CEMC at Home

Grade 4/5/6 - Monday, April 20, 2020

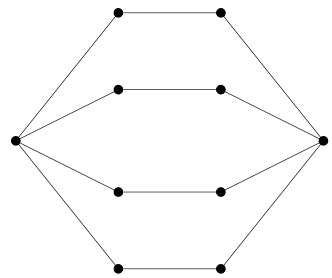
Tag, and That's It!

In this activity, we will play a game of tag on a graph!

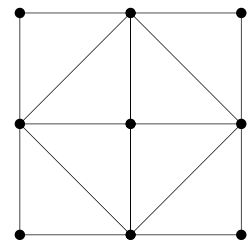
You Will Need:

- Two players
- A piece of paper and a pencil
- Two counters

  A different small object for each player.



Board 1



Board 2

How to Play:

1. Choose one of the two game boards shown above (Board 1 or Board 2) for the game. Notice that each board consists of dots and line segments drawn between certain pairs of dots.

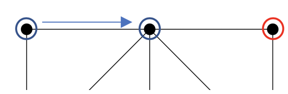
   Larger versions of these game boards are provided on the next page.

2. Players alternate turns. Decide which player will go first (Player 1) and which player will go second (Player 2). Just like a game of tag, Player 1 is "it", and Player 2 must avoid being caught by Player 1.

3. On the first turn, Player 1 puts their counter on any dot they wish. Next, Player 2 puts their counter on any *other* dot on the game board.

4. Next, Player 1 can move their counter from their current dot to another dot by following a single line segment on the game board. Player 1 can also choose to "pass", and not move their counter at all. Player 2 then moves according to the same rules.

   For example, on Board 2, a player can move from the top left dot to the top middle dot on a single turn, but cannot move from the top left dot to the top right dot, because that means moving across two line segments.

5. On all remaining turns, Player 1 and Player 2 take turns moving their counter following the rules outlined in 4. At all times, Player 1 is trying to catch Player 2, and Player 2 is trying to stay away from Player 1.

6. Player 1 can "catch" Player 2 by occupying the same dot as Player 2. If this happens, then Player 1 wins. If Player 1 is unable to catch Player 2 and gives up, then Player 2 wins.

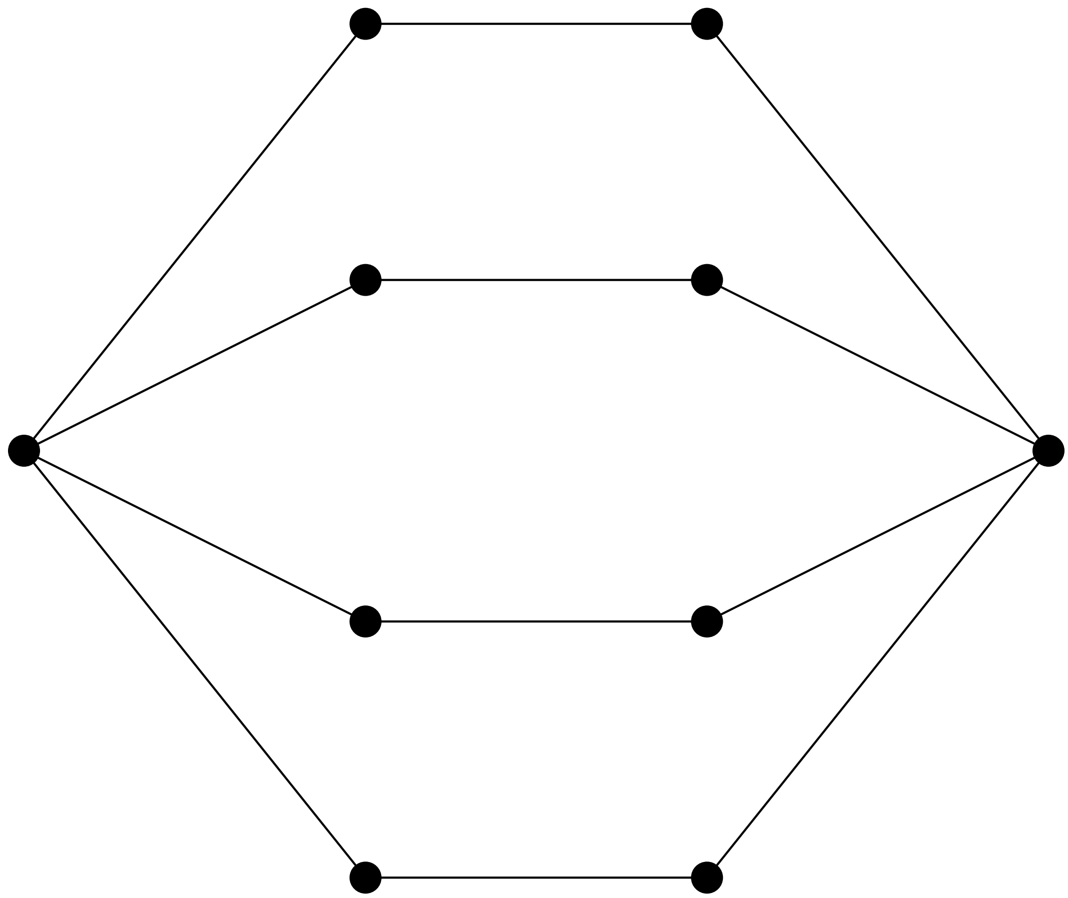
Play this game a number of times using each of the game boards (Board 1 and Board 2). Alternate who goes first and who goes second. As you play, think about the following questions:

- Who seems to win most often: Player 1 or Player 2? For each of the game boards (Board 1 and Board 2), can you come up with a strategy that will allow you to win every time?
- The game boards for this game are called *graphs*. A graph is made up of dots (called *vertices*), along with lines (called *edges*) that connect certain pairs of vertices. Can you build a new game board (or graph) which gives Player 1 an advantage in the game? What about Player 2?

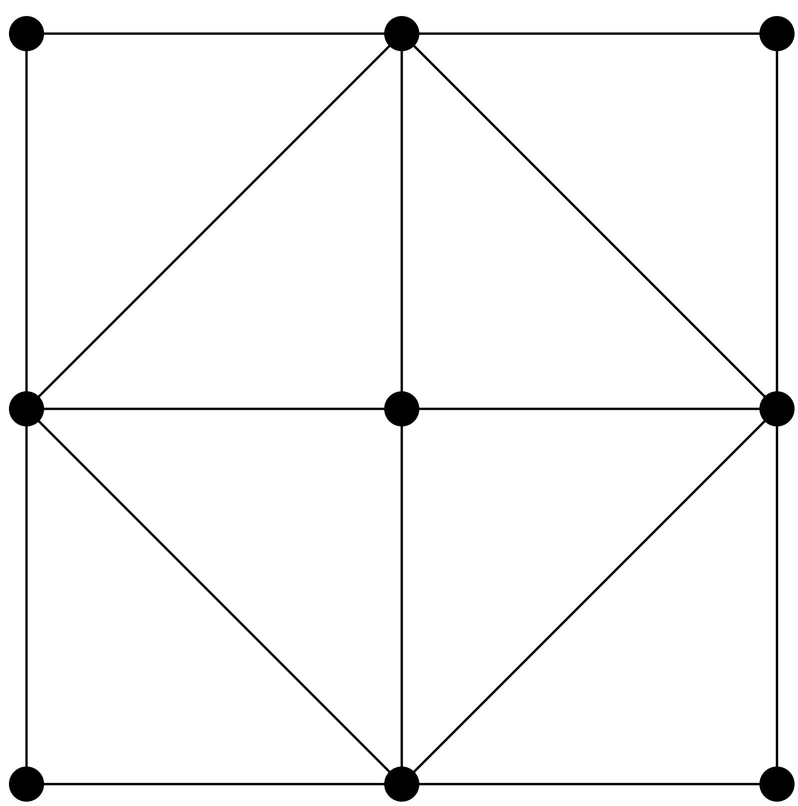
More Info:

Check the CEMC at Home webpage on Monday, April 27 for a discussion of Tag, and That's It!.

Board 1



Board 2



CEMC at Home

Grade 4/5/6 - Monday, April 20, 2020
Tag, and That's It! - Solution

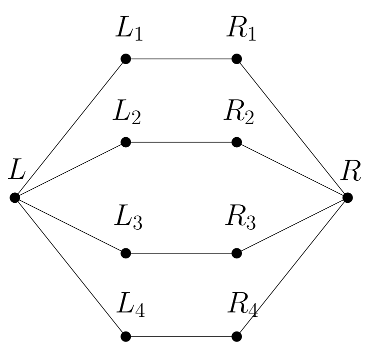
You may have noticed while playing the game that if you play on Board 1, then there is always a way for Player 2 to avoid Player 1 for the entire game. Also, if you play on Board 2, then there is a way for Player 1 to catch Player 2 at some point in the game. We will explain the strategy for each board.

How can Player 2 avoid getting caught by Player 1 on Board 1?

Player 1 has ten different choices for where to place their counter at the start of the game. Player 2 should then place their counter as far as possible from Player 1's counter. This means placing their counter so that it is three line segments away from Player 1's counter.

Can you see why? You can get from one dot to any other dot by crossing at most 3 line segments.

For example, if Player 1 places their counter at R, then Player 2 can place their counter at L. One way for Player 2 to place their counter (based on Player 1's placement) is given in the table below.



Player 1's Vertex	Vertex Player 2 Should Choose
R	L
R_1	L_2
R_2	L_1
R_3	L_4
R_4	L_3
L_1	R_2
L_2	R_1
L_3	R_4
L_4	R_3
L	R

The table above also tells Player 2 how to move in order to stay away from Player 1 throughout the game. For example, if Player 1 moves from R to R_1 on their first move of the game, then Player 2 should move from L to L_2. We see that R_1 and L_2 are paired in the second row in the table above and so this is the right move for Player 2.

Notice that after these two moves, Player 2 is again three line segments away from Player 1.

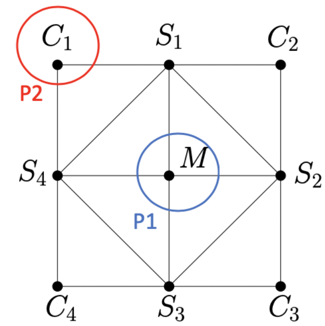
Since Player 2 can always make sure to be three line segments away from Player 1 (at the end of Player 2's turn), Player 1 can never catch Player 2 if Player 2 uses this strategy!

See the next page for a discussion of Board 2.

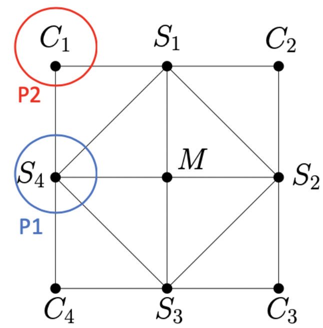
How can Player 1 catch Player 2 on Board 2?

One strategy is for Player 1 to start by placing their counter on the middle dot, marked with M for "middle".

Once Player 1 is occupying M, the only safe move for Player 2 is to occupy one of the corner dots, labelled C_1, C_2, C_3 and C_4. (C is for "corner"). If Player 2 starts anywhere else, then Player 1 can catch Player 2 on the first move. Let's say Player 2 places their counter on C_1 as shown in the diagram on the right.



From M, Player 1 can then move to one of the two dots connected to C_1, labelled S_1 and S_4 in the diagram to the right. Let's say Player 1 moves to S_4.

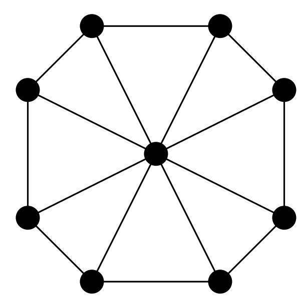


Player 2 now has the choice to pass or to move to S_1. No matter how Player 2 moves, Player 1 can catch Player 2 on the next move and win the game.

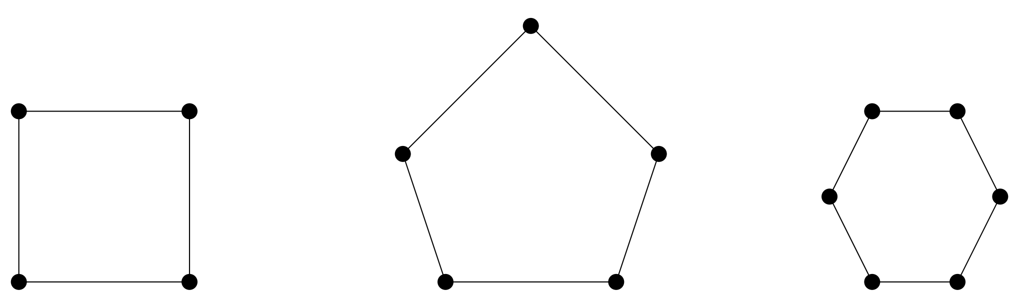
(The strategy is similar if Player 2 moves to S_1 instead, or chooses an entirely different corner to start the game.)

Can you build a new game board (or graph) which gives Player 1 an advantage in the game? What about Player 2?

One easy way to guarantee Player 1 can always win is to include a *universal vertex* in the graph. This is a vertex that is connected to every other vertex by an edge. The "wheel" game board shown on the right gives an example where the vertex in the centre of the wheel is the universal vertex. If Player 1 occupies the universal vertex, they can catch Player 2 on the next turn, no matter where Player 2 goes.



On the other hand, if the game board is a *cycle* of four or more vertices, then Player 2 will always be able to win. Examples of cycles of 4, 5, and 6 vertices are given below to illustrate. The idea is that Player 2 should always stay on the "opposite side" of the cycle from Player 1, keeping as many vertices between the two players as possible. When the cycle has at least 4 vertices, Player 2 will always keep at least two vertices away from Player 1, and so Player 1 will have to give up.



What other examples can you come up with?

CEMC at Home

Grade 4/5/6 - Tuesday, April 21, 2020
Four Square and Ten

Problems:

(a) Using only the four digits 1, 2, 3, and 4, place a digit in each blank square in the grid to the right so that every row, column, *and* diagonal in the grid uses each of the four digits exactly once.

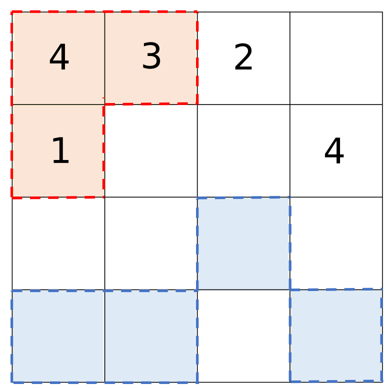
4	3	2	
1			4

(b) Notice that when you add up all 16 digits in your solution grid from part (a), you get a total of 40. In other words, the *sum* of all of the digits in the grid is 40.

Since $40 = 5 \times 8$, it may be possible to divide your solution grid into 5 groups of squares for which the sum of the digits in each group of squares is 8. Can you divide your solution grid from part (a) into 5 groups of squares so that

– the sum of the digits in the squares in each group is 8, *and*
– the squares in each group are *connected*?

A group of squares is connected if each square in the group is touching another square in the group. Squares can touch by sharing an edge or by sharing a vertex as shown in the picture to the right.



Extra Challenges:

1. Can you find more than one solution for part (b)? How many different solutions can you find?
2. Notice that the digits in each row, column, and diagonal in your solution grid from part (a) add to 10. Complete the starting grid from part (a) again, using only the four digits 1, 2, 3, 4, but following these new rules:
   - Each digit can be used more than once in each row, column, or diagonal.
   - The digits in each row, column, and diagonal in your grid must add to 10.

   In how many different ways can you complete the starting grid from part (a) according to these new rules? Are there more possible solutions here than there were for part (a)?

More info:

Check out the CEMC at Home webpage on Tuesday, April 28 for a solution to Four Square and Ten.

CEMC at Home

Grade 4/5/6 - Tuesday, April 21, 2020
Four Square and Ten - Solution

(a) Choosing the correct digits for the blank squares can be done following the steps below. Which squares are filled in during each step is indicated in the diagram by placing the corresponding step numbers in the upper left corner of the squares. *Note that there is only one way to complete the grid following the rules.*

Step 1. Since each row must use each of the digits 1, 2, 3, and 4 exactly once, the last digit in the first row must be 1.

Step 2. The middle two digits of the second row must be 2 and 3, in some order, since 1 and 4 already appear in this row. Since each column also needs to use each of the four digits exactly once, by looking at the first row, we see that 2 must be placed in the second column and 3 must be placed in the third column.

Step 3. The diagonal from lower left to upper right already has a 1 and a 3, so the digit in the third row and second column must be either a 2 or a 4. Since there is already a 2 in the second column, we must place a 4 in this square.

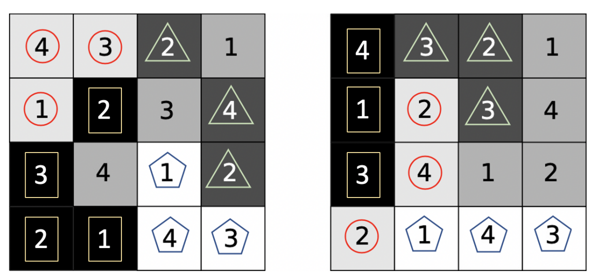
4	3	2	1. 1
1	2. 2	2. 3	4
4. 3	3. 4	5. 1	5. 2
4. 2	4. 1	5. 4	5. 3

Step 4. We now know that the remaining square in the diagonal we worked with in Step 3. must be filled with a 2. This means that the square directly above this 2 must be filled with a 3 (to complete the first column) and the square to the right must be filled with a 1 (to complete the second column).

Step 5. The square in the third row and third column must be filled with a 1 since the third column already has a 2 and a 3, and the third row already has a 4. Now, the only way to complete the diagonal from top left to bottom right is to place a 3 in the square in the bottom right corner. Finally, we see that we have to place a 2 and a 4 as shown to complete the grid.

(b) In the figure to the right, you will see two different ways to divide the solution grid from part (a) into 5 groups of connected squares with the sum of the digits in each group equal to 8.

Are there any other ways to divide the grid?



Extra Challenges:

1. See above for two different ways to divide the grid from part (a) following the rules given in part (b).
2. One possible solution is shown on the right.
   How many different solutions can you find?
   Try to divide these grids as in part (b).

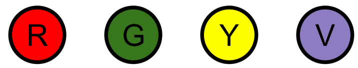
4	3	2	1
1	2	3	4
1	2	3	4
4	3	2	1

CEMC at Home

Grade 4/5/6 - Wednesday, April 22, 2020

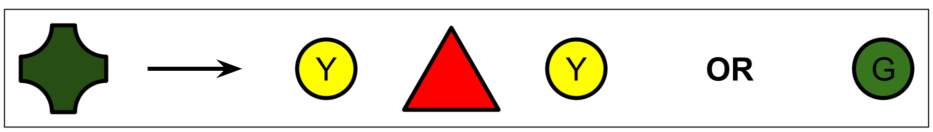
Jessie's Jewelry

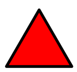
Introduction: Jessie likes to make bracelets and necklaces using four different types of coloured beads.



Although the patterns of the beads in the jewelry may look random, Jessie actually follows strict rules to make the jewelry. Each rule mentions *shapes* and *beads*, and explains how to *replace* a certain shape with a new pattern of shapes and beads.

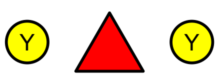
For example, one of Jessie's replacement rules for making jewelry is shown below.

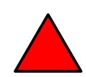


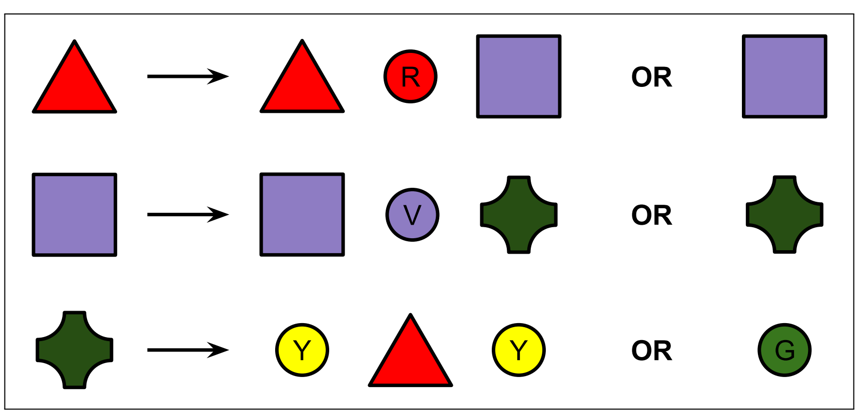
The symbols  and  in the rule represent *placeholder shapes*.

The symbols  and  represent two of the four different types of *beads*.

The rule shown above says that whenever Jessie sees the shape  in a pattern, this shape can be replaced with one of two different things:

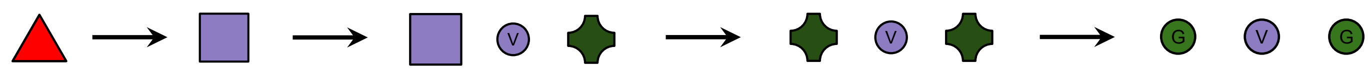
- the pattern , or
- the single bead .

When making a new piece of jewelry, Jessie always starts out with the single placeholder shape  and then uses the three replacement rules shown below.



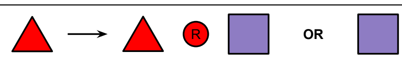
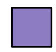
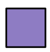
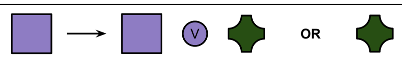
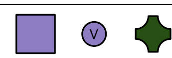
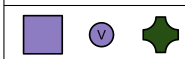
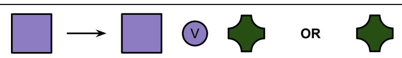
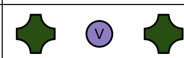
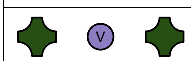
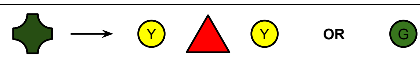
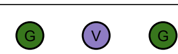
Jessie applies these rules until a final pattern consisting of *only beads* is reached.

Example: Let's see Jessie's rules in action in an example. Here is an outline of one of Jessie's attempts to create a simple piece of jewelry.



Notice how Jessie started with the shape and ended with a pattern of only beads.

See the table below for an explanation of how this final pattern of beads was created using the rules.

Current Stage of Pattern	Rule Applied	Next Stage of Pattern	Explanation
			*Jessie had two choices for what pattern to substitute for the shape and chose the second option.*
			*Jessie had two choices for what pattern to substitute for the shape and chose the first option.*
			*Jessie had many choices here. Jessie could replace the with one of two patterns and/or replace the with one of two patterns. Jessie chose to apply the rule for only and chose the second pattern option this time.*
			*Jessie had two choices for what pattern to substitute for each of the shapes and chose the second option for each of the two shapes.*

Problems:

1. Start with a shape and work through the rules yourself. Make different choices than Jessie did in the example above, and create a different pattern of beads than the one shown above.
2. Explain how Jessie can start with a shape and apply the rules to end up with the following final pattern of beads:

3. No matter how Jessie chooses to apply the rules, there is no way to create the final pattern of beads shown below by starting with a shape. Can you explain why Jessie cannot create this final pattern?

More Info:

Check out the CEMC at Home webpage on Wednesday, April 29 for a solution to Jessie's Jewelry.

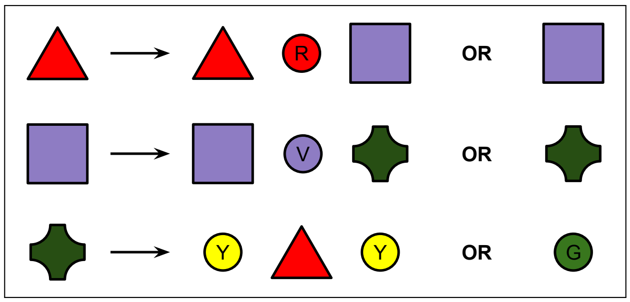
See the next page for an extra challenge problem to try!

Extra Challenge: Which of the following patterns of beads can be made according to Jessie's rules? Explain your answers.

CEMC at Home

Grade 4/5/6 - Wednesday, April 22, 2020
Jessie's Jewelry - Solution

Summary of Jessie's Jewelry: When making a new piece of jewelry, Jessie always starts out with the single placeholder shape and then uses the three replacement rules shown below.



Jessie applies these rules until a final pattern consisting of *only beads* is reached.

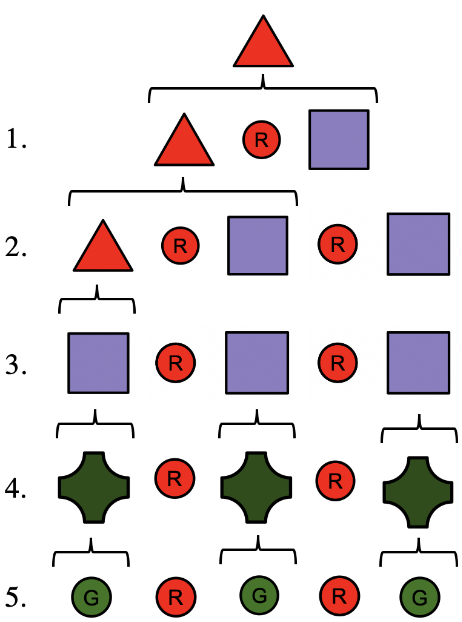
Problems and Solutions:

(a) Start with a shape and work through the rules yourself.

Solution: There are many different possible patterns you can make here. You will see a few patterns that can be made while reading the solutions that follow.

(b) Explain how Jessie can start with a shape and apply the rules to end up with the following final pattern of beads:

Solution: Here is one possible way to end up with the pattern above:



Step 1. The is replaced with the pattern R.

Step 2. The is replaced with the pattern R.

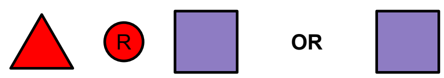
Step 3. The is replaced with a .

Step 4. Each is replaced with a .

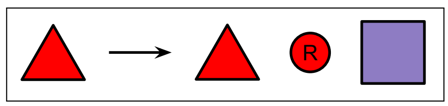
Step 5. Each is replaced with a G.

(c) No matter how Jessie chooses to apply the rules, there is no way to create the final pattern of beads shown below by starting with a shape. Can you explain why Jessie cannot create this final pattern?

Solution: Jessie starts with a single . In the first step, Jessie must replace this with one of two things:



Suppose that Jessie replaced the with the first option:

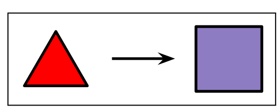


We know that we need to get a V bead immediately to the right of the R bead.

Looking at the rules, the only way this could possibly happen is if a shape appearing to the right of the R bead is replaced with the pattern V at some point. But this would mean there would have to be at least one object between the R bead and the V bead (no matter what replaces the shape in the pattern V).

This tells us that this strategy cannot possibly produce the correct piece of jewelry since we cannot get the R V part of the pattern.

Now suppose that Jessie instead goes with the second option at the beginning:



This means that the only way Jessie could introduce a R bead into the pattern is to reintroduce a shape at some point. Looking at the rules, the only way to introduce a shape would be to replace a shape with the Y Y pattern.

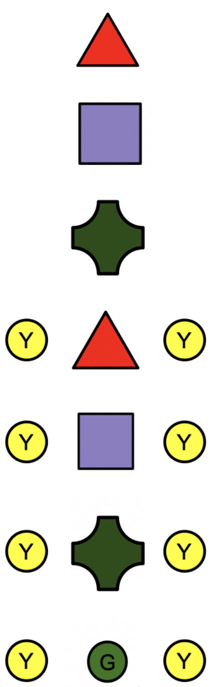
But this means the only way we could get a R bead into the pattern is to have it appear between the two Y beads.

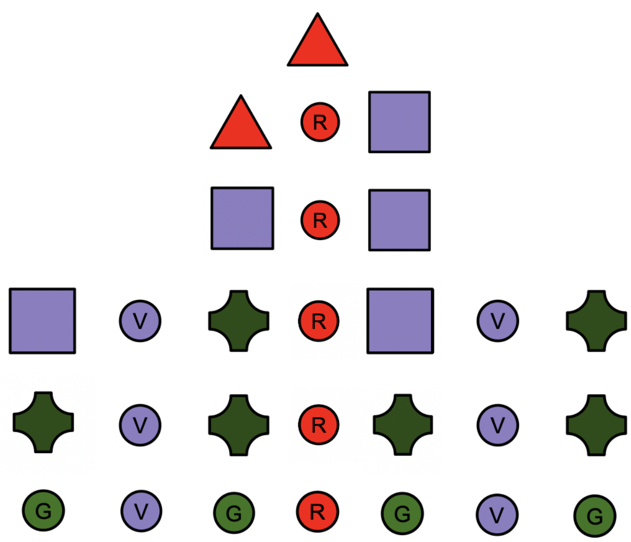
This tells us that this strategy cannot possibly produce the correct piece of jewelry either since the piece of jewelry has the R bead to the left of both Y beads.

Now we have reached a problem: there are no other strategies left to try! This means we can be sure that Jessie cannot make this piece of jewelry following these rules.

Extra Challenge: Which of the following patterns of beads can be made according to Jessie's rules? Explain your answers.

Solution: All three patterns can be made using the rules. We outline the intermediate patterns for the first two pieces of jewelry below, and we leave the third piece of Jewelry for you to think about on your own.





Computer Science Connections:

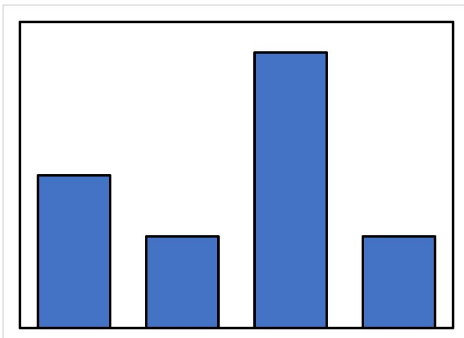
The rules in this problem are similar to rules that are used to describe a *context-free grammar*. In Computer Science we use context-free grammars to define the rules of programming languages.

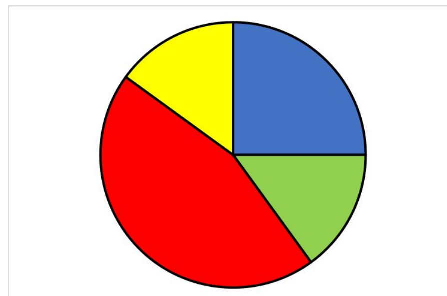
The rules in this problem actually describe simple mathematical expressions. If you substitute $+$ or $-$ operators where you see R, and you substitute $\times$ or $\div$ where you see V, and you substitute matching parenthesis where you see a pair of Y beads surrounding a triangle, then you will see that the sequences that are valid for Jessie's jewelry are also valid mathematical expressions. The sequence that does not follow the rules is not a valid mathematical expression.

CEMC at Home features Problem of the Week

Grade 4/5/6 - Thursday, April 23, 2020
Lost Data

Tanner randomly surveyed 40 students in his school about their ages. The ages given were six, seven, eight, and nine. After gathering the answers, he drew a bar chart and a pie chart to show the results. Unfortunately, before he labelled each chart, he lost the original data. The charts are shown below:





Tanner did remember that 6 students in the survey answered six for their age. He also remembered that $\frac{1}{4}$ of the students surveyed answered eight for their age, and that the most popular answer was age seven. Based on this information, complete the table below:

Age	Number of Students
six	6
seven	
eight	
nine	

More Info:

Check the CEMC at Home webpage on Thursday, April 30 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 30.

This CEMC at Home resource is the current grade 3/4 problem from Problem of the Week (POTW). This problem was developed for students in grades 3 and 4, but is also appropriate for students in grades 5 and 6. POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week, to view this week's grade 5/6 problem, and to find many more past problems and their solutions, visit the Problem of the Week webpage.

Problem of the Week

Problem A and Solution

Lost Data

Problem

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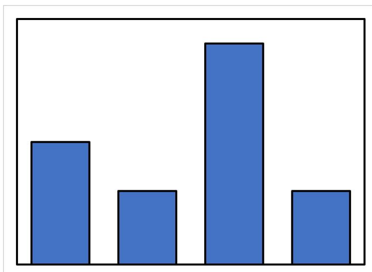

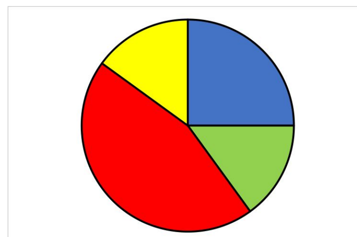

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six	6
seven	
eight	
nine	

Solution

We compute $\frac{1}{4}$ of 40 is equal to 10. So 10 of the students who were surveyed are eight.

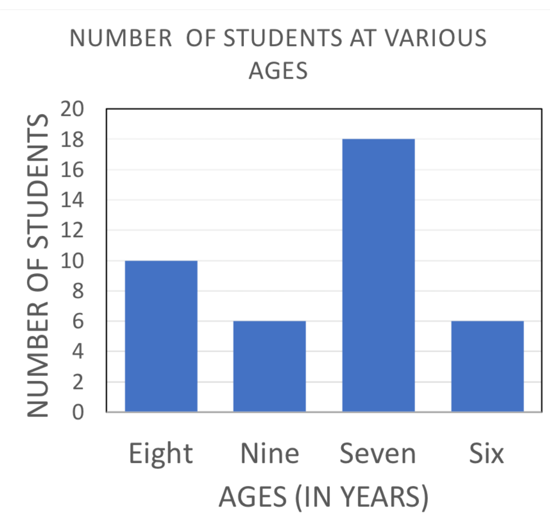
From the pie chart, we can see that there are two data values that are less than $\frac{1}{4}$ of the total number responses, and one data value is greater than $\frac{1}{4}$ of the total number of responses. We can also see from both charts that two of the data values are the same and less than one quarter of the total number of responses. From these two observations, along with the fact that one of the data values is 6, we can conclude that there is a second data value that is also 6.

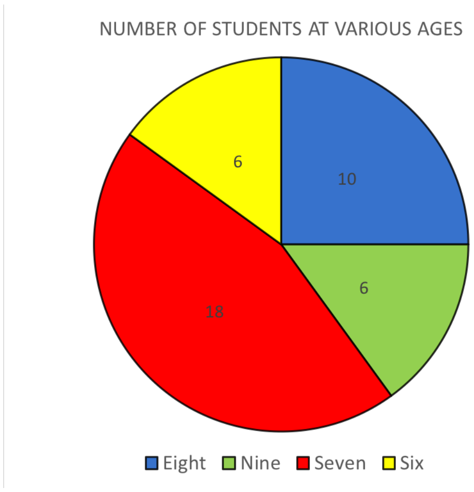
Now we can calculate the sum of three of the data values: $6 + 6 + 10 = 22$. Since 40 students were surveyed, the last data value must be $40 - 22 = 18$. This is the largest data value. Since seven was the most popular answer, there must be 18 students who are seven.

Now we know how many students answered six, seven, and eight for their ages. We also know that there is one data value (6) that we have not assigned to an age. So there must have been 6 students who answered age 9 in the survey. So the completed table is:

Age	Number of Students
six	6
seven	18
eight	10
nine	6

Note: For some reason, Tanner decided to use word labels on the horizontal axis of the bar chart, in alphabetical order. That is, the labels on the horizontal axis were eight, nine, seven and six. His completed graphs are shown below.

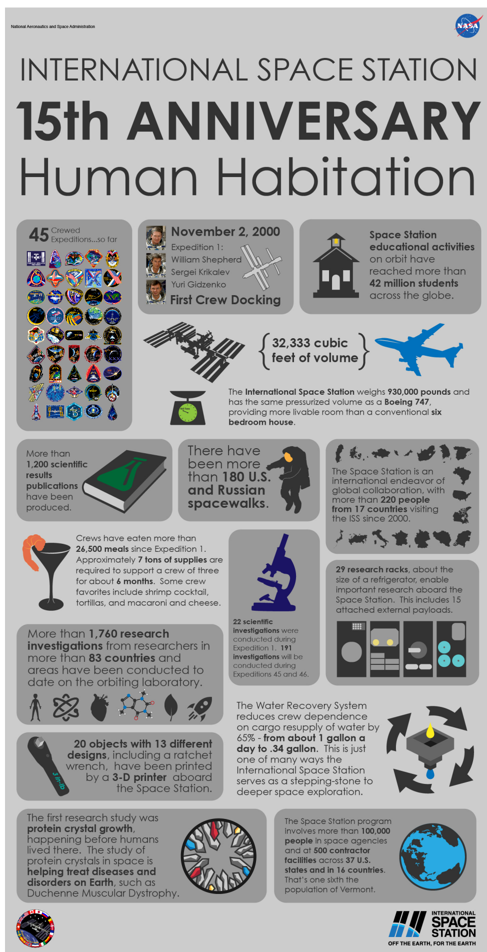




Teacher's Notes

The charts in this problem were generated by an Excel spreadsheet with the same underlying data values: 6, 18, 10, and 6. Different graphical representations of data may make it easier to see relationships among the data values. For example, although it appears that the yellow and green sections of the pie chart are the same size, it is clearer in the bar chart (especially in the solution with grid lines) that these values are the same. However, in the pie chart, it is much easier to see that the blue section is one quarter of the total surveyed.

We see more and more sophisticated examples of using images to represent data. People have been using *infographics* as a way of capturing people's attention - often in an attempt to sell things, but also in an attempt to emphasize important information - for a long time. However with today's technology we see these kinds of images everywhere.



(Image provided by NASA / Public domain)

CEMC at Home

Grade 4/5/6 - Friday, April 24, 2020
Metric Pentathlon

You Will Need:

- Two or more participants
  Enlist your family - this is for all ages! The score cards included are for four participants, but if you have more, you can add extra rows or make your own.
- A metre stick or measuring tape
  If you do not have a metre stick, then can you make one? Only one event uses a metre stick.
- A balloon inflated to about 4 cm across, or a soft, lightweight cloth ball about that size
- Two straws for each participant
  To make a straw, you can use a small strip of paper, roll it into a tube, and tape it into place. The straw should have an opening around 0.5 cm wide and be around 20 cm long. Try to make all of the straws similar in size.
- A cotton ball for each participant (about the size that comes in the top of a pill bottle)
- A paper plate
  If you don't have a paper plate, then you can cut a disk out of cardboard or something similar.
- A watch or clock with a second hand
- Tape
- A pen or pencil

What To Do: For each event, each participant in turn will do the following:

1. Enter your name on the score card for that event.
2. Estimate the distance or time you think you can achieve for the event, and enter that estimate on the score card.
3. Perform the task required by the event.
4. With the help of another participant, measure the actual value you achieve for the time or distance, and enter that measure on the score card.
5. Calculate your score, which is the difference between your estimate and the measured value, and enter it on the score card.

   Subtract the smaller value from the larger value so that the difference is a positive number.

For fairness, vary the order of the participants from event to event. *Think about why!*

The winner of each event is the participant with the *lowest score*, that is the participant with the smallest difference between their estimate and the actual measure.

Event 1: Balloon Toss

For this event you will need:

- The metre stick or measuring tape
- A long piece of tape
- The balloon or cloth ball

Make a line by laying the piece of tape on the floor in a hallway or a room with the most space.

Standing with your toes at the line, estimate how far, in metres, you can toss the balloon or cloth ball and enter your estimate on the score card.

Then toss the balloon underhand, and ask another participant to measure the distance from the line to where the balloon or ball lands on the floor.

Enter their measurement and determine your score.

Score Card for Event 1

Names	Estimate	Measure	Score

NOTE: If a soft cloth ball is used for this event, the distances may be longer.

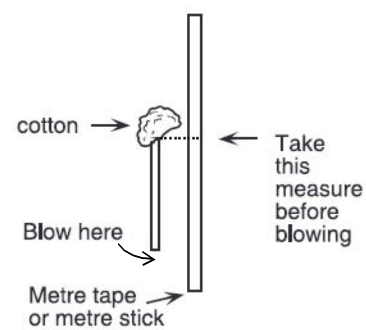
Event 2: Cotton Ball Blow

For this event you will need:

- The metre stick
- One straw for each participant
- The cotton ball(s)

Ask another participant to hold the metre stick vertically against a wall while you complete your turn at the event.

As shown in the diagram to the right, hold a straw vertically and balance your cotton ball on top of the straw. The goal is to blow through the bottom of the straw to move the cotton ball upward.



Estimate how far, in centimetres, you think you can blow the cotton ball vertically, and enter your estimate on the score card.

Ask the other participant to mark the initial and final position of the ball on the metre stick as you blow it upward. Enter the difference in centimetres between those two marks as your measure, and determine your score.

Score Card for Event 2

Names	Estimate	Measure	Score

If you find it hard to point the straw upward and balance the cotton ball while standing, how about trying this event by instead lying on your back!

Event 3: Paper Plate Discus Throw

For this event you will need:

- The metre stick or measuring tape
- A long piece of tape
- The paper plate

Make a line on the floor as you did for Event 1.

Standing with your toes at the line, estimate how far, in metres, you can throw the paper plate, and enter your estimate on the score card.

Then toss the paper plate like a frisbee, and ask another participant to measure the distance from the line to where the plate lands on the floor.

Enter their measurement and determine your score.

Score Card for Event 3

Names	Estimate	Measure	Score

Event 4: Straw Javelin Throw

For this event you will need:

- The metre stick or measuring tape
- A long piece of tape
- One straw for each participant

Make a line on the floor as you did for Event 1.

Standing with your toes at the line, estimate how far, in centimetres, you can throw a second straw, and enter your estimate on the score card.

Then throw the straw, and ask another participant to measure the distance from the line to where the straw lands on the floor.

Enter their measurement and determine your score.

Score Card for Event 4

Names	Estimate	Measure	Score

Event 5: Heel-Toe Walk

For this event you will need:

- The watch or clock
- Two pieces of tape (or other markers)

Place two markers on the floor, approximately 5 metres apart.

Standing with your toes at the first marker, estimate how many seconds it will take you to walk in heel-to-toe fashion to the second marker, and enter your estimate on the score card.

Then ask another participant to use a clock or watch to measure the time in seconds it takes you to do the actual walk.

Enter their measurement and determine your score.

Score Card for Event 5

Names	Estimate	Measure	Score

Once all participants have completed all events, we are ready to declare a winner! The overall winner of the Metric Pentathlon should be the participant who did the best on the events overall, but there are many different ways to measure this. An extra activity might be to decide, as a group, the fairest way to use all of the scores (some in different units) to declare the winner of the Metric Pentathlon.