CEMC at Home

Grade 9/10 - Monday, April 20, 2020

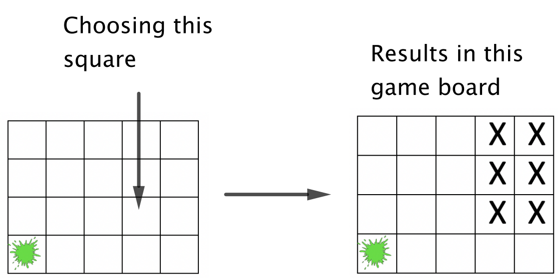
Splat

You Will Need:

- Two players
- A Splat game board which is a rectangular grid with the bottom left corner marked with a Splat

How to Play:

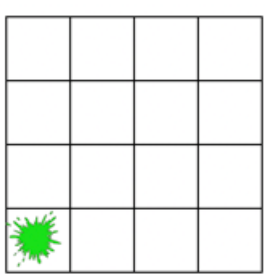
1. Start with a Splat game board.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On each turn, the current player must select a square and removes that square and all of the squares above and it and to the right of it (see example below).



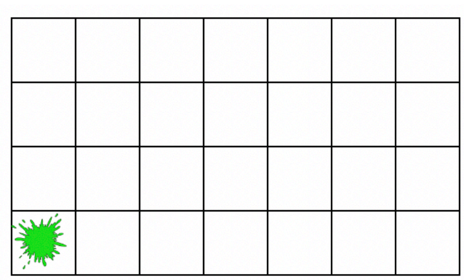
Any square that is removed (marked with an X*) cannot be chosen on any turns that follow.*

4. The player who is forced to remove the square with the Splat loses the game.

Play this game a number of times using the game boards below. Alternate which player goes first. For each of the sample game boards (Game Board 1 and Game Board 2), is there a strategy that guarantees one of the players (Player 1 or Player 2) a win every time?



Game Board 1



Game Board 2

More Info:

Check out the CEMC at Home webpage on Monday, April 27 for a discussion of a strategy for Splat. We encourage you to discuss your ideas online using any forum you are comfortable with.

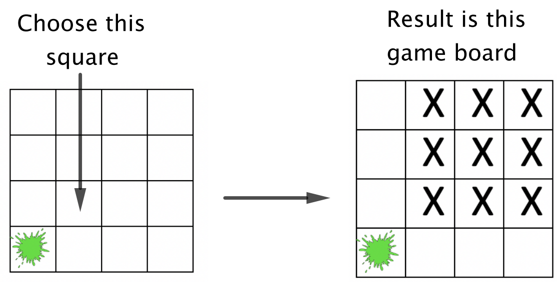
This game is often called *Chomp*.

CEMC at Home

Grade 9/10 - Monday, April 20, 2020
Splat - Solution

Game Board 1

Player 1 has a winning strategy for this game board. Player 1 should choose the square one over and one up from the square with the Splat as shown in the diagram below.



From here, Player 1 will mimic whatever Player 2 does in the following way:

- If Player 2 chooses a square from the remaining column, then Player 1 will choose the corresponding square in the remaining row.
- If Player 2 chooses a square from the remaining row, then Player 1 will choose the corresponding square in the remaining column.

For example, if Player 2 chooses the square marked as P2 (in Diagram 1 below), then Player 1 should chose the square marked as P1. The resulting game board is shown in Diagram 2 below.

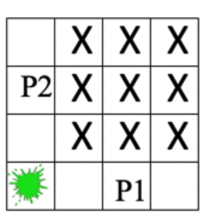


Diagram 1

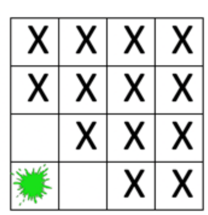


Diagram 2

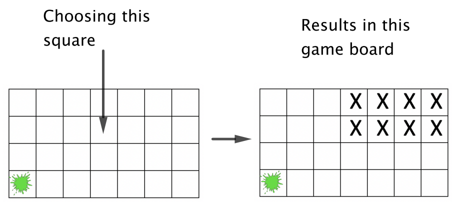
If Player 1 keeps mimicking Player 2 in this way, eventually the square with the Splat will be the only square left after Player 1 makes a move. Then Player 2 must choose that square and lose the game.

Note: This solution is valid for any game board with an equal number of rows and columns (except for a 1 by 1 game board). To start, Player 1 should always choose the square one over and one up from the Splat. Then Player 1 mimics Player 2 as described above.

The strategy for Game Board 2 is on the next page.

Game Board 2

It turns out that Player 1 has a winning strategy for this game board but it is not easy to find or to describe. In one strategy, Player 1 can start with the following play.



From here, the strategy depends on what move Player 2 chooses to make. Player 1 will then have to react to this move and there are many different ways this can unfold. We will not present the strategy here but we encourage you to think about it and look into it further on your own. (Remember that this game is often called *Chomp*.)

It is actually possible to argue that Player 1 must have a winning strategy without actually finding one! An argument would involve explaining the following two facts:

1. Exactly one player (Player 1 or Player 2) must have a winning strategy for this game board.
2. It is not possible for Player 2 to have a winning strategy for this game board.

For point 1., it turns out that since there is no way for the game to end in a tie, we can argue that at least one player must have a winning strategy (although this argument is not as easy to explain carefully as you might think). By the definition of a winning strategy, it is not possible for both players to have a winning strategy for the same game board.

To argue point 2. we can use what is sometimes called a *strategy-stealing argument*. The idea here is as follows: If Player 2 had a winning strategy, then Player 1 could use that strategy before Player 2 has a chance to use it. The name "strategy stealing argument" is slightly misleading. A winning strategy must work regardless of the moves made by the other player and so cannot be "stolen" (at least not without a mistake!). That Player 1 can "steal" the winning strategy simply means that Player 1 must have had the winning strategy in the first place.

Let's assume that Player 2 has a winning strategy. This means that no matter what first move Player 1 makes, Player 2 is guaranteed to win the game following this strategy.

In particular, this means that if Player 1 starts the game by selecting the top right hand square in the grid, then Player 2 must have a winning strategy from this point. This means Player 2 has a move in response to Player 1 removing the top right hand square that leaves Player 1 with a game board from which they cannot possibly win.

No matter what the game board looks like after Player 2 responds, Player 1 could have made their first move in a way that leaves this exact same game board! (Can you see why?) In this way, Player 1 can "steal" the strategy by making their first move to leave Player 2 with this game board from which they cannot win.

This argument may take a few times through to understand, but it achieves something quite remarkable. It explains that Player 1 has a winning strategy, but gives no hint whatsoever as to what the strategy is. In fact, it doesn't even tell us what the first move should be!

The above explanation that Player 2 cannot have a winning strategy involves a type of argument called a proof by contradiction.

CEMC at Home

Grade 9/10 - Tuesday, April 21, 2020
Some Counting

Consider the integers from 10 to 30, inclusive.

If you wrote down all of these integers, you would use a total of twelve 2s: one 2 for the units (ones) digit of the integer 12, then ten 2s for the tens digit in each of the integers from 20 to 29, and an additional 2 for the units digit of 22.

If you were asked how many of the integers from 10 to 30 contain the digit 2, you would answer eleven, not twelve. While the number 22 contains two copies of the digit 2, we only count this number once.

Solve the following problems that involve counting digits and numbers in a longer list of integers.

Problem 1: Consider the integers from 300 to 600, inclusive.

(a) If you wrote down all of the integers from 300 to 600, inclusive, how many times would you write the digit 4?

(b) How many of the integers from 300 to 600, inclusive, contain the digit 4?

Problem 2: If you wrote down all of the integers from 300 to 600, inclusive, what is the sum of all of the digits that you would write?

Discussion for Problem 2: To answer part (a) above, you need to count how many times you would write the digit 4 when writing the integers from 300 to 600. Since there would be 301 integers in this list, actually writing them all down is not the best approach.

We have a similar issue in Problem 2. One way to solve this problem would be to write out the 301 integers from 300 to 600, inclusive, and then add the resulting 903 digits together. However, there are nicer ways to solve this problem. Do you see how your work in Problem 1 may help you in answering this question?

If you need help getting started, you can always return to our first example, the list of integers from 10 to 30, which can be written down quickly:

$$10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30$$

What do you notice when calculating the sum of the 42 digits in this list of integers? You might find it helpful to consider the 21 units digits and the 21 tens digits separately.

More Info:

Check out the CEMC at Home webpage on Tuesday, April 28 for a solution to Some Counting.

CEMC at Home

Grade 9/10 - Tuesday, April 21, 2020
Some Counting - Solution

Problem 1: Consider the integers from 300 to 600, inclusive.

(a) If you wrote down all of the integers from 300 to 600, inclusive, how many times would you write the digit 4?

(b) How many of the integers from 300 to 600, inclusive, contain the digit 4?

Solution to Problem 1:

(a) There are a number of ways to approach this question, but we will use an organized count.

First, consider the hundreds digits of the 301 integers from 300 to 600. None of the integers from 300 to 399 have a 4 in the hundreds position. The 100 integers from 400 to 499 each have a 4 in the hundreds position. None of the integers from 500 to 600 have a 4 in the hundreds position. So there are a total of 100 4s appearing in the hundreds position in the list of integers from 300 to 600.

Next, consider the tens digits. Only the integers from 340 to 349, 440 to 449, and 540 to 549 have a 4 in the tens position. So there are $3 \times 10 = 30$ 4s appearing in the tens position in the list of integers from 300 to 600.

Finally, consider the units (ones) digits. In the integers from 300 to 399 there are 10 integers with a 4 in the units position. These integers are as follows:

$$304, 314, 324, 334, 344, 354, 364, 374, 384, 394$$

Similarly, there are 10 integers from 400 to 499 with a 4 in the units position and 10 integers from 500 to 600 with a 4 in the units position. So there are $3 \times 10 = 30$ 4s appearing in the units position in the list of integers from 300 to 600.

Combining the counts for the three different positions, when writing the integers from 300 to 600 the digit 4 will be written a total of $100 + 30 + 30 = 160$ times.

(b) In our solution to part (a), we determined that there are 100 integers from 300 to 600 with a 4 in the hundreds position, 30 integers with a 4 in the tens position, and 30 integers with a 4 in the units position, but we note that $100 + 30 + 30 = 160$ is not a correct count of the number of integers that contain a 4. Some of these integers have more than one digit equal to 4 and so were counted more than once in the total of 160.

We now need to determine how many integers were counted more than once, and exactly how many times they were counted.

How many of the integers have a 4 in both the hundreds and the tens positions? These integers will begin with 44, and thus are from 440 to 449, and there are 10 of them. These 10 integers have been counted at least twice.

How many of the integers have a 4 in both the hundreds and the units positions? These integers will begin with a 4 and end with a 4. These integers have the form 4X4, where the X can be any of the ten digits, and there are 10 of them. These 10 integers have been counted at least twice.

How many of the integers have a 4 in both the tens and the units positions? These integers have the form X44, where X can be 3, 4 or 5. Thus there are 3 of these integers. These 3 integers have been counted at least twice.

There is only one integer with a 4 in all three positions, 444, and it was counted three times because it belongs to each of the three groups we just examined. All other integers from these groups were counted exactly two times because they each belong to two of the three groups.

To count the number of integers that contain at least one digit 4 we do the following calculation. Start with the total of 160 and subtract the number of integers that were counted at least twice (due to having a digit 4 in at least two positions). This results in the following:

$$160 - 10 - 10 - 3 = 137$$

But there is one last thing to consider before we obtain the final answer. The integer 444 was counted (or included) three times and then removed (or excluded) three times in our calculation above. (Do you see why?) This means it is not included in the count of 137. Adding the integer 444 back into our count we get a final answer of

$$160 - (10 + 10 + 3) + 1 = 138$$

Therefore, there are 138 integers between 300 and 600 that contain the digit 4.

Note: In this solution, the method of *inclusion-exclusion* has been demonstrated.

Problem 2: If you wrote down all of the integers from 300 to 600, inclusive, what is the sum of all of the digits that you would write?

Solution to Problem 2:

In the solution to this problem, it is again necessary to be organized in your approach. We give two possible ways to calculate this sum. The second approach uses our work from Problem 1(a).

Approach 1:

First, consider the units digits of the integers from 300 to 399. In each set of ten consecutive integers in this range, each of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ appears once (in the units position) and their sum is 45. There are ten distinct sets of 10 consecutive integers in the range from 300 to 399, and so the sum of the units digits in this range is $10 \times 45 = 450$. Similarly, in both the ranges from 400 to 499 and 500 to 599, the sum of the units digits will be 450. Thus, from 300 to 599, the sum of the units digits will be $3 \times 450 = 1350$.

Now consider the tens digits of the integers from 300 to 399. We note that each of the digits $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ appears 10 times in the tens position in this list of 100 integers. For example, the digit 0 appears as the tens digit in the integers from 300 to 309. The sum of these tens digits is $10 \times 45 = 450$. As with the units digits, the same number of each digit occurs again in the ranges 400 to 499 and 500 to 599. Thus, from 300 to 599, the sum of the tens digits will be $3 \times 450 = 1350$.

In the range 300 to 599 there are 100 integers with a 3 in the hundreds position, 100 integers with a 4 in the hundreds position, and 100 integers with a 5 in the hundreds position. The sum of these 300 digits is $100 \times 3 + 100 \times 4 + 100 \times 5 = 100 \times (3 + 4 + 5) = 100 \times 12 = 1200$.

Note that the only digits we have not yet included in our sum are the digits of the integer 600. Therefore, the sum of all digits used in writing down the integers from 300 to 600, inclusive, is

$$1350 + 1350 + 1200 + 6 + 0 + 0 = 3906$$

Approach 2:

You may have noticed that you can use your work from Problem 1(a) to calculate the sum in Problem 2. We outline how to do this here.

Recall that when writing the integers from 300 to 600, inclusive, the digit 4 will be written a total of 160 times. A similar argument can be used to show that the digits 3 and 5 will also appear a total of 160 times. Convince yourself that the digits 3 and 5 will appear exactly as often as the digit 4 in each of the three positions: hundreds, tens, and units.

The counts for the remaining digits are a bit different because of the particular range of integers. Notice that the digits 1, 2, 7, 8 and 9 do not appear as hundreds digits, and the digit 6 appears only once as a hundreds digit (in 600).

Using the work done in Problem 1(a), we can see that the digit 9 will appear $30 + 30 = 60$ times. This is because the digit 9 appears exactly as often as the digit 4 in the units position and in the tens position, but does not appear at all in the hundreds position. (Remember that the digit 4 appears 30 times in the tens position and 30 times in the units position.) Similar reasoning shows that the digits 1, 2, 7, and 8 also appear 60 times each, and that the digit 6 appears 61 times. (Remember that the digit 6 appears once in the hundreds position.)

When we add up all of the digits, each digit 4 contributes 4 to the sum, and so the 160 4s contribute a total of $160 \times 4 = 640$ to the sum of the digits. Dealing with the other digits in a similar way, we can calculate the sum of all of the digits as follows:

$$60 \times 1 + 60 \times 2 + 160 \times 3 + 160 \times 4 + 160 \times 5 + 61 \times 6 + 60 \times 7 + 60 \times 8 + 60 \times 9 = 3906$$

CEMC at Home

Grade 9/10 - Wednesday, April 22, 2020

Sunken Treasure

Video

Watch the following presentation on *The Knapsack Problem*, a problem involving optimization:

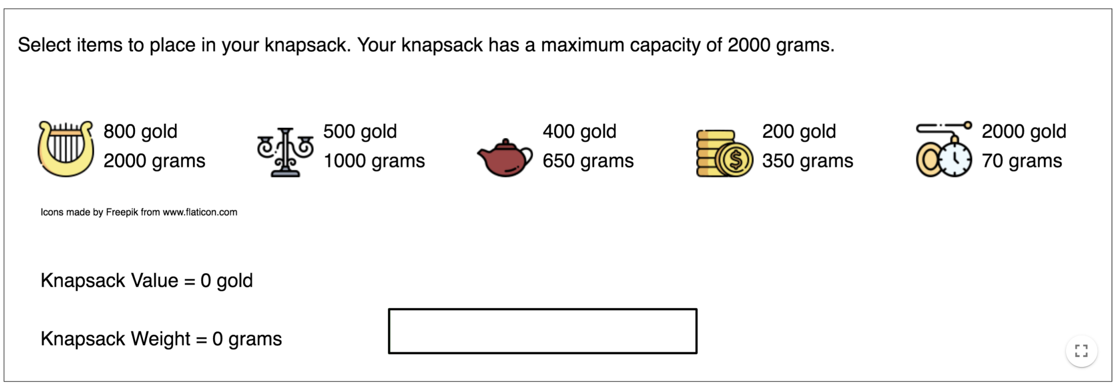
https://youtu.be/qihjUx8Qakk

The two scenarios from the presentation, along with the questions, are included for your reference.

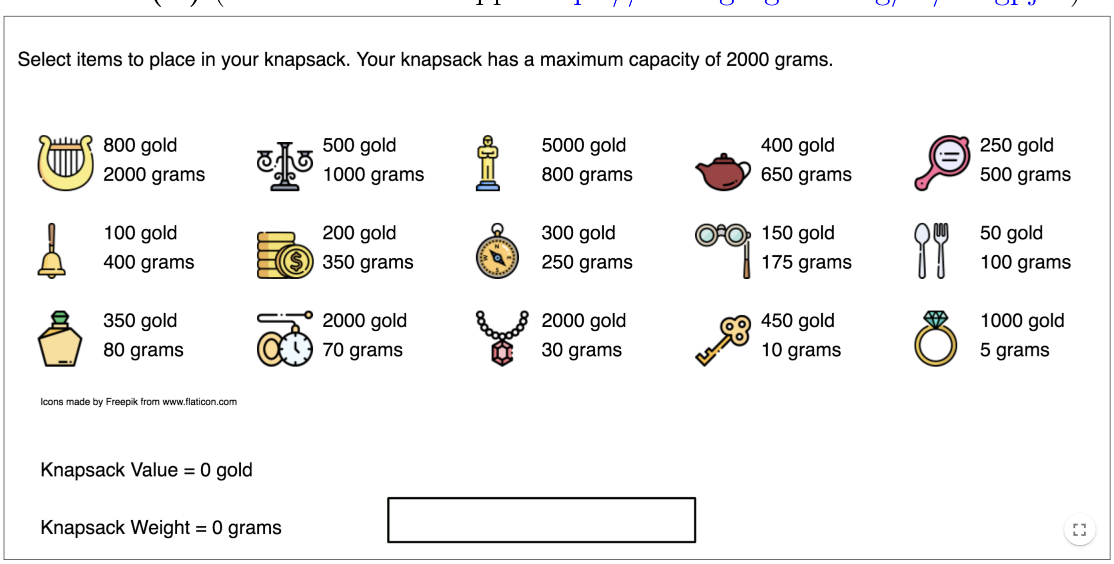
Questions:

1. What is the maximum knapsack value you can achieve?
2. Which subset of items achieves this maximum value?
3. What was your process?
4. How do you know for sure that your solution is correct?

Sunken Treasure (A) (with Interactive App: https://www.geogebra.org/m/hvbnqhzg)



Sunken Treasure (B) (with Interactive App: https://www.geogebra.org/m/dnrgpjcd)



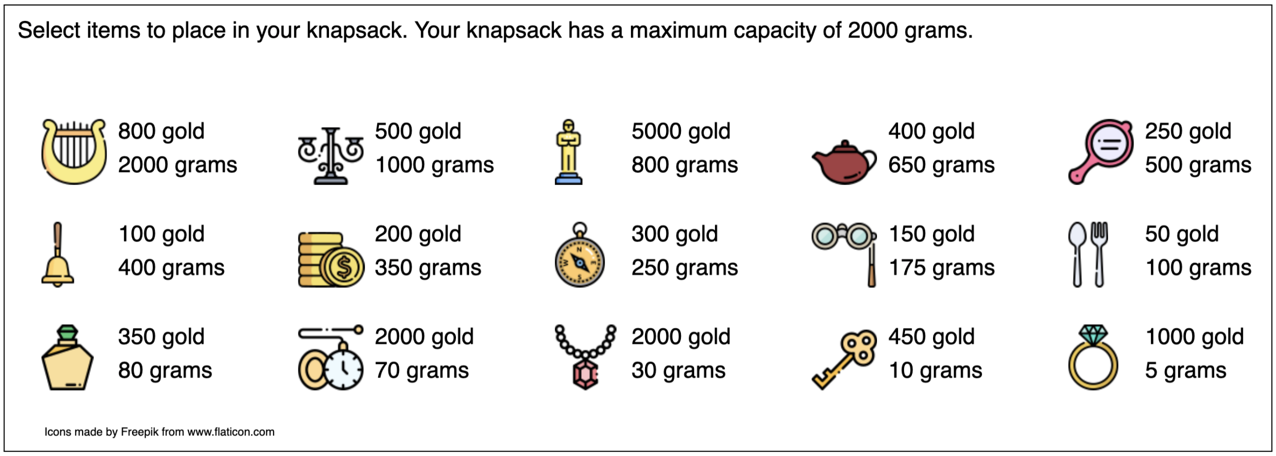
More Info:

Check out the CEMC at Home webpage on Wednesday, April 29 for a solution to Sunken Treasure.

CEMC at Home

Grade 9/10 - Wednesday, April 22, 2020
Sunken Treasure - Solution

Recall the set up for the activity Sunken Treasure (B):

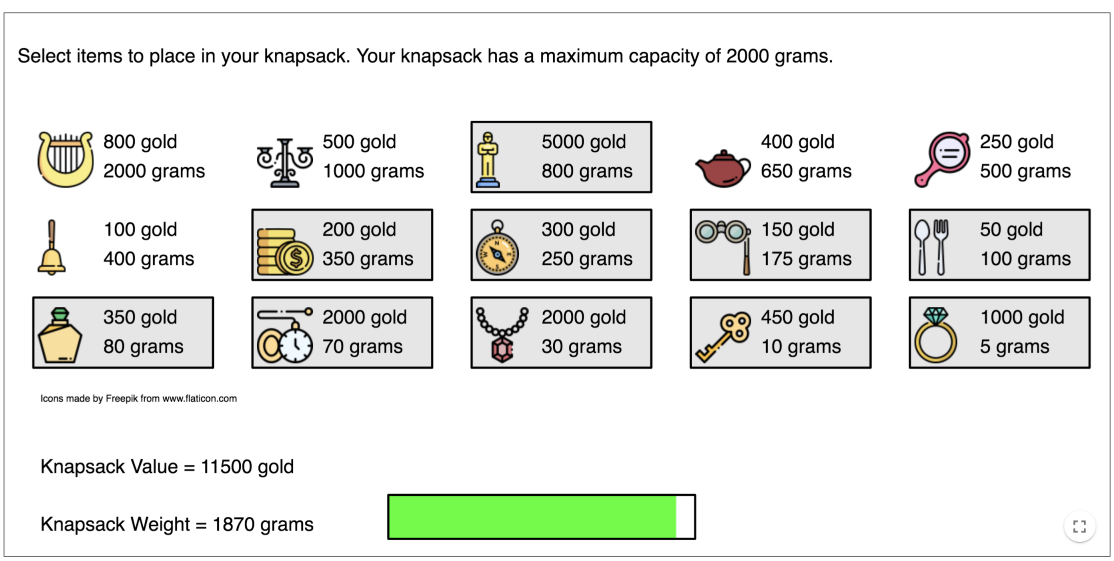


What is the maximum knapsack value you can achieve?

The maximum knapsack value you can achieve is 11 550 gold, although this value may have been difficult for you to obtain.

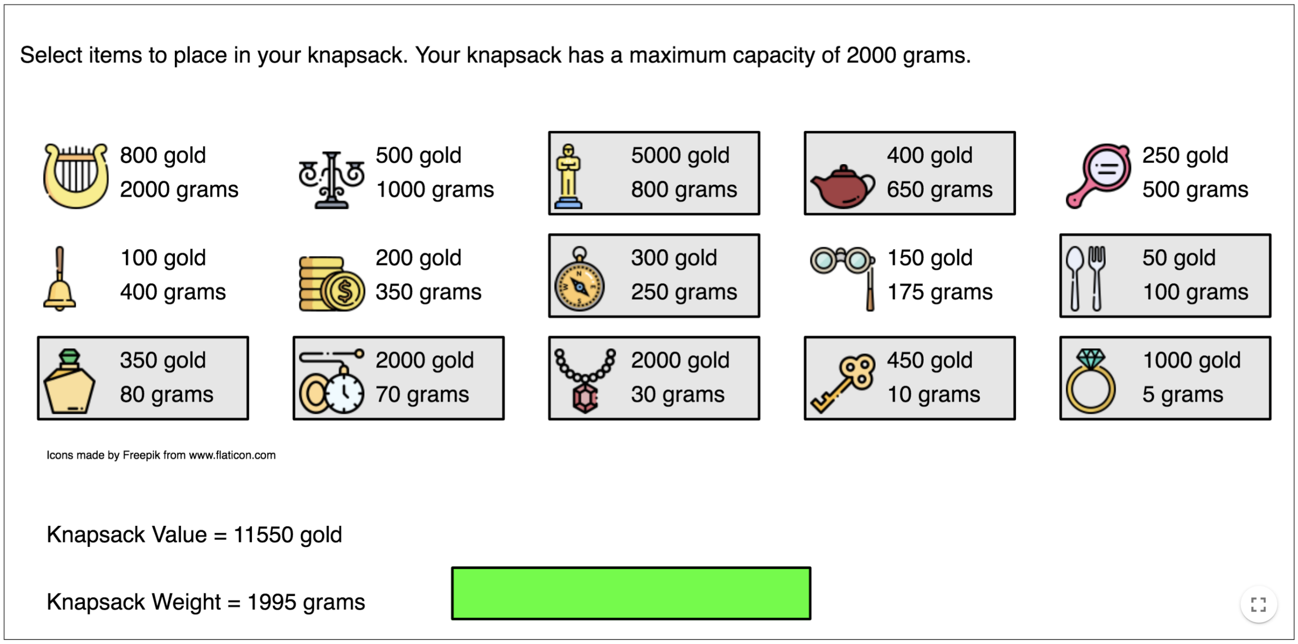
If you tried to fill your knapsack by choosing items with the greatest rate of *gold per gram*, then you most likely achieved a value of 11 500.

If you calculate each item's rate of gold per gram, sort the items from largest to smallest rate, and then place items in your bag in this same order as long as they fit in your knapsack, then you will end up with the following subset of the available items:



This algorithm, where you choose the best item at each step in the hope of getting the best outcome overall, is known as a *greedy algorithm*. Greedy algorithms do not look at the bigger picture and do not plan ahead.

If we step back and look at the bigger picture, we can see that if we remove the coins and the opera glasses and instead take the teapot, we can increase the knapsack's value by 50 gold as shown below.



It turns out that this solution is optimal, but how do we know for sure that this is the case? We could check the value attained by every possible subset of items, and make sure our value is at least as high as all of the rest. This approach is often called a *brute force algorithm*.

How long would this approach take? When forming a subset, each item is either in the subset or not. So there are 2 possibilities for each of the 15 available items: in or out. This means there are $2^{15} = 32\,768$ subsets we would have to check.

Of course, there are many subsets that we can eliminate immediately, or at least very quickly, because it is clear they will not produce an optimal result.

Checking all of these subsets is not feasible by hand, but suppose a computer can check 1 million subsets every second. In this case, it would take a computer less than 1 second to check all of the subsets and find the optimal solution. The problem is, this approach does not scale well. If the number of available items is increased from 15 to 100, then there would be 2^{100} subsets to check which would take a computer approximately 40 quadrillion years to consider.

Using brute force to find the optimal solution is often not feasible, but unfortunately the best known algorithms to solve this type of problem are also not always feasible. This explains why the knapsack problem is a hard problem in general. For hard problems, it is often good enough to find a solution which is *close to the optimal solution*.

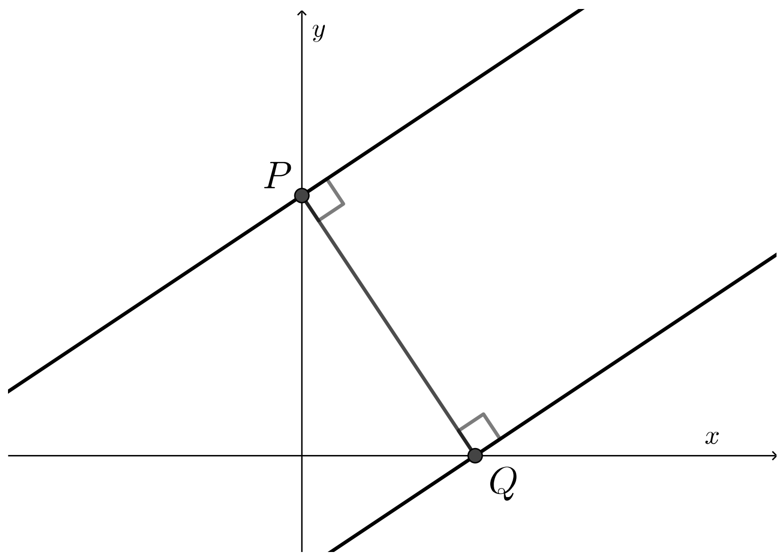
Can you find a way to reason that the value given above is optimal without the help of a computer? If you have some programming experience, can you write a computer program that can check this for you?

CEMC at Home features Problem of the Week

Grade 9/10 - Thursday, April 23, 2020
Crossing Points in General

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines.

If the line through P has equation $y = mx + k$, then determine the y-intercept of the line through Q in terms of m and k.



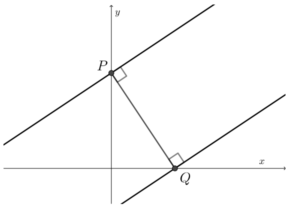
If you are finding the general problem difficult to start, consider first solving a problem with a specific example for the line through P, like $y = 4x + 3$, and then attempt the more general problem.

More Info:

Check the CEMC at Home webpage on Thursday, April 30 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 30.

This CEMC at Home resource is the current grade 9/10 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php



Problem of the Week

Problem D and Solution

Crossing Points in General

Problem

Two distinct lines are drawn such that the first line passes through point P on the y-axis and the second line passes through point Q on the x-axis. Line segment PQ is perpendicular to both lines. If the line through P has equation $y = mx + k$, then determine the y-intercept of line through Q in terms of m and k.

Solution

For ease of reference, we will call the first line l_1 and the second line l_2.

Let b represent the y-intercept of l_2.

Since l_1 has equation $y = mx + k$, we know that the slope of l_1 is m and the y-intercept is k. Therefore, P is the point $(0, k)$.

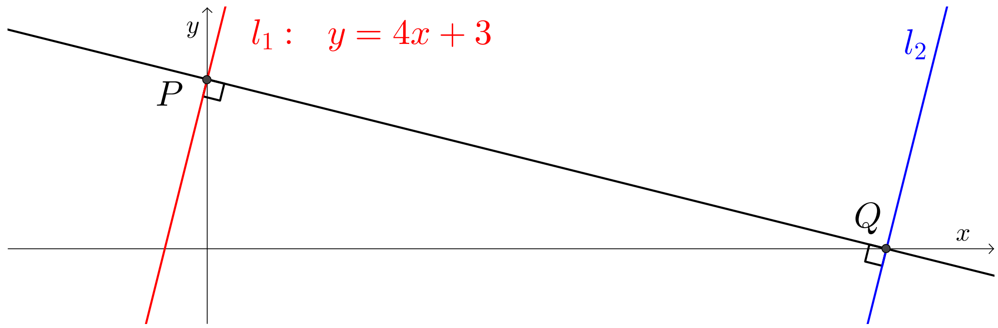
Since PQ is perpendicular to both lines, it follows that l_1 is parallel to l_2. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, $\text{slope}(PQ) = -\frac{1}{m}$. Since k is the y-intercept of the perpendicular segment PQ and the slope of PQ is $-\frac{1}{m}$, the equation of the line through PQ is $y = -\frac{1}{m}x + k$.

To find the x-intercept of $y = -\frac{1}{m}x + k$, set $y = 0$ and solve for x. If $y = 0$, then $0 = -\frac{1}{m}x + k$ and $\frac{1}{m}x = k$. The result $x = mk$ follows. Therefore, the x-intercept of $y = -\frac{1}{m}x + k$ is mk and the coordinates of Q are $(mk, 0)$.

We can now find the y-intercept of l_2 since we know $Q(mk, 0)$ is on l_2 and the slope of l_2 is m. Substituting into the slope-intercept form of the line, $y = mx + b$, we obtain $0 = (m)(mk) + b$ which simplifies to $b = -m^2k$.

Therefore, the y-intercept of l_2, the line through Q, is $-m^2k$.

For the student who solved the problem using $y = 4x + 3$ as the equation of l_1, you should have obtained the answer -48 for the y-intercept of l_2, the line through Q. A full solution to this problem is provided on the next page.



Let l_1 represent the line $y = 4x + 3$. Let l_2 represent the second line, the line through Q.

From the equation of l_1 we know that the slope is 4 and the y-intercept is 3. Therefore P is the point (0,3).

Since $PQ \perp l_1$ and $PQ \perp l_2$, it follows that $l_1 \parallel l_2$. Also, the slope of PQ is the negative reciprocal of the slope of l_1. Therefore, slope$(PQ) = -\frac{1}{4}$. Since 3 is the y-intercept of PQ and the slope of PQ is $-\frac{1}{4}$, the equation of the line through PQ is $y = -\frac{1}{4}x + 3$.

The x-intercepts of the line through perpendicular PQ and the line l_2 are the same since both lines intersect at Q on the x-axis. To find this x-intercept, set $y = 0$ in $y = -\frac{1}{4}x + 3$. Then $0 = -\frac{1}{4}x + 3$ and $\frac{1}{4}x = 3$. The result, $x = 12$, follows. The x-intercept of the line through perpendicular PQ and the line l_2 is 12 and point Q is $(12, 0)$.

We can now find equation of l_2 since $Q(12, 0)$ is on l_2 and the slope of l_2 is 4. Substituting $x = 12$, $y = 0$ and $m = 4$ into $y = mx + b$, we obtain $0 = (4)(12) + b$ which simplifies to $b = -48$. The equation of l_2 is $y = 4x - 48$ and the y-intercept is -48.

This is the same result we obtained from the general solution on the previous page.

CEMC at Home

Grade 9/10 - Friday, April 24, 2020
Colouring Fun

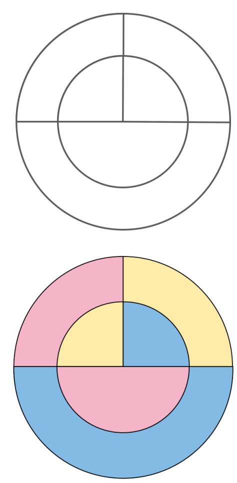
You Will Need:

- Pieces of paper
- Pencil crayons or markers for colouring

Colouring Figures

In the following activities, we will be colouring figures using different colours. Here we outline the properties of a *valid colouring* for the purposes of today's activities:

- All regions in the figure are coloured.
- Any pair of neighbouring regions in the figure (that is, regions sharing an edge or border) are coloured with two different colours.
- Regions that meet only at a single point can be coloured with the same colour.

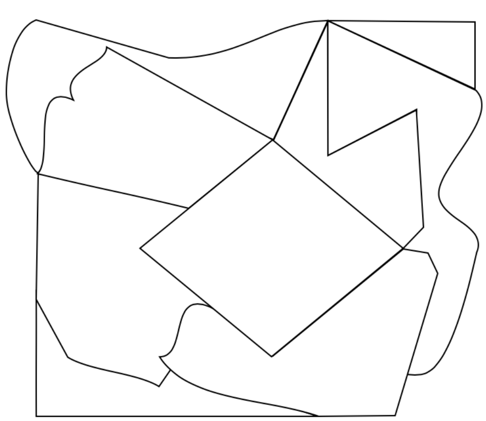


Consider the example shown to the right. Since there are six different regions in the top figure shown, one easy way to produce a valid colouring of the figure is to use six different colours and colour each region using one of these colours. However, the figure can be coloured according to the rules using fewer than six colours. Can you figure out how many colours are actually *needed* to colour this figure?

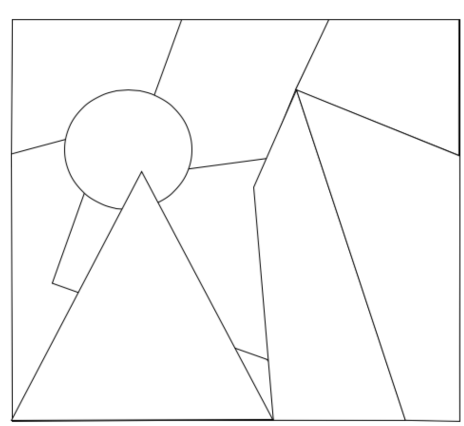
It turns out that there are valid colourings of this figure that only use three different colours, but no valid colouring that uses fewer than three colours. This means that the *minimum* number of colours needed to colour this figure is 3. The bottom image to the right shows one possible valid colouring using three colours. Can you explain why this figure cannot be coloured using only two colours?

Activity 1: Find a valid colouring for each of the figures shown below that uses the fewest colours possible. Can you explain why the figure cannot be coloured using fewer colours than what you have?

(a)



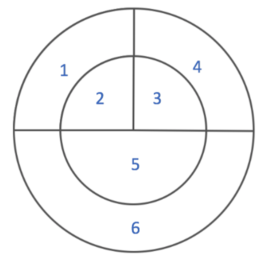
(b)

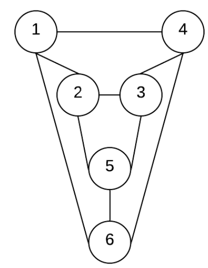


Modelling Figures Using Graphs

As figures become more complex, it can become difficult to see how to colour them using the fewest colours possible. To help with this, we can translate all of the necessary information needed for colouring from the original figure into a simpler figure called a *graph*. In other words, we *model* the figure using a graph. We can then colour the graph instead of the figure, and then translate this colouring back to the figure. Let's investigate this idea by revisiting the circle figure from earlier.

To model a figure using a graph for the purposes of colouring, follow these steps:





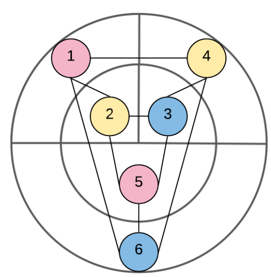
i. Label each region of the figure with a unique positive integer.

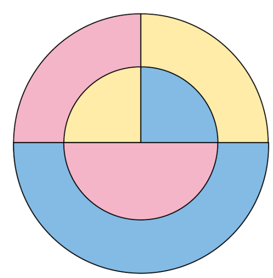
ii. Model the picture from i. using a *graph*. Graphs consist of points (called *vertices*) and lines (called *edges*). Create a point (vertex) in the graph for each region of the picture. We will actually use a circle (as shown) as we would like to include the region number with the point. Next, connect various pairs of vertices using lines (edges). Two vertices should be connected by an edge exactly if they represent neighbouring regions in the original figure.

   For example, since regions 1 and 2 share a border in the figure, there is an edge between vertices 1 and 2 in the graph, and since regions 4 and 5 do not share a border in the figure, there is no edge between vertices 4 and 5 in the graph.

iii. Colour all of the vertices in the graph so that no adjacent vertices (vertices connected by an edge) share the same colour.

iv. Transfer the colours from the vertices of your graph to the corresponding regions in the original figure. This must correspond to a valid colouring of the figure. (Can you see why?)





Since the original goal was to colour figures using the fewest colours possible, the goal when modelling these problems with graphs is to colour graphs using the fewest colours possible.

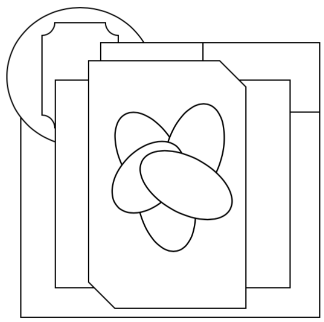
For example, here is the method that was used to colour the graph above: Start with vertex 1, and assign it a colour (pink). Next, identify all of the vertices that are adjacent to this vertex, which in this example are 2, 4 and 6. If possible, colour all of these vertices with the same second colour (yellow). If this is not possible, colour as many as you can with the second colour. In this example, we colour 2 and 4 yellow, but not 6 as it is connected to 4. We now know we need a third colour (blue). It turns out that we can colour all remaining vertices with these three colours.

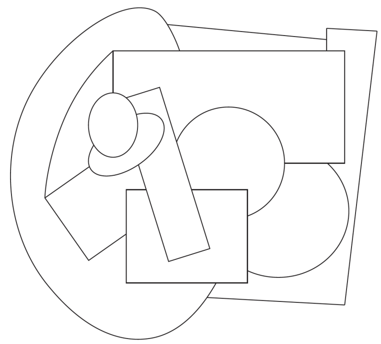
We can also argue that three colours is the best we can do. Notice that we have no choice but to use three different colours for the three vertices 2, 3, and 5. You need two different colours for 2 and 3, but neither of these colours can then be used for 5.

Activity 2:

(a) Consider the two figures shown below. For each figure, do the following:

- Model the figure using a graph as outlined in i. and ii. on the previous page.
- Find a colouring of the graph as outlined in iii. on the previous page that uses the fewest colours possible.
- Explain why the graph cannot be coloured according to the guidelines using fewer colours than what you have.





If you complete these steps, you will have determined the minimum number of colours needed for a valid colouring of each figure. If you would like, you can now colour the original figures as outlined by your graph colourings!

(b) Can you create a two-dimensional figure that requires five colours in order to achieve a valid colouring? Spend some time thinking about whether or not you think this is possible.

Graph Colouring in Action!

Many real-world problems can be translated into graph colouring problems. These problems often involve resources (colours) and conflicts (two regions that cannot be coloured the same), and you are tasked with assigning resources in an optimal way (using the fewest colours possible) while ensuring that no conflicts arise. This type of problem often arises when attempting to make schedules and timetables, and large scale versions are famously difficult to solve!

Check out the problems Timetabling and Aircraft Scheduling from past Beaver Computing Challenges. Read the story for the problem and try to figure out how to model this problem as a graph colouring problem. Ask yourself the following questions:

- What should the vertices represent?
- How do you decide whether or not an edge should be drawn between a particular pair of vertices?
- What do the colours represent?
- Does finding a valid colouring of your graph which uses the fewest colours possible provide a solution to the problem?

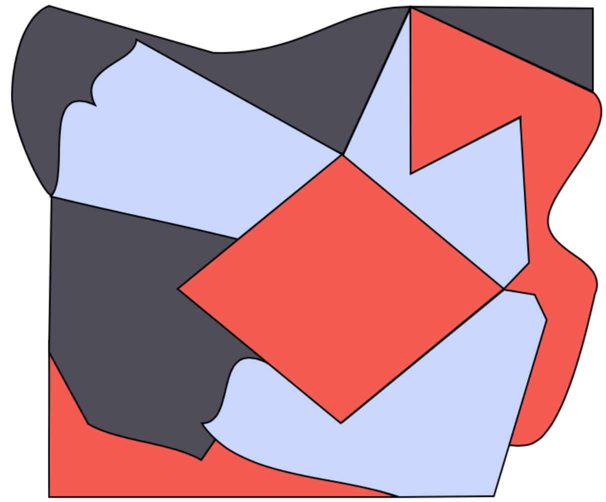
More Info:

Check out the CEMC at Home webpage on Friday, May 1 for a solution to Colouring Fun.

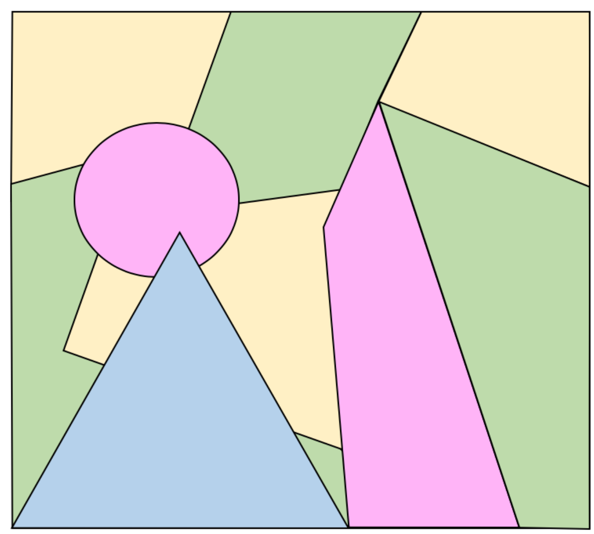
CEMC at Home

Grade 9/10 - Friday, April 24, 2020
Colouring Fun - Solution

1(a) The colouring to the right shows one way this figure can be coloured using three colours. Is there a valid colouring that uses fewer colours? Consider the region in the bottom left corner that is coloured red. This region has exactly two neighbours and these two neighbours are also neighbours of each other. Thus, in a valid colouring, these three regions must all be given different colours. That is, at least three colours are required. Since at least three colours are required and we have found a colouring that uses exactly three colours, the minimum number of colours required to colour this figure is three.



1(b) The colouring to the right shows one way this figure can be coloured using four colours. Is there a valid colouring that uses fewer colours? Consider the following four regions: the circle coloured pink along with the green region to its left, the yellow region to its bottom-left, and the blue region below it. Notice that every pair of regions in this group of four are neighbours. This means that all four of these regions must be given a different colour in a valid colouring. That is, at least four colours are required.
Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this figure is four.



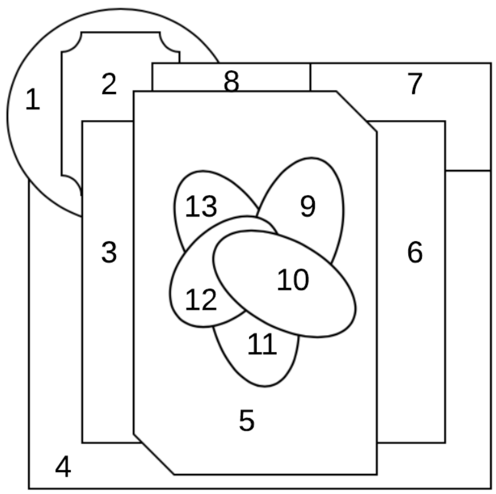
Graph colouring is the assignment of labels, in this case colours, to each vertex of a graph, G, such that no adjacent vertices (i.e. connected by an edge) share the same colour. The goal is to colour G with the minimum number of colours possible. This minimum number is known as the *chromatic number* of G.

There are many applications of graph colouring in the real world. Here are a few examples:

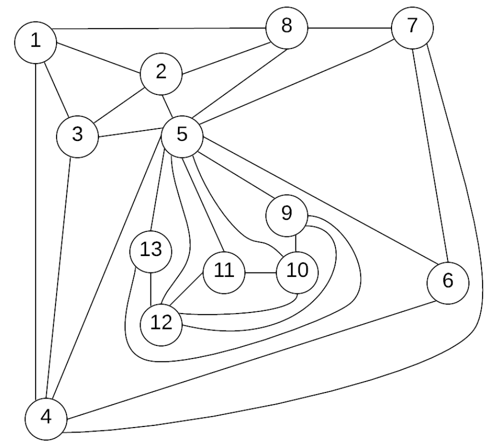
- Scheduling: classes, exams, meetings, sports, flights
- Creating and solving Sudoku puzzles
- Internet bandwidth allocation

2(a) We first work through the activity for the first figure.

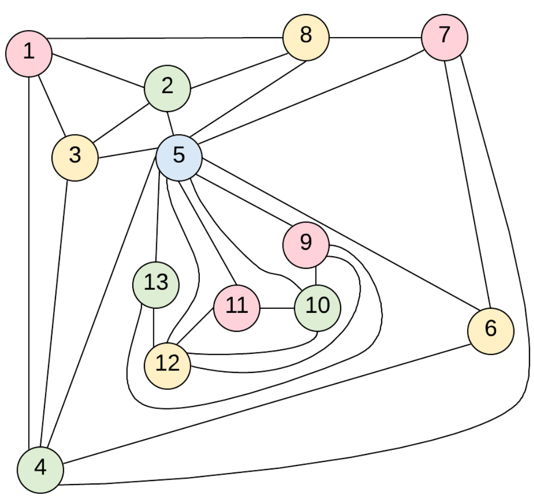
i. Label the regions



ii. Translate into a graph

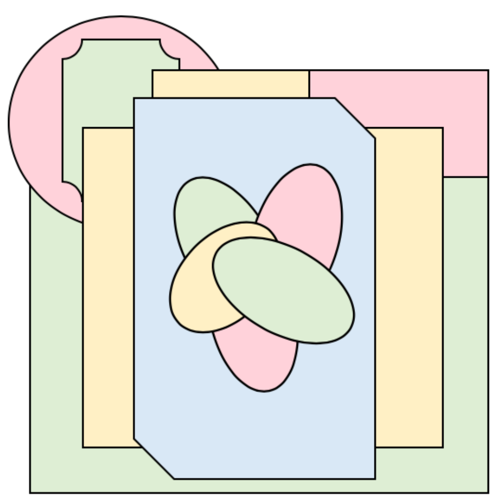


iii. Colour the graph



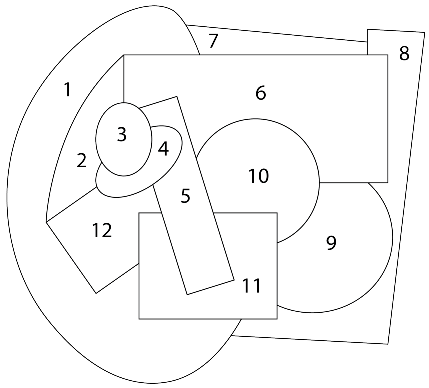
The colouring to the left shows one way this graph can be coloured using four colours. Is there a valid colouring that uses fewer colours? Notice that the group of vertices labelled 5, 10, 11, and 12 has the property that all pairs of vertices from this group are adjacent. Therefore, vertices 5, 10, 11, and 12 all require a different colour in a valid colouring, so at least four colours are required. Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this graph is four. That is, the *chromatic number* of this graph is 4.

Finally, we use translate the colouring of the graph to a colouring of the figure:

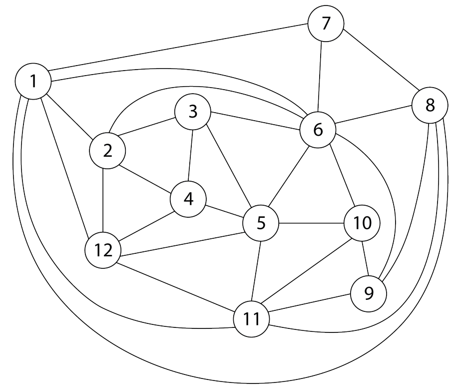


Next we work through the activity for the second figure.

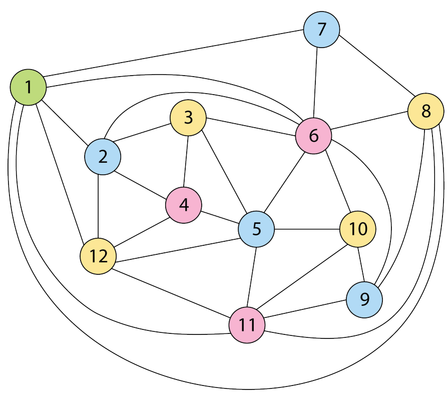
i. Label the regions



ii. Translate into a graph



iii. Colour the graph

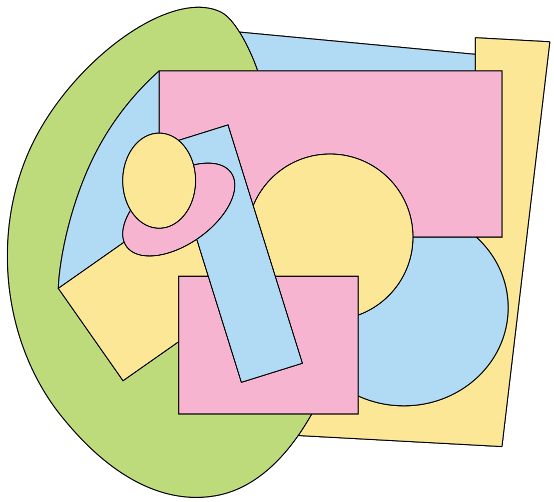


The colouring above shows one way this graph can be coloured using four colours. Is there a valid colouring that uses fewer colours? You can spend a bit of time trying to find a group of four vertices for which each pair of vertices is adjacent, but how do you proceed if you cannot find a group like this? (*Note that vertices 1, 6, 7, and 8 satisfy this property, but we will proceed as though we have not spotted these.*)

Let's try to colour this graph using only three colours: pink, yellow, and blue. Consider the vertex labelled 4 and assign this vertex a colour, say pink. (Note that exactly what colour we choose is not important here.) The vertex 4 is adjacent to the vertices labelled 2, 3, 5, and 12 and so these vertices cannot be coloured pink. Looking at the connections between these four vertices, convince yourself that since only two colours remain, vertices 2 and 5 must be coloured with the same colour, say blue, and vertices 3 and 12 must be coloured with the other colour, yellow. Now consider the vertex numbered 11, which is adjacent to vertex 5 (coloured blue) and vertex 12 (coloured yellow), and so must be coloured pink. Finally, consider the vertex numbered 1. This vertex is adjacent to vertex 2 (coloured blue), vertex 12 (coloured yellow), and vertex 11 (coloured pink). In order to produce a valid colouring, vertex 1 must be assigned a colour, but that colour cannot be pink, yellow, or blue! This means we cannot possibly finish our colouring without adding a fourth colour.

Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this graph is four. That is, the *chromatic number* of this graph is 4.

Finally, we use translate the colouring of the graph to a colouring of the figure:



2(b) It turns out that, as hard as you try, you will not be able to create a two-dimensional figure that requires five colours in order to achieve a valid colouring. This result follows from a famous theorem in graph theory known as the *Four Colour Theorem*. The Four Colour Theorem states that any two-dimensional map, like the figures we have been colouring in this activity, always has a valid colouring with four colours. That is, it can always be coloured with four or fewer colours.

The proof of the Four Colour Theorem is not trivial. In fact, this theorem was first conjectured in 1852, but was not proven until the 1970s, and the proof required the aid of a computer!