# CEMC at Home
## Grade 9/10 - Tuesday, April 14, 2020
## Some Loose Change

In the little-known country of Galbrathia, there are only two types of money: a 5 $\mathbb{J}$ coin and a 9 $\mathbb{J}$ bill. Using this money, you can make some amounts, but not others. For example, you can make 19 $\mathbb{J}$ using two 5 $\mathbb{J}$ coins and one 9 $\mathbb{J}$ bill, but there is no way to make 17 $\mathbb{J}$ using these coins and bills.

*Our goal is to determine the largest amount of money that cannot be made using these bills and coins.*

1. Show how you can make 23 $\mathbb{J}$ using these bills and coins.

2. Explain why you cannot make 17 $\mathbb{J}$ using these bills and coins.

3. The table below shows one way to make each value from 40 $\mathbb{J}$ to 44 $\mathbb{J}$, inclusive:

   | Amount (in $\mathbb{J}$) | # of 5 $\mathbb{J}$ coins | # of 9 $\mathbb{J}$ bills |
   | --- | --- | --- |
   | 40 | 8 | 0 |
   | 41 | 1 | 4 |
   | 42 | 3 | 3 |
   | 43 | 5 | 2 |
   | 44 | 7 | 1 |

   Explain how we could use this information to make each value from 45 $\mathbb{J}$ to 49 $\mathbb{J}$, inclusive.

   *To start, think about how the information in the table about 40 $\mathbb{J}$ can help you make 45 $\mathbb{J}$.*

4. Explain how the information in 3. allows us to say that we can make *every* amount that is at least 40 $\mathbb{J}$.

   *To start, think about how you might make the values from 50 $\mathbb{J}$ to 54 $\mathbb{J}$.*

5. What is the largest amount $N$ $\mathbb{J}$ that cannot be made using these bills and coins?

   *To answer this question fully, you will need to do some work to figure out what you think the value of the positive integer $N$ is. Next, you'll need to argue that $N$ $\mathbb{J}$ cannot in fact be made. Finally, you'll need to argue that every value from $(N+1)$ $\mathbb{J}$ onward can be made. You'll likely want to "work backwards" using what you learned from 3. and 4..*

**Extensions:**

A. There are three ways to make 90 $\mathbb{J}$ using these bills and coins: using 18 coins, using 9 coins and 5 bills, or using 10 bills. (To move from one way to the next way, we have traded 9 coins for 5 bills, since these have equal value.) How many ways are there to make 900 $\mathbb{J}$ using these bills and coins?

B. In the neighbouring country of Pnoll, stamps are issued in amounts of 7 $\mathbb{K}$, 9 $\mathbb{K}$, and 11 $\mathbb{K}$. What is the largest amount of postage that *cannot* be made using these stamps?

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**More Info:**

Check out the CEMC at Home webpage on Tuesday, April 21 for a solution to Some Loose Change.

# CEMC at Home

## Grade 9/10 - Tuesday, April 14, 2020
## Some Loose Change - Solution

1. Since 23 $\mathbb{J}$ = 2 × 9 $\mathbb{J}$ + 1 × 5 $\mathbb{J}$, then we can can use 2 bills and 1 coin.

2. Since 2 × 9 is greater than 17, then to make 17 $\mathbb{J}$, we must use either zero bills or one bill.
   If zero 9 $\mathbb{J}$ bills are used, then the entire 17 $\mathbb{J}$ must be made using 5 $\mathbb{J}$ coins.
   Since 17 is not divisible by 5, this is not possible.
   If one 9 $\mathbb{J}$ bill is used, then the remaining 8 $\mathbb{J}$ must be made using 5 $\mathbb{J}$ coins.
   Since 8 is not divisible by 5, this is not possible.
   Therefore, it is not possible to make 17 $\mathbb{J}$ using these bills and coins.

3. We start with the combinations of coins and bills that make up each of 40 $\mathbb{J}$ through 44 $\mathbb{J}$, inclusive.
   If we add one 5 $\mathbb{J}$ coin to each of these combinations, we add 5 $\mathbb{J}$ to each total to get totals from 45 $\mathbb{J}$ through 49 $\mathbb{J}$, inclusive.
   We can summarize these in a table:

   | Amount (in $\mathbb{J}$) | # of 5 $\mathbb{J}$ coins | # of 9 $\mathbb{J}$ bills |
   |---|---|---|
   | 40 | 8 | 0 |
   | 41 | 1 | 4 |
   | 42 | 3 | 3 |
   | 43 | 5 | 2 |
   | 44 | 7 | 1 |
   | 45 | 9 | 0 |
   | 46 | 2 | 4 |
   | 47 | 4 | 3 |
   | 48 | 6 | 2 |
   | 49 | 8 | 1 |

4. Adding one 5 $\mathbb{J}$ coin to each of the combinations that make five consecutive amounts (like 40 $\mathbb{J}$ to 44 $\mathbb{J}$) allows us to make the next five consecutive amounts (in this case, 45 $\mathbb{J}$ to 49 $\mathbb{J}$).
   Because we can continue to do this indefinitely, we can make any amount that is at least 40 $\mathbb{J}$.
   Given any such amount, can you describe a quick way to determine a number of bills and coins that can be used to make this amount?

5. The table below shows that we can make each amount from 32 $\mathbb{J}$ to 36 $\mathbb{J}$, inclusive:

   | Amount (in $\mathbb{J}$) | # of 5 $\mathbb{J}$ coins | # of 9 $\mathbb{J}$ bills |
   |---|---|---|
   | 32 | 1 | 3 |
   | 33 | 3 | 2 |
   | 34 | 5 | 1 |
   | 35 | 7 | 0 |
   | 36 | 0 | 4 |

   Repeating the argument from 4., this table shows that we can in fact make every amount that is at least 32 $\mathbb{J}$.

It is impossible to make 31 $\mathbb{J}$ using only these coins and bills. (Can you modify the argument from 2. to show this?)

Since we cannot make 31 $\mathbb{J}$ and we can make every amount that is at least 32 $\mathbb{J}$, then 31 $\mathbb{J}$ is the largest amount that we cannot make using these bills and coins.

*Extensions:*

A. We can make 900 $\mathbb{J}$ using 100 bills and 0 coins.

Since 5 bills and 9 coins have the same value, we can trade 5 bills for 9 coins.

This means that we can make 900 $\mathbb{J}$ using 95 bills and 9 coins.

How many times can you repeat this process before no further trades are possible?

B. Try working with just 7 $\mathbb{K}$ and 9 $\mathbb{K}$ stamps first and work through similar steps to 3., 4., and 5. from earlier.

Once you know the largest amount that cannot be made with just 7 $\mathbb{K}$ and 9 $\mathbb{K}$ stamps, try introducing 11 $\mathbb{K}$ stamps to see what amounts that you previously couldn't make can now be made.

# CEMC at Home
## Grade 9/10 - Wednesday, April 15, 2020
## Flipping Glasses

In this activity, we will flip glasses according to a set of rules and try to achieve certain targets.

**You Will Need:** Three empty glasses.

**Rules:**

- On each turn, **you must flip two adjacent glasses.**

- Flipping a glass that is facing up changes it to facing down.

- Flipping a glass that is facing down changes it to facing up.



Up    Down
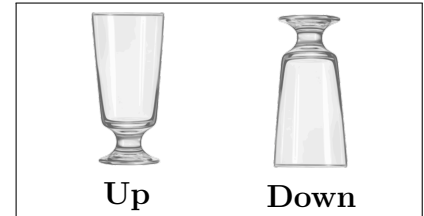
**Activities:**



1. Line up your three glasses as shown in the picture to the right. Is it possible to flip the glasses, according to the rules, so that all three glasses are facing up? If so, how many turns does it take?

2. There are exactly eight different starting positions for the three glasses.

   (a) Two possible starting positions are shown below. Complete the rest of the table.

   

   (b) From which of these eight starting positions is it possible to flip the glasses, according to the rules, so that all three glasses are eventually facing up?

   (c) From part (b), what do you notice about the original number of glasses facing down in the scenarios where you *can* flip all three glasses so that they are eventually facing up?

**Extension:** Suppose you now have eight glasses lined up as shown. Is it possible to flip the glasses, according to the rules, so that all eight glasses are eventually facing up?



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**More Info:**

Check out the CEMC at Home webpage on Wednesday, April 22 for a solution to Flipping Glasses.

You can model your solutions using a *finite state machine* as shown with the *Glasses* question on the 2012 Beaver Computing Challenge and the CEMC at Home resource *Where Am I?* from April 1.

# CEMC at Home

## Grade 9/10 - Wednesday, April 15, 2020
## Flipping Glasses - Solution

1. It is possible to flip the three glasses in two turns so that all three glasses are facing up.

   Here are images describing one possible solution:

   

   **Start**          **Turn 1**          **Turn 2**

   To visualize the transformation of the glasses after each turn (i.e. facing up or down), we can use a *finite state machine*:

   – Each circle represents a possible *state* of the three glasses, where D represents a glass facing down and U represents a glass facing up.

   – The *starting state* is the circle with the arrow "out of nowhere" pointing to it (DUD).

   – To *transition* between states, follow an arrow by performing the action describing it.

   – The desired state is the circle with the double border around it (*final state*).
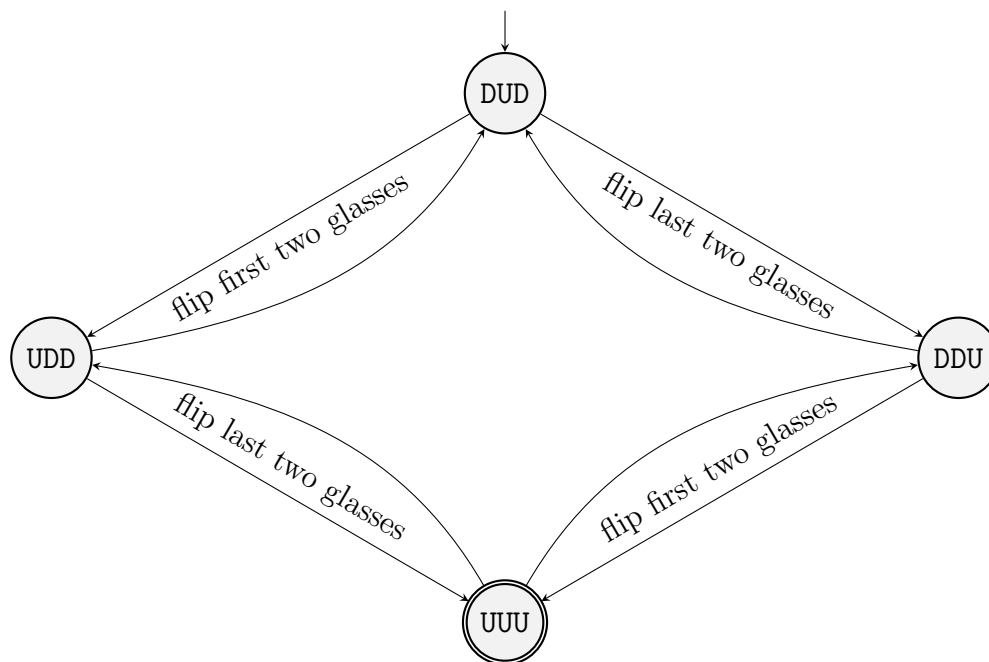


From the diagram above, we see that there are two different ways to arrive at the desired state (UUU) in exactly two turns (or two transitions from the starting state). One of these two solutions was outlined earlier (using images).

**Note:** If we flip the first and the third glass from the starting state, this would also result in the desired state of UUU. However, this is not a valid transition since these glasses are *not* adjacent.

2. (a) Here are the eight different starting positions for the three glasses.
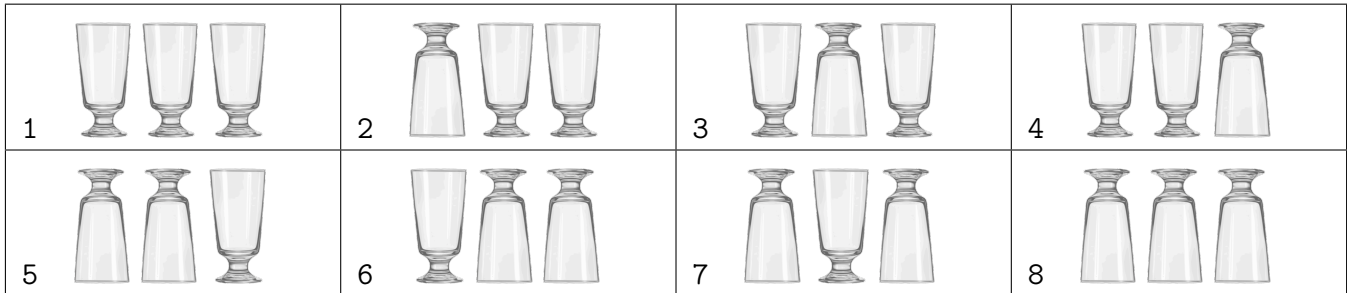


(b) You are able to flip all three glasses so that they are facing up only if you start with position 1, 5, 6, or 7. Starting position 1 already has all three glasses facing up, so we don't have to do anything. In question 1, we showed that the glasses starting in position 7 can be flipped so that all three glasses are facing up. Positions 5 and 6 can be flipped to UUU in one turn. Another way to think about this is to notice that states 1, 5, 6, and 7 are exactly the states in the finite state machine from Question 1. By following arrows, it is possible to move between any two of these four states. In particular, it is possible to get from any of states 5, 6, and 7 to state 1.

You may have also figured out that it is impossible to reach UUU from the starting states 2, 3, 4, or 8. An explanation of this will be provided in part (c).

(c) In each of the starting positions for which it is possible to flip the glasses so that all three are facing up, there are either 0 or 2 glasses facing down.

On each turn, two adjacent glasses are flipped causing one of the following three outcomes:

* If the two glasses were facing down, the number of glasses facing up increases by two.
* If the two glasses were facing up, the number of glasses facing up decreases by two.
* If one glass was facing up and the other was facing down, the number of glasses facing up does not change.

Therefore, on each turn, the number of glasses facing down either increases by two, decreases by two, or stays the same. A consequence of this is that if the number of glasses facing down at the beginning is odd, then the number of glasses facing down will always be odd, and hence can never be 0.

In starting positions 2, 3, 4, and 8, an odd number of glasses are facing down. This means that no sequence of valid flips can change any of them to the state UUU (which has 0 glasses facing down). Put differently, a state must have an even number of glasses facing down for there to be a chance to change it to UUU. In part (b), we we confirmed that it is actually possible to do this for all states starting with an even number of glasses facing down.

**Extension**: It is not possible to have all eight glasses facing up. We can generalize our reasoning from 2(c) beyond three glasses. The observation at the end of part (c) is true for any number of glasses. That is, if there are an odd number of glasses facing down at the beginning, then there will always be an odd number of glasses facing down. Since we start with an odd number of glasses (five) facing down, the number of glasses facing down will always be odd, and hence will never be 0. At best, we can reach a state where 1 glass is facing down and 7 glasses are facing up. Can you convince yourself that it is indeed possible to reach a state where exactly one glass is facing down?

# CEMC at Home features Problem of the Week

## Grade 9/10 - Thursday, April 16, 2020
## Mind Your Ps and Qs and Rs

Three digit numbers $PQR$, $QQP$ and $PQQ$ are formed using the single digits $P$, $Q$ and $R$ such that:

$$PQR + 2 = PQQ$$

AND

$$PQR \times 2 = QQP$$

Determine all possible combinations of values of $P$, $Q$ and $R$ that satisfy the two equations.

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**More Info:**

Check the CEMC at Home webpage on Thursday, April 23 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 23.

$PQR + 2 = PQQ$

## Problem of the Week
## Problem D and Solution

$PQR \times 2 = QQP$

## Mind Your Ps and Qs and Rs

### Problem

The letters $P$, $Q$, and $R$ represent single digits. Determine all possible combinations of values of $P$, $Q$ and $R$, given that: $PQR + 2 = PQQ$ and $PQR \times 2 = QQP$

### Solution

From $PQR + 2 = PQQ$, we note that the number $PQQ$ is 2 more than the number $PQR$ and that the first two digits of each number are the same. From this, $Q = R + 2$ follows. We can prove this using place value. The number $PQR = 100 \times P + 10 \times Q + R = 100P + 10Q + R$. The number $PQQ = 100 \times P + 10 \times Q + Q = 100P + 10Q + Q$.
Then,

$$\begin{aligned}(100P + 10Q + R) + 2 &= 100P + 10Q + Q \\ 100P + 10Q + R + 2 &= 100P + 11Q \\ R + 2 &= Q, \text{ as above.}\end{aligned}$$

From this expression we can obtain a restriction on the possible integer values of $R$. $R$ must be an integer from 0 to 7, inclusive. If $R = 8$, then $R = Q + 2 = 10$ and $Q$ is not a single digit.

From here, we will show two possible solutions.

### <u>Solution 1</u>

Representing $PQR \times 2 = QQP$ using place value,

$$\begin{aligned}(100P + 10Q + R) \times 2 &= 100Q + 10Q + P \\ 200P + 20Q + 2R &= 110Q + P \\ 199P &= 90Q - 2R \\ \text{Substituting } R + 2 \text{ for } Q: \quad 199P &= 90(R + 2) - 2R \\ 199P &= 90R + 180 - 2R \\ P &= \frac{88R + 180}{199}\end{aligned}$$

We are looking for an integer value of $R$ from 0 to 7 such that $88R + 180$ is a multiple of 199. The only value of $R$ that produces a multiple of 199 when substituted into $88R + 180$ is $R = 7$. When $R = 7$, $P = \frac{88R+180}{199} = \frac{88(7)+180}{199} = 4$ and $Q = R + 2 = 9$.

Therefore, the only values that satisfy the system of equations are $P = 4$, $Q = 9$ and $R = 7$.

We can verify this result:

$$PQR + 2 = 497 + 2 = 499 = PQQ \text{ and}$$

$$PQR \times 2 = 497 \times 2 = 994 = QQP.$$

## Solution 2

If $R$ must be an integer from 0 to 7, inclusive, then $Q$ must be an integer from 2 to 9, inclusive.
From the second equation, we also know that $P$ would be the units digit of $2R$.
We can now use a table to determine all possible values for $P$, $Q$ and $R$

| $Q$ | $R$ | $2R$ | $P$ | $PQR$ | $QQP$ | Does $2 \times PQR = QQP$? |
|---|---|---|---|---|---|---|
| 2 | 0 | 0 | 0 | 020 | 220 | No |
| 3 | 1 | 2 | 2 | 231 | 332 | No |
| 4 | 2 | 4 | 4 | 442 | 444 | No |
| 5 | 3 | 6 | 6 | 653 | 556 | No |
| 6 | 4 | 8 | 8 | 864 | 668 | No |
| 7 | 5 | 10 | 0 | 075 | 770 | No |
| 8 | 6 | 12 | 2 | 286 | 882 | No |
| 9 | 7 | 14 | 4 | 497 | 994 | Yes |

We will now need to verify that the final row also works for $PQR + 2 = PQQ$.

$$PQR + 2 = 497 + 2 = 499 = PQQ.$$

Therefore, the only values that satisfy the system of equations are $P = 4$, $Q = 9$ and $R = 7$.

# CEMC at Home

## Grade 9/10 - Friday, April 17, 2020
## Toothpick Challenges

In the following toothpick puzzles, you will be presented with some toothpicks as well as some instructions. Your task is to arrange, rearrange, or remove toothpicks in order to produce the desired result.
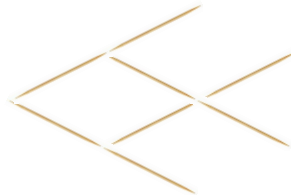
## Example

Starting with 10 toothpicks arranged into 3 squares as shown, move exactly 2 toothpicks to create a figure that has exactly 2 squares. (The solution is on the second page.)
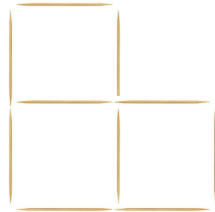
## Reversing Fish

Starting with 8 toothpicks arranged into a fish facing left as shown, move exactly 3 toothpicks in order to produce an identical fish, but now facing right.

## Many Squares to No Squares

Start with 40 toothpicks arranged into a 4 by 4 grid as shown. Notice that this grid contains many squares of various sizes. Remove exactly 9 toothpicks so that no square, of any size, remains.
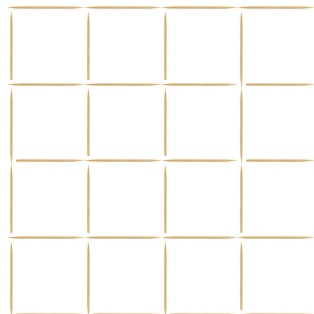
*Can you count the total number of squares in this figure? Hint: There are more than 17.*
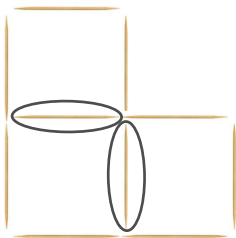
## Triangles

Using exactly 6 toothpicks, arrange the toothpicks to create exactly 4 equilateral triangles. Each triangle will have side lengths equal to the length of one toothpick.
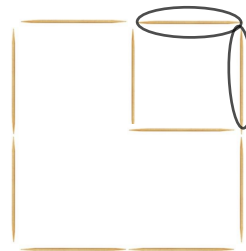


*Hint: You will not be able to lay your solution flat on a table.*

## Solution for Example



Move circled toothpicks

We now have two squares: a large 2 by 2 square, and a small 1 by 1 square.

**More Info:**

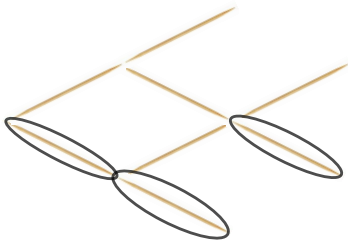Check out the CEMC at Home webpage on Friday, April 24 for a solution to Toothpick Challenges.

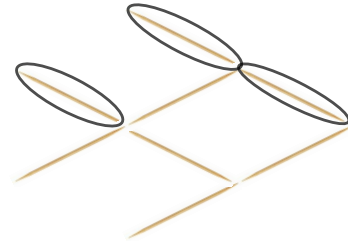# CEMC at Home
## Grade 9/10 - Friday, April 17, 2020
## Toothpick Challenges - Solutions

### Reversing Fish

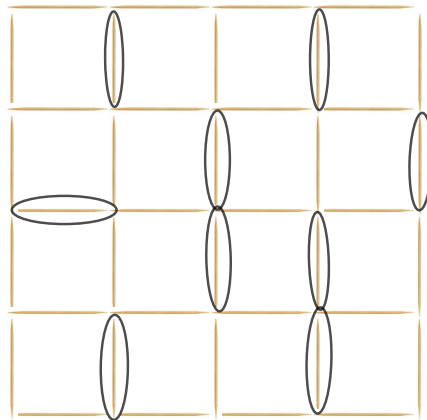Move the circled toothpicks to their corresponding places noted.

Move circled toothpicks

### Many Squares to No Squares

Remove the circled toothpicks.

### Triangles

Make a triangular based pyramid. (A regular tetrahedron.)