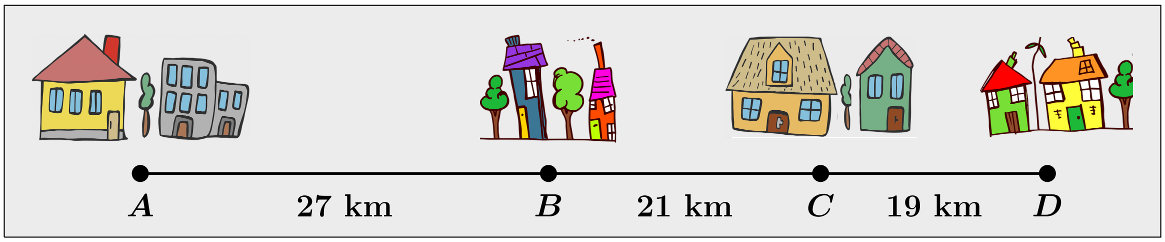
CEMC at Home

Grade 4/5/6 - Tuesday, April 14, 2020
Roadmaps to Success

Question 1: Four towns, A, B, C, and D, lie in that order along a straight road. We are told that the distance from A to B is 27 km, the distance from B to C is 21 km, and the distance from C to D is 19 km. This information is shown in the *roadmap* below.

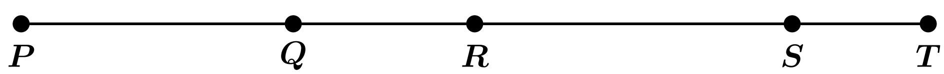


Using these distances, we can see that the distance from A to C equals the sum of the distance from A to B and the distance from B to C, which gives 27 km + 21 km = 48 km. We can keep track of the distances between each pair of towns in the chart shown below. For example, notice that the distance from A to B (27 km) is placed in the same column as A and the same row as B in the chart.

- What is the distance from B to D? Put your answer in the correct place in the chart to the right.
- What is the distance from A to D? Put your answer in the correct place in the chart to the right.

A			
27	***B***		
48	21	***C***	
__	__	19	***D***

Question 2: Five towns, P, Q, R, S, and T, lie in that order along a different straight road.



Since there are five towns, there are 10 different pairs of towns and these are listed below:

$$PQ, PR, PS, PT, QR, QS, QT, RS, RT, ST$$

Look at the chart below, which is meant to give all the distances between pairs of towns, in kilometres. Four of the distances are given: PR, PS, QS, and RT. For example, the distance from Q to S is 31 km. The six remaining distances are missing: PQ, PT, QR, QT, RS, and ST. It may be surprising to you that we can use the just the four pieces of information given here to complete the entire chart!

(a) What is the distance from R to S?

(b) What is the distance from Q to R?

(c) What is the distance from P to T?

(d) Complete the rest of the chart.

P				
__	***Q***			
29	__	***R***		
48	31	__	***S***	
__	__	29	__	***T***

To solve this problem, you might find it helpful to draw the distances given in the chart onto the roadmap above and add new distances when you find them. For example, what does the distance of 31 in the chart above represent on the roadmap? Note that the roadmap is not drawn exactly to scale.

Question 3: Suppose that five towns, U, V, W, X, and Y, lie in that order along a different straight road. Four of the distances between pairs of towns are given in the chart and six distances are missing. Unfortunately, someone made a mistake when measuring and recording the distances. Explain why it is impossible for all four of these distances to be correct.

U				
28	***V***			
102	__	***W***		
__	__	__	***X***	
__	200	125	__	***Y***

More Info:

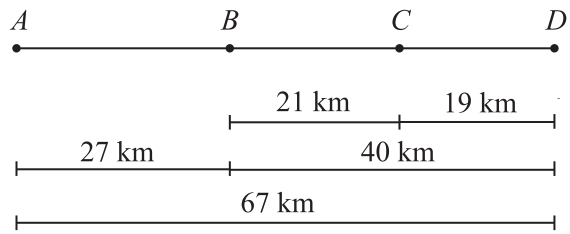
Check the CEMC at Home webpage on Tuesday, April 21 for a solution to Roadmaps to Success.

CEMC at Home

Grade 4/5/6 - Tuesday, April 14, 2020
Roadmaps to Success - Solution

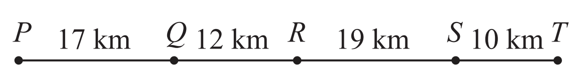
1. Since the distance from B to C is 21 km and the distance from C to D is 19 km, then the distance from B to D is 21 km + 19 km = 40 km.
   Since the distance from A to B is 27 km and the distance from B to D is 40 km, then the distance from A to D is 27 km + 40 km = 67 km.

A			
27	B		
48	21	C	
67	40	19	D

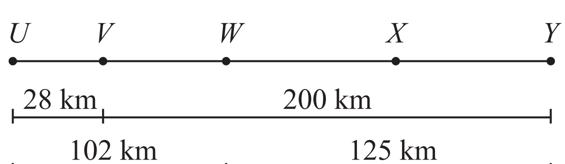


2. (a) Since the distance from P to S is 48 km and the distance from P to R is 29 km, then the distance from R to S is 48 km − 29 km = 19 km.
   (b) Since the distance from Q to S is 31 km and the distance from R to S is 19 km, then the distance from Q to R is 31 km − 19 km = 12 km.
   (c) Since the distance from P to R is 29 km and the distance from R to T is 29 km, then the distance from P to T is 29 km + 29 km = 58 km.
   (d) Since the distance from P to R is 29 km and the distance from Q to R is 12 km, then the distance from P to Q is 29 km − 12 km = 17 km.
   Since the distance from P to T is 58 km and the distance from P to Q is 12 km, then the distance from Q to T is 58 km − 17 km = 41 km.
   Finally, the distance from S to T is 10 km. Can you see why?

P				
17	Q			
29	12	R		
48	31	19	S	
58	41	29	10	T



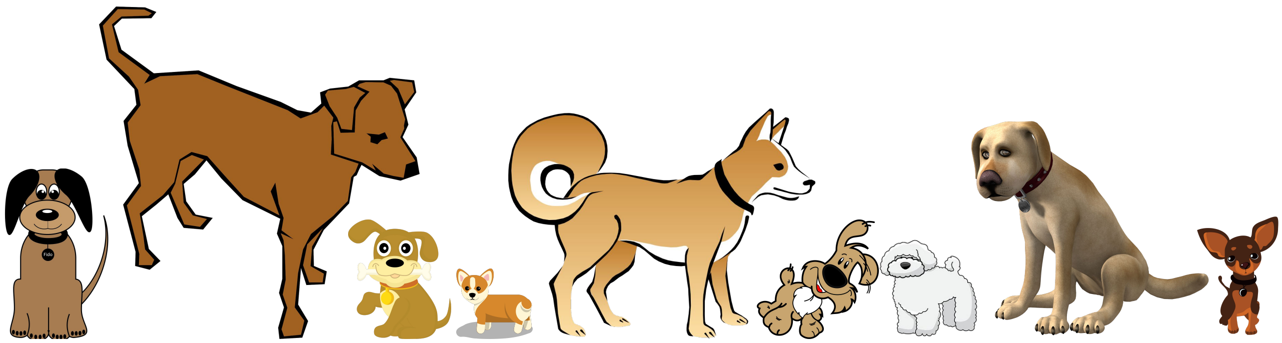
3. The distance from U to Y equals the distance from U to V plus the distance from V to Y.
   This means that the distance from U to Y equals 28 km + 200 km = 228 km.
   The distance from U to Y also equals the distance from U to W plus the distance from W to Y.
   This means that the distance from U to Y equals 102 km + 125 km = 227 km.
   We got two different answers for the distance from U to Y. This cannot be the case, so there must be a mistake in the four distances given.



CEMC at Home

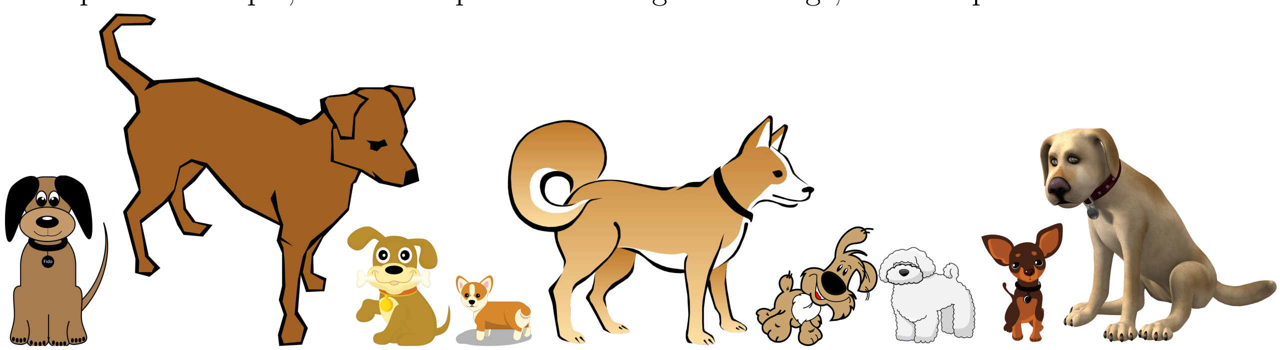
Grade 4/5/6 - Wednesday, April 15, 2020
Doggies Swapped

Consider this lineup of nine dogs of various sizes:



First lineup

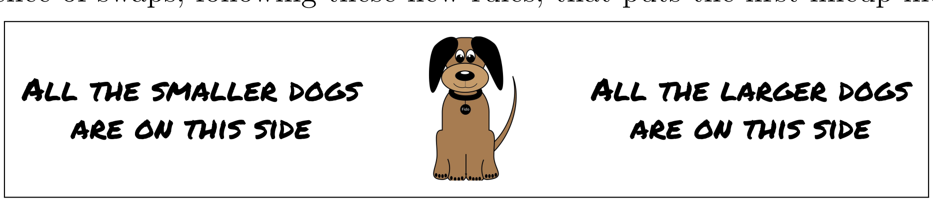
We can move dogs in the lineup by swapping them. A *swap* means two dogs exchange positions in the lineup. For example, after a swap of the two rightmost dogs, the lineup looks like this:



After one possible swap

Problem 1: The goal in this problem is to move the biggest dog (2nd from the left) to the rightmost position, and the smallest dog (4th from the left) to the leftmost position of the lineup, and to do so using the fewest swaps possible. In this problem, we can only swap two dogs that are *right beside each other*, but a dog may get swapped again after it moves into a new position in the lineup. What is the minimum number of swaps required to get the *first lineup* into this form following these rules?

Problem 2: The name of the dog in the leftmost position in the first lineup is **Spot**. The goal in this problem is to rearrange the dogs so that all dogs that are smaller than **Spot** are to **Spot**'s left and all dogs that are larger than **Spot** are to **Spot**'s right. (Otherwise, the dogs can be in any order.) In this problem, we can swap two dogs in *any positions* in the lineup, but each dog can be involved in *at most one swap*. For example, we could choose to swap the dog in the first position with the dog in the last position in the lineup, but then neither dog can be swapped again later. Can you find a sequence of swaps, following these new rules, that puts the first lineup into this form?



More Info:

Check out the CEMC at Home webpage on Wednesday, April 22 for a solution to Doggies Swapped.

CEMC at Home

Grade 4/5/6 - Wednesday, April 15, 2020
Doggies Swapped - Solution

Problem 1: The goal in this problem is to move the biggest dog (2nd from the left) to the rightmost position, and the smallest dog (4th from the left) to the leftmost position of the lineup, and to do so using the fewest swaps possible. In this problem, we can only swap two dogs that are *right beside each other*, but a dog may get swapped again after it moves into a new position in the lineup. What is the minimum number of swaps required to get the *first lineup* into this form following these rules?

Solution: Here is one way get the lineup into this form using exactly 9 swaps that follow the rules:

Remember that we can only swap two dogs that are right beside each other.

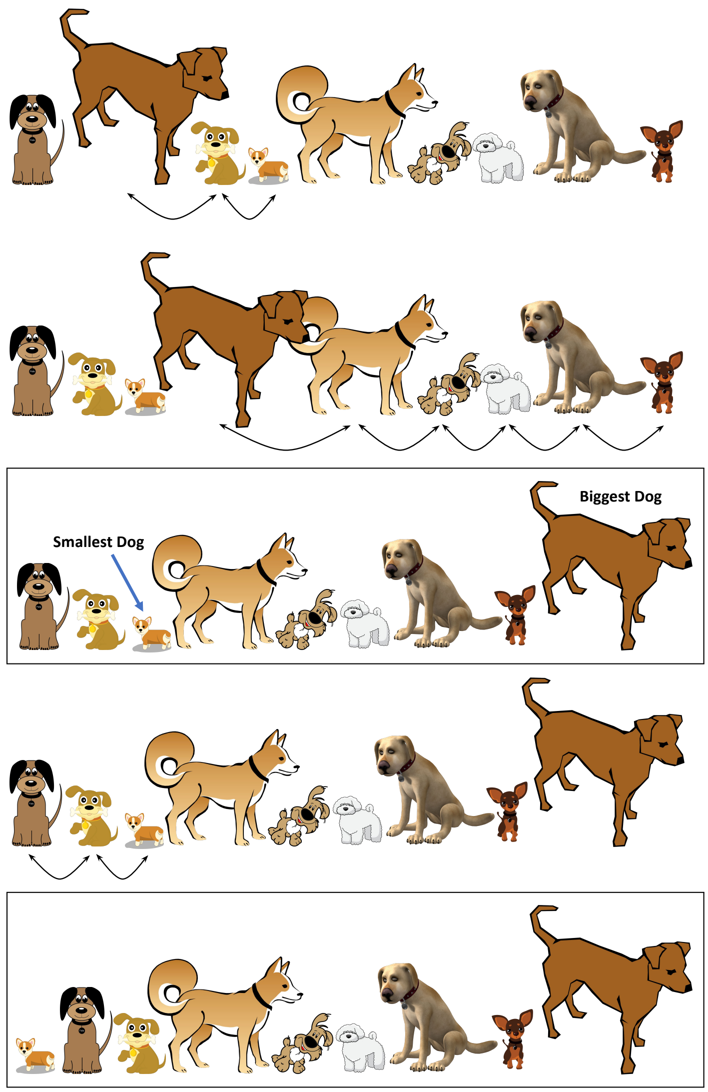
To get the biggest dog to the rightmost position, we could swap the biggest dog, again and again, with the dog immediately to its right, until the biggest dog makes it all the way to the rightmost position of the lineup.

Since the first lineup has 7 dogs to the right of the biggest dog, we can move the biggest dog to the rightmost position using 7 swaps in a row.

Notice that after we have finished moving the biggest dog in the way outlined above, the smallest dog has ended up 3rd from the left in the new lineup. This means we can move the smallest dog to the leftmost position by making 2 more swaps.

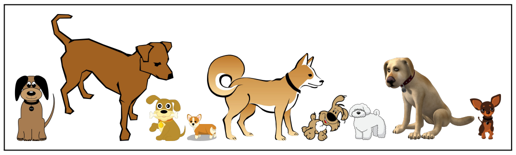
We swap the smallest dog with the dog immediately to its left, and then repeat this once more.

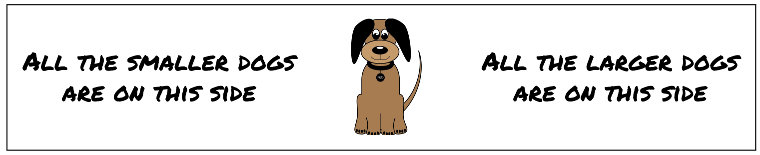
Finally, after these 9 swaps, we end up with the lineup of dogs to the right which is in the correct form.



You may have thought about this question long enough to be convinced that you cannot achieve the goal in fewer than 9 swaps. It turns out that 9 swaps is the best we can do. Can you explain why? Why can it not be done in 8 (or fewer) swaps? Think about how many swaps the biggest dog must take part in (at least 7), the number of swaps the smallest dog must take part in (at least 3), and how many of these swaps could involve *both* the biggest dog and the smallest dog.

Problem 2: The name of the dog in the leftmost position in the first lineup is **Spot**. The goal in this problem is to rearrange the dogs so that all dogs that are smaller than **Spot** are to **Spot**'s left and all dogs that are larger than **Spot** are to **Spot**'s right. In this problem, we can swap two dogs in *any positions* in the lineup, but each dog can be involved in *at most one swap*. Can you find a sequence of swaps, following these new rules, that puts the first lineup into this form?





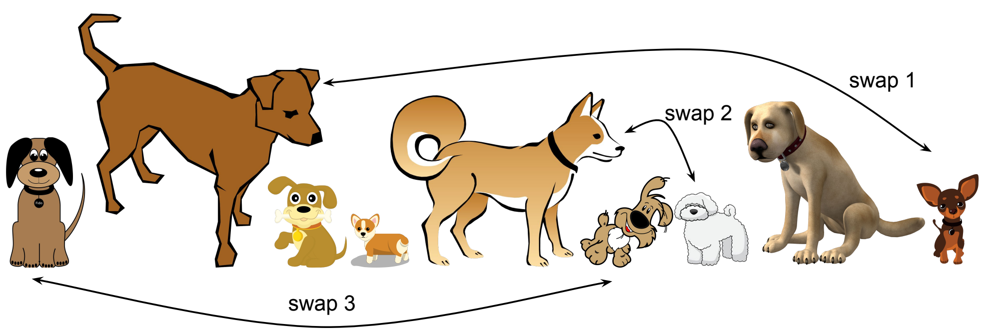
Solution:

Here is one way get the lineup into this form using exactly 3 swaps that follow the rules:

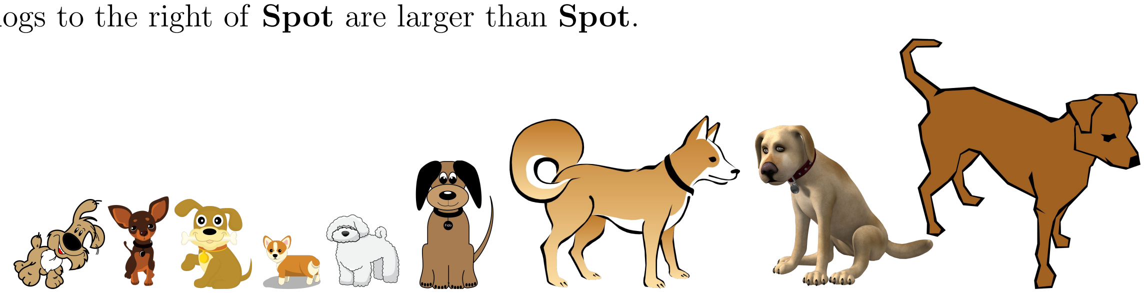
In the starting lineup, find the dog furthest to the right that is smaller than **Spot** and the dog furthest to the left that is bigger than **Spot**. Swap these two dogs (swap 1).

In the new lineup (after swap 1), find the dog furthest to the right that is smaller than **Spot** and the dog furthest to the left that is bigger than **Spot**. Swap these two dogs (swap 2).

In the new lineup (after swap 2), find the dog furthest to the right that is smaller than **Spot**. Swap **Spot** and this dog (swap 3).



Notice that each dog is involved in at most one of these three swaps. After these three swaps, we get the following lineup of dogs where all of the dogs to the left of **Spot** are smaller than **Spot** and all of the dogs to the right of **Spot** are larger than **Spot**.



Computer Science Connections:

In Computer Science, it is important to be able to sort lists of data, that is, arrange lists into a meaningful order. There are many techniques (also known as *algorithms*) for sorting lists. Some algorithms are better than others.

One straightforward algorithm to sort lists is known as *bubble sort*. This technique passes over the list of values multiple times. On each pass, pairs of values that are beside each other are compared. The algorithm starts by comparing the first two values, then moves on to the second and third values, then the third and fourth values, and so on. If the values in a pair are already in the correct order compared to each other, then nothing is done. If the values in the pair are out of order compared to each other, then they are swapped. After the first pass over the list, you are guaranteed to have the largest value at the end of the list. After multiple passes, you end up with a sorted list. This tends to be a very slow way to sort a list.

Another algorithm to sort lists is known as *quicksort*. This technique chooses a value known as a *pivot*, and then swaps values around until the pivot ends up in a spot in the list where all values before the pivot are smaller than the pivot and all values after the pivot are larger than the pivot. Next, the list is split into two parts: the part of the list before the pivot and the part after the pivot. The process is then repeated on each of these smaller lists. The process of moving pivots and splitting lists is repeated until the list is sorted. In most cases, this algorithm is much faster than bubble sort.

There are many other ways you can sort a list. Can you come up with your own algorithm?

CEMC at Home features Problem of the Week

Grade 4/5/6 - Thursday, April 16, 2020
What's My Number?

I am a 3-digit even number.

The sum of my three digits is 20.

I am greater than 40×10.

I am less than $1000 \div 2$.

What number am I?

More Info:

Check the CEMC at Home webpage on Thursday, April 23 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 23.

This CEMC at Home resource is the current grade 3/4 problem from Problem of the Week (POTW). This problem was developed for students in grades 3 and 4, but is also appropriate for students in grades 5 and 6. POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week, to view this week's grade 5/6 problem, and to find many more past problems and their solutions, visit the Problem of the Week webpage.

Problem of the Week
Problem A and Solution
What's My Number?

Problem

I am a 3-digit even number.

The sum of my three digits is 20.

I am greater than 40×10.

I am less than $1000 \div 2$.

What number am I?

Solution

Since $40 \times 10 = 400$, we know the number is greater than 400. Since the number is even, the smallest possible number is 402.

Since $1000 \div 2 = 500$, we know the number is less than 500. Since the number is even, the largest possible number is 498.

So we are looking for an even number between 402 and 498, inclusive. We could check each of the numbers in that range, to see which one has digits that add up to 20. However, that would mean checking 49 numbers. It would be better to reduce the range of numbers to check, if possible.

Here is one way of thinking about the solution:

Since the possible numbers are from 402 to 498, the first digit of the number is 4. Since the sum of the digits is 20, then the middle and last digit must sum to $20 - 4 = 16$. Since the number is even, its last digit is 0, 2, 4, 6, or 8.

If the last digit is 0, then the middle digit must be $16 - 0 = 16$, which is not a digit from 0 to 9.
If the last digit is 2, then the middle digit must be $16 - 2 = 14$, which is not a digit from 0 to 9.
If the last digit is 4, then the middle digit must be $16 - 4 = 12$, which is not a digit from 0 to 9.
If the last digit is 6, then the middle digit must be $16 - 6 = 10$, which is not a digit from 0 to 9.
If the last digit is 8, then the middle digit must be $16 - 8 = 8$, which is a valid digit.

We have examined all possible cases and the only number satisfying all of the conditions is 488. The number we are looking for is 488.

Teacher's Notes

Here is another way of thinking about the solution:

> Since the numbers from 402 to 498 all start with the digit 4, and the sum of all three digits must be 20, then the sum of the last two digits of the number must be $20 - 4 = 16$.
>
> Since the largest possible digit is 9, then the other digit of the number must be at least $16 - 9 = 7$. So each of the last two digits must be in the range 7 to 9.
>
> Since we are only considering even numbers, then the solution must be a number that ends with an 8.
>
> So we can look at even numbers, that start with a 4, end with an 8, and where the middle digit is at least 7. The possibilities are: 478, 488, and 498. Now we can check the sum of the digits of these numbers:
>
> - $4 + 7 + 8 = 19$
> - $4 + 8 + 8 = 20$
> - $4 + 8 + 9 = 21$

So the only number that satisfies all of the requirements is 488.

CEMC at Home

Grade 4/5/6 - Friday, April 17, 2020
Toothpick Triangles

In this activity, we will explore how many different triangles we can make if our triangles can only have side lengths that are equal to whole number factors of the number 24.

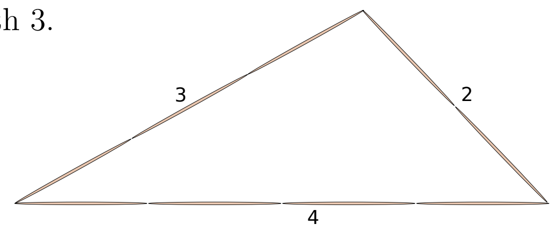
The whole number *factors* of any whole number N are the whole numbers which divide evenly into N. For example, 12 has six whole number factors. They are 1, 2, 3, 4, 6, and 12.

You Will Need:

- A pencil and a piece of paper
- A flat surface to work on
- Around 30 toothpicks

How To Play:

1. Make a list of all whole number factors of 24.
2. Think of each toothpick as having a length of 1 unit. We will line up different numbers of toothpicks in straight lines, end-to-end, to make sides of triangles. If a side is made up of 3 toothpicks, then we will say this side has length 3.
3. Choose three different factors of 24. Using the toothpicks, try and create a triangle that has side lengths equal to these three factors of 24.



   For example, think about the three factors 2, 3 and 4. You will need 2 toothpicks to form a side of length 2, 3 toothpicks to form a side of length 3, and 4 toothpicks to form a side of length 4. These three sides can be arranged on a flat surface to form a triangle as shown above. If you start with the factors 2, 3, and 6 instead, a triangle cannot be formed. Can you figure out why?

4. Form as many different triangles as you can which have side lengths equal to three different factors of 24. Make a list of the side lengths of each of the triangles you form.

   You will have to play with the positions of the toothpick sides to see whether they fit together to form a triangle. You might find it helpful to tape the toothpicks together while you work with a particular side length.

Follow-up Questions:

- There are groups of three factors of 24 that *cannot* be used to form a triangle. Did you discover some of these groups of factors? Why could you not use them to form triangles?
- Think about what you discovered while exploring the factors of 24. Can you find three other numbers between 10 and 50 with factors that can be used to form many different triangles? Are there numbers for which no triangles can be formed?

  If you are allowed to use the same factor more than once to make the side lengths, then you can always make triangles! (Why?) If we want three different factors then it is not always possible.

More info:

Check out the CEMC at Home webpage on Friday, April 24 for a solution to Toothpick Triangles.

CEMC at Home

Grade 4/5/6 - Friday, April 17, 2020
Toothpick Triangles - Solution

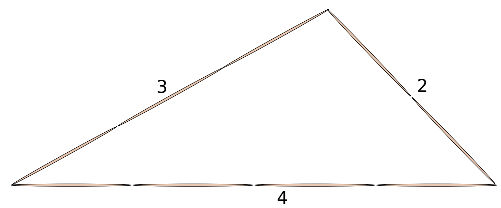
Activity: Using toothpicks, each with a length of 1 unit, form as many different triangles as you can which have side lengths equal to three different factors of 24.

Discussion: The number 24 has eight whole number factors: 1, 2, 3, 4, 6, 8, 12, 24.

If we choose three different factors from this list, then we may or may not be able to form a triangle having these side lengths. Here are some examples:

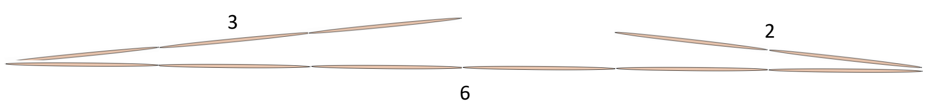
A triangle can be formed using the factors 2, 3, and 4.

If we form sides using 2, 3 and 4 toothpicks, we can arrange them so that each pair of sides meets at a point and a triangle is formed.



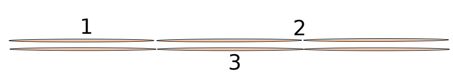
A triangle cannot be formed using the factors 2, 3, and 6.

If we form sides using 2, 3 and 6 toothpicks, then we cannot arrange the sides so that each pair of sides meets at a point. The sides having 2 and 3 toothpicks must touch the ends of the side with 6 toothpicks, but they also must tough each other. As the picture shows, the sides 2 and 3 are not long enough to do so. Since $2 + 3$ is less than 6, we cannot "close" the triangle.



A triangle cannot be formed using the factors 1, 2, and 3.

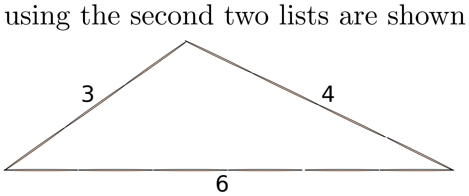
The sides having 1 and 2 toothpicks must touch the ends of the side with 3 toothpicks, but they also must touch each other. As the picture shows, the only way to do this is to lay the toothpicks directly on top of each other, but this does not form a triangle. Since $1 + 2$ is equal to 3 we cannot form a triangle.

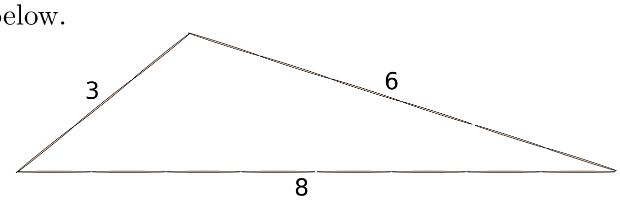


Triangles can only be formed if you start with one the five lists of factors in the box shown on the right.

We have already shown that the first list of factors works, and the triangles formed using the second two lists are shown below.

2, 3, 4
3, 4, 6
3, 6, 8
4, 6, 8
6, 8, 12





The triangles that are formed using the last two lists are just scaled versions of two triangles shown earlier, with sides lengths all doubled. *Which ones?*

No other choices of three factors will allow you to form triangles. *Can you figure out why?*

It turns out that the following is true:

Any group of three factors for which one of the numbers is greater than or equal to the sum of the other two numbers cannot be used to form a triangle. Otherwise, a triangle can be formed.

- The list 1, 2, 3 *cannot* be used to form a triangle because 3 is equal to 1 + 2.
- The list 2, 3, 6 *cannot* be used to form a triangle because 6 is greater than 2 + 3.
- The list 2, 3, 4 *can* be used to form a triangle because 2 + 3 is greater than 4, 3 + 4 is greater than 2, and 2 + 4 is greater than 3.

Follow-up Questions:

Here are some facts about the factors of the numbers between 10 and 50:

- If you work with the factors of 12, 20, 24, 30, 36, 40, 42, or 48, then you can make at least one triangle. The factors of all other numbers result in no triangles.

  Since the numbers 12, 36, and 48 also have factors 1, 2, 3, 4, and 6 (among other factors), we know from our earlier work with the factors of 24 that we can form triangles using these factors (and possibly using others as well).

  If you use the factors of a prime number, like 11, then since there are only two factors to work with (1 and 11), you have no chance at making a triangle following the rules. However, if you allow for triangles with equal side lengths, then you can always make triangles! For example, you can make an equilateral triangle with side length 1.

- You can make the most triangles if you work with the factors of 48. The next highest number comes from using the factors of 36, and then 24 (which we worked with).
- You can make only one triangle if you work with the factors of 20.