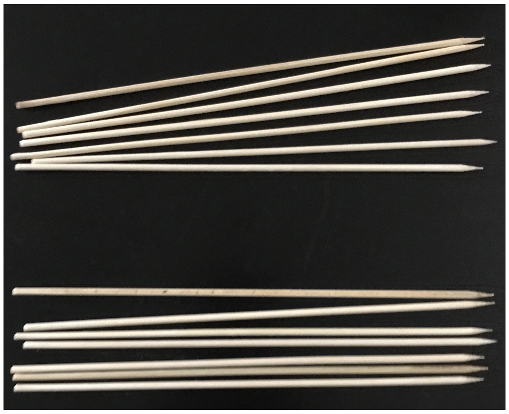
CEMC at Home

Grade 11/12 - Monday, April 6, 2020
Pick up Sticks



You Will Need:

- Two players
- 14 sticks (these could be chopsticks, popsicle sticks, toothpicks, pencils, etc.)

How to Play:

1. Arrange the 14 sticks into two piles with 7 sticks each.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On your turn, you can pick up as many sticks as you want, as long as all of the sticks you pick up come from the same pile.
4. The player who picks up the last stick wins the game!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy for this game? If so, which player (Player 1 or Player 2) has a winning strategy?

Is there a connection between this game and the game we played on March 23 (Rook to the Top)?

Variations:

A. Change the number of sticks that you have in the two piles. There should still be an equal number of sticks in each pile. How does this variation affect your strategy for the game?

B. Play the game instead starting with two piles with a different number of sticks. How does this variation affect your strategy for the game?

More Info:

Check out the CEMC at Home webpage on Tuesday, April 14 for a discussion of a strategy for Pick Up Sticks. We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Monday, April 6, 2020
Pick Up Sticks - Strategy

As you might have expected based on our first two games in this game series, Pick Up Sticks is just another variation of the first game Rook to the Top (from March 23).

Please take a minute to remind yourself of the Rook to the Top game and its winning strategy.

How are these games related? To see the connection, we will first describe the game Pick Up Sticks in terms of ordered pairs. Let's label our two piles A and B. The ordered pair (a, b) will refer to the state of the game where we have a sticks remaining in pile A and b sticks remaining in pile B. Therefore, we start with the state corresponding to the ordered pair $(7, 7)$ and the winning state corresponds to the ordered pair $(0, 0)$. There are 64 different possible states during the game and they correspond to the ordered pairs of integers (a, b), where $0 \le a \le 7$ and $0 \le b \le 7$.

A valid move in the game takes us from an ordered pair (a, b) to another ordered pair (a^*, b^*) where either $a^* < a$ and $b^* = b$, or $a^* = a$ and $b^* < b$. The first case represents removing some number of sticks from pile A (decreasing the first component and leaving the second component unchanged), and the second case represents removing some number of sticks from pile B (decreasing the second component and leaving the first component unchanged).

So how does this help us see the connection to the game Rook to the Top? Remember that Rook to the Top is played on an 8 by 8 grid with 64 squares. We begin by labelling the 64 squares on our grid with the ordered pairs as shown. (Notice that the convention in this table is not the same as the normal labelling of ordered pairs in the plane.)

(0,7)	(0,6)	(0,5)	(0,4)	(0,3)	(0,2)	(0,1)	(0,0)
(1,7)	(1,6)	(1,5)	(1,4)	(1,3)	(1,2)	(1,1)	(1,0)
(2,7)	(2,6)	(2,5)	(2,4)	(2,3)	(2,2)	(2,1)	(2,0)
(3,7)	(3,6)	(3,5)	(3,4)	(3,3)	(3,2)	(3,1)	(3,0)
(4,7)	(4,6)	(4,5)	(4,4)	(4,3)	(4,2)	(4,1)	(4,0)
(5,7)	(5,6)	(5,5)	(5,4)	(5,3)	(5,2)	(5,1)	(5,0)
(6,7)	(6,6)	(6,5)	(6,4)	(6,3)	(6,2)	(6,1)	(6,0)
(7,7)	(7,6)	(7,5)	(7,4)	(7,3)	(7,2)	(7,1)	(7,0)

We know that each ordered pair corresponds to a possible state in a game of Pick Up Sticks. A valid move where the first component does not change and the second component decreases corresponds to a move to the right in the table. A valid move where the second component does not change and the first component decreases corresponds to a move upward in the table.

As with the grid in our solution of Rook to the Top, we highlight an important diagonal on the grid, called the *winning diagonal*. Again, the strategy is to make moves, if possible, to end up on the winning diagonal. The ordered pairs on the winning diagonal correspond to the states where the two piles have an equal number of sticks.

From our knowledge of the winning strategy for Rook to the Top, we know that the second player has a winning strategy and this strategy is to always move the rook back onto the winning diagonal. Similarly, the second player in Pick Up Sticks has a winning strategy, and this strategy is to always "move" the game into a state located on the winning diagonal. This means always picking up sticks in a way that leaves the two piles with an equal number of sticks.

Since we start with two equal piles, the first player has no choice but to make the piles unequal. The second player can then make the piles equal and the total number of sticks in the piles will have decreased. The first player again must make the piles unequal and the second player can again make the piles equal. Repeating this process, the second player will always be able to return to a state on the winning diagonal with fewer sticks in the piles, in total. Since there is a finite number of sticks, the second player will eventually make the piles equal with no sticks left and win the game. Therefore, the second player has a winning strategy for this game.

Variations:

A. *Different number of sticks in the piles, but piles still equal in size.*

   Now that we know the winning strategy for the original version of Pick Up Sticks, we can see that the number of sticks we start with has no effect on the winning strategy, provided that we still start with piles that are equal in size. The second player will still have a winning strategy and it will be the same as described above.

B. *Piles that are not equal in size.*

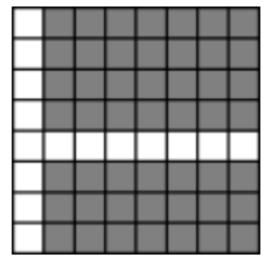
   If we start the game with two piles with different numbers of sticks, then the strategy is similar, but the first player in the game would have the winning strategy. On the first move, the first player can remove sticks from one pile in order to leave the piles with an equal number of sticks. The second player then has no choice but to make the piles unequal, giving the first player the chance to make them equal again (with fewer sticks in total). The argument continues as it did for the strategy in the original game.

CEMC at Home

Grade 11/12 - Tuesday, April 7, 2020
Counting and Probability

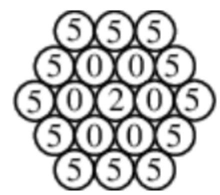
Today's problems are from some of our past Fermat Contests. These problems were compiled using our Problem Set Generator. The Problem Set Generator can be used to create a randomly generated problem set or a problem set focussed on a specific topic and/or a specific level of difficulty. We chose the topic of *counting and probability* to generate this problem set. Try using the Problem Set Generator to create your own problem set!

1. In the diagram, how many 1×1 squares are shaded in the 8×8 grid?
   (A) 53 (B) 51 (C) 47
   (D) 45 (E) 49



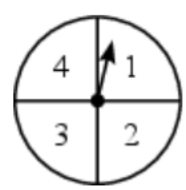
(Source: 2017 Fermat (Grade 11), #2)
Primary Topics: **Geometry and Measurement | Counting and Probability**
Secondary Topics: **Area | Counting**

2. Starting with the 2 in the centre, the number 2005 can be formed by moving from circle to circle only if the two circles are touching. How many different paths can be followed to form 2005?
   (A) 36 (B) 24 (C) 12
   (D) 18 (E) 6



(Source: 2005 Fermat (Grade 11), #12)
Primary Topics: **Counting and Probability**
Secondary Topics: **Counting | Digits**

3. On each spin of the spinner shown, the arrow is equally likely to stop on any one of the four numbers. Deanna spins the arrow on the spinner twice. She multiplies together the two numbers on which the arrow stops. Which product is most likely to occur?
   (A) 2 (B) 4 (C) 6
   (D) 8 (E) 12



(Source: 2014 Fermat (Grade 11), #15)
Primary Topics: **Counting and Probability | Number Sense**
Secondary Topics: **Probability | Counting**

4. Amina and Bert alternate turns tossing a fair coin. Amina goes first and each player takes three turns. The first player to toss a tail wins. If neither Amina nor Bert tosses a tail, then neither wins. What is the probability that Amina wins?
   (A) $\frac{21}{32}$ (B) $\frac{5}{8}$ (C) $\frac{3}{7}$ (D) $\frac{11}{16}$ (E) $\frac{9}{16}$

(Source: 2015 Fermat (Grade 11), #21)
Primary Topics: **Counting and Probability**
Secondary Topics: **Probability**

More Info: Visit our webpage to find past Fermat Contests and their solutions.

CEMC at Home

Grade 11/12 - Wednesday, April 8, 2020
Silly Square Roots

The connections between mathematics and computer science run deep. Here we explore these connections through the following problem which was Question 5a on the 2019 Euclid Contest. No programming experience is needed to follow along but there is also a challenge for you if you do have some programming experience.

Problem 1

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{50}$.

Solution to Problem 1 Using Mathematics

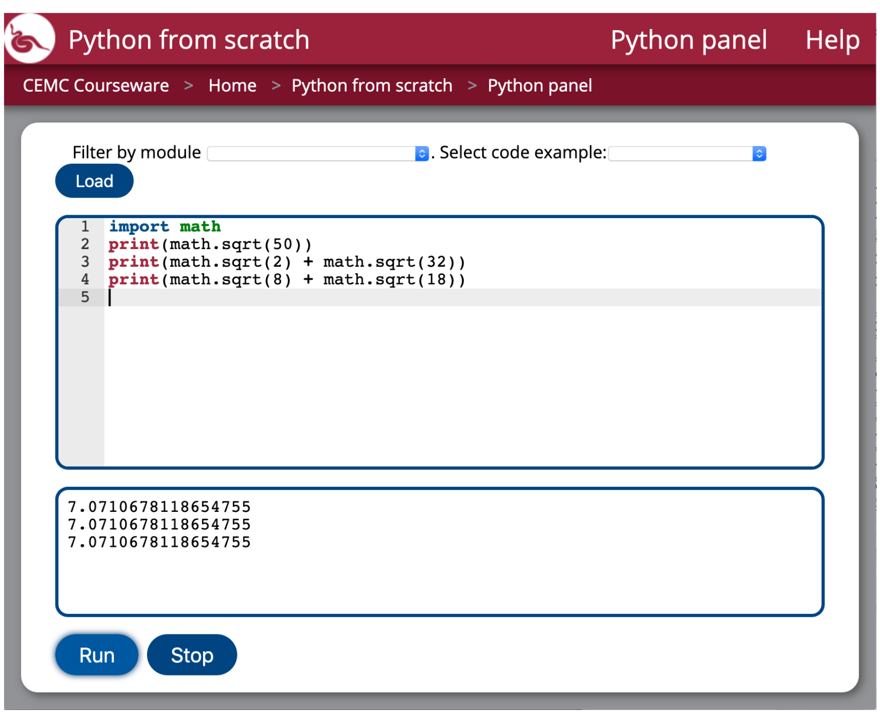
On a math contest or test, we would normally use known mathematical facts, such as the properties of square roots, to find the two pairs. Attempt the problem above this way and then check your answer. Note that your solution can rely on the fact that we were told there are exactly two pairs satisfying the equation.

Solution to Problem 1 Using Computer Programming

Now visit our Python from scratch panel and enter the following code in the upper code box.

```
import math
print(math.sqrt(50))
print(math.sqrt(2) + math.sqrt(32))
print(math.sqrt(8) + math.sqrt(18))
```

Hit the Run button to see the result of this program. Your screen should look something like this:



Congratulations if you just entered a computer program and ran it for the first time!

The key thing to observe is that this program uses a library function called `math.sqrt` to display the three values $\sqrt{50}$, $\sqrt{2}+\sqrt{32}$ and $\sqrt{8}+\sqrt{18}$. They all appear to be equal suggesting that the two pairs are $(2, 32)$ and $(8, 18)$. But where did these numbers come from?

In order to discover these two pairs, we could generate possible pairs (a, b) and check if $\sqrt{a}+\sqrt{b} = \sqrt{50}$ for each pair. Think about how you would generate possible pairs systematically and check whether the equality holds. The next program will simulate one way to do this. Replace your previous program with the following program and run it to see what it displays.

```
import math
for a in range(1, 50):
    for b in range(a + 1, 50):
        if (math.sqrt(a) + math.sqrt(b)) == math.sqrt(50):
            print((a,b))
```

Try to get a rough understanding of why this produces us the right answer. One advantage of programs written in Python is that they tend to be readable by beginners. However, if you are new to programming or new to the language Python, here are a few notes to help explain some of the details:

- The first line gives the rest of the program access to a library of mathematical functions such as `math.sqrt`. If we do not include this line, Python will produce an error message.
- The second line tells us that a *variable* `a` will take on the integer values starting at 1 and ending at $50 - 1 = 49$. (Python does not include the second number in the range.) The block of three lines indented below this will be executed once for each of these values of `a`. Together, this is typically called a *loop*.
- The third line is similar to the second line except the values of `b` begin at the value `a + 1`. The result is that we have one loop nested inside another. For each value of `a`, the variable `b` will take on its own range of values.
- To see the how the values of the variables `a` and `b` change, you can `print` (or display) them at a strategic point in your program. Here is an example using 5 in place of 50:

```
for a in range(1, 5):
    for b in range(a + 1, 5):
        print((a,b))
```

```
(1, 2)
(1, 3)
(1, 4)
(2, 3)
(2, 4)
(3, 4)
```

- The fourth line executes a conditional test involving variables `a` and `b`. The `==` operator attempts to determine if the two expressions are equal. If they are, the indented line below is executed. Otherwise, it is skipped.

Now ask yourself why 50 is used as an upper bound in `range(1,50)` and `range(a + 1, 50)`. Can this be improved? Why is 1 used as a lower bound in `range(1, 50)` but not in `range(a + 1, 50)`? We will discuss these details when solutions to the next two problems are presented.

Problem 2

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{75}$.

Attempt to Solve Problem 2

The two pairs that solve this problem are $(3, 48)$ and $(12, 27)$. However, our programming solution to Problem 1 where each 50 is replaced by 75, does not work! You should try it yourself to see the result. Why is nothing displayed? We get a clue by doing this:

```python
import math
print(math.sqrt(75))
print(math.sqrt(3) + math.sqrt(48))
print(math.sqrt(12) + math.sqrt(27))
```

```
8.660254037844387
8.660254037844386
8.660254037844386
```

The fundamental problem is that the `math.sqrt` function does not compute exact values. We can see that the displayed results above are extremely close to each other but the first one, the purported value of $\sqrt{75}$, is just a bit larger.

This is not a flaw with Python! It is impossible to store let alone calculate irrational numbers like $\sqrt{75}$, π and e exactly. An infinite amount of memory would be needed to do this. We already know this from mathematics where we understand that an irrational number is one for which its decimal expansion does not terminate or end with a repeating sequence.

Can you find a way to use a combination of mathematical insight and Python to solve this second problem?

Problem 3

Determine *all* pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a} + \sqrt{b} = \sqrt{147}$.

More Info:

Check out the CEMC at Home webpage on Wednesday, April 15 for a solution to Silly Square Roots.

Our webpage Computer Science and Learning to Program is the best place to find the CEMC's computer science resources. Two resources through which you can explore Python further are:

Python from scratch
A gentle introduction to programming designed with the beginner in mind.

CS Circles
Interactive lessons teaching the basics of writing computer programs in Python. This is also an introduction but moves at a bit of a faster pace. The CS Circles console is an alternative tool that can be used to enter and run all the code explored in this resource.

CEMC at Home

Grade 11/12 - Wednesday, April 8, 2020
Silly Square Roots - Solution

Here we expand on our use of writing Python programs to solve Question 5a on the 2019 Euclid Contest and two variations of this problem.

Problem 1

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{50}$.

Discussion of Programming Solution

We saw that the following program gives the two correct answers.

```
import math
for a in range(1, 50):
    for b in range(a + 1, 50):
        if (math.sqrt(a) + math.sqrt(b)) == math.sqrt(50):
            print((a,b))
```

Here are some observations about the mathematics used to inform and write the code:

- The smallest value to consider for a is 1 because it must be a positive integer.
- The smallest value to consider for b is $a + 1$ because we want $a < b$ and both a and b must be integers.
- If $a \geq 50$, then $\sqrt{a} + \sqrt{b} \geq \sqrt{50} + \sqrt{b} \geq \sqrt{50}$. This means we don't need to consider values of a that are greater than or equal to 50. The same applies for b.
- Since we want $a < b$, we have $2\sqrt{a} = \sqrt{a} + \sqrt{a} < \sqrt{a} + \sqrt{b}$. So if $\sqrt{a} + \sqrt{b} = \sqrt{50}$, we get $2\sqrt{a} < \sqrt{50}$. Squaring both sides and rearranging gives $a < \frac{50}{4} < 13$. This means that we could have replaced `range(1,50)` with `range (1,13)`, and hence checked fewer pairs.

Problem 2

Determine the two pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b} = \sqrt{75}$.

Programming Solution

We noticed that changing 50 to 75 in the programming solution to Problem 1, does not produce the correct answer for Problem 2. In this sense, our solution to Problem 1 was "lucky". One approach that does give us the right answer is the following:

```
import math
for a in range(1,75):
    for b in range(a+1,75):
        if abs(math.sqrt(a) + math.sqrt(b) - math.sqrt(75)) < 0.0001:
            print((a,b))
```

Since the `math.sqrt` function gives close approximate values, we can test if $\sqrt{a}+\sqrt{b}$ is approximately equal to $\sqrt{75}$. To do this, we look at the positive difference of these two values. This is called the absolute value and is written $|\sqrt{a}+\sqrt{b}-\sqrt{75}|$ in mathematics and computed using

```
abs((math.sqrt(a) + math.sqrt(b)) - math.sqrt(75))
```

in Python. It turns out that being within 0.0001 is "close enough". That is, if we use Python to search for pairs (a, b) satisfying $|\sqrt{a}+\sqrt{b}-\sqrt{75}| < 0.0001$ as outlined earlier, then we happen to find exactly two pairs: $(3, 48)$ and $(12, 27)$. We were told that there are exactly two solutions to the equation, and so we can be sure that our program has found all of the solutions. On the other hand, if we had used Python to find pairs (a, b) satisfying $|\sqrt{a}+\sqrt{b}-\sqrt{75}| < 0.001$, then we would have instead found three pairs: the two solutions as well as one extraneous pair, $(8, 34)$. If we changed the bound to 0.01 or 0.1, then the true solutions would be hidden among an even larger number of extraneous pairs. You can explore the number of pairs in these cases on your own.
How would you use the results of these searches to help determine the complete set of solutions if you were not told in advance that there were exactly two solutions?

Problem 3

Determine all pairs of positive integers (a, b) with $a < b$ that satisfy the equation $\sqrt{a}+\sqrt{b}=\sqrt{147}$.

Programming Solution

Here we do not know how many pairs of positive integers to look for. A solution to this problem uses a nice combination of mathematics and computer science.

Consider positive integers a and b and suppose that

$$\sqrt{a}+\sqrt{b}=\sqrt{147}.$$

By squaring both sides we get

$$a+2\sqrt{ab}+b=147.$$

Rearranging gives

$$2\sqrt{ab}=147-a-b.$$

Squaring both sides again yields

$$4ab=(147-a-b)^2.$$

This is an equation that does not involve square roots. But we do have to be a bit careful because squaring equations can introduce solutions. Now, we are only considering cases where a and b are positive, so the first time we squared both sides of the equation above did not introduce solutions. It is also true that the solutions we are looking for must satisfy $a + b < 147$ (try to show this!) which means the second time we squared both sides of the equation also did not introduce solutions. Now we can test values for a and b that satisfy this equation only using integer operations, which give exact values in Python:

```
import math
for a in range(1,147):
    for b in range(a+1,147):
        if 4*a*b == (147-a-b)*(147-a-b) and a + b < 147:
            print((a,b))
```

Running this program gives us the answers of $(3, 108)$, $(12, 75)$ and $(27, 48)$.

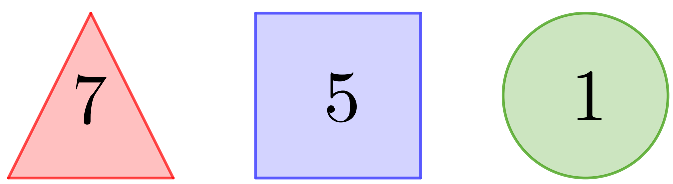
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 9, 2020
What's Your Unlucky Number?

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky.

Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $20 \times 12 \times 8 = 1920$ different combinations of tokens that can be created by selecting one token from each bag. Note that the order of selection does not matter. Also note that selecting the 7 red token, the 5 blue token and 1 green token is different than selecting the 5 red token, 7 blue token and the 1 green token.



Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?

More Info:

Check the CEMC at Home webpage on Thursday, April 16 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 16.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week
Problem E and Solution
What's Your Unlucky Number?

Problem

Everyone has a lucky number. Sue Perstitious does not have a lucky number but considers the number 13 to be unlucky. Three bags each contain tokens. The green bag contains 20 round green tokens, each with a different integer from 1 to 20. The red bag contains 12 triangular red tokens, each with a different integer from 1 to 12. The blue bag contains 8 square blue tokens, each with a different integer from 1 to 8.

Any token in a specific bag has the same chance of being selected as any other token from that same bag. There is a total of $20 \times 12 \times 8 = 1920$ different combinations of tokens that can be created by selecting one token from each bag. Note that the order of selection does not matter. Also note that selecting the 7 red token, the 5 blue token and 1 green token is different than selecting the 5 red token, 7 blue token and the 1 green token.

Sue selects one token from each bag. What is the probability that the sum of the numbers selected is divisible by 13, her unlucky number?

Solution

There are 20 different numbers which can be selected from the green bag, 12 different numbers which can be selected from the red bag, and 8 different numbers which can be selected from the blue bag. So there are $20 \times 12 \times 8 = 1920$ different combinations of numbers which can be selected from the three bags.

Let (g, r, b) represent the outcome of a selection where g is the number on the token selected from the green bag, r is the number on the token selected from the red bag and b is the number on the token selected from the blue bag. Also, $1 \leq g \leq 20$, $1 \leq r \leq 12$, and $1 \leq b \leq 8$, for integers g, r, b.

Numbers that are divisible by 13 are $13, 26, 39, 52, \cdots$. The maximum sum that can be reached any selection is $8 + 12 + 20 = 40$. To count the number of possibilities for sums which are divisible by 13, we will consider three cases: a sum of 13, a sum of 26 and a sum of 39. Within the first two cases, we will look at sub-cases based on the possible outcome for the 8 possible selections from the blue bag.

1. **The sum of the numbers on the 3 tokens is 13.**
   - **1 is on the token selected from the blue bag**
     The sum of the numbers on the other two tokens is 12. Selecting the 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 11, 1), (2, 10, 1), (3, 9, 1), \cdots, (11, 1, 1)$, 11 possibilities in total.
   - **2 is on the token selected from the blue bag**
     The sum of the numbers on the other two tokens is 11. Selecting the 11 or 12 token from the red bag is not possible since the number on the token selected from the green bag must be at least 1. The possibilities for (g, r, b) are $(1, 10, 2), (2, 9, 2), (3, 8, 2), \cdots, (10, 1, 2)$, 10 possibilities in total.

- **3 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 10. Using reasoning similar to the preceding two cases, the possibilities for (g, r, b) are $(1, 9, 3), (2, 8, 3), (3, 7, 3), \cdots, (9, 1, 3)$, 9 possibilities in total.
- **4 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 9. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 8, 4), (2, 7, 4), (3, 6, 4), \cdots, (8, 1, 4)$, 8 possibilities in total.
- **5 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 8. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 7, 5), (2, 6, 5), (3, 5, 5), \cdots, (7, 1, 5)$, 7 possibilities in total.
- **6 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 7. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 6, 6), (2, 5, 6), (3, 4, 6), \cdots, (6, 1, 6)$, 6 possibilities in total.
- **7 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 6. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 5, 7), (2, 4, 7), (3, 3, 7), (4, 2, 7), (5, 1, 7)$, 5 possibilities in total.
- **8 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 5. Using reasoning similar to the first two cases, the possibilities for (g, r, b) are $(1, 4, 8), (2, 3, 8), (3, 2, 8), (4, 1, 8)$, 4 possibilities in total.

Summing the results from the 8 cases, there are $11 + 10 + 9 + \cdots + 5 + 4 = 60$ combinations so that the sum of numbers on the 3 tokens is 13.

2. **The sum of the numbers on the 3 tokens is 26.**

- **1 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 25. The largest number possible from the green bag is 20 so the smallest possible number from the red bag would be 5. The largest number possible from the red bag is 12 so the smallest possible number from the green bag would be 13. The numbers from the green bag go from 20 to 13 while the numbers from the red bag go from 5 to 12. The possibilities for (g, r, b) are $(20, 5, 1), (19, 6, 1), (18, 7, 1), \cdots, (13, 12, 1)$, 8 possibilities in total.
- **2 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 24. Using similar reasoning to the first case in this section, the possibilities for (g, r, b) are $(20, 4, 2), (19, 5, 2), (18, 6, 2), \cdots, (12, 12, 2)$, 9 possibilities in total.
- **3 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 23. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are $(20, 3, 3), (19, 4, 3), (18, 5, 3), \cdots, (11, 12, 3)$, 10 possibilities in total.

- **4 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 22. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are
  $(20, 2, 4), (19, 3, 4), (18, 4, 4), \cdots, (10, 12, 4)$, 11 possibilities in total.
- **5 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 21. Using reasoning similar to the first case in this section, the possibilities for (g, r, b) are
  $(20, 1, 5), (19, 2, 5), (18, 3, 5), \cdots, (9, 12, 5)$, 12 possibilities in total.
- **6 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 20. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 8. The highest number possible from the green bag is 19 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are
  $(19, 1, 6), (18, 2, 6), (17, 3, 6), \cdots, (8, 12, 6)$, 12 possibilities in total.
- **7 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 19. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 7. The highest number possible from the green bag is 18 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are
  $(18, 1, 7), (17, 2, 7), (16, 3, 7), \cdots, (7, 12, 7)$, 12 possibilities in total.
- **8 is on the token selected from the blue bag**
  The sum of the numbers on the other two tokens is 18. The highest number possible from the red bag is 12 so the smallest possible number from the green bag would be 6. The highest number possible from the green bag is 17 since the number from the red bag must be at least 1. Therefore, the possibilities for (g, r, b) are
  $(17, 1, 8), (16, 2, 8), (15, 3, 8), \cdots, (6, 12, 8)$, 12 possibilities in total.

Summing the results from the cases, there are $8 + 9 + 10 + 11 + 4(12) = 86$ combinations so that the sum of the numbers on the 3 tokens is 26.

3. **The sum of the numbers on the 3 tokens is 39.**

   The maximum sum that can be obtained is $8 + 12 + 20 = 40$. A sum of 39 can only be achieved by keeping two of the three tokens at their maximum and reducing the third token to 1 less than its maximum. The possibilities for (g, r, b) are $(20, 12, 7)$, $(20, 11, 8)$ and $(19, 12, 8)$, 3 possibilities in total.

The total number of combinations in which the sum of the numbers on the three tokens is divisible by 13 is $60 + 86 + 3 = 149$.

Therefore, the probability of selecting three tokens with a sum which is divisible by 13 is $\frac{149}{1920}$.

There is less than an 8% chance that the three tokens selected by Sue will sum to a multiple of her unlucky number.