CEMC at Home

Grade 11/12 - Monday, March 23, 2020

Rook to the Top

Do you have a chessboard at home? Get it out, grab another person and let's play a game! If you don't know how to play chess, don't worry!

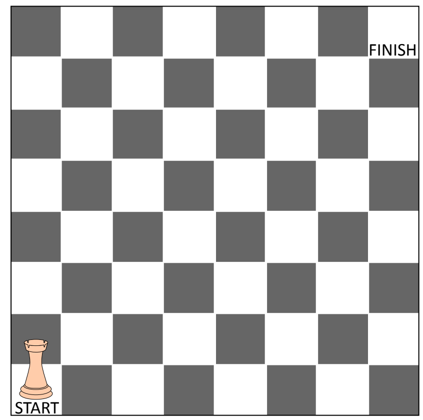
You Will Need:

- Two players
- A chessboard or checkerboard

  If you can't find a board, then you can draw an 8×8 *grid on a piece of paper.*

- A rook (as shown)

  If you can't find a rook, then you can use a coin or any small object in place of the rook.



How to Play:

1. Place the rook in the bottom left corner of the board.
2. The two players will alternate turns moving the rook. Decide which player will go first.
3. On your turn, you can move the rook as many squares as you want either to the right or up.

   You must move the rook at least one square and you cannot move the rook both right and up on the same turn. And of course you cannot run the rook off the board!

4. The player to place the rook in the top right corner of the board wins the game!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy* for this game? Does the winning strategy depend on whether you move the rook first or second?

> * A *strategy* is a pre-determined set of rules that a player will use to play the game. The strategy dictates what the player will do for every possible situation in the game. It's a *winning strategy*, if the strategy allows the player to always win, regardless of what the other player does.

Variation:

- Cover up the bottom 3 rows of the chessboard and start with the rook in the new bottom left corner. Play the game with the same rules. Does this change the winning strategy?

More Info:

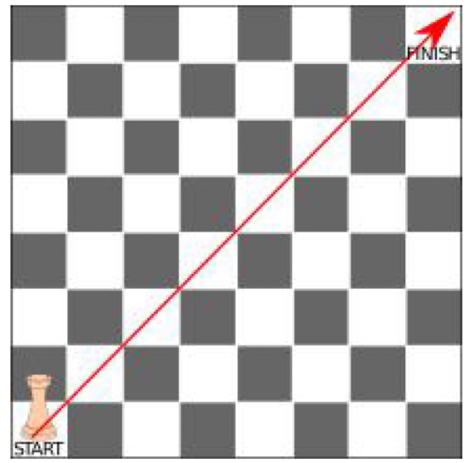
Check out the CEMC at Home webpage on Monday, March 30 for a discussion of a strategy for this game. We encourage you to discuss your ideas online using any forum you are comfortable with.

We sometimes put games on our math contests! Check out Question 2 on the 2003 Hypatia Contest for another game where we are looking for a winning strategy.

CEMC at Home

Grade 11/12 - Monday, March 23, 2020
Rook to the Top - Solution

We are going to call the diagonal line of white squares indicated in diagram, the *main diagonal*. Playing this game, you probably realized that the main diagonal is important to the strategy of this game.

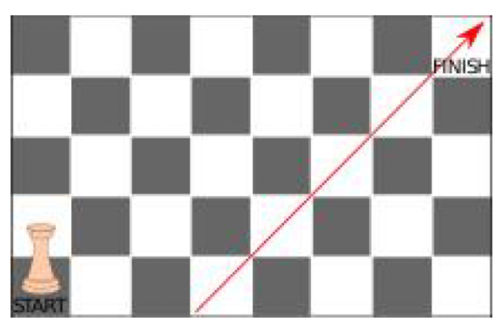


The rook begins on the main diagonal. The first player moves the rook and no matter what move they make, they will have to move the rook off of the main diagonal. If the first player moves the rook n squares to the right, then the second player can move the rook n squares up and the rook will be back on the main diagonal. If the first player moves the rook n squares up, then the second player can move the rook n squares to the right and the rook will be back on the main diagonal. In such a way the second player can guarantee that the rook will be on the main diagonal after their turn and the rook will be closer to the top right square (and maybe even at this square)!

Since the rook is back on the main diagonal, the first player must again move the rook off of the main diagonal and the second player can again put it back on to the main diagonal. Repeating this process, the second player will always be able to place the rook on the main diagonal closer to the top right square. Since there are a finite number of squares on the chessboard, the second player will eventually place the rook in the square at the top right corner.

Thus, we can see that the second player has a winning strategy for this game.

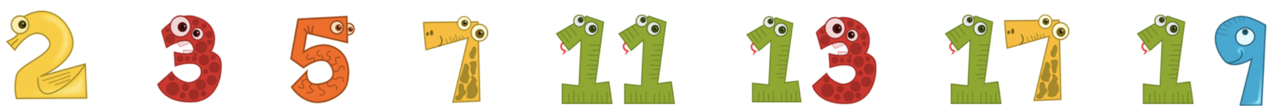
Variation:
In the variation of this game, we have a board with only five rows. We refer to the diagonal shown as the main diagonal. In this variation, the rook does not start on the main diagonal. If the first player moves the rook three spaces to the right, the rook will then be on the main diagonal. After this first move, the second player has no choice but to move the rook off of the main diagonal, leaving the first player the opportunity to place it back on the diagonal. Then the strategy continues as described for the first version. Therefore, the first player has the winning strategy in this variation of the game.



Extension: Consider a chessboard with any number of rows and any number of columns. For what size of chessboard will the first player have a winning strategy? For what size of chessboard will the second player have a winning strategy?

CEMC at Home

Grade 11/12 - Tuesday, March 24, 2020
Divisors and Primes



There are lots of problems that involve divisors of integers: counting divisors, looking for particular divisors, identifying common divisors, and more. For the following problems it might be helpful to review what a prime number is and how to find the prime factorization of an integer. Let's practice:

1. Find the prime factorization of 72 600.

   To help towards a solution, think about the following questions:

   - What are prime numbers?
   - Is 2 a divisor of 72 600? Is 3 a divisor of 72 600?
   - For each prime divisor p of 72 600, how many copies of p can we factor out of 72 600?

2. For how many integers n is $72\left(\frac{3}{2}\right)^n$ equal to an integer?

   To help towards a solution, think about the following questions:

   - Try some values of n. What if $n = 1$? What if $n = 10$? What if $n = -4$?
   - How big can n be? How small can n be?
   - Could prime factorizations help us here?

3. Determine the number of positive divisors of the integer 14!.

   Note: The *factorial* of a positive integer n, denoted by $n!$, is the product of all positive integers less than or equal to n. For example, $4! = 1 \times 2 \times 3 \times 4 = 24$.

4. For a positive integer n, $f(n)$ is defined as the largest power of 3 that is a divisor of n.

   What is $f\left(\dfrac{100!}{50!20!}\right)$?

More Info:

Check the CEMC at Home webpage on Wednesday, March 25 for a solution to Divisors and Primes. We encourage you to discuss your ideas online using any forum you are comfortable with.

These problems were taken from the CEMC's free online course *Problem Solving and Mathematical Discovery*. Check it out here: `https://courseware.cemc.uwaterloo.ca/40`

CEMC at Home

Grade 11/12 - Tuesday, March 24, 2020
Divisors and Primes - Solution

1. Find the prime factorization of 72 600.

   Solution:

   First, we factor 72 600 into two factors, $72\,600 = 726 \times 100$. Next, we factor each of these factors into a product of two factors, $726 = 2 \times 363$ and $100 = 10 \times 10$. We repeat this process until all the factors are prime numbers, (some of these prime factors will be repeated),

$$\begin{aligned} 72\,600 &= 726 \times 100 \\ &= 2 \times 363 \times 10 \times 10 \\ &= 2 \times 3 \times 121 \times 2^2 \times 5^2 \\ &= 2 \times 3 \times 11^2 \times 2^2 \times 5^2 \\ &= 2^3 \times 3 \times 5^2 \times 11^2. \end{aligned}$$

2. For how many integers n is $72\left(\frac{3}{2}\right)^n$ equal to an integer?

   Solution:

   Notice that the prime factorization of 72 is $2^3 \times 3^2$, so the expression $72\left(\frac{3}{2}\right)^n$ can be written as

$$72\left(\frac{3}{2}\right)^n = 2^{3-n} \times 3^{2+n}.$$

   The expression will be an integer whenever the exponents $3 - n$ and $2 + n$ are non-negative integers. So, $3 - n \geq 0$ and $2 + n \geq 0$ imply that $n \leq 3$ and $n \geq -2$. Hence, there are six possible values of n, which are $-2, -1, 0, 1, 2, 3$.

3. Determine the number of positive divisors of the integer 14!.

   Solution:

   First, we find the prime factorization of 14!, which is the following:

$$14! = 2^{11} \times 3^5 \times 5^2 \times 7^2 \times 11 \times 13.$$

   Any divisor of 14! has a prime factorization of the form $2^p \times 3^q \times 5^r \times 7^s \times 11^t \times 13^u$, where p, q, r, s, t, u are integers and $0 \leq p \leq 11$, $0 \leq q \leq 5$, $0 \leq r \leq 2$, $0 \leq s \leq 2$, $0 \leq t \leq 1$, and $0 \leq u \leq 1$.

Hence, the total number of choices for the exponents p, q, r, s, t, u are $12, 6, 3, 3, 2, 2$ respectively, and therefore, the total number of possible divisors is $12 \times 6 \times 3 \times 3 \times 2 \times 2 = 2592$.

4. For a positive integer n, $f(n)$ is defined as the largest power of 3 that is a divisor of n. What is $f\left(\frac{100!}{50!20!}\right)$?

Solution:

First, we count the number of factors of 3 included in 100!. Every multiple of 3 includes at least 1 factor of 3. The product 100! includes 33 multiples of 3 (since $33 \times 3 = 99$). Counting one factor of 3 from each of the multiples of 3 (these are $3, 6, 9, 12, 15, 18, \cdots, 93, 96, 99$), we see that 100! includes at least 33 factors of 3.

However, each multiple of $3^2 = 9$ includes a second factor of 3 (since $9 = 3^2, 18 = 3^2 \times 2$, etc.) which was not counted in the previous 33 factors. The product 100! includes 11 multiples of 9 (since $11 \times 9 = 99$), and thus there are at least 11 additional factors of 3 in 100!.

Similarly, 100! includes 3 multiples of $3^3 = 27$, each of which contribute an additional factor of 3 (these are $27 = 3^3, 54 = 3^3 \times 2$, and $81 = 3^4$).

Finally, there is one multiple of $3^4 = 81$ which contributes one more factor of 3.

Since $3^5 > 100$, then 100! does not include any multiples of 3^5 and so we have counted all possible factors of 3. Thus, 100! includes exactly $33 + 11 + 3 + 1 = 48$ factors of 3, and so $100! = 3^{48} \times t$ for some positive integer t that is not divisible by 3.

Counting in a similar way, the product 50! includes 16 multiples of 3, 5 multiples of 9, and 1 multiple of 27, and thus includes $16+5+1 = 22$ factors of 3. Therefore, $50! = 3^{22} \times r$ for some positive integer r that is not divisible by 3.

Also, 20! includes $6 + 2 = 8$ factors of 3, and thus $20! = 3^8 \times s$ for some positive integer s that is not divisible by 3. Therefore,

$$\frac{100!}{50!20!} = \frac{3^{48} \times t}{(3^{22} \times r)(3^8 \times s)} = \frac{3^{48} \times t}{3^{30} \times rs} = 3^{18} \times \frac{t}{rs}.$$

Since $\frac{t}{rs}$ is an integer (as $\frac{100!}{50!20!}$ is an integer), and each of r, s and t does not include any factors of 3, then 3 is not a divisor of the integer $\frac{t}{rs}$.

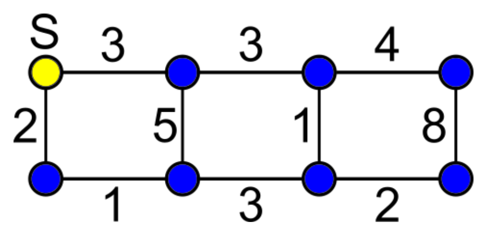
Therefore, the largest power of 3 which divides $\frac{100!}{50!20!}$ is 3^{18}, and so $f\left(\frac{100!}{50!20!}\right) = 18$.

Note: Solutions to these problems can also be found in the Number Theory section of the CEMC's free online course *Problem Solving and Mathematical Discovery*. Check out these problems and more by visiting `https://courseware.cemc.uwaterloo.ca/40`

CEMC at Home

Grade 11/12 - Wednesday, March 25, 2020
Byber Path

You drive for a delivery service called Byber. You start at location S and you have to drop off a package at each of the seven other locations, shown as circles. The locations are joined by roads shown as lines. You cannot visit any location more than once on your route. You can finish at any location that you wish. The number beside a line is the toll you need to pay for taking the corresponding road.



What is the least total amount you need to pay in order to drop off all seven packages?

You may end up with the correct answer for this problem without being completely convinced that your answer is indeed correct! Think about what it would take to completely justify the validity of your answer. You would need to show that there is a route that will result in exactly your amount, and explain why every other route would cost at least as much as yours.

More Info:

Check the CEMC at Home webpage on Wednesday, April 1 for the correct answer with justification.

This problem almost ended up on the 2019 Beaver Computing Challenge (BCC) which is a problem solving contest with a focus on computational and logical thinking.

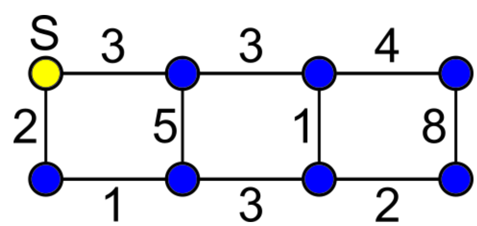
While only officially open to students in Grades 5 to 10, students in Grades 11 and 12 can have fun and learn something by trying the BCC problems. You can find more problems like this on past BCC contests.

CEMC at Home

Grade 11/12 - Wednesday, March 25, 2020
Byber Path - Solution

Problem

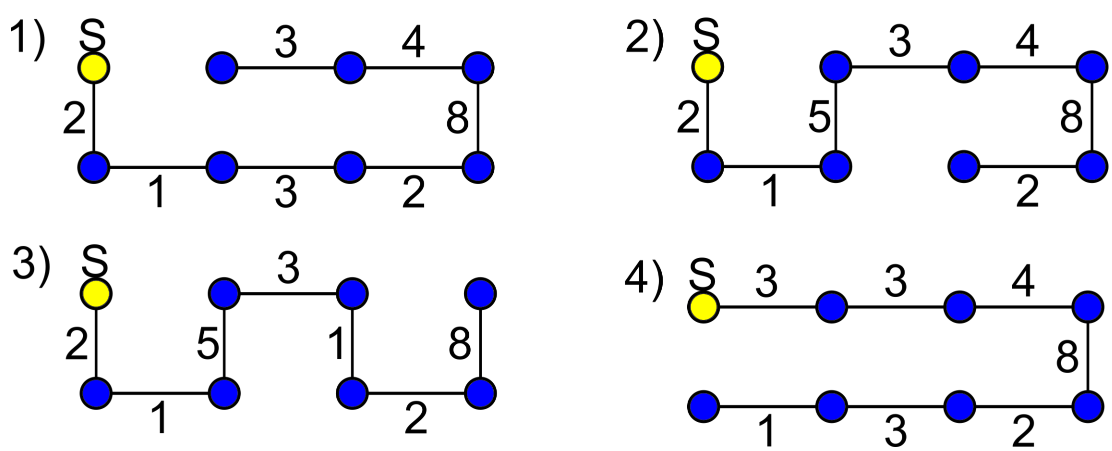
You drive for a delivery service called Byber. You start at location S and you have to drop off a package at each of the seven other locations, shown as circles. The locations are joined by roads shown as lines. You cannot visit any location more than once on your route. You can finish at any location that you wish. The number beside a line is the toll you need to pay for taking the corresponding road.



What is the least total amount you need to pay in order to drop off all seven packages?

Solution

There are only four possible paths that start at S and visit each of the locations exactly once:



To justify this, notice that we have two choices for where to travel from S: down or right.

- If we travel right first, then the rest of our path is determined. If we move down before reaching the right-most location in the top row, then we would need to backtrack to reach all of the bottom locations. This would mean revisiting at least one location on the way. Therefore, we must travel right until we reach the right-most location in the top row, and then must follow path 4) for the remainder.

- If we travel down first, then we must travel right next, but then there we again have two options: up or right.
  - If we travel right, then the rest of our path is determined. We cannot move up next, as then we cannot reach the right-most locations without backtracking, and so we must move right to get to the right-most bottom location. From here we would have to follow path 1) for the remainder.
  - If we travel up, then we must travel right next. From here we have two choices: down or right.
    - If we travel down, then the rest of our path is determined. We must follow path 3).
    - If we travel right, then the rest of our path is determined. We must follow path 2).

These four possible paths have total paid amounts of:

1) $2+1+3+2+8+4+3=23$
2) $2+1+5+3+4+8+2=25$
3) $2+1+5+3+1+2+8=22$
4) $3+3+4+8+2+3+1=24$

(where the sum is shown starting at S and moving through the path).

The least total paid amount of these four paths is 22.

Connections to Computer Science

For each Beaver Computing Challenge problem, we include a short description of its connections to computer science. The italicized keywords emphasize terminology that can be used to search online, if you are interested in learning more.

In this particular problem, the locations and roads can be modelled by a *graph*. Locations are the *vertices* and roads are the *edges* of the graph. To use a computer to solve a problem like this, we need to

- figure out how to represent the graph, and
- discover and implement an *algorithm* to produce the final answer.

In a *programming language*, different *data structures* exist or can be built to represent virtually anything we can image. Amazingly, everything is ultimately modelled by a long *binary* sequence of 0s and 1s.

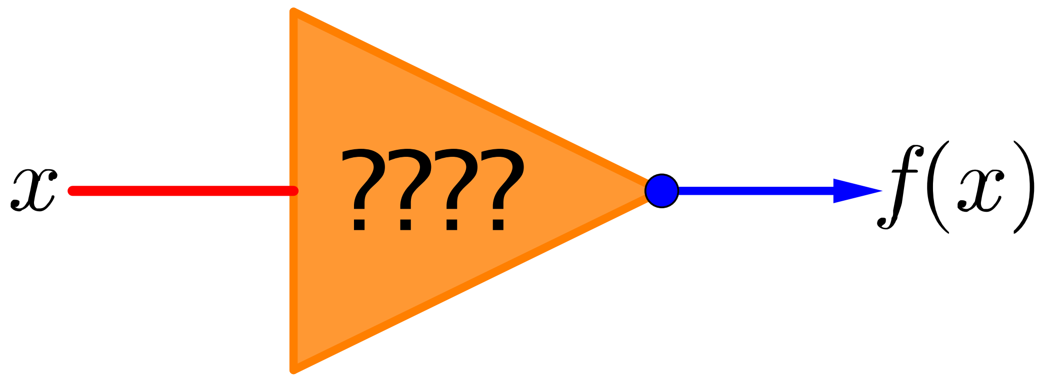
Different computer algorithms are used for finding the best or the worst path through a graph. In this problem, one of the restrictions is to find a path which visits all the vertices exactly once. This is called a *Hamiltonian path*. Problems involving Hamiltonian paths are well-known and considered very difficult. They are closely related to what is known as the famous *Travelling salesman problem*. You can read more about this problem here.

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, March 26, 2020
Functionally Possible

The function $f(x) = x^5 - 3x^4 + ax^3 - x^2 + bx - 2$ has a value of 5 when $x = 3$.

Determine the value of the function when $x = -3$.

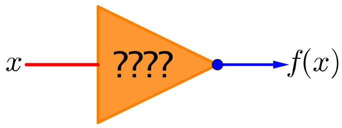


More Info:

Check the CEMC at Home webpage on Thursday, April 2 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 2.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week
Problem E and Solution
Functionally Possible

Problem

The function $f(x) = x^5 - 3x^4 + ax^3 - x^2 + bx - 2$ has a value of 5 when $x = 3$.

Determine the value of the function when $x = -3$.

Solution

We know that the function has a value of 5 when $x = 3$. Therefore, $f(3) = 5$.

$$\begin{aligned} f(3) &= 5 \\ (3)^5 - 3(3)^4 + a(3)^3 - (3)^2 + b(3) - 2 &= 5 \\ 243 - 243 + 27a - 9 + 3b - 2 &= 5 \\ 27a + 3b &= 16 \qquad (1) \end{aligned}$$

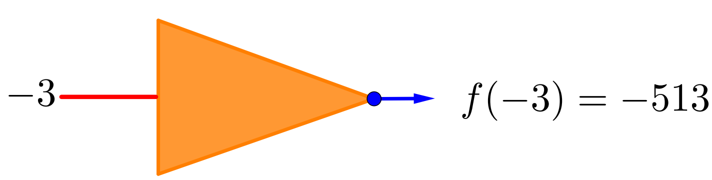
At this point we seem to have used up the given information. Maybe we can learn more by looking at precisely what we are asked to determine.

In this problem, we want the value of the function when $x = -3$. In other words, we want $f(-3)$.

$$\begin{aligned} f(-3) &= (-3)^5 - 3(-3)^4 + a(-3)^3 - (-3)^2 + b(-3) - 2 \\ &= -243 - 243 - 27a - 9 - 3b - 2 \\ &= -27a - 3b - 497 \end{aligned}$$

But from (1) above, $27a + 3b = 16$ so
$f(-3) = -27a - 3b - 497 = -(27a + 3b) - 497 = -16 - 497 = -513$.

Therefore, the value of the function is -513 when $x = -3$.



We are not given enough information to find the precise values of a and b but enough information is given to solve the problem.

CEMC at Home

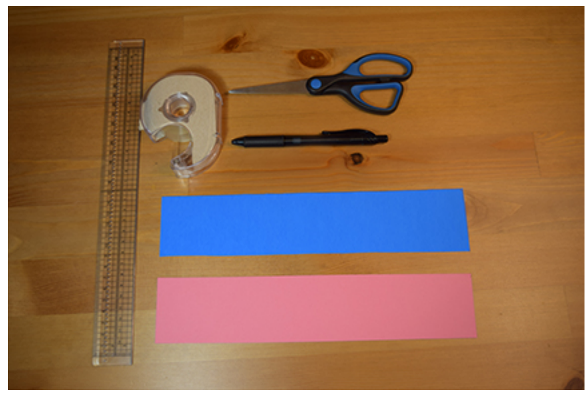
Grade 11/12 - Friday, March 27, 2020
The Möbius Strip

A Möbius strip is a mathematical object that has interesting properties. It is a surface with only one face (or side) and only one edge (or boundary). In this activity we will build a Möbius strip and investigate its curious properties. The purpose of this activity is not to fully understand the mathematics of a Möbius strip, but rather to hopefully surprise and intrigue you!

You will need:

- A pencil
- A ruler
- Scissors
- Tape
- Two rectangular strips of paper of different colours.

  Strips of around 6 cm wide and 30 cm long will work well. We will use one blue strip and one pink strip, but you can use any colours you want.



How to construct a Möbius strip:

1. Use your ruler to draw a line along the blue strip that divides the strip into two equal parts. Do the same on the other side of the strip.
2. Use your ruler to draw two lines along the pink strip that divides the strip into three equal parts. Do the same on the other side of the strip.
3. Grab the blue strip by the two short edges. Twist one end of the strip half of the way around and join the two short edges together. (Make sure this is a "half twist" and not a "full twist".) Line up the short edges and tape them together from end to end.
4. Repeat the same process with the pink strip. You now have two Möbius strips.

Let's explore some properties of our Möbius strips!

1. Take one of the strips you made and answer the following questions:

   (a) How many faces does the Möbius strip have?

   You might need to spend some time thinking about what is meant by a "face" here.

   (b) How many edges does the Möbius strip have?

   You might need to spend some time thinking about what is meant by an "edge" here.

   (c) Does the Möbius strip have an "inside" and an "outside"?

2. Take the blue Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along the line drawn in the middle of the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the blue strip along the middle line. You will need to carefully cut or puncture the strip somewhere along this line in order to start the cut. What happens once you cut along this line? Is it what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

3. Take the pink Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along one of the two lines that we drew down down the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the pink strip along one of the lines. When you cut, you might notice that it doesn't actually matter which of the two lines you chose to cut along. Is the result what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

More Info:

Check the CEMC at Home webpage on Friday, April 3 for further discussion on The Möbius Strip.

CEMC at Home

Grade 11/12 - Friday, March 27, 2020
The Möbius Strip - Solution

1. Take one of the strips you made and answer the following questions:

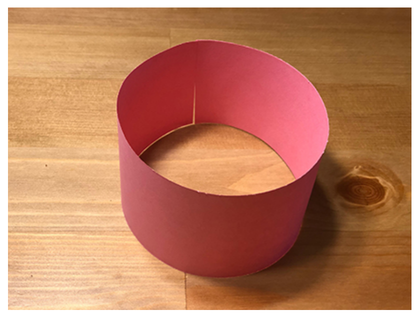
   (a) How many faces does the Möbius strip have?

   (b) How many edges does the Möbius strip have?

   (c) Does the Möbius strip have an "inside" and an "outside"?

   Discussion:

   We will compare our Möbius strip with a cylinder. A cylinder can be made by taking the two short edges of a strip, lining them up, and taping them together from end to end.

   A cylinder has an "outside" face and an "inside" face. An ant walking on one of these faces must cross an edge (or boundary) to get to the other face. The Möbius strip has only one face (or side). An ant can walk along the entire surface of the Möbius strip without crossing an edge (or boundary). In particular, an ant that begins on any part of either line we drew, can follow this line and will end up back where it started. In doing this, it will have travelled the full length of *both* lines drawn on the two sides of the original blue strip.

   Similarly, a cylinder has a "top" edge and "bottom" edge, but a Möbius strip has only one edge (or boundary). Imagine the ant walking along the edge of the Möbius strip. The ant will travel along the entire edge of the strip and will end up back where it started.

   In summary, a cylinder has two faces and two edges, but a Möbius strip only has one face and one edge. Both can be created from a single strip of paper.

2. Take the blue Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along the line drawn in the middle of the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the blue strip along the middle line. You will need to carefully cut or puncture the strip somewhere along this line in order to start the cut. What happens once you cut along this line? Is it what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

Discussion:

Following our intuition with the cylinder, we know that if we cut the cylinder along a middle line parallel to its two edges, then we will obtain two smaller detached cylinders. However, if we cut the Möbius strip along the middle line, the result might surprise us in two ways:

- We get one strip instead of two detached pieces.
- The strip we get is not a Möbius strip!

This result is illustrated in Figure 1.

Figure 1

One way to help us understand this is to think about the process of "gluing and cutting" in a different order. We can *begin* by cutting along the middle line and taping the two pieces together leaving a gap to show our cut. *After* doing this, we can do a "half twist" and join the short edges together as in the original instructions. This is illustrated in Figure 2. Notice that the green ends connect to each other and the black ends also connect to each other.

Figure 2

Try to use the analogy of an ant walking on the surface and edges of the resulting strip to convince yourself that this strip is an attached single piece with two faces and two sides. An effective way to do this is to do the construction shown in Figure 2 yourself and follow the path an ant might take with your finger. Interestingly, this resulting strip is what we would get if we followed the original instructions using two "full twists" instead of a "half twist".

3. Take the pink Möbius strip and answer the following questions:

   (a) What do you think will happen if you cut the strip along one of the two lines that we drew down down the strip? How many detached pieces do you think you will you get? Will they be Möbius strips? Make your predictions.

   (b) Let's verify your predictions. Cut the pink strip along one of the lines. When you cut, you might notice that it doesn't actually matter which of the two lines you chose to cut along. Is the result what you predicted? How many edges does each detached piece have? How many faces?

   (c) After making your cut, you may have ended up with an object that surprised you. Looking back, can you explain why you ended up with this object?

Discussion:

As before, Figure 3 is an illustration of the result and Figure 4 illustrates what happens if we *begin* by cutting and taping before doing a "half twist" and joining the short edges together.

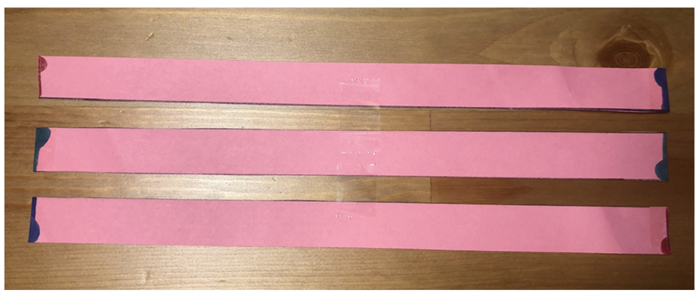


Figure 3

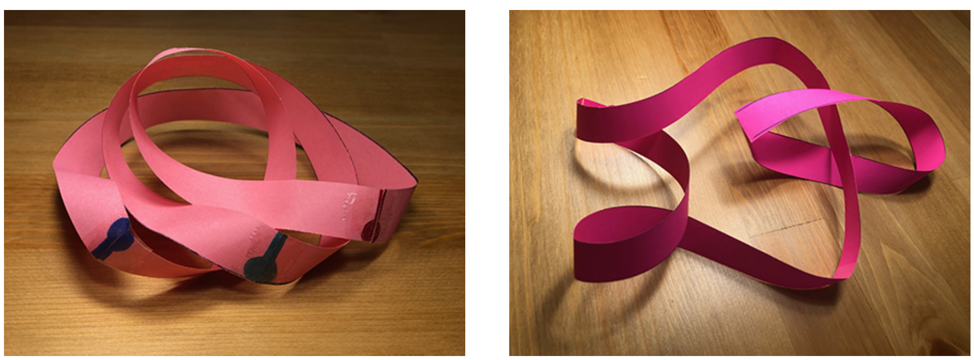


Figure 4

Some interesting and surprising things happen:

- The opposite ends of the upper and lower strips connect to each other making a longer strip which is similar in structure (but narrower) than the one we obtained from the blue strip.
- We also get a shorter strip which is a Möbius strip.
- These two strips are linked together!