CEMC at Home

Grade 11/12 - Monday, April 27, 2020
Pick Up Sticks - Part 3

In Pick Up Sticks Part 1 and Part 2 we looked at a game where we removed sticks from piles starting with two piles of sticks and three piles of sticks, respectively.

Refresh your memory on the rules for these games and the winning strategies for these games.

Pick Up Sticks is actually an ancient game called Nim that is said to have originated in China. In Pick Up Sticks Part 2 we were playing Nim with piles of 3, 5 and 7 sticks. In our discussion of the winning strategy for this game we came up with a table of winning positions. If a player could remove sticks so that the piles matched a winning position, then the player had a winning strategy for the game. Here is the table of winning positions:

Winning Positions
$(0,0,0)$
$(0,k,k)$
$(1,2,3)$
$(1,4,5)$
$(2,4,6)$
$(2,5,7)$
$(3,4,7)$
$(3,5,6)$

In this activity, we will explore a different way to describe the winning positions in this game. The hope is that we can generalize this new description in order to study the winning positions of a more general version of Nim: a game with n piles, each with an arbitrary number of sticks.

A different view of the winning positions

Can you see a pattern in the table of winning positions shown above? Is there a simple condition that only the winning positions satisfy? At first glance, you may look at the table and think that for a position to be a winning position, the sum of the numbers of sticks in the first two piles must be equal to the number of sticks in the third pile. But we can see that $(3,5,6)$ is a winning position and $3+5 \neq 6$. We also note that $(1,3,4)$ satisfies $1+3=4$, but $(1,3,4)$ is not a winning position.

We need to dive deeper, and we will find the answer in a surprising place! The first step in our analysis is to write the numbers representing the pile sizes in *binary*. (If you have not seen binary numbers before, you can check out this Math Circles lesson before continuing.) The table of winning positions with the pile sizes given in binary is given below.

Winning Positions with Pile Sizes in Binary
$(0,0,0)$
$(0,k,k)$
$(1,10,11)$
$(1,100,101)$
$(10,100,110)$
$(10,101,111)$
$(11,100,111)$
$(11,101,110)$

Notice that we have not transformed the winning position $(0,k,k)$ as we do not know the value of k.

Next, we introduce something which is called the *digital sum*. To calculate the digital sum of two binary numbers, we add the digits in the same position (or place value), but follow the rule that $1 + 1 = 0$. For example, the following shows that the digital sum of 100101 and 110011 is 10110:

$$\begin{array}{r} 1\,0\,0\,1\,0\,1 \\ \oplus\ 1\,1\,0\,0\,1\,1 \\ \hline 0\,1\,0\,1\,1\,0 \end{array}$$

We will call the digital binary sum the *Nim-sum*. We will use the notation $x \oplus y$ to refer to the Nim-sum of x and y as shown above. Thus we write $100101 \oplus 110011 = 10110$.

(As a side note to those of you who study computer science: The digital sum calculated in binary is also known as *exclusive or (XOR)*.)

Investigation 1: Nim with piles of size 3, 5, and 7

1. For each winning position (a, b, c) given in our table of winning positions with pile sizes in binary, calculate the Nim-sum, $a \oplus b \oplus c$.

   For example, consider the winning position $(10, 100, 110)$. *We calculate* $10 \oplus 100 = 110$ *and then add* 110 *to the result and get* $(10 \oplus 100) \oplus 110 = 110 \oplus 110 = 0$ *as shown below:*

$$\begin{array}{r} 1\ \ 0 \\ \oplus\ 1\ \ 0\ \ 0 \\ \hline 1\ \ 1\ \ 0 \end{array} \qquad \begin{array}{r} 1\ \ 1\ \ 0 \\ \oplus\ 1\ \ 1\ \ 0 \\ \hline 0\ \ 0\ \ 0 \end{array}$$

2. We know that the positions $(1, 2, 4)$, $(2, 4, 4)$ and $(3, 4, 5)$ are losing positions for this game. For each of these positions, calculate the Nim-sum of their three pile sizes.
3. Based on your work in questions 1 and 2, can you come up with a guess at a condition that can be checked in order to determine whether or not a particular position is a winning position for Nim starting with piles of 3, 5 and 7 sticks?
4. Can you prove that your condition is correct? Your explanation should include these two arguments:
   (a) Show that if position P satisfies your condition, and position Q can be obtained from P in a single move, then Q does not satisfy your condition.
   (b) Show that if position P does not satisfy your condition, then there exists a move that will take position P to some position Q that does satisfy your condition.

Investigation 2: More general Nim games

5. Do you think your condition from Investigation 1 can also be used to determine the winning positions in a game of Nim with 3 piles, each with *any number of sticks*?
6. Do you think your condition from Investigation 1 can also be used to to determine the winning positions in a general game of Nim staring with n *piles, each with any number of sticks*?

See the next page for some properties of the digital binary sum that you might find useful.

More Info:

Check out the CEMC at Home webpage on Monday, May 4 for a solution to Pick Up Sticks - Part 3. We encourage you to discuss your ideas online using any forum you are comfortable with.

Properties of the Nim-sum

In your proofs you might want to use some of the properties of the Nim-sum given below. For extra fun, you can try to prove these properties yourself. You can also investigate the words "commutative", "associative", "identity", and "inverse".

- $a \oplus b = b \oplus a$ (Nim-sum is commutative)
- $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (Nim-sum is associative)
- $0 \oplus a = a$ (0 is the identity associated with Nim-sum)
- $a \oplus a = 0$ (the inverse of a is itself under Nim-sum)

CEMC at Home

Grade 11/12 - Monday, April 27, 2020
Pick up Sticks - Part 3 - Solution

1. The Nim-sum of the three pile sizes of each of these winning positions is 0. For example, for the position $(3, 5, 6)$, or $(11, 101, 110)$ in binary, Nim-sum of the pile sizes to be:

$$\begin{array}{r} 1\,1 \\ 1\,0\,1 \\ \oplus\ 1\,1\,0 \\ \hline 0\,0\,0 \end{array}$$

2. For the losing position $(1, 2, 4)$, or $(1, 10, 100)$ in binary, we calculate the Nim-sum of the piles sizes to be:

$$\begin{array}{r} 1 \\ 1\,0 \\ \oplus\ 1\,0\,0 \\ \hline 1\,1\,1 \end{array}$$

   For the losing position $(2, 4, 4)$, or $(10, 100, 100)$ in binary, we calculate the Nim-sum of the piles sizes to be:

$$\begin{array}{r} 1\,0 \\ 1\,0\,0 \\ \oplus\ 1\,0\,0 \\ \hline 0\,1\,0 \end{array}$$

   For the losing position $(3, 4, 5)$, or $(11, 100, 101)$ in binary, we calculate the Nim-sum of the pile sizes to be:

$$\begin{array}{r} 1\,1 \\ 1\,0\,0 \\ \oplus\ 1\,0\,1 \\ \hline 0\,1\,0 \end{array}$$

3. Based on our answers to 1. and 2. we may hypothesize that a position is a winning position if the Nim-sum of its pile sizes is 0 and it is a losing position if the Nim-sum of its pile sizes is not 0. Mathematicians might also express this condition as follows: a position is a winning position if and only if the Nim-sum of its pile sizes is 0.

4. Let's try to justify that our hypothesis from 3. is correct. We will argue the following:

   (a) Show that if position P satisfies your condition, and position Q can be obtained from P in a single move, then Q does not satisfy your condition.

   (b) Show that if position P does not satisfy your condition, then there exists a move that will take position P to some position Q that does satisfy your condition.

Suppose one of the players is faced with a game position P (other than $(0, 0, 0)$) having a Nim-sum equal to 0. Below we show an illustration of the Nim-sum calculation, where each letter represents either a 0 or a 1.

$$\begin{array}{rccc} & A & B & C \\ & D & E & F \\ \oplus & G & H & I \\ \hline & 0 & 0 & 0 \end{array}$$

Note that every positive integer less than 8 (and hence every pile size in this particular game) can be represented using a binary number with at most three digits. Remember that leading 0s are often omitted, for example if $A = 0$ then we may write the first number as simply BC.

We will show that if some number of sticks are removed from the "top pile" on this player's turn, then the resulting position Q cannot have Nim-sum 0. (Since the order of the sum doesn't matter, we can put whatever pile we want as the "top pile" and all we need to do is to make sure that this pile has at least one stick to proceed.)

Since the Nim-sum of position P is 0, this means there must be an even number of 1s in each column in the Nim-sum calculation. Let's set a_1 to be the number of sticks in the top pile before this player's turn and a_2 to be the number of sticks in the top pile after this player's turn. This means $a_2 < a_1$, and so the binary representations of a_1 and a_2 must differ in at least one place (digit).

Now we consider the Nim-sum of position P (on the left) and the new position Q (on the right) after this player's turn. Note that we are assuming a_1 is ABC and a_2 is $A^*B^*C^*$.

$$\begin{array}{rccc} & A & B & C \\ & D & E & F \\ \oplus & G & H & I \\ \hline & 0 & 0 & 0 \end{array} \qquad \begin{array}{rccc} & A^* & B^* & C^* \\ & D & E & F \\ \oplus & G & H & I \\ \hline & & & \end{array}$$

We know that we must have $A \neq A^*$, $B \neq B^*$, or $C \neq C^*$. Let's suppose we have $A \neq A^*$. Since the first column of the Nim-sum on the left must have an even number of 1s, we must have that the Nim-sum on the right must have an odd number of 1s. This happens because changing the digit from A to A^* will either add exactly one 1 to this column, or remove exactly one 1 from this column. In this case we see that the Nim-sum of Q (on the right) must have at least one digit of 1, and hence cannot be 0. The argument is similar if we have $B \neq B^*$, $C \neq C^*$ (or any combination of the three). Therefore, Q cannot have Nim-sum 0.

Now suppose one of the players is faced with a game position P having a Nim-sum *not* equal to 0. We will show that there is a move this player can make in order to bring the game to a position Q that *has* Nim-sum 0. We know that at least one column in the Nim-sum of position P totals to 1. Find the leftmost column in this Nim-sum that totals to 1, and pick any pile whose pile number has a 1 in this column. In the illustration below, we assume this pile has a_1 (or ABC) sticks.

$$\begin{array}{rccc} & A & B & C \\ & D & E & F \\ \oplus & G & H & I \\ \hline & & & \end{array} \qquad \begin{array}{rccc} & A^* & B^* & C^* \\ & D & E & F \\ \oplus & G & H & I \\ \hline & & & \end{array}$$

The player will change this "top pile" to a_2 (or $A^*B^*C^*$) sticks to get to the next position, Q. How should the player choose a_2 in order to achieve a Nim-sum of 0 on the right?

If the leftmost column in the Nim-sum of position P (on the left) has an even number of 1s, then we set $A^* = A$. If the leftmost column in the Nim-sum of position P has an odd number of 1s, then we set $A^* \neq A$, that is, if $A = 1$ then we set $A^* = 0$ and if $A = 0$ then we set $A^* = 1$. We choose the other digits of a_2 in a similar way. There is certainly a number a_2 with these binary digits, and choosing $A^*B^*C^*$ in this way will ensure a Nim-sum of 0 (on the right). But, we need to know that this a_2 is *less than* a_1 so that changing the pile from a_1 to a_2 is actually a legal move.

Can you see why this a_2 must be less than a_1? The idea is to argue that the numbers a_1 and a_2 must differ in at least one place (or digit), and the leftmost place (or digit) where they differ must have a 1 in a_1 and a 0 in a_2. Can you see why this is true? This will ensure that no matter what other digits may differ to the right in these representations, a_1 must be larger than a_2.

The General Game of Nim

It turns out that for any game of Nim (with n piles, each with an arbitrary number of sticks) a position is a winning position if and only if the Nim-sum of the n pile sizes is equal to 0.

We encourage you to think about this yourself, or read on if you are interested in a formal proof of this result.

First, consider the following result.

The Nim-sum Theorem
Suppose that the pile sizes before a move are $a_1, a_2, \ldots, a_n$ and that the pile sizes after the move are $b_1, b_2, \ldots, b_n$.
Let s be the Nim-sum of the pile sizes before the move, that is

$$s = a_1 \oplus a_2 \oplus \cdots \oplus a_n.$$

Let t be the Nim-sum of the pile sizes after the move, that is

$$t = b_1 \oplus b_2 \oplus \cdots \oplus b_n.$$

If the move changed the size of pile k, then $t = s \oplus (a_k \oplus b_k)$.

Proof: We begin by noting that $a_i = b_i$ for all $i \neq k$.

We will prove the statement using the properties of Nim-sum which were given in Question 5.

$$\begin{aligned}
t &= 0 \oplus t && \text{(0 is the identity)} \\
&= (s \oplus s) \oplus t && (s \oplus s = 0) \\
&= s \oplus (s \oplus t) && \text{(associativity)} \\
&= s \oplus [(a_1 \oplus a_2 \oplus \cdots \oplus a_n) \oplus (b_1 \oplus b_2 \oplus \cdots \oplus b_n)] && \\
&= s \oplus (a_1 \oplus b_1) \oplus (a_2 \oplus b_2) \oplus \cdots \oplus (a_n \oplus b_n) && \text{(associativity and commutativity)} \\
&= s \oplus (a_1 \oplus a_1) \oplus \cdots \oplus (a_k \oplus b_k) \oplus \cdots \oplus (a_n \oplus a_n) && (a_i = b_i \text{ for } i \neq k) \\
&= s \oplus 0 \oplus \cdots \oplus (a_k \oplus b_k) \oplus \cdots \oplus 0 && (a_i \oplus a_i = 0) \\
&= s \oplus (a_k \oplus b_k) && \text{(0 is the identity)}
\end{aligned}$$

which is our desired result.

Now, using the Nim-sum Theorem we will prove the two necessary results. In each of these results, s and t are defined as in the Nim-sum Theorem.

Result 1: If $s = 0$, then $t \neq 0$.

Proof: Assume $s = 0$. Then, using the Nim-sum Theorem we get that $t = 0 \oplus (a_k \oplus b_k) = a_k \oplus b_k$. Is it possible that $a_k \oplus b_k = 0$? Using the properties of Nim-sum we would obtain:

$$\begin{aligned} a_k \oplus b_k &= 0 \\ (a_k \oplus b_k) \oplus b_k &= 0 \oplus b_k \\ a_k \oplus (b_k \oplus b_k) &= b_k && \text{(associativity and 0 is the identity)} \\ a_k \oplus 0 &= b_k && \text{(the inverse of } b_k \text{ is itself)} \\ a_k &= b_k && \text{(0 is the idenity)} \end{aligned}$$

However, we know that $a_k \neq b_k$ because we changed the pile size of pile k. So it must be that $a_k \oplus b_k \neq 0$ and therefore, $t \neq 0$.

Result 2: If $s \neq 0$, then there exists a move for which $t = 0$.

Proof: Let 2^j be the largest power of 2 such that $2^j \leq s$. This power of 2 corresponds to the leftmost non-zero digit in the binary representation of s. This means that there must be at least one pile i, for which the digit corresponding to 2^j in the binary representation of a_i is a 1.

We will remove sticks from pile i so that the resulting pile size b_i has the property that $b_i = a_i \oplus s$. For this to be a legal move, we must be able to choose b_i so that $b_i < a_i$.

Why can this be done? We leave this part of the proof for you to think about on your own. Recall how we made this argument in the case of the three piles.

After we do this move we get

$$\begin{aligned} t &= s \oplus (a_i \oplus b_i) && \text{(using the Nim-sum Theorem)} \\ &= s \oplus (a_i \oplus (a_i \oplus s)) && \text{(substituting } b_i = a_i \oplus s\text{)} \\ &= s \oplus ((a_i \oplus a_i) \oplus s) && \text{(associativity)} \\ &= s \oplus (0 \oplus s) \\ &= s \oplus s && \text{(0 is the identiy)} \\ &= 0 \end{aligned}$$

as required.

The General Winning Strategy

Since you know how to calculate the winning positions in any game of Nim, you can form a winning strategy for this game. However, changing the pile sizes into binary and computing the Nim-sum of these pile sizes is a lot of mental math to do if there are many piles and/or large piles in the game. One trick that makes things easier is to think of the piles sizes as the sum of powers of 2 that would be used to determine their binary representation. Then "cancel out" any pairs of powers of 2 that match. If they all cancel out, then you have a winning position. If you have any odd number of any power, then you have a losing position.

CEMC at Home

Grade 11/12 - Tuesday, April 28, 2020
Think Before You Solve

There is often more than one possible approach to solving a particular math problem, and some approaches may take more time and effort than others. Before jumping into the algebra or arithmetic involved in a problem, it can be helpful to take a few minutes to think about some initial steps you might take that could greatly simplify your work overall. Is it helpful to simplify any expressions before substituting values for the variables? Is it helpful to either factor or expand any expressions before trying to solve the given equation(s)? Think about this while solving the problems given below.

---

1. Let $x = \frac{1}{3}$, $y = \frac{1}{7}$, and $z = \frac{1}{11}$. Determine the exact value of

$$\frac{xy + xz + yz}{xyz}$$

2. Determine the x-intercepts and y-intercepts of the graph with equation

$$y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4).$$

3. Determine all pairs (x, y) of real numbers that satisfy the following system of equations:

$$x^2 + x + y - y^2 = 20$$
$$(x+y)^2(x-y) = 75$$

   You may find that this problem is more challenging than the first two. Think about how you might transform this system into a simpler system, perhaps by changing your view of what the "variables" are.

More Info:

Check the CEMC at Home webpage on Tuesday, May 5 for a solution to Think Before You Solve.

Problem 2 is from a past Euclid Contest, and problem 3 is from a past Canadian Team Mathematics Contest (CTMC). You can find past CEMC contests here.

CEMC at Home

Grade 11/12 - Tuesday, April 28, 2020
Think Before You Solve - Solution

1. Let $x = \frac{1}{3}$, $y = \frac{1}{7}$, and $z = \frac{1}{11}$. Determine the exact value of

$$\frac{xy + xz + yz}{xyz}$$

Solution: It is tempting to immediately substitute the values for x, y, and z into the given expression, however it is helpful to first observe that this expression can be simplified. The arithmetic involved in this question is made easier if the expression is first simplified as shown below:

$$\begin{aligned}
\frac{xy + xz + yz}{xyz} &= \frac{xy}{xyz} + \frac{xz}{xyz} + \frac{yz}{xyz} \\
&= \frac{1}{z} + \frac{1}{y} + \frac{1}{x} \\
&= \frac{1}{\left(\frac{1}{11}\right)} + \frac{1}{\left(\frac{1}{7}\right)} + \frac{1}{\left(\frac{1}{3}\right)} \qquad \text{substituting } x = \tfrac{1}{3}, y = \tfrac{1}{7}, z = \tfrac{1}{11} \\
&= 11 + 7 + 3 \\
&= 21
\end{aligned}$$

2. Determine the x-intercepts and y-intercepts of the graph with equation

$$y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4).$$

Solution: It is tempting to expand each product above and add the two resulting cubic polynomials. Since the result will be a quadratic polynomial (notice that the x^3 terms will cancel) you can proceed like this to determine the intercepts. However, a useful observation to make first is that both cubic polynomials have a common factor of $(x-2)(x-3)$ and so you can completely avoid expanding the products. The equation of the graph can be rewritten as

$$\begin{aligned}
y &= (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4) \\
&= (x-2)(x-3)[(x-1) - (x-4)] \\
&= 3(x-2)(x-3)
\end{aligned}$$

This is a quicker way to get to this resulting quadratic, and it is already in factored form! From the factorization above, we can see that the x-intercepts are 2 and 3, and the y-intercept, which is the value of y when $x = 0$, is $3(-2)(-3) = 18$.

3. Solve the following system of equations

$$x^2 + x + y - y^2 = 20$$
$$(x + y)^2(x - y) = 75$$

Solution: You might think of either starting by expanding the second equation, or by factoring the first equation. It turns out that factoring is the best approach here, but exactly how to factor may not be immediately clear. One useful observation to make first is that, rather than trying to solve for x and y, we should try to solve for $x+y$ and $x-y$. How did we realize this? First, notice that the second equation involves a product of two copies of $x+y$ and one copy of $x-y$. Furthermore, with a bit of rearrangement and factoring of the first equation, we obtain:

$$\begin{aligned} x^2 + x + y - y^2 &= (x^2 - y^2) + (x + y) \\ &= (x + y)(x - y) + (x + y) \\ &= (x + y)(x - y + 1) \end{aligned}$$

Thus, the first equation may be rewritten as $(x + y)(x - y + 1) = 20$, which is starting to look similar to the second equation. Let $A = x + y$ and let $B = x - y$. With this substitution, we are now trying to solve the following related system in A and B:

$$A(B + 1) = 20$$
$$A^2B = 75$$

Notice that the second equation tells us that $A \neq 0$ and $B \neq 0$. Since B occurs only to the exponent 1 in both equations, we will solve for B in each equation. Solving for B in the first equation gives $B = \frac{20}{A} - 1$ and solving for B in the second equation gives $B = \frac{75}{A^2}$, and so we must have

$$\frac{20}{A} - 1 = \frac{75}{A^2}$$

Multiplying both sides by A^2 gives us the (equivalent) equation

$$20A - A^2 = 75 \text{ or } A^2 - 20A + 75 = 0$$

By looking at the different ways to factor 75, or using the quadratic formula, we can obtain the two solutions to the quadratic equation: $A = 5$ and $A = 15$.

When $A = 5$, the equation $A^2B = 75$ tells us that $B = 3$ and when $A = 15$, the same equation tells us that $B = \frac{1}{3}$.

Now that we have solved for A and B we need to solve for x and y. Remember that $A = x + y$ and $B = x - y$.

When $A = 5$ and $B = 3$ we have that $x + y = 5$ and $x - y = 3$. Adding the two equations we get $2x = 8$ and so $x = 4$. This means $y = 1$.

When $A = 15$ and $B = \frac{1}{3}$ we have that $x + y = 15$ and $x - y = \frac{1}{3}$. Adding the two equations we get $2x = \frac{46}{3}$ and so $x = \frac{23}{3}$. This means $y = \frac{22}{3}$.

Therefore, there are two solutions to the system: $(4, 1)$ and $(\frac{23}{3}, \frac{22}{3})$.

You should verify that these two pairs are indeed solutions to the original system.

CEMC at Home

Grade 11/12 - Wednesday, April 29, 2020
Comparison Machine Strikes Again

Your last mission was a success. So once again, you have been asked to develop algorithms to complete tasks involving the relative order of n distinct integers.

If you are a newcomer to the mission or a veteran that needs some reminders about the set-up and details, see the April 22 CEMC at Home resource for Grade 11/12 called Comparison Machine.

The Tasks

Complete the tasks below. For each task, you are told the number of integers you will be given and the limit on the number of times you can use the machine.

Remember that the machine will take in the indices of two integers in the list and will output the index of the larger integer.

n	Task	Limit
4	List the integers from smallest to largest.	5
9	You are given the additional information that • $a_1 < a_2$ and, • $a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$. List all 9 integers from smallest to largest.	6
5	Determine the median integer.	6

A Python computer program has also been built to help you test your solutions for this second mission.

Here are instructions for using the tool:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open this free online Python interpreter in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. You will interact with the tool using the right black panel, and you might want to widen this panel.
5. After completing a test, or if you encounter an error, you can hit *run* to begin another test. If you want to start over during a test, you can hit *stop* and then *run*.

More Info:

Check out the CEMC at Home webpage on Wednesday, May 6 for a solution to Comparison Machine Strikes Again.

CEMC at Home

Grade 11/12 - Wednesday, April 29, 2020
Comparison Machine Strikes Again - Solutions

Summary of the Tasks

Develop algorithms to complete tasks involving the relative order of n distinct integers.

- The integers are random and unknown to you. All you know is that they are named $a_1, a_2, \ldots a_n$.
- For each task, your approach must work no matter what the order of the integers is.
- A helpful machine M is available. The machine knows the relative order of these integers. To use it, you enter the index of two integers into the machine and it will tell you which of the two corresponding integers is larger.
- For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.
- Your memory is perfect and you can remember (or record) the result every time you use the machine.

Here are the three tasks:

n	**Task**	**Limit**
4	List the integers from smallest to largest.	5
9	You are given the additional information that • $a_1 < a_2$ and, • $a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$. List all 9 integers from smallest to largest.	6
5	Determine the median integer.	6

Solution for Task 1

Compute $M(1, 2)$ and record the answer as a. If $a = 1$, then set $b = 2$. Otherwise set $b = 1$.
That is, we set a equal to the index of the larger integer (or "winner") and set b equal to the index of the smaller integer (or "loser").
Compute $M(3, 4)$ and record the answer as c. If $c = 3$, then set $d = 4$. Otherwise set $d = 3$.
Compute $M(a, c)$ and record the answer as e. If $e = a$, then set $f = c$. Otherwise set $f = a$.
Compute $M(b, d)$ and record the answer as g. If $g = b$, then set $h = d$. Otherwise set $h = b$.
At this point we have used the machine 4 times. We know the integer at index e is the largest overall because it is the "winner of the winners". We know the integer at index h is the smallest overall because it is the "loser of the losers". Finally, compute $M(f, g)$ to determine the order of the other two integers. This process allows us to list all four integers from smallest to largest and we have used the machine 5 times.

Solution for Task 2

First we find the correct place for a_1 among the items in the ordered list $a_3, a_4, a_5, a_6, a_7, a_8, a_9$. That is, we find an index i such that $a_i < a_1 < a_{i+1}$. To do this, we compute M(1,6). If we learn that $a_1 > a_6$, then we compute $M(1, 8)$. Otherwise, we compute $M(1, 4)$. Now we have used the machine twice and we know one of following four things is true (and we know which one is true):

- $a_1 > a_6$ and $a_1 > a_8$, or
- $a_1 > a_6$ and $a_1 < a_8$, or
- $a_1 < a_6$ and $a_1 > a_4$, or
- $a_1 < a_6$ and $a_1 < a_4$.

If the first case above is true, then we know that either $a_8 < a_1 < a_9$ or $a_1 > a_9$. We next compute $M(1, 9)$ to determine which of these is true. In particular, we determine which of $a_3, \ldots, a_8, a_9, a_1$ or $a_3, \ldots, a_8, a_1, a_9$ is a list of integers from smallest to largest. The other three cases shown above are similar. In summary, we can place a_1 correctly using the machine 3 times. Now we simply repeat this process to find the correct place for a_2 in the ordered list $a_3, a_4, a_5, a_6, a_7, a_8, a_9$. Finally, since we know that $a_1 < a_2$, we can list all 9 integers from smallest to largest and we have used the machine $3 + 3 = 6$ times.

(Note that the process of placing a_2 can use the machine less than 6 times in some cases by only placing it within the integers among $a_3, a_4, a_5, a_6, a_7, a_8$ and a_9 that are greater than a_1. However, in the worst-case, we will still need to use the machine 6 times.)

Part of this algorithm is well-known and very important. It is called binary search. Its key idea is to find the correct place for an item in an ordered list by repeatedly comparing it to a "middle item".

Solution for Task 3

A critical piece of this solution is to note that we can use the first 3 comparisons of the solution to Task 1 in order to find the largest of any four integers. Specifically, this involves comparing the integers in pairs and then determining the "winner of the winners". We will call this our *subroutine*.

We begin by using our subroutine with any four of the five integers. The largest of these four integers cannot be the median of the five integers (because it is larger than at least three others) so we discard it. We now need to find the second largest of the remaining four integers since this integer must be the median integer.

Use the subroutine with these four integers. However, from our first use of the subroutine, we already know how two of these integers compare so we can reuse this comparison as our first comparison in the second use of the subroutine. So to complete the second use of the subroutine we only need to do 2 new comparisons. We can discard the largest of the remaining four integers as before. Three integers remain. The largest of these three integers must be the median of the original five integers.

As before, from our previous use of the subroutine, we already know how two of the three remaining integers compare. So we compare the third integer with the larger of these two integers to determine the largest of these three integers. So we have completed our task using the machine $3 + 2 + 1 = 6$ times.

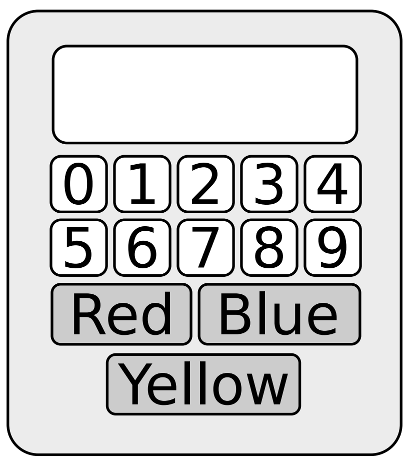
CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 30, 2020
More Digit Swapping

Ali programs three buttons in a machine to swap digits in a 4-digit integer.

Red button:	swaps the thousands and tens digits
Blue button:	swaps the thousands and hundreds digits
Yellow button:	swaps the hundreds and units (ones) digits

Ali types the integer 1234 into the machine. Using only the Red, Blue, and Yellow buttons, determine all outputs she can produce with exactly 5 more button presses that she cannot produce using fewer than 5 more button presses.

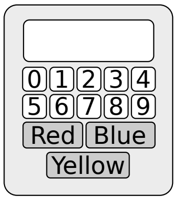


More Info:

Check the CEMC at Home webpage on Thursday, May 7 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, May 7.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week
Problem E and Solution
More Digit Swapping

Problem

Ali programs three buttons in a machine to swap digits in a 4-digit integer.

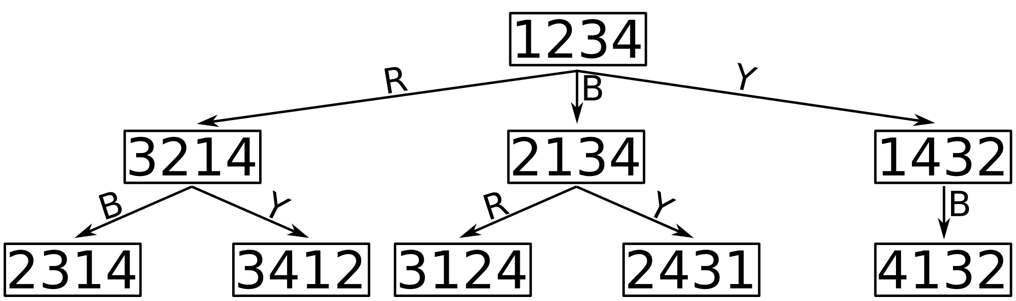
Red button:	swaps the thousands and tens digits
Blue button:	swaps the thousands and hundreds digits
Yellow button:	swaps the hundreds and units (ones) digits

Ali types the integer 1234 into the machine. Using only the Red, Blue, and Yellow buttons, determine all outputs she can produce with exactly 5 more button presses that she cannot produce using fewer than 5 more button presses.

Solution

We make a tree diagram to show all the possible outputs. We create this tree diagram one row at a time, moving from left to right. When we get a number that already exists in the tree, we do not write it down so our tree does not contain any duplicates. This will ensure that the tree shows only the shortest path to reach each of the possible different outputs.

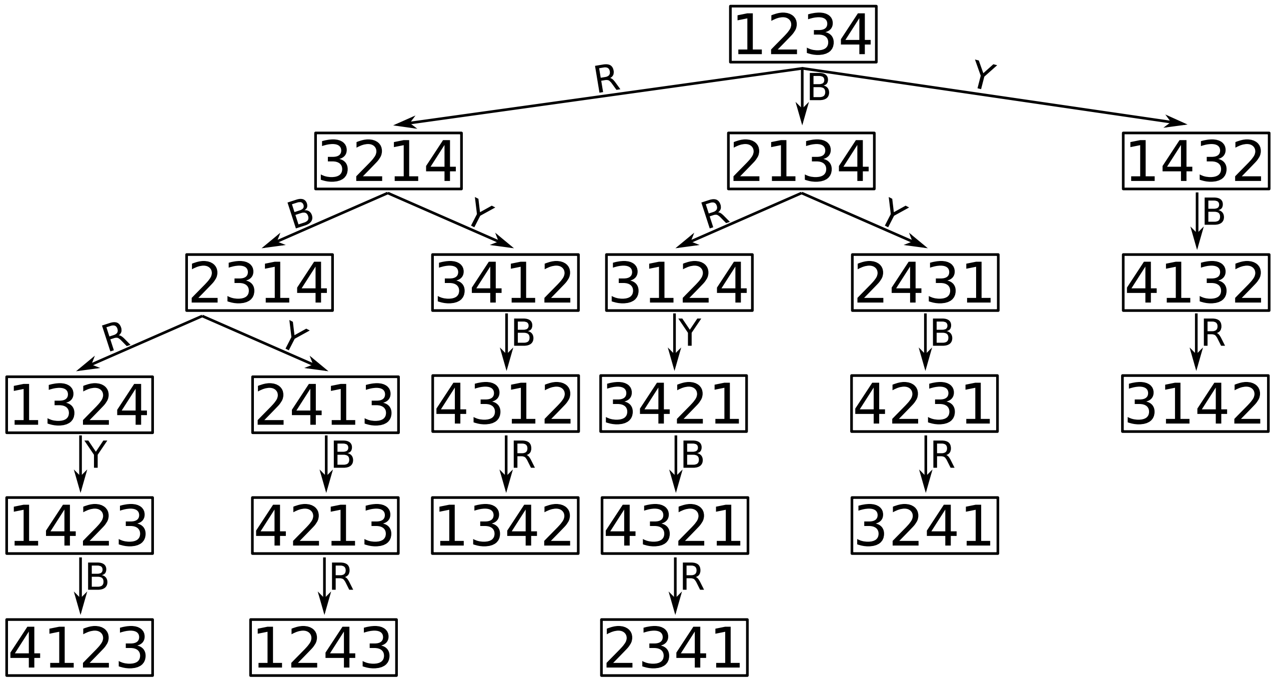
We can then use this tree diagram to find all the outputs that require exactly 5 more button presses to produce and cannot be produced in less.



Notice that we did not write down the output for pressing the Red button twice. This is because we would have ended up with 1234, which the number we were at before the first Red button push, and so is already in our tree. The same argument applies for pressing any button twice in a row.

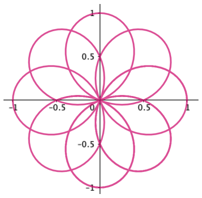
Notice also that we did not write down the output for pressing the Yellow button followed by the Red button. This is because we would have ended up with 3412, which is already in our tree.

We continue with the tree diagram, stopping after we have gone through all possible outputs from pressing 5 buttons.



Therefore, the outputs which can be produced by exactly five more button presses and cannot be produced with fewer are 4123, 1243, and 2341.

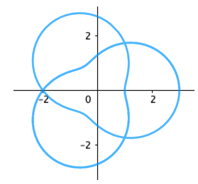
<u>Extension:</u> For the input of 1234, which output requires the most presses of the Red, Blue, and Yellow buttons to produce?



CEMC at Home

Grade 11/12 - Friday, May 1, 2020

Polar Coordinates



The *Cartesian Coordinate System* is the most familiar system that we use to represent points in the plane. Today, we will learn about a different system, the *Polar Coordinate System*. Next Friday, we will learn how to graph interesting curves like the two above. Equations for graphs like these are often very complicated using Cartesian coordinates, but can be much simpler using polar coordinates.

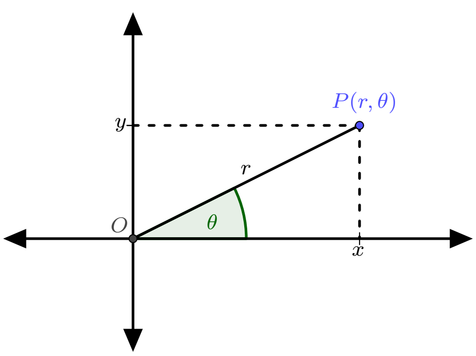
We will be using radians in this activity. If you have never measured angles in radians before, then either see the last page of this resource for an introduction to radians, or check out this lesson from the CEMC courseware.

Polar Coordinates

In Cartesian coordinates, a point P in the plane is given as $P(x, y)$, where x and y are real numbers.

Remind yourself of exactly what the values x and y represent here.

The point P can also be described using *polar coordinates* (r, θ). Here, r is the distance between the point P and the origin O. Also, θ is the angle (in radians) measured from the x-axis. (*Like when we look at the unit circle, positive angles are measured counter-clockwise from the positive x-axis.*) In polar coordinates, we call the positive x-axis the *polar axis*.



Suppose that P is in the first quadrant. Consider the right-angled triangle formed by the point P, the origin O, and the vertical line from P to the x-axis. This triangle has base x, height y and hypotenuse r.

By the Pythagorean Theorem, $r^2 = x^2 + y^2$.

We have $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$ from the definitions of sine and cosine in right-angled triangles.

Manipulating these equations, we obtain the three equations below that help us to relate the polar and Cartesian coordinates of the point P.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

Example

Consider the Cartesian point $Q(1, 1)$. Since $x = 1$ and $y = 1$, then $r = \sqrt{1^2 + 1^2} = \sqrt{2}$. Also, the line segment joining the origin O to Q makes an angle of $\frac{\pi}{4}$ with the positive x-axis. This means that polar coordinates for Q are $(\sqrt{2}, \frac{\pi}{4})$.

Try drawing a picture and clearly labelling x, y, r, and θ. Make sure you understand why these values of r and θ are correct.

Question 1

Plot the points with Cartesian coordinates $A(8\sqrt{3}, 8)$ and $B(\frac{5}{4}, \frac{5\sqrt{3}}{4})$ and then convert them to polar coordinates.

Question 2

Plot the points with Cartesian coordinates $C(8, -8\sqrt{3})$ and $D(-\frac{5\sqrt{3}}{4}, -\frac{5}{4})$ and then convert them to polar coordinates.

Example

Consider the point with polar coordinates $\left(4, \frac{3\pi}{2}\right)$. Since $r = 4$ and $\theta = \frac{3\pi}{2}$, we have that

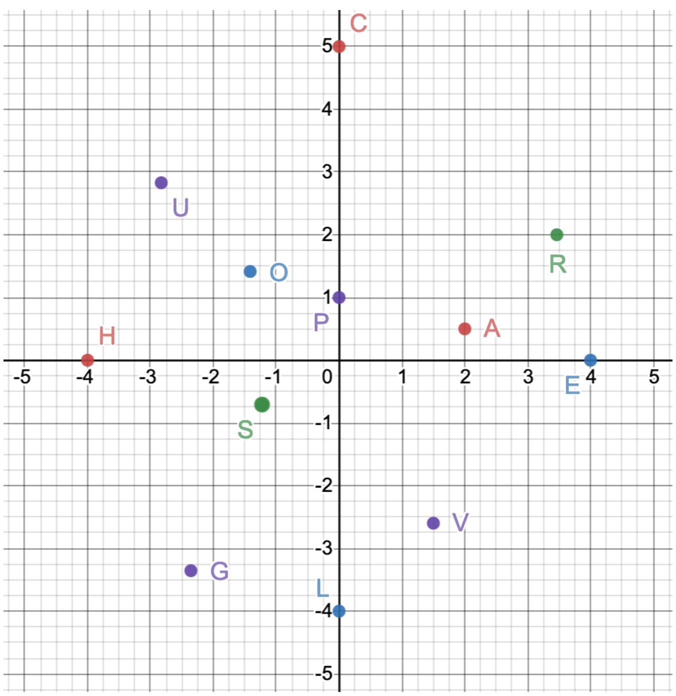
$$x = r\cos\theta = 4\cos\left(\frac{3\pi}{2}\right) = 4(0) = 0$$
$$y = r\sin\theta = 4\sin\left(\frac{3\pi}{2}\right) = 4(-1) = -4$$

This means that the Cartesian coordinates of the point are $(0, -4)$.

Can you see why these must be the correct Cartesian coordinates by visualizing the point?

Activity

Consider the polar coordinates (r, θ), with $0 \leq \theta < 2\pi$, of each of the 12 points plotted in the graph below. Exactly one of these points satisfies each of the following properties, and each point is labelled with a different letter. Determine which point best matches each property and use this information to complete the phrase below.



1. This point has polar coordinates $(4, 0)$.
2. This point has polar coordinates $(4, \frac{3\pi}{2})$.
3. This point has polar coordinates $(4, \frac{3\pi}{4})$.
4. This point could also be described using polar coordinates $(2, \frac{11\pi}{4})$.
5. This point's first coordinate, r, satisfies $r^2 = 2$.
6. This point has the largest first coordinate, r, out of all of the points.
7. This point has the smallest *positive* second coordinate, θ, out of all of the points.
8. This point's second coordinate, θ, satisfies $2\sin\theta = 1$.
9. This point's second coordinate, θ, satisfies $\cos\theta = -1$.
10. This point's first coordinate, r, satisfies $r = 3$.
11. This point's coordinates satisfy $r = \sin\theta$. *Remember that* $-1 \leq \sin\theta \leq 1$.
12. This point's coordinates satisfy $r = \theta$.

In next Friday's activity we will learn how to...

___	___	___	___	___	___	___	___	___	___	___	___	___	___	___	___	!
12	8	7	11	9	11	4	2	7	8	6	3	8	10	1	5	

More Info:

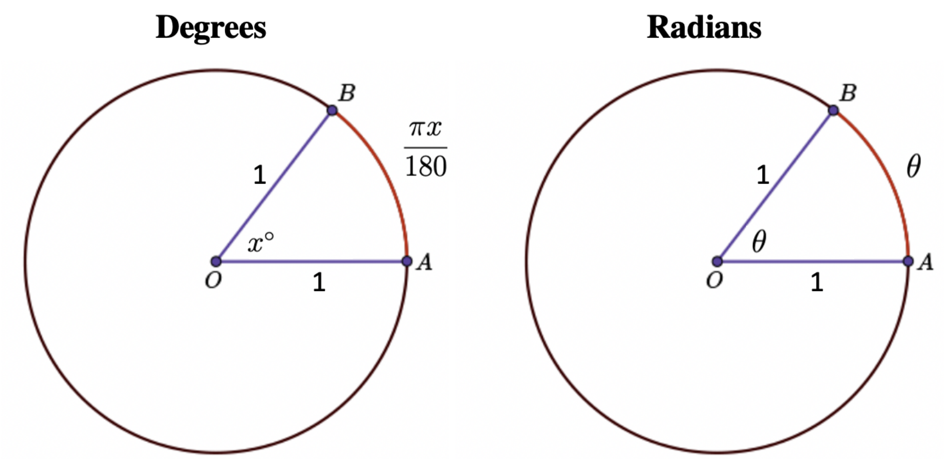
Check the CEMC at Home webpage on Friday, May 8 for a solution to Polar Coordinates.

Radians

When we first learn about angles, we write their measures (that is, their "sizes") using degrees. For example, a complete circular angle measures 360°, a straight angle measures 180°, and a right angle measures 90°. Angles like 30°, 45°, and 60° are also familiar.

A second way of measuring angles is in *radians*. In this case, a complete circular angle measures 2π. What connection can you see between 2π and the unit circle? The circumference of the unit circle is 2π. Radians are defined so that an angle of measure $x°$ measures $\dfrac{\pi x}{180}$ radians.

The value $\dfrac{\pi x}{180}$ *is actually the arc length of a sector of the unit circle defined by the angle with measure* $x°$ *so radians are in some sense measuring the arc length corresponding to the angle, which is one way of measuring the angle itself.*



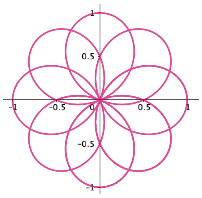
Questions:

(a) Convert the angles with the following measures from degrees to radians: 180°, 90°, 60°, 45°, 30°, 48°.

(b) Convert the angles with the following measures from radians to degrees: $\frac{\pi}{5}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{7\pi}{4}$.

(c) Complete the chart below. *The angles are given in radians.*

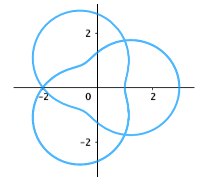
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$					
$\cos\theta$					



CEMC at Home

Grade 11/12 - Friday, May 1, 2020

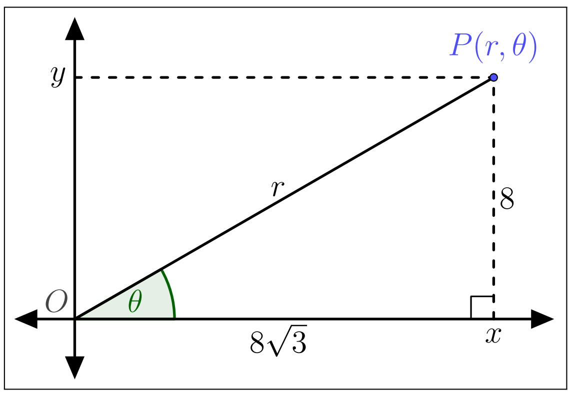
Polar Coordinates - Solution



Question 1

Plot the points with Cartesian coordinates $A\left(8\sqrt{3}, 8\right)$ and $B\left(\frac{5}{4}, \frac{5\sqrt{3}}{4}\right)$ and then convert them to polar coordinates.

Solution: We first plot the point $A\left(8\sqrt{3}, 8\right)$ in the plane.



Since $x = 8\sqrt{3}$ and $y = 8$, we have

$$r = \sqrt{x^2 + y^2} = \sqrt{\left(8\sqrt{3}\right)^2 + 8^2} = \sqrt{192 + 64} = 16$$

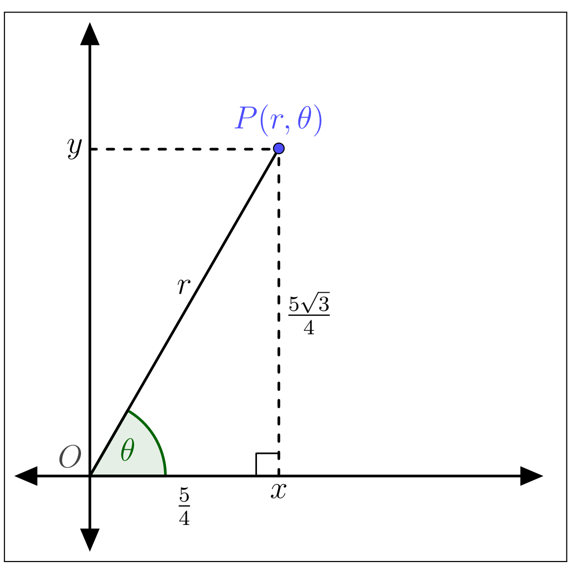
From the right-angled triangle in the diagram, we see we are looking for an angle θ in the first quadrant that satisfies

$$\sin(\theta) = \frac{y}{r} = \frac{8}{16} = \frac{1}{2}$$

One possible choice is $\theta = \frac{\pi}{6}$.

This means the point with Cartesian coordinates $(x, y) = \left(8\sqrt{3}, 8\right)$ can be described using polar coordinates $(r, \theta) = \left(16, \frac{\pi}{6}\right)$.

Now we plot the point $B\left(\frac{5}{4}, \frac{5\sqrt{3}}{4}\right)$ in the plane.



Since $x = \frac{5}{4}$ and $y = \frac{5\sqrt{3}}{4}$, we have

$$x^2 + y^2 = \left(\frac{5}{4}\right)^2 + \left(\frac{5\sqrt{3}}{4}\right)^2 = \frac{25}{16} + \frac{75}{16} = \frac{100}{16} = \frac{25}{4}$$

and so $r = \sqrt{x^2 + y^2} = \frac{5}{2}$. From the right-angled triangle in the diagram, we see we are looking for an angle θ in the first quadrant that satisfies

$$\cos(\theta) = \frac{x}{r} = \frac{\left(\frac{5}{4}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{2}$$

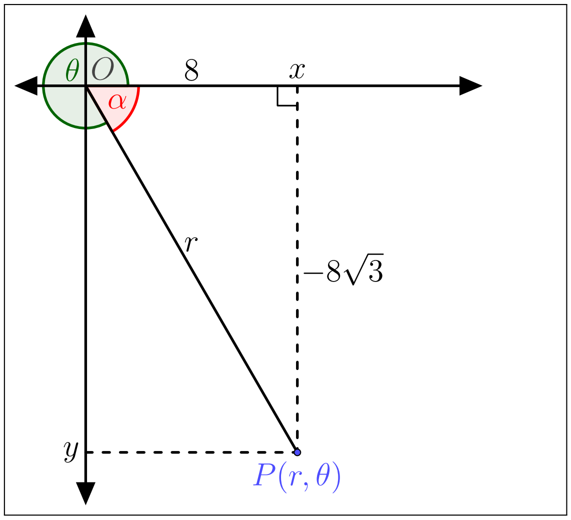
One possible choice is $\theta = \frac{\pi}{3}$.

This means the point with Cartesian coordinates $(x, y) = \left(\frac{5}{4}, \frac{5\sqrt{3}}{4}\right)$ can be described using polar coordinates $(r, \theta) = \left(\frac{5}{2}, \frac{\pi}{3}\right)$.

Question 2

Plot the points with Cartesian coordinates $C\left(8, -8\sqrt{3}\right)$ and $D\left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ and then convert them to polar coordinates.

Solution: We first plot the point $C\left(8, -8\sqrt{3}\right)$ in the plane.



Since $x = 8$ and $y = -8\sqrt{3}$, we have

$$r = \sqrt{x^2 + y^2} = \sqrt{8^2 + \left(-8\sqrt{3}\right)^2} = \sqrt{64 + 192} = 16$$

From the right-angled triangle in the diagram, we see we are looking for an angle θ in the fourth quadrant for which the associated acute angle α satisfies

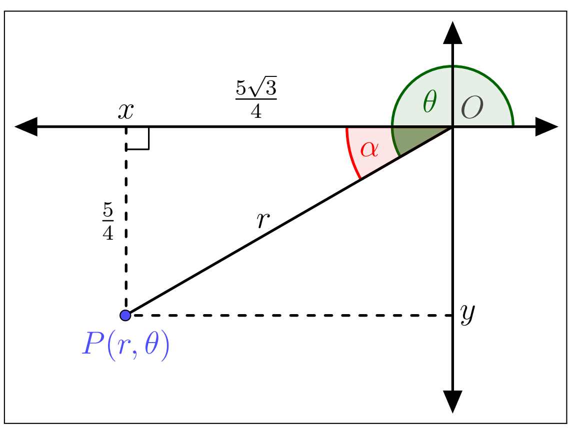
$$\cos(\alpha) = \frac{8}{16} = \frac{1}{2}$$

This means $\alpha = \frac{\pi}{3}$ and so one possible choice is $\theta = \frac{5\pi}{3}$.

This means the point with Cartesian coordinates $(x, y) = \left(8, -8\sqrt{3}\right)$ can be described using polar coordinates $(r, \theta) = \left(16, \frac{5\pi}{3}\right)$.

Note: We could have instead observed that point C is related to point A. They are the same distance from the origin, and their angles are complementary.

Now we plot the point $D\left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ in the plane.



Since $x = -\frac{5\sqrt{3}}{4}$ and $y = -\frac{5}{4}$, we have

$$x^2 + y^2 = \left(-\frac{5\sqrt{3}}{4}\right)^2 + \left(-\frac{5}{4}\right)^2 = \frac{75}{16} + \frac{25}{16} = \frac{25}{4}$$

and so $r = \sqrt{x^2 + y^2} = \frac{5}{2}$. From the right-angled triangle in the diagram, we see we are looking for an angle θ in the third quadrant for which the associated acute angle α satisfies

$$\sin(\alpha) = \frac{\left(\frac{5}{4}\right)}{\left(\frac{5}{2}\right)} = \frac{1}{2}$$

This means $\alpha = \frac{\pi}{6}$ and one possible choice is $\theta = \frac{7\pi}{6}$.

This means the point with Cartesian coordinates $(x, y) = \left(-\frac{5\sqrt{3}}{4}, -\frac{5}{4}\right)$ can be described using polar coordinates $(r, \theta) = \left(\frac{5}{2}, \frac{7\pi}{6}\right)$.

Activity Answers:

In next Friday's activity we will learn how to...

G	R	A	P	H	P	O	L	A	R	C	U	R	V	E	S	!
12	8	7	11	9	11	4	2	7	8	6	3	8	10	1	5	

See the next page for an explanation of each matching.

1. This point has polar coordinates $(4, 0)$. ***(E)***

Since $r = 4$ and $\theta = 0$, this is a point that is 4 units from the origin and lies on the ray defined by $\theta = 0$ which is the positive x-axis. This describes only point E.

2. This point has polar coordinates $\left(4, \frac{3\pi}{2}\right)$. ***(L)***

This is a point that is 4 units from the origin and lies on the ray defined by $\theta = \frac{3\pi}{2}$ which is the negative y-axis. This describes only point L.

3. This point has polar coordinates $\left(4, \frac{3\pi}{4}\right)$. ***(U)***

This is a point that is 4 units from the origin lies on the ray defined by $\theta = \frac{3\pi}{4}$. This describes only point U.

4. This point could also be described using polar coordinates $\left(2, \frac{11\pi}{4}\right)$. ***(O)***

Note that $\frac{11\pi}{4}$ and $\frac{11\pi}{4} - 2\pi = \frac{3\pi}{4}$ are equivalent angles. So we are looking for the point with polar coordinates $\left(2, \frac{3\pi}{4}\right)$. This is on the same ray as U above, but 2 units from the origin. This describes only point O.

5. This point's first coordinate, r, satisfies $r^2 = 2$. ***(S)***

This means $r = \pm\sqrt{2} \approx \pm 1.4$. It looks like the only point that is around 1.4 units from the origin is S. You can draw a circle of radius 1.4 on the graph to confirm. This is describing point S.

6. This point has the largest first coordinate, r, out of all of the points. ***(C)***

The point with the largest first coordinate will be the farthest from the origin. The point C is 5 units away and every other point appears to be closer than that. You can draw a circle of radius 5 on the graph to confirm! This is describing point C.

7. This point has the smallest *positive* second coordinate, θ, out of all of the points. ***(A)***

The point with the smallest positive second coordinate will make the smallest angle with the positive x-axis. This describes the point A.

8. This point's second coordinate, θ, satisfies $2 \sin\theta = 1$. ***(R)***

If $0 \leq \theta < 2\pi$ and $\sin\theta = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$. The only point that lies on the ray defined by $\theta = \frac{\pi}{6}$ is R and there are no points that lie on the ray defined by $\theta = \frac{5\pi}{6}$. This describes R.

9. This point's second coordinate, θ, satisfies $\cos\theta = -1$. ***(H)***

If $0 \leq \theta < 2\pi$ and $\cos\theta = -1$, then $\theta = \pi$. The only point that lies on the ray defined by $\theta = \pi$ (the negative x-axis) is H.

10. This point's first coordinate, r, satisfies $r = 3$. ***(V)***

The only point that appears to be 3 units from the origin is V. You can draw a circle of radius 3 on the graph to confirm. This is describing point V.

11. This point's coordinates satisfy $r = \sin\theta$. ***(P)***

Since $-1 \leq \sin\theta \leq 1$, any coordinates that satisfy this equality must have $-1 \leq r \leq 1$. The only point within 1 unit of the origin is P. In fact, P appears to have polar coordinates $r = 1$ and $\theta = \frac{\pi}{2}$ which do satisfy $\sin\theta = \sin\left(\frac{\pi}{2}\right) = 1 = r$.

12. This point's coordinates satisfy $r = \theta$. ***(G)***

We are now left with one property (12) and one point (G). This means G must be the point satisfying $r = \theta$. Using the distance formula, you can check that G is around 4 units from the origin. The ray through G is near the ray defined by $\theta = \frac{5\pi}{4} \approx 4$, which provides some evidence that $r \approx \theta$. (The actual point plotted has $r = \theta = 4.1$.)