CEMC at Home

Grade 7/8 - Monday, March 30, 2020
Sum the Dice

You Will Need:

- At least two players
- Two dice
- Paper and a pencil for keeping score

How to Play:

1. Players alternate turns rolling the two dice at once.
2. On your turn, you start by rolling the dice once. Here is how the score for this turn is decided:
   - If you rolled a 1 on either die, then your turn is over and you score 0 on this turn.
   - If you did not roll a 1 on either die, then you add up the two numbers rolled and record this number.
   - You now decide if you would like to roll again, or take the this recorded number as your final score for this turn.
   - If you decide to roll again, then you keep adding the sum of the two dice on each roll to your running total, until you either decide to stop or you roll a 1 on either die.
   - If you decide to stop on your own, then you take the total you have accumulated as your score for this turn, but if you roll a 1 at any time, then your turn is over and you score a 0 on this turn.
3. The first player whose total score (on all of their turns) reaches 100 wins the game!

Play the game a number of times. Can you come up with a good strategy for this game?

Follow-up Questions (*You can use the aids on the next page to help you answer these questions.*)

1. Let's say you decide in advance that you will stop after one roll no matter what you see on the dice. In this case, there are 10 different possible scores that you could achieve on this turn. We will call these scores the *possible outcomes*. Fill in the boxes below with the possible outcomes.

2. Explain why the 10 possible outcomes in 1. are not all *equally likely* to occur.
3. Write the outcomes in 1. in order from *most likely* to *least likely* to occur. Explain your thinking.

most likely ⟶ **least likely**

4. Does this exploration lead you to believe that you should change your strategy for this game?

More Info:

Check out the CEMC at Home webpage on Tuesday, March 31 for a solution to Sum the Dice.

Aids for the follow-up questions:

Each cell in the table below represents a possible result when two dice are rolled. Notice that the row and column with a 1 have already been filled in. This is because if you roll a 1 on either die, then your score for the turn is zero.

Fill in the rest of the table with the score for each possible roll and then use it to answer the follow-up questions.

		Die #1 Roll					
		1	2	3	4	5	6
Die #2 Roll	1	0	0	0	0	0	0
	2	0					
	3	0					
	4	0					
	5	0					
	6	0					

An *outcome* is the result of a probability experiment, such as rolling a die.
Outcomes are considered *equally likely* if they have the same chance of happening.

If you want more practice exploring outcomes in a game or an experiment, then check out this lesson in the CEMC Courseware.

CEMC at Home

Grade 7/8 - Monday, March 30, 2020
Sum the Dice - Solution

		Die #1					
		1	2	3	4	5	6
Die #2	1	0	0	0	0	0	0
	2	0	4	5	6	7	8
	3	0	5	6	7	8	9
	4	0	6	7	8	9	10
	5	0	7	8	9	10	11
	6	0	8	9	10	11	12

Follow-up Questions:

1. Let's say you decide in advance that you will stop after one roll no matter what you see on the dice. In this case, there are 10 different possible scores that you could achieve on this turn. We will call these scores the *possible outcomes*. Fill in the boxes below with the possible outcomes.

0	4	5	6	7	8	9	10	11	12

2. Explain why the 10 possible outcomes in 1. are not all *equally likely* to occur.

   This is because there are more ways to get some outcomes than others. For example, there are two ways to get a score of 5 but only one way to get a score of 4.

3. Write the outcomes in 1. in order from *most likely* to *least likely* to occur. Explain your thinking.

0	8	7 & 9	6 &10	5 & 11	4 & 12

most likely ⟶ **least likely**

Using the table, we can count the number of ways we can roll each score. The greater the number of ways we can roll a particular score, the more likely that score is. We noticed that some scores are equally likely (for example 7 and 9), so we wrote them in the same box.

There are 11 ways to roll a score of 0, so it is the most likely. There are 5 ways to roll a score of 8. There are 4 ways to roll a score of 7 or a score of 9. There are 3 ways to roll a score of 6 or a score of 10. There are two ways to roll a score of 5 or a score of 11. There is only 1 way to roll a score of 4 or a score of 12, so those are the least likely to be rolled.

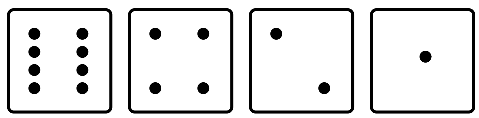
CEMC at Home

Grade 7/8 - Tuesday, March 31, 2020
Building Numbers

In this activity we will explore a way to build the counting numbers. Let's think about the following sequence of numbers:

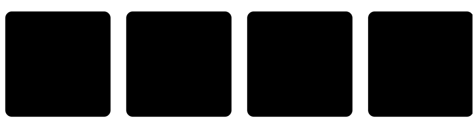
$$1, 2, 4, 8$$

Notice that each number is twice as big as the number to its left. (You may recognize these numbers as powers of 2.) We will represent each of the numbers in the list using a card with the correct number of dots on its face. Notice that we have placed the cards in order from most to fewest dots.

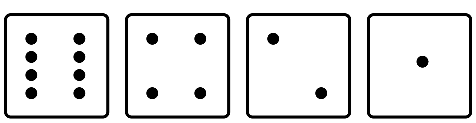


We will build different numbers by choosing which of these cards to put face up and which of these cards to put face down. The number represented by the cards will be the total number of dots that are showing.

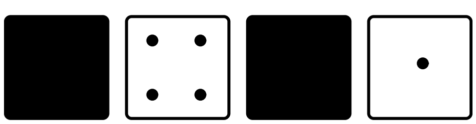
Let's look at some examples.



If we put all of the cards face down, this would represent the number 0 because there are no dots showing.



If we put all of the cards face up, this would represent the number 15 because there are $8+4+2+1 = 15$ dots showing in total.



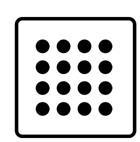
This would represent the number 5 because there are $4 + 1 = 5$ dots showing in total.

Question 1: How many different numbers can you represent using these four cards? List all of the numbers that can be represented and show how to represent each one using the cards.

See the next page for tools to help organize your solution.

Question 2: Now add a fifth card that has 16 dots on its face. Can you determine which numbers can be represented using the five cards (following the same rules)?

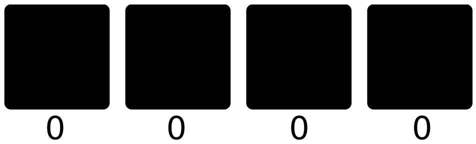
How might you use your work in Question 1 to help answer Question 2?



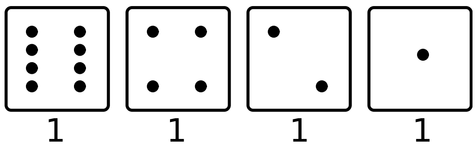
Variation: Make four cards with 1, 3, 9, and 27 dots on them. What numbers can you make with these cards? What do you notice about building numbers with these cards that is different from building numbers with the original cards 1, 2, 4, 8?

On the last page, you will find cards you can cut out and use while you explore these questions.

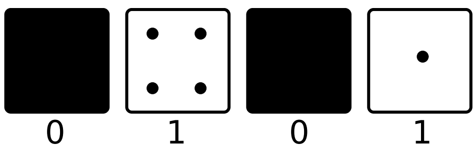
Drawing the cards for each number will take a lot of time, so let's use a simple "code" to keep track of our work. We will indicate that a card is face up by a 1, and indicate that a card is face down with a 0. This means we can communicate what our cards look like using just 0s and 1s.



The code 0000 represents all cards being face down (the number 0).



The code 1111 represents all cards being face up (the number 15).



The code 0101 represents this configuration of cards (the number 5).

Record all of the numbers you can represent here.

Code	Number
0000	0

Code	Number

Code	Number

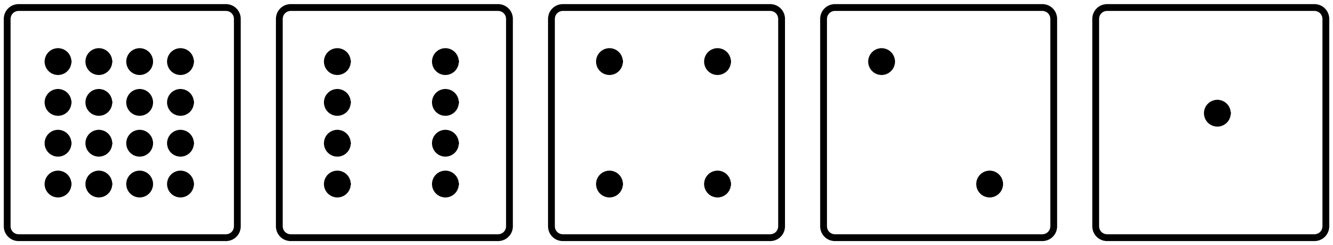
Code	Number
1111	15

More Info:

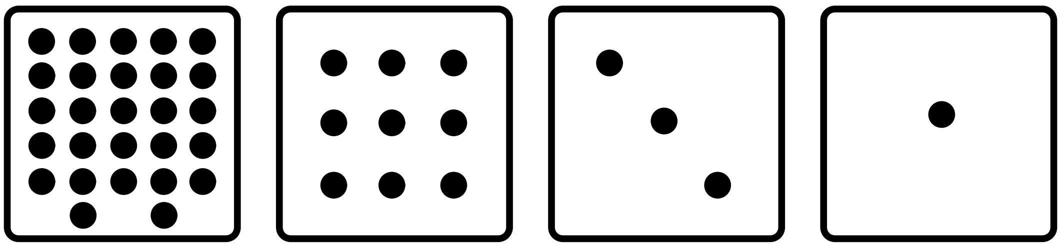
Check out the CEMC at Home webpage on Wednesday, April 1 for the solution to Building Numbers.

We will explore Building Numbers further on Wednesday, April 1 with Secret Messages.

Cards for the main activity



Cards for the variation



CEMC at Home

Grade 7/8 - Tuesday, March 31, 2020
Building Numbers - Solution

Solution to Question 1: We can represent each of the integers from 0 to 15 using the four cards. Here are the codes for these 16 integers.

Code	Number
0000	0
0001	1
0010	2
0011	3

Code	Number
0100	4
0101	5
0110	6
0111	7

Code	Number
1000	8
1001	9
1010	10
1011	11

Code	Number
1100	12
1101	13
1110	14
1111	15

Solution to Question 2: We can represent each of the integers from 0 to 31 using the five cards. We can represent each of the numbers from Question 1 by using the same configuration for the cards 1, 2, 4, and 8, as given above and then placing the new card (with 16 dots) face down. We can make the numbers 16 through 31 in a similar way but by placing the new card face up. For example, since 1001 is the code for 9 in Question 1, 01001 will be the code for 9 in Question 2, and 11001 will be the code for $9+16 = 25$ in Question 2. Here are the codes for all 32 integers that can be represented.

Code	Number
00000	0
00001	1
00010	2
00011	3
00100	4
00101	5
00110	6
00111	7

Code	Number
01000	8
01001	9
01010	10
01011	11
01100	12
01101	13
01110	14
01111	15

Code	Number
10000	16
10001	17
10010	18
10011	19
10100	20
10101	21
10110	22
10111	23

Code	Number
11000	24
11001	25
11010	26
11011	27
11100	28
11101	29
11110	30
11111	31

Variation: Using the cards with 1, 3, 9, and 27 dots, you can make the following numbers:

$$0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, 36, 37, 39, 40$$

Notice that we can represent exactly 16 numbers just like we could with the cards with 1, 2, 4, and 8 dots. The main difference here is that there are gaps in our list. For example, we cannot represent any number from 14 to 26. Is it surprising that there are gaps in this list, while the lists in Questions 1 and 2 had no gaps?

CEMC at Home

Grade 7/8 - Wednesday, April 1, 2020
Secret Messages

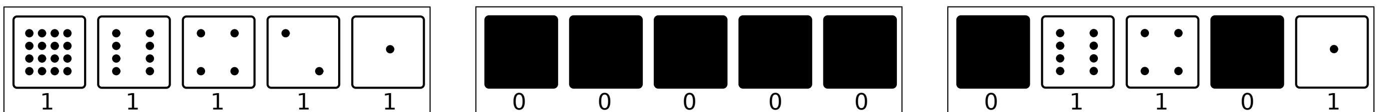
In yesterday's activity Building Numbers, we used cards and codes to represent counting numbers using only the digits 0 and 1. (If you didn't do this activity, please try it now. You may want the cards for today's activity too.) Today we will use these same ideas to write secret messages!

Our secret messages will have two "levels of secrecy" which are explained below.

First, we pair each letter of the alphabet with an integer from 1 to 26 as shown in the table.

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	**O**	**P**	**Q**	**R**	**S**	**T**	**U**	**V**	**W**	**X**	**Y**	**Z**
14	15	16	17	18	19	20	21	22	23	24	25	26

Second, we pair each number between 1 and 26 with a sequence of five digits, all either 0 or 1. We can set up the five cards as in Building Numbers to help us with this part of the code. Remember that to determine the number represented by the cards, we count the total number of dots showing.



11111 codes the number 31 | 00000 codes the number 0 | 01101 codes the number 13

Activity 1: Decode the following secret messages.

1. 01001 00001 01101 10111 10010 01001 10100 01001 01110 00111 01001 01110 00011 01111 00100 00101.

2. 00011 00001 01110 11001 01111 10101 10010 00101 00001 00100 10100 01000 01001 10011?

Activity 2: Write your own secret messages for your friends and family to decode. Can they read your messages without knowing your coding plan? Explain to them how to decode your messages.

Extension: Can you make a similar coding scheme instead using the cards with 1, 3, and 9 dots from Building Numbers? You may notice that you cannot represent all of the integers from 1 to 26 by placing these cards face up or face down as usual. Can you fix this problem by using two copies of each card?

More Info:

Check out the CEMC at Home webpage on Thursday, April 2 to see a solution to Secret Messages.

Did you know that the codes we have been using (the sequences of 0s and 1s) are called *binary numbers*? Every counting number can be represented using a binary number and binary numbers are used by computers to store and share information.

Cryptography is the study of reading and writing secret messages. To learn more about cryptography, check out this Math Circles lesson.

CEMC at Home

Grade 7/8 - Wednesday, April 1, 2020
Secret Messages - Solution

Answers for Activity 1: Here are the secret messages decoded.

1. [01001] [00001][01101] [10111][10010][01001][10100][01001][01110][00111] [01001][01110] [00011][01111][00100][00101].

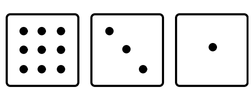
   I AM WRITING IN CODE.

2. [00011][00001][01110] [11001][01111][10101] [10010][00101][00001][00100] [10100][01000][01001][10011]?

   CAN YOU READ THIS?

Discussion of the Extension:

You can only represent eight numbers using the following three cards.

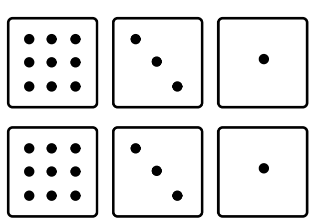


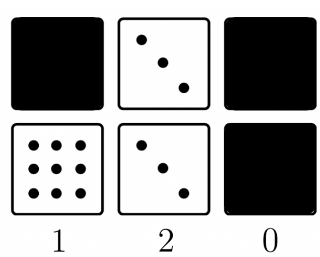
Remember that the digit 1 indicates that the card is face up and 0 indicates that the card is face down.

Code	Number
000	0
001	1
010	3
011	4

Code	Number
100	9
101	10
110	12
111	13

This is not enough to assign a different code to each letter in the alphabet, but we can fix this issue if we instead use two of each card to make our codes.





Again, let's use sequences of three digits to represent numbers, but this time we will use the digits 0, 1, and 2 according to the following rules:

- If both cards (of the same type) are face down, then the digit is 0.
- If exactly one card (of the two cards of the same type) is face up, then the digit is 1.
- If both cards (of the same type) are face up then the digit is 2.

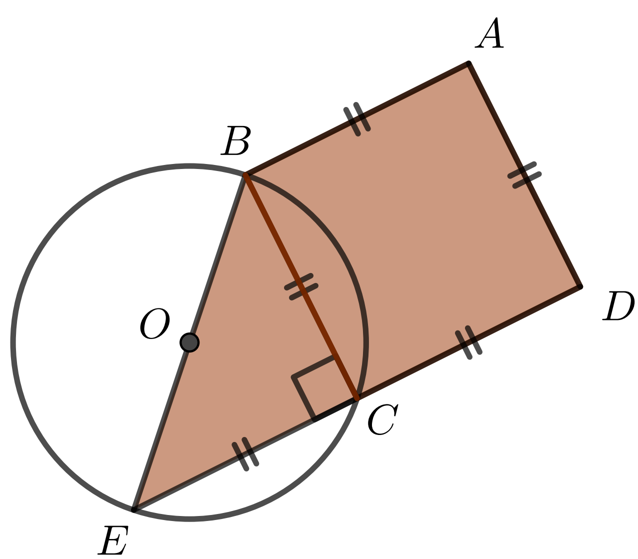
In the example above, we see that the code 120 represents the number $1 \times 9 + 2 \times 3 + 0 \times 1 = 15$. Can you represent all of the integers from 1 to 26 using this new coding strategy?

CEMC at Home features Problem of the Week

Grade 7/8 - Thursday, April 2, 2020
A Circle and Other Shapes

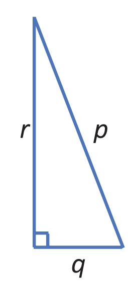
Quadrilateral $ABED$ is made up of square $ABCD$ and right isosceles $\triangle BCE$. BE is a diameter of the circle with centre O. Point C is also on the circle.

If the area of $ABED$ is 24 cm^2, what is the length of BE?



The *Pythagorean Theorem* states, "In a right triangle, the square of the length of hypotenuse (the side opposite the right angle) equals the sum of the squares of the lengths of the other two sides".

In the right triangle to the right, $p^2 = r^2 + q^2$.

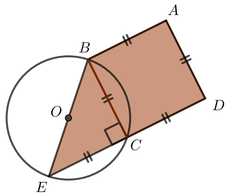


More Info:

Check the CEMC at Home webpage on Thursday, April 9 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 9.

This CEMC at Home resource is the current grade 7/8 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

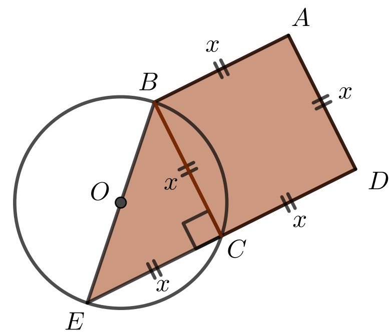


Problem of the Week
Problem C and Solution
A Circle and Other Shapes

Problem

Quadrilateral $ABED$ is made up of square $ABCD$ and right isosceles $\triangle BCE$. BE is a diameter of the circle with centre O. Point C is also on the circle. If the area of $ABED$ is 24 cm^2, what is the length of BE?

Solution



Let $AB = AD = DC = CB = CE = x$.
Therefore, the area of square $ABCD$ is x^2 and the area of $\triangle BCE$ is $\frac{1}{2}(x)(x) = 0.5x^2$.

Therefore,

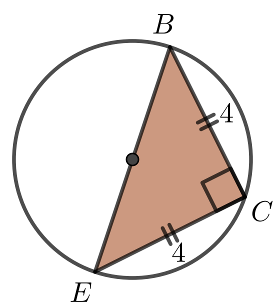
$$\begin{aligned} \text{total area of quadrilateral } ABED &= \text{area of square } ABCD + \text{ area of } \triangle BCE \\ &= x^2 + 0.5x^2 \\ &= 1.5x^2 \end{aligned}$$

Now we also know that the area of $ABED$ is 24 cm^2. Therefore,

$$\begin{aligned} 1.5x^2 &= 24 \\ \frac{1.5x^2}{1.5} &= \frac{24}{1.5} \\ x^2 &= 16 \\ x &= 4, \text{ since } x > 0 \end{aligned}$$

Now let's look at $\triangle BCE$.
We know $BC = CE = 4$.
Using the Pythagorean Theorem,

$$\begin{aligned} BE^2 &= 4^2 + 4^2 \\ &= 16 + 16 \\ &= 32 \\ BE &= \sqrt{32}, \text{ since } BE > 0 \end{aligned}$$



Therefore, $BE = \sqrt{32}$ cm, or approximately 5.7 cm.

CEMC at Home

Grade 7/8 - Friday, April 3, 2020
Toothpick Patterns

A sequence is a list of numbers or other objects. In math, we often study sequences that follow a pattern rule. A pattern rule is used to determine how to form or continue a sequence. This activity will explore the importance of having a clear pattern rule.

Before you start:

Look at the first three terms of a sequence with six terms in total: $2, 3, 5$.
Are you confident that you know which three numbers come next in this sequence? Why or why not?

Without the context (or a pattern rule), we cannot be sure exactly how a sequence was meant to continue. Here are a few continuations of this sequence based on some possible pattern rules. See if the sequence you first imagined is among these:

- $2, 3, 5, \mathbf{8}, \mathbf{13}, \mathbf{21}$

  Rule: Add the two previous terms to get the next term.

- $2, 3, 5, \mathbf{9}, \mathbf{17}, \mathbf{33}$

  Rule: Double the previous term and subtract 1 to get the next term.

- $2, 3, 5, \mathbf{8}, \mathbf{12}, \mathbf{17}$

  Rule: Add 1 to the first term to get the second term, add 2 to the second term to get the third term, and continue in this way, adding n to the n^{th} term to get the term after that.

These are just three of the many different pattern rules that could describe the sequence. Can you come up with some more? We will explore the idea of pattern rules further in the following activities.

You Will Need: As many toothpicks as you can find!
Any small straight objects, similar in length, can be used in place of toothpicks if needed.

Question 1: Use toothpicks to build the following two figures, side by side as shown.

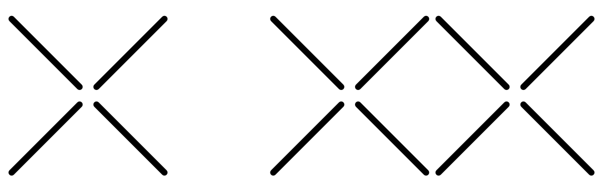


Figure 1 Figure 2

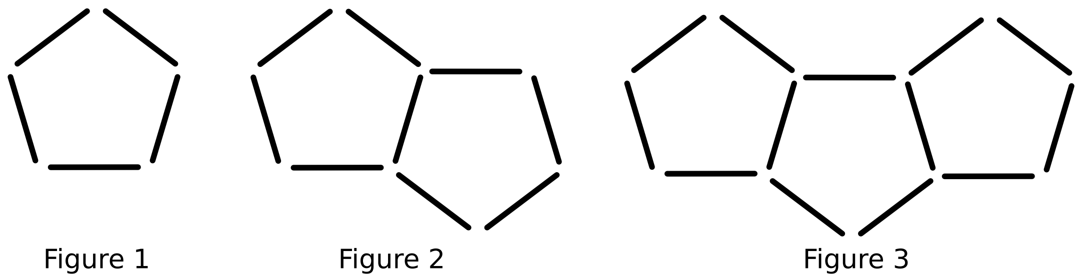
(a) Describe a possible pattern rule for a sequence of figures starting with Figure 1 and Figure 2.
We are looking for a pattern rule that describes how to create the next shapes in the sequence.

(b) Use your pattern rule to build Figure 3 and Figure 4 in the sequence.

(c) Without sharing your pattern rule, have a friend or family member build what they think Figure 3 and Figure 4 in the sequence should be. Are they the same as your figures?

(d) You may have found that everyone you asked in (c) built exactly the figures you expected. Does this mean that the pattern rule is clear? Can you come up with a few different possible pattern rules that also make sense with the first two figures? Try and be as creative as possible!

Question 2: Use toothpicks to build the following three figures, side by side as shown.



(a) Determine a possible pattern rule for a sequence of figures starting with Figures 1, 2, and 3.

(b) Use your pattern rule to build Figures 4, 5, and 6 in the sequence.

(c) Without sharing your pattern rule, have a friend or family member build what they think Figures 4, 5, and 6 in the sequence should be. Are they the same as your figures?

(d) Using your pattern rule, can you determine

- how many toothpicks are needed to build Figure 10 in the sequence?
- how many toothpicks are needed to build Figure 25 in the sequence?

Extension: Create a toothpick pattern activity yourself!

Come up with a new idea for a pattern that can be formed using toothpicks. For an extra twist, why not throw in two or three different types of objects to be used in the pattern you build? Make the first few figures in the sequence with your pattern rule, and see if your friends and family can figure our your intended pattern. If they get it wrong on their first attempt then ask them to try again, or help them out by adding one extra figure and have them try again!