CEMC at Home

Grade 9/10 - Monday, April 6, 2020

Sprouts

You Will Need:

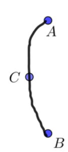
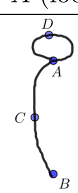
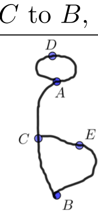
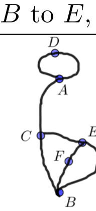
- Two players
- A piece of paper and a pencil

How to Play:

1. Start with two or three dots on the page, reasonably spaced out.
2. Players alternate turns. Decide which player will go first.
3. On your turn, do the following (if possible, according to the restrictions given in 4):

   Draw a curve joining two existing dots and add a dot to the newly drawn curve.
   Note that this curve can be drawn between two different dots, or in the form of a loop from one dot back to itself.
4. Here are the restrictions on the moves performed in 3:
   - You cannot draw a curve if it will result in a dot having more than three curve segments coming in or out of the dot. In particular, you cannot draw a loop on a dot that already has more than one curve segment coming in or out.
   - You cannot draw a curve if it will have to cross an existing curve.
   - The added dot cannot be placed on top of an existing dot.
5. The last person to successfully draw a new curve according to the rules wins the game!

An example of a complete game starting with 2 dots:

Start	Player 1 Joins A to B, adds C	Player 2 Joins A to A (loop), adds D	Player 1 Joins C to B, adds E	Player 2 Joins B to E, adds F
				

Notice that, after these four turns, Player 1 cannot draw a new curve. Player 1 cannot draw a curve from A since there are already three curve segments coming in or out of A (with two from the loop). This is the same for dots B, C, and E. Player 1 cannot join D to F since the curve would have to cross an existing curve, and cannot draw a loop on D or F as they each already have two curve segments coming in or out. Therefore, Player 2 wins!

Play the game a number of times starting with 2 dots. Keep track of the total number of turns it takes for each game to be won. Is there a certain number of turns after which the game is guaranteed to have ended?

Play the game a number of times starting with 3 dots. Is there a certain number of turns after which the game is guaranteed to have ended? How does this answer compare to your answer for the game starting with 2 dots?

More Info: Check the CEMC at Home webpage on Tuesday, April 14 for a discussion of Sprouts.

CEMC at Home

Grade 9/10 - Monday, April 6, 2020
Sprouts - Solution

Sprouts starting with 2 dots

While playing games of Sprouts starting with 2 dots, you may have noticed that each game ended after at most 5 turns. Did you also notice that each game included at least 4 turns?

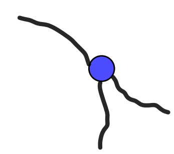
Sprouts starting with 3 dots

While playing games of Sprouts starting with 3 dots, you may have noticed that each game ended after at most 8 turns. Did you also notice that each game included at least 6 turns?

We encourage you to think about each of these observations and see if you can explain why this happened. We provide a discussion below to explain why any game starting with 3 dots must end after at most 8 turns.

A game of Sprouts starting with 3 dots must end after at most 8 turns

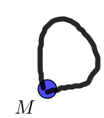
First, we note that the game is over as soon as no dots in the game can be part of a newly drawn curve. Remember that all dots in the game can have at most three curve segments attached to them. We will think of each dot as having three "slots" that can be filled. Curve segments can be attached to a dot M in a few different ways as shown below.



Curve drawn from M to another dot

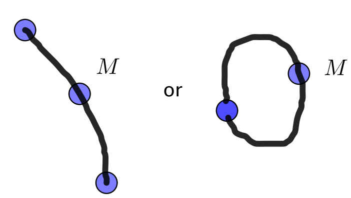
This curve takes up one slot on dot M

Loop drawn at M



This loop takes up two slots on dot M

M added to a newly drawn curve



This curve/loop takes up two slots on dot M

The game starts with three dots and no curves. Since there are three dots in total, each having three slots, the game starts with nine available slots.

After one turn is complete, one of two things has happened:

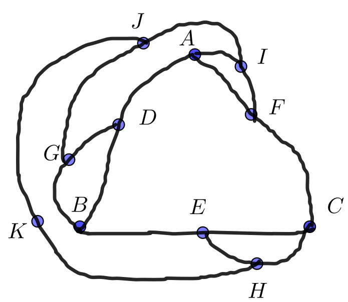
- a loop was added to one of the three dots, and a fourth dot was added to this loop, or
- a curve was drawn between two of the three dots, and a fourth dot was added to this curve.

In either case, after the first turn, there will be eight available slots remaining. Here we explain why:

Drawing a loop on a dot fills two of the three available slots on that particular dot, reducing us to $9 - 2 = 7$ slots available for the three original dots. However, a fourth dot is also added to this loop. Since two slots of this new dot are already taken, there is exactly one slot open on this new dot. This means there are $7 + 1 = 8$ slots available among the four dots now in the game. Notice that the situation is similar if a curve is drawn from one dot to another dot: we fill two slots (one on each dot at the ends of this curve), but gain one new slot from the fourth dot that is added.

In a similar way, we can argue that for *each turn that follows*, there is a net total loss of one slot per turn. After turn 2 there are 7 slots left, after turn 3 there are 6 slots left, and so on. If the game makes it to turn 8, then there can be only 1 slot left after turn 8 is complete. Since there must be at least 2 slots available in order for a new curve to be drawn, we can be sure that there is no legal move to make on turn 9. Therefore, we see that any game starting with 3 dots must end after at most 8 turns.

Did one of your games last exactly 8 turns? Note that our discussion above does not argue that a game can actually make it all the way to 8 turns, just that it is impossible for a game to make it to 9 turns. Below is an example of a game that lasted exactly 8 turns, which shows that 8 is the *maximum* number of turns attainable in a game starting with 3 dots.



Move	Endpoints	Added Point
1	A and B	D
2	B and C	E
3	A and C	F
4	B and D	G
5	E and C	H
6	A and F	I
7	G and I	J
8	H and J	K

Now try the following on your own:

- Explain why a game of Sprouts starting with 3 dots must last for at least 6 turns.
- Determine if a game of Sprouts starting with 3 dots can end in exactly 6 turns.

The game of Sprouts was invented by mathematicians John H. Conway (who died recently) and Michael S. Paterson at Cambridge University.

CEMC at Home

Grade 9/10 - Tuesday, April 7, 2020
Sum Code

In this puzzle, every letter of the alphabet represents a different integer from 1 to 26. Your task is to figure out which number is assigned to each letter. To get you started, you are given that H = 20 and N = 17. Use the algebraic equations to crack the code and figure out the remaining assignments.

A	B	C	D	E	F	G	H	I	J	K	L	M
							20					
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
17												

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 ~~17~~ 18 19 ~~20~~ 21 22 23 24 25 26

Algebraic Equations

$E = D \times D$

$B = E - D$

$H = E + D$

$B = T \times D$

$H = D \times C$

$J = C - T$

$V = C \times C$

$Y \times Y = P + I$

$Y + M = P - Y$

$P = V + 1$

$R = F - R$

$S = R - J$

$A = K + L$

$U = K \times T$

$Z = O + W - K$

$O = W + C$

$X = T \times C$

$Q = G - N + U$

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for a solution to Sum Code.

CEMC at Home

Grade 9/10 - Tuesday, April 7, 2020
Sum Code - Solution

Answers

A	B	C	D	E	F	G	H	I	J	K	L	M
8	12	5	4	16	22	19	20	10	2	7	1	14
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
17	18	26	23	11	9	3	21	25	13	15	6	24

Explanation

Since we are told that H = 20, and there are two equations involving the letter H, a good place to start is with these equations.

E = D × D	V = C × C	A = K + L
B = E − D	Y × Y = P + I	U = K × T
H = E + D	Y + M = P − Y	Z = O + W − K
B = T × D	P = V + 1	O = W + C
H = D × C	R = F − R	X = T × C
J = C − T	S = R − J	Q = G − N + U

The equation H = D × C tells us that D and C are a factor pair of 20. This means they could be 2 and 10 (in some order) or 4 and 5 (in some order). Note that they cannot be 1 and 20. (Why not?)

The equations E = D × D and V = C × C tell us more about this factor pair. If the factor pair is 2 and 10, then E and V are 4 and 100 (in some order). This is not possible since the numbers in this code only range from 1 to 26. Therefore, it must be the case that the factor pair D and C are 4 and 5 (in some order) which means that E and V are 16 and 25 (in some order).

Suppose D = 5 and C = 4. Then E = 25 and V = 16. Using the equation H = E + D we get that H = 25 + 5 = 30 which we know must be false. Since this is not the correct order of the factor pair, we know we must have D = 4 and C = 5. In this case, we get E = 16 and V = 25. We confirm with equation H = E + D that we get H = 20 as expected.

We now know for certain the values of D, C, E and V. By substituting these values into all of the relevant equations above, we can also determine the values for B, T, J, P, and X.

To proceed further, consider the equation Y × Y = P + I. This tells us that P + I is a perfect square. What does this tell us about possible values for I and Y? What does this information, combined with the equation Y + M = P − Y, reveal about the value of M?

By substituting values we already know into equations, and combining equations that contain common letters, we can proceed to crack the rest of the code, as indicated in the answer key above.

CEMC at Home

Grade 9/10 - Wednesday, April 8, 2020
Buying Local

Last week five people (Charlie, Manuel, Priya, Sal, and Tina) shopped at a local farmers' market. Each person went on a different day (Monday through Friday), bought a different item (carrots, blueberries, tomatoes, apples, or potatoes), and spent a different amount of money ($1.50, $2.00, $2.50, $3.50, or $3.75).

Use the clues below to determine who went on each day, what they bought, and for how much.

1. Sal went to the market two days before Priya.
2. Manuel spent $3.75 at the market the day before someone bought tomatoes.
3. Charlie paid $2.50 for carrots the day after someone spent $3.50.
4. The person who went on Wednesday bought apples.
5. Someone bought potatoes for $2.00 on Monday

You may find the following table useful in organizing your solution.

	Monday	Tuesday	Wednesday	Thursday	Friday	Carrots	Blueberries	Tomatoes	Apples	Potatoes	$1.50	$2.00	$2.50	$3.50	$3.75
Charlie															
Manuel															
Priya															
Sal															
Tina															
$1.50															
$2.00															
$2.50															
$3.50															
$3.75															
Carrots															
Blueberries															
Tomatoes															
Apples															
Potatoes															

More Info:

Check out the CEMC at Home webpage on Wednesday, April 15 for a solution to Buying Local.

This type of puzzle is known as a logic puzzle and we sometimes include them in Problem of the Week. Here is one called Winter Carnival Event.

CEMC at Home

Grade 9/10 - Wednesday, April 8, 2020
Buying Local - Solution

Answer

- Charlie bought carrots for \$2.50 on Friday
- Manuel bought apples for \$3.75 on Wednesday
- Priya bought tomatoes for \$3.50 on Thursday
- Sal bought blueberries for \$1.50 on Tuesday
- Tina bought potatoes for \$2.00 on Monday

Explanation

There are many different ways to arrive at the answers above. You may have used the chart provided with the problem to keep track of matches that were confirmed or deemed impossible while examining and combining the different clues. Below we present an explanation in words only. It may be helpful to follow along by filling out the chart given with the problem as you read.

The apples were purchased on Wednesday (clue 4). The potatoes were purchased for \$2.00 on Monday (clue 5). The carrots were not purchased on Tuesday because then the potatoes would cost \$3.50 instead of \$2.00 (clue 3). So the carrots were purchased for \$2.50 on either Thursday or Friday. The tomatoes were not purchased on Tuesday because then the potatoes would cost \$3.75 instead of \$2.00 (clues 2 and 5). The tomatoes were not purchased on Friday because if they were, then the carrots were purchased on Thursday, and the carrots would cost \$3.75 instead of \$2.50 (clues 2 and 3).

So the potatoes were purchased for \$2.00 on Monday, the blueberries were purchased on Tuesday, the apples were purchased on Wednesday, the tomatoes were purchased on Thursday, and the carrots were purchased for \$2.50 on Friday.

Manuel spent \$3.75 on Wednesday purchasing apples (clue 2). Charlie spent \$2.50 on Friday purchasing carrots, and someone spent \$3.50 on Thursday purchasing tomatoes (clue 3). Since the potatoes cost \$2.00 (clue 5) someone spent \$1.50 on Tuesday purchasing blueberries. Priya, Sal, and Tina went to the market on Monday, Tuesday, and Thursday in some order. Since Sal went two days before Priya (clue 1), then it must be the case that Sal went on Tuesday, Priya went on Thursday, and Tina went on Monday.

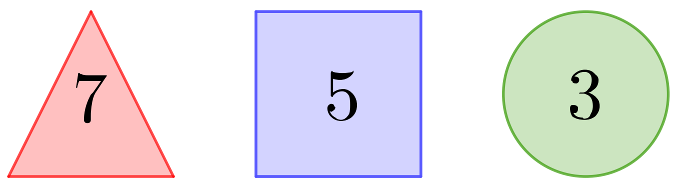
CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 9, 2020

Tokens Taken

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.



You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

More Info:

Check the CEMC at Home webpage on Thursday, April 16 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 16.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

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Problem of the Week
Problem D and Solution
Tokens Taken

Problem

Three bags each contain tokens. The green bag contains 22 round green tokens, each with a different integer from 1 to 22. The red bag contains 15 triangular red tokens, each with a different integer from 1 to 15. The blue bag contains 10 square blue tokens, each with a different integer from 1 to 10.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $22 \times 15 \times 10 = 3300$ different combinations of tokens created by selecting one token from each bag. Note that selecting the 7 red token, the 5 blue token and 3 green token is different than selecting the 5 red token, 7 blue token and the 3 green token. The order of selection does not matter.

You select one token from each bag. What is the probability that two or more of the selected tokens have the number 5 on them?

Solution

Solution 1

There are 22 different numbers which can be chosen from the green bag, 15 different numbers which can be chosen from the red bag, and 10 different numbers which can be chosen from the blue bag. So there are a total of $22 \times 15 \times 10 = 3300$ different combinations of numbers which can be produced by selecting one token from each bag.

To count the number of possibilities for a 5 to appear on at least two of the tokens, we will consider cases.

1. Each of the selected tokens has a 5 on it.

   This can only occur in 1 way.
2. A 5 appears on the green token and on the red token but not on the blue token.

   There are 9 choices for the blue token excluding the 5. A 5 can appear on the green token and on the red token but not on the blue token in 9 ways.
3. A 5 appears on the green token and on the blue token but not on the red token.

   There are 14 choices for the red token excluding the 5. A 5 can appear on the green token and on the blue token but not on the red token in 14 ways.
4. A 5 appears on the red token and on the blue token but not on the green token.

   There are 21 choices for the green token excluding the 5. A 5 can appear on the red token and on the blue token but not on the green token in 21 ways.

Summing the results from each of the cases, the total number of ways for a 5 to appear on at least two of the tokens is $1 + 9 + 14 + 21 = 45$. The probability of 5 appearing on at least two of the tokens is $\frac{45}{3300} = \frac{3}{220}$.

Solution 2

This solution uses a known result from probability theory. If the probability of event A occurring is a, the probability of event B occurring is b, the probability of event C occurring is c, and the results are not dependent on each other, then the probability of all three events happening is $a \times b \times c$.

The probability of a specific number being selected from the green bag is $\frac{1}{22}$ and the probability of any specific number not being selected from the green bag is $\frac{21}{22}$.

The probability of a specific number being selected from the red bag is $\frac{1}{15}$ and the probability of any specific number not being selected from the red bag is $\frac{14}{15}$.

The probability of a specific number being selected from the blue bag is $\frac{1}{10}$ and the probability of any specific number not being selected from the blue bag is $\frac{9}{10}$.

In the following we will use $P(p, q, r)$ to mean the probability of p being selected from the green bag, q being selected from the red bag, and r being selected from the blue bag. So, $P(5, 5,\ \text{not } 5)$ means that we want the probability of a 5 being selected from the green bag, a 5 being selected from the red bag, and anything but a 5 being selected from the blue bag.

$$
\begin{aligned}
& \text{Probability of 5 being selected from at least two of the bags} \\
= \ & \text{Probability of 5 from each bag} + \text{Probability of 5 from exactly 2 bags} \\
= \ & \text{P(5, 5, 5)} + \text{P(5, 5, not 5)} + \text{P(5, not 5, 5)} + \text{P(not 5, 5, 5)} \\
= \ & \frac{1}{22} \times \frac{1}{15} \times \frac{1}{10} + \frac{1}{22} \times \frac{1}{15} \times \frac{9}{10} + \frac{1}{22} \times \frac{14}{15} \times \frac{1}{10} + \frac{21}{22} \times \frac{1}{15} \times \frac{1}{10} \\
= \ & \frac{1}{3300} + \frac{9}{3300} + \frac{14}{3300} + \frac{21}{3300} \\
= \ & \frac{45}{3300} \\
= \ & \frac{3}{220}
\end{aligned}
$$

The probability of 5 appearing on at least two of the tokens is $\dfrac{3}{220}$.