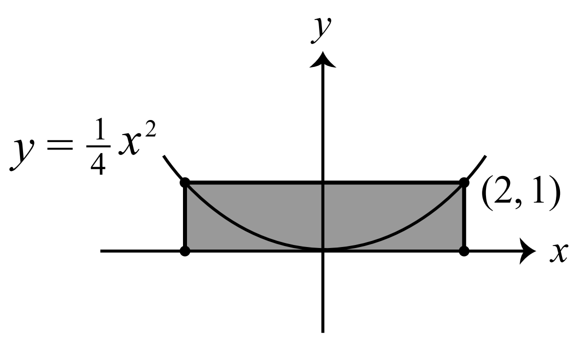
CEMC at Home

Grade 11/12 - Monday, June 1, 2020
Contest Day 5

Today's resource features one question from the recently released 2020 CEMC Mathematics Contests.

2020 Hypatia Contest, #2

The parabola with equation $y = \frac{1}{4}x^2$ has its vertex at the origin and the y-axis as its axis of symmetry. For any point (p, q) on the parabola (not at the origin), we can form a *parabolic rectangle*. This rectangle will have one vertex at (p, q), a second vertex on the parabola, and the other two vertices on the x-axis. A parabolic rectangle with area 4 is shown.



(a) A parabolic rectangle has one vertex at $(6, 9)$. What are the coordinates of the other three vertices?

(b) What is the area of the parabolic rectangle having one vertex at $(-3, 0)$?

(c) Determine the areas of the two parabolic rectangles that have a side length of 36.

(d) Determine the area of the parabolic rectangle whose length and width are equal.

More Info:

Check out the CEMC at Home webpage on Monday, June 8 for a solution to the Contest Day 5 problem.

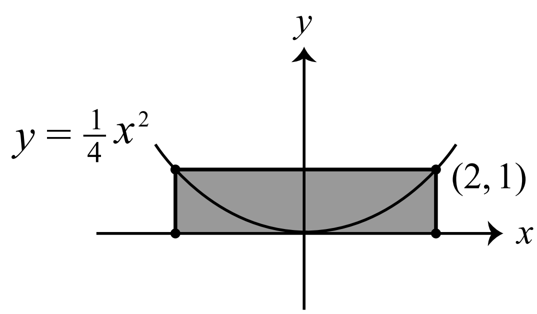
CEMC at Home

Grade 11/12 - Monday, June 1, 2020
Contest Day 5 - Solution

A solution to the contest problem is provided below.

2020 Hypatia Contest, #2

The parabola with equation $y = \frac{1}{4}x^2$ has its vertex at the origin and the y-axis as its axis of symmetry. For any point (p, q) on the parabola (not at the origin), we can form a *parabolic rectangle*. This rectangle will have one vertex at (p, q), a second vertex on the parabola, and the other two vertices on the x-axis. A parabolic rectangle with area 4 is shown.



(a) A parabolic rectangle has one vertex at $(6, 9)$. What are the coordinates of the other three vertices?

(b) What is the area of the parabolic rectangle having one vertex at $(-3, 0)$?

(c) Determine the areas of the two parabolic rectangles that have a side length of 36.

(d) Determine the area of the parabolic rectangle whose length and width are equal.

Solution:

(a) The parabola $y = \frac{1}{4}x^2$ and the parabolic rectangle are each symmetrical about the y-axis, and thus a second vertex of the rectangle lies on the parabola and has coordinates $(-6, 9)$.
A third vertex of the parabolic rectangle lies on the x-axis vertically below $(6, 9)$, and thus has coordinates $(6, 0)$.
Similarly, the fourth vertex also lies on the x-axis vertically below $(-6, 9)$, and thus has coordinates $(-6, 0)$.

(b) If one vertex of a parabolic rectangle is $(-3, 0)$, then a second vertex has coordinates $(3, 0)$, and so the rectangle has length 6.
The vertex that lies vertically above $(3, 0)$ has x-coordinate 3.
This vertex lies on the parabola $y = \frac{1}{4}x^2$ and thus has y-coordinate equal to $\frac{1}{4}(3)^2 = \frac{9}{4}$.
The width of the rectangle is equal to this y-coordinate $\frac{9}{4}$, and so the area of the parabolic rectangle having one vertex at $(-3, 0)$ is $6 \times \frac{9}{4} = \frac{54}{4} = \frac{27}{2}$.

(c) Let a vertex of the parabolic rectangle be the point $(p, 0)$, with $p > 0$.
A second vertex (also on the x-axis) is thus $(-p, 0)$, and so the rectangle has length $2p$.
The width of this rectangle is given by the y-coordinate of the point that lies on the parabola vertically above $(p, 0)$, and so the width is $\frac{1}{4}p^2$.
The area of a parabolic rectangle having length $2p$ and width $\frac{1}{4}p^2$ is $2p \times \frac{1}{4}p^2 = \frac{1}{2}p^3$.
If such a parabolic rectangle has length 36, then $2p = 36$, and so $p = 18$.
The area of this rectangle is $\frac{1}{2}(18)^3 = 2916$.
If such a parabolic rectangle has width 36, then $\frac{1}{4}p^2 = 36$ or $p^2 = 144$, and so $p = 12$ (since $p > 0$).
The area of this rectangle is $\frac{1}{2}(12)^3 = 864$.
The areas of the two parabolic rectangles that have side length 36 are 2916 and 864.

(d) Let a vertex of the parabolic rectangle be the point $(m, 0)$, with $m > 0$.
A second vertex (also on the x-axis) is thus $(-m, 0)$, and so the rectangle has length $2m$.
The width of this rectangle is given by the y-coordinate of the point that lies on the parabola vertically above $(m, 0)$, and so the width is $\frac{1}{4}m^2$.
The area of a parabolic rectangle having length $2m$ and width $\frac{1}{4}m^2$ is $2m \times \frac{1}{4}m^2 = \frac{1}{2}m^3$.
If the length and width of such a parabolic rectangle are equal, then

$$\begin{aligned} \tfrac{1}{4}m^2 &= 2m \\ m^2 &= 8m \\ m^2 - 8m &= 0 \\ m(m-8) &= 0 \end{aligned}$$

Thus $m = 8$ (since $m > 0$), and so the area of the parabolic rectangle whose length and width are equal is $\frac{1}{2}(8)^3 = 256$.

CEMC at Home

Grade 11/12 - Tuesday, June 2, 2020
Famous Mathematicians

Throughout human history, many mathematicians have made significant contributions to the subject. These important historical figures often lead fascinating lives filled with interesting stories. Five of these mathematicians are listed below.

Euclid	Best known for his work in geometry, he was a mathematician from ancient Greece whose work the *Elements* may be the most influential writing in the history of mathematics.
Hypatia	From Alexandria, Egypt, she is considered by many to be the greatest mathematician of her time. She was also an astronomer and philosopher.
Srinivasa Ramanujan	His immense talent allowed him to accumulate an incredible amount of mathematical knowledge despite coming from very modest means in India and having very limited access to formal education.
John Fields	This Canadian mathematician is best known for establishing a global award for outstanding contributions to mathematics.
Terence Tao	He was a child prodigy and a recipient of the the prestigious Fields Medal. He is currently an active mathematician doing research in many areas including number theory and probability.

Choose two of these five mathematicians and for each one you choose:

1. Do some online research to determine an additional interesting fact about the mathematician.
2. Try to find a connection between something you have studied in a recent mathematics class and the mathematical work of this historical figure.
3. If you had the chance to go back in time and meet this mathematician, what question would you ask them?

More Info:

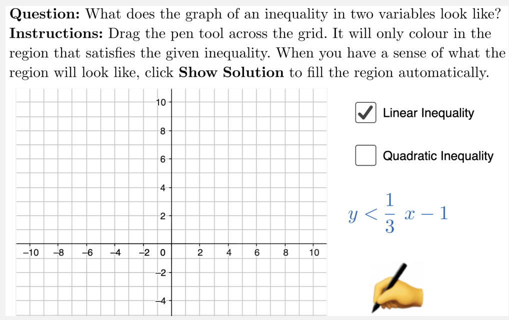
The CEMC Hypatia Math Contest and Euclid Math Contest are named in honour of two of these five mathematicians.

CEMC at Home

Grade 11/12 - Wednesday, June 3, 2020
Interact with Mathematics

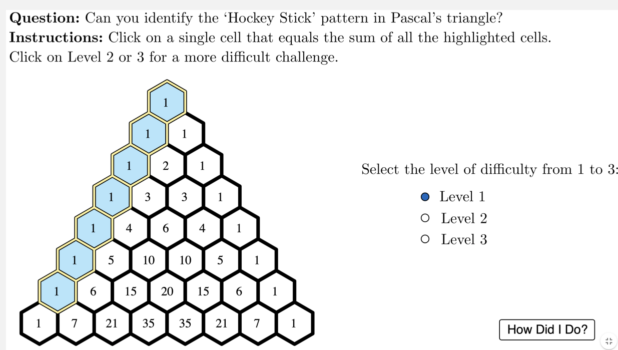
Technology can help us make mathematical discoveries and learn about mathematical objects. Three online examples of this from different areas of mathematics are featured below.

Inequalities: What does the graph of an inequality in two variables look like?



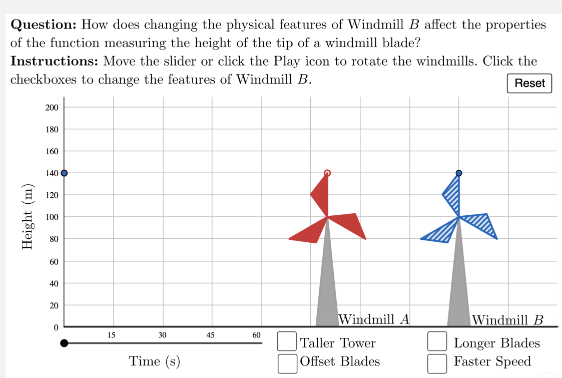
Link to App: https://www.geogebra.org/m/nqurv5yr

Pascal's Triangle: Explore the *Hockey Stick* pattern found in Pascal's Triangle.



Link to App: https://www.geogebra.org/m/bzvdfqtk

Modelling Periodic Behaviour: Discover how the physical properties of a windmill relate to the sinusoidal function that models the movement of its blades.



Link to App: https://www.geogebra.org/m/bmmrh2ap

More Info: CEMC courseware lessons feature hundreds of interactive mathematics applications. For the Grade 9/10/11 CEMC courseware, an interactive library has been built which allows you to perform a keyword search and/or display only the applications from a given strand, unit or lesson.

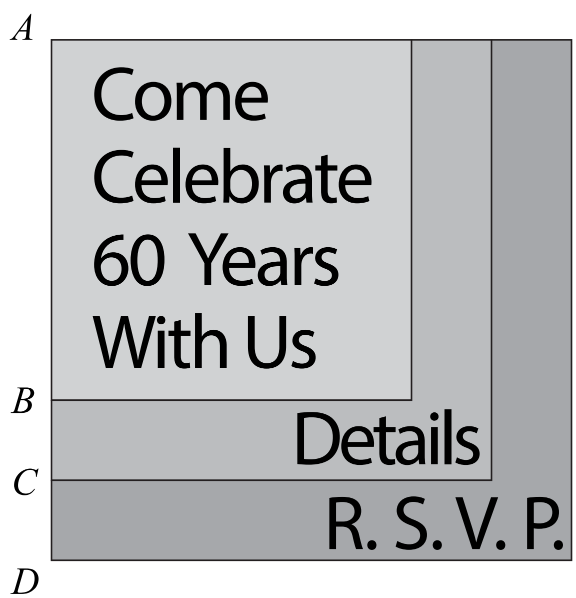
CEMC at Home

Grade 11/12 - Thursday, June 4, 2020
An Inviting Problem

An invitation to a 60$^{\text{th}}$ anniversary party is made by overlapping three squares, as shown below.

Each square has a positive integer side length. Side AB of the smallest square lies along side AC of the middle square, which lies along side AD of the largest square. The area of the middle square not covered by the smallest square is 33 cm^2.

If $BC = CD$, determine all possible side lengths of each square.



More Info:

Check out the CEMC at Home webpage on Friday, June 5 for a solution to An Inviting Problem.

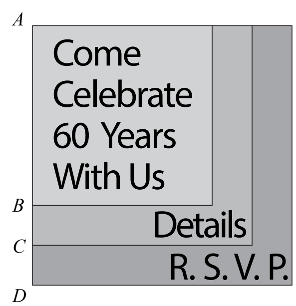
This CEMC at Home resource is a past problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students during the school year. POTW is wrapped up for the current school year and will resume on September 17, 2020. To subscribe to POTW and to find more past problems and their solutions visit:
https://www.cemc.uwaterloo.ca/resources/potw.php

CEMC at Home

Grade 11/12 - Thursday, June 4, 2020
An Inviting Problem - Solution

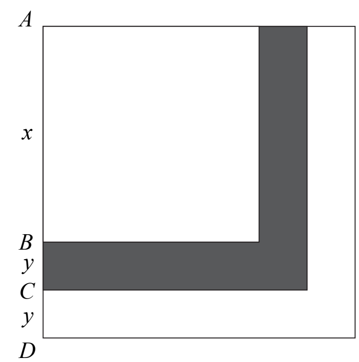
Problem:

An invitation to a 60^{th} anniversary party is made by overlapping three squares, as shown below. Each square has a positive integer side length. Side AB of the smallest square lies along side AC of the middle square, which lies along side AD of the largest square. The area of the middle square not covered by the smallest square is 33 cm^2.
If $BC = CD$, determine all possible side lengths of each square.



Solution:

Let $AB = x$ and $BC = y$. Therefore $CD = BC = y$.
Also, since the side lengths of the squares are integers, x and y are integers.



The shaded region has area 33. The shaded region is equal to the area of the square with side length AC minus the area of the square with side length AB. Since $AB = x$ and $AC = AB + BC = x + y$, we have

$$\begin{aligned} 33 &= (\text{area of square with side length } AC) \ - (\text{area of square with side length } AB) \\ &= (x+y)^2 - x^2 \\ &= x^2 + 2xy + y^2 - x^2 \\ &= 2xy + y^2 \\ &= y(2x+y) \end{aligned}$$

Since x and y are integers, so is $2x+y$. Therefore, $2x+y$ and y are two positive integers that multiply to give 33. Therefore, we must have $y = 1$ and $2x + y = 33$ or $y = 3$ and $2x + y = 11$ or $y = 11$ and $2x + y = 3$ or $y = 33$ and $2x + y = 1$. The last two would imply that $x < 0$, which is not possible. Therefore, $y = 1$ and $2x + y = 33$ or $y = 3$ and $2x + y = 11$.

When $y = 1$ and $2x + y = 33$, it follows that $x = 16$. Then the small square has side length $x = 16$ cm, the middle square has side length $x + y = 17$ cm, and the largest square has side length $x + 2y = 18$ cm.

When $y = 3$ and $2x + y = 11$, it follows that $x = 4$. Then the small square has side length $x = 4$ cm, the middle square has side length $x+y = 7$ cm, and the largest square has side length $x+2y = 10$ cm.

Therefore, there are two possible sets of squares: 16 cm $\times$ 16 cm, 17 cm $\times$ 17 cm and 18 cm $\times$ 18 cm or 4 cm $\times$ 4 cm, 7 cm $\times$ 7 cm and 10 cm $\times$ 10 cm. Each set of squares satisfies the conditions of the problem.

CEMC at Home

Grade 11/12 - Friday, June 5, 2020
Math and CS in the News

Most weeks, our CEMC Homepage provides a link to a story in the media about mathematics and/or computer science. These stories show us how important mathematics and computer science are in today's world. They are a great source for discussions.

Using this article from Phys.org, think about the following questions. (URL also provided below.)

1. What is a cryptocurrency? Can you describe this to a friend? Can you name two examples of cryptocurrencies?

2. What advantages and disadvantages do you see to cryptocurrencies?

3. It turns out that the cryptocurrencies require enormous energy consumption. Do some research on this issue and think about the implications.

4. Predict the future: How much do you think cryptocurrencies will be used in 20 years?

URL of the article:
https://phys.org/news/2020-04-decrypting-cryptocurrencies.html

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More Info:

A full archive of past posts can be found in our Math and CS in the News Archive. Similar resources for other grades may also be of interest.