	Ally must take 1, 3, or 4 clips, leaving 8, 6, or 5 clips for Bri's turn. As above, if a pile has 8, 6, or 5 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).
10	Player 1	Ally takes 1 clip, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
11	Player 1	Ally takes 4 clips, leaving 7 clips for Bri's turn. As above, if a pile has 7 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
12	Player 1	Ally takes 3 clips, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
13	Player 1	Ally takes 4 clips, leaving 9 clips for Bri's turn. As above, if a pile has 9 clips, the second player will win. Since it is Bri's turn, the second player is Ally (Player 1).
14	Player 2	Ally must take 1, 3, or 4 clips, leaving 13, 11, or 10 clips for Bri's turn. As above, if a pile has 13, 11, or 10 clips, the first player will win. Since it is Bri's turn, the first player is Bri (Player 2).

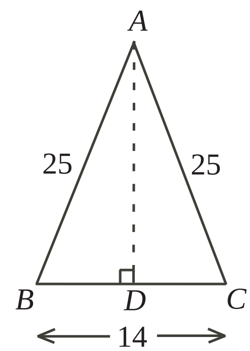
CEMC at Home

Grade 9/10 - Tuesday, March 31, 2020
Splitting Triangles

The CEMC offers many contests to inspire the next generation of mathematicians and computer scientists. Below is a favourite question from a past Fryer Contest (aimed at Grade 9 students). You can find a link to an additional question from a past Galois Contest (aimed at Grade 10 students) at the bottom of the page.

Fryer 2011 Contest, Question 2

In any isosceles triangle ABC with $AB = AC$, the altitude AD bisects the base BC so that $BD = DC$.



(a) (i) As shown in $\triangle ABC$, $AB = AC = 25$ and $BC = 14$. Determine the length of the altitude AD.

(ii) Determine the area of $\triangle ABC$.

(b) Triangle ABC from part (a) is cut along its altitude from A to D (Figure 1). Each of the two new triangles is then rotated 90° about point D until B meets C directly below D (Figure 2). This process creates a new triangle which is labelled PQR (Figure 3).

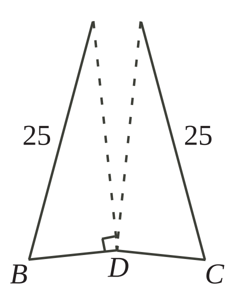


Figure 1

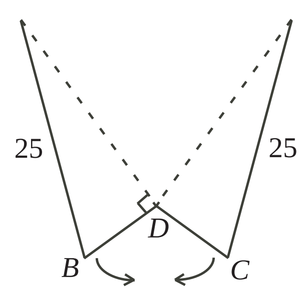


Figure 2

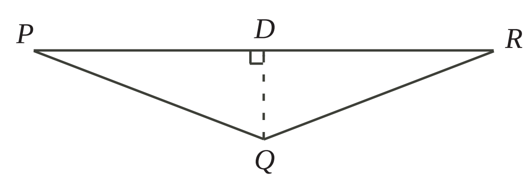
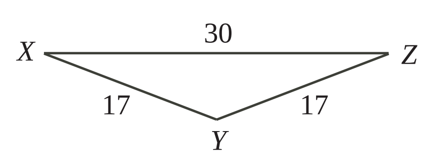


Figure 3

(i) In $\triangle PQR$, determine the length of the base PR.

(ii) Determine the area of $\triangle PQR$.

(c) There are two different isosceles triangles whose side lengths are integers and whose areas are 120. One of these two triangles, $\triangle XYZ$, is shown. Determine the lengths of the three sides of the second triangle.



More Info:

Check out the CEMC at Home webpage on Tuesday, April 7 for the solution to Splitting Triangles.

For an extra question from a past Galois Contest try Question 1 from the 2016 Galois Contest.

CEMC at Home

Grade 9/10 - Tuesday, March 31, 2020
Splitting Triangles - Solution

Fryer 2011 Contest, Question 2

(a) (i) Since $AB = AC$, then $\triangle ABC$ is isosceles. Therefore, the altitude AD bisects the base BC so that $BD = DC = \frac{14}{2} = 7$. Since $\angle ADB = 90^\circ$, then $\triangle ADB$ is right angled. By the Pythagorean Theorem, $25^2 = AD^2 + 7^2$ or $AD^2 = 25^2 - 7^2$ or $AD^2 = 625 - 49 = 576$, and so $AD = \sqrt{576} = 24$, since $AD > 0$.

(ii) The area of $\triangle ABC$ is $\frac{1}{2} \times BC \times AD$ or $\frac{1}{2} \times 14 \times 24 = 168$.

(b) (i) Through the process described, $\triangle ADB$ is rotated 90° counter-clockwise about D to become $\triangle PDQ$. Similarly, $\triangle ADC$ is rotated 90° clockwise about D to become $\triangle RDQ$. Through both rotations, the lengths of the sides of the original triangles remain unchanged. Thus, $PD = AD = 24$ and $RD = AD = 24$. Since P, D and R lie in a straight line, then base $PR = PD + RD = 24 + 24 = 48$.

(ii) When $\triangle ADC$ is rotated 90° clockwise about D, side DC becomes altitude DQ in $\triangle PQR$. Therefore, $DQ = DC = 7$. Thus, the area of $\triangle PQR$ is $\frac{1}{2} \times PR \times DQ$ or $\frac{1}{2} \times 48 \times 7 = 168$.

Note: The area of $\triangle PQR$ is equal to the area of $\triangle ABC$ from part (a)(ii). This is because $\triangle ABC$ is composed of $\triangle ADB$ and $\triangle ADC$, and $\triangle PQR$ is composed of rotated copies of these two right triangles.

(c) Since $XY = YZ$, then $\triangle XYZ$ is isosceles. Draw altitude YW from Y to W on XZ. Altitude YW bisects the base XZ so that $XW = WZ = \frac{30}{2} = 15$, as shown. Since $\angle YWX = 90^\circ$, then $\triangle YWX$ is right angled. By the Pythagorean Theorem, $17^2 = YW^2 + 15^2$ or $YW^2 = 17^2 - 15^2$ or $YW^2 = 289 - 225 = 64$, and so $YW = \sqrt{64} = 8$, since $YW > 0$. By reversing the process described in part (b), we rotate $\triangle XWY$ clockwise 90° about W and similarly rotate $\triangle ZWY$ counter-clockwise 90° about W. By the note at the end of the solution to part (b), the new isosceles triangle and the given isosceles triangle will have the same area. The new triangle formed has two equal sides of length 17 (since XY and ZY form these sides) and a third side having length twice that of YW or $2 \times 8 = 16$ (since the new base consists of two copies of YW).

CEMC at Home

Grade 9/10 - Wednesday, April 1, 2020
Where Am I?

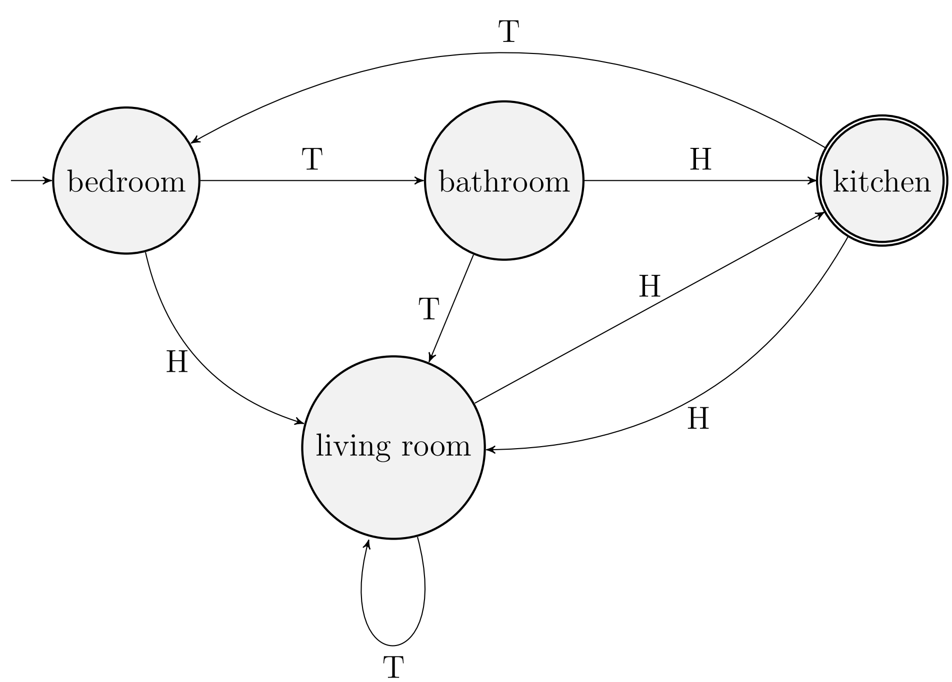
Let's play a game! This is a game without a *strategy* which makes it different from the other games we have played so far. All of our moves will be decided by tossing a coin. For this game you will need to label four different spaces as "bedroom", "bathroom", "kitchen", and "living room". The chosen spaces could be actual rooms, or four pieces of paper spread out around a single room, each having one of these labels. You will also need a coin.

Start in the bedroom (or standing on the paper labelled "bedroom") and toss the coin. Depending on the result of the coin toss, move according to the following rules:

- If you are in the bedroom and the coin lands "heads", move to the living room. If the coin lands "tails", move to the bathroom.
- If you are in the bathroom and the coin lands "heads", move to the kitchen. If the coin lands "tails", move to the living room.
- If you are in the kitchen and the coin lands "heads", move to the living room. If the coin lands "tails", move to the bedroom.
- If you are in the living room and the coin lands "heads", move to the kitchen. If the coin lands "tails", remain in the living room.

Continue the process of tossing the coin and moving from room to room until you have tossed the coin 10 times. The goal of this game is to end up in the kitchen. Which of the four rooms did you end up in?

The rules of this game can be illustrated using the following diagram.



This diagram is known as a *finite-state machine* (FSM). The circles (rooms) are the *states*. The arrows between circles are called *transitions* and they describe how to change states depending on input. The input in our game is a coin toss which resulted in either "heads" (`H`) or "tails" (`T`). The arrow coming from "nowhere" to a circle indicates the *start state*. In our game the start state is the bedroom. The double circle indicates an *accepting state*. An accepting state identifies a desired outcome. In our game, the desired outcome is the kitchen. Note that depending on your sequence of inputs (coin tosses) you may or may not have finished the game in the kitchen.

Finite-state machines are models. Using a finite-state machine to model a process allows you to analyze the process without having to actually implement the process.

Use the FSM model of our game to help you answer the following questions about our game.

Questions:

1. Biyu tosses the coin 6 times with the following results: `T H T H T H`. Which room does Biyu finish in?

2. Salmaan tosses the coin 10 times with the following results: `T T H T H H T T H T`. Which room does Salmaan finish in?

3. Leticia tosses the coin 7 times with the following results: `H H T T ? T T`. If Leticia finishes in the bathroom, what was the result of her 5th coin toss?

4. Pablo tossed the coin 12 times. His last coin toss landed "tails". Did Pablo finish in the kitchen? Yes, no, or maybe (depending on the actual sequence)?

5. Rashida tossed the coin 9 times. Her last coin landed "heads". Did Rashida finish in the kitchen? Yes, no, or maybe (depending on the actual sequence)?

6. Armando tossed the coin 3 times. Which room is it *not* possible for Armando to finish in?

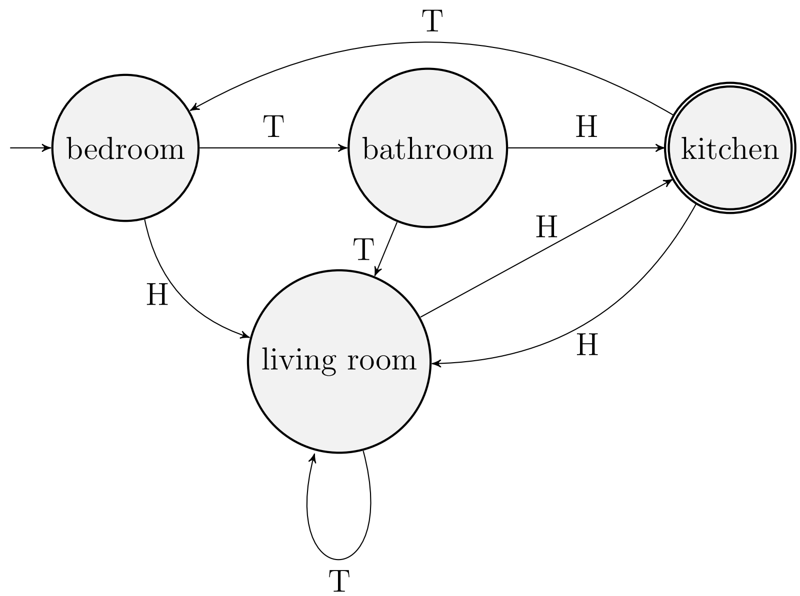
More Info:

Check out the CEMC at Home webpage on Wednesday, April 8 for the solution to Where Am I?

At a very abstract level, all computers are finite-state machines, moving from state to state depending on input received. To learn more about this topic, you can view videos of past Math Circles presentations such as the ones on Finite Automata recorded in Fall 2018.

CEMC at Home

Grade 9/10 - Wednesday, April 1, 2020
Where Am I? - Solution



Answers:

1.

START	T	H	T	H	T	H
bedroom	bathroom	kitchen	bedroom	living room	living room	kitchen

Biyu finishes in the kitchen.

2.

START	T	T	H	T	H	H	T	T	H	T
bed	bath	living	kitchen	bed	living	kitchen	bed	bath	kitchen	bed

Salmaan finishes in the bedroom.

3. After the first 4 coin tosses, Leticia must have ended up in the bathroom. Her 5th coin toss must have landed either "heads" or "tails". If it landed "heads", then she would have moved to the kitchen, and from the kitchen, her remaining 2 coin tosses would have taken her to the bathroom. If it landed "tails", then she would have moved to the living room, and her remaining 2 coin tosses would have kept her in the living room. Since we are told that Leticia finished in the bathroom, her 5th coin toss must have landed "heads".

4. Notice that the only way to arrive in the kitchen is for a coin toss to land "heads". Since Pablo's last coin toss landed on "tails", it is not possible for Pablo to finish in the kitchen.

5. It is not possible to tell which room Rashida finishes in without knowing the results of her other coin tosses. For example, suppose her first 8 coin tosses all land "heads". In this case, Rashida **will not** finish in the kitchen. However, if her first 8 coin tosses all land "tails", then Rashida **will** finish in the kitchen.

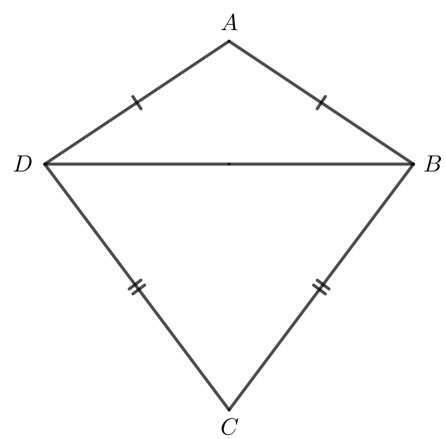
6. Armando's 3 coin tosses will result in one of 8 different combinations of "heads" and "tails". Trying each combination and tracking which room Armando finishes in reveals that for no combination does Armando finish in the bathroom.

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 2, 2020
Go Fly a Kite

Amanda wants to fly a kite. The kite is composed of two isosceles triangles, $\triangle ABD$ and $\triangle BCD$. The height of $\triangle BCD$ is 2 times the height of $\triangle ABD$, and the width of the kite, BD, is 1.5 times the height of the larger triangle.

If the area of the kite is 1800 cm^2, what is the perimeter of the kite?



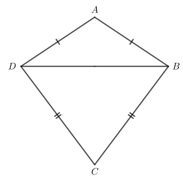
Did you know that in an isosceles triangle the altitude to the unequal side of the triangle bisects that unequal side?

More Info:

Check the CEMC at Home webpage on Thursday, April 9 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 9.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem D and Solution

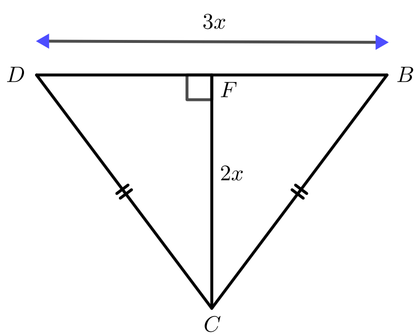
Go Fly a Kite

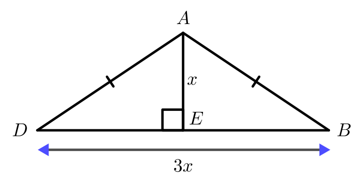
Problem

Amanda wants to fly a kite. The kite is composed of two isosceles triangles, $\triangle ABD$ and $\triangle BCD$. The height of $\triangle BCD$ is 2 times the height of $\triangle ABD$, and the width of the kite, BD, is 1.5 times the height of the larger triangle. If the area of the kite is 1800 cm^2, what is the perimeter of the kite?

Solution

Let the height of $\triangle ABD$ be $AE = x$. Therefore, the height of $\triangle BCD$ is $CF = 2x$. Also, the width of the kite is $BD = 3x$. Therefore, the base of each triangle is $3x$.





The area of $\triangle BCD = \dfrac{(3x)(2x)}{2} = 3x^2$ and the area of $\triangle ABD = \dfrac{(3x)(x)}{2} = \dfrac{3x^2}{2}$.
Also,

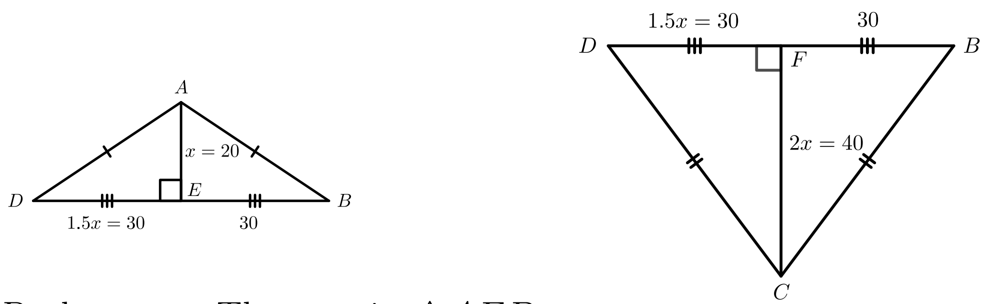
$$\begin{aligned}\text{area of kite } ABCD &= \text{area of } \triangle BCD + \text{area of } \triangle ABD \\ &= 3x^2 + \frac{3x^2}{2} \\ &= \frac{9x^2}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{9x^2}{2} &= 1800 \\ 9x^2 &= 3600 \\ x^2 &= 400 \\ x &= 20, \quad \text{since } x > 0\end{aligned}$$

Now, to find the perimeter of the kite, we need to find the lengths of the sides of the kite.

Since $\triangle ABD$ is isosceles, E will bisect BD and therefore
$DE = BE = 1.5x = 30$. This is shown below in the diagram to the left.
Similarily, $\triangle BCD$ is isosceles, F will bisect BD, and therefore
$DF = BF = 1.5x = 30$. This is shown below in the diagram to the right.



Using the Pythagorean Theorem in $\triangle AED$,

$$\begin{aligned} AD^2 &= 20^2 + 30^2 \\ &= 400 + 900 \\ &= 1300 \\ AD &= \sqrt{1300}, \text{ since } AD > 0. \end{aligned}$$

Also $AB = AD = \sqrt{1300}$ cm.

Similiarily in $\triangle DFC$,

$$\begin{aligned} DC^2 &= 30^2 + 40^2 \\ &= 2500 \\ DC &= 50, \text{ since } DC > 0. \end{aligned}$$

Also, $BC = DC = 50$ cm.

Now,

$$\begin{aligned} \text{the perimeter of the kite} &= \sqrt{1300} + \sqrt{1300} + 50 + 50 \\ &= 2\sqrt{1300} + 100 \\ &\approx 172.1 \end{aligned}$$

Therefore, the exact perimeter is $2\sqrt{1300} + 100$ cm or approximately 172.1 cm.

Note:

The expression $\sqrt{1300}$ can be simplified as follows:

$$\sqrt{1300} = \sqrt{100 \times 13} = \sqrt{100} \times \sqrt{13} = 10\sqrt{13}.$$

Therefore, the exact perimeter is

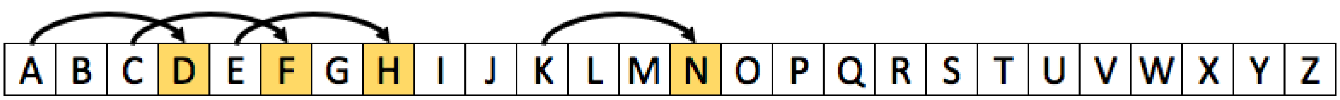
$$2\sqrt{1300} + 100 = 2(10\sqrt{13}) + 100 = 20\sqrt{13} + 100 \text{ cm}.$$

CEMC at Home

Grade 9/10 - Friday, April 3, 2020
Surprise Party

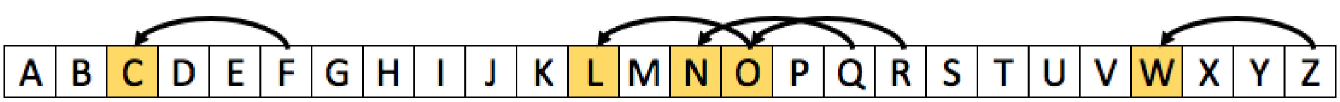
You are planning a surprise party for your friend, Eve. To prevent Eve from finding out about the details of the party, you and the other party planners have agreed to communicate in code. You have chosen to code your messages using a *substitution cipher* known as the *Caesar cipher*. A substitution cipher works by systematically replacing each letter (or symbol) in a message with a different letter (or symbol). A Caesar cipher involves "shifting" the alphabet.

In order to code messages using a Caesar cipher, your group first needs to choose an integer k from 1 to 25, inclusive. This integer k is called the *key* for the cipher, and determines by how many places the alphabet will be shifted. To *encrypt* a message (that is, to change the message from regular text to code) each letter in the message is replaced with the letter that appears k positions to the right in the alphabet. For example, to encrypt the message `C A K E` using a key of 3, the letter `C` is replaced with the letter `F`, which is 3 positions to the right, and the original message `C A K E` becomes the *encrypted* (or coded) message `F D N H`.



Note that if you cannot move k places to the right in the alphabet, then you wrap around to the beginning. For example, the letter 3 places to the right of `Y` is `B`.

To *decrypt* a message (that is, to change the code back to regular text) each letter in the coded message is replaced with the letter that appears k positions to the left, wrapping around if necessary. For example, to decrypt the message `F O R Z Q` using the same key of 3, the letter `F` is replaced with the letter `C`, and the coded message `F O R Z Q` can be revealed to be the message `C L O W N`.



For the questions below, consider making your own *Caesar Cipher Decoding Wheel* (see last page) to help you encrypt and decrypt. Alternatively, if you have some programming knowledge you can create a computer program that can encrypt and decrypt messages given some text and a key as input.

Questions:

1. Using a key of 6, encrypt the message `P A R T Y  S T A R T S  A T  S E V E N`.
2. Using a key of 24, decrypt the message `R F C  R F C K C  G Q  D Y L R Y Q W`.
3. The other party planners sent you the following message but the key got lost. Can you still decrypt the message? *Hint: What is the most commonly used letter in the English language?*

   `P F R P Y P  H T W W  M C T Y R  E S P  N L V P  L Y O  O P N Z C L E T Z Y D`

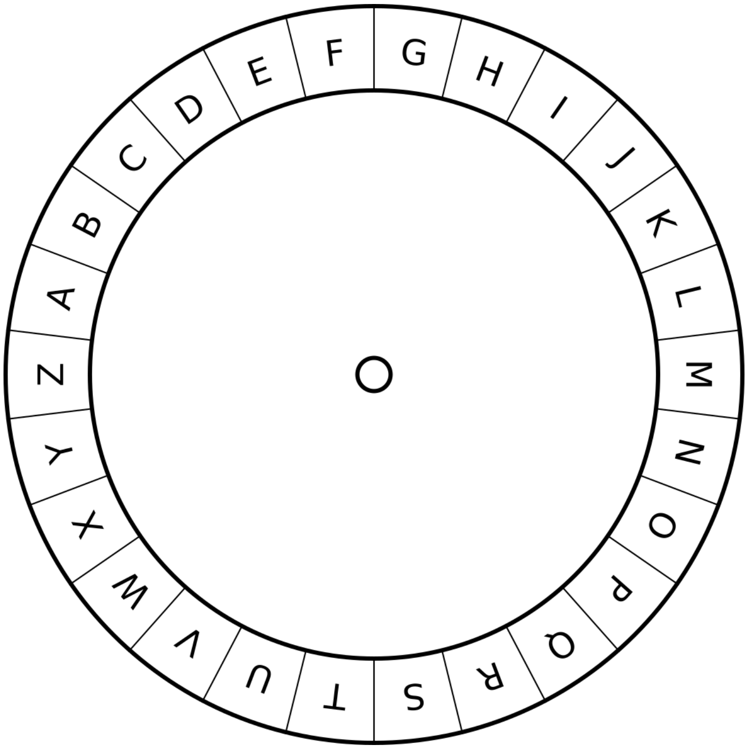
More Info:

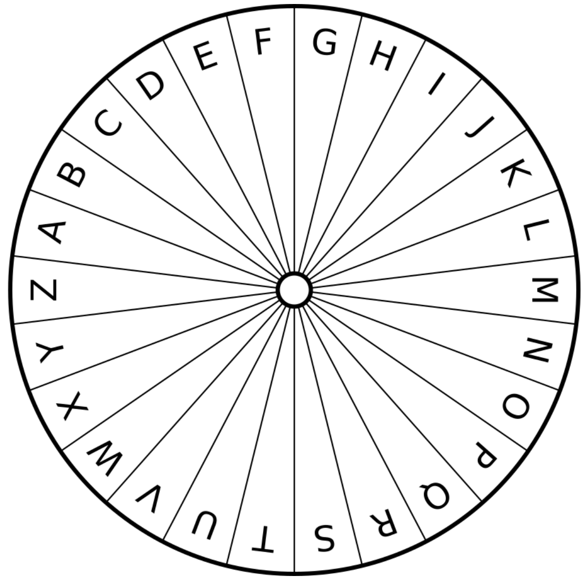
Check out the CEMC at Home webpage on Thursday, April 9 for the solution to Surprise Party.

For a slightly more challenging substitution cipher, check out the Vatsyayana Encryption Scheme.

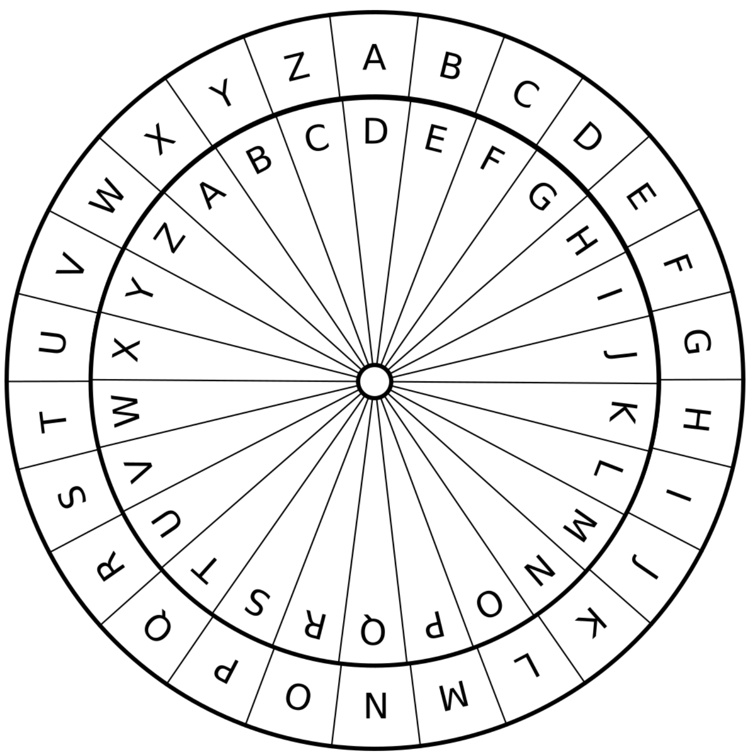
Caesar Cipher Decoding Wheel

Print and cut out the following two circles. Place the smaller circle on top of the larger circle and attach them through the middle using a paper fastener (brad).





Rotate the circles so that the `A`'s are aligned. Then set your key by rotating the inner circle **counter clockwise**. In the diagram below the key is set to 3.



You are now ready to encrypt and decrypt! To encrypt, replace each letter on the outer circle with the corresponding letter on the inner circle. To decrypt, replace each letter on the inner circle with the corresponding letter on the outer circle.

CEMC at Home

Grade 9/10 - Friday, April 3, 2020
Surprise Party - Solution

Answers:

1. The encrypted message is `V G X Z E  Y Z G X Z Y  G Z  Y K B K T`.

2. The decrypted message is `T H E  T H E M E  I S  F A N T A S Y`.

3. One way to decode a message that is encrypted using a Caesar Cipher, when the key is unknown, is to try all possible keys until one key produces a message that makes sense. There are only 25 possible keys, so this wouldn't take too long.

   A more clever way is to take advantage of letter frequencies in the English language. The most common letter in the English language is `E`. The most common letter in the encrypted message is `P`. This means that a good guess might be that the letter `E` has been shifted to the letter `P`. This would make the key equal to 11. Using a key of 11, the decrypted message is:

   `E U G E N E  W I L L  B R I N G  T H E  C A K E  A N D  D E C O R A T I O N S`

Note: *An attempt to break a substitution cipher by using knowledge of commonly used letters or phrases in a language, as we did above, is an example of what is called frequency analysis. For frequency analysis to be as reliable as possible, we want to study as much text, encrypted using the same cipher, as we can. If we have only a short message to work with, then it is very possible that the letter* `E` *will not be the most frequently occurring letter in the original message (and we will be tricked). If we have a very long message, or a very large quantity of messages, chances are good that within a few tries we will have found the right match for the letter* `E`.