CEMC at Home

Grade 4/5/6 - Monday, April 27, 2020
Keepin' It Square

In today's game, we will compete to connect the dots to draw squares on dot paper.

You Will Need:

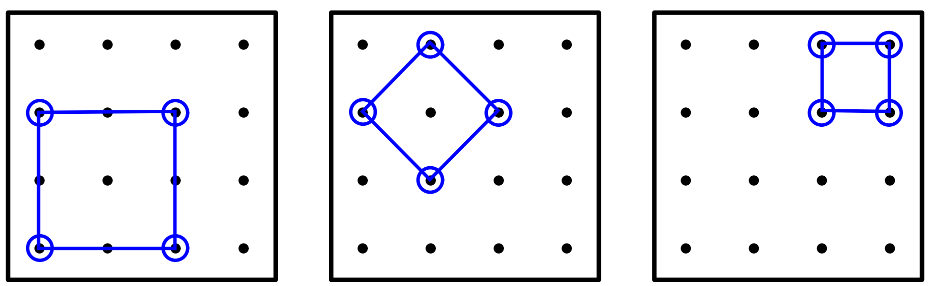
- Two players
- Many different pieces of dot paper

  The game board we will use is 4 dots by 4 dots, but you can play with other boards as well.
- A pencil

How to Play:

1. Start with a piece of dot paper that is 4 dots by 4 dots.
2. The two players take turns circling dots on the dot paper. Decide which player will go first.
3. On your turn, you start by circling one dot that has not yet been circled. You then try to make a square by connecting 4 circled dots on the board.

   Below are some examples of squares. Notice that squares can be different sizes and orientations.



4. The first player to successfully make a square wins.

Variations:

A. Play the same game but keep track (using different coloured pens or pencil crayons) of which dot is circled by which player. In this variation, each player can only make a square using their *own* circled dots.

B. Play the same game but instead of stopping after the first square is drawn, continue until all dots have been circled. The player who made the most squares in total wins the game.

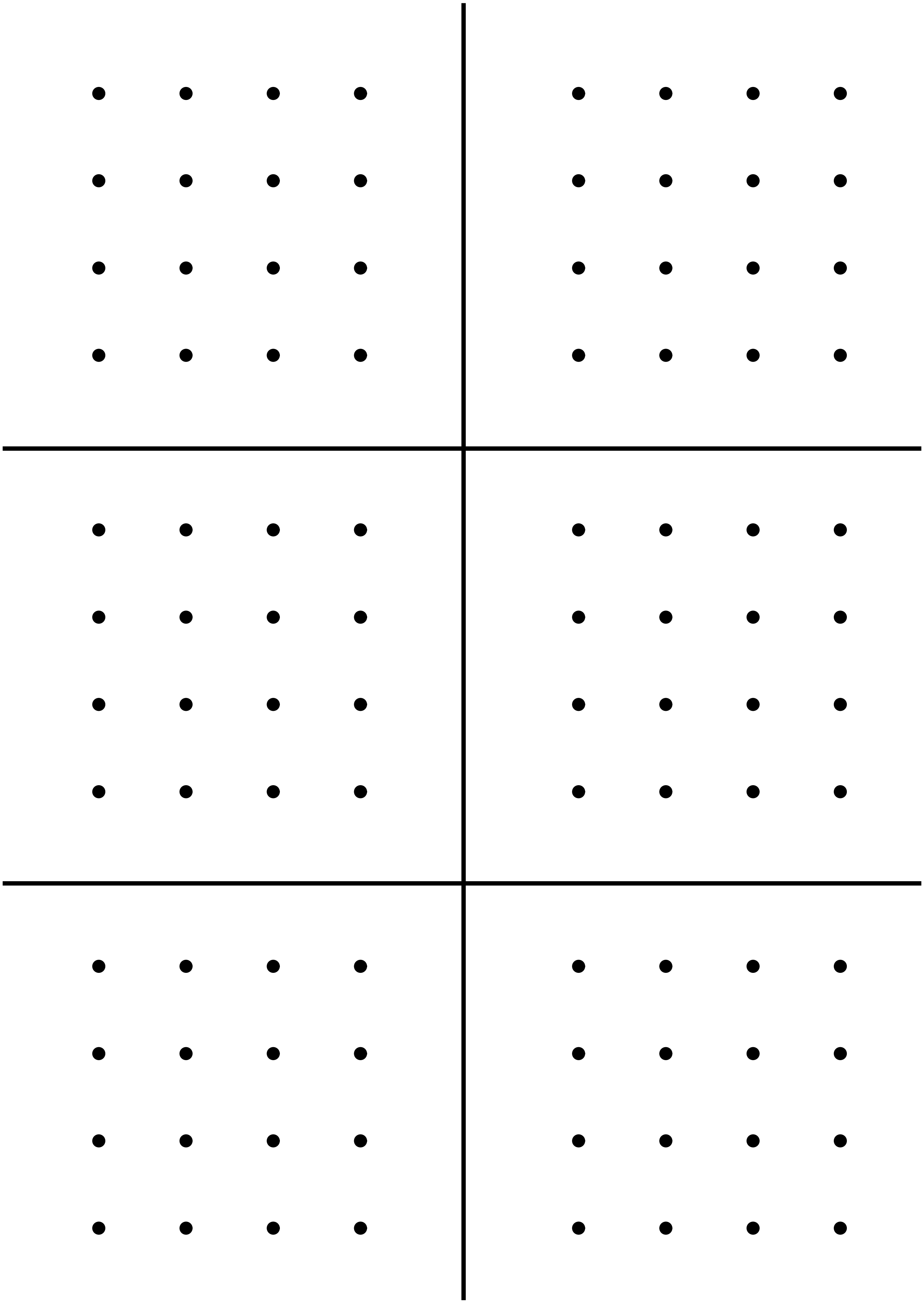
Note that players can make more than one square on a turn. Each square drawn must be a new square, but can share vertices with an existing square and/or overlap with an existing square.

Follow-up Questions:

1. How many different squares are there on a 4 dots by 4 dots game board?
2. How many different squares are there on a 5 dots by 5 dots game board?

More Info:

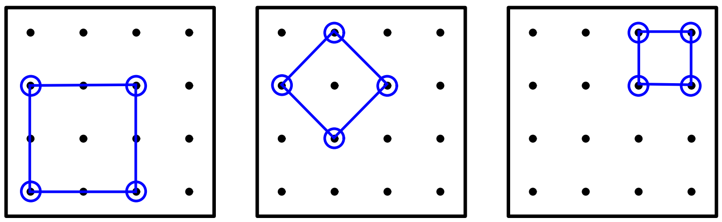
Check out the CEMC at Home webpage on Monday, May 4 for a solution to Keepin' It Square.



CEMC at Home

Grade 4/5/6 - Monday, April 27, 2020
Keepin' It Square - Solution

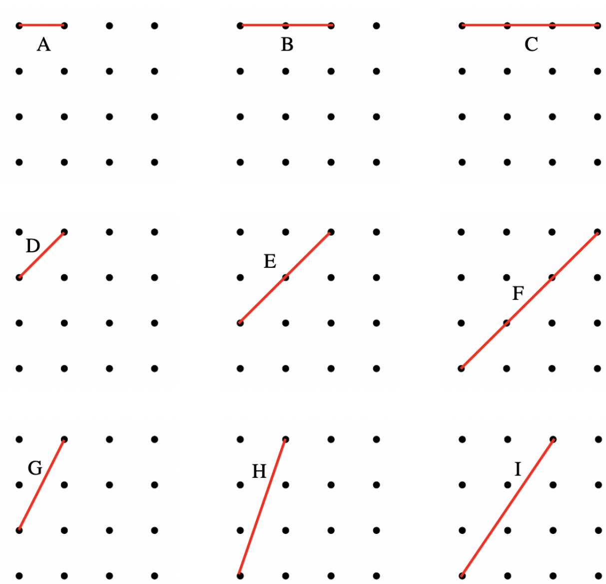
There are many different squares that can be drawn on the following dot paper. A few examples are shown below.



- How many different squares can be drawn that have the same side length as the square shown in the leftmost image?
- How many different squares can be drawn that have the same side length as the square shown in the middle image?
- How many different squares can be drawn that have the same side length as the square shown in the rightmost image?
- What other side lengths are possible for squares drawn on this dot paper?

To make sure that we get a correct count, we need to organize our thinking. We will group the different possible squares based on their side lengths.

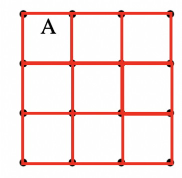
First, convince yourself that there are exactly nine different lengths that a line segment on this board could have. Examples of line segments with each of these lengths are shown below and labelled A - I.



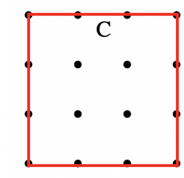
Remember that all vertices of a square drawn must lie on dots. This means the images above show the only possible side lengths of a square on this board.

Now let's count how many different squares of each side length can be drawn.

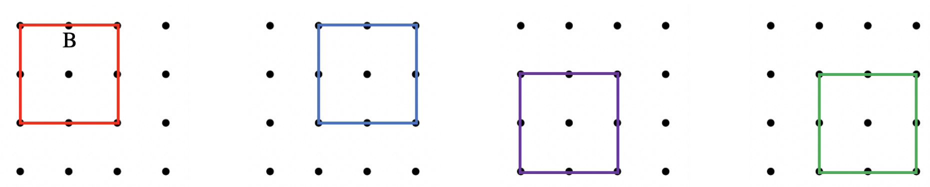
There are 9 squares with side lengths equal to the length of A.



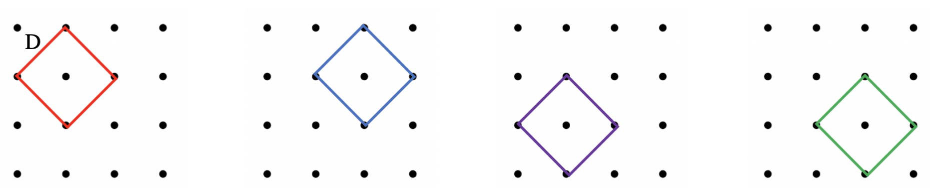
There is 1 square with side lengths equal to the length of C.



There are 4 squares with side lengths equal to the length of B.

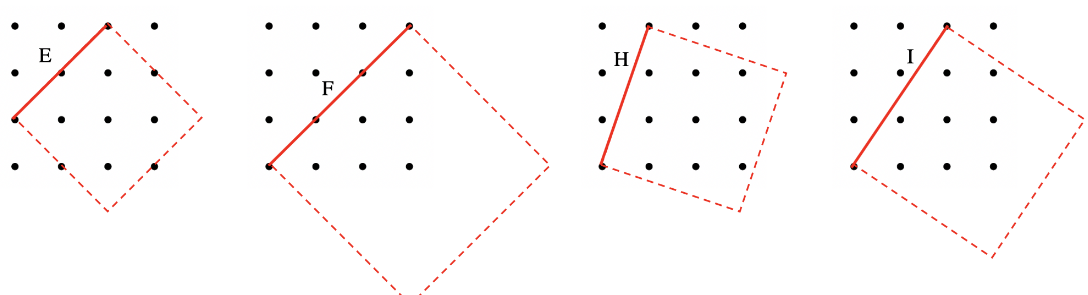


There are 4 squares with side lengths equal to the length of D.

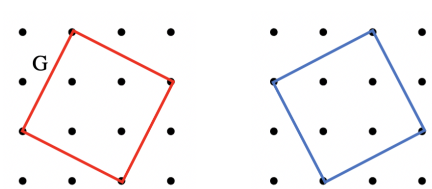


There are 0 squares with side lengths equal to the length of E, F, H, or I.

No matter how you place these line segments on the grid, the squares formed must extend past the edges of the grid. Some examples are shown below.



There are 2 squares with side lengths equal to the length of G.



This means there are $9 + 1 + 4 + 4 + 2 = 20$ different squares that can be drawn on the dot paper.

Can you do a similar count to figure out how many different squares can be drawn on dot paper that is 5 dots by 5 dots (instead of 4 dots by 4 dots)?

CEMC at Home

Grade 4/5/6 - Tuesday, April 28, 2020
Figure Out These Rectangles

Problem 1: A rectangle has length ℓ and width w, in centimetres, and a perimeter of 16 cm.

(a) Explain how you know it must be true that $\ell + w = 8$ cm.

(b) Suppose further that the side lengths ℓ and w are positive whole numbers of centimetres with ℓ greater than w. Given this, what are the possible pairs of values for ℓ and w?

Don't use the exact shape of the diagram given to determine the values of ℓ and w. This is just an illustration and does not represent exactly what the rectangle must look like. It can be wider or longer than what is shown here. For example, w could be 1 cm. In this case, what must ℓ be?

Problem 2: Five identical rectangles each with length ℓ and width w are arranged to form the larger rectangle below and to the right. The side lengths ℓ and w are positive whole numbers of centimetres.

(a) How many widths w make up one length ℓ?

Where in the diagram can we see the relationship between these two values?

(b) Suppose each identical smaller rectangle has perimeter 16 cm and ℓ greater than w. Using your work from Problem 1 and the relationship between ℓ and w from Problem 2 (a), can you figure out the values of ℓ and w?

You found different possibilities for the pair ℓ and w in Problem 1 (b). Which of these pairs of values satisfies the relationship you found in Problem 2 (a)?

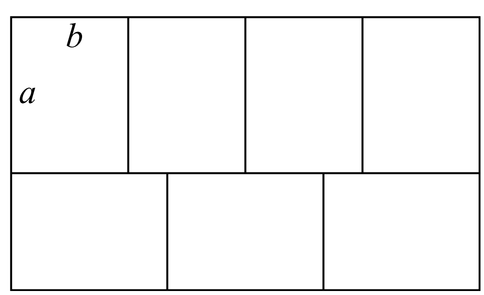
(c) If ℓ and w have the values you found in part (b) directly above, what is the total area of the larger rectangle?

Challenge Problem: Seven identical rectangles each with length a and width b are arranged to form the larger rectangle below and to the right. The side lengths a and b are positive whole numbers of centimetres and the total area of the larger rectangle is $84\,\text{cm}^2$.

(a) What is the area, in square centimetres, of each of the seven smaller identical rectangles?

(b) Explain why 3 times the value of a must be equal to 4 times the value of b. Using this relationship, can you find what the values of a and b must be?

(c) What is the perimeter of the larger rectangle?



More info:

Check the CEMC at Home webpage on Tuesday, May 5 for a solution to Figure Out These Rectangles.

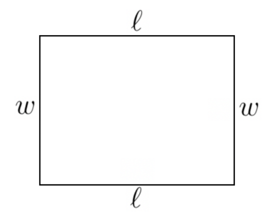
CEMC at Home

Grade 4/5/6 - Tuesday, April 28, 2020
Figure Out These Rectangles - Solution

Problem 1

(a) The perimeter consists of two lengths ℓ and two widths w. Since adding two widths and two lengths together gives 16 cm, adding one width and one length together must give $16 \div 2 = 8$ cm. This means the sum of the length and the width must be 8 cm, or

$$\ell + w = 8 \text{ cm}$$

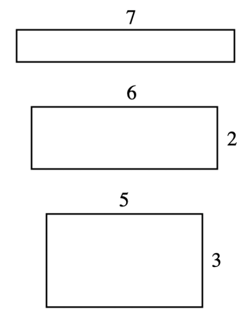


(b) There are only three ways to make 8 by adding two different positive whole numbers:

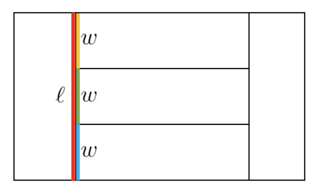
$$1 + 7 = 8,\ 2 + 6 = 8,\ 3 + 5 = 8$$

Since the length is greater than the width, the only possibilities for ℓ and w, in centimetres, are shown in the table below.

width w	length ℓ
1	7
2	6
3	5



Problem 2

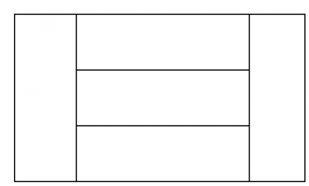


(a) With the sides labelled as shown in the top diagram on the right, we see that the length ℓ is equal to three widths w.

(b) The only pair from Problem 1(b) for which the length is three times the width is the pair $\ell = 6$ cm and $w = 2$ cm.

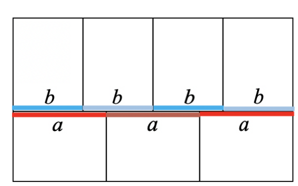
(c) If each smaller identical rectangle has length 6 cm and width 2 cm, then the area of each smaller rectangle is $6 \times 2 = 12$ cm^2.

Since there are 5 smaller rectangles making up the larger rectangle, the area of the larger rectangle must be 5×12 cm^2 $= 60$ cm^2.



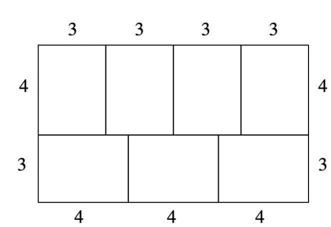
Challenge Problem

(a) Since the total area of the larger rectangle is 84 cm^2 and it is formed using seven smaller identical rectangles, the area of each smaller rectangle must be $84 \div 7 = 12$ cm^2.



(b) With the sides labelled as shown in the top diagram on the right, we see that three lengths a are equal to four widths b. Since the area of each smaller rectangle is 12 cm^2 we know that a times b must be 12. The factor pairs of 12 are 1 and 12, 2 and 6, and 3 and 4. The only pair that satisfies the correct relationship is 3 and 4. This means $a = 4$ cm and $b = 3$ cm.

(c) Using the labelled diagram, we see that the larger rectangle has length 12 cm and width 7 cm. This means its perimeter is



$$12 \text{ cm} + 7 \text{ cm} + 12 \text{ cm} + 7 \text{ cm} = 38 \text{ cm}$$

CEMC at Home

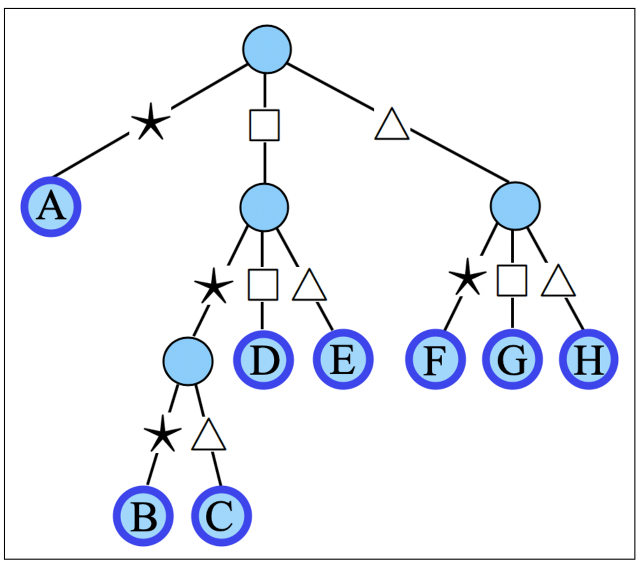
Grade 4/5/6 - Wednesday, April 29, 2020
Cool Codes

Introduction: Charlotte and Jacques want to send secret messages to each other. They will use three different symbols in their messages: ★, □, and △.

To *encode* their secret messages they use a *key* that only they know. Encoding a message means changing the message from regular text to code involving the symbols ★, □, and △.

This same key can be used to *decode* their secret messages. Decoding a message means changing the message from code involving the symbols ★, □, and △ back to regular text.

The key they use to encode and decode messages is the *tree* shown below.



Notice that each of the letters A, B, C, D, E, F, G, and H appear as *leaves* on this tree. This means that this tree can be used to encode any message that contains only these letters.

Each letter has its own code that is determined by the letter's placement on the tree. The code for a letter can be found by following the path on the tree from the top circle to the circle containing the letter, and picking up all of the symbols on the *branches* that you travel along on this path.

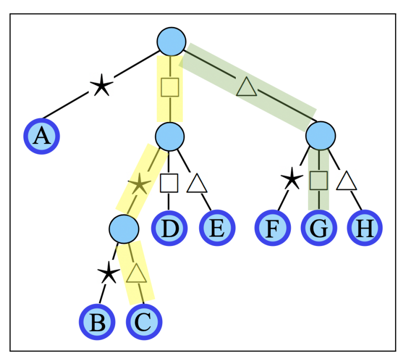
Notice that each *left branch* has the symbol ★, each *middle branch* has the symbol □, and each *right branch* has the symbol △.

For example, the code for the letter **C** is □ ★ △.

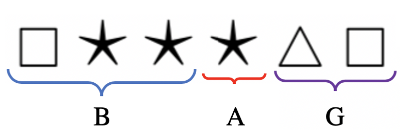
We find this code by starting at the top circle and taking a middle branch (picking up a □), followed by a left branch (picking up a ★), followed by a right branch (picking up a △), to end up at **C**.

As another example, the code for the letter **G** is △ □.

We find this code by starting at the top circle and taking a right branch (picking up a △), followed by a middle branch (picking up a □), to end up at **G**.



To encode a message, we replace each letter in the message with its code from the tree. For example, the encoded version of **BAG** is □ ★ ★ ★ △ □ as shown below.



We have already seen the code for the letter **G**, but you should use the tree to check that these are the correct codes for **B** and **A**. Notice that codes for different letters can be of different lengths.

Example: A message was encoded using the given tree and the resulting code is shown below. Use the tree to decode the message.

□ ★ △ ★ △ □ □ △

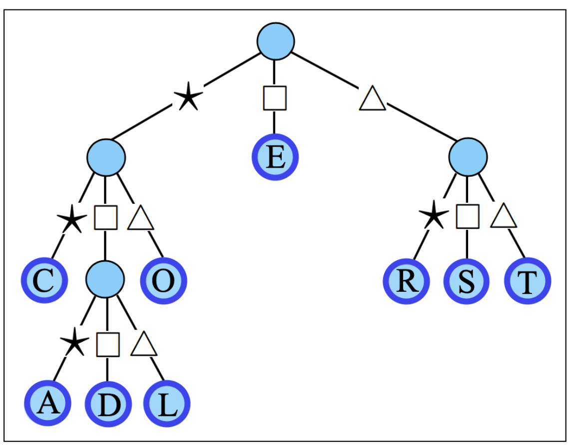
Explanation: You can decode this message letter by letter. Start at the top circle of the tree and follow the path indicated by the symbols until you end up at a letter.

To follow the path indicated by these symbols you must follow a middle branch (□), then a left branch (★), then a right branch (△), and this takes you to the the letter **C**. Once you reach a letter you stop, and restart the process at the top circle of the tree with the next part of the sequence.

To follow the path indicated by the remaining symbols ★ △ □ □ △, we must follow a left branch (★) which leads us straight to the letter **A**, and so we stop and reset again. Continuing in this way until we reach the end of the code, we reveal the original message of **CAGE** as shown below:

Starting at the top circle,	□ → middle, ★ → left, △ → right	⇒	**C**
Going back to the top,	★ → left	⇒	**A**
Going back to the top again,	△ → right, □ → middle	⇒	**G**
Going back to the top again,	□ → middle, △ → right	⇒	**E**

Problem: It is always a good idea to change your key often to keep your messages safe. To send their messages tomorrow, Charlotte and Jacques will use the following different tree.



(a) Using their tree as the key, what is the code for the word **LOST**?

(b) Jacques sent Charlotte the following secret message.

★ □ △ □ △

Explain why Jacques must have made a mistake when encoding the message.

(c) Using their tree as the key, decode the message displayed below.

△ □ □ ★ ★ △ ★ □ △ △ ★ ★ ★ △ ★ □ □ □ △ □ ★ □ ★ △ ★ □ ★ ★ ★ △ ★ △ ★ □ △

Extension:

There are different trees that will encode exactly the same letters, but in a different way. Suppose that the codes for the letters in a tree are as shown in the table on the right.

1. Draw the tree that matches the codes in the table.
2. Charlotte and Jacques think that the tree they used in the problem above is better than the tree that matches the codes that you found in 1. Do you agree with this? Why or why not?

A	★
C	□ ★ ★
D	△ ★
E	□ ★ □
L	△ □
O	□ ★ △
R	□ □
S	□ △
T	△ △

More Info:

Check out the CEMC at Home webpage on Wednesday, May 6 for a solution to Cool Codes.

CEMC at Home

Grade 4/5/6 - Wednesday, April 29, 2020
Cool Codes - Solution

Problem:

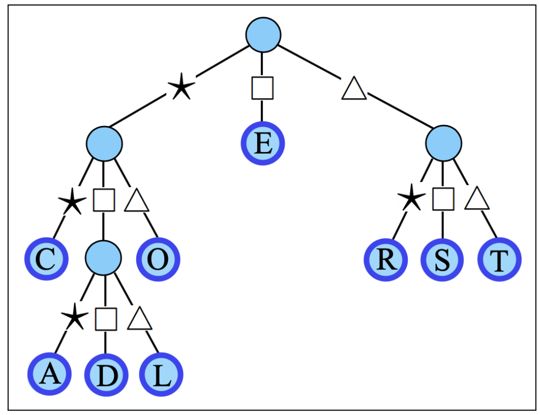
(a) Using their tree as the key, what is the code for the word **LOST**?

(b) Jacques sent Charlotte the following secret message.

★ □ △ □ △

Explain why Jacques must have made a mistake when encoding the message.

(c) Using their tree as the key, decode the message displayed below.



△ □ □ ★ ★ △ ★ □ △ △ ★ ★ ★ △ ★ □ □ □ △ □ ★ □ ★ △ ★ □ ★ ★ ★ △ ★ △ ★ □ △

Solution:

(a) The code for **LOST** is: ★ □ △ ★ △ △ □ △ △

(b) Here is what we get when we try to decode this message:

Starting at the top circle,	★ → left, □ → middle, △ → right	⇒	**L**
Going back to the top,	□ → middle	⇒	**E**
Going back to the top,	△ → right	⇒	?

We have now run out of symbols and are unable to complete the process. Since △ alone is not the code for a letter, this message cannot have been encoded correctly.

(c) The decoded message is: **SECRETCODESARECOOL** (or "secret codes are cool").

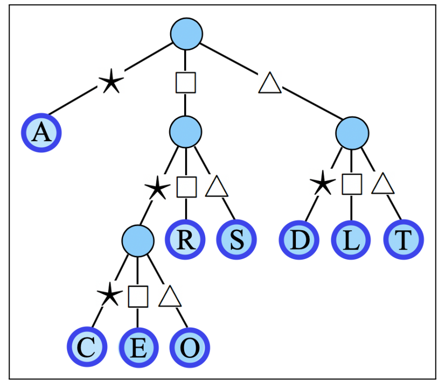
Extension:

1. Draw the tree that matches the codes in the table.
2. Charlotte and Jacques think that the tree they used in the problem above is better than the tree that matches the codes that you found in 1. Do you agree with this? Why or why not?

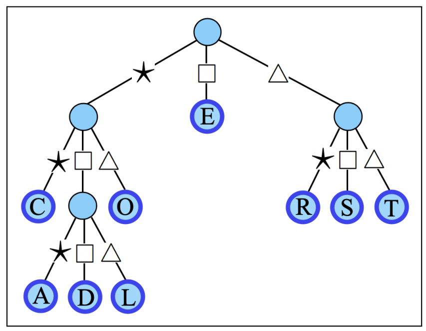
A	★
C	□ ★ ★
D	△ ★
E	□ ★ □
L	△ □
O	□ ★ △
R	□ □
S	□ △
T	△ △

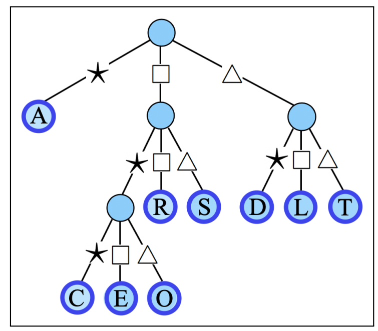
Solution:

1. The tree that matches this encoded table is:



2. Let's compare the two trees:





When using the tree on the left (created by Charlotte and Jacques) to encode the message

SECRETCODESARECOOL

the encoded message has 35 symbols. *Go back and count them for yourself.*

This is already pretty long! However, if we use the new tree on the right to encode the same message, the encoded message will have *even more* symbols, which means we would have to send a longer message. The encoded message in this case would have 45 symbols in total. Can you see why?

We can determine the number of symbols the encoded message will have by counting how many symbols are needed to code each letter, and how many times each letter appears in the original message.

The number of times a letter appears is sometimes called the frequency of the letter.

The rightmost column of the table shows how many symbols are needed to code each letter, in total. When we add up the numbers in this column, we get the total length of the encoded message. The total is 45.

Letter	Code	Frequency of Letter	Total Symbols
A	★	1	1
C	□ ★ ★	3	9
D	△ ★	1	2
E	□ ★ □	4	12
L	△ □	1	2
O	□ ★ △	3	9
R	□ □	2	4
S	□ △	2	4
T	△ △	1	2

The reason it takes more symbols to encode the message with the second tree is that the letters that appear most often in the original message (C, E, and O) happen to be the letters with the longest codes (three symbols instead of one or two).

If you look at the first tree, you will see that the letters C, E, and O have codes of length one or two, rather than three.

So it looks like the first tree (created by Charlotte and Jacques) is a better choice for encoding this particular message since it takes fewer symbols to do so. *What about for other messages?*

CEMC at Home features Problem of the Week

Grade 4/5/6 - Thursday, April 30, 2020

Fishing for Thermoclines

Paige is headed to Lac Nilgault to do some fishing for lake trout and pike. She knows that small pike generally like 18°C water, large pike like 12°C water, and lake trout like 10°C water.
She measures the water temperature in the lake at different depths, and collects the following data.

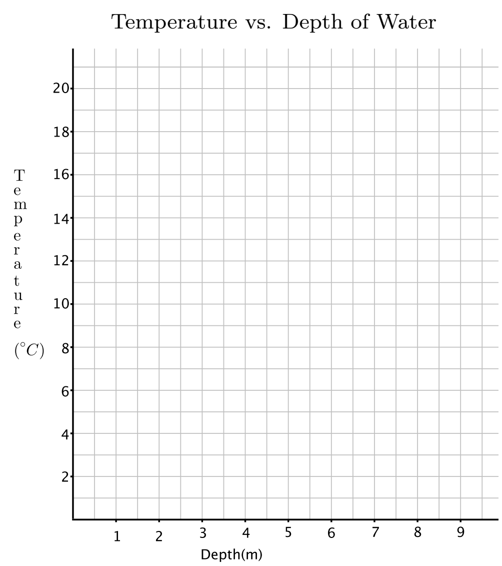
Depth (m)	**Temp** (°C)	**Depth** (m)	**Temp** (°C)
0.5	20	5	15
1	19	5.5	15
1.5	18	6	15
2	18	6.5	12
2.5	18	7	11
3	17	7.5	10
3.5	17	8	10
4	16	8.5	10
4.5	16	9	9

On the grid to the right, plot points to make a broken-line graph to illustrate this data.

a) At what depths should Paige be fishing for small pike?

b) At what depths should Paige be fishing for lake trout?

c) Paige knows that most lakes have a thermocline where the water will rapidly change temperature. At what depth is the top of the thermocline in Lac Nilgault?



More Info:

Check the CEMC at Home webpage on Thursday, May 7 for the solution to this problem.
Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, May 7.

This CEMC at Home resource is the current grade 5/6 problem from Problem of the Week (POTW). This problem was developed for students in grades 5 and 6, but is also accessible to students in grade 4. POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week, to view this week's grade 3/4 problem, and to find many more past problems and their solutions, visit the Problem of the Week webpage.

Problem of the Week

Problem B and Solution

Fishing for Thermoclines

Problem

Paige is headed to Lac Nilgault to do some fishing for lake trout and pike. She knows that small pike generally like 18°C water, large pike like 12°C water, and lake trout like 10°C water. She measures the water temperature in the lake at different depths, and collects the following data.

Depth m	Temp °C	Depth m	Temp °C
0.5	20	5	15
1	19	5.5	15
1.5	18	6	15
2	18	6.5	12
2.5	18	7	11
3	17	7.5	10
3.5	17	8	10
4	16	8.5	10
4.5	16	9	9

Plot points to make a broken-line graph to illustrate this data.

a) At what depths should Paige be fishing for small pike?

b) At what depths should Paige be fishing for lake trout?

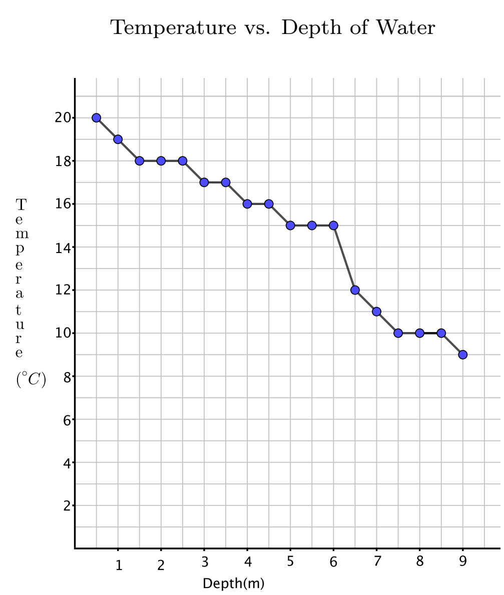
c) Paige knows that most lakes have a thermocline where the water will rapidly change temperature. At what depth is the top of the thermocline in Lac Nilgault?

Solution

a) According to the data, Paige should fish for small pike at a depth between 1.5 m and 2.5 m.

b) She should fish for lake trout at a depth between 7.5 m and 8.5 m.

c) We can see from the graph that the temperature starts to quickly drop at a depth of about 6 m. Therefore, the top of the thermocline is at a depth of about 6 m.



CEMC at Home

Grade 4/5/6 - Friday, May 1, 2020
This Farmer is No Square

Long ago, there was a farmer whose land was in the shape of a square, each side being exactly 100 metres long.

One day, a woman knocked on the farmer's door and begged for something to eat. Being a kind person, the farmer fed her a nice lunch.

After the woman had eaten, she said "Kind farmer, I am your Queen! As a reward for your kindness, I will grant you enough additional land that you may double the area of your farm. However, your land must remain in the shape of a square."

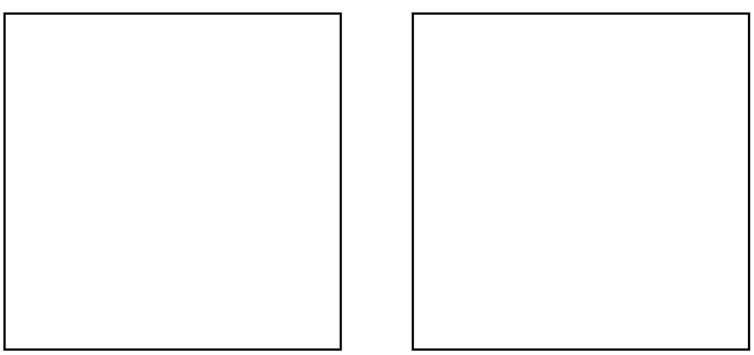
Let's help the farmer figure out how he might determine the side length of his new square of land.

(a) A first thought might be to simply double the length of each side of the original square of land. That is, the new square of land would have sides that are each 200 metres long.

Draw a diagram of his original square of land and of this proposed new one. Explain why this solution does not give the desired result of a new square of land with twice the area of the original one.

(b) Suppose each square below represents a copy of the farmer's original square of land, so their combined area represents the area of the new square of land. Can you divide the given squares in such a way that the pieces can be reassembled in order to form a single larger square? Can you convince yourself that the shape you created is, in fact, a square?

You may cut and arrange these two squares in any way you would like, but your rearrangement must use all of these pieces, be in the shape of a square, and not change the overall area.



HINT:

Young Geo had thoughts analytical
About cutting out shapes most polygonal;
"These boxes," she said,
"Don't get me ahead...
I shall try, instead, some diagonals."

(c) Can you relate the side length of the larger square you created in part (b) to either the side length or the diagonal length of the smaller square?

Challenge: If the farmer's land had been a rectangle, but not necessarily a square, would your technique from part (b) produce a single larger rectangle twice the area of the original rectangle?

More info:

Check out the CEMC at Home webpage on Friday, May 8 for a solution to This Farmer is No Square.

CEMC at Home

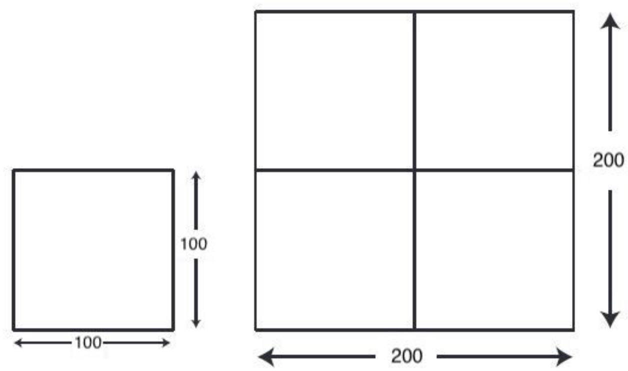
Grade 4/5/6 - Friday, May 1, 2020
This Farmer is No Square - Solution

Problem Summary:

In return for a kind act, the Queen has granted a farmer enough land to double the area of his existing farm, which is square-shaped with each side exactly 100 metres long. However, the new farm must also be in the shape of a square. Let's help the farmer figure out how he might determine the side length of his new square of land.

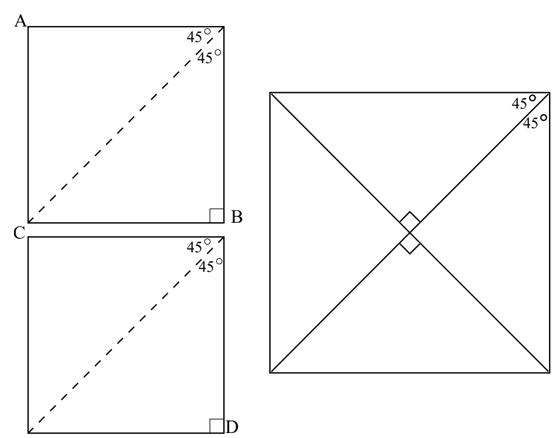
Solution:

(a) The diagram to the right reveals why doubling the side length of the square does not work. The new square that is formed can be divided up into four smaller squares, each with side length 100 metres, as shown. Hence, this plan yields a farm that has four times the original area, not twice the area.



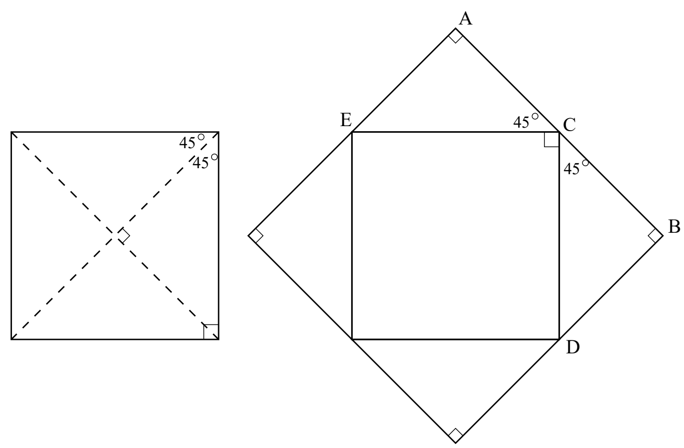
(b) While it is tempting to try to combine the two squares by cutting rectangular pieces and fitting them together, it is not very fruitful. Also, convincing yourself that the figure you end up forming is, in fact, a square, can prove to be quite difficult.

One solution is to cut each square along a diagonal, creating four congruent right-angled triangles. Then line up vertices A, B, C, and D so they become the centre of a new square whose sides are the cut edges.
To justify that this new shape is a square, notice that each side of the new shape corresponds to a diagonal of a smaller square, and so the sides are all equal in length. Also, each new corner angle is formed by two halves of the previous corner angle, and so is a right angle. Thus, the new shape is a square. Also, since it is formed from two smaller squares, it has area that is twice the area of a smaller square.



Another solution is to cut one square along both diagonals, as shown below. This will create four congruent right-angled triangles. Then place each triangle onto one side of the other small square, with the longer side of the triangle (which is also the side of the original square) lined up along the side of the square.

To justify that this new shape is a square, we need to show that the sides of the new shape are all straight lines of the same length. Since ACB is formed by two halves of the previous corner angle and another square's corner angle, can you see why ACB will form a straight line? Also notice that each side of the new shape is formed by two halves of the diagonal of the original smaller square. Therefore, each side length of the new shape is equal to the diagonal length of the original smaller square. So the shape that results from this construction is a square. Also, since it is formed from two smaller squares, it has area that is twice the area of a smaller square.



(c) As established in both solutions to part (b), the side length of the larger square is equal to the diagonal length of the smaller square.

Challenge:

If the farmer's land is a rectangle, but not a square, then both approaches presented in the solutions to part (b) will give a shape that has twice the area, but the shape will not be a square. It will instead be a *rhombus*. Try out both approaches on two identical rectangles and see if you can figure out why!