CEMC at Home

Grade 11/12 - Monday, April 20, 2020
Pick up Sticks - Part 2

You Will Need:

- Two players
- 15 sticks (these could be chopsticks, popsicle sticks, toothpicks, pencils, etc.)

How to Play:

1. Arrange the 15 sticks into three piles: one with 3 sticks, one with 5 sticks, and one with 7 sticks.
2. Players alternate turns.
   Decide which player will go first (Player 1) and which player will go second (Player 2).
3. On your turn, you can pick up as many sticks as you want, as long as all of the sticks you pick up come from the same pile.
4. The player who picks up the last stick wins the game!

Play this game a number of times. Alternate which player goes first. Is there a winning strategy for this game? If so, which player (Player 1 or Player 2) has a winning strategy?

Can you use your winning strategy from the first game of Pick Up Sticks (from April 6) to develop a winning strategy for this game?

Variations: Change the number of sticks that you have in each of the three piles. (You will need to find more sticks, if you want to start with more than 15 sticks in total.) Does the number of sticks in each pile affect your winning strategy?

More Info:

Check out the CEMC at Home webpage on Monday, April 27 for a discussion of a strategy for this game. We encourage you to discuss your ideas online using any forum you are comfortable with.

CEMC at Home

Grade 11/12 - Monday, April 20, 2020
Pick Up Sticks - Part 2 - Solution

The approach we will use to figure out the winning strategy for this game will be different than the approach used in the similar game of Pick Up Sticks with two piles. The winning strategy will be built around an understanding of what we will call *winning positions*.

A position of the game will be represented by an ordered triple of non-negative integers. The ordered triple (a, b, c) will refer to the position where one pile has a sticks, one pile has b sticks and one pile has c sticks. We will always represent a position with an ordered triple (a, b, c) satisfying $a \leq b \leq c$. This can be done because the order of the piles does not matter. For example, if the three piles at some point have 1, 3, and 5 sticks, then the current position of the game is represented by $(1, 3, 5)$. If 3 sticks are now removed from the pile of 5, then the three piles have 1, 3, and 2 sticks but the new position is represented by $(1, 2, 3)$ rather than $(1, 3, 2)$.

A *winning position* is a position with the property that if we make a move to bring the game to that position, we have a winning strategy from that point forward. In other words, a winning position is a position from which our partner cannot win (unless we make a mistake later). If we move the game to a winning position, our partner cannot make a move to change it to a winning position. Furthermore, no matter what move our partner makes, there will be a move available on our next turn that changes the game back to a winning position.

By the definition of the game, we win if we pick up the last stick. Therefore $(0, 0, 0)$ is a winning position. We are going to develop a table of winning positions for our game. Since the game starts in the position $(3, 5, 7)$, we will not consider any positions (a, b, c) where $a > 3$, $b > 5$, or $c > 7$. Here are two observations.

Observation 1: From our work with Pick Up Sticks with two piles, we know that if a position has exactly two piles and these piles are equal, then it is a winning position. Also, a position having exactly two unequal piles is not a winning position. Therefore, the position $(0, k, k)$ is a winning position and the position $(0, k, \ell)$ with $k < \ell$ is not a winning position.

Observation 2: Suppose we have two position P and Q and that P is a winning position. If two of the piles in Q are the same as two of the piles of P *and* the other pile of Q has more sticks than the other pile of P, then Q is *not* a winning position. This is because there is a move from Q to the winning position P. For example, $(3, 3, 5)$ is not a winning position because $(0, 3, 3)$ is a winning position (see previous observation), the two positions $(0, 3, 3)$ and $(3, 3, 5)$ have two pile sizes in common, and the other pile in $(3, 3, 5)$ is larger than the other pile in $(0, 3, 3)$.

This video uses these two observations to develop a table of winning positions for our game.

Here are the winning positions: $(0, 0, 0)$, $(0, k, k)$, $(1, 2, 3)$, $(1, 4, 5)$, $(2, 4, 6)$, $(2, 5, 7)$, $(3, 4, 7)$, and $(3, 5, 6)$.

Since our starting position is $(3, 5, 7)$ we can see that Player 1 has a winning strategy. Player 1 can remove one stick from any of the three piles to move the game to a winning position.

If you change the starting position of the game, you will change which player has a winning strategy. In general, if the starting position is not a winning position, then Player 1 has a winning strategy, and if the starting position is a winning position, then Player 2 has a winning strategy. The reasoning from the video can be extended to identify the winning positions in a game starting with different numbers of sticks.

CEMC at Home

Grade 11/12 - Tuesday, April 21, 2020
Six Numbers

Problem:

There is a list of 6 unknown numbers p, q, r, s, t, u and they are ordered $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. Adding together the two numbers in each pair we get 15 sums. These 15 sums, in increasing order, are as follows

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$$

Determine the value of $r + s$.

Discussion: Try solving this problem! We will present different approaches for a solution next week, but for now let's look at the problem more closely and discuss some approaches that do not work.

Let's first understand what is being asked. The 15 sums mentioned are obtained by choosing two different numbers from the list p, q, r, s, t, u, and adding them together. For example, the values of the expressions $q + t$ and $u + r$ are among the 15 numbers listed in the question. One of the 15 numbers is $r + s$ and we need to determine which one.

One approach that someone might take is to make the following assumption: Since r and s are the middle two numbers in the list p, q, r, s, t, u, the sum $r + s$ should be the middle number in the list $25, 30, \ldots, 103, 117$, that is $r + s = 63$. This approach does not work. The problem is that the sum $r + s$ might not actually be equal to the middle number in the list of sums. For example, if we had the list of 6 numbers $1, 2, 4, 7, 100, 110$, you can check that the sum $r + s = 4 + 7 = 11$ is not the 8^{th} number in the list of the 15 sums, but is rather the 6^{th} number. The full list of sums is as follows:

$$3, 5, 6, 8, 9, \boxed{11}, 101, 102, 104, 107, 111, 112, 114, 117, 210$$

Expanding on the idea in this first (incorrect) approach, you could decide to write down the expressions for all 15 sums, involving the 6 unknowns, and start assigning values to particular expressions based on what we *do* know about their relative sizes. For example, since p and q represent the smallest two numbers, $p + q$ must represent the smallest sum and so we must have that

$$p + q = 25$$

You can argue that $p + r$ must represent the next smallest sum, so it must be the case that

$$p + r = 30$$

Unfortunately, using only this line of reasoning you will get stuck quite quickly. For example, you cannot be immediately sure of which expression represents the next smallest sum, 38. Can you see why? You can use similar reasoning to assign the largest values in the list of sums, but then will run into a similar problem there as well.

Think about other ways you might approach this problem.

More Info:

Check the CEMC at Home webpage on Tuesday, April 28 for two different solutions to Six Numbers.

CEMC at Home

Grade 11/12 - Tuesday, April 21, 2020
Six Numbers - Solution

Problem:

There is a list of 6 unknown numbers p, q, r, s, t, u and they are ordered $p < q < r < s < t < u$. There are exactly 15 pairs of numbers that can be formed by choosing two different numbers from this list. Adding together the two numbers in each pair we get 15 sums. These 15 sums, in increasing order, are as follows

$$25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$$

Determine the value of $r + s$.

We will show two different solutions to this problem. For both solutions, we note that besides the original list of 6 numbers p, q, r, s, t, u, there are two additional lists of interest to examine here:

List 1: $25, 30, 38, 41, 49, 52, 54, 63, 68, 76, 79, 90, 95, 103, 117$
List 2: $p+q, p+r, \ldots, t+u$

What is the relationship between List 1 and List 2? They contain the exact same 15 numbers, just possibly in a different order. In fact, we have not even specified an order for List 2. You can do this if you wish by explicitly writing down the 15 sums.

Solution 1

First, note that since $p < q < r < s < t < u$, the expression $p + q$ must represent the smallest of the 15 sums. Also, $p + r$ must represent the second smallest sum (think about why). This means $p+q = 25$ and $p+r = 30$ and so $(p+r)-(p+q) = 30-25 = 5$. However, $(p+r)-(p+q) = r-q$, so it must be the case that $r - q = 5$. Using this, observe that $(s + r) - (s + q) = r - q = 5$, and similarly $(t + r) - (t + q) = r - q = 5$, and finally $(u + r) - (u + q) = r - q = 5$. That is, each of the following 4 pairs of sums from List 2 has a difference of 5 when the first number in the pair is subtracted from the second: $(p + q, p + r), (s + q, s + r), (t + q, t + r), (u + q, u + r)$.

List 1 contains exactly 4 pairs of numbers with a difference of 5. They are $(25, 30)$,$(49, 54)$, $(63, 68)$, and $(90, 95)$. Therefore, these 4 pairs must be the 4 pairs $(p + q, p + r)$, $(s + q, s + r)$, $(t + q, t + r)$, and $(u+q, u+r)$ in some order. Focusing on second coordinates, this means the numbers 30, 54, 68, and 95 are the sums $p + r$, $s + r$, $t + r$, and $u + r$ in some order. Since $p < s < t < u$, we must have $p+r < s+r < t+r < u+r$, which means $s+r$ is the second smallest of 30, 54, 68, and 95. Thus, $r + s = s + r = 54$.

See the next page for Solution 2.

Solution 2

We start by stating the key to this approach: The sum of all of the numbers in List 1 is equal to the sum of all of the numbers in List 2. This is because addition is a commutative operation. In other words, the order in which you add numbers does not change the sum.

So let's add up all the values in List 1 and then add up all the values in List 2 and compare these two sums. Adding up the values in List 1 gives

$$25 + 30 + 38 + 41 + 49 + 52 + 54 + 63 + 68 + 76 + 79 + 90 + 95 + 103 + 117 = 980$$

What about the sum of the values in List 2? If we add up the 15 expressions, how many times does each variable appear in the sum? Let's start with variable p. There are 5 p's because p appears in exactly five of the 15 expressions in the list. In particular, p appears exactly once with each of the 5 other variables. Similarly, there are 5 q's, 5 r's, 5 s's, 5 t's, and 5 u's. Therefore, adding up all expressions in List 2 results in the expression

$$5p + 5q + 5r + 5s + 5t + 5u$$

Since List 1 and List 2 have the same sum, we have the equality

$$5p + 5q + 5r + 5s + 5t + 5u = 980$$

Dividing both sides by 5 gives us that

$$p + q + r + s + t + u = 196$$

which we rewrite in the following helpful way

$$(p + q) + (r + s) + (t + u) = 196$$

Since p and q are the smallest two numbers in our list of 6 integers, $p+q$ must be equal to the smallest number in List 1 and so $p + q = 25$. Similarly, since t and u are the two largest numbers in our list of 6 integers, $t + u$ must be equal to the largest number in List 1 and so $t + u = 117$. Therefore,

$$r + s = 196 - (p + q) - (t + u) = 196 - 25 - 117 = 54$$

How neat is that! Nobody told us to add up all 15 numbers, but once we do it (in the two different ways, using each of Lists 1 and 2), a nice solution reveals itself!

CEMC at Home

Grade 11/12 - Wednesday, April 22, 2020
Comparison Machine

Your mission, should you choose to accept it, is to develop algorithms to complete tasks involving the the relative order of n distinct integers. The problem is that the integers are random and unknown to you! All you know is that they are named $a_1, a_2, \ldots, a_n$. Note that we call i the *index* of the integer a_i.

For each task, your approach must work no matter what the order of the integers is.

Some good news: A helpful machine is available. The machine knows the relative order of these integers. To use it, you enter the indices of two integers into the machine and it will tell you which of the two corresponding integers is larger. For example, if $a_4 = 5$, $a_2 = 7$ and you enter 4 and 2 into the machine, it will tell you that a_2 is larger. We name the machine M and in this case we have $M(4, 2) = 2$ and $M(2, 4) = 2$. Either of these application of M tells you that the integer with index 2 is larger than the integer with index 4.

Some bad news: For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.

Some more good news: Your memory is perfect and you can remember (or record) the result every time you use the machine.

Example

Suppose $n = 4$ and you want to determine which of the integers, a_1, a_2, a_3, or a_4, is the largest, while limiting yourself to only 3 uses of the machine. Here is one way to do this:

1. Compute $M(1, 2)$ and record this answer as the index x.
2. Compute $M(3, 4)$ and record this answer as the index y.
3. The largest integer is the integer with index $M(x, y)$.

The Tasks and a Fun Tool

Below are the three different tasks to be completed. Each task outlines how many integers there will be in the list (n), what you are attempting to answer about the list (Task), and how many times you can ask the machine for help (Limit).

n	Task	Limit
7	Determine the largest integer.	6
8	Determine both the largest integer and the smallest integer.	10
8	Determine the second largest integer.	9

Important: We have written a Python computer program that will generate random integers and simulate the helpful machine. It is a lot of fun to use this interactive tool to test if your solutions are correct. See the next page for instructions on how to use the tool.

Using the Tool

The tool works by repeatedly asking you what you want to enter into the machine and then displaying the result. After the number of times you have used the machine reaches the limit, it will ask you for the index of the largest integer in the list. It will then tell you whether or not you are correct.

You do not need to know anything about Python in order to use the tool.

Getting the correct answer for a few lists does not mean that you have a correct algorithm for the task.

Your algorithm has to work for any choice of integers, regardless of their order. The more you test your algorithm, the more evidence you have that it is correct. After testing out your algorithm using the tool, try to explain why your algorithm will work on all possible lists.

Here are instructions for using the tool:

1. Open this webpage in one tab of your internet browser. You should see Python code.
2. Open this free online Python interpreter in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. You will interact with the tool using the right black panel, and you might want to widen this panel.
5. After completing a test, or if you encounter an error, you can hit *run* to begin another test. If you want to start over during a test, you can hit *stop* and then *run*.

More Info:

Check out the CEMC at Home webpage on Wednesday, April 29 for solutions to the three tasks.

Our webpage Computer Science and Learning to Program is the best place to find the CEMC's computer science resources. Two resources through which you can explore Python further are:

Python from scratch
A gentle introduction to programming designed with the beginner in mind.

CS Circles
Interactive lessons teaching the basics of writing computer programs in Python. This is also an introduction but moves at a bit of a faster pace.

CEMC at Home

Grade 11/12 - Wednesday, April 22, 2020
Comparison Machine - Solutions

Summary of the Tasks

Develop algorithms to complete tasks involving the relative order of n distinct integers.

- The integers are random and unknown to you. All you know is that they are named $a_1, a_2, \ldots a_n$.
- For each task, your approach must work no matter what the order of the integers is.
- A helpful machine M is available. The machine knows the relative order of these integers. To use it, you enter the index of two integers into the machine and it will tell you which of the two corresponding integers is larger.
- For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.
- Your memory is perfect and you can remember (or record) the result every time you use the machine.

Details of the Three Tasks

n	Task	Limit
7	Determine the largest integer.	6
8	Determine both the largest integer and the smallest integer.	10
8	Determine the second largest integer.	9

Solution for Task 1

We will keep track of an index named c. The "current maximum" will be at index c. Begin by setting $c = 1$. Then, repeatedly set $c = M(c, i)$ for $i = 2, 3, \ldots, 7$. (Setting $c = M(c, i)$ means we compute the value of $M(c, i)$ and then update c to be this value.) This uses the machine exactly 6 times. We can then declare a_c to be the largest integer because the machine has told us either directly or indirectly that it is larger than each of the other integers.

Can you see why this algorithm works? If the index of the largest integer is 1, then we will see $M(1, i) = 1$ at each stage and the value of c will never change. If the index of the largest integer is not 1, then at some point the value of c will change. In particular, the value of c will change from 1 to i the first time we see a comparison of the form $M(1, i) = i$ for some $i \geq 2$. The value of c will continue to change each time we find an integer that is larger than our "current maximum".

An example using the provided tool is on the next page.

```
Enter i such that 1 <= i <= 7 : 1
Enter j such that 1 <= j <= 7 : 2
M(i,j) = 2
----------------------------------------
You have used the machine, 1 time(s).
----------------------------------------
Enter i such that 1 <= i <= 7 : 2
Enter j such that 1 <= j <= 7 : 3
M(i,j) = 2
----------------------------------------
You have used the machine, 2 time(s).
----------------------------------------
Enter i such that 1 <= i <= 7 : 2
Enter j such that 1 <= j <= 7 : 4
M(i,j) = 4
----------------------------------------
You have used the machine, 3 time(s).
----------------------------------------
Enter i such that 1 <= i <= 7 : 4
Enter j such that 1 <= j <= 7 : 5
M(i,j) = 5
----------------------------------------
You have used the machine, 4 time(s).
----------------------------------------
Enter i such that 1 <= i <= 7 : 5
Enter j such that 1 <= j <= 7 : 6
M(i,j) = 5
----------------------------------------
You have used the machine, 5 time(s).
----------------------------------------
Enter i such that 1 <= i <= 7 : 5
Enter j such that 1 <= j <= 7 : 7
M(i,j) = 7
----------------------------------------
You have used the machine, 6 time(s).
----------------------------------------

Enter the index of the largest integer: 7

Correct!
The integers in order are: [119, 123, 117, 130, 140, 106, 190]
```

The value of c just before each use of the machine and right after the last use of the machine will be

$$1, 2, 2, 4, 5, 5, 7.$$

Solution for Task 2

Begin by computing $M(1,2)$, $M(3,4)$, $M(5,6)$, and $M(7,8)$ and recording the answers as indices a, b, c, d. Name the other indices e, f, g, h. At this point we have used the machine 4 times and know the largest integer must be at index a, b, c or d. This is because the machine has told us that the integers at indices e, f, g and h are each smaller than at least one of the integers at indices a, b, c or d. Similarly, the smallest integer must be at index e, f, g or h.

Now notice that our solution for Task 1 generalizes to give a way to find the largest of n integers using the machine $n-1$ times. Moreover, if we adjust things by using c to keep track of the index of a "current *minimum*", we can also use it to find the smallest of n integers using the machine $n-1$ times.

Using this generalization, we can find the largest integer among those at indices a, b, c and d using the machine 3 more times. We can also use it to find the smallest integer among those at indices e, f, g and h using the machine 3 more times. These two values are the largest and smallest integers overall and we used the machine $4 + 3 + 3 = 10$ times.

Solution for Task 3

Begin by using the machine 4 times as in the solution for Task 2, and defining a, b, c, and d in the same way. Then compute $M(a, b)$ and $M(c, d)$, recording the answers as the indices x and y. Next, compute $M(x, y)$ and record the answer as the index z. You can think of this approach as a standard "tournament" or "bracket" in a competition where only the winners move on at each stage.

The integer a_z must be the largest integer because it is the only integer that has not been declared smaller than some other integer by the machine. Moreover, the second largest integer can only have been declared smaller than a_z. So to find the second largest integer, we need only find the largest integer among those at indices entered into the machine with z. Note that z was entered into the machine 3 times so we can use our generalized solution to Task 1 by using the machine 2 more times to determine the second largest integer. In total, we used the machine $4 + 2 + 1 + 2 = 9$ times.

Note

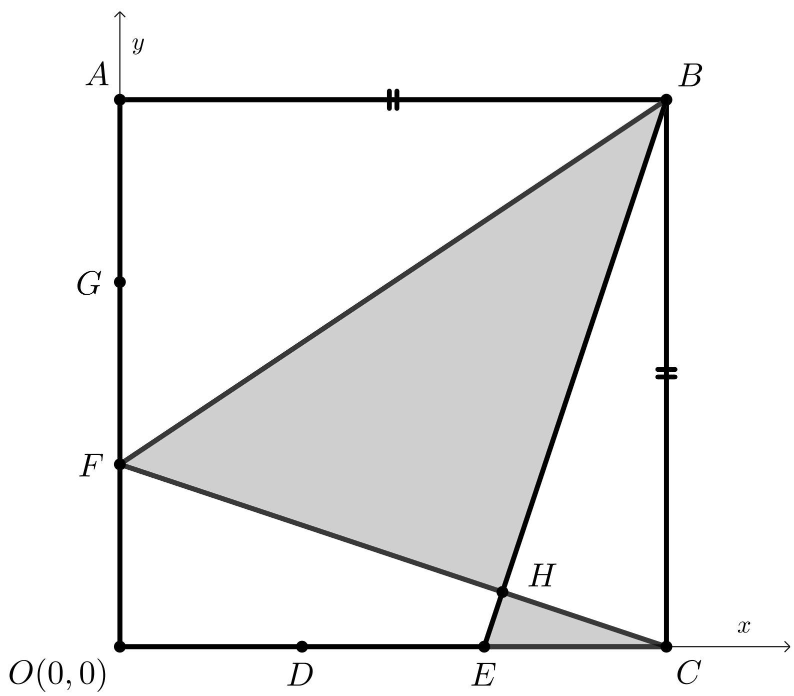
We have only shown that we can complete these tasks within the limits of 6, 10 and 9. Do we need the limits to be this high or can they be lowered? It turns out that they cannot be lowered; they are optimal. That is, for each task, there do not exist correct algorithms that always use the machine strictly fewer times than the given limit. Proving this is very challenging.

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 23, 2020
Maybe One-Third?

In the diagram, square $OABC$ is positioned with O at the origin $(0, 0)$, A on the positive y-axis, C on the positive x-axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H.

If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?



More Info:

Check the CEMC at Home webpage on Thursday, April 30 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 30.

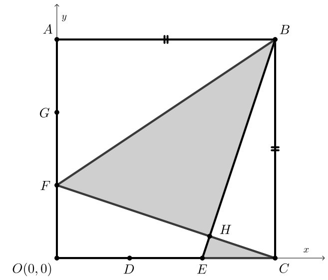
This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week

Problem E and Solution

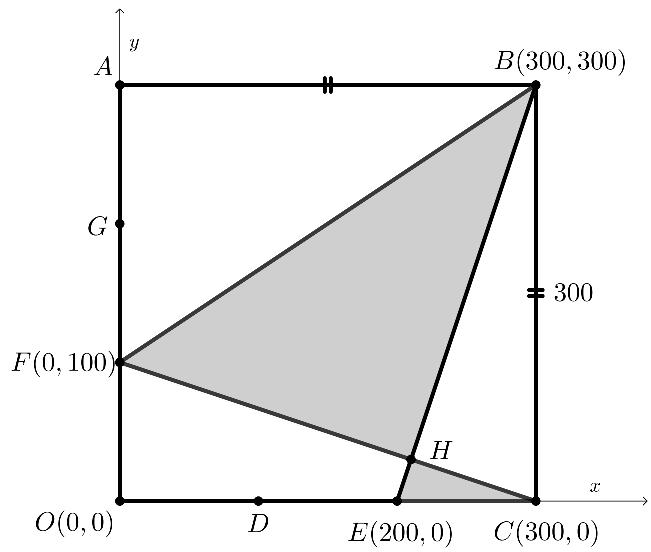
Maybe One-Third?



Problem

In the diagram, square $OABC$ is positioned with O at the origin $(0,0)$, A on the positive y-axis, C on the positive x-axis, and B in the first quadrant. Side OA is trisected by points F and G so that $OF = FG = GA = 100$. Side OC is trisected by points D and E so that $OD = DE = EC = 100$. Line segment BE intersects line segment CF at H. If the interiors of $\triangle BHF$ and $\triangle CHE$ are both shaded, then what fraction of the total area of the square is shaded?

Solution



Since $OF = 100$ and F is on the positive y-axis, the coordinates of F are $(0, 100)$.

Since $OD = DE = 100$, it follows that $OE = 200$. Since E is on the positive x-axis, the coordinates of E are $(200, 0)$.

Since $OD = DE = EC = 100$, it follows that the side length of the square is $OC = 300$. Since C is on the positive x-axis, the coordinates of C are $(300, 0)$.

It then follows that the coordinates of B are $(300, 300)$.

The diagram has been updated to reflect the new information.

We will proceed to find the coordinates of H.

Find the equation of the line through $B(300, 300)$ and $E(200, 0)$.

The slope of $BE = \frac{300-0}{300-200} = 3$. We substitute $x = 200$, $y = 0$ and $m = 3$ into $y = mx + b$ to find b. Then $0 = 3(200) + b$ and $b = -600$ follows. The equation of the line through BE is $y = 3x - 600$. (1)

Find the equation of the line through $C(300, 0)$ and $F(0, 100)$.

The slope of $CF = \frac{100-0}{0-300} = -\frac{1}{3}$. Since $F(0, 100)$ is on the y-axis, the y-intercept is 100. It follows that the equation of the line through C and F is $y = -\frac{1}{3}x + 100$. (2)

Find the coordinates of H, the intersection of the two lines.

At the intersection, the x-coordinates are equal and the y-coordinates are equal. In (1) and (2), since $y = y$, then

$$3x - 600 = -\frac{1}{3}x + 100 \;\Rightarrow\; 9x - 1800 = -x + 300 \;\Rightarrow\; 10x = 2100 \;\Rightarrow\; x = 210$$

Substituting $x = 210$ into (1), $y = 3(210) - 600 = 30$. Therefore, the coordinates of H are $(210, 30)$.

At this point we could follow one of two approaches. The first approach would be to find the area of the shaded regions indirectly, by first determining the area of the unshaded regions and then subtracting this from the area of the square. We will leave this approach to the solver.

Our second approach, which is below, is to calculate the areas of the shaded triangles directly.

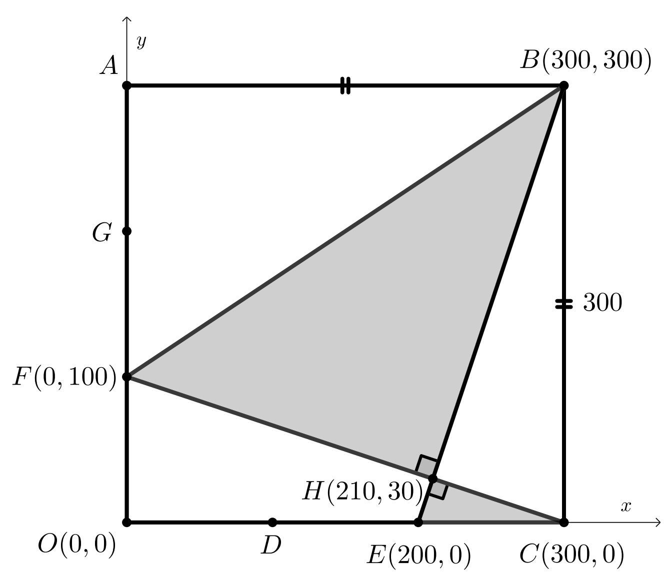
The slope of BE is 3 and the slope of CF is $-\frac{1}{3}$. Since these slopes are negative reciprocals, we know that $BE \perp CF$. It follows that $\triangle BHF$ and $\triangle CHE$ are right-angled triangles.

We will now proceed with finding the side lengths necessary to calculate the area of each shaded triangle.

We first find the area of $\triangle BHF$.

$$\begin{aligned} BH &= \sqrt{(300-210)^2 + (300-30)^2} \\ &= \sqrt{90^2 + 270^2} \\ &= \sqrt{90^2(1 + 3^2)} \\ &= 90\sqrt{10} \end{aligned}$$

$$\begin{aligned} HF &= \sqrt{(0-210)^2 + (100-30)^2} \\ &= \sqrt{210^2 + 70^2} \\ &= \sqrt{70^2(3^2 + 1)} \\ &= 70\sqrt{10} \end{aligned}$$

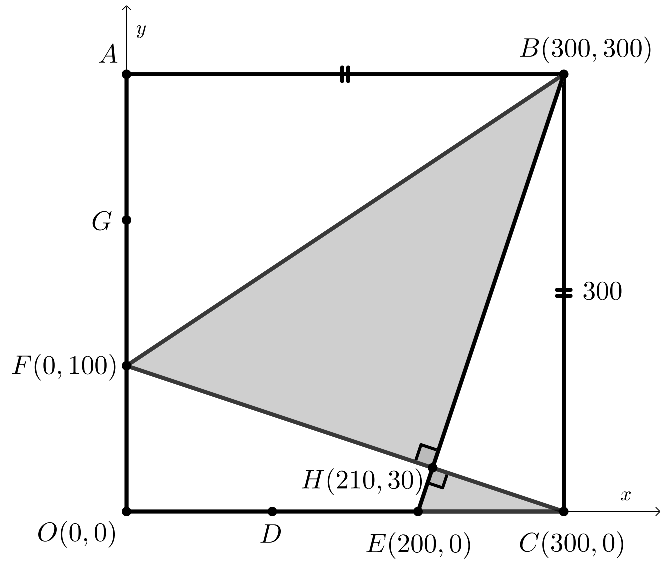


$$\begin{aligned} \text{Area } \triangle BHF &= BH \times HF \div 2 \\ &= 90\sqrt{10} \times 70\sqrt{10} \div 2 \\ &= 31\,500 \end{aligned}$$

Next we find the area of $\triangle CHE$.

$$\begin{aligned} CH &= \sqrt{(300-210)^2+(0-30)^2} \\ &= \sqrt{90^2+30^2} \\ &= \sqrt{30^2(3^2+1)} \\ &= 30\sqrt{10} \end{aligned}$$

$$\begin{aligned} HE &= \sqrt{(210-200)^2+(30-0)^2} \\ &= \sqrt{10^2+30^2} \\ &= \sqrt{10^2(1+3^2)} \\ &= 10\sqrt{10} \end{aligned}$$



$$\begin{aligned} \text{Area } \triangle CHE &= CH \times HE \div 2 \\ &= 30\sqrt{10} \times 10\sqrt{10} \div 2 \\ &= 1500 \end{aligned}$$

We can now calculate the total area shaded, the area of square $OABC$, and the fraction of the area of the square that is shaded.

$$\begin{aligned} \text{Total Area Shaded} &= \text{Area } \triangle BHF + \text{Area } \triangle CHE \\ &= 31\,500 + 1500 \\ &= 33\,000 \end{aligned}$$

$$\begin{aligned} \text{Area } OABC &= OA \times OC \\ &= 300 \times 300 \\ &= 90\,000 \end{aligned}$$

$$\begin{aligned} \text{Fraction of Total Area Shaded} &= \frac{\text{Area } \triangle BHF}{\text{Area } OABC} \\ &= \frac{33\,000}{90\,000} \\ &= \frac{11}{30} \end{aligned}$$

Therefore, $\frac{11}{30}$ of the total area of the square is shaded. This, in fact, is more than one-third.

CEMC at Home

Grade 11/12 - Friday, April 24, 2020
TSP

The Travelling Salesperson Problem (also known as the Travelling Salesman Problem or TSP) is a famous problem in mathematics and computer science. It is widely known because it has applications to many problems that affect our everyday lives and has instances that have remained unsolved for years! Here is one way to state the problem:

> A salesperson wants to visit n cities. What is the shortest route that the salesperson can take in order to visit each of the n cities exactly once and return to their starting point?

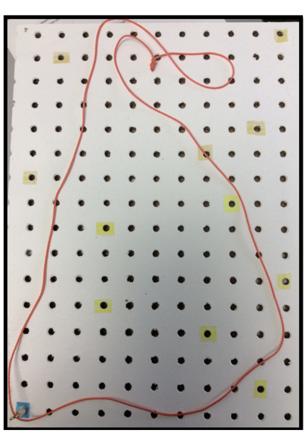
For each pair of cities, the distance between the cities is known. There are many different ways to measure this distance for the purposes of this problem. For example, this distance could be the length of a winding road joining the two cities. Alternatively, it could be the length of the straight line segment joining the two cities. This second option is the measurement of distance that we will use in this activity.

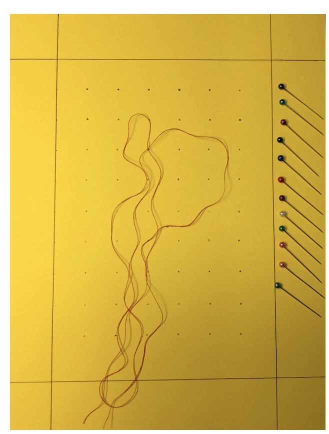
You Will Need:

- Two or more players
- A rectangular pegboard

  We use a board which is 24 cm wide and 30 cm long. See below for alternatives to the pegboard.
- 12 pegs
- A string of length 1 m
- A ruler or measuring tape

If you don't have a pegboard and pegs at home, you can make something similar for yourself. Some good options might include cardboard with pins, foam with toothpicks, or wood with screws. Be as creative as you like.



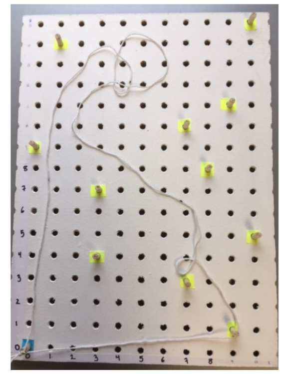


Set Up

We will play a game using the pegboard, the pegs and the string. Here we explain how to setup for the game.

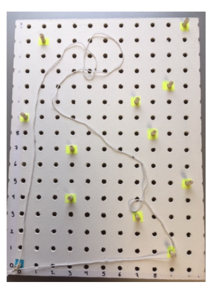
- Attach one of the ends of the string to one of the pegs. You can glue the string to the peg or tie the string around the peg.
- Place the peg with the string attached in the bottom left corner of the board.

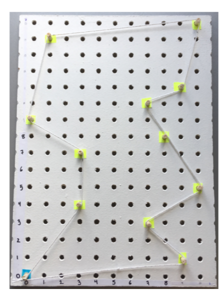
During the game, the players will place the remaining 11 pegs in various places on the board. A possible arrangement is shown in the figure.



How To Play:

1. Start with the pegboard as outlined in the set up.
2. To start the game, the players first need to place the remaining 11 pegs into the board.
   The pegs can be placed anywhere on the board. You can decide whether to have one player place all of the pegs, or have the players take turns placing the pegs until all pegs are in place.
3. Players alternate turns working with the board. Decide which player will go first.
4. The first player does the following:
   - Arrange the string so that it touches each peg exactly once and returns to the bottom left peg. Make sure the string is pulled tight. *An example of this is shown below.*

   - Place a finger to mark the place on the string where the string meets itself at the bottom left peg.
   - Unravel the string, making sure to keep the correct mark on the string. Measure the length of the string between the attached peg and the place marked by the finger. Record this measurement as the score for the first player's first turn.
5. Now all other players take a turn doing what the first player did. All players are trying to find a route for the string that touches each peg exactly once and returns to the bottom left peg, *using the smallest possible length of string.*
6. The game ends after each player has had three turns with the string. The winner of the game is the player who achieved the *smallest* score (measurement) on a single turn.

When you are finished a game, you can pull out all but the bottom left peg and try again! New placements of the pegs will lead to whole new games.

Revisiting the Travelling Salesperson Problem (TSP)

The possible routes players can make with the string in this game model the possible routes in particular instances of the TSP with 12 cities. Each time you set up the peg board, with new distances between pairs of pegs, you are setting up a new version of the TSP, with new distances between the pairs of cities.

It is important to note that winning the game does not mean that you have solved the instance of the TSP associated with this particular board. To win the game, you just need to have the shortest route out of all the routes found in the game. But there may be a shorter route that no player found during the game! Think about how you might convince yourself that you have actually found the *shortest route* of *all possible routes*. In general this is hard to do, and gets even harder the more pegs you add to your game (or cities you add to your problem).

Think about the following questions:

- How many different possible routes are there to choose from in each round of the game using 12 pegs?
- If you change the game to include more pegs, say n pegs in total, how many different possible routes are there to choose from in each round of this game?

The TSP has been studied for many decades, yet there is no known efficient algorithm to solve this problem in general. We encourage you to look into this problem more on your own.

More Info:

The TSP can be modelled using *graphs*. Check out this resource about the TSP to learn more!