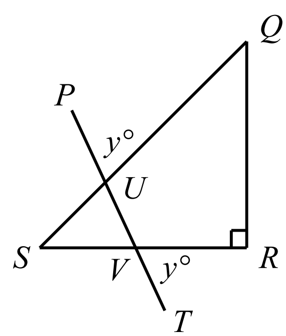
CEMC at Home

Grade 7/8 - Monday, June 8, 2020
Contest Day 6

Today's resource features two questions from the recently released 2020 CEMC Mathematics Contests.

2020 Gauss Contest, #16

In the diagram, $\triangle QRS$ is an isosceles right-angled triangle with $QR = SR$ and $\angle QRS = 90°$. Line segment PT intersects SQ at U and SR at V. If $\angle PUQ = \angle RVT = y°$, the value of y is



(A) 72.5 **(B)** 60 **(C)** 67.5
(D) 62.5 **(E)** 70

2020 Gauss Contest, #20

If a and b are positive integers and $\frac{20}{19} = 1 + \cfrac{1}{1 + \frac{a}{b}}$, what is the least possible value of $a + b$?

(A) 16 **(B)** 19 **(C)** 20 **(D)** 38 **(E)** 39

More Info:

Check out the CEMC at Home webpage on Monday, June 15 for solutions to the Contest Day 6 problems.

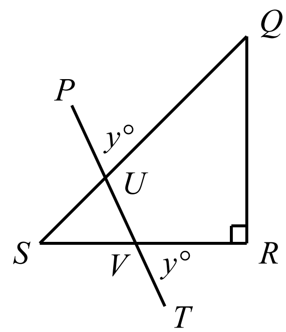
CEMC at Home

Grade 7/8 - Monday, June 8, 2020
Contest Day 6 - Solution

Solutions to the two contest problems are provided below.

2020 Gauss Contest, #16

In the diagram, $\triangle QRS$ is an isosceles right-angled triangle with $QR = SR$ and $\angle QRS = 90^\circ$. Line segment PT intersects SQ at U and SR at V. If $\angle PUQ = \angle RVT = y^\circ$, the value of y is



(A) 72.5 **(B)** 60 **(C)** 67.5

(D) 62.5 **(E)** 70

Solution:

Since $\triangle QRS$ is an isosceles right-angled triangle with $QR = SR$, then $\angle RQS = \angle RSQ = 45^\circ$. Opposite angles are equal in measure, and so $\angle SUV = \angle PUQ = y^\circ$ and $\angle SVU = \angle RVT = y^\circ$. In $\triangle SVU$, $\angle VSU + \angle SUV + \angle SVU = 180^\circ$ or $45^\circ + y^\circ + y^\circ = 180^\circ$ or $2y = 135$ and so $y = 67.5$.

ANSWER: (C)

2020 Gauss Contest, #20

If a and b are positive integers and $\frac{20}{19} = 1 + \cfrac{1}{1 + \frac{a}{b}}$, what is the least possible value of $a + b$?

(A) 16 **(B)** 19 **(C)** 20 **(D)** 38 **(E)** 39

Solution:

We begin by expressing $\frac{20}{19}$ in a form that is similar to the right side of the given equation.
Converting $\frac{20}{19}$ to a mixed fraction we get, $\frac{20}{19} = 1\frac{1}{19} = 1 + \frac{1}{19}$.
Since $\frac{20}{19} = 1 + \cfrac{1}{1 + \frac{a}{b}}$ and $\frac{20}{19} = 1 + \frac{1}{19}$, then $1 + \cfrac{1}{1 + \frac{a}{b}} = 1 + \frac{1}{19}$ and so $\cfrac{1}{1 + \frac{a}{b}} = \frac{1}{19}$.
The numerators of $\cfrac{1}{1 + \frac{a}{b}}$ and $\frac{1}{19}$ are each equal to 1, and since these fractions are equal to one another, their denominators must also be equal.
That is, $1 + \frac{a}{b} = 19$ and so $\frac{a}{b} = 18$.
Since a and b are positive integers, then the fractions $\frac{a}{b}$ which are equal to 18 are $\frac{18}{1}$, $\frac{36}{2}$, $\frac{54}{3}$, and so on.
Thus, the least possible value of $a + b$ is $18 + 1 = 19$.

ANSWER: (B)

CEMC at Home

Grade 7/8 - Tuesday, June 9, 2020
All the Marbles

Problem 1: There are two boxes of marbles. Box A contains ____ red marbles and ____ green marbles. Box B contains ____ red marbles and ____ green marbles. A marble is drawn from each box at random.

(a) Fill in the blanks using four different digits from 1 to 9 so that the probability of drawing a red marble from Box A is higher than the probability of drawing a red marble from Box B.

To get a low probability you want a large number of marbles in the box in total, and a small number of the specific marble in the box. How can this help you choose your numbers?

(b) Fill in the blanks using four different digits from 1 to 9 so that the probability of drawing a red marble is the same for each box.

Need help getting started? Try out your answers on our interactive tool. Note that there are many possible answers for each question. Can you find more than one?

Problem 2: A box contains ____ red, ____ blue, ____ green, and ____ yellow marbles. A marble is drawn from the box at random.

(a) Fill in the blanks using four different digits from 1 to 9 so that the probability of drawing each colour of marble matches the information below.

$$\text{Red: } \frac{5}{24} \qquad \text{Blue: } \frac{1}{8} \qquad \text{Green: } \frac{3}{8} \qquad \text{Yellow: } \frac{7}{24}$$

Is your solution unique? Try to find a second solution, or explain why one does not exist.

(b) Is it possible to fill in the blanks using four different digits from 1 to 9 so that the probability of drawing each colour of marble matches the information below? Explain.

$$\text{Red: } \frac{1}{12} \qquad \text{Blue: } \frac{1}{4} \qquad \text{Green: } \frac{5}{12} \qquad \text{Yellow: } \frac{1}{3}$$

More Info:

Check out the CEMC at Home webpage on Wednesday, June 10 for a solution to All the Marbles.

For more practice with probability, check out this lesson in the CEMC Courseware.

CEMC at Home

Grade 7/8 - Tuesday, June 9, 2020
All the Marbles - Solution

Problem 1: Box A contains _____ red marbles and _____ green marbles. Box B contains _____ red marbles and _____ green marbles. A marble is drawn from each box at random.

(a) Fill in the blanks using four different digits from 1 to 9 so that the probability of drawing a red marble from Box A is higher than the probability of drawing a red marble from Box B.

Solution:

Here it makes sense to have many red marbles and few green marbles in Box A and few red marbles and many green marbles in Box B.

Let's place 9 red marbles and 1 green marble in Box A. This would mean $9 + 1 = 10$ marbles in total in Box A, 9 of which are red. This means that the probability of drawing a red marble from Box A is $\frac{9}{10}$.

Let's place place 2 red marbles and 8 green marbles in Box B. (Remember that we cannot use the digits 1 and 9 again.) This would mean $2 + 8 = 10$ marbles in total in Box B, 2 of which are red. This means that the probability of drawing a red marble from Box B is $\frac{2}{10}$

Since $\frac{9}{10} > \frac{2}{10}$, the probability of drawing a red marble from Box A is higher than the probability of drawing a red marble from Box B.

Note that there are many other strategies and combinations of digits that will also work here. Can you find a few more? Can you find them all?

(b) Fill in the blanks using four different digits from 1 to 9 so that the probability of drawing a red marble from each box is the same.

Solution:

One strategy is to choose, in advance, the probability we want for each of the two boxes, and then find digits that result in this probability.

Suppose we want the probability of drawing a red marble from the boxes to be $\frac{1}{3}$. That means for every red marble in each box, there are 2 green marbles. So the boxes contain twice as many green marbles as red marbles. There are four pairs of digits where one is twice the other:

1 and 2, 2 and 4, 3 and 6, 4 and 8

We want to use these pairs as the numbers of red and green marbles in the boxes. Since we cannot use the same digit more than once, we must pair these in one of the following ways:

- One box: 1 and 2; other box: 3 and 6
- One box: 1 and 2; other box: 4 and 8
- One box: 2 and 4; other box: 3 and 6
- One box: 3 and 6; other box: 4 and 8

Let's choose the first pairing: 1 red marble and 2 green marbles in Box A; 3 red marbles and 6 green marbles in Box B. This means the probability of drawing a red marble from Box A is $\frac{1}{1+2} = \frac{1}{3}$ and the probability of drawing a red marble from Box B is $\frac{3}{3+6} = \frac{3}{9} = \frac{1}{3}$ as well.

It is possible to use this strategy with a different target probability, however some probabilities will not work. For example, can you see why $\frac{1}{2}$ would not work out for this question?

Problem 2: A box contains _____ red, _____ blue, _____ green, and _____ yellow marbles. A marble is drawn from the box at random.

(a) Fill in the blanks using four different digits from 1 to 9 so that the probability of drawing each colour of marble matches the information below.

$$\text{Red: } \frac{5}{24} \qquad \text{Blue: } \frac{1}{8} \qquad \text{Green: } \frac{3}{8} \qquad \text{Yellow: } \frac{7}{24}$$

Solution:

Let's start by writing the probabilities with a common denominator. Since $8 \times 3 = 24$, the lowest common denominator is 24.

$$\text{Red: } \frac{5}{24} \qquad \text{Blue: } \frac{1}{8} = \frac{3}{24} \qquad \text{Green: } \frac{3}{8} = \frac{9}{24} \qquad \text{Yellow: } \frac{7}{24}$$

Notice that the sum of the numerators is $5 + 3 + 9 + 7 = 24$. This mean that if we have 5 red marbles, 3 blue marbles, 9 green marbles, and 7 yellow marbles in the box then we will have 24 marbles in total, and we would get the correct probability for each colour of marble.

There is only one solution to this problem. Can you convince yourself that there is no other way to fill in the blanks according to the rules and get these probabilities?

(b) Is it possible to fill in the blanks using four different digits from 1 to 9 so that the probability of drawing each colour of marble matches the information below? Explain.

$$\text{Red: } \frac{1}{12} \qquad \text{Blue: } \frac{1}{4} \qquad \text{Green: } \frac{5}{12} \qquad \text{Yellow: } \frac{1}{3}$$

Solution:

It is not possible to fill in the blanks in a way that produces the four probabilities given. Here is one way to explain why there is no solution:

We start by writing the probabilities with a common denominator. We can see that the lowest common denominator is 12.

$$\text{Red: } \frac{1}{12} \qquad \text{Blue: } \frac{1}{4} = \frac{3}{12} \qquad \text{Green: } \frac{5}{12} \qquad \text{Yellow: } \frac{1}{3} = \frac{4}{12}$$

Notice that the sum of the numerators is $1+3+5+4 = 13$ and so the sum of these probabilities is

$$\frac{1}{12} + \frac{3}{12} + \frac{5}{12} + \frac{4}{12} = \frac{13}{12}$$

This is a problem! Can you see why? The probabilities of the four possible outcomes should add up to 1, but these four probabilities add up to a value greater than 1.

The probability of drawing each colour of marble represents the fraction of the marbles that are that particular colour. For example, if the probability of drawing a red marble is $\frac{1}{12}$, then it must be the case that $\frac{1}{12}$ of the marbles are red. Similarly, if the other three probabilities are as given, it must be the case that $\frac{3}{12}$ of the marbles are blue, $\frac{5}{12}$ of the marbles are green, and $\frac{4}{12}$ of the marbles are yellow. These fractions cannot possibly all be correct because we cannot have four parts of a whole that add up to more than the whole.

Therefore, we cannot fill in the blanks so that the probability of drawing each marble matches the information given.

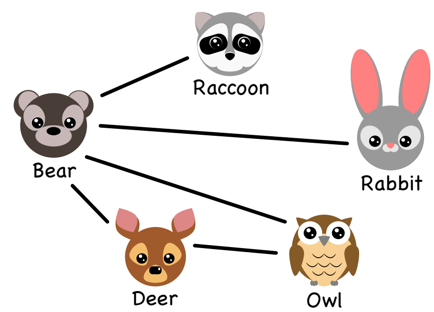
There are other ways to argue that these probabilities cannot be achieved.

CEMC at Home

Grade 7/8 - Wednesday, June 10, 2020
Do I Know You?

A bear, raccoon, rabbit, owl, and deer live in the same forest. Some of the five animals have met before, and some have not. How many of the other animals they have each met before is recorded in the table below. From the information in the table, we can draw a diagram that shows one possibility for which pairs of animals have and have not met. A line is drawn between two animals in the diagram if the two animals have met before, otherwise there is no line drawn.

Animal	Number of Animals Met
Bear	4
Rabbit	1
Raccoon	1
Owl	2
Deer	2



Since the bear has met 4 animals, it must have met all of the other animals. This means there must be a line between the bear and each of the other four animals. Since the rabbit and the racoon have each only met one other animal, it must have been the bear. This means there cannot be any more lines drawn from the rabbit or the racoon. Since the owl has met 2 other animals, it must have also met the deer as shown. Notice that this diagram matches the information in the table.

Problem 1: Five different animals recorded how many of the other animals they had each met before in a table. Which of the following tables are possible? Explain your answers.

Need help getting started? Try drawing a diagram like the one above for each table.

A.

Animal	Number
Elephant	3
Zebra	2
Monkey	4
Tiger	2
Snake	1

B.

Animal	Number
Elephant	1
Zebra	1
Monkey	1
Tiger	1
Snake	1

C.

Animal	Number
Elephant	2
Zebra	3
Monkey	1
Tiger	1
Snake	3

Problem 2: Five different animals recorded how many of the other animals they had each met before in the table shown. Find all possible values for the missing number in the table.

Animals	Number of Animals Met
Cat	3
Dog	3
Hamster	?
Gecko	4
Bird	4

More Info:

Check out the CEMC at Home webpage on Thursday, June 11 for a solution to Do I Know You?

A variation of this problem appeared on a past Beaver Computing Challenge (BCC).

CEMC at Home

Grade 7/8 - Wednesday, June 10, 2020
Do I Know You? - Solution

Problem 1: Five different animals recorded how many of the other animals they had each met before in a table. Which of the following tables are possible? Explain your answers.

A.

Animal	Number
Elephant	3
Zebra	2
Monkey	4
Tiger	2
Snake	1

B.

Animal	Number
Elephant	1
Zebra	1
Monkey	1
Tiger	1
Snake	1

C.

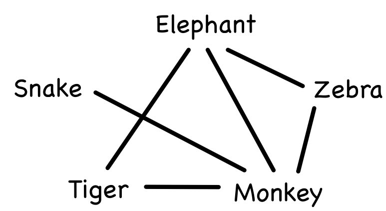
Animal	Number
Elephant	2
Zebra	3
Monkey	1
Tiger	1
Snake	3

Solution:

Let's try drawing a diagram for each table.

A. This table is possible.

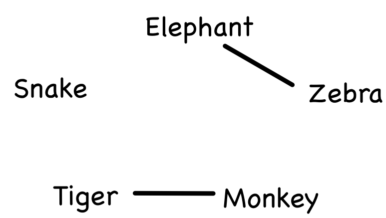
We see that the monkey has met all 4 of the other animals. That means the snake has met only the monkey. The elephant has met 3 animals, so must have also met the zebra and tiger, in addition to the monkey, as it could not have met the snake. The completed diagram given on the right shows a scenario that would result in the numbers in this table. (Is this the only possible diagram?)



B. This table is *not* possible.

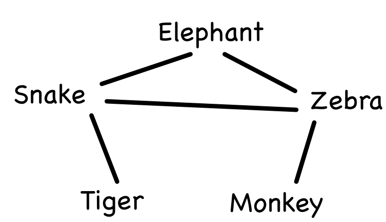
Each animal has met exactly one other animal. This means that each animal must be a part of exactly one line in our diagram and so the animals must be "paired". But there are an odd number of animals so this is impossible.

For example, suppose that we draw a line between the elephant and the zebra and another line between the monkey and the tiger, so that each of these four animals has met exactly one other animal (as in the table). Now, to get the right number for the snake, we need to draw a line from the snake to another animal, but we cannot do so without raising the other animal's number to 2. We will run into a similar problem no matter how we try to pair up the animals.



C. This table is possible.

Suppose that the zebra has met the elephant, the snake, and the monkey. Then the snake could have also met the elephant and the tiger. The completed diagram given on the right shows a scenario that would result in the numbers in this table. (Is this the only possible diagram?)



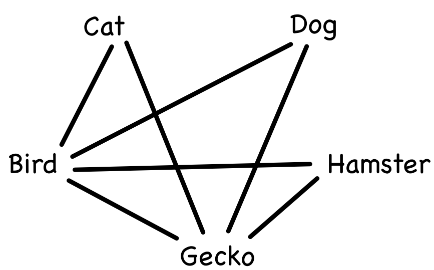
Problem 2: Five different animals recorded how many of the other animals they had each met before in the table shown. Find all possible values for the missing number in the table.

Animals	**Number of Animals Met**
Cat	3
Dog	3
Hamster	?
Gecko	4
Bird	4

Solution:

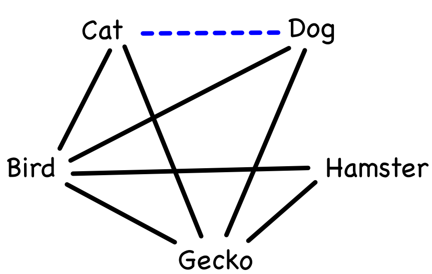
Let's try drawing a diagram to figure out how many other animals the hamster could have met.

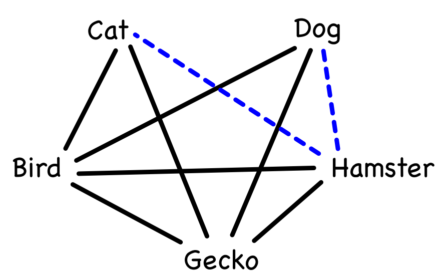
From the table, we can see that the gecko and the bird have each met all 4 of the other animals. So we start off by creating the following diagram that displays this information.



Notice that the above diagram is not complete as the cat and the dog have each met 3 animals, which is 1 more each than is shown in this diagram.

We now have two choices for how to complete the diagram. The cat and the dog could either have met each other and not the hamster (as in the diagram below on the let) or they could have each met the hamster and not each other (as in the diagram below on the right).





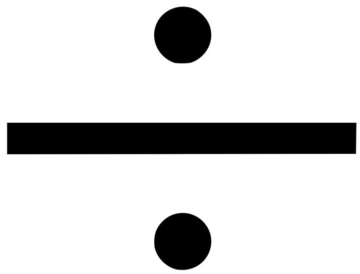
In the first case, the hamster has met 2 other animals, and in the second case, the hamster has met all 4 of the other animals.

This means there are two possible values for the missing number in the table: 2 or 4.

CEMC at Home

Grade 7/8 - Thursday, June 11, 2020
Mystery Number

A positive integer has exactly eight positive factors. If two of the factors are 21 and 35, what is the positive integer?



For some integer n, a factor of n is a non-zero integer that divides evenly into n.
For example, 3 is a factor of 18 since $18 \div 3 = 6$, but 4 is not a factor of 18 since $18 \div 4 = 4.5$.

More Info:

Check out the CEMC at Home webpage on Friday, June 12 for a solution to Mystery Number.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 7/8 - Thursday, June 11, 2020
Mystery Number - Solution

Problem:

A positive integer has exactly eight positive factors. If two of the factors are 21 and 35, what is the positive integer?

For some integer n, a factor of n is a non-zero integer that divides evenly into n.
For example, 3 is a factor of 18 since $18 \div 3 = 6$, but 4 is not a factor of 18 since $18 \div 4 = 4.5$.

Solution:

Let n represent the number we are looking for.

We know that four of the positive factors of n are 1, 21, 35 and n. In our solution we will first find the remaining four positive factors and then determine n.

Since 21 is a factor of n and $21 = 3 \times 7$, 3 and 7 must also be factors of n.

Since 35 is a factor of n and $35 = 5 \times 7$, 5 must also be a factor of n.

Since 3 is a factor of n and 5 is a factor of n, and since 3 and 5 have no common factors, $3 \times 5 = 15$ must also be a factor n.

We have found all eight of the positive factors of the unknown number. The positive factors are 1, 3, 5, 7, 15, 21, 35 and n. We now need to determine n.

From the list of factors, we see that the prime factors of n are 3, 5 and 7, and it follows that $n = 3 \times 5 \times 7 = 105$.

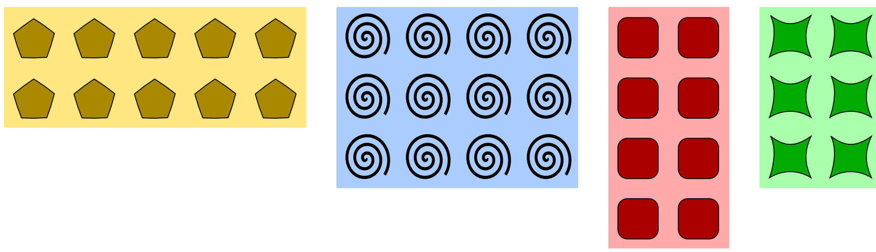
Therefore, the positive integer is 105.

CEMC at Home

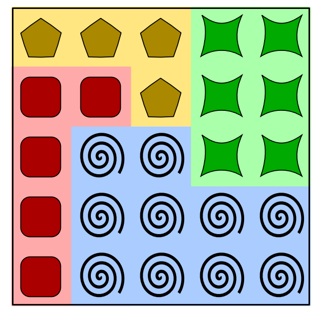
Grade 7/8 - Friday, June 12, 2020
So Many Layers

Problem 1: Kouji covered a wall with four overlapping rectangular sheets of wallpaper as shown. Each sheet of wallpaper is designed using a different image in a repeating pattern. In what order did Kouji place the wallpaper sheets on the wall?

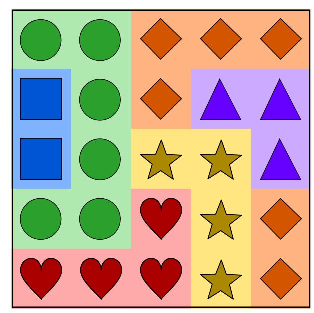
Wallpaper Pieces Used



Result



Problem 2: Tanu covered a wall with six overlapping rectangular sheets of wallpaper as shown. Each sheet of wallpaper is designed using a different image in a repeating pattern.



(a) Give possible dimensions for the six sheets of wallpaper Tanu used.

(b) Determine in what order Tanu placed the wallpaper sheets on the wall.

To help get started, cut out pieces of wallpaper using the images provided on the next page and arrange your pieces to help you solve the problem. Which wallpaper sheets have only one possible set of dimensions and which have more than one?

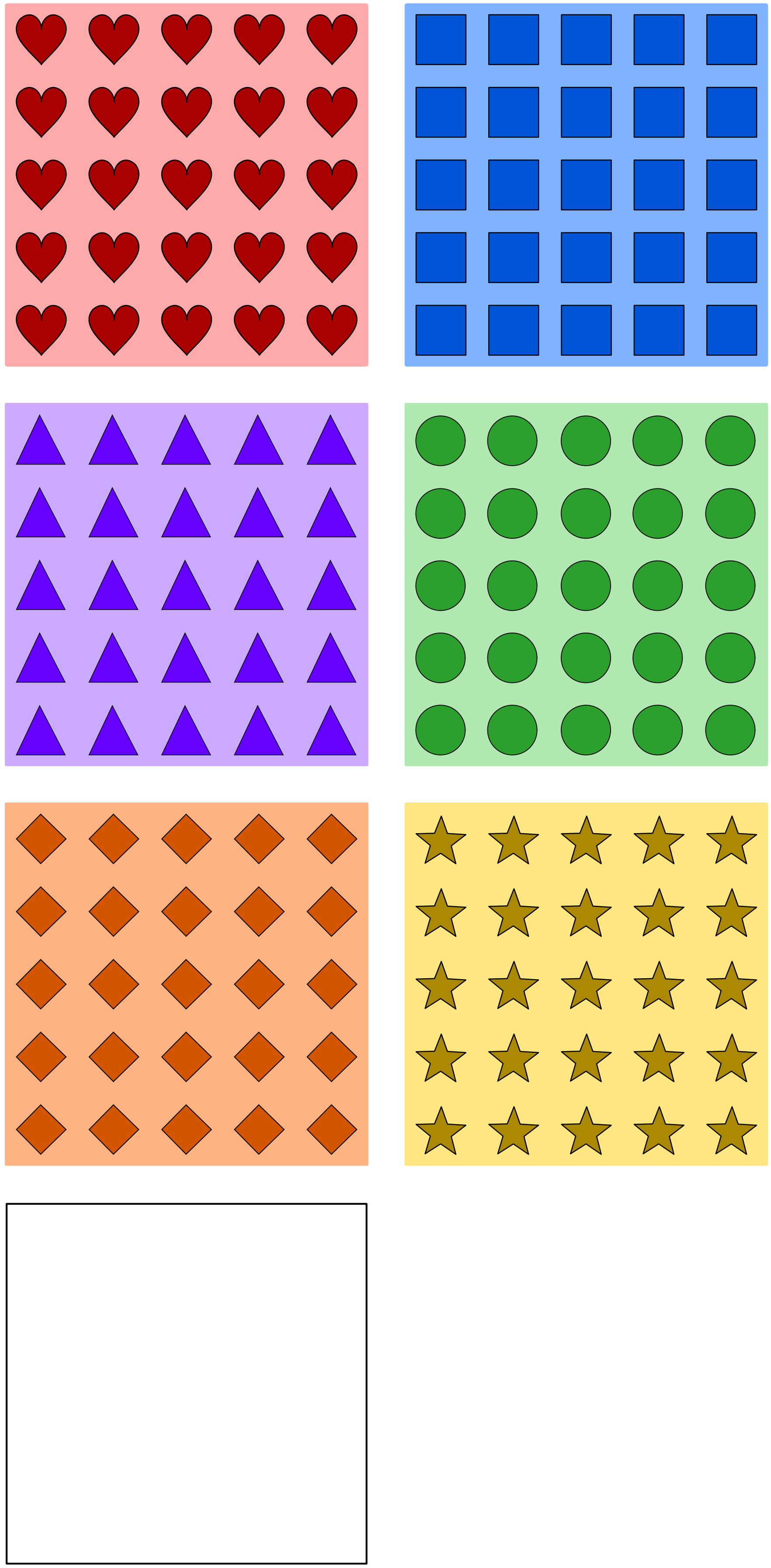
Problem 3: Tanu changes her mind and decides she wants to place the six sheets of wallpaper (from Problem 2) on the wall so that you cannot tell that the wallpaper is overlapping. Each sheet of wallpaper should be visible, but the visible piece should be in the shape of a rectangle so that it looks like the wallpaper pieces were cut to fit right next to each other. Draw one way that Tanu could cover the wall.

There are different possibilities for the dimensions of some of the sheets from Problem 2. You can use the dimensions you found in Problem 2 here, or experiment with different possibilities!

More Info:

Check out the CEMC at Home webpage on Monday, June 15 for a solution to So Many Layers.

A variation of this problem appeared on a past Beaver Computing Challenge (BCC). The BCC is a problem solving contest with a focus on computational and logical thinking.

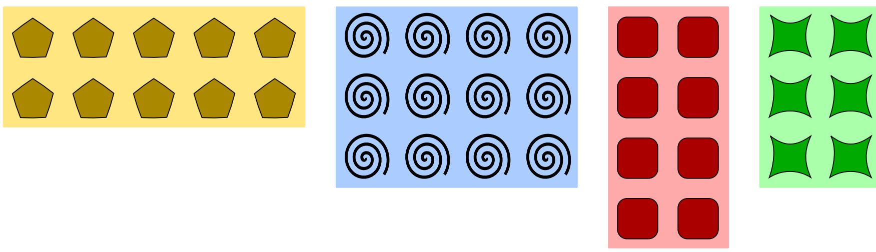


CEMC at Home

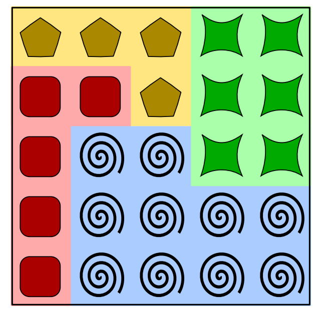
Grade 7/8 - Friday, June 12, 2020
So Many Layers - Solution

Problem 1: Kouji covered a wall with four overlapping rectangular sheets of wallpaper as shown. Each sheet of wallpaper is designed using a different image in a repeating pattern. In what order did Kouji place the wallpaper sheets on the wall?

Wallpaper Pieces Used

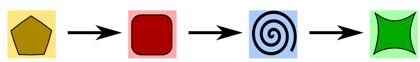


Result



Solution:

The four wallpaper sheets were placed in the following order, from first to last:

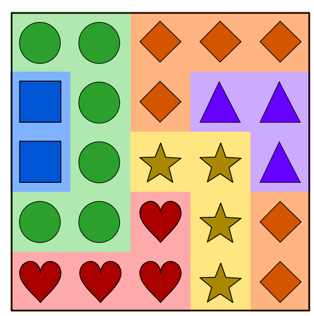


To see why this is true, first observe that the wallpaper with the green star-like squares is the only wallpaper that is entirely visible, so it must have been placed last. It remains to justify the order in which the other three sheets were placed. The wallpaper with the red rounded squares is cut off by the blue spirals, so it must have been placed before the blue spirals. The wallpaper with the yellow pentagons is cut off by the red rounded squares, so it must have been placed before the red rounded squares. In order from first to last, the first three sheets to be placed must have been the yellow pentagons, the red rounded squares, and the blue spirals.

Problem 2: Tanu covered a wall with six overlapping rectangular sheets of wallpaper as shown. Each sheet of wallpaper is designed using a different image in a repeating pattern.

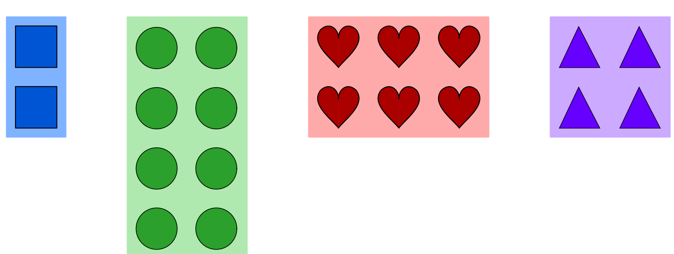
(a) Give possible dimensions for the six sheets of wallpaper Tanu used.

(b) Determine in what order Tanu placed the wallpaper sheets on the wall.

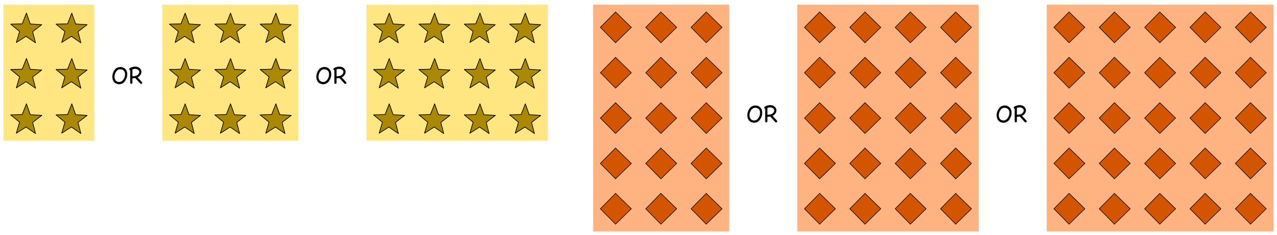


Solution:

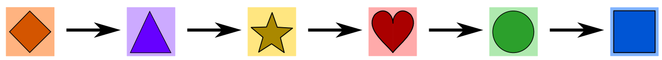
(a) Since each sheet is rectangular, we can see enough to determine that four of the sheets have the dimensions shown below:



We cannot be sure of the dimensions of the remaining two sheets. Here are the different possibilities for these sheets:



(b) The wallpaper sheets were placed in the following order, from first to last:



The wallpaper with the blue squares is the only wallpaper that is entirely visible, so it must have been placed last. The wallpaper with the green circles is cut off by the blue squares, so it must have been placed before the blue squares. By similar reasoning, the red hearts were placed before the green circles, the yellow stars were placed before the red hearts, the purple triangles were placed before the yellow stars, and the orange diamonds were placed before the purple triangles. Therefore, the sheets must have been placed in the order indicated above.

Note that we do not need to know the dimensions of all six wallpaper sheets to determine the order in which they must have been placed. For example, we cannot be sure whether the orange sheet and the green sheet overlap or are placed side-by-side, but this does not stop us from figuring out in which order they were placed.

Problem 3: Tanu changes her mind and decides she wants to place the six sheets of wallpaper (from Problem 2) on the wall so that you cannot tell that the wallpaper is overlapping. Each sheet of wallpaper should be visible, but the visible piece should be in the shape of a rectangle so that it looks like the wallpaper pieces were cut to fit right next to each other. Draw one way that Tanu could cover the wall.

Solution:

Your solution may depend on the dimensions you chose for the wallpapers with yellow stars and orange diamonds. Here are two possible solutions. Can you find others?