CEMC at Home

Grade 11/12 - Monday, June 8, 2020
Contest Day 6

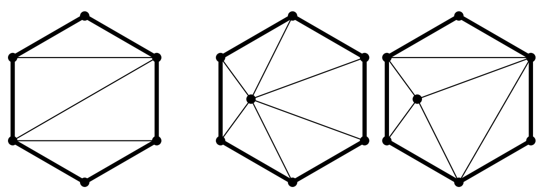
Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Hypatia Contest, #3

A *triangulation* of a regular polygon is a division of its interior into triangular regions. In such a division, each vertex of each triangle is either a vertex of the polygon or an interior point of the polygon. In a triangulation of a regular polygon with $n \geq 3$ vertices and $k \geq 0$ interior points with no three of these $n + k$ points lying on the same line,

- no two line segments connecting pairs of these points cross anywhere except at their endpoints, and
- each interior point is a vertex of at least one of the triangular regions.

Every regular polygon has at least one triangulation. The number of triangles formed by any triangulation of a regular polygon with n vertices and k interior points is constant and is denoted $T(n, k)$. For example, in every possible triangulation of a regular hexagon and one interior point, there are exactly 6 triangles. That is, $T(6, 1) = 6$.



$T(6, 0) = 4$ $T(6, 1) = 6$

(a) What is the value of $T(3, 2)$?

(b) Determine the value of $T(4, 100)$.

(c) Determine the value of n for which $T(n, n) = 2020$.

More Info:

Check out the CEMC at Home webpage on Monday, June 15 for a solution to the Contest Day 6 problem.

CEMC at Home

Grade 11/12 - Monday, June 8, 2020
Contest Day 6 - Solution

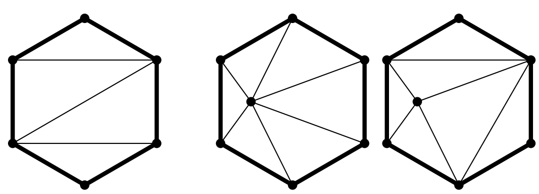
A solution to the contest problem is provided below.

2020 Hypatia Contest, #3

A *triangulation* of a regular polygon is a division of its interior into triangular regions. In such a division, each vertex of each triangle is either a vertex of the polygon or an interior point of the polygon. In a triangulation of a regular polygon with $n \geq 3$ vertices and $k \geq 0$ interior points with no three of these $n + k$ points lying on the same line,

- no two line segments connecting pairs of these points cross anywhere except at their endpoints, and
- each interior point is a vertex of at least one of the triangular regions.

Every regular polygon has at least one triangulation. The number of triangles formed by any triangulation of a regular polygon with n vertices and k interior points is constant and is denoted $T(n, k)$. For example, in every possible triangulation of a regular hexagon and one interior point, there are exactly 6 triangles. That is, $T(6, 1) = 6$.



$T(6, 0) = 4$ $\qquad$ $T(6, 1) = 6$

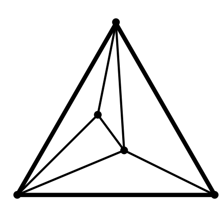
(a) What is the value of $T(3, 2)$?

(b) Determine the value of $T(4, 100)$.

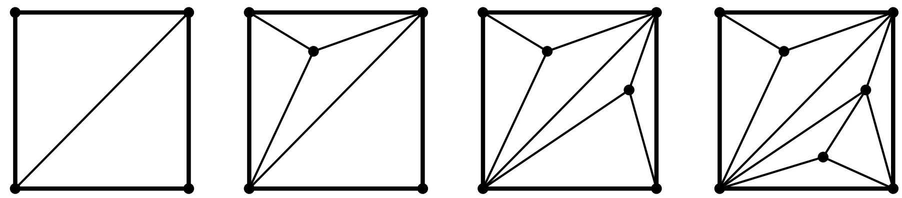
(c) Determine the value of n for which $T(n, n) = 2020$.

Solution:

(a) For $n \geq 3$ and $k \geq 0$, the value of $T(n, k)$ is constant for all possible locations of the k interior points and all possible triangulations.
Thus, we may use the triangulation shown to determine that $T(3, 2) = 5$.

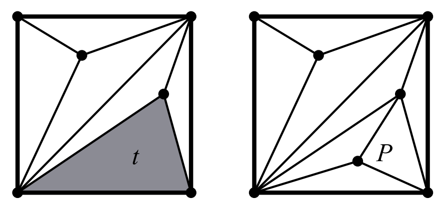


(b) We begin by drawing triangulations to determine the values of $T(4,k)$ for $k = 0, 1, 2, 3$.



$T(4,0)=2$ $T(4,1)=4$ $T(4,2)=6$ $T(4,3)=8$

Although we would obtain these same four answers by positioning the interior points in different locations, or by completing the triangulations in different ways, the diagrams above were created to help visualize a pattern.
From the answers shown, we see that $T(4, k+1) = T(4, k) + 2$, for $k = 0, 1, 2$.
We must justify why this observation is true for all $k \geq 0$ so that we may use the result to determine the value of $T(4, 100)$.
Notice that each triangulation (after the first) was created by placing a new interior point inside the previous triangulation.
Further, each square is divided into triangles, and so each new interior point is placed *inside* a triangle of the previous triangulation (since no 3 points may lie on the same line).
For example, in the diagrams shown to the right, we observe that P lies in triangle t of the previous triangulation.
Also, each of the triangles outside of t is untouched by the addition of P, and thus they continue to contribute the same number of triangles (5) to the value of $T(4, 3)$ as they did to the value of $T(4, 2)$.



$T(4,2)=6$ $T(4,3)=8$

Triangle t contributes 1 to the value of $T(4, 2)$.
To triangulate the region defined by triangle t, P must be joined to each of the 3 vertices of triangle t (no other triangulation of this region is possible).
Thus, the placement of P divides triangle t into 3 triangles for every possible location of P inside triangle t.
That is, t contributes 1 to the value of $T(4, 2)$, but the region defined by t contributes 3 to the value of $T(4, 3)$ after the placement of P.
To summarize, the value of $T(4, k+1)$ is 2 more than the value of $T(4, k)$ for all $k \geq 0$ since:

- the $(k+1)^{st}$ interior point may be placed anywhere inside the triangulation for $T(4, k)$ (provided it is not on an edge)
- specifically, the $(k+1)^{st}$ interior point lies inside a triangle of the triangulation which gives $T(4, k)$
- this triangle contributed 1 to the value of $T(4, k)$
- after the $(k+1)^{st}$ interior point is placed inside this triangle and joined to each of the 3 vertices of the triangle, this area contributes 3 to the value of $T(4, k+1)$
- this is a net increase of 2 triangles, and thus $T(4, k+1) = T(4, k) + 2$, for all $k \geq 0$.

$T(4, 0) = 2$ and each additional interior point increases the number of triangles by 2.
Thus, k additional interior points increases the number of triangles by $2k$, and so $T(4, k) = T(4, 0) + 2k = 2 + 2k$ for all $k \geq 0$.
Using this formula, we get $T(4, 100) = 2 + 2(100) = 202$.

(c) In the triangulation of a regular n-gon with no interior points, we may choose any one of the n vertices and join this vertex to each of the remaining $n-3$ non-adjacent vertices.
All such triangulations of a regular n-gon with no interior points creates $n-2$ triangles, and so $T(n,0) = n-2$ for all $n \geq 3$ (since $T(n,0)$ is constant).
The reasoning used in part (b) extends to any regular polygon having $n \geq 3$ vertices.
That is, each additional interior point that is added to the triangulation for $n \geq 3$ vertices and $k \geq 0$ interior points gives a net increase of 2 triangles.
Thus, $T(n,k+1) = T(n,k)+2$ for all regular polygons having $n \geq 3$ vertices and $k \geq 0$ interior points.
So then k additional interior points increases the number of triangles by $2k$, and so $T(n,k) = T(n,0) + 2k = (n-2) + 2k$ for all $k \geq 0$.
Using this formula $T(n,k) = (n-2) + 2k$, we get $T(n,n) = (n-2) + 2n = 3n-2$ and $3n-2 = 2020$ when $n = \frac{2022}{3} = 674$.

CEMC at Home

Grade 11/12 - Tuesday, June 9, 2020
It's True Because it Isn't False!

Background

What do you think of the following multiple choice question?

> Exactly one of the following statements is true. Which one is it?
>
> (a) 6 is a prime number.
>
> (b) $x = 2$ is a solution to the equation $x + 3 = 4$.
>
> (c) The point $(1, 2)$ lies on the line with equation $y = x + 10$.
>
> (d) The CEMC was founded in 1995 with origins dating back to the 1960s.
>
> (e) $2^0 = 0$

The correct answer is (d) and you probably got it right without knowing anything about the history of the CEMC. How did you do that? You may have used a useful trick for answering a multiple choice question: elimination!

Answer (a) is false because $6 = 2 \times 3$ so 6 is not prime.
Answer (b) is false because when $x = 2$, the value of $x + 3$ is $2 + 3 = 5$ and $5 \neq 4$.
Answer (c) is false because when $x = 1$, the value of $x + 10$ is $1 + 10 = 11$ and $11 \neq 2$.
Answer (e) is false because $2^0 = 1$ and $1 \neq 0$.

Now that we have eliminated four of the five answers, we know that (d) must be the true statement. Elimination works here because we are told that exactly one of the statements is true. Without that information, we cannot confidently answer the question (unless we are CEMC history buffs!).

Example 1: Consider the statement "There is no greatest positive integer."

How can we eliminate the possibility that this statement is false?

We know that the statement is either true or false, but let's say that we do not know which is the case. Let's suppose that the statement is false, and see where this leads.

Suppose that there is a greatest positive integer.
Let's call this greatest positive integer k.

Now, we proceed as we normally would in a mathematical argument, using sound logic and facts that we know to be true.

Consider the number $k + 1$.
We know that the number $k + 1$ must be a positive integer and must satisfy $k + 1 > k$.

Here we see that something is wrong. We have shown that if the given statement is false, then we can deduce that the following are both true about the number k:

- k is the greatest positive integer, and
- $k + 1$ is a positive integer that is greater than k.

But these cannot both be true of k. We claim that this means that the given statement could not possibly be false. Can you see why we can conclude this?

Explanation of a Proof Method (Proof by Contradiction)

In mathematics, we deal with sentences that have a definite state of being either true or false. We call these sentences *statements*. Since a statement is either true or false, we can use elimination to argue that a statement must be true by eliminating the possibility that it is false.

How can we eliminate the possibility that a statement is false? We can suppose that it is false and show that this assumption leads us to *a contradiction*. A contradiction is a combination of ideas that are opposed to one another, and hence cannot be simultaneously true. (For example, if we deduce that a number x must satisfy $x > 1$ *and* $x < -1$, then we have reached a contradiction.)

If we reach a contradiction in our argument, then we can be sure that there is at least one flaw in our argument. If the only possible error in our argument was our initial assumption ("the statement is false") then this is the only thing that could have caused us to reach a contradiction. If we are sure that the assumption "the statement is false" is wrong, then the only remaining option is that the statement is actually true!

We call this logical method a *proof by contradiction*.

Example 2: Here is an example of a proof by contradiction.

Statement: The sum of a rational number and an irrational number is an irrational number.

Proof:

Suppose, for a contradiction, that there is a rational number r and an irrational number α for which the sum $r + \alpha$ is rational.

Since r is rational, we can write $r = \frac{a}{b}$ for integers a, b $(b \neq 0)$.

Since $r+\alpha$ is rational, we can write $r+\alpha = \frac{c}{d}$ for integers c, d $(d \neq 0)$.

This means we have $\frac{a}{b} + \alpha = \frac{c}{d}$.

It follows that $\alpha = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{bd}$.

This means α is rational.

We have reached a contradiction.

Therefore, it cannot be the case that there is a rational number and an irrational number whose sum is rational.

We conclude that it must be the case that the sum of a rational number and an irrational number is an irrational number.

A rational number is a real number that can be expressed as a fraction of two integers (with non-zero denominator). An irrational number is a real number that cannot be expressed in this way.

We suppose that the given statement is false.

We proceed using logic and mathematical facts we know to be true.

We have deduced the following:

- α is irrational
- α is rational

But of course, they cannot both be true!

We have eliminated the possibility that our statement is false.

Practice: Your turn! Prove each of the following statements using a proof by contradiction approach.

1. There do not exist integers x and y such that $10x - 25y = 6$.
   Start by supposing that there do exist integers x and y that satisfy $10x - 25y = 6$. Then think about the factors of the integer $10x - 25y$.
2. If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.
   Start by supposing that there are positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.
3. Extra Challenge: If the parabola $y = ax^2 + bx + c$ (with a, b, c non-zero real numbers) touches or crosses the x-axis, then a, b, c cannot form a geometric sequence, in that order.

More Info:

Check out the CEMC at Home webpage on Tuesday, June 16 for proofs of the above statements.

CEMC at Home

Grade 11/12 - Tuesday, June 9, 2020
It's True Because it Isn't False! - Solution

Prove each of the following statements using a proof by contradiction approach.

1. There do not exist integers x and y such that $10x - 25y = 6$.

   Proof:

   Suppose, for a contradiction, that there do exist integers x and y such that $10x - 25y = 6$.

   Since x and y are integers, the quantity $10x - 25y$ is an integer as well.
   Since $10x - 25y = 5(2x - 5y)$, the integer 5 must be a factor of the integer $10x - 25y$.
   Since $10x - 25y = 6$, this means the integer 5 must also be a factor of 6.
   This is a contradiction because 5 is not a factor of 6.

   Therefore, we conclude that there cannot exist integers x and y such that $10x - 25y = 6$.

2. If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

   Proof:

   Suppose, for a contradiction, that there are positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   Squaring both sides of the equation gives

$$\begin{aligned}
\left(\sqrt{x+y}\right)^2 &= \left(\sqrt{x} + \sqrt{y}\right)^2 \\
x + y &= \left(\sqrt{x}\right)^2 + 2\sqrt{x}\sqrt{y} + \left(\sqrt{y}\right)^2 \\
x + y &= x + 2\sqrt{x}\sqrt{y} + y \\
0 &= 2\sqrt{xy} \qquad \text{since } x, y > 0 \\
0 &= \sqrt{xy} \\
0 &= xy
\end{aligned}$$

   Since $xy = 0$ we must have $x = 0$ or $y = 0$. This is a contradiction. Since x and y are both positive real numbers, it cannot be the case that $x = 0$ and it cannot be the case that $y = 0$.

   This means there cannot be positive real numbers x and y for which $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$.

   Therefore, we conclude that if x and y are positive real numbers, then we must have $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

3. Extra Challenge: If the parabola $y = ax^2 + bx + c$ (with a, b, c non-zero real numbers) touches or crosses the x-axis, then a, b, c cannot form a geometric sequence, in that order.

   Solution:

   Let $y = ax^2 + bx + c$ be a parabola, with a, b, c non-zero real numbers, that crosses or touches the x-axis. We want to prove that a, b, c cannot form a geometric sequence, in that order.

   Suppose, for a contradiction, that a, b, c is a geometric sequence, in that order.

   Since a, b, c is a geometric sequence, it has a common ratio, r, such that $b = ar$ and $c = ar^2$.
   Since we know that $b \neq 0$, we know that $r \neq 0$.

Since the parabola crosses or touches the x-axis, the quadratic equation $ax^2 + bx + c = 0$ has at least one real solution. So the discriminant of the quadratic must be at least zero.

Since the discriminant is

$$b^2 - 4ac = (ar)^2 - 4a(ar^2) = (ar)^2 - 4(ar)^2 = -3(ar)^2$$

we must have $-3(ar)^2 \geq 0$.

However, since $a \neq 0$ and $r \neq 0$, we must have $ar \neq 0$. It follows that $(ar)^2 > 0$ and so $-3(ar)^2 < 0$. This is a contradiction since we cannot have both $-3(ar)^2 \geq 0$ and $-3(ar)^2 < 0$.

Therefore, it cannot be the case that a, b, c is a geometric sequence, in that order.

Discussion: Proving statements can be one of the most rewarding parts of mathematics, but it can also be challenging. There are a variety of proof techniques that we can use, one of which is proof by contradiction. While there are no fixed rules about which technique to use to prove a statement (one of the reasons why writing a proof can be challenging!), there are some statements that lend themselves well to a proof by contradiction approach. The statements from this resource fall into this category.

Our first problem here involved showing that a pair of integers with some property *cannot* exist, and the third problem here involved showing that a triple of real numbers *cannot* satisfy a certain condition. Taking a proof by contradiction approach allowed us to see what would happen if the pair of integers *did* exist and if the real numbers *did* satisfy the condition. We were able to do some algebra and, quite quickly, we discovered that these situations lead to contradictions and so we could rule them out.

Think about how you would prove these statements without taking a proof by contradiction approach. For example, for the first statement, you would need to argue directly that every possible pair of integers fails to satisfy the equality. How would you make this argument? Sometimes, it can be challenging to argue that objects *do not* exist or *do not* satisfy a certain condition *directly* (especially when there are infinitely many objects to rule out!). You may not need to do a proof by contradiction, but thinking about the problem in this way will often lead you to a nice argument.

In contrast, you might find it very natural to prove the second statement here without using a proof by contradiction approach. Given two positive real numbers x and y, can you check directly that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$?

Here is an example of a *direct proof of statement 2:*

Let x and y be positive real numbers. Then we have

$$(\sqrt{x+y})^2 - (\sqrt{x} + \sqrt{y})^2 = (x+y) - (x + 2\sqrt{x}\sqrt{y} + y) = -2\sqrt{x}\sqrt{y}$$

Since $\sqrt{x} > 0$ and $\sqrt{y} > 0$ we have $\sqrt{x}\sqrt{y} > 0$ and so $-2\sqrt{x}\sqrt{y} < 0$. It follows that

$$(\sqrt{x+y})^2 - (\sqrt{x} + \sqrt{y})^2 < 0$$

and hence

$$(\sqrt{x+y})^2 < (\sqrt{x} + \sqrt{y})^2$$

Since $\sqrt{x+y} > 0$ and $\sqrt{x} + \sqrt{y} > 0$, it must be the case that $\sqrt{x+y} < \sqrt{x} + \sqrt{y}$. In particular, this means that $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$ as desired.

CEMC at Home

Grade 11/12 - Wednesday, June 10, 2020

Trugs and Falths

You are an ambassador of The Republic of Logica and you have been sent on a mission to Duoterra, one of the islands of Logica. This island is entirely inhabited by two societies of people: the Trugs, who always tell the truth and the Falths, who always lie. You must figure out to which society the people you encounter belong. To do so you will have to use the skills you learned at the University of Logica.

One of these skills is the process of elimination which is more formally known as "Proof by Contradiction". To learn about this method, see the CEMC at Home Grade 11/12 activity from June 9.

On each day, you meet at least two new people on the island. Let's see what happens on Day 1.

Day 1: You meet Algorn and Birk.

Algorn: Birk and I are Trugs.
Birk: Algorn is a Falth.

From this information you must determine to which society Algorn and Birk belong.

Solution for Day 1: Algorn must be either a Trug or a Falth.

- *Suppose that Algorn is a Trug.* This means that Algorn is telling the truth and so both Algorn and Birk must be Trugs. Therefore, Birk is also telling the truth and so Algorn is a Falth. This contradicts our initial assumption that Algorn is a Trug. So Algorn must be a Falth.
- *Algorn is a Falth.* This means Algorn is lying, which is already clear because she said that both Birk and her are Trugs. Birk's statement is then true and so Birk must be a Trug.

In summary, Algorn is a Falth and Birk is a Trug.

Logical Connectives

During your travels you will encounter sentences that involve the words "and", "or", and "not". Here is a summary of how to interpret the truth of statements like this:

AND	A statement of the form "P and Q" is true if *both* P and Q are true. A statement of the form "P and Q" is false if *at least one* of P and Q is false.
OR	A statement of the form "P or Q" is true if *at least one* of P and Q is true. A statement of the form "P or Q" is false if *both* P and Q are false.

Note that the word "or" is sometimes used differently than described above in everyday English. Sometimes people interpret "P or Q" as being true when *exactly one* of P and Q is true. We will not use this interpretation in what follows. You will also encounter statements involving "not". This has the usual meaning of the "opposite". In particular, "not P" is true exactly when P is false.

Day 2: You meet Crozul and Dek.

Crozul: I am a Trug or Dek is a Falth.
Dek: Exactly one of Crozul and I is a Trug.

From this information you must determine to which society Crozul and Dek belong.

Solution for Day 2: Crozul must be either a Trug or a Falth.

- *Suppose that Crozul is a Trug.* Therefore, Crozul is telling the truth. Since Crozul is a Trug, then Dek could be a Trug or a Falth and Crozel's statement would still be true.
  - *Suppose Dek is a Trug.* Then they are both Trugs and Dek's statement is false, which is not possible since Dek is a Trug. So Dek is a Falth.
  - *Dek is a Falth.* Then exactly one of Crozul and Dek is a Trug and so Dek's statement is true, which is not possible since Dek is a Falth.

  So, if Crozul is a Trug, then there are no possibilities for Dek to belong to either society, which is not possible. So it must be the case that Crozul is a Falth.
- *Crozul is a Falth.* Then Crozul is lying and so his statement tells us that he is not a Trug (this is consistent with our assumption) and that Dek is not a Falth (he is a Trug). We need to verify that this assignment of societies is consistent with Dek's statement. Exactly one of Crozul and Dek is a Trug and so Dek is telling the truth, which is consistent with the fact that he is a Trug.

In summary, Cozul is a Falth and Dek is a Trug.

Your journey continues

On your third, fourth and fifth days you have three more interactions. Use these interactions to determine to which society the people you meet belong.

Day 3: You meet Gup and Hoken.

Gup: Hoken is a Falth.
Hoken: I am a Trug or Gup is a Trug.

Day 4: You meet Ized and Jeke.

Ized: Jeke is not a Falth and I am a Trug.
Jeke: Ized and I are from the same society.

Day 5: You meet Kip, Lolo and Moy.

Kip: I am not a Falth and Lolo is not a Falth.
Lolo: Kip is a Falth.
Moy: Lolo is a Trug.

More Info:

Check out the CEMC at Home webpage on Wednesday, June 17 for a solution to Trugs and Falths.

These problems are based on a famous type of logic puzzles called "Knights and Knaves". They were made popular in the late 1970s by Raymond Smullyan, an American mathematician. An internet search for "knights and knaves" will lead to many other problems of this type.

CEMC at Home

Grade 11/12 - Wednesday, June 10, 2020
Trugs and Falths - Solution

Here are the solutions for the remaining days of your travels.

Day 3: You meet Gup and Hoken.

Gup: Hoken is a Falth.
Hoken: I am a Trug or Gup is a Trug.

Solution:

Gup must be either a Trug or a Falth.

- *Suppose that Gup is a Trug.*
  This means that Gup is telling the truth and so Hoken is a Falth. Therefore, Hoken is lying and so Hoken is not a Trug and Gup is not a Trug. (Recall that if "P or Q" is false, then P is false and Q is false.) But Gup is a Trug and so we have reached a contradiction. Therefore, Gup cannot be a Trug and hence Gup must be a Falth.
- *Gup is a Falth.*
  This means that Gup is lying and so Hoken must be a Trug. Note that this information is consistent with Hoken's statement as well. Since Hoken is a Trug, Hoken must be telling the truth and indeed at least one of Gup and Hoken is a Trug. (Recall that if "P or Q" is true, then at least one of P and Q is true.)

In summary, Gup is a Falth and Hoken is a Trug.

Day 4: You meet Ized and Jeke.

Ized: Jeke is not a Falth and I am a Trug.
Jeke: Ized and I are from the same society.

Solution:

Jeke must be either a Trug or a Falth.

- *Suppose that Jeke is a Trug.*
  This means that Jeke is telling the truth. Therefore, Ized and Jeke are from the same society and so Ized is also a Trug. Note that this is consistent with what Ized said. If Ized is a Trug, then Ized is telling the truth which means Jeke is not a Falth (and so is a Trug) and Ized is also a Trug.
  Note that we did not reach a contradiction here. This means we cannot eliminate the possibility that Jeke is a Trug. Does this mean we can be sure that Jeke is a Falth? Let's confirm that the other possibility leads to a contradiction.
- *Suppose that Jeke is a Falth.*
  This means that Jeke is lying and so Jeke and Ized are from different societies. Therefore, Ized is a Trug. If Ized is a Trug, Ized is telling the truth. But this means Jeke is not a Falth. This is a contradiction. Therefore, Jeke cannot be a Falth and so must be a Trug.

Since Jeke is a Trug, we already know from our work above that Ized must also be a Trug.
In summary, Jeke and Ized are both Trugs.

Day 5: You meet Kip, Lolo and Moy.

Kip: I am not a Falth and Lolo is not a Falth.
Lolo: Kip is a Falth.
Moy: Lolo is a Trug.

Solution:

Lolo must be either a Trug or a Falth.

- *Suppose that Lolo is a Falth.*
  This means Lolo is lying and so Kip is not a Falth and is hence a Trug. If Kip is a Trug, then Kip is telling the truth and so Lolo is not a Falth. This is a contradiction. Therefore, Lolo cannot be a Falth and hence must be a Trug.
- *Lolo is a Trug.*
  This means Lolo is telling the truth and so Kip is a Falth. It also means that Moy is telling the truth about Lolo's society and hence Moy is a Trug. Note that this information is consistent with Kip's statement as well. Since Kip is a Falth, Kip must be lying and indeed the first part of Kip's statement is false.

In summary, Kip is a Falth, and Lolo and Moy are both Trugs.

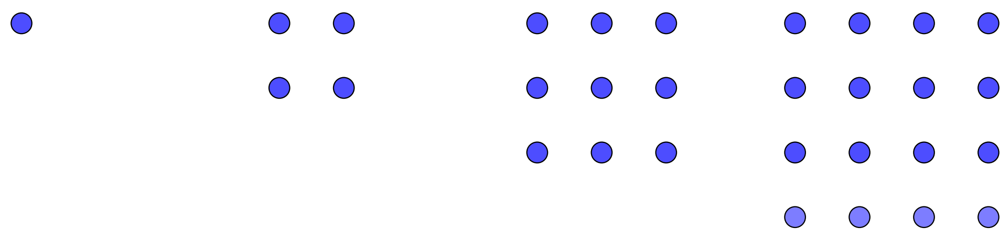
CEMC at Home

Grade 11/12 - Thursday, June 11, 2020
That Number Makes it Perfect

A *perfect square* is an integer that can be expressed as the product of two equal integers. The integer 25 is a perfect square since it can be expressed as the product 5×5 or 5^2.

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.



$1 \times 1 = 1^2 = 1$ $\quad 2 \times 2 = 2^2 = 4$ $\quad 3 \times 3 = 3^2 = 9$ $\quad 4 \times 4 = 4^2 = 16$

Did you know that the sum, S, of the positive integers from 1 to some positive integer n can be calculated using the formula $S = \dfrac{n \times (n+1)}{2}$?
For example,

$$1+2+3+4+5+6+7+8+9+10 = \frac{10 \times 11}{2} = 55.$$

This result may be helpful in this problem.

More Info:

Check out the CEMC at Home webpage on Friday, June 12 for a solution to That Number Makes it Perfect.

This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Thursday, June 11, 2020
This Number Makes it Perfect - Solution

Problem:

A *perfect square* is an integer that can be expressed as the product of two equal integers. The integer 25 is a perfect square since it can be expressed as the product 5×5 or 5^2.

The positive even integers 2 to 1600, inclusive, are each multiplied by the same positive integer, n. All of the products are then added together and the resulting sum is a perfect square.

Determine the value of the smallest positive integer n that makes this true.

Solution:

We are asked to determine the smallest positive integer, n, such that

$$2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n \qquad (1)$$

is a perfect square.

Factoring (1), we obtain

$$\begin{aligned} 2n + 4n + 6n + \cdots + 1596n + 1598n + 1600n &= 2n(1 + 2 + 3 + \cdots + 798 + 799 + 800) \\ &= 2n\left(\frac{800 \times 801}{2}\right) \\ &= n(800)(801) \qquad (2) \\ &= n[(2)(2)(2)(2)(2)(5)(5)][(3)(3)(89)] \\ &= n(2^5)(5^2)(3^2)(89) \qquad (3) \end{aligned}$$

In going from (2) to (3), we have expressed the 800×801 as the product of prime factors.

We need to determine what additional factors are required to make the quantity in (3) a perfect square such that n is as small as possible.

A positive integer larger than one is a perfect square exactly when each of its prime factors occurs an even number of times in its prime factorization. In order for each prime in the prime factorization to occur an even number of times, we need n to be $2 \times 89 = 178$. Then the quantity in (3) becomes

$$n(2^5)(5^2)(3^2)(89) = (2)(89)(2^5)(5^2)(3^2)(89) = (2^6)(5^2)(3^2)(89^2) = \left[(2^3)(5)(3)(89)\right]^2,$$

a perfect square.

Therefore, the smallest positive integer is $n = 178$ and the perfect square is

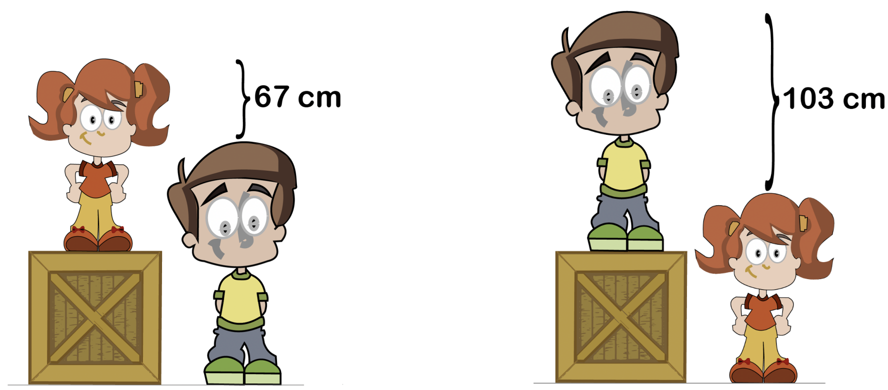
$$178 \times 800 \times 801 = 114\,062\,400 = (10\,680)^2.$$

CEMC at Home

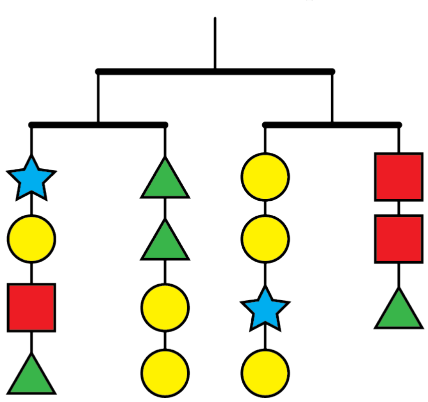
Grade 11/12 - Friday, June 12, 2020
Picture This

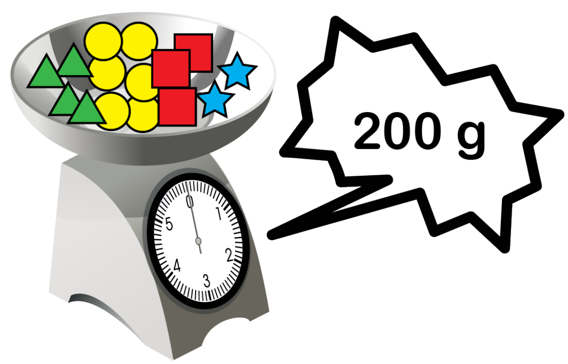
Problem 1: What is the mass of the baby penguin?

Problem 2: How tall is the crate?



Problem 3: A mobile of shapes is hanging perfectly balanced. What is the mass of each shape?





All of the individual shapes of the same type (star, circle, square, or triangle) have the same mass. All of the shapes from the mobile are placed on the scale.

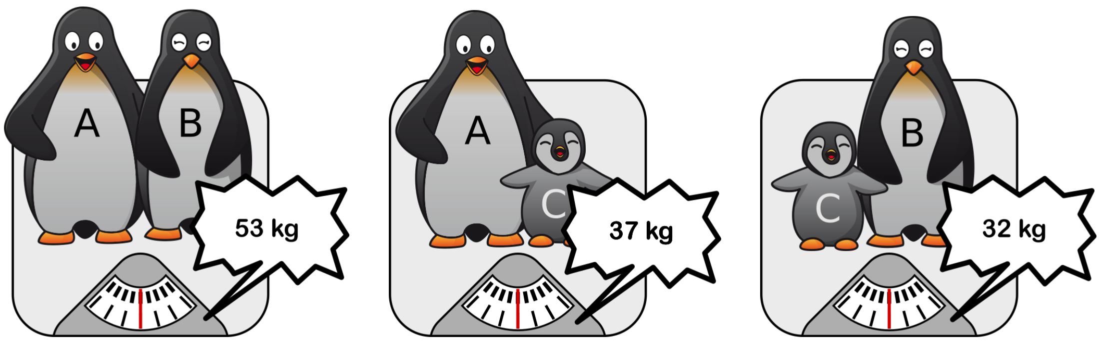
More Info:

Check out the CEMC at Home webpage on Friday, June 19 for a solution to Picture This.

CEMC at Home

Grade 11/12 - Friday, June 12, 2020
Picture This - Solution

Problem 1: What is the mass of the baby penguin?



Solution 1:

Let A, B, and C represent the masses, in kg, of the three penguins (as indicated in the diagrams). The mass of the baby penguin is C kg. The three images give us the following three equations, in order:

$$A + B = 53$$
$$A + C = 37$$
$$B + C = 32$$

Adding these three equations together, we see that $2A + 2B + 2C = 53 + 37 + 32 = 122$. This means $2(A + B + C) = 122$ and hence $A + B + C = 61$. Finally, we have

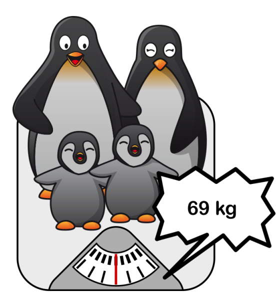
$$C = (A + B + C) - (A + B) = 61 - 53 = 8$$

Therefore, the baby penguin has a mass of 8 kg.

Solution 2:

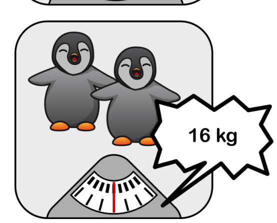
First, combine the penguins from the last two images.

This produces a scale containing the two adult penguins and two "copies" of the baby penguin, for a total mass of $37 + 32 = 69$ kg.



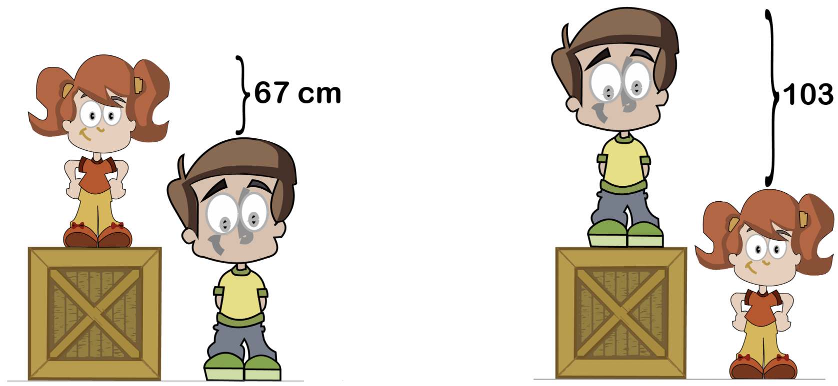
Now remove the penguins from the first image.

This produces a scale containing two baby penguins, with a combined mass of $69 - 53 = 16$ kg.



Therefore, one baby penguin has a mass of $\frac{16}{2} = 8$ kg.

Problem 2: How tall is the crate?



Solution 1:

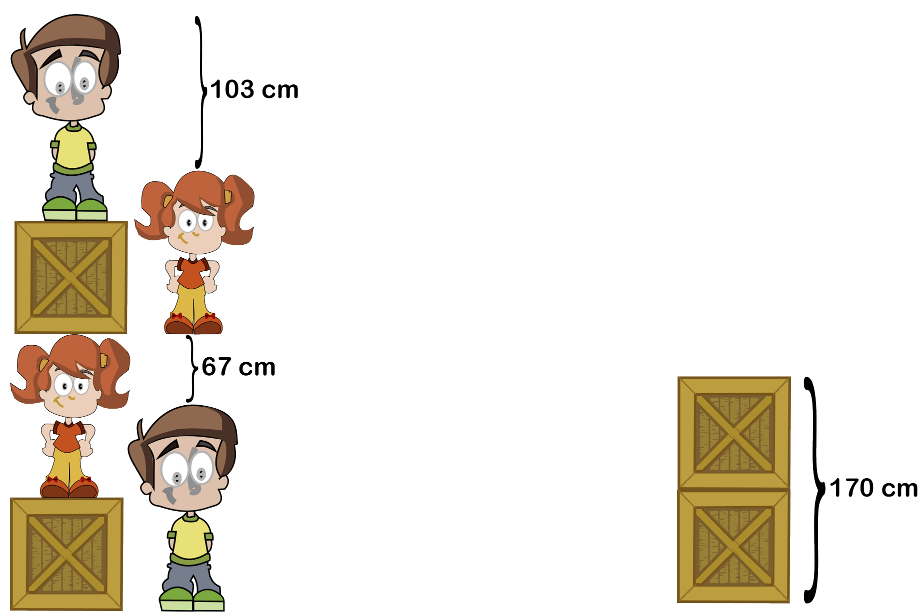
Let a denote the height of the child on the crate in the left image, b denote the height of the child on the crate in the right image, and c denote the height of the crate (in each image), all in cm.

Looking at the first image, we have $c + a = b + 67$.
Looking at the second image, we have $c + b = a + 103$.

Rearranging the first equation gives $c + a - b = 67$, and rearranging the second equation gives $c + b - a = 103$. Adding the two equations together we get $2c + a - b + b - a = 103 + 67$ which simplifies to $2c = 170$. This means $c = 85$ and therefore, the height of the crate is 85 cm.

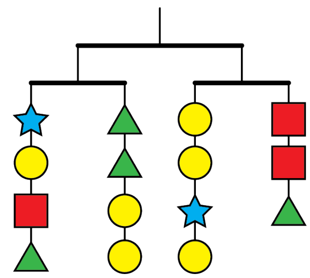
Solution 2:

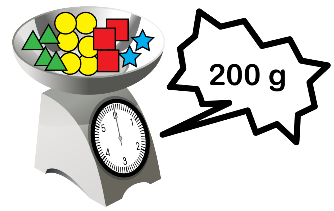
Stack the two images vertically as shown below on the left. The leftmost stack consists of two crates and both children. The rightmost stack consists both children and $67 + 103 = 170$ cm of "space". These two stacks must have the same total height. (Why?)



Now remove the children from each side of the stack. This tells us that two crates stacked one on top of the other have a height of 170 cm. Therefore, one crate is $\frac{170}{2} = 85$ cm tall.

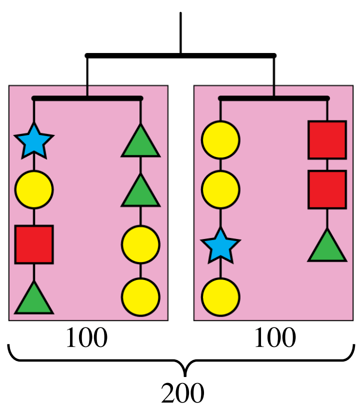
Problem 3: A mobile of shapes is hanging perfectly balanced. What is the mass of each shape?

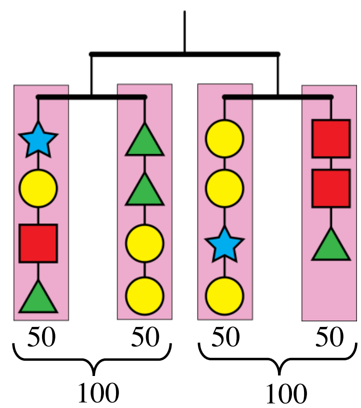




Solution 1:

Recall that the total mass of all of the shapes in the mobile is 200 grams. Since the mobile is perfectly balanced, these 200 grams must be evenly distributed between the left portion and the right portion of the mobile, making the mass of the shapes in each portion 100 grams as shown in the image below on the left. Using similar reasoning, we see that the total mass of the shapes on each of the four bottom strings must be 50 grams as shown in the image below on the right.





Let b be the mass of one blue star, y be the mass of one yellow circle, r be the mass of one red square, and g be the mass of one green triangle, all in grams. The first, second, third, and fourth strings show the following relationships, in order:

(1) $b+y+r+g=50$ (2) $2g+2y=50$ (3) $3y+b=50$ (4) $2r+g=50$

There are many ways to proceed from here. One way (that may be hard to find!) is to notice the following relationship involving the four lefthand expressions above:

$$(2g+2y)-4(b+y+r+g)+4(3y+b)+2(2r+g)=10y$$

Using equations (1), (2), (3), and (4), we have

$$(2g+2y)-4(b+y+r+g)+4(3y+b)+2(2r+g)=50-4(50)+4(50)+2(50)=150$$

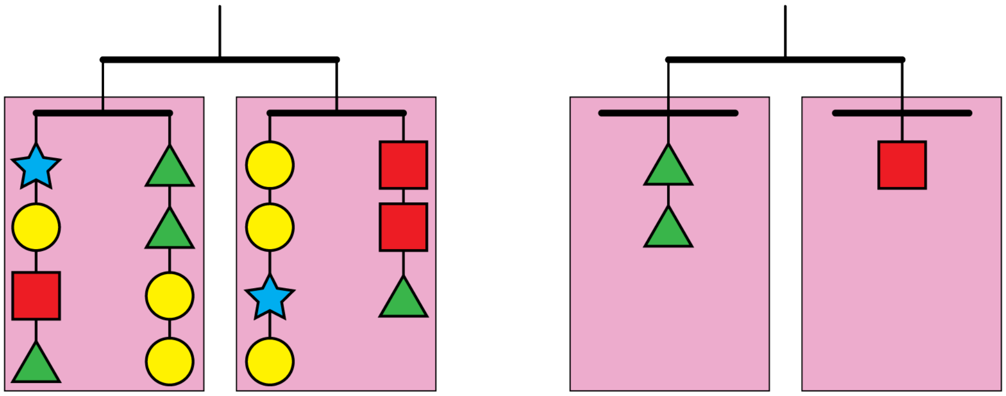
and so $10y=150$ or $y=15$.

Since $y=15$ and $g+y=25$, from (2), we have $g=10$. Since $g=10$ and $2r+g=50$, from (4), we have $r=20$. Since $y=15$ and $3y+b=50$, from (3), we have $b=5$.

Therefore, each blue star has a mass of 5 grams, each yellow circle has a mass of 15 grams, each red square has a mass of 20 grams, and each green triangle has a mass of 10 grams.

Solution 2:

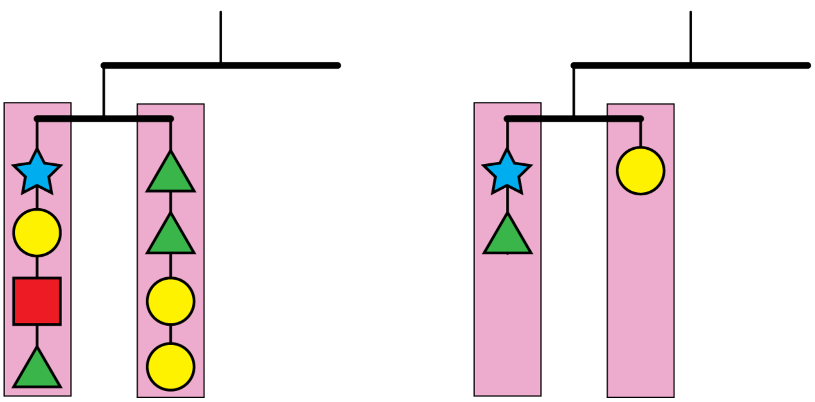
Since the mobile is perfectly balanced, the total mass of the shapes on the left portion of the mobile must be equal to the total mass of the shapes on the right portion. We can remove identical shapes from each side of the mobile, while still keeping it balanced.



After removing 1 ★, 3 ●, 1 ■ and 1 ▲ from each side of the mobile, we are left with:

1 ■ = 2 ▲ (Equation 1)

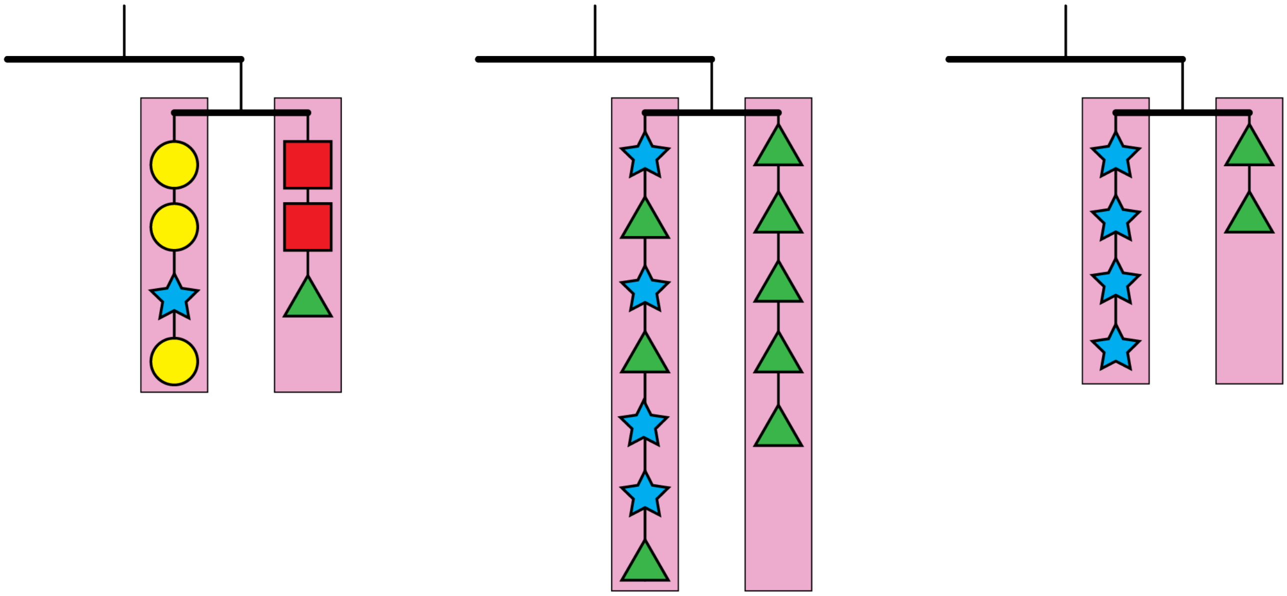
Next, since the left portion of the mobile is also perfectly balanced, the total mass of the shapes on the first string is equal to the total mass of the shapes on the second string. We can remove identical shapes, or groups of shape that have the same total mass, while still keeping it balanced.

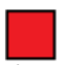


We can remove 1 ● from each string, and we can also remove 1 ■ from the first string and 2 ▲ from the second string since we have learned they are equivalent in mass. Now we are left with:

1 ● = 1 ★ + 1 ▲ (Equation 2)

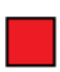
Similarly, the total mass of the shapes on the third string is equal to the total mass of the shapes on the fourth string. Since the strings don't have any shapes in common, let's first use Equations 1 and 2 to make some shape substitutions.



Using Equation 1, replace every  on the fourth string with 2 . Using Equation 2, replace every  on the third string with 1  and 1 . Then, after removing 3  from each string, we are left with:

4  = 2  or 2  = 1  (Equation 3)

Using our three equations we can express the mass of each shape in terms of its mass in :

Shape	Equivalent mass in 	Reason
	1 	by definition
	2 	Equation 3
	3 	Equation 2 & 3
	4 	Equation 1 & 3

Since the total mass of all the shapes is 200 g we can then conclude the following:

$$
\begin{aligned}
4\,\triangle + 6\,\bigcirc + 3\,\square + 2\,\bigstar &= 200\\
4\left(2\,\bigstar\right) + 6\left(3\,\bigstar\right) + 3\left(4\,\bigstar\right) + 2\,\bigstar &= 200\\
8\,\bigstar + 18\,\bigstar + 12\,\bigstar + 2\,\bigstar &= 200\\
40\,\bigstar &= 200\\
1\,\bigstar &= 5
\end{aligned}
$$

Since 1 $\bigstar$ has a mass of 5 g we can now determine the mass of each shape:

Shape	Equivalent mass in $\bigstar$	Mass in g
$\bigstar$	1 $\bigstar$	5 g
$\triangle$	2 $\bigstar$	10 g
$\bigcirc$	3 $\bigstar$	15 g
$\square$	4 $\bigstar$	20 g