CEMC at Home

Grade 11/12 - Monday, March 30, 2020
Check Your Calendar

You Will Need:

- Two players

How to Play:

1. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).

2. Player 1 starts the game by saying the date "January 1".

3. Player 2 then says a date later in the year which has either the same month or the same day as the date said by Player 1 ("January 1"). For example, Player 2 could say "January 5" (same month) or "March 1" (same day), but not "February 4" (different month and day).

   Note: If you are playing with the 2020 calendar, then February 29 may be used!

4. The players now alternate saying dates, based on the date said previously, following the same rules as given in #3.

5. The player who says "December 31" wins the game!

An example of a complete game:
Alexis and David are playing the game. They decide that Alexis will be Player 1.

Alexis: January 1
David: January 16
Alexis: April 16
David: July 25
Alexis: David!!! You can't change the month and day!
David: Oh yeah, right. July 16
Alexis: July 31
David: December 31! I win!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy for this game? Does the winning strategy depend on whether you are Player 1 or Player 2?

Is there a connection between this game and the game we played on March 23 (Rook to the Top)?

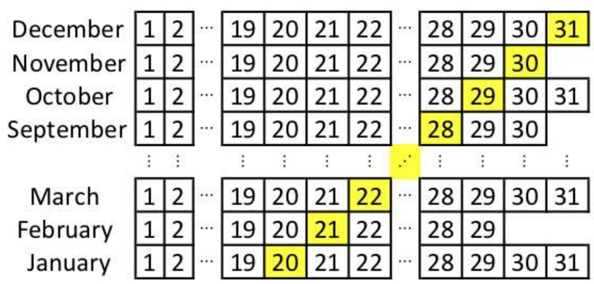
More Info:
Check out the CEMC at Home webpage on Monday, April 6 for a solution to Calendar Game.
We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Monday, March 30, 2020
The Calendar Game - Solution

It turns out that the strategy for winning the Calendar Game is similar to the strategy for winning the game Rook to the Top (that we played on March 23). You might want to refresh your memory by having a look at the strategy for Rook to the Top.

In Rook to the Top, we played on an 8 by 8 grid. In some sense, we can also think of the Calendar Game as being played on a "grid". In this case it will be a 12 by 31 grid with some spaces not open for play.



Notice that the grid has a row for each of the 12 months, and the rows contain either 29, 30 or 31 squares, depending on how many days are in that particular month (in 2020). The diagonal that is highlighted on the grid is the one we will focus on for the winning strategy of this game. We will refer to it as the *winning diagonal*.

The winning diagonal consists of the following dates:

January 20	July 26
February 21	August 27
March 22	September 28
April 23	October 29
May 24	November 30
June 25	December 31

Since Player 1 must say "January 1", we see that the first "move" lands off of the winning diagonal. Player 2 can then "move" onto the winning diagonal by saying "January 20". Now Player 1 must change either the month or the day (but not both) and so any allowed date will represent a move off of the winning diagonal. If they change the month, this corresponds to a vertical move upwards and if they change the day, this corresponds to a horizontal move to the right. Player 2 can now choose the appropriate date from the table above to move back onto the winning diagonal. For example, if Player 1 changes the month and says "May 20", Player 2 can then change the day and say "May 24" (from the table above). If instead Player 1 changes the day and says "January 27", Player 2 can then change the month and say "August 27" (from the table above).

Repeating this process, Player 1 will always have to move off of the winning diagonal, and Player 2 will always be able to return to the winning diagonal, closer to December 31. Since there are a finite number of dates to choose from, Player 2 will eventually say December 31.

Thus, Player 2 has a winning strategy for this game.

CEMC at Home

Grade 11/12 - Tuesday, March 31, 2020
Making a List

Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. Ellie makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.)

1. Suppose that Ellie starts with $a = 5$ and $b = 16$. Can you determine the four numbers that each appear twice in the third list without writing out all 36 numbers in the third list?

2. Suppose that Ellie's third list has the following properties:

   (i) the integer 49 appears in the third list,

   (ii) there is no number in the third list that is a multiple of 64, and

   (iii) there is at least one number in the third list that is larger than 75.

   Determine all possibilities for the pair (a, b).

More Info:

Check out the CEMC at Home webpage on Tuesday, April 7 for the solution to Making a List.

Part of this question appeared on a past Euclid Contest. You can see the original question and the rest of the contest here: 2015 Euclid Contest.

CEMC at Home

Grade 11/12 - Tuesday, March 31, 2020
Making a List - Solution

Problem:

Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. Ellie makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.)

1. Suppose that Ellie starts with $a = 5$ and $b = 16$. Can you determine the four numbers that each appear twice in the third list without writing out all 36 numbers in the third list?
2. Suppose that Ellie's third list has the following properties:
   (i) the integer 49 appears in the third list,
   (ii) there is no number in the third list that is a multiple of 64, and
   (iii) there is at least one number in the third list that is larger than 75.

   Determine all possibilities for the pair (a, b).

Solution to 1.

The first list is 5, 6, 7, 8, 9, 10 and the second list is 16, 17, 18, 19, 20, 21.

The prime numbers that are a factor of at least one number in the first two lists are as follows:

$$2, 3, 5, 7, 17, 19$$

We examine the 36 products in the third list based on their prime factors.

Products with a factor of 17 or 19

Note that 17 is the only number in the first or second list that has a factor of 17. This means that there will be exactly six numbers in the third list that have a factor of 17 (obtained by multiplying each number in the first list by 17) and these six numbers will be all be different. The situation is similar for the six numbers in the third list that have a factor of 19. We can see that all twelve of these numbers only appear once in the third list.

Products with a factor of 7

Note that 7 and 21 are the only numbers in the first or second list that have a factor of 7. Suppose we have $7 \times m = n \times 21$ where $m \neq 21$ comes from the second list and $n \neq 7$ comes from the first list. Since m must have a factor of 3, the only possibility for m is 18, which forces n to be 6, and gives

$$7 \times 18 = 6 \times 21 = 126$$

Products with a factor of 5

Note that 5, 10, and 20 are the only numbers in the first or second list that have a factor of 5. Suppose we have $5 \times m = n \times 20$ where $m \neq 20$ comes from the second list and $n \neq 5$ comes from the first list. Since m must have a factor of 4, the only possibility for m is 16, which would force n to be 4, which is not possible as 4 is not in the first list. Now suppose we have $10 \times m = n \times 20$ where $m \neq 20$ comes from the second list and $n \neq 10$ comes from the first list. Since m must have a factor of 2, the only possibilities for m are 16 and 18, which would force n to be 8 and 9, respectively, giving us

$$10 \times 16 = 8 \times 20 = 160$$

$$10 \times 18 = 9 \times 20 = 180$$

Products whose only prime factors are 2 or 3

All remaining numbers in the list must be formed by multiplying one of 6, 8, or 9 by one of 16, or 18. There is only one final duplication here, formed by

$$8 \times 18 = 9 \times 16 = 144$$

Therefore the four numbers that each appear twice in the third list are as follows:

$$\begin{aligned} 126 &= 7 \times 18 = 6 \times 21 \\ 144 &= 8 \times 18 = 9 \times 16 \\ 160 &= 8 \times 20 = 10 \times 16 \\ 180 &= 9 \times 20 = 10 \times 18 \end{aligned}$$

Solution to 2.

We will start by considering what condition (i) tells us about the values of a and b as it seems to be the most restrictive of the three conditions.

Condition (i) tells us that 49 must be a product of an integer from the first list and an integer from the second list. Since $49 = 7^2$, 7 is prime, and all integers in the two lists are positive, these integers must be either 1 and 49 or 7 and 7. We will find all possible values of a and b by considering two cases separately:

- Case 1: 49 was obtained in the third list by multiplying 1 and 49
- Case 2: 49 was obtained in the third list by multiplying 7 and 7

Note: It is not possible for 49 to be obtained in both of these ways at once because if a list contains 49 then it cannot also contain 7. However, knowing this will not be important for our solution.

Case 1: 49 was obtained by multiplying 1 and 49.

Since the number 1 is in one of the lists, we must have either $a = 1$ or $b = 1$. The condition of $a < b$ means we must have $a = 1$. This means that the first list must be

$$1, 2, 3, 4, 5, 6$$

and the number 49 must appear somewhere in the second list.

Therefore, the second list is one of the following six lists (with each list appearing horizontally):

$$\begin{array}{cccccc} 44, & 45, & 46, & 47, & 48, & 49 \\ 45, & 46, & 47, & 48, & 49, & 50 \\ 46, & 47, & 48, & 49, & 50, & 51 \\ 47, & 48, & 49, & 50, & 51, & 52 \\ 48, & 49, & 50, & 51, & 52, & 53 \\ 49, & 50, & 51, & 52, & 53, & 54 \end{array}$$

Notice that $4 \times 48 = 192 = 64 \times 3$. Since 4 is in the first list, and no number in the third list can be a multiple of 64, the second list cannot contain the number 48. This leaves just one possibility for the second list (the last one above):

$$49, 50, 51, 52, 53, 54$$

This case leads us to exactly one possibility for the pair (a, b), namely $(1, 49)$.

We can verify that the third list for the pair $(a, b) = (1, 49)$ actually satisfies conditions (ii) and (iii). For (ii), we note that $64 = 2^6$ and that we can get at most two factors of 2 from a number in the first list and at most two factors of 2 from a number in the second list. It follows that any product in the third list will have at most 4 factors of 2, and hence cannot be a multiple of 64. For (iii), we note that $2 \times 49 = 98$ is in the third list and is greater than 75.

Case 2: 49 was obtained by multiplying 7 and 7.

In this case, we know that the number 7 must appear in both the first list and the second list. In order for this to happen we need to have $2 \leq a \leq 7$ and $2 \leq b \leq 7$. Since $a < b$, we actually must have $3 \leq b \leq 7$. (The smallest a can be is 2 and so b must be at least one more than that.)

Since $3 \leq b \leq 7$, the second list *must* contain the number 8. This means that to satisfy condition (ii), the first list *cannot* contain the number 8. Therefore, we must have $a = 2$, making the first list

$$2, 3, 4, 5, 6, 7$$

Since $7 \times 10 = 70$ and $7 \times 11 = 77$, the third list can only satisfy condition (iii) if the second list contains a number at least as large as 11. This means we cannot have $b = 3$, $b = 4$, or $b = 5$, leaving the only possible values to be $b = 6$ or $b = 7$. These values produce the following second lists:

$$b = 6: \quad 6, 7, 8, 9, 10, 11$$

$$b = 7: \quad 7, 8, 9, 10, 11, 12$$

Therefore, this case leads us to two additional possibilities for the pair (a, b), namely $(2, 6)$ and $(2, 7)$.

We can verify that the third list for each of the the pairs $(a, b) = (2, 6)$ and $(a, b) = (2, 7)$ satisfies conditions (ii) and (iii) using a similar argument to the one given in Case 1.

Combining the two cases, we conclude that there are exactly three pairs, (a, b), that satisfy all three conditions. They are as follows:

$$(1, 49), (2, 6), (2, 7)$$

CEMC at Home

Grade 11/12 - Wednesday, April 1, 2020
Collecting Pollen and Wood

Video

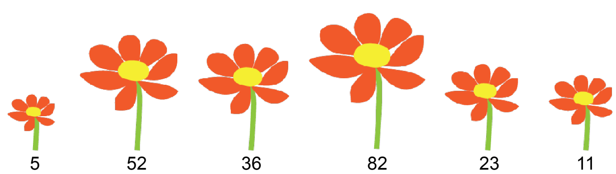
Watch the following presentation on algorithmic paradigms, based on two past problems from the Beaver Computing Challenge:

https://youtu.be/XuR1a_9orJQ

The two problems discussed in the presentation are included below for your reference. Links to the two apps used in the video are also provided should you wish to do some exploration on your own.

Collecting Pollen

Beever the Bee flies to a field of flowers to collect pollen. On each flight Beever visits only one flower and can collect up to 10 mg of pollen. Beever may return to the same flower more than once. The field contains 6 flowers, each containing a different amount of pollen (in mg) as shown.



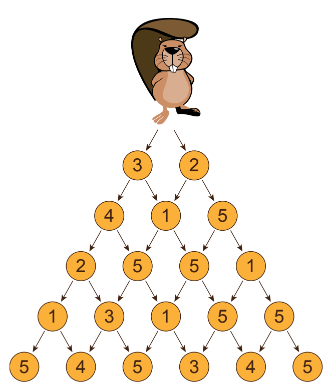
If Beever flies to the field 20 times, what is the maximum total amount of pollen Beever can collect?

Note: This problem was also given as a Grade 4/5/6 exercise last week, but our focus will be different. Our goal is not just to arrive at the answer to the problem, but rather to discuss the algorithm used to arrive at the optimal solution.

App for exploration: https://www.geogebra.org/m/guzzeqn4

Collecting Wood

A beaver collects wood while descending from a mountaintop. Each stop contains a different amount of wood as shown.



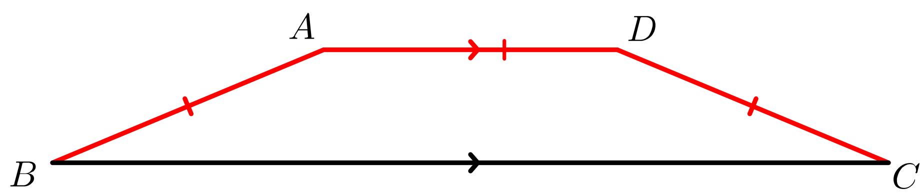
The beaver can only follow the arrows down. What is the maximum total amount of wood the beaver can collect?

App for exploration: https://www.geogebra.org/m/nsmtks3u

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 2, 2020
Three Equal Sides

In trapezoid $ABCD$, the lengths of AB, AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides.



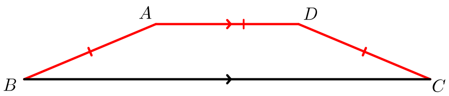
If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

More Info:

Check the CEMC at Home webpage on Thursday, April 9 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 9.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem E and Solution

Three Equal Sides

Problem

In trapezoid $ABCD$, the lengths of AB, AD and DC are equal and the length of BC is 2 units less than the sum of the lengths of the other three sides. If the distance between the parallel sides AD and BC is 5 units, what is the area of the trapezoid?

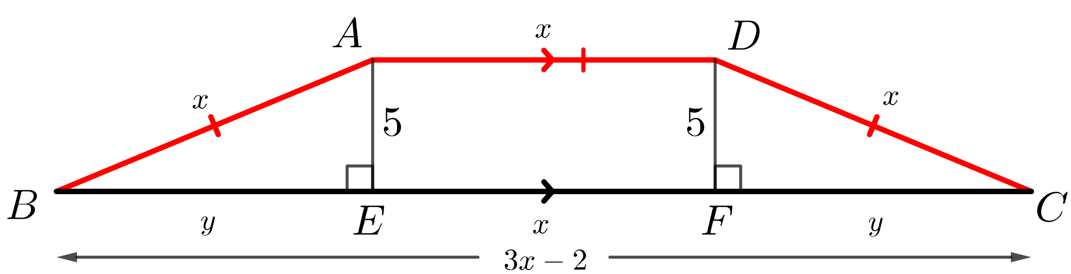
Solution

Let x represent the length of AB. Then $AB = AD = DC = x$. Since the base BC is two less than the sum of the three equal sides, $BC = 3x - 2$.

Construct altitudes from A and D meeting BC at E and F, respectively. Then $AE = DF = 5$, the distance between the two parallel sides.

Let y represent the length of BE. We can show that $BE = FC$ using the Pythagorean Theorem as follows: $BE^2 = AB^2 - AE^2 = x^2 - 5^2 = x^2 - 25$ and $FC^2 = DC^2 - DF^2 = x^2 - 5^2 = x^2 - 25$. Then $FC^2 = x^2 - 25 = BE^2$, so $FC = BE = y$ since $FC > 0$.

Since $\angle AEF = \angle DFE = 90^\circ$ and AD is parallel to EF, it follows that $\angle DAE = \angle ADF = 90^\circ$ and $AEFD$ is a rectangle so $EF = AD = x$. The following diagram contains all of the given and found information.



We can now determine a relationship between x and y.

$$\begin{aligned} BC &= BE + EF + FC \\ 3x - 2 &= y + x + y \\ 2x - 2 &= 2y \\ x - 1 &= y \end{aligned}$$

In right $\triangle ABE$, $AB^2 = BE^2 + AE^2$ gives $x^2 = y^2 + 5^2$. Substituting $y = x - 1$ we get $x^2 = (x-1)^2 + 25$. Solving, $x^2 = x^2 - 2x + 1 + 25$, or $2x = 26$, or $x = 13$.

Since $x = 13$, $3x - 2 = 3(13) - 2 = 37$. Therefore, $AD = x = 13$ and $BC = 3x - 2 = 37$.

Therefore,

$$\begin{aligned} \text{area of trapezoid } ABCD &= AE \times (AD + BC) \div 2 \\ &= 5 \times (13 + 37) \div 2 \\ &= 125 \text{ units}^2 \end{aligned}$$

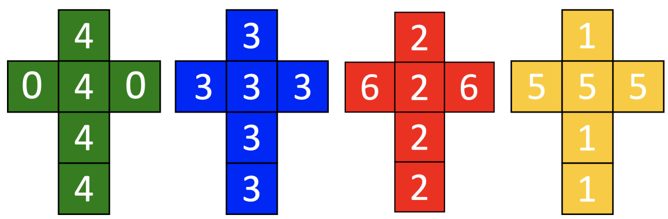
CEMC at Home

Grade 11/12 - Friday, April 3, 2020
A Dicey Situation

In this activity, we will investigate the properties of various non-standard dice.

You Will Need: Two players and four standard six-sided dice.

Description of the Dice: Alter your dice so that the sides of the dice are labelled according to the following diagrams. For example, you could put stickers on each side of the dice.



If you do not have an easy way to alter your dice, you can make use of the following conversion table for each of the dice. This table will allow you to roll a standard die as if it were one of the dice shown above. For example, if you roll a 3 on the standard die representing the green die, then you interpret this as rolling a 4 on the green die (using the second "Green" row in the table).

Die Colour	Number Rolled on Standard Die	Number Rolled on Altered Die
Green	1, 6	0
	2, 3, 4, 5	4
Blue	1, 2, 3, 4, 5, 6	3
Red	1, 6	6
	2, 3, 4, 5	2
Yellow	1, 4, 6	5
	2, 3, 5	1

If all of your dice are identical, you will need a way to keep track of which die is which "colour".

Investigation:

To start a game, each player chooses a die to use for the entire game. One game consists of 20 rounds. In each round, each player rolls their die and the player that rolls the higher number wins the round. The winner of the game is the person who won the most rounds. No round can end in a tie, but the game may end in a tie. Play the game a number of times and alternate the pair of dice used for each game. Keep track of the scores of each game, along with which player had which die for each game.

Follow-up Questions:

1. If you choose your die first, is there a best choice for your die?
2. If you choose your die second, is there a best choice for your die?
   Your answers can depend on what die the other player chooses.
3. Try to justify your answers for 1. and 2. by analyzing the four dice and calculating probabilities.

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for the solution to A Dicey Situation.

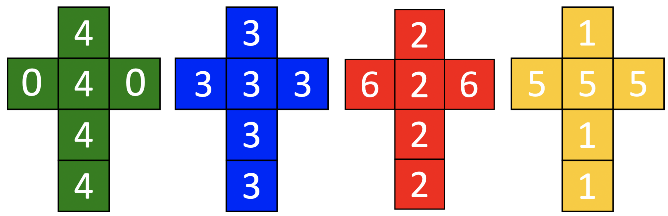
To refresh your knowledge of probabilities, check out this lesson from the CEMC courseware.

CEMC at Home

Grade 11/12 - Friday, April 3, 2020
A Dicey Situation - Solution

Investigation:

To start a game, each player chooses a die to use for the entire game.



One game consists of 20 rounds. In each round, each player rolls their die and the player that rolls the higher number wins the round. The winner of the game is the person who won the most rounds. No round can end in a tie, but the game may end in a tie. Play the game a number of times and alternate the pair of dice used for each game. Keep track of the scores of each game, along with which player had which die for each game.

Follow-up Questions:

1. If you choose your die first, is there a best choice for your die?
2. If you choose your die second, is there a best choice for your die?
   Your answers can depend on what die the other player chooses.
3. Try to justify your answers for 1. and 2. by analyzing the four dice and calculating probabilities.

Solution: You may have answered 1. and 2. using information gathered during your investigation. We will give answers to 1. and 2. that are based upon theoretical probabilities related to the four dice. We start by examining the game that arises if certain pairs of dice are chosen by the two players. Note that the rolls of the two dice are independent.

Green Versus Blue

What happens if the two players pick the green and blue dice (in some order)? Since each of the dice has 6 faces, and each face is equally likely to end up as the top face on a roll, there are $6 \times 6 = 36$ equally likely outcomes when these two dice are rolled. We indicate which die wins in each of the 36 cases the table below:

	3	3	3	3	3	3
0	B	B	B	B	B	B
0	B	B	B	B	B	B
4	G	G	G	G	G	G
4	G	G	G	G	G	G
4	G	G	G	G	G	G
4	G	G	G	G	G	G

Based on this table, we can see that the probability of green winning is

$$\text{P(Green Winning Against Blue)} = \frac{24}{36} = \frac{2}{3}$$

and the probability of blue winning is

$$\text{P(Blue Winning Against Green)} = \frac{12}{36} = \frac{1}{3}$$

Since rolls of the two dice are independent, we can calculate probabilities without listing the possible outcomes. We know that the blue die will always roll a 3, and so we only need to consider the roll of the green die. The probability that the green die rolls a 0 is $\frac{2}{6} = \frac{1}{3}$, and the probability that the green die rolls a 4 is $\frac{4}{6} = \frac{2}{3}$. If the green die rolls a 0, then the blue die wins, and if the green die rolls a 4, then the green die wins. Therefore, we have

$$\text{P(Green Winning Against Blue)} = \frac{2}{3}$$

$$\text{P(Blue Winning Against Green)} = \frac{1}{3}$$

So between green and blue, green is a better choice.

Blue Versus Red

What happens if the two players pick the blue and red dice (in some order)? We know that the blue die will always roll a 3, and so we only need to consider the roll of the red die. If the red die rolls a 2, which happens with probability $\frac{4}{6} = \frac{2}{3}$, then the blue die wins. If the red die rolls a 6, which happens with probability $\frac{2}{6} = \frac{1}{3}$, then the red die wins. Therefore, we have

$$\text{P(Blue Winning Against Red)} = \frac{2}{3}$$

$$\text{P(Red Winning Against Blue)} = \frac{1}{3}$$

So between blue and red, blue is the better choice.

Red Versus Yellow

What happens if the two players pick the red and yellow dice (in some order)? In the table below we indicate which die wins for each of the 36 equally likely rolls.

	1	1	1	5	5	5
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
2	R	R	R	Y	Y	Y
6	R	R	R	R	R	R
6	R	R	R	R	R	R

From this table, we see that

$$\text{P(Red Winning Against Yellow)} = \frac{24}{36} = \frac{2}{3}$$

$$\text{P(Yellow Winning Against Red)} = \frac{12}{36} = \frac{1}{3}$$

Since the rolls of the two dice are independent, we can calculate probabilities without listing all of the outcomes. There is only one type of roll that will result in a win for the yellow die: 5 rolled on the yellow die and 2 rolled on the red die. The probability that a 5 is rolled on the yellow die is $\frac{3}{6} = \frac{1}{2}$, and the probability that a 2 is rolled on the red die is $\frac{4}{6} = \frac{2}{3}$.
Multiplying these two probabilities we determine that

$$\text{P(Yellow Winning Against Red)} = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$$

and since there are only two possibilities (yellow wins or red wins), the two probabilities must sum to 1, and so we have that

$$\text{P(Red Winning Against Yellow)} = 1 - \frac{1}{3} = \frac{2}{3}.$$

So between red and yellow, red is the better choice.

Yellow Versus Green

What happens if the two players pick the yellow and green dice (in some order)? Similar reasoning to the case above allows us to determine that

$$\begin{aligned}\text{P(Green Winning Against Yellow)} &= \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}\\ \text{P(Yellow Winning Against Green)} &= 1 - \frac{1}{3} = \frac{2}{3}\end{aligned}$$

So between yellow and green, yellow is the better choice.

At this point there are two more pairs to consider (green and red, and blue and yellow) which are left to you. Let's summarize our findings for the four pairs we considered:

- In a game played with the green and blue dice, the better die is green.
- In a game played with the blue and red dice, the better die is blue.
- In a game played with the red and yellow dice, the better die is red.
- In a game played with the yellow and green dice, the better die is yellow.

One interesting thing that our results show is that the dice form a kind of "cycle", rather than an ordering from "best" to "worst". For every choice of die, there exists another die which is a better choice, and another die that is a worse choice. Now we discuss the follow-up questions:

If you choose your die first, then no matter what die you choose, the other player will be able to choose a die that beats your die with probability $\frac{2}{3}$. If you choose your die second, then the best choice of die is outlined above: if the first player chooses blue, then you should choose green; if the first player chooses red, then you should choose blue; if the first player chooses yellow then you should choose red, and if the first player chooses green then you should choose yellow. Can you convince yourself that these are the best choices? For example, if your opponent chooses the red die, then we know the blue die is a good choice (from bullet 2) and the yellow die is a bad choice (from bullet 3), but what about the green die? Since the blue die will win against the red die with a probability of $\frac{2}{3}$, you need to convince yourself that the green die has worse odds than this against the red die.

Background of the dice: These dice were invented by Bradley Efron, an American statistician. The dice provide an example of four *non-transitive** dice. You can create different sets of non-transitive dice. For example, you can create a set of three 6-sided non-transitive dice using the values from 1 to 9 on the faces of the dice. Non-transitive dice are an interesting topic about which you might want to do more reading.

*You may have heard the word *transitive* used in math before, and even if you have not, you have likely worked with many comparisons that are transitive: for example, if $x < y$ and $y < z$ then we must have $x < z$. In the case of Efron's dice, we can compare the dice, but these comparisons may not be transitive: for example, if "green is better than blue" and "yellow is better than green", must "yellow be better than blue"?