CEMC at Home

Grade 7/8 - Monday, April 20, 2020
Placing Polygons

In today's game, we will take turns drawing different shapes on a playing board.

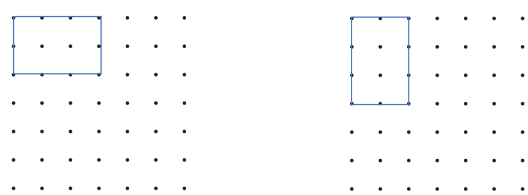
You Will Need:

- Two players
- Many different pieces of dot paper
- A pencil
- An eraser
- A ruler
- A pair of dice

How to Play:

1. Start with a piece of dot paper that is 15 dots by 15 dots (some is provided on the last page).
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. Player 1 starts by rolling the pair of dice. The two positive integers rolled represent the dimensions of a rectangle (width and length, in some order). This player draws a rectangle on the dot paper with the four vertices located at dots, and one vertex located in the top left corner of the grid, that has the appropriate dimensions.

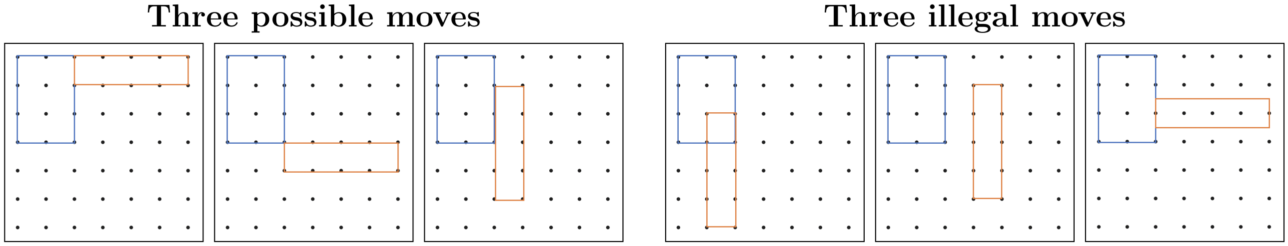
   For example, if a 2 and a 3 are rolled, then there are two options for the rectangle drawn as shown below. Note that 1 unit is the horizontal/vertical distance between two adjacent dots.

4. Player 2 now rolls the dice and draws a rectangle with length and width based on the numbers rolled, and following these rules:
   - The rectangle drawn must have its four vertices located on dots.
   - The rectangle drawn must touch an existing rectangle.
   - The rectangle drawn must not overlap with any existing rectangle.

   For example, various legal (and illegal) moves are shown below for a roll of 1 and 4.

   Three possible moves **Three illegal moves**

5. The two players alternate rolling the dice and drawing shapes following the rules outlined in 4.
6. If at any time during the game the current player cannot make a legal move based on their roll of the dice, then the other player wins the game.

Play this game a number of times and then think about these questions:

- Is there a minimum number of turns that must be played in this game before it can be won?
- Is there a maximum number of turns that can be played before a game must be won?

Variation 1

Play the same game, but with the following variation at the end: Once the game ends (because the current player cannot make a legal move based on their roll of the dice), each player adds up the total area covered by all of the rectangles they drew on their turns. The player with the largest area wins.

You will need to find a way to clearly indicate which rectangles belong to which player as you play. Will your strategy for playing the game change based on this variation?

Variation 2

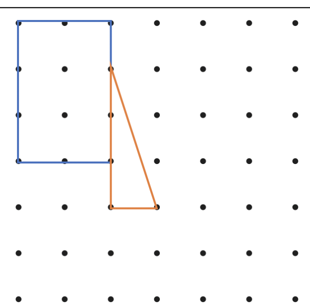
Play another game with the same rules as the original game, but with the following variation:

The two numbers rolled can be interpreted as either

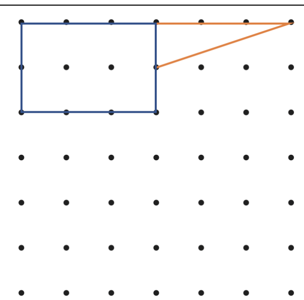
- the base and height of a parallelogram, *or*
- the base and height of a triangle.

On each turn, the current player has the option of drawing any parallelogram or triangle with an appropriate base and height based on their roll of the dice. As before, the shape drawn must have all of its vertices located at dots, and must touch an existing shape on the paper without overlapping with any existing shapes.

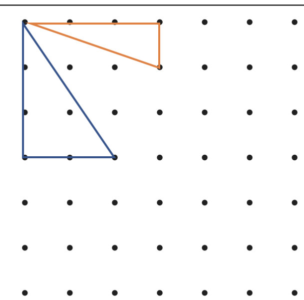
Below are some possibles starts to this variation of the game. Note that you have to be extra careful to draw the shapes accurately in this game, as it will be harder to tell whether or not two shapes are overlapping or just touching. Remember that two triangles (or parallelograms) that have the same base and height can still look very different!



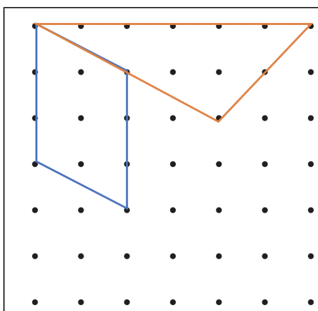
Player 1 rolls 2 and 3
Player 2 rolls 1 and 3



Player 1 rolls 2 and 3
Player 2 rolls 1 and 3



Player 1 rolls 2 and 3
Player 2 rolls 1 and 3

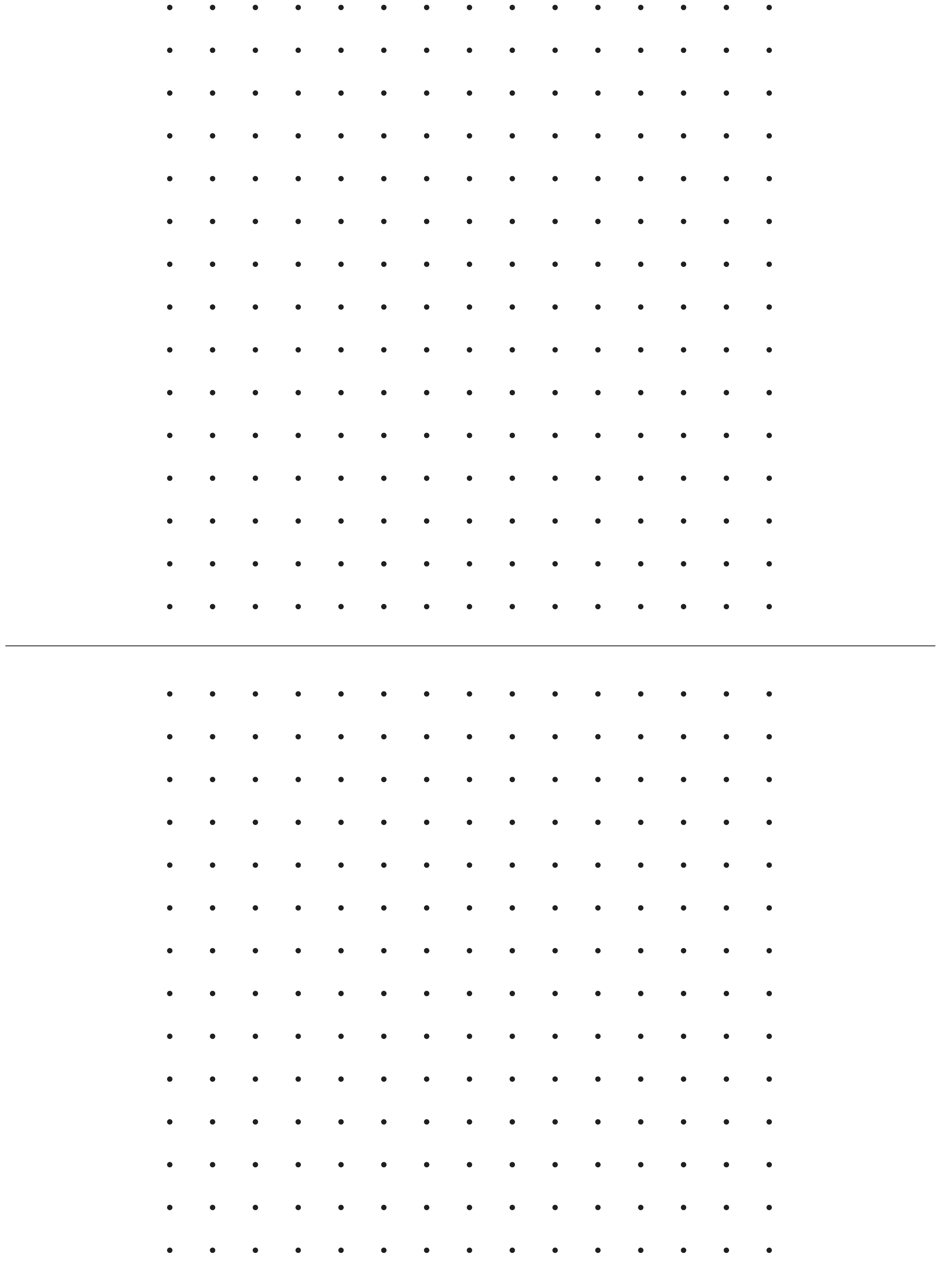


Player 1 rolls 2 and 3
Player 2 rolls 2 and 6

There is a lot of choice involved in this variation of the game. Given a roll of the dice, you get to decide the type of shape (parallelogram or triangle), the type of angles (right angle(s) or not), and the orientation of your shape (the shapes can have up to four different orientations).

More Info:

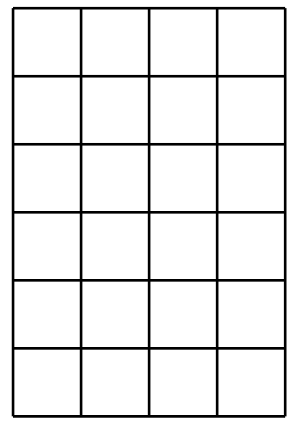
In these games, we played around with different *polygons*. To learn more about polygons, check out this lesson in the CEMC Courseware. You can also view lessons specifically about triangles, or parallelograms and other quadrilaterals.



CEMC at Home

Grade 7/8 - Tuesday, April 21, 2020
Square Doodles

Problem 1: The grid below is divided into 24 small squares. Follow the steps to doodle in some of the squares. Read the steps carefully and do them in the order given. No square can have more than one doodle in it.



1. Draw a smiley face in $\frac{1}{4}$ of the squares.
2. Draw a cloud in $\frac{1}{6}$ of the remaining blank squares.
3. Draw a spiral in $\frac{2}{5}$ of the remaining blank squares.
4. Draw wavy lines in $\frac{1}{3}$ of the remaining blank squares.

How many squares have *not* been doodled in after you have finished?

Problem 2: Shreya started with a different blank grid, and was given similar instructions as in Problem 1. Shreya was given one instruction for drawing each of the same four shapes but *the fractions* in Shreya's instructions may have been different and the *order in which Shreya was asked to draw the shapes* may have been different. Shreya ended up with 6 squares with wavy lines, 4 squares with spirals, 5 squares with clouds, 10 squares with smiley faces, and 5 blank squares.

We cannot be sure of exactly what list of instructions Shreya was given. Can you give one possibility for the list of instructions that Shreya could have been working from?

To start, determine the total number of squares in Shreya's grid.

Problem 3: Aryan was given a grid of blank squares and the following instructions, in order.

1. Draw a cloud in $\frac{1}{4}$ of the squares.
2. Draw a spiral in $\frac{1}{6}$ of the remaining blank squares.
3. Draw a smiley face in $\frac{2}{5}$ of the remaining blank squares.

After following the steps, Aryan ended up with exactly 6 blank squares. How many blank squares could Aryan have started with? Is there more than one possibility?

Can you work "backwards"? You know there were 6 blank squares after the third step was completed. Can you figure out how many blank squares there were right before Aryan started the third step?

More Info:

Check out the CEMC at Home webpage on Wednesday, April 22 for a solution to Square Doodles.

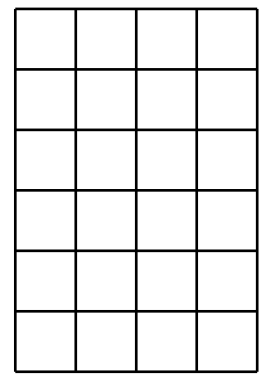
For more practice working with fractions, check out this lesson in the CEMC Courseware.

CEMC at Home

Grade 7/8 - Tuesday, April 21, 2020
Square Doodles - Solution

Problem 1 Summary: Follow the steps below to doodle in some of the squares. How many squares have *not* been doodled in after you have finished?

1. Draw a smiley face in $\frac{1}{4}$ of the squares.
2. Draw a cloud in $\frac{1}{6}$ of the remaining blank squares.
3. Draw a spiral in $\frac{2}{5}$ of the remaining blank squares.
4. Draw wavy lines in $\frac{1}{3}$ of the remaining blank squares.



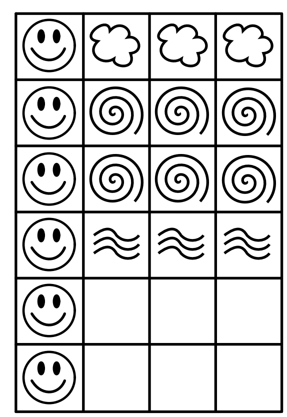
Solution: We will go through each step below. One possibility for the final grid is also shown.

Step 1. Since $\frac{1}{4}$ of 24 is 6, smiley faces are drawn in 6 blank squares.
There are $24 - 6 = 18$ squares left blank after Step 1.

Step 2. Since $\frac{1}{6}$ of 18 is 3, clouds are drawn in 3 blank squares.
There are $18 - 3 = 15$ squares left blank after Step 2.

Step 3. Since $\frac{2}{5}$ of 15 is 6, spirals are drawn in 6 blank squares.
There are $15 - 6 = 9$ squares left blank after Step 3.

Step 4. Since $\frac{1}{3}$ of 9 is 3, wavy lines are drawn in 3 blank squares.
There are $9 - 3 = 6$ squares left blank after Step 4.



Therefore, 6 squares have not been doodled in after finishing these steps.

Problem 2 Summary: Shreya started with a different blank grid, and was given four instructions as in Problem 1, but possibly with a different order of shapes drawn, and different fractions for the shapes. Shreya ended up with 6 squares with wavy lines, 4 squares with spirals, 5 squares with clouds, 10 squares with smiley faces, and 5 blank squares. Can you give one possibility for the list of instructions that Shreya could have been working from?

Solution: We know each square in Shreya's grid must end up either blank or with one of the four shapes doodled in it. Since $6 + 4 + 5 + 10 + 5 = 30$, we determine that Shreya's grid must have had 30 squares in total.

Let's try and find possible instructions Shreya could have been working from. Let's assume that she was asked to draw the shapes in the order they were listed in the question: 6 wavy lines, 4 spirals, 5 clouds, and then 10 smiley faces, and see if we can find the correct fractions.

- *Shreya drew 6 wavy lines:* Notice that 6 is $\frac{1}{5}$ of 30. If Shreya drew 6 wavy lines during Step 1, then the instruction must have been "Draw wavy lines in $\frac{1}{5}$ of the squares."
  After this step there would be $30 - 6 = 24$ remaining blank squares.
- *Shreya drew 4 spirals:* Notice that 4 is $\frac{1}{6}$ of 24. If Shreya drew 4 spirals during Step 2, then the instruction must have been "Draw spirals in $\frac{1}{6}$ of the remaining blank squares."
  After this step there would be $24 - 4 = 20$ remaining blank squares.
- *Shreya drew 5 clouds:* Notice that 5 is $\frac{1}{4}$ of 20. If Shreya drew 5 clouds during Step 3, then the instruction must have been "Draw clouds in $\frac{1}{4}$ of the remaining blank squares."
  After this step, there would be $20 - 5 = 15$ remaining blank squares.
- *Shreya drew 10 smiley faces:* Notice that 10 is $\frac{2}{3}$ of 15. If Shreya drew 10 smiley faces during Step 4, then the instruction must have been "Draw smiley faces in $\frac{2}{3}$ of the remaining blank squares."
  After this step, there would be $15 - 10 = 5$ remaining blank squares, as expected.

We have found one set of instructions that Shreya could have been working from, however there are many other possibilities. If you assume Shreya drew the shapes in a different order, then you will get a different solution if you follow similar steps above.

There are 24 different correct answers to this problem, because there are 24 different orders in which Shreya could have been asked to draw the shapes. Depending on what order you choose, you may not get fractions that are as nice as the ones in our work above.

Problem 3 Summary: Aryan was given a grid of blank squares and the following instructions and Aryan ended up with exactly 6 blank squares at the end. How many blank squares could Aryan have started with?

1. Draw a cloud in $\frac{1}{4}$ of the squares.
2. Draw a spiral in $\frac{1}{6}$ of the remaining blank squares.
3. Draw a smiley face in $\frac{2}{5}$ of the remaining blank squares.

Solution: It turns out that there is only one possibility for the number of squares in Aryan's grid. To find this number, we will work backwards, starting with what must have happened during Step 3.

During Step 3, Aryan drew smiley faces in $\frac{2}{5}$ of the blank squares and so left the other $\frac{3}{5}$ of these squares blank. We know that exactly 6 squares were still blank at the end of this step. Since 6 is $\frac{3}{5}$ of 10, Aryan's grid must have had 10 blank squares right before Step 3 began.

Using similar reasoning, we can determine that Aryan's grid must have had 12 blank squares right before Step 2 began (since 10 is $\frac{5}{6}$ of 12), and 16 blank squares right before Step 1 began (since 12 is $\frac{3}{4}$ of 16). Try to work out these details for yourself.

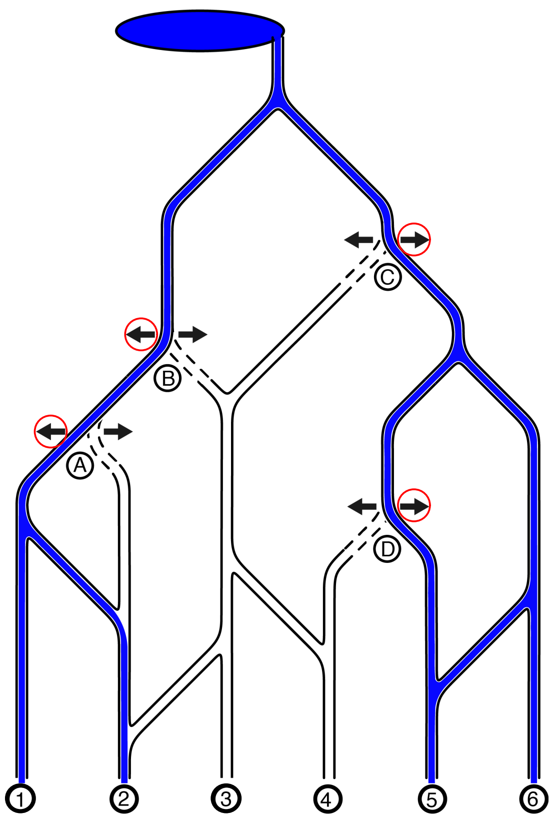
Therefore, Aryan must have started with a grid of 16 blank squares.

CEMC at Home

Grade 7/8 - Wednesday, April 22, 2020
Go with the Flow

Martin has created an irrigation system to water the fields in his farm. The water flows from a lake at the top of the hill all the way down to six fields numbered 1 to 6 at the bottom. Along the water canals, Martin has installed four water gates (A, B, C, and D), where he can direct the water to flow either to the left or to the right, but not in both directions.

An example showing how these gates can be set to have the water flow to fields 1, 2, 5, and 6 is shown below.



Problem 1: Explain how Martin can set the water gates so that water flows to fields 2, 3, and 4.

Want to check your answer? Use this online exploration to set each gate and see if you are correct.

Problem 2: Martin wants to set the gates so that water flows to fields 2, 3, 5, and 6.

(a) Explain why this is not possible based on how the farm is currently set up.

(b) Explain how the water canals in the farm can be adjusted in order to make this possible.

i. Can you achieve this by removing one existing canal from the irrigation system?

ii. Can you achieve this by adding one new canal to the irrigation system?

Do these changes affect your solution to Problem 1?

More Info:

Check out the CEMC at Home webpage on Thursday, April 24 for solutions to Go with the Flow.

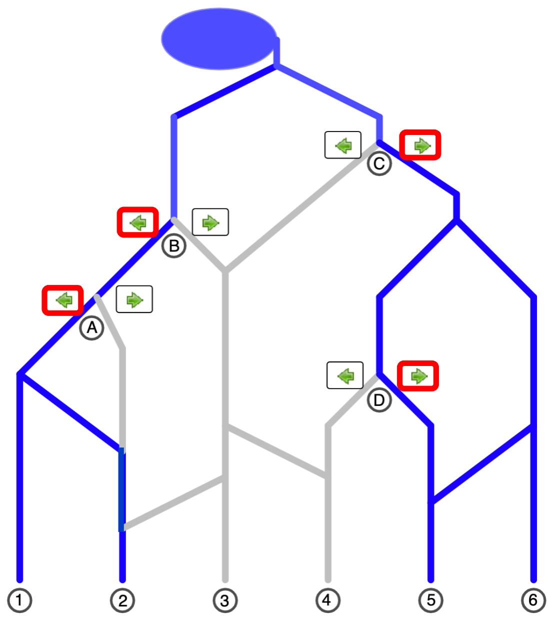
A variation of this problem appeared on a past Beaver Computing Challenge (BCC). The BCC is a problem solving contest with a focus on computational and logical thinking.

CEMC at Home

Grade 7/8 - Wednesday, April 22, 2020
Go with the Flow - Solutions

Martin has created an irrigation system to water the fields in his farm. The water flows from a lake at the top of the hill all the way down to six fields numbered 1 to 6 at the bottom. Along the water canals, Martin has installed four water gates (A, B, C, and D), where he can direct the water to flow either to the left or to the right, but not in both directions.

An example showing how these gates can be set to have the water flow to fields 1, 2, 5, and 6 is shown to the right.



Problem 1: Explain how Martin can set the water gates so that water flows to fields 2, 3, and 4.

Solution: Notice that we do not want water to flow to field 6. That tells us gate C must be set to the left. Setting gate C to the left means it does not matter which direction we set gate D because no water will be flowing through gate D. Gate B can also be set either to the left or to the right. If gate B is set to the left, then gate A must be set to the right because we do not want water to flow to field 1. If gate B is set to the right, then it does not matter how we set gate A.

The tables below give a summary of all possible ways Martin could set the water gates so that water flows to fields 2, 3, and 4. You should check for yourself that each of these settings achieves the result.

Gate A	Gate B	Gate C	Gate D
Left or Right	Right	Left	Left or Right

Gate A	Gate B	Gate C	Gate D
Right	Left	Left	Left or Right

Problem 2: Martin wants to set the gates so that water flows to fields 2, 3, 5, and 6.

(a) Explain why this is not possible based on how the farm is currently set up.

(b) Explain how the water canals in the farm can be adjusted in order to make this possible.

  i. Can you achieve this by removing one existing canal from the irrigation system?

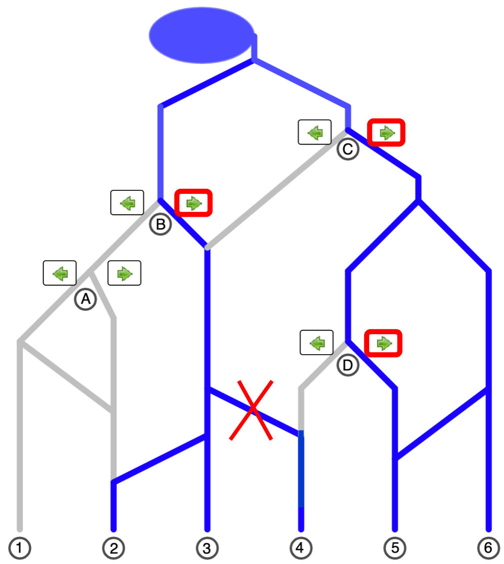
  ii. Can you achieve this by adding one new canal to the irrigation system?

Solution:

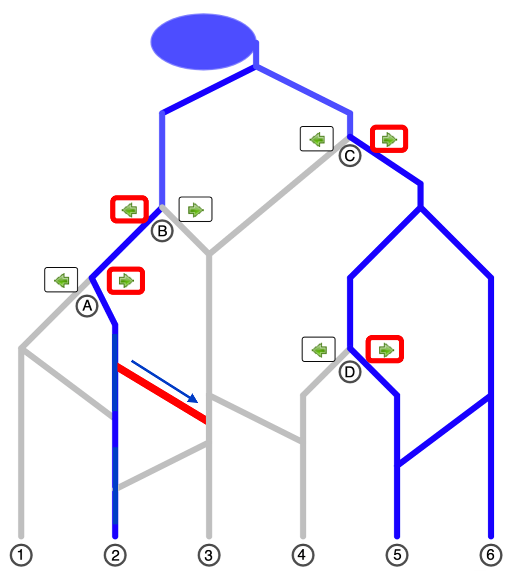
(a) To get water to field 6, gate C must be set to the right. With gate C set to the right, we have no choice but to set gate D to the right, to ensure that water does not flow to field 4. With gates C *and* D set to the right, we have no choice but to set gate B to the right as well to ensure water gets to field 3. However, setting gate B to the right will also result in water flowing into field 4. Therefore, it is not possible to have water flow to only fields 2, 3, 5, and 6.

In fact, you might have noticed from the diagram that whenever water flows into field 3, it must also flow into field 4. This means it is impossible to achieve the goal in Problem 2.

(b) i. By removing the canal shown below, we can have water flow into fields 2, 3, 5, and 6. Removing this canal would allow water to flow into field 3 without also flowing into field 4. This was the problem we had with the original canal setup. Following the logic in the solution to part (a), we set gates B, C, and D to the right, but with this canal removed, we will now get water flowing into field 3 and *not* field 4. (It does not matter which direction we set for gate A because no water will be flowing through it.) After doing this, we will have water flowing to only fields 2, 3, 5, and 6, as desired.



ii. By adding the canal shown below, we can have water flow into fields 2, 3, 5, and 6. Adding this canal would allow water to flow into field 3 without also flowing into field 4. In this case, we should set gate A to the right, gate B to the left, and gates C and D to the right. After doing this, we will have water flowing to only fields 2, 3, 5, and 6, as desired.

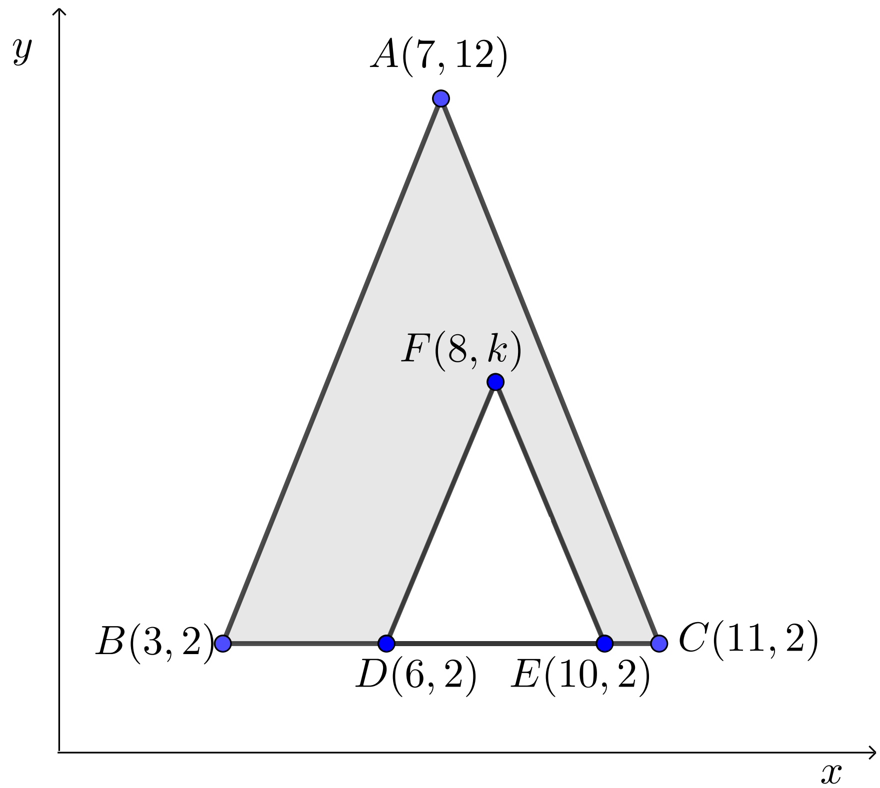


CEMC at Home features Problem of the Week

Grade 7/8 - Thursday, April 23, 2020
Up to a Certain Point

Points $A(7, 12)$, $B(3, 2)$, $C(11, 2)$, $D(6, 2)$ and $E(10, 2)$ are placed on the Cartesian plane, as shown below. The point F is placed inside $\triangle ABC$ so that the area of the shaded region is 32 units^2.

If the x-coordinate of F is 8, what is the y-coordinate of F?



More Info:

Check the CEMC at Home webpage on Thursday, April 30 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 30.

This CEMC at Home resource is the current grade 7/8 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

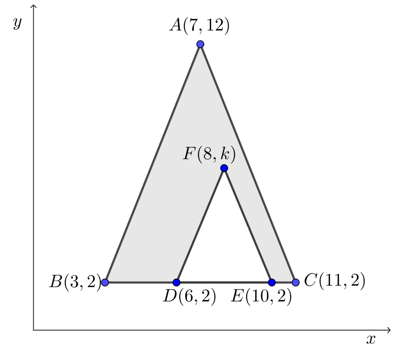
Problem of the Week

Problem C and Solution

Up to a Certain Point

Problem

Points $A(7,12)$, $B(3,2)$, $C(11,2)$, $D(6,2)$ and $E(10,2)$ are placed on the Cartesian plane, as shown to the right. The point F is placed inside $\triangle ABC$ so that the area of the shaded region is 32 units^2. If the x-coordinate of F is 8, what is the y-coordinate of F?

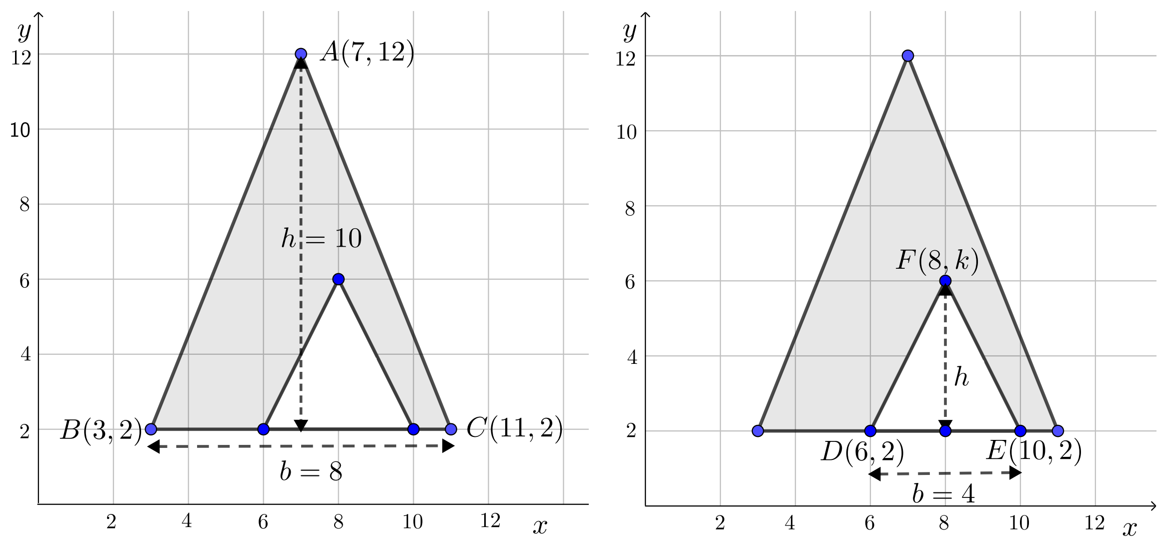


Solution

We will solve this problem in two ways. The first solution uses grid lines, the second solution does not. In both solutions, to find the area of the shaded region we will take the area of the large triangle, $\triangle ABC$, subtract the area of the small triangle, $\triangle DEF$, and then use the given information that this area is equal to 32 units^2.

Solution 1

To determine the height (h) and base (b) of each triangle, we use grid paper:



The area of $\triangle ABC = \frac{b \times h}{2} = \frac{8 \times 10}{2} = 40$ (see above left).

The area of $\triangle DEF = \frac{b \times h}{2} = \frac{4 \times h}{2} = 2h$ (see above right).

Therefore, the area of the shaded region is $40 - 2h$, which is also equal to 32. So $2h$ must equal 8, and therefore $h = 4$.

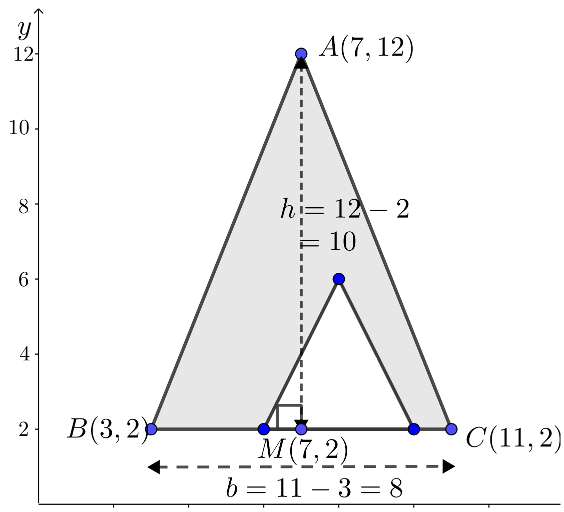
Now point F is $h = 4$ units higher than the base of the triangle, which is 2 units above the x-axis.

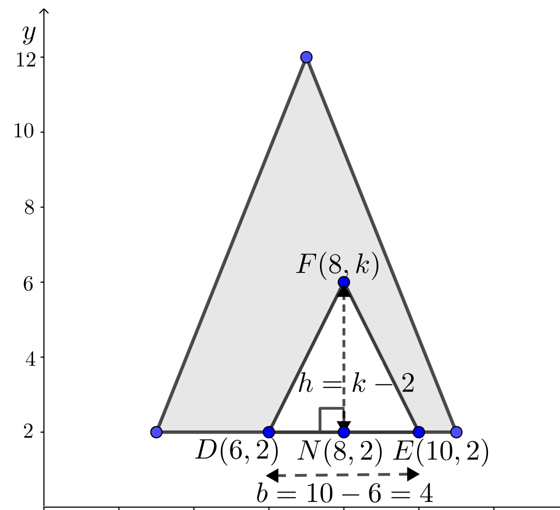
Therefore, the y-coordinate of point F is $2 + 4 = 6$.

Solution 2

In this solution, we use the fact that the distance between two points that have the same x-coordinate is the positive difference between their y-coordinates. We will also use the fact that the distance between two points that have the same y-coordinate is the positive difference between their x-coordinates.

In $\triangle ABC$, drop a perpendicular from vertex A to M on BC. Since BC is horizontal, then AM is vertical. Since every point on a vertical line has the same x-value, M has x-coordinate 7. Similarly, since M is on the horizontal line through $B(3,2)$ and $C(11,2)$, M has y-coordinate 2. Therefore, the base of $\triangle ABC$ is $b = 11 - 3 = 8$ and the height is $h = 12 - 2 = 10$. Now, the area of $\triangle ABC = \frac{b \times h}{2} = \frac{8 \times 10}{2} = 40$ (see below left).





In $\triangle DEF$, drop a perpendicular from vertex F to N on DE. Since DE is horizontal, then FN is vertical. Since every point on a vertical line has the same x-value, N has x-coordinate 8. Similarly, since N is on the horizontal line through $D(6,2)$ and $E(10,2)$, N has y-coordinate 2. Therefore, the base of $\triangle DEF$ is $b = 10 - 6 = 4$ and the height is $h = k - 2$. Now, the area of $\triangle DEF = \frac{b \times h}{2} = \frac{4 \times (k-2)}{2} = 2(k-2)$ (see above right).

We can now solve for k:

$$\begin{array}{rrcl} & 40 - 2(k-2) & = & 32 \\ \text{Subtracting 32 from each side:} & 8 - 2(k-2) & = & 0 \\ \text{Adding } 2(k-2) \text{ to each side:} & 8 & = & 2(k-2) \\ & 4 & = & k-2 \\ & 6 & = & k \end{array}$$

Therefore, the y-coordinate of point F is 6.

CEMC at Home

Grade 7/8 - Friday, April 24, 2020
Order Those Volumes

How can we compare the sizes of three-dimensional objects?

You Will Need:

- A ruler or measuring tape
- Some paper
- A pen or pencil

Introduction

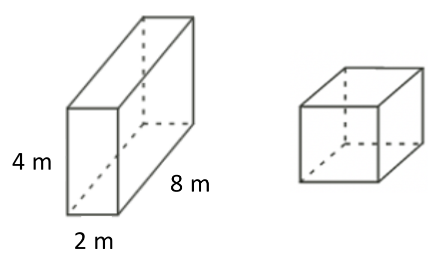
The volume of an object is a measurement of how much space the object takes up. A long time ago, the metric system was proposed and from this system, we gained a basic volume unit: 1 m^3. (Do you know how long ago this system was proposed and first adopted?) This is the space taken up by a cube that has edge lengths of 1 m. If you have a ruler or measuring tape, then you can get a pretty good idea of how much space this is.

Having a standard system of measurement like this allows us to compare volumes of many three-dimensional objects of different shapes.

Comparing rectangular prisms

Since we have a formula for calculating the volume of a rectangular prism, we can compare the volumes of any two rectangular prisms by measuring their dimensions. Remember that a rectangular prism with length ℓ, width w, and height h has volume $V = \ell \times w \times h$.

Problem 1: The rectangular prism and the cube shown below have equal volumes. What is the length of each edge of the cube?



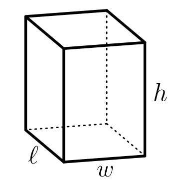
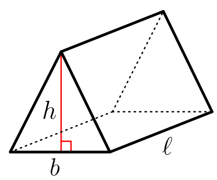
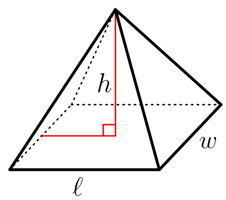
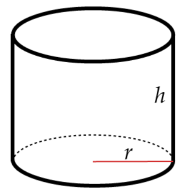
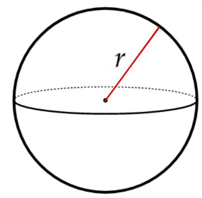
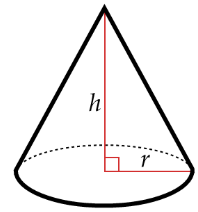
Activity 1: Find five objects that are all rectangular prisms and appear to be reasonably close in size. Order these objects on a table from smallest volume to largest volume, based only on a visual assessment. Once you have done this, get out your ruler or measuring tape and measure the dimensions of each prism. Calculate the approximate volume of each prism based on your measurements and check if your order was correct!

Rectangular Prism	Dimensions	Volume
1		
2		
3		
4		
5		

Comparing other familiar objects

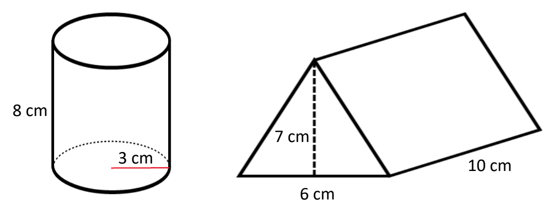
What other volume formulas do you know? Do you know formulas for calculating the volume of some of the three-dimensional objects that we explored last week in the activity Cut It Out? Below are several volume formulas for objects that may or may not be familiar to you.

If you would like to learn more about these objects or formulas, see the links at the bottom of this page.

Rectangular Prism	**Triangular Prism**	**Rectangular Pyramid**	**Cylinder**	**Sphere**	**Cone**
					
$V = \ell \times w \times h$	$V = \dfrac{b \times h \times \ell}{2}$	$V = \dfrac{\ell \times w \times h}{3}$	$V = \pi(r \times r \times h)$	$V = \dfrac{4\pi(r \times r \times r)}{3}$	$V = \dfrac{\pi(r \times r \times h)}{3}$

Some of these volume formulas involve the number π (pi). The number π is a decimal number that is non-terminating and non-repeating. If you have not worked with π before, then for the purposes of this activity, you can replace π in each formula with the number 3.14. This is an approximation of the value of π.

Problem 2: A cylinder has radius 3 cm and height 8 cm. A triangular prism has triangular face with base 6 cm and height 7 cm and has a length of 10 cm. Which object has a larger volume?



Activity 2: Try to find a rectangular prism, a triangular prism, a cylinder, a sphere, and one other type of object in the table above (pyramid or cone). Ideally, the five objects are all reasonably close in size. Order these objects on a table from smallest volume to largest volume, based only on a visual assessment. Once you have done this, get out your ruler or measuring tape and try to measure the necessary dimensions of each object. Calculate the approximate volume of each object based on your measurements and check if your order was correct!

Did you gather any objects for which you cannot easily measure the necessary dimensions? Can you approximate the volumes of your five objects well enough to be sure of the correct order of their volumes? Why or why not?

Object	Dimensions	Volume
1		
2		
3		
4		
5		

Comparing volumes without measuring dimensions

Think about an activity where you are asked to gather five objects of *any* shape and compare their volumes. What are some ways that you might accurately compare volumes of objects that have irregular shapes, dimensions that cannot be easily measured, or for which you do not have a volume formula?

More Info: For more on the solids from Activity 2, check out the CEMC Courseware to learn about volumes of prisms, cylinders, pyramids and cones, and spheres.