CEMC at Home

Grade 11/12 - Tuesday, April 14, 2020
One Equation and Two Unknowns

In this problem, we will look at finding solutions to the equation $x^2 - y^2 = n$ where x, y and n are all positive integers.

1. The equation $x^2 - y^2 = 75$ is satisfied by three pairs (x, y) of positive integers. Can you find these three pairs?

   Two of the pairs are not too hard to find by writing out a list of perfect squares or by trial and error. You'll need to be a lot more patient to find the third pair using these methods; you might try using a spreadsheet or writing a program to do this instead.

2. Determine the pair (x, y) of positive integers for which $x + y = 175$ and $x - y = 3$.

   Using our knowledge of differences of squares, $x^2 - y^2 = (x + y)(x - y)$. This means that if $x + y = 175$ and $x - y = 3$, then $x^2 - y^2 = (x + y)(x - y) = 175 \cdot 3 = 525$.

3. There are six pairs (x, y) of positive integers for which $x^2 - y^2 = 525$. Determine all such pairs.

   Note that one way to factor 525 is $525 = 175 \cdot 3$. In question 2. we found one solution to the given equation corresponding to this factor pair of 525. Can you use other factor pairs of 525 to find other solutions the given equation?

Extensions

A. Are there pairs (x, y) of positive integers for which $x^2 - y^2 = 210$. Why or why not?

B. Determine the number of pairs (x, y) of positive integers for which $x^2 - y^2 = 7!$.

Remember that 7! is the product of the positive integers from 1 to 7, inclusive; that is,

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

More Info:

Check out the CEMC at Home webpage on Tuesday, April 21 for a solution to One Equation and Two Unknowns.

CEMC at Home

Grade 11/12 - Tuesday, April 14, 2020
One Equation and Two Unknowns - Solution

1. We start by writing out the squares of the first 15 positive integers:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225$$

From this list, we might see that
$$100 - 25 = 75$$
and
$$196 - 121 = 75$$
From these, we see that $10^2 - 5^2 = 75$ and $14^2 - 11^2 = 75$, which tell us that $(x, y) = (10, 5)$ and $(x, y) = (14, 11)$ are solutions to the equation $x^2 - y^2 = 75$.
The third solution is $(x, y) = (38, 37)$, since $38^2 - 37^2 = 1444 - 1369 = 75$.
The solutions to 2. and 3. below will demonstrate one way to find this last solution.
Alternatively, we can rearrange the given equation to obtain $x^2 = y^2 + 75$.
Since $x > 0$, then $x = \sqrt{y^2 + 75}$. We could then write a computer program or use a spreadsheet to calculate the values of $\sqrt{y^2 + 75}$ starting from $y = 1, 2, 3, \ldots$ until we get a third value of $\sqrt{y^2 + 75}$ that is an integer.

2. If $x + y = 175$ and $x - y = 3$, then $(x + y) + (x - y) = 175 + 3$ or $2x = 178$, which gives $x = 89$.
If $x + y = 175$ and $x = 89$, then $y = 175 - x = 175 - 89 = 86$.
Therefore, $(x, y) = (89, 86)$.

3. First, we write 525 as a product of prime numbers:

$$525 = 105 \cdot 5 = 21 \cdot 5 \cdot 5 = 7 \cdot 3 \cdot 5 \cdot 5 = 3 \cdot 5^2 \cdot 7$$

This means the positive divisors of 525 are

$$1, 3, 5, 7, 15, 21, 25, 35, 75, 105, 175, 525$$

(We can find these by combining the prime factors of 525 in various ways.)
Since 525 has 12 positive divisors, then there are 6 ways of factoring 525 as the product of two positive integers:

$$525 = 525 \cdot 1 = 175 \cdot 3 = 105 \cdot 5 = 75 \cdot 7 = 35 \cdot 15 = 25 \cdot 21$$

Since $x^2 - y^2 = (x + y)(x - y)$, we can use the factorizations above to determine values for $x + y$ and $x - y$ and from them, find x and y as in 2.
For example, suppose that $x + y = 525$ and $x - y = 1$.
If these equations are true, then $x^2 - y^2 = (x + y)(x - y) = 525 \cdot 1 = 525$.
If $x + y = 525$ and $x - y = 1$, then $(x + y) + (x - y) = 525 + 1$ which gives $2x = 526$ and so $x = 263$.
Using $x + y = 525$ and $x = 263$, we obtain $y = 525 - x = 525 - 263 = 262$.

We can verify that $(x, y) = (263, 262)$ is a solution to the equation $x^2 - y^2 = 525$.
All 6 cases can be seen in the following table:

$x+y$	$x-y$	$2x = (x+y) + (x-y)$	x	$y = (x+y) - x$
525	1	526	263	262
175	3	178	89	86
105	5	110	55	50
75	7	82	41	34
35	15	50	25	10
25	21	46	23	2

We can verify that each of the pairs in this table is a solution to the equation $x^2 - y^2 = 525$.

Can you explain why there cannot be any other positive integer solutions?

Extensions:

A. There are no such pairs. If you make a table like the one in the solution to 3. using each factorization of 210, you will see that x and y are never integers (in fact, each of x and y is always halfway between two integers).
It turns out that when x and y are integers, then $x + y$ and $x - y$ have to be both even or both odd. (Try thinking about the various combinations of x and y being even and odd.)
This means that $(x + y)(x - y)$ is either odd (if both factors are odd) or a multiple of 4 (if both factors are even). Since 210 is neither odd nor a multiple of 4, then there cannot be any integer solutions.

B. We note that 7! is a multiple of 4. One way to solve this problem is to count the number of ways of factoring 7! as the product of two positive even integers. Using the thinking from A. and 3., this will allow us to find all of the positive integer solutions to $x^2 - y^2 = 7!$. Can you think through the steps to convince yourself that this is true?

CEMC at Home

Grade 11/12 - Wednesday, April 15, 2020

Escape Room

In the April 8 resource Silly Square Roots, we explored connections between mathematics and computer science. We continue that here, but instead of using computer science to investigate a past math contest problem, we will use mathematics to investigate Problem J5/S2 from the 2020 Canadian Computing Competition (CCC). No computing experience is needed.

Escape Room

An *escape room* is an M-by-N grid with each position (cell) containing a positive integer. The rows are numbered $1, 2, \ldots, M$ and the columns are numbered $1, 2, \ldots, N$. We use (r, c) to refer to the cell in row r and column c.

You start in the cell at $(1, 1)$ and can escape when you reach the cell at (M, N).

If you are in a cell containing the value x, then you can jump to any cell (a, b) with $1 \leq a \leq M$ and $1 \leq b \leq N$ that satisfies $a \times b = x$. For example, from a cell containing a 6, there are up to four cells you can jump to: $(2, 3)$, $(3, 2)$, $(1, 6)$, or $(6, 1)$. If the room is a 5-by-6 grid, there is no row 6 and therefore, only the first three jumps would be possible.

Consider the following example of a 3-by-4 escape room.

	Column 1	Column 2	Column 3	Column 4
Row 1	3	10	8	14
Row 2	1	11	12	12
Row 3	6	2	3	9

Starting in the cell at $(1, 1)$ which contains a 3, one possibility is to jump to the cell at $(1, 3)$. This cell contains an 8 so from it, you could jump to the cell at $(2, 4)$. This jump brings you to a cell containing 12 from which you can jump to $(3, 4)$ and escape.

Questions

1. Give a different sequence of jumps that allows you to escape the room in the example above.
2. Place positive integers in the empty cells of the room below so that it is possible to escape the room visiting each of the six cells along the way. Cells can be visited more than once.

	Column 1	Column 2	Column 3
Row 1	3		
Row 2	1		5

3. In this question, we will consider the 2-by-2 escape rooms for which each entry in the grid is a positive integer less than or equal to 4.
   (a) How many escape rooms of this form are there in total?
   (b) Describe all escape rooms of this form from which it is actually possible to escape.
   (c) One escape room of this form is chosen at random. What is the probability that it is possible to escape from this room?

More Info:

Check out the CEMC at Home webpage on Wednesday, April 22 for a solution to Escape Room.

If you have some programming experience (not much is needed!), you can try programming solutions to past problems from the CCC and test them the using CCC Online Grader.

CEMC at Home

Grade 11/12 - Wednesday April 15 2020
Escape Room - Solutions

Solution to Question 1

Another way to reach the cell at $(3, 4)$ and escape is to start in the cell at $(1, 1)$ which contains a 3, jump to the cell at $(3, 1)$ which contains a 6, jump to the cell at $(2, 3)$ which contains a 12, and jump to the cell at $(3, 4)$.

Solution to Question 2

It is possible to escape the room below by visiting each cell at least once.

	Column 1	Column 2	Column 3
Row 1	3	6	4
Row 2	1	2	5

One way to do this is to visit cells in the following order:

$$(1, 1), (1, 3), (2, 2), (2, 1), (1, 1), (1, 3), (2, 2), (1, 2), (2, 3)$$

Solution to Question 3

(a) Each 2-by-2 grid has exactly 4 entries. Since each entry in the grid must be a positive integer less than or equal to 4, it must be one of 1, 2, 3, or 4. Since there are 4 choices for each of the 4 entries in the grid, there are $4^4 = 256$ possible rooms.

(b) Consider *any* 2-by-2 room. If the cell at $(1, 1)$ contains a 4, then it is possible to immediately jump to $(2, 2)$ and escape. The only other way to escape is to jump to $(2, 2)$ from $(1, 2)$ or $(2, 1)$. This means that at least one of these cells must contain a 4 and $(1, 1)$ must contain a 2.

In summary, it is possible to escape from a 2-by-2 room exactly if it matches at least one of the three possibilities given below where a, b, c, d, e, f and g are positive integers.

	Column 1	Column 2
Row 1	4	a
Row 2	b	c

	Column 1	Column 2
Row 1	2	4
Row 2	d	e

	Column 1	Column 2
Row 1	2	f
Row 2	4	g

This describes *all* 2-by-2 rooms from which it is possible to escape, not just the ones that have entries that are at most 4. To describe only the rooms relevant to this question, we just restrict the parameters a, b, c, d, e, f, and g above to be positive integers less than or equal to 4.

(c) We are considering 2-by-2 rooms such that each integer in the grid is less than or equal to 4. We want to determine the probability $\frac{a}{b}$, where b is the total number of rooms of this form and a is the number of rooms of this form from which it is possible to escape.

From part (a), we have $b = 256$.

From part (b), since the parameters a, b, c, d, e, f, and g are each 1, 2, 3, or 4, there are

- $4 \times 4 \times 4 = 64$ ways to choose a, b and c,
- $4 \times 4 = 16$ ways to choose d and e, and
- $4 \times 4 = 16$ ways to choose d and e.

This gives a total of $64 + 16 + 16 = 96$ ways to pick the values for the parameters, but since there is some overlap between the bottom two sets of rooms from part (b), shown below, we have over counted the number of rooms.

	Column 1	Column 2
Row 1	2	4
Row 2	d	e

	Column 1	Column 2
Row 1	2	f
Row 2	4	g

More precisely, we have counted rooms with $d = f = 4$ twice. To account for this overlap, we must subtract 4 because there are exactly 4 rooms that appear in both sets above.

Therefore, the probability is $\frac{96 - 4}{256} = \frac{23}{64}$.

More Info:

If you have some programming experience (not much is needed!), you can try programming solutions to past problems from the Canadian Computing Competition and test them the using CCC Online Grader.

CEMC at Home features Problem of the Week

Grade 11/12 - Thursday, April 16, 2020

CUBES

Certain numbers have interesting properties. For example, $1^3 + 5^3 + 3^3 = 153$. That is, the sum of the cubes of the individual digits of the positive integer 153 is the number itself. This may lead you to ask a question like, "Are there other such numbers?" (Yes there are, but that is not our concern today.)

The number 512 stands alone as a three-digit positive integer with three different digits such that the cube of the sum of the digits equals the number itself. That is, $(5 + 1 + 2)^3 = 512$. This is the only three-digit positive integer with three distinct digits that has this property.

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number.

That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that

$$(C+U+B+E+S)^3 = CUBES$$

More Info:

Check the CEMC at Home webpage on Thursday, April 23 for the solution to this problem. Alternatively, subscribe to Problem of the Week at the link below and have the solution, along with a new problem, emailed to you on Thursday, April 23.

This CEMC at Home resource is the current grade 11/12 problem from Problem of the Week (POTW). POTW is a free, weekly resource that the CEMC provides for teachers, parents, and students. Each week, problems from various areas of mathematics are posted on our website and e-mailed to our subscribers. Solutions to the problems are e-mailed one week later, along with a new problem. POTW is available in 5 levels: A (grade 3/4), B (grade 5/6), C (grade 7/8), D (grade 9/10), and E (grade 11/12).

To subscribe to Problem of the Week and to find many more past problems and their solutions visit: https://www.cemc.uwaterloo.ca/resources/potw.php

Problem of the Week

$(C+U+B+E+S)^3 = CUBES$

Problem E and Solution

CUBES

Problem

Find all five-digit positive integers with distinct digits such that the cube of the sum of the digits equals the original number. That is, find all five-digit positive integers of the form $CUBES$ with distinct digits such that $(C+U+B+E+S)^3 = CUBES$.

Solution

A straightforward approach to solving this problem is to determine the smallest possible number and the largest possible number. Then, work at finding the numbers in that range that satisfies the given property.

The smallest five-digit number with distinct digits is 10 234. Since $\sqrt[3]{10\,234} \doteq 21.7$, the smallest number to consider is $22^3 = 10\,648$. The largest sum of five distinct digits is $9+8+7+6+5 = 35$, so the largest number to consider is $35^3 = 42\,875$. The possibilities, if any exist, are from 22^3 to 35^3. We need to examine these cubes to find the solution.

Number	Number^3	Sum of the Digits	Has the Property?
22	10 648	19	no, $22 \neq 19$
23	12 167		no, digits not distinct
24	13 824	18	no, $24 \neq 18$
25	15 625		no, digits not distinct
26	17 576		no, digits not distinct
27	19 683	27	Yes, $(1+9+6+8+3)^3 = 19683$
28	21 952		no, digits not distinct
29	24 389	26	no, $29 \neq 26$
30	27 000		no, digits not distinct
31	29 791		no, digits not distinct
32	32 768	26	no, $32 \neq 26$
33	35 937		no, digits not distinct
34	39 304		no, digits not distinct
35	42 875	26	no, $35 \neq 26$

Since we have examined all possibilities, we can conclude that 19 683 is the only five-digit positive integer with distinct digits such that the cube of the sum of the digits of the number equals the original number.

CEMC at Home

Grade 11/12 - Friday, April 17, 2020
Order from Randomness

Today we will explore a famous geometric figure known as the *Sierpinski triangle*. The Sierpinski triangle is an example of an object called a *fractal* and is formed by the repeated application (or iteration) of a certain process.

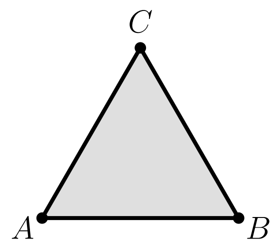
You will need:

- A blue pen and a black pen (or two different coloured pens)
- A ruler
- A six-sided die

Set up:

- On a piece of paper, draw an equilateral triangle with sides of length 16 cm. Label the vertices A, B, and C. *What tools might you use to make sure you draw an accurate triangle?*
- At various times, you will need to randomly choose one of the three vertices of the triangle. To do so, use a standard six-sided die and the following conversion table.

Number Rolled	Vertex
1 or 2	A
3 or 4	B
5 or 6	C



Instructions for Part I:

1. Start at the point A.

   We will call the point we are working with the "current point".

2. Roll the die to choose a second point according to the table above.

   If you roll a 1 or a 2 here, roll the die again until you get a number which is not 1 or 2.

3. Use your ruler to locate the midpoint between the current point and the point corresponding to your roll. Use your blue pen to draw a dot at this midpoint.

4. Repeat Steps 2 and 3 now starting with the midpoint most recently drawn (instead of A). Roll the die to choose a second point (A, B, or C). Draw the midpoint between the current point and the point corresponding to your roll. This new point may not be on the perimeter of $\triangle ABC$.

   Notice that from now on you will not need to re-roll the die if you get a 1 or a 2.

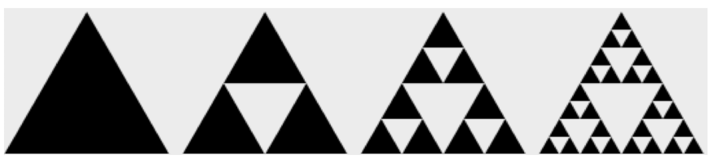
5. Repeat this process until you have drawn at least 10 new points.

   It is important to carefully draw the midpoints in Step 3 since accuracy will be important for our follow-up questions.

Look at the blue dots you drew in and on $\triangle ABC$. You might think that they are drawn not following any particular pattern but let's investigate this further!

Instructions for Part II:

1. Starting with a new copy of $\triangle ABC$, locate the midpoints of the sides of the triangle. Using your ruler and the black pen draw a line between each pair of midpoints. (There are 3 pairs.)
2. You should now have four smaller triangles: one in each corner of the original triangle, which we will call the *corner triangles*, and one *upside-down triangle* in the middle.
3. Perform Step 1 on each of the 3 corner triangles. After this, you should have twelve new even smaller triangles: 9 smaller corner triangles and 3 smaller upside-down triangles.
4. Perform Step 1 on each of the twelve corner triangles. After this, you should have 36 new smaller triangles: 27 smaller corner triangles and 9 smaller upside-down triangles.
5. With your black pen shade all of the 27 smaller corner triangles.

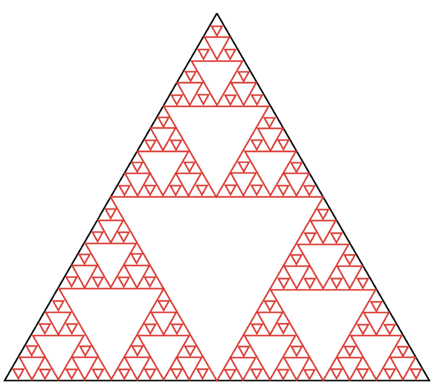


What do you notice when you compare the figure you created in Part I with the figure you created in Part II? If you were precise enough when placing the blue dots for the midpoints in Part I, then you should notice that all your dots ended up in or on the shaded regions in the figure from Part II, and that none of the dots are in the non-shaded regions!

Can you explain why all of the blue dots ended up in the shaded regions?

Follow-up Questions:

Imagine that at each iteration we remove (or cut out) the upside-down triangles from the figure.



1. After the first iteration, what is the total area of the three triangles remaining after the removal of the upside-down triangle? (Recall that our initial triangle has side lengths of 16 cm.)
2. After the second iteration, what is the total area of the nine triangles remaining after the removal of the upside-down triangles?
3. After the nth iteration, what is the total area of the triangles remaining after the removal of the upside-down triangles?

Try to visualize the figure we would obtain if we repeated this process a very large number of times. Imagine what would happen if we could repeat this process infinitely many times. The figure obtained is known as the *Sierpinski triangle*. It is impossible to draw the Sierpinski triangle, but with a bit of imagination, it is possible to visualize it and explore some of its interesting properties. Mathematicians have ways of formalizing various properties of this figure, including its area. What do you think the area of the Sierpinski triangle should be? You can explore this question on your own.

More Info:

Check the CEMC at Home webpage on Friday, April 24 for further discussion on this activity.

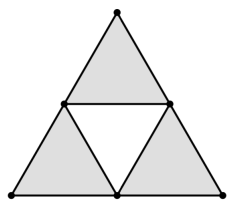
To learn more about fractals, check out this Math circles lesson.

CEMC at Home

Grade 11/12 - Friday, April 17, 2020
Order from Randomness - Solutions

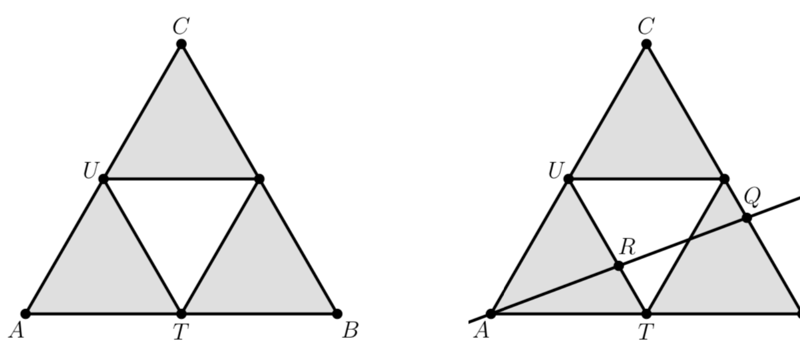
Question: Why did all of the blue dots drawn in Part I of the activity end up in the shaded regions of the figure drawn in Part II?

Discussion: Consider the figure obtained after performing steps 1 and 2 from Part II of the activity. We count the boundaries of the triangular regions as being part of the shaded regions, which we have coloured light grey instead of black for this explanation.



Assume that P is the most recently drawn blue dot. Assume also that after we draw P, we roll a 1 on the die. This means that we have chosen vertex A for the next iteration, and thus we will place a new blue dot at point P', the midpoint of line segment AP. Our goal is to show that P' must land in one of the shaded regions (rather than in the upside-down triangle). In particular, we will show that in this case P' must lie in the bottom leftmost shaded region.

Label the points U and T as shown below and draw the line passing through the points A and P.



You should think about how the picture would change if we rolled something other than a 1 (or 2). Also note that P does not appear in the diagram and we do not know exactly what this line would look like. If P is on the boundary of $\triangle ABC$, then the argument is a bit different. Here we have an illustration of what a "typical line" might look like and we work from here.

This line must intersect UT and CB somewhere; we call these points R and Q, respectively.

We know that P, P', and R all lie on line segment AQ.

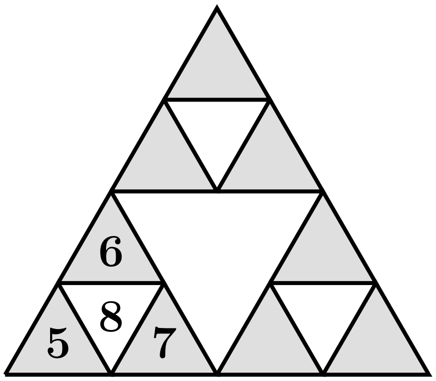
We know that $AP' = \frac{1}{2}AP$, by construction, and we can prove that $AR = \frac{1}{2}AQ$.

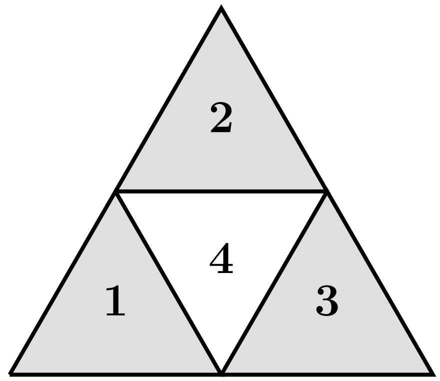
(See the end of the next page for a proof of this fact.)

Since $AP \leq AQ$, we must have $\frac{1}{2}AP \leq \frac{1}{2}AQ$ which means $AP' \leq AR$. This means that point P' must lie in the shaded region $\triangle AUT$.

Think about which region P' would end up in if we used vertex B or C instead of A.

We now know that P' must lie in the shaded region labelled with a 1 in the image below on the right. Now consider the figure obtained after performing Step 3 from Part II, shown below on the left. We will try to argue that P' must actually be in one of the smaller shaded regions, labelled 5, 6, and 7, rather than in the unshaded region labelled 8. This gives us "one more step" in our argument.





Using similar reasoning as before, we can show the following:

i. If P is in region 1, then P' must be in region 5.

ii. If P is in region 2, then P' must be in region 6.

iii. If P is in region 3, then P' must be in region 7.

iv If P is in region 4, then P' must be in region 8.

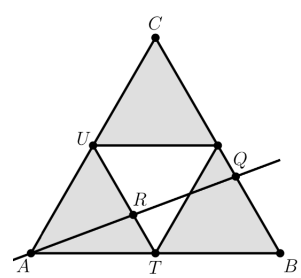
In particular, this shows that if P started off in one of the three larger shaded regions, then P' must end up in one of the three smaller shaded regions. This argument can be repeated to show that P' must lie in one of the even smaller shaded regions in the next figure in the pattern, and so on.

We have now outlined the main building blocks of an argument. A justification of the fact in question involves arguing that the first blue point we draw lies in a shaded region in *every figure* in the pattern. From this we can deduce from our work above that this is also true of every subsequent blue point drawn. This argument can be formalized using something called the principle of mathematical induction.

Proof that $AR = \frac{1}{2}AQ$.

Consider $\triangle AQB$ and $\triangle ART$. Since the segment TU is parallel to the segment BC, $\angle ART = \angle AQB$. As well, $\angle RAT = \angle QAB$ since they are the same angle, which means $\triangle ART \sim \triangle AQB$. Using this similarity and the fact that T is the midpoint of AB, we have the ratios:

$$\frac{AR}{AQ} = \frac{AT}{AB} = \frac{1}{2}.$$



This means R is the midpoint of the segment AQ. Put another way, Q is twice as far away from A as R is. If we let V be the point on ℓ that is twice as far from A as P is, then V will lie farther from A than Q, and hence, V cannot be inside the triangle. Remember, we are trying to show that none of the blue dots are in the upside down triangle. Let us return to this problem.

Follow-up Questions:

Imagine that at each iteration we remove (or cut out) the upside-down triangles from the figure.

1. After the first iteration, what is the total area of the three triangles remaining after the removal of the upside-down triangle? (Recall that our initial triangle has side lengths of 16 cm.)

   Solution:

   Since the three remaining equilateral triangles have the same side length of 8 cm, we will find the area of one of them and multiply it by 3.

   Using the Pythagorean theorem or trigonometry, you can check that an equilateral triangle of side length 8 cm has height of $4\sqrt{3}$ cm. Therefore, the area of one of the remaining triangles is

   $$\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(8 \times 4\sqrt{3}) = 16\sqrt{3}\text{ cm}^2.$$

   Therefore, the total area of the remaining three triangles is $48\sqrt{3}$ cm^2.

2. After the second iteration, what is the total area of the nine triangles remaining after the removal of the upside-down triangles?

   Solution:

   The nine remaining equilateral triangles have the same side length of 4 cm. Similar to the previous solution, we will find the area of one of them and multiply it by 9.

   An equilateral triangle of side length 4 cm has a height of $2\sqrt{3}$ cm, so its area is

   $$\frac{1}{2}(\text{base} \times \text{height}) = \frac{1}{2}(4 \times 2\sqrt{3}) = 4\sqrt{3}\text{ cm}^2.$$

   Therefore, the total area of the nine remaining triangles is $36\sqrt{3}$ cm^2.

3. After the nth iteration, what is the total area of the triangles remaining after the removal of the upside-down triangles?

   Solution:

   In each iteration, every remaining triangle is divided into four smaller equilateral triangles of equal area and the middle triangle is removed. This means in each iteration, each remaining triangle has $\frac{1}{4}$ of its area removed. It follows that the total area of the remaining triangles after each iteration is $\frac{3}{4}$ the total area before that iteration.

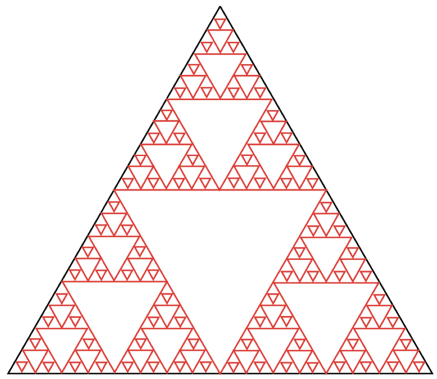
   This agrees with the previous solutions. Indeed, after the first iteration the area is $48\sqrt{3}$ and after the second iteration the area is

   $$36\sqrt{3} = \frac{3}{4}(48\sqrt{3})$$

   After the third iteration, the area will be $\left(\frac{3}{4}\right)^2 48\sqrt{3}$, after the fourth iteration it will be $\left(\frac{3}{4}\right)^3 48\sqrt{3}$, and so on. Continuing in this way, for a positive integer n, the area after the nth iteration we will be

   $$\left(\frac{3}{4}\right)^{n-1} 48\sqrt{3}$$

The Sierpinski Triangle



Since the Sierpinski triangle is obtained by continuing these iterations forever, the area of the Sierpinski triangle should be less than the area remaining after any finite number of iterations. An area should be nonnegative, so if we are to assign an area to this strange object, it should be some nonnegative number A with the property that

$$A < \left(\frac{3}{4}\right)^{n-1} 48\sqrt{3}$$

for *every* positive integer n. However, since $0 < \frac{3}{4} < 1$, the quantity $\left(\frac{3}{4}\right)^{n-1} 48\sqrt{3}$ will eventually be smaller than any positive number. Therefore, the only number that A could be is 0.

You might have come across this kind of idea before if you have studied *limits*.