The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2024 Pascal Contest

(Grade 9)

Wednesday, February 28, 2024
(in North America and South America)

Thursday, February 29, 2024
(outside of North America and South America)

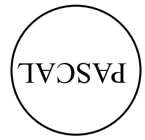
Solutions

1. Calculating, $2 - 0 + 2 - 4 = 2 + 2 - 4 = 0$.

   Answer: (B)

2. The distance between two numbers on the number line is equal to their positive difference. Here, this distance is $6 - (-5) = 11$.

   Answer: (D)

3. Since a turn of 180° is a half-turn, the resulting figure is 

. (Note that we would obtain the same result rotating by 180° clockwise or 180° counterclockwise.)

   Answer: (C)

4. Since July 1 is a Wednesday, then July 8 and July 15 are both Wednesdays.
   Since July 15 is a Wednesday, then July 17 is a Friday.

   Answer: (D)

5. The first rhombus and the last rhombus each have three edges that form part of the exterior of the figure, and so they each contribute 3 to the perimeter.
   The inner four rhombi each have two edges that form part of the exterior of the figure, and so they each contribute 2 to the perimeter.
   Thus, the perimeter is $2 \times 3 + 4 \times 2 = 14$.

   Answer: (B)

6. On Monday, Narsa ate 4 cookies.
   On Tuesday, Narsa ate 12 cookies.
   On Wednesday, Narsa ate 8 cookies.
   On Thursday, Narsa ate 0 cookies.
   On Friday, Narsa ate 6 cookies.
   This means that Narsa ate $4 + 12 + 8 + 0 + 6 = 30$ cookies.
   Since the package started with 45 cookies, there are $45 - 30 = 15$ cookies left in the package after Friday.

   Answer: (D)

7. For there to be equal numbers of each colour of candy, there must be at most 3 red candies and at most 3 yellow candies, since there are 3 blue candies to start.
   Thus, Shuxin ate at least 7 red candies and at least 4 yellow candies.
   This means that Shuxin ate at least $7 + 4 = 11$ candies.
   We note that if Shuxin eats 7 red candies, 4 yellow candies, and 0 blue candies, there will indeed be equal numbers of each colour.

   Answer: (C)

8. Since 10 students have black hair and 3 students have black hair and wear glasses, then a total of $10 - 3 = 7$ students have black hair but do not wear glasses.

   Answer: (A)

9. Since 25% is equivalent to $\frac{1}{4}$, then the fraction of the trail covered by the section along the river and the section through the forest is $\frac{1}{4}+\frac{5}{8}=\frac{2}{8}+\frac{5}{8}=\frac{7}{8}$.
This means that the final section up a hill represents $1-\frac{7}{8}=\frac{1}{8}$ of the trail.
Since $\frac{1}{8}$ of the trail is 3 km long, then the entire trail is 8×3 km $= 24$ km long.
ANSWER: (A)

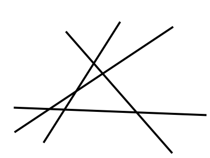
10. Using the definition, $(5\nabla 2)\nabla 2 = (4 \times 5 + 2)\nabla 2 = 22\nabla 2 = 4 \times 22 + 2 = 90$.
ANSWER: (E)

11. *Solution 1*
If all of Lauren's 10 baskets are worth 2 points, she would have $10 \times 2 = 20$ points in total.
Since she has 26 points in total, then she scores $26 - 20 = 6$ more points than if all of her baskets are worth 2 points.
This means that, if 6 of her baskets are worth 3 points, she would gain 1 point for each of these 6 baskets and so have $20 + 6 = 26$ points.
Thus, she makes 6 baskets worth 3 points.
(We note that $6 \times 3 + 4 \times 2 = 26$.)

Solution
Suppose that Lauren makes x baskets worth 3 points each.
Since she makes 10 baskets, then $10 - x$ baskets that she made are worth 2 points each.
Since Lauren scores 26 points, then $3x + 2(10 - x) = 26$ and so $3x + 20 - x = 26$ which gives $x = 6$.
Therefore, Lauren makes 6 baskets worth 3 points.
ANSWER: (B)

12. From the given list, the numbers 11 and 13 are the only prime numbers, and so must be Karla's and Levi's numbers in some order.
From the given list, 16 is the only perfect square; thus, Glen's number was 16.
The remaining numbers are 12, 14, 15.
Since Hao's and Julia's numbers were even, then their numbers must be 12 and 14 in some order.
Thus, Ioana's number is 15.
ANSWER: (B)

13. Each of the 4 lines can intersect each of the other 3 lines at most once.
This might appear to create $4 \times 3 = 12$ points of intersection, but each point of intersection is counted twice – one for each of the 2 lines.
Thus, the maximum number of intersection points is $\dfrac{4 \times 3}{2} = 6$.
The diagram below demonstrates that 6 intersection points are indeed possible:



ANSWER: (D)

14. When 10 numbers have an average of 17, their sum is $10 \times 17 = 170$.
When 9 numbers have an average of 16, their sum is $9 \times 16 = 144$.
Therefore, the number that was removed was $170 - 144 = 26$.
ANSWER: (A)

15. Since $CD = DE = EC$, then $\triangle CDE$ is equilateral, which means that $\angle DEC = 60^\circ$.
Since $\angle DEB$ is a straight angle, then $\angle CEB = 180^\circ - \angle DEC = 180^\circ - 60^\circ = 120^\circ$.
Since $CE = EB$, then $\triangle CEB$ is isosceles with $\angle ECB = \angle EBC$.
Since $\angle ECB + \angle CEB + \angle EBC = 180^\circ$, then $2 \times \angle EBC + 120^\circ = 180^\circ$, which means that $2 \times \angle EBC = 60^\circ$ or $\angle EBC = 30^\circ$.
Therefore, $\angle ABC = \angle EBC = 30^\circ$.

Answer: (A)

16. Since $x^2 < x$ and $x^2 \geq 0$, then $x > 0$ and so it cannot be the case that x is negative.
Thus, neither (D) nor (E) is the answer.
Since $x^2 < x$, then we cannot have $x > 1$. This is because when $x > 1$, we have $x^2 > x$.
Thus, (A) is not the answer and so the answer is (B) or (C).
If $x = \frac{1}{3}$, then $x^2 = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ and $\frac{x}{2} = \frac{1/3}{2} = \frac{1}{6}$.
Since $\frac{1}{6} > \frac{1}{9}$, then (B) cannot be the answer.
Therefore, the answer must be (C).
Checking, when $x = \frac{3}{4}$, we have $x^2 = \frac{9}{16}$ and $\frac{x}{2} = \frac{3}{8}$.
Since $\frac{x}{2} = \frac{3}{8} = \frac{6}{16} < \frac{9}{16} = x^2$, then $\frac{x}{2} < x^2$.
Also, $x^2 = \frac{9}{16} < \frac{12}{16} = \frac{3}{4} = x$.
This confirms that $x = \frac{3}{4}$ does satisfy the required conditions.

Answer: (C)

17. In 2 hours travelling at 100 km/h, Melanie travels $2 \text{ h} \times 100 \text{ km/h} = 200 \text{ km}$.
When Melanie travels 200 km at 80 km/h, it takes $\dfrac{200 \text{ km}}{80 \text{ km/h}} = 2.5 \text{ h}$.
Melanie travels a total of $200 \text{ km} + 200 \text{ km} = 400 \text{ km}$.
Melanie travels for a total of $2 \text{ h} + 2.5 \text{ h} = 4.5 \text{ h}$.
Therefore, Melanie's average speed is $\dfrac{400 \text{ km}}{4.5 \text{ h}} \approx 88.89 \text{ km/h}$.
Of the given choices, this is closest to 89 km/h.

Answer: (B)

18. From the given information, we know that

$$\mathrm{S} + \mathrm{E} + \mathrm{T} = 2 \qquad \mathrm{H} + \mathrm{A} + \mathrm{T} = 7 \qquad \mathrm{T} + \mathrm{A} + \mathrm{S} + \mathrm{T} + \mathrm{E} = 3 \qquad \mathrm{M} + \mathrm{A} + \mathrm{T} = 4$$

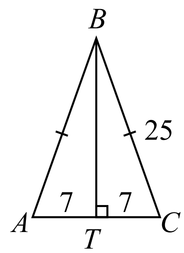
Since $\mathrm{T} + \mathrm{A} + \mathrm{S} + \mathrm{T} + \mathrm{E} = 3$ and $\mathrm{S} + \mathrm{E} + \mathrm{T} = 2$, then $\mathrm{T} + \mathrm{A} = 3 - 2 = 1$.
Since $\mathrm{H} + \mathrm{A} + \mathrm{T} = 7$ and $\mathrm{T} + \mathrm{A} = 1$, then $\mathrm{H} = 7 - 1 = 6$.
Since $\mathrm{M} + \mathrm{A} + \mathrm{T} = 4$ and $\mathrm{H} = 7$, then $\mathrm{M} + (\mathrm{A} + \mathrm{T}) + \mathrm{H} = 4 + 6 = 10$.
Therefore, the value of the word MATH is 10.
We note that it is also possible to find specific values for S, E, T, A that give the correct values to the words. One such set of values is $\mathrm{A} = 1$, $\mathrm{T} = 0$, $\mathrm{S} = 4$, and $\mathrm{E} = -2$. These values are not unique, even though the values assigned to M and H (namely, 3 and 6) are unique.

Answer: (E)

19. The perimeter of $\triangle ABC$ is equal to $(3x+4)+(3x+4)+2x = 8x+8$.
The perimeter of rectangle $DEFG$ is equal to

$$2 \times (2x-2) + 2 \times (3x-1) = 4x - 4 + 6x - 2 = 10x - 6$$

Since these perimeters are equal, we have $10x - 6 = 8x + 8$ which gives $2x = 14$ and so $x = 7$.
Thus, $\triangle ABC$ has $AC = 2 \times 7 = 14$ and $AB = BC = 3 \times 7 + 425$.
We drop a perpendicular from B to T on AC.



Since $\triangle ABC$ is isosceles, then T is the midpoint of AC, which gives $AT = TC = 7$.
By the Pythagorean Theorem, $BT = \sqrt{BC^2 - TC^2} = \sqrt{25^2 - 7^2} = \sqrt{625 - 49} = \sqrt{576} = 24$.
Therefore, the area of $\triangle ABC$ is equal to $\frac{1}{2} \cdot AC \cdot BT = \frac{1}{2} \times 14 \times 24 = 168$.

ANSWER: (C)

20. Since N is between 1 000 000 and 10 000 000, inclusive, then $25 \times N$ is between 25 000 000 and 250 000 000, inclusive, and so $25 \times N$ has 8 digits or it has 9 digits.
We consider the value of $25 \times N$ as having 9 digits, with the possibility that the first digit could be 0.
Since $25 \times N$ is a multiple of 25, its final two digits must be 00, 25, 50, or 75.
For a fixed set of leftmost three digits, xyz, the multiple of 25 that has the largest sum of digits must be $xyz\,999\,975$ since the next four digits are as large as possible (all 9s) and the rightmost two digits have the largest possible sum among the possible endings for multiples of 25.
So to answer the question, we need to find the integer of the form $xyz\,999\,975$ which is between 25 000 000 and 250 000 000 and has the maximum possible sum $x+y+z$.
We know that the maximum possible value of x is 2, the maximum possible value of y is 9, and the maximum possible value of z is 9.
This means that $x+y+z \leq 2+9+9 = 20$.
We cannot have 299 999 975 since it is not in the given range.
However, we could have $x+y+z = 19$ if $x = 1$ and $y = 9$ and $z = 9$.
Therefore, the integer 199 999 975 is the multiple of 25 in the given range whose sum of digits is as large as possible. This sum is $1 + 6 \times 9 + 7 + 5 = 67$.
We note that $199\,999\,975 = 25 \times 7\,999\,999$ so it is a multiple of 25. Note that $N = 7\,999\,999$ is between 1 000 000 and 10 000 000.

ANSWER: (C)

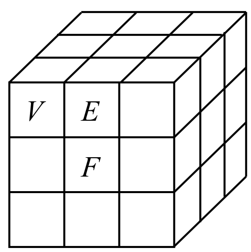
21. Since the second column includes the number 1, then step (ii) was never used on the second column, otherwise each entry would be at least 2.
To generate the 1, 3 and 2 in the second column, we thus need to have used step (i) 1 time on row 1, 3 times on row 2, and 2 times on row 3.
This gives:

1	1	1
3	3	3
2	2	2

We cannot use step (i) any more times, otherwise the entries in column 2 will increase. Thus, $a = 1 + 3 + 2 = 6$.
To obtain the final grid from this current grid using only step (ii), we must increase each entry in column 1 by 6 (which means using step (ii) 3 times) and increase each entry in column 3 by 4 (which means using step (ii) 2 times). Thus, $b = 3 + 2 = 5$.
Therefore, $a + b = 11$.

Answer: 11

22. The 27 small cubes that make up the larger $3 \times 3 \times 3$ can be broken into 4 categories: 1 small cube in the very centre of the larger cube (not seen in the diagram), 8 small cubes at the vertices of larger cube (an example is marked with V), 12 small cubes on the edges not at vertices (an example is marked with E), and 6 small cubes at the centre of each face (an example is marked with F).



The centre cube contributes 0 to the surface area of the cube.
Each of the 8 vertex cubes contributes 3 to the surface area of the larger cube, as 3 of the 6 faces of each such cube are on the exterior of the larger cube.
Each of the 12 edge cubes contributes 2 to the surface area of the larger cube.
Each of the 6 face cubes contributes 1 to the surface area of the larger cube.
There are 10 small red cubes that need to be placed as part of the larger cube.
To minimize the surface area that is red, we place the red cubes in positions where they will contribute the least to the overall surface area. To do this, we place 1 red cube at the centre (contributing 0 to the surface area), 6 red cubes at the centres of the faces (each contributing 1 to the surface area), and the remaining 3 red cubes on the edges (each contributing 2 to the surface area).
In total, the surface area that is red is $1 \times 0 + 6 \times 1 + 3 \times 2 = 12$.

Answer: 12

23. We want to count the number of four-digit codes $abcd$ that satisfy the given rules.
From the first rule, at least one of the digits must be 4, but $b \neq 4$ and $d \neq 4$.
Therefore, either $a = 4$ or $c = 4$. The fourth rule tells us that we could have both $a = 4$ and $c = 4$.

Suppose that $a = 4$ and $c = 4$.
The code thus has the form $4b4d$.
The second and third rules tell us that the remaining digits are 2 and 7, and that there are no further restrictions on where the 2 and 7 are placed.
Therefore, in this case, the code is either 4247 or 4742, and so there are 2 possible codes.

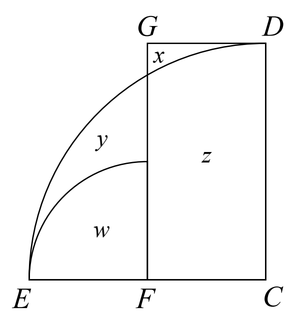
Suppose that $a = 4$ and $c \neq 4$. (Recall that $b \neq 4$ and $d \neq 4$.)
The code thus has the form $4bcd$.
The remaining digits include a 2 (which can be placed in any of the remaining positions), a 7, and either a 1 or a 6.
There are 3 positions in which the 2 can be placed, after which there are 2 positions in which the 7 can be placed, after which there are 2 digits that can be placed in the remaining position.
Therefore, in this case, there are $3 \times 2 \times 2 = 12$ possible codes.

Suppose that $c = 4$ and $a \neq 4$.
The code thus has the form $ab4d$.
The remaining digits include a 2 (with the restriction that $a \neq 2$), a 7, and either a 1 or a 6.
There are 2 positions in which the 2 can be placed, after which the 7 can be placed in either of the 2 remaining positions, after which there are 2 digits that can be placed in the remaining position.
Therefore, in this case, there are $2 \times 2 \times 2 = 8$ possible codes.

In total, there are $2 + 12 + 8 = 22$ possible codes.

ANSWER: 22

24. We label the two other regions w and z as shown:



If we start with the area of the larger quarter circle (which is equal to $y + w + z$) and then subtract the area of the smaller quarter circle (which is equal to w), we are left $y + z$.
If we then subtract the area of the rectangle (which is equal to $x + z$), we are left with $y - x$.
In other words, $y - x$ is equal to the area of the larger quarter circle minus the area of the smaller quarter circle minus the area of the retangle.
The larger quarter circle has radius 30 and so its area is $\frac{1}{4}\pi \times 30^2 = 225\pi$.
The radius of the smaller quarter circle is half of that of the larger quarter circle, because F is the midpoint of CE.
Thus, the smaller quarter circle has radius 15 and so its area is $\frac{1}{4}\pi \times 15^2 = \frac{225}{4}\pi$.
The width of the rectangle is equal to FC, which is half of CE or 15.

The height of the rectangle is 30, and so its area is $15 \times 30 = 450$.
Therefore, $y - x = 225\pi - \frac{225}{4}\pi - 450 = \frac{900}{4}\pi - \frac{225}{4}\pi - 450 = \frac{675}{4}\pi - 450 \approx 80.1$.
This tells us that $y - x$ is positive (the diagram certainly makes it look positive), which means that $d = y - x$ and the closest integer to d is 80.

Answer: 80

25. We write $a = 3^r$, $b = 3^s$ and $c = 3^t$ where each of r, s, t is between 1 and 8, inclusive.
Since $a \leq b \leq c$, then $r \leq s \leq t$.
Next, we note that

$$\frac{ab}{c} = \frac{3^r 3^s}{3^t} = 3^{r+s-t} \qquad \frac{ac}{b} = \frac{3^r 3^t}{3^s} = 3^{r+t-s} \qquad \frac{bc}{a} = \frac{3^s 3^t}{3^r} = 3^{s+t-r}$$

Since $t \geq s$, then $r + t - s = r + (t - s) \geq r > 0$ and so $\dfrac{ac}{b}$ is always an integer.
Since $t \geq r$, then $s + t - r = s + (t - r) \geq s > 0$ and so $\dfrac{bc}{a}$ is always an integer.
Since $\dfrac{ab}{c} = 3^{r+s-t}$, then $\dfrac{ab}{c}$ is an integer exactly when $r + s - t \geq 0$ or $t \leq r + s$.
This means that we need to count the number of triples (r, s, t) where $r \leq s \leq t$, each of r, s, t is an integer between 1 and 8, inclusive, and $t \leq r + s$.

Suppose that $r = 1$. Then $1 \leq s \leq t \leq 8$ and $t \leq s + 1$.
If $s = 1$, t can equal 1 or 2. If $s = 2$, t can equal 2 or 3. This pattern continues so that when $s = 7$, t can equal 7 or 8. When $s = 8$, though, t must equal 8 since $t \leq 8$.
In this case, there are $2 \times 7 + 1 = 15$ pairs of values for s and t that work, and so 15 triples (r, s, t).

Suppose that $r = 2$. Then $2 \leq s \leq t \leq 8$ and $t \leq s + 2$.
This means that, when $2 \leq s \leq 6$, t can equal s, $s + 1$ or $s + 2$.
When $s = 7$, t can equal 7 or 8, and when $s = 8$, t must equal 8.
In this case, there are $5 \times 3 + 2 + 1 = 18$ triples.

Suppose that $r = 3$. Then $3 \leq s \leq t \leq 8$ and $t \leq s + 3$.
This means that, when $3 \leq s \leq 5$, t can equal s, $s + 1$, $s + 2$, or $s + 3$.
When $s = 6, 7, 8$, there are 3, 2, 1 values of t, respectively.
In this case, there are $3 \times 4 + 3 + 2 + 1 = 18$ triples.

Suppose that $r = 4$. Then $4 \leq s \leq t \leq 8$ and $t \leq s + 4$.
This means that when $s = 4$, there are 5 choices for t.
As in previous cases, when $s = 5, 6, 7, 8$, there are 4, 3, 2, 1 choices for t, respectively.
In this case, there are $5 + 4 + 3 + 2 + 1 = 15$ triples.

Continuing in this way, when $r = 5$, there are $4 + 3 + 2 + 1 = 10$ triples, when $r = 6$, there are $3 + 2 + 1 = 6$ triples, when $r = 7$, there are $2 + 1 = 3$ triples, and when $r = 8$, there is 1 triple.
The total number of triples (r, s, t) is $15 + 18 + 18 + 15 + 10 + 6 + 3 + +1 = 86$.
Since the triples (r, s, t) correspond with the triples (a, b, c), then the number of triples (a, b, c) is $N = 86$.

Answer: 86

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2023 Pascal Contest

(Grade 9)

Wednesday, February 22, 2023
(in North America and South America)

Thursday, February 23, 2023
(outside of North America and South America)

Solutions

1. Since 110 003 is greater than 110 000 and each of the other four choices is less than 110 000, the integer 110 003 is the greatest of all of the choices.

   Answer: (B)

2. From left to right, the number of shaded squares in each column with shaded squares is 1, 3, 5, 4, 2.
   Thus, the number of shaded squares is $1 + 3 + 5 + 4 + 2 = 15$.
   Alternatively, we could note that exactly one-half of the 30 squares are shaded since each column with shaded squares can be paired with a column of the same number of unshaded squares. (The 1st column is paired with the 8th, the 2nd with the 7th, the 3rd with the 6th, and the 4th with the 5th.) Thus, again there are $\frac{1}{2} \times 30 = 15$ shaded squares.

   Answer: (C)

3. Evaluating, $2^3 - 2 + 3 = 2 \times 2 \times 2 - 2 + 3 = 8 - 2 + 3 = 9$.

   Answer: (C)

4. Since $3 + \triangle = 5$, then $\triangle = 5 - 3 = 2$.
   Since $\triangle + \square = 7$ and $\triangle = 2$, then $\square = 5$.
   Thus, $\triangle + \triangle + \triangle + \square + \square = 3 \times 2 + 2 \times 5 = 6 + 10 = 16$.

   Answer: (E)

5. Evaluating, $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} = 0.3 + 0.03 + 0.003 = 0.333$.

   Answer: (A)

6. Since $\frac{1}{3}$ of x is equal to 4, then x is equal to 3×4 or 12. Thus, $\frac{1}{6}$ of x is equal to $12 \div 6 = 2$.
   Alternatively, since $\frac{1}{6}$ is one-half of $\frac{1}{3}$, then $\frac{1}{6}$ of x is equal to one-half of $\frac{1}{3}$ of x, which is $4 \div 2$ or 2.

   Answer: (C)

7. Jurgen takes $25 + 35 = 60$ minutes to pack and then walk to the bus station.
   Since Jurgen arrives 60 minutes before the bus leaves, he began packing $60 + 60 = 120$ minutes, or 2 hours, before the bus leaves.
   Since the bus leaves at 6:45 p.m., Jurgen began packing at 4:45 p.m.

   Answer: (A)

8. Since the letters of RHOMBUS take up 7 of the 31 spaces on the line, there are $31 - 7 = 24$ spaces that are empty.
   Since the numbers of empty spaces on each side of RHOMBUS are the same, there are $24 \div 2 = 12$ empty spaces on each side.
   Therefore, the letter R is placed in space number $12 + 1 = 13$, counting from the left.

   Answer: (B)

9. The digits to the right of the decimal place in the decimal represenation of $\frac{1}{7}$ occur in blocks of 6, repeating the block of digits 142857.
   Since $16 \times 6 = 96$, then the 96th digit to the right of the decimal place is the last in one of these blocks; that is, the 96th digit is 7.
   This means that the 97th digit is 1, the 98th digit is 4, the 99th digit is 2, and the 100th digit is 8.

   Answer: (D)

10. The path that the ant walks from A to B is vertical and has length 5.
The path that the ant walks from B to C is horizontal and has length 8.
The path that the ant walks from C to A does not follow the gridlines. To determine the length of CA, we can use the Pythagorean Theorem because AB and BC meet at a right angle.
This gives $CA^2 = AB^2 + BC^2 = 5^2 + 8^2 = 25 + 64 = 89$.
Since $CA > 0$, then $CA = \sqrt{89}$.
Thus, the total distance that the ant walks is $5 + 8 + \sqrt{89}$ or $13 + \sqrt{89}$.

ANSWER: (D)

11. Suppose that the original prism has length ℓ cm, width w cm, and height h cm.
Since the volume of this prism is 12 cm^3, then $\ell wh = 12$.
The new prism has length 2ℓ cm, width $2w$ cm, and height $3h$ cm.
The volume of this prism, in cm^3, is $(2l) \times (2w) \times (3h) = 2 \times 2 \times 3 \times lwh = 12 \times 12 = 144$.

ANSWER: (E)

12. Since $31 = 3 \times 10 + 1$ and $94 = 3 \times 31 + 1$ and $331 = 3 \times 110 + 1$ and $907 = 3 \times 302 + 1$, then each of 31, 94, 331, and 907 appear in the second column of Morgan's spreadsheet.
Thus, 131 must be the integer that does not appear in Morgan's spreadsheet. (We note that 131 is 2 more than $3 \times 43 = 129$ so is not 1 more than a muliple of 3.)

ANSWER: (C)

13. The total decrease in temperature between these times is $16.2°\text{C} - (-3.6°\text{C}) = 19.8°\text{C}$.
The length of time between 3:00 p.m. one day and 2:00 a.m. the next day is 11 hours, since it is 1 hour shorter than the length of time between 3:00 p.m. and 3:00 a.m.
Since the temperature decreased at a constant rate over this period of time, the rate of decrease in temperature was $\dfrac{19.8°\text{C}}{11\text{ h}} = 1.8°\text{C/h}$.

ANSWER: (B)

14. There are 2 possible "states" for each door: open or closed.
Therefore, there are $2 \times 2 \times 2 \times 2 = 2^4 = 16$ possible combinations of open and closed for the 4 doors.
If exactly 2 of the 4 doors are open, these doors could be the 1st and 2nd, or 1st and 3rd, or 1st and 4th, or 2nd and 3rd, or 2nd and 4th, or 3rd and 4th. Thus, there are 6 ways in which 2 of the 4 doors can be open.
Since each door is randomly open or closed, then the probability that exactly 2 doors are open is $\frac{6}{16}$ which is equivalent to $\frac{3}{8}$.

ANSWER: (A)

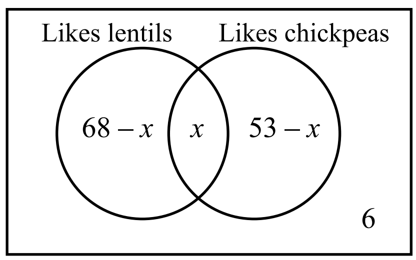
15. Nasim can buy 24 cards by buying three 8-packs ($3 \times 8 = 24$).
Nasim can buy 25 cards by buying five 5-packs ($5 \times 5 = 25$).
Nasim can buy 26 cards by buying two 5-packs and two 8-packs ($2 \times 5 + 2 \times 8 = 26$).
Nasim can buy 28 cards by buying four 5-packs and one 8-pack ($4 \times 5 + 1 \times 8 = 28$).
Nasim can buy 29 cards by buying one 5-pack and three 8-packs ($1 \times 5 + 3 \times 8 = 29$).
Nasim cannot buy exactly 27 cards, because the number of cards in 8-packs that he buys would be 0, 8, 16, or 24, leaving 27, 19, 11, or 3 cards to buy in 5-packs. None of these are possible, since none of 27, 19, 11, or 3 is a multiple of 5.
Therefore, for 5 of the 6 values of n, Nasim can buy exactly n cards.

ANSWER: (A)

16. Suppose that Mathilde had m coins at the start of last month and Salah had s coins at the start of last month.
From the given information, 100 is 25% more than m, so $100 = 1.25m$ which means that $m = \dfrac{100}{1.25} = 80$.
From the given information, 100 is 20% less than s, so $100 = 0.80s$ which means that $s = \dfrac{100}{0.80} = 125$.
Therefore, at the beginning of last month, they had a total of $m + s = 80 + 125 = 205$ coins.

Answer: (E)

17. Suppose that x students like both lentils and chickpeas.
Since 68 students like lentils, these 68 students either like chickpeas or they do not.
Since x students like lentils and chickpeas, then x of the 68 students that like lentils also like chickpeas and so $68 - x$ students like lentils but do not like chickpeas.
Since 53 students like chickpeas, then $53 - x$ students like chickpeas but do not like lentils.
We know that there are 100 students in total and that 6 like neither lentils nor chickpeas.
We use a Venn diagram to summarize this information:



Since there are 100 students in total, then $(68 - x) + x + (53 - x) + 6 = 100$ which gives $127 - x = 100$ and so $x = 27$.
Therefore, there are 27 students that like both lentils and chickpeas.

Answer: (B)

18. Since $\angle ABD = 180^\circ$ and $\angle ABC = x^\circ$, then $\angle CBD = 180^\circ - x^\circ$.
Since the measures of the angles in $\triangle BCD$ add to 180°, then

$$\angle BDC = 180^\circ - (180^\circ - x^\circ) - 90^\circ = x^\circ - 90^\circ$$

Similarly, $\angle GFD = 180^\circ$ and $\angle FDE = y^\circ - 90^\circ$. Finally, $\angle BDF = 180^\circ$ and so

$$\begin{aligned} \angle BDC + \angle CDE + \angle FDE &= 180^\circ \\ (x^\circ - 90^\circ) + 80^\circ + (y^\circ - 90^\circ) &= 180^\circ \\ x + y - 100 &= 180 \end{aligned}$$

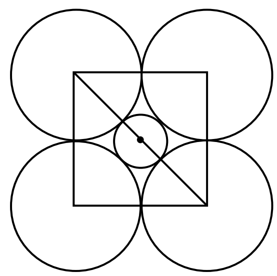
and so $x + y = 280$.

Answer: (D)

19. Before Kyne removes hair clips, Ellie has 4 red clips and $4 + 5 + 7 = 16$ clips in total, so the probability that she randomly chooses a red clip is $\frac{4}{16}$ which equals $\frac{1}{4}$.
After Kyne removes the clips, the probability that Ellie chooses a red clip is $2 \times \frac{1}{4}$ or $\frac{1}{2}$.
Since Ellie starts with 4 red clips, then after Kyne removes some clips, Ellie must have 4, 3, 2, 1, or 0 red clips.
Since the probability that Ellie chooses a red clip is larger than 0, she cannot have 0 red clips.
Since the probability of her choosing a red clip is $\frac{1}{2}$, then the total number of clips that she has after k are removed must be twice the number of red clips, so could be 8, 6, 4, or 2.
Thus, the possible values of k are $16 - 8 = 8$ or $16 - 6 = 10$ or $16 - 4 = 12$ or $16 - 2 = 14$.
Of these, 12 is one of the given possibilities. (One possibility is that Kyne removes 2 of the red clips, 5 of the blue clips and 5 of the green clips, leaving 2 red clips and 2 green clips.)

Answer: (C)

20. Draw one of the diagonals of the square. The diagonal passes through the centre of the square.



By symmetry, the centre of the smaller circle is the centre of the square. (If it were not the centre of the square, then one of the four larger circles would have to be different from the others somehow, which is not true.)
Further, the diagonals of the square pass through the points where the smaller circle is tangent to the larger circles. (The line segment from each vertex of the square to the centre of the smaller circle passes through the point of tangency. These four segments are equal in length and meet at right angles since the diagram can be rotated by 90 degrees without changing its appearance. Thus, each of these is half of a diagonal.)
Since each of the larger circles has radius 5, the side length of the square is $5 + 5 = 10$.
Since the square has side length 10, its diagonal has length $\sqrt{10^2 + 10^2} = \sqrt{200}$ by the Pythagorean Theorem.
Therefore, $5 + 2r + 5 = \sqrt{200}$ which gives $2r = \sqrt{200} - 10$ and so $r \approx 2.07$.
Of the given choices, r is closest to 2.1, or (C).

Answer: (C)

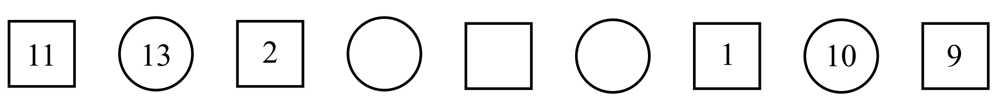
21. We follow Alicia's algorithm carefully:

- Step 1: Alicia writes down $m = 3$ as the first term.
- Step 2: Since $m = 3$ is odd, Alicia sets $n = m + 1 = 4$.
- Step 3: Alicia writes down $m + n + 1 = 8$ as the second term.
- Step 4: Alicia sets $m = 8$.
- Step 2: Since $m = 8$ is even, Alicia sets $n = \frac{1}{2}m = 4$.
- Step 3: Alicia writes down $m + n + 1 = 13$ as the third term.
- Step 4: Alicia sets $m = 13$.
- Step 2: Since $m = 13$ is odd, Alicia sets $n = m + 1 = 14$.
- Step 3: Alicia writes down $m + n + 1 = 28$ as the fourth term.
- Step 4: Alicia sets $m = 28$.
- Step 2: Since $m = 28$ is even, Alicia sets $n = \frac{1}{2}m = 14$.
- Step 3: Alicia writes down $m + n + 1 = 43$ as the fifth term.
- Step 5: Since Alicia has written down five terms, she stops.

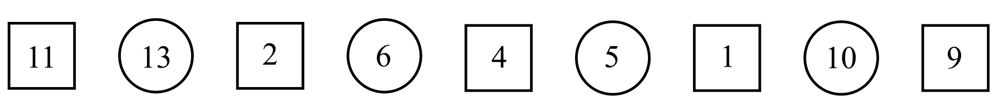
Therefore, the fifth term is 43.

ANSWER: 43

22. From the given information, if a and b are in two consecutive squares, then $a + b$ goes in the circle between them.
Since all of the numbers that we can use are positive, then $a + b$ is larger than both a and b.
This means that the largest integer in the list, which is 13, cannot be either x or y (and in fact cannot be placed in any square). This is because the number in the circle next to it must be smaller than 13 (which is the largest number in the list) and so cannot be the sum of 13 and another positive number from the list.
Thus, for $x + y$ to be as large as possible, we would have x and y equal to 10 and 11 in some order. But here we have the same problem: there is only one larger number from the list (namely 13) that can go in the circles next to 10 and 11, and so we could not fill in the circle next to both 10 and 11.
Therefore, the next largest possible value for $x + y$ is when $x = 9$ and $y = 11$. (We could also swap x and y.)
Here, we could have $13 = 11 + 2$ and $10 = 9 + 1$, giving the following partial list:



The remaining integers (4, 5 and 6) can be put in the shapes in the following way that satisfies the requirements.



This tells us that the largest possible value of $x + y$ is 20.

ANSWER: 20

23. Suppose that Dewa's four numbers are w, x, y, z.
The averages of the four possible groups of three of these are

$$\frac{w+x+y}{3}, \frac{w+x+z}{3}, \frac{w+y+z}{3}, \frac{x+y+z}{3}$$

These averages are equal to 32, 39, 40, 44, in some order.
The sums of the groups of three are equal to 3 times the averages, so are 96, 117, 120, 132, in some order.
In other words, $w+x+y$, $w+x+z$, $w+y+z$, $x+y+z$ are equal to 96, 117, 120, 132 in some order.
Therefore,

$$(w+x+y)+(w+x+z)+(w+y+z)+(x+y+z) = 96+117+120+132$$

and so

$$3w+3x+3y+3z = 465$$

which gives

$$w+x+y+z = 155$$

Since the sum of the four numbers is 155 and the sums of groups of 3 are 96, 117, 120, 132, then the four numbers are

$$155-96=59 \qquad 155-117=38 \qquad 155-120=35 \qquad 155-132=23$$

and so the largest number is 59.

ANSWER: 59

24. Triangular-based pyramid $APQR$ can be thought of as having triangular base $\triangle APQ$ and height AR.
Since this pyramid is built at a vertex of the cube, then $\triangle APQ$ is right-angled at A and AR is perpendicular to the base.
The area of $\triangle APQ$ is $\frac{1}{2} \times AP \times AQ = \frac{1}{2}x(x+1)$. The height of the pyramid is $\frac{x+1}{2x}$.
Thus, the volume of the pyramid is $\frac{1}{3} \times \frac{1}{2}x(x+1) \times \frac{x+1}{2x}$ which equals $\frac{(x+1)^2}{12}$.
Since the cube has edge length 100, its volume is 100^3 or 1 000 000.
Now, 1% of 1 000 000 is $\frac{1}{100}$ of 1 000 000 or 10 000.
Thus, 0.01% of 1 000 000 is $\frac{1}{100}$ of 10 000 or 100.
This tells us that 0.04% of 1 000 000 is 400, and 0.08% of 1 000 000 is 800.
We want to determine the number of integers x for which $\frac{(x+1)^2}{12}$ is between 400 and 800.
This is equivalent to determining the number of integers x for which $(x+1)^2$ is between $12 \times 400 = 4800$ and $12 \times 800 = 9600$.
Since $\sqrt{4800} \approx 69.28$ and $\sqrt{9600} \approx 97.98$, then the perfect squares between 4800 and 9600 are $70^2, 71^2, 72^2, \ldots, 96^2, 97^2$.
These are the possible values for $(x+1)^2$ and so the possible values for x are $69, 70, 71, \ldots, 95, 96$.
There are $96-69+1=28$ values for x.

ANSWER: 28

25. Since the median of the list a, b, c, d, e is 2023 and $a \le b \le c \le d \le e$, then $c = 2023$.
Since 2023 appears more than once in the list, then it appears 5, 4, 3, or 2 times.

Case 1: 2023 appears 5 times
Here, the list is 2023, 2023, 2023, 2023, 2023.
There is 1 such list.

Case 2: 2023 appears 4 times
Here, the list would be 2023, 2023, 2023, 2023, x where x is either less than or greater than 2023.
Since the mean of the list is 2023, the sum of the numbers in the list is 5×2023, which means that $x = 5 \times 2023 - 4 \times 2023 = 2023$, which is a contradiction.
There are 0 lists in this case.

Case 3: 2023 appears 3 times
Here, the list is a, b, 2023, 2023, 2023 (with $a < b < 2023$) or a, 2023, 2023, 2023, e (with $a < 2023 < e$), or 2023, 2023, 2023, d, e (with $2023 < d < e$).
In the first case, the mean of the list is less than 2023, since the sum of the numbers will be less than 5×2023.
In the third case, the mean of the list is greater than 2023, since the sum of the numbers will be greater than 5×2023.
So we need to consider the list a, 2023, 2023, 2023, e with $a < 2023 < e$.
Since the mean of this list is 2023, then the sum of the five numbers is 5×2023, which means that $a + e = 2 \times 2023$.
Since a is a positive integer, then $1 \le a \le 2022$. For each such value of a, there is a corresponding value of e equal to $4046 - a$, which is indeed greater than 2023.
Since there are 2022 choices for a, there are 2022 lists in this case.

Case 4A: 2023 appears 2 times; $c = d = 2023$
(We note that if 2023 appears 2 times, then since $c = 2023$ and $a \le b \le c \le d \le e$, we either have $c = d = 2023$ or $b = c = 2023$.)
Here, the list is a, b, 2023, 2023, e with $1 \le a < b < 2023 < e$.
This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean.
For this to be the case, the sum of its numbers equals 5×2023, which means that $a + b + e = 3 \times 2023 = 6069$.
Every pair of values for a and b with $1 \le a < b < 2023$ will give such a list by defining $e = 6069 - a - b$. (We note that since $a < b < 2023$ we will indeed have $e > 2023$.)
If $a = 1$, there are 2021 possible values for b, namely $2 \le b \le 2022$.
If $a = 2$, there are 2020 possible values for b, namely $3 \le b \le 2022$.
Each time we increase a by 1, there will be 1 fewer possible value for b, until $a = 2021$ and $b = 2022$ (only one value).
Therefore, the number of pairs of values for a and b in this case is

$$2021 + 2020 + \cdots + 2 + 1 = \tfrac{1}{2} \times 2021 \times 2022 = 2021 \times 1011$$

This is also the number of lists in this case.

Case 4B: 2023 appears 2 times; $b = c = 2023$
Here, the list is a, 2023, 2023, d, e with $1 \le a < 2023 < d < e$.
This list has median 2023 and no other integer appears more than once. Thus, it still needs to satisfy the condition about the mean.
For this to be the case, the sum of its numbers equals 5×2023, which means that $a + d + e = 3 \times 2023 = 6069$.
If $d = 2024$, then $a + e = 4045$. Since $1 \le a \le 2022$ and $2025 \le e$, we could have $e = 2025$ and $a = 2020$, or $e = 2026$ and $a = 1019$, and so on. There are 2020 such pairs, since once a reaches 1, there are no more possibilities.
If $d = 2025$, then $a + e = 4044$. Since $1 \le a \le 2022$ and $2026 \le e$, we could have $e = 2026$ and $a = 2018$, or $e = 2027$ and $a = 1017$, and so on. There are 2018 such pairs.
As d increases successively by 1, the sum $a + e$ decreases by 1 and the minimum value for e increases by 1, which means that the maximum value for a decreases by 2, which means that the number of pairs of values for a and e decreases by 2. This continues until we reach $d = 3033$ at which point there are 2 pairs for a and e.
Therefore, the number of pairs of values for a and e in this case is

$$2020 + 2018 + 2016 + \cdots + 4 + 2$$

which is equal to

$$2 \times (1 + 2 + \cdots + 1008 + 1009 + 1010)$$

which is in turn equal to $2 \times \frac{1}{2} \times 1010 \times 1011$ which equals1010×1011.

Combining all of the cases, the total number of lists a, b, c, d, e is

$$N = 1 + 2022 + 2021 \times 1011 + 1010 \times 1011 = 1 + 1011 \times (2 + 2021 + 1010) = 1 + 1011 \times 3033$$

and so $N = 3\,066\,364$.
The sum of the digits of N is $3 + 0 + 6 + 6 + 3 + 6 + 4$ or 28.

ANSWER: 28

The CENTRE for EDUCATION
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2022 Pascal Contest

(Grade 9)

Wednesday, February 23, 2022
(in North America and South America)

Thursday, February 24, 2022
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{20+22}{2} = \dfrac{42}{2} = 21$.

   Answer: (D)

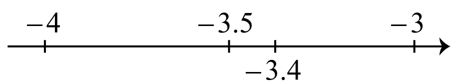
2. From the graph, we see that Haofei donated \$2, Mike donated \$6, Pierre donated \$2, and Ritika donated \$8.
   In total, the four students donated $\$2 + \$6 + \$2 + \$8 = \$18$.

   Answer: (B)

3. In the given sum, each of the four fractions is equivalent to $\dfrac{1}{2}$.
   Therefore, the given sum is equal to $\dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} = 2$.

   Answer: (E)

4. On a number line, -3.4 is between -4 and -3.
   This means that -3.4 is closer to -4 and -3 than to any of 0, 3 or 4, and so the answer must be -4 or -3.
   If we start at -3 and move in the negative direction (that is, to the left), we reach -3.4 after moving 0.4 units.
   It then takes an additional 0.6 units to move in the negative direction from -3.4 to -4.
   Therefore, -3.4 is closer to -3 than to -4, and so the answer is (B) or -3.
   Alternatively, when comparing -3, -4 and -3.4, we could note that -3.4 is between -3.5 and -3 and so is closer to -3:

   Answer: (B)

5. From the diagram, $PR = 10 - 3 = 7$ and $QS = 17 - 5 = 12$ and so $PR : QS = 7 : 12$.

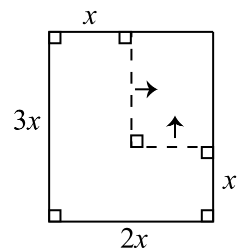
   Answer: (A)

6. Between them, Robyn and Sasha have $4 + 14 = 18$ tasks to do.
   If each does the same number of tasks, each must do $18 \div 2 = 9$ tasks.
   This means that Robyn must do $9 - 4 = 5$ of Sasha's tasks.

   Answer: (C)

7. Because all of the angles in the figure are right angles, each line segment is either horizontal or vertical.
   The height of the figure is $3x$ and the width of the figure is $2x$.
   This means that the length of the unmarked vertical segment must equal $3x - x = 2x$.
   Also, the length of the unmarked horizontal segment must equal $2x - x = x$.
   Starting in the top left corner and adding lengths in a clockwise direction, the perimeter is $x + 2x + x + x + 2x + 3x = 10x$.

Alternatively, we can "complete the rectangle" by sliding the shortest horizontal side and the shortest vertical side as shown to form a rectangle with height $3x$ and width $2x$:



The perimeter of this rectangle is $2 \times 2x + 2 \times 3x = 10x$.

Answer: (E)

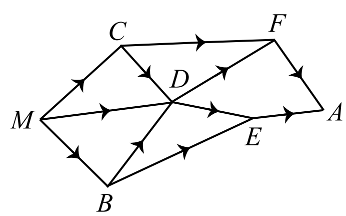
8. The total central angle in a circle is 360°.
Since the Green section has an angle at the centre of the circle of 90°, this section corresponds to $\frac{90^\circ}{360^\circ} = \frac{1}{4}$ of the circle.
This means that when the spinner is spun once, the probability that it lands on the Green section is $\frac{1}{4}$.
Similarily, the probability that the spinner lands on Blue is also $\frac{1}{4}$.
Since the spinner lands on one of the four colours, the probability that the spinner lands on either Red or Yellow is $1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$.

Answer: (D)

9. Since the line with equation $y = 2x + b$ passes through the point $(-4, 0)$, the coordinates of the point must satisfy the equation of the line.
Substituting $x = -4$ and $y = 0$ gives $0 = 2(-4) + b$ and so $0 = -8 + b$ which gives $b = 8$.

Answer: (E)

10. We label Mathville as M, Algebratown as A, and the other intersection points of roads as shown.



There is 1 route from M to each of C and B: $M \to C$ and $M \to B$.
There are 3 routes to D: $M \to D$ and $M \to C \to D$ and $M \to B \to D$.
This means that there are 4 routes to F:

$$M \to C \to F \qquad M \to D \to F \qquad M \to C \to D \to F \qquad M \to B \to D \to F$$

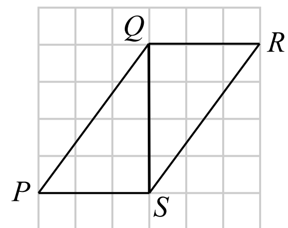
Similarly, there are 4 routes to E:

$$M \to B \to E \qquad M \to D \to E \qquad M \to C \to D \to E \qquad M \to B \to D \to E$$

Finally, there are $4 + 4 = 8$ routes to A, since every route comes through either E or F, no route goes through both E and F, and there are 4 routes to each of E and F.

Answer: (C)

11. Since the given grid is 6×6, the size of each of the small squares is 1×1.
This means that $QR = PS = 3$.
Join Q to S.



Since QS is vertical, and QR and PS are both horizontal, then $\angle RQS = 90^\circ$ and $\angle PSQ = 90^\circ$.
We note further that $QS = 4$.
Since $\triangle RQS$ is right-angled at Q, by the Pythagorean Theorem,

$$RS^2 = QR^2 + QS^2 = 3^2 + 4^2 = 25$$

Since $RS > 0$, then $RS = 5$.
Similarly, $PQ = 5$.
Thus, the perimeter of $PQRS$ is $PQ + QR + RS + PS = 5 + 3 + 5 + 3 = 16$.

Answer: (C)

12. The integers between 1 and 100 that have a ones digit equal to 6 are

$$6, 16, 26, 36, 46, 56, 66, 76, 86, 96$$

of which there are 10.
The additional integers between 1 and 100 that have a tens digits equal to 6 are

$$60, 61, 62, 63, 64, 65, 67, 68, 69$$

of which there are 9. (Note that 66 was included in the first list and not in the second list since we are counting integers rather than total number of 6s).
Since the digit 6 must occur as either the ones digit or the tens digit, there are $10 + 9 = 19$ integers between 1 and 100 with at least 1 digit equal to 6.

Answer: (C)

13. Suppose that Rosie runs x metres from the time that they start running until the time that they meet.
Since Mayar runs twice as fast as Rosie, then Mayar runs $2x$ metres in this time.
When Mayar and Rosie meet, they will have run a total of 90 m, since between the two of them, they have covered the full 90 m.
Therefore, $2x + x = 90$ and so $3x = 90$ or $x = 30$.
Since $2x = 60$, this means that Mayar has run 60 m when they meet.

Answer: (D)

14. We use A, B, C, D, and E to represent Andy, Bev, Cao, Dhruv, and Elcim, respectively.
We use the notation $D > B$ to represent the fact "Dhruv is older than Bev".
The five sentences give $D > B$ and $B > E$ and $A > E$ and $B > A$ and $C > B$. These show us that Dhruv and Cao are older than Bev, and Elcim and Andy are younger than Bev.
This means that two people are older than Bev and two people are younger than Bev, which means that Bev must be the third oldest.

ANSWER: (B)

15. We note that all of the given possible sums are odd, and also that every prime number is odd with the exception of 2 (which is even).
When two odd integers are added, their sum is even.
When two even integers are added, their sum is even.
When one even integer and one odd integer are added, their sum is odd.
Therefore, if the sum of two integers is odd, it must be the sum of an even integer and an odd integer.
Since the only even prime number is 2, then for an odd integer to be the sum of two prime numbers, it must be the sum of 2 and another prime number.
Note that

$$19 = 2 + 17 \qquad 21 = 2 + 19 \qquad 23 = 2 + 21 \qquad 25 = 2 + 23 \qquad 27 = 2 + 25$$

Since 17, 19 and 23 are prime numbers and 21 and 25 are not prime numbers, then 3 of the given integers are the sum of two prime numbers.

ANSWER: (A)

16. Since 60 games are played and each of the 3 pairs plays the same number of games, each pair plays $60 \div 3 = 20$ games.
Alvin wins 20% of the 20 games that Alvin and Bingyi play, so Alvin wins $\frac{20}{100} \times 20 = \frac{1}{5} \times 20 = 4$ of these 20 games and Bingyi wins $20 - 4 = 16$ of these 20 games.
Bingyi wins 60% of the 20 games that Bingyi and Cheska play, so Bingyi wins a total of $\frac{60}{100} \times 20 = \frac{3}{5} \times 20 = 12$ of these 20 games.
The games played by Cheska and Alvin do not affect Bingyi's total number of wins.
In total, Bingyi wins $16 + 12 = 28$ games.

ANSWER: (C)

17. Since $a + 5 = b$, then $a = b - 5$.
Substituting $a = b - 5$ and $c = 5 + b$ into $b + c = a$, we obtain

$$\begin{aligned} b + (5 + b) &= b - 5 \\ 2b + 5 &= b - 5 \\ b &= -10 \end{aligned}$$

(If $b = -10$, then $a = b - 5 = -15$ and $c = 5 + b = -5$ and $b + c = (-10) + (-5) = (-15) = a$, as required.)

ANSWER: (C)

18. Starting with the balls in the order `1 2 3 4 5`, we make a table of the positions of the balls after each of the first 10 steps:

Step	Ball that moves	Order after step
1	Rightmost	`1 2 5 3 4`
2	Leftmost	`2 5 1 3 4`
3	Rightmost	`2 5 4 1 3`
4	Leftmost	`5 4 2 1 3`
5	Rightmost	`5 4 3 2 1`
6	Leftmost	`4 3 5 2 1`
7	Rightmost	`4 3 1 5 2`
8	Leftmost	`3 1 4 5 2`
9	Rightmost	`3 1 2 4 5`
10	Leftmost	`1 2 3 4 5`

After 10 steps, the balls are in the same order as at the beginning. This means that after each successive set of 10 steps, the balls will be returned to their original order.
Since 2020 is a multiple of 10, then after 2020 steps, the balls will be in their original order.
Steps 2021 through 2025 will repeat the outcomes of steps 1 through 5 above, and so after 2025 steps, the balls will be in the reverse of their original order.
Therefore, 2025 is a possible value of N. This argument can be adapted to check that none of 2028, 2031 and 2027 are possible values of N.

ANSWER: (E)

19. The six-digit integer that Miyuki sent included the digits 2022 in that order along with two 3s. If the two 3s were consecutive digits, there are 5 possible integers:

332022 233022 203322 202332 202233

We can think about the pair of 3s being moved from left to right through the integer.
If the two 3s are not consecutive digits, there are 10 possible pairs of locations for the 3s: 1st/3rd, 1st/4th, 1st/5th, 1st/6th, 2nd/4th, 2nd/5th, 2nd/6th, 3rd/5th, 3rd/6th, 4th/6th.
These give the following integers:

323022 320322 320232 320223 230322 230232 230223 203232 203223 202323

(We can think about moving the leftmost 3 from left to right through the integer and finding all of the possible locations for the second 3.)
In total, there are thus $5 + 10 = 15$ possible six-digit integers that Miyuki could have texted.

ANSWER: (E)

20. *Solution 1*

Each of the n friends is to receive $\frac{1}{n}$ of the pizza.

Since there are two pieces that are each $\frac{1}{6}$ of the pizza and these pieces cannot be cut, then each friend receives at least $\frac{1}{6}$ of the pizza. This means that there cannot be more than 6 friends; that is, $n \leq 6$.

Therefore, $n = 7, 8, 9, 10$ are not possible. The sum of these is 34.

The value $n = 2$ is possible. We show this by showing that the pieces can be divided into two groups, each of which totals $\frac{1}{2}$ of the pizza.

Note that $\frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.

This also means that the other 6 pieces must also add to $\frac{1}{2}$.

We show that the value of $n = 3$ is possible by finding 3 groups of pieces, with each group totalling $\frac{1}{3}$ of the pizza.

Since $2 \times \frac{1}{6} = \frac{1}{3}$ and $4 \times \frac{1}{12} = \frac{1}{3}$, then the other 4 pieces must also add to $\frac{1}{3}$ (the rest of the pizza) and so $n = 3$ is possible.

The value $n = 4$ is possible since $2 \times \frac{1}{8} = \frac{1}{4}$ and $\frac{1}{6} + \frac{1}{12} = \frac{2}{12} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$ (which can be done twice). The other 4 pieces must also add to $\frac{1}{4}$.

The value $n = 6$ is possible since two pieces are $\frac{1}{6}$ on their own, two groups of size $\frac{1}{6}$ can be made from the four pieces of size $\frac{1}{12}$, and $\frac{1}{8} + \frac{1}{24} = \frac{3}{24} + \frac{1}{24} = \frac{4}{24} = \frac{1}{6}$ (which can be done twice), which makes 6 groups of size $\frac{1}{6}$.

The sum of the values of n that are not possible is either 34 (if $n = 5$ is possible) or 39 (if $n = 5$ is not possible). Since 34 is not one of the choices, the answer must be 39.

(We can see that $n = 5$ is not possible since to make a portion of size $\frac{1}{5}$ that includes a piece of size $\frac{1}{6}$, the remaining pieces must total $\frac{1}{5} - \frac{1}{6} = \frac{6}{30} - \frac{5}{30} = \frac{1}{30}$. Since every piece is larger than $\frac{1}{30}$, this is not possible.)

Solution 2

The pizza is cut into 2 pieces of size $\frac{1}{24}$, 4 of $\frac{1}{12}$, 2 of $\frac{1}{8}$, and 2 of $\frac{1}{6}$.

Each of these fractions can be written with a denominator of 24, so we can think of having 2 pieces of size $\frac{1}{24}$, 4 of $\frac{2}{24}$, 2 of $\frac{3}{24}$, and 2 of $\frac{4}{24}$.

To create groups of pieces of equal total size, we can now consider combining the integers 1, 1, 2, 2, 2, 2, 3, 3, 4, and 4 into groups of equal size. (These integers represent the size of each piece measured in units of $\frac{1}{24}$ of the pizza.)

Since the largest integer in the list is 4, then each group has to have size at least 4.

Since $4 = 24 \div 6$, then the slices cannot be broken into more than 6 groups of equal size, which

means that $n = 7, 8, 9, 10$ are not possible.
Here is a way of breaking the slices into $n = 6$ equal groups, each with total size $24 \div 6 = 4$:

$$4 \qquad 4 \qquad 3+1 \qquad 3+1 \qquad 2+2 \qquad 2+2$$

Here is a way of breaking the slices into $n = 4$ equal groups, each with total size $24 \div 4 = 6$:

$$4+2 \qquad 4+2 \qquad 3+3 \qquad 2+2+1+1$$

Here is a way of breaking the slices into $n = 3$ equal groups, each with total size $24 \div 3 = 8$:

$$4+4 \qquad 2+2+2+2 \qquad 3+3+1+1$$

Here is a way of breaking the slices into $n = 2$ equal groups, each with total size $24 \div 2 = 12$:

$$4+4+2+2 \qquad 3+3+2+2+1+1$$

Since 24 is not a multiple of 5, the pieces cannot be broken into 5 groups of equal size.
Therefore, the sum of the values of n that are not possible is $5 + 7 + 8 + 9 + 10 = 39$.

ANSWER: (D)

21. A 10 cm by 10 cm board has 9 rows of 9 holes, or $9 \times 9 = 81$ pegs in total.
Each hole on the 2 main diagonals has a peg in it.
There are 9 holes on each diagonal, with the centre hole on both diagonals, since there is an odd number of holes in each row.
Therefore, the total number of holes on the two diagonals is $9 + 9 - 1 = 17$.
This means that the number of empty holes is $81 - 17 = 64$.

ANSWER: 64

22. We start by looking for patterns in the rightmost two digits of powers of 4, powers of 5 and powers of 7.
The first few powers of 5 are

$$5^1 = 5 \qquad 5^2 = \mathbf{25} \qquad 5^3 = 1\mathbf{25} \qquad 5^4 = 6\mathbf{25} \qquad 5^5 = 31\mathbf{25}$$

It appears that, starting with 5^2, the rightmost two digits of powers of 5 are always 25.
To see this, we want to understand why if the rightmost two digits of a power of 5 are 25, then the rightmost two digits of the next power of 5 are also 25.
The rightmost two digits of a power of 5 are completely determined by the rightmost two digits of the previous power, since in the process of multiplication, any digits before the rightmost two digits do not affect the rightmost two digits of the product.
This means that the rightmost two digits of every power of 5 starting with 5^2 are 25, which means that the rightmost two digits of 5^{129} are 25.

The first few powers of 4 are

$$4^1 = 4 \qquad 4^2 = \mathbf{16} \qquad 4^3 = \mathbf{64} \qquad 4^4 = 2\mathbf{56} \qquad 4^5 = 10\mathbf{24} \qquad 4^6 = 40\mathbf{96} \qquad 4^7 = 16\,3\mathbf{84}$$

$$4^8 = 65\,5\mathbf{36} \qquad 4^9 = 262\,1\mathbf{44} \qquad 4^{10} = 1\,048\,5\mathbf{76} \qquad 4^{11} = 4\,194\,3\mathbf{04} \qquad 4^{12} = 16\,777\,2\mathbf{16}$$

We note that the rightmost two digits repeat after 10 powers of 4. This means that the rightmost two digits of powers of 4 repeat in a cycle of length 10.
Since 120 is a multiple of 10 and 127 is 7 more than a multiple of 10, the rightmost two digits

of 4^{127} are the same as the rightmost two digits of 4^7, which are 84.

The first few powers of 7 are

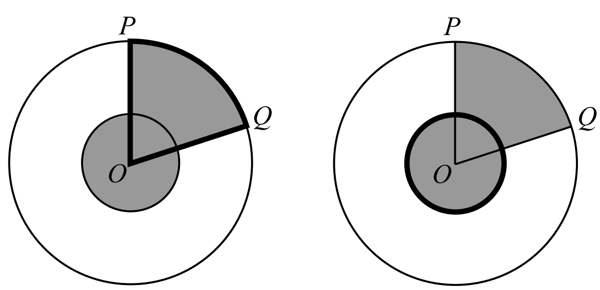
$$7^1 = 7 \qquad 7^2 = \mathbf{49} \qquad 7^3 = 3\mathbf{43} \qquad 7^4 = 24\mathbf{01} \qquad 7^5 = 16\,8\mathbf{07} \qquad 7^6 = 117\,6\mathbf{49}$$

We note that the rightmost two digits repeat after 4 powers of 7. This means that the rightmost two digits of powers of 7 repeat in a cycle of length 4.
Since 128 is a multiple of 4 and 131 is 3 more than a multiple of 4, the rightmost two digits of 7^{131} are the same as the rightmost two digits of 7^3, which are 43.

Therefore, the rightmost two digits of $4^{127} + 5^{129} + 7^{131}$ are the rightmost two digits of the sum $84 + 25 + 43 = 152$, or 52. (This is because when we add integers with more than two digits, any digits to the left of the rightmost two digits do not affect the rightmost two digits of the sum.)

ANSWER: 52

23. Since the shaded regions are equal in area, then when the unshaded sector in the small circle is shaded, the area of the now fully shaded sector of the larger circle must be equal to the area of the smaller circle.



The smaller circle has radius 1 and so it has area $\pi \times 1^2 = \pi$.
The larger circle has radius 3 and so it has area $\pi \times 3^2 = 9\pi$.
This means that the area of the shaded sector in the larger circle is π, which means that it must be $\frac{1}{9}$ of the larger circle.
This means that $\angle POQ$ must be $\frac{1}{9}$ of a complete circular angle, and so $\angle POQ = \frac{1}{9} \times 360^\circ = 40^\circ$.
Thus, $x = 40$.

ANSWER: 40

24. Since a Pretti number has 7 digits, it is of the form $a\,bcd\,efg$.
From the given information, the integer with digits abc is a perfect square.
Since a Pretti number is a seven-digit positive integer, then $a > 0$, which means that abc is between 100 and 999, inclusive.
Since $9^2 = 81$ (which has two digits) $10^2 = 100$ (which has three digits) and $31^2 = 961$ (which has three digits) and $32^2 = 1024$ (which has four digits), then abc (which has three digits) must be one of $10^2, 11^2, \ldots, 30^2, 31^2$, since 32^2 has 4 digits..
From the given information, the integer with digits $defg$ is a perfect cube.
Since the thousands digit of a Pretti number is not 0, then $d > 0$.
Since $9^3 = 729$ and $10^3 = 1000$ and $21^3 = 9261$ and $22^3 = 10\,648$, then $defg$ (which has four digits) must be one of $10^3, 11^3, \ldots, 20^3, 21^3$, since 22^3 has 5 digits.
Since the ten thousands digit and units digit of the original number are equal, then $c = g$.
In other words, the units digits of abc and $defg$ are equal.

The units digit of a perfect square depends only on the units digit of the integer being squared, since in the process of multiplication no digit to the left of this digit affects the resulting units digit.
The squares 0^2 through 9^2 are $0, 1, 4, 9, 16, 25, 36, 49, 64, 81$.
This gives the following table:

Units digit of n^2	Possible units digits of n
0	0
1	1, 9
4	2, 8
5	5
6	4, 6
9	3, 7

Similarly, the units digit of a perfect cube depends only on the units digit of the integer being cubed.
The cubes 0^3 through 9^3 are $0, 1, 8, 27, 64, 125, 216, 343, 512, 729$.
This gives the following table:

Units digit of m^3	Possible units digits of m
0	0
1	1
2	8
3	7
4	4
5	5
6	6
7	3
8	2
9	9

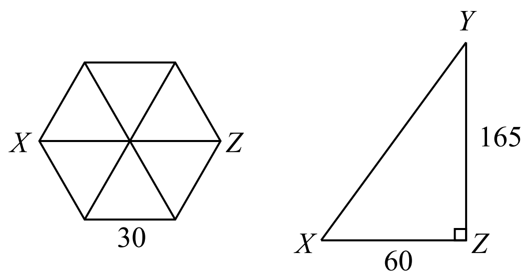
We combine this information to list the possible values of $c = g$ (from the first table, these must be 0, 1, 4, 5, 6, 9), the squares between 10^2 and 31^2, inclusive, with this units digit, and the cubes between 10^3 and 21^3 with this units digit:

Digit $c = g$	Possible squares	Possible cubes	Pretti numbers
0	10^2, 20^2, 30^2	10^3, 20^3	$3 \times 2 = 6$
1	11^2, 19^2, 21^2, 29^2, 31^2	11^3, 21^3	$5 \times 2 = 10$
4	12^2, 18^2, 22^2, 28^2	14^3	$4 \times 1 = 4$
5	15^2, 25^2	15^3	$2 \times 1 = 2$
6	14^2, 16^2, 24^2, 26^2	16^3	$4 \times 1 = 4$
9	13^2, 17^2, 23^2, 27^2	19^3	$4 \times 1 = 4$

For each square in the second column, each cube in the third column of the same row is possible. (For example, 19^2 and 11^3 give the Pretti number 3 611 331 while 19^2 and 21^3 give the Pretti number 3 619 261.) In each case, the number of Pretti numbers is thus the product of the number of possible squares and the number of possible cubes.
Therefore, the number of Pretti numbers is $6 + 10 + 4 + 2 + 4 + 4 = 30$.

ANSWER: 30

25. Throughout this solution, we remove the units (cm) as each length is in these same units.
First, we calculate the distance flown by the fly, which we call f.
Let Z be the point on the base on the prism directly underneath Y.
Since the hexagonal base has side length 30, then $XZ = 60$.
This is because a hexagon is divided into 6 equilateral triangles by its diagonals, and so the length of the diagonal is twice the side length of one of these triangles, which is twice the side length of the hexagon.

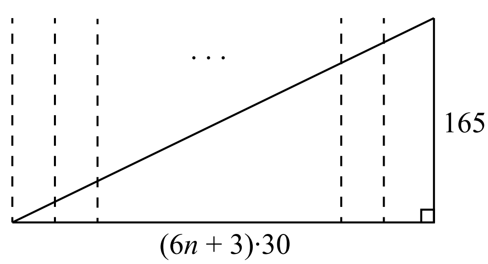


Also, $\triangle XZY$ is right-angled at Z, since XZ lies in the horizontal base and YZ is vertical.
By the Pythagorean Theorem, since $XY > 0$, then

$$XY = \sqrt{XZ^2 + YZ^2} = \sqrt{60^2 + 165^2}$$

Therefore, $f = XY = \sqrt{60^2 + 165^2}$.

Next, we calculate the distance crawled by the ant, which we call a.
Since the ant crawls $n + \frac{1}{2}$ around the prism and its crawls along all 6 of the vertical faces each time around the prism, then it crawls along a total of $6(n + \frac{1}{2}) = 6n + 3$ faces.
To find a, we "unwrap" the exterior of the prism.
Since the ant passes through $6n + 3$ faces, it travels a "horizontal" distance of $(6n + 3) \cdot 30$.
Since the ant moves from the bottom of the prism to the top of the prism, it passes through a vertical distance of 165.
Since the ant's path has a constant slope, its path forms the hypotenuse of a right-angled triangle with base of length $(6n + 3) \cdot 30$ and height of length 165.



By the Pythagorean Theorem, since $a > 0$, then $a = \sqrt{((6n + 3) \cdot 30)^2 + 165^2}$.

Now, we want a to be at least $20f$. In other words, we want to find the smallest possible value of n for which $a > 20f$.
Since these quantities are positive, the inequality $a > 20f$ is equivalent to the inequality $a^2 > 20^2 f^2$.

The following inequalities are equivalent:

$$\begin{aligned}
a^2 &> 20^2 f^2 \\
((6n+3)\cdot 30)^2 + 165^2 &> 400(60^2 + 165^2) \\
(6n+3)^2 \cdot 30^2 + 165^2 &> 400(60^2 + 165^2) \\
(6n+3)^2 \cdot 2^2 + 11^2 &> 400(4^2 + 11^2) \qquad \text{(dividing both sides by } 15^2\text{)} \\
4(6n+3)^2 + 121 &> 400 \cdot 137 \\
4(6n+3)^2 &> 54\,679 \\
(6n+3)^2 &> \tfrac{54\,679}{4} \\
6n+3 &> \sqrt{\tfrac{54\,679}{4}} \qquad \text{(since both sides are positive)} \\
6n &> \sqrt{\tfrac{54\,679}{4}} - 3 \\
n &> \tfrac{1}{6}\left(\sqrt{\tfrac{54\,679}{4}} - 3\right) \approx 18.986
\end{aligned}$$

Therefore, the smallest positive integer n for which this is true is $n = 19$.

Answer: 19

The CENTRE for EDUCATION
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2021 Pascal Contest

(Grade 9)

Tuesday, February 23, 2021
(in North America and South America)

Wednesday, February 24, 2021
(outside of North America and South America)

Solutions

1. Since Q is between P and R, then $PQ + QR = PR$.
   Since $PR = 12$ and $PQ = 3$, then $QR = PR - PQ = 12 - 3 = 9$.

   Answer: (D)

2. The fraction $\frac{4}{8}$ is equivalent to the fraction $\frac{1}{2}$.
   Therefore, the number 4 should be placed in the $\square$.

   Answer: (C)

3. Elena works for 4 hours and earns \$13.25 per hour.
   This means that she earns a total of $4 \times \$13.25 = \$53.00$.

   Answer: (E)

4. The perimeter of each of the squares of side length 1 is $4 \times 1 = 4$.
   The perimeters of the 7 squares in the diagram do not overlap, and so the perimeter of the entire figure is $7 \times 4 = 28$.

   Answer: (D)

5. Since there are 60 seconds in 1 minute, the number of seconds in 1.5 minutes is $1.5 \times 60 = 90$.
   Thus, Wesley's times were 63 seconds, 60 seconds, 90 seconds, 68 seconds, and 57 seconds.
   When these times in seconds are arranged in increasing order, we obtain 57, 60, 63, 68, 90.
   Thus, the median time is 63 seconds.

   Answer: (A)

6. The area of the original rectangle is $13 \times 10 = 130$.
   When the dimensions of the original rectangle are each increased by 2, we obtain a rectangle that is 15 by 12.
   The area of the new rectangle is $15 \times 12 = 180$, and so the area increased by $180 - 130 = 50$.

   Answer: (A)

7. *Solution 1*
   10% of 500 is $\frac{1}{10}$ of 500, which equals 50.
   Thus, 110% of 500 equals $500 + 50$, which equals 550.

   Solution 2
   110% of 500 is equal to $\frac{110}{100} \times 500 = 110 \times 5 = 550$.

   Answer: (E)

8. *Solution 1*
   We undo each of the operations in reverse order.
   The final result, 85, was obtained by multiplying a number by 5. This number was $85 \div 5 = 17$.
   The number 17 was obtained by decreasing n by 2. Thus, $n = 17 + 2 = 19$.

   Solution 2
   When n is decreased by 2, we obtain $n - 2$.
   When $n - 2$ is multiplied by 5, we obtain $5 \times (n - 2)$.
   From the given information, $5 \times (n - 2) = 85$ which means that $5n - 10 = 85$.
   From this, we obtain $5n = 95$ and so $n = \frac{95}{5} = 19$.

   Answer: (B)

9. Because 2 circles balance 1 triangle and 1 triangle balances 3 squares, then 2 circles balance 3 squares.
Because 2 circles balance 3 squares, then $2 + 2 = 4$ circles balance $3 + 3 = 6$ squares, which is choice (E).
(Can you argue that none of the other choices is equivalent to 6 squares?)

Answer: (E)

10. The integers that are multiples of both 5 and 7 are the integers that are multiples of 35.
The smallest multiple of 35 greater than 100 is $3 \times 35 = 105$. (The previous multiple of 35 is $2 \times 35 = 70$.)
Starting at 105 and counting by 35s, we obtain

$$105, 140, 175, 210, 245, 280, 315$$

The integers in this list that are between 100 and 300 and are not multiples of 10 (that is, whose units digit is not 0) are 105, 175, 245, of which there are 3.

Answer: (C)

11. Since $a\nabla b = a^b \times b^a$, then $2\nabla 3 = 2^3 \times 3^2 = 8 \times 9 = 72$.

Answer: (B)

12. Since $\triangle PQR$ is right-angled at Q, then $\angle PRQ = 90^\circ - \angle QPR = 90^\circ - 54^\circ = 36^\circ$.
Since $\angle PRS = \angle QRS$, then $\angle QRS = \frac{1}{2}\angle PRQ = \frac{1}{2}(36^\circ) = 18^\circ$.
Since $\triangle RQS$ is right-angled at Q, then $\angle RSQ = 90^\circ - \angle QRS = 90^\circ - 18^\circ = 72^\circ$.

Answer: (E)

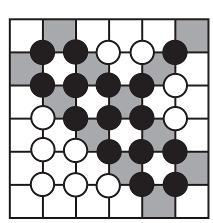
13. Since $m + 1 = \dfrac{n-2}{3}$, then $3(m + 1) = n - 2$.
This means that $3m + 3 = n - 2$ and so $3m - n = -2 - 3 = -5$.

Answer: (B)

14. Starting at 38, the robot moves 2 squares forward to 36, then rotates 90° clockwise to face 29 and then moves to 29.
Starting at 29, the robot moves 2 squares forward to 15, then rotates 90° clockwise to face 16 and then moves to 16.

Answer: (A)

15. There are 25 possible locations for the disc to be placed.
In the diagram below, each of these locations is marked with a small black disc if it is touching 2 shaded and unshaded squares (an equal number) and a small white disc if it is touching different numbers of shaded and unshaded squares.



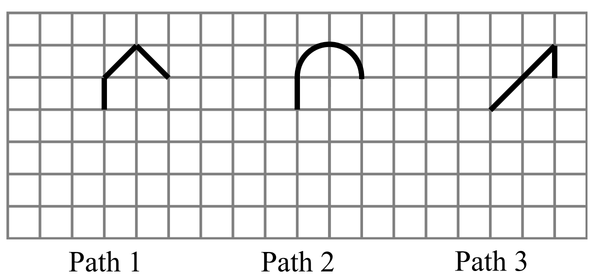
Therefore, there are 15 locations where the disc is touching an equal number of shaded and unshaded squares.
This means that the desired probability is $\frac{15}{25}$, which equals $\frac{3}{5}$.

ANSWER: (E)

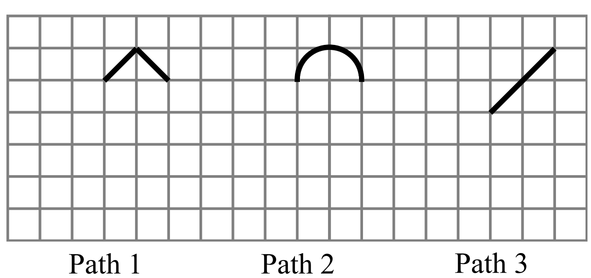
16. Perfect cubes have the property that the number of times that each prime factor occurs is a multiple of 3. This is because its prime factors can be separated into three identical groups; in this case, the product of each group is the cube root of the original number.
In particular, if $n^3 = 2^4 \times 3^2 \times 5^5 \times k$ where n is an integer, then the number of times that the prime factors 2, 3 and 5 occur in the integer n^3 must be multiples of 3.
Since n^3 already includes 4 factors of 2, then k must include at least 2 additional factors of 2, so that n^3 has a factor of 2^6. (k could also include more factors of 2, as long as the total number of factors of 2 is a multiple of 3.)
Since n^3 already includes 2 factors of 3, then k must include at least 1 additional factor of 3.
Since n^3 already includes 5 factors of 5, then k must include at least 1 additional factor of 5.
Therefore, k includes at least 2 factors of 2, at least 1 factor of 3, and at least 1 factor of 5.
This means that the smallest possible value of k is $2^2 \times 3 \times 5 = 60$. In principle, k could also include other prime factors, but to make k as small as possible, we do not need to consider this further.

ANSWER: (C)

17. To compare the lengths of these Paths, we begin by removing identical portions. In particular, we remove the horizontal segment of length 2, a vertical segment of length 1 from the left, and a vertical segment of length 4 from the right to obtain the following images:

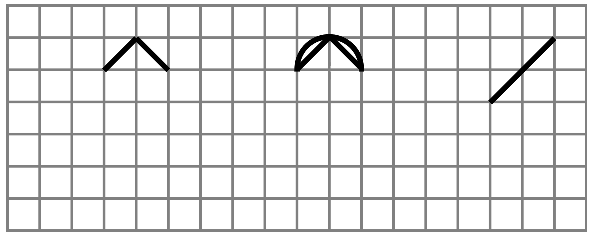


By removing the same lengths, we do not change the *relative* lengths of the Paths.
Each of the Paths still has a vertical segment of length 1, so we remove each of these segments, again maintaining the relative lengths of the Paths.



Each of Path 1 and Path 3 now consists of the diagonals of two of the grid squares. Thus, their original lengths were equal and so $x = z$.
This means that the final answer must equal (C) or (E), depending on whether $x = z$ is less than y or greater than y.

To answer this question, we re-draw the remaining segments of Path 1 under Path 2:



Since a straight line path between two points is shorter than any other path between these two points, the length of the semi-circle is longer than the total length of the two straight line segments.
This means that $x = z$ and $z < y$.
(As an alternate approach, can you determine the length of each of the original Paths and compare these numerically?)

ANSWER: (C)

18. The length of time between 10:10 a.m. and 10:55 a.m. is 45 minutes.
The length of time between 10:55 a.m. and 11:58 a.m. is 1 hour and 3 minutes, or 63 minutes.
Since trains arrive at each of these times and we are told that trains arrive every x minutes, then both 45 and 63 must be multiples of x. (In other words, if we count forward repeatedly by x minutes starting at 10:10 a.m., we will eventually count 10:55 a.m. and then eventually count 11:58 a.m.)
Of the given choices (9, 7, 5, 10, 11), only 9 is a factor of each of 45 and 63.

ANSWER: (A)

19. *Solution 1*
We work backwards through the given information.
At the end, there is 1 candy remaining.
Since $\frac{5}{6}$ of the candies are removed on the fifth day, this 1 candy represents $\frac{1}{6}$ of the candies left at the end of the fourth day.
Thus, there were $6 \times 1 = 6$ candies left at the end of the fourth day.
Since $\frac{4}{5}$ of the candies are removed on the fourth day, these 6 candies represent $\frac{1}{5}$ of the candies left at the end of the third day.
Thus, there were $5 \times 6 = 30$ candies left at the end of the third day.
Since $\frac{3}{4}$ of the candies are removed on the third day, these 30 candies represent $\frac{1}{4}$ of the candies left at the end of the second day.
Thus, there were $4 \times 30 = 120$ candies left at the end of the second day.
Since $\frac{2}{3}$ of the candies are removed on the second day, these 120 candies represent $\frac{1}{3}$ of the candies left at the end of the first day.
Thus, there were $3 \times 120 = 360$ candies left at the end of the first day.
Since $\frac{1}{2}$ of the candies are removed on the first day, these 360 candies represent $\frac{1}{2}$ of the candies initially in the bag.
Thus, there were $2 \times 360 = 720$ in the bag at the beginning.

Solution 2
Suppose that there were x candies in the bag at the beginning.

On the first day, $\frac{1}{2}$ of the candies are eaten, which means that $1 - \frac{1}{2} = \frac{1}{2}$ of the candies remain. Since there were x candies at the beginning of the first day, there are $\frac{1}{2}x$ candies at the end of the first day.
On the second day, $\frac{2}{3}$ of the remaining candies are eaten, which means that $1 - \frac{2}{3} = \frac{1}{3}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{2}x$ candies at the beginning of the second day, there are $\frac{1}{3} \times \frac{1}{2}x = \frac{1}{6}x$ candies at the end of the second day.
On the third day, $\frac{3}{4}$ of the remaining candies are eaten, which means that $1 - \frac{3}{4} = \frac{1}{4}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{6}x$ candies at the beginning of the third day, there are $\frac{1}{4} \times \frac{1}{6}x = \frac{1}{24}x$ candies at the end of the third day.
On the fourth day, $\frac{4}{5}$ of the remaining candies are eaten, which means that $1 - \frac{4}{5} = \frac{1}{5}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{24}x$ candies at the beginning of the fourth day, there are $\frac{1}{5} \times \frac{1}{24}x = \frac{1}{120}x$ candies at the end of the fourth day.
On the fifth day, $\frac{5}{6}$ of the remaining candies are eaten, which means that $1 - \frac{5}{6} = \frac{1}{6}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{120}x$ candies at the beginning of the fifth day, there are $\frac{1}{6} \times \frac{1}{120}x = \frac{1}{720}x$ candies at the end of the fifth day.
Since 1 candy remains, then $\frac{1}{720}x = 1$ which gives $x = 720$, and so there were 720 candies in the bag before the first day.

ANSWER: (B)

20. We make a chart of the possible integers, building their digits from left to right. In each case, we could determine the required divisibility by actually performing the division, or by using the following tests for divisibility:

- An integer is divisible by 3 when the sum of its digits is divisible by 3.
- An integer is divisible by 4 when the two-digit integer formed by its tens and units digits is divisible by 4.
- An integer is divisible by 5 when its units digit is 0 or 5.

$8R$	$8RS$	$N = 8RST$
81	812	8120
		8125
	816	8160
		8165
84	840	8400
		8405
	844	8440
		8445
	848	8480
		8485
87	872	8720
		8725
	876	8760
		8765

In the first column, we note that the integers between 80 and 89 that are multiples of 3 are 81, 84 and 87. In the second column, we look for the mutiples of 4 between 810 and 819, between

840 and 849, and between 870 and 879. In the third column, we add units digits of 0 or 5.
This analysis shows that there are 14 possible values of N.

Answer: (E)

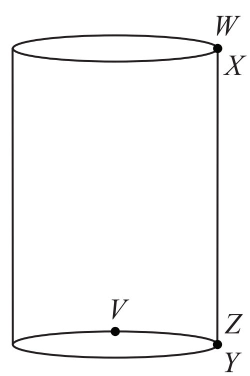
21. Since the average volume of three cubes is 700 cm^3, their total volume is 3×700 cm^3 or 2100 cm^3.
The volume of a cube with edge length s cm is s^3 cm^3.
Therefore, $3^3 + 12^3 + x^3 = 2100$ and so $27 + 1728 + x^3 = 2100$ or $x^3 = 345$.
Since $x^3 = 345$, then $x \approx 7.01$, which is closest to 7.

Answer: (E)

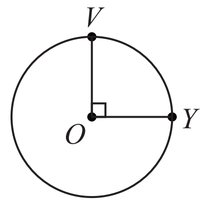
22. The height of each block is 2, 3 or 6.
Thus, the total height of the tower of four blocks is the sum of the four heights, each of which equals 2, 3 or 6.
If 4 blocks have height 6, the total height equals $4 \times 6 = 24$.
If 3 blocks have height 6, the fourth block has height 3 or 2.
Therefore, the possible heights are $3 \times 6 + 3 = 21$ and $3 \times 6 + 2 = 20$.
If 2 blocks have height 6, the third and fourth blocks have height 3 or 2.
Therefore, the possible heights are $2\times6+3+3 = 18$ and $2\times6+3+2 = 17$ and $2\times6+2+2 = 16$.
If 1 block has height 6, the second, third and fourth blocks have height 3 or 2.
Therefore, the possible heights are $6+3+3+3 = 15$ and $6+3+3+2 = 14$ and $6+3+2+2 = 13$ and $6 + 2 + 2 + 2 = 12$.
If no blocks have height 6, the possible heights are $3 + 3 + 3 + 3 = 12$ and $3 + 3 + 3 + 2 = 11$ and $3 + 3 + 2 + 2 = 10$ and $3 + 2 + 2 + 2 = 9$ and $2 + 2 + 2 + 2 = 8$.
The possible heights are thus $24, 21, 20, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8$.
There are 14 possible heights.

Answer: (B)

23. When the cylinder is created, W and X touch and Z and Y touch.



This means that WY is vertical and so is perpendicular to the plane of the circular base of the cylinder.
This means that $\triangle VYW$ is right-angled at Y.
By the Pythagorean Theorem, $WV^2 = WY^2 + VY^2$.
Note that WY equals the height of the rectangle, which is 3 (the length of WZ) and that VY is now measured *through* the cylinder, not along the line segment ZY.
Let O be the centre of the circular base of the cylinder.
In the original rectangle, $ZY = WX = 4$ and $ZV = 3$, which means that $VY = 1 = \frac{1}{4}ZY$.
This means that V is one-quarter of the way around the circumference of the circular base from Y back to Z.



As a result, $\angle YOV = 90^\circ$, since 90° is one-quarter of a complete circular angle.
Thus, $\triangle YOV$ is right-angled at O.
By the Pythagorean Theorem, $VY^2 = VO^2 + OY^2$.
Since YO and OV are radii of the circular base, then $VO = OY$ and so $YV^2 = 2VO^2$.
Since the circumference of the circular base is 4 (the original length of ZY), then if the radius of the base is r, we have $2\pi r = 4$ and so $r = \dfrac{4}{2\pi} = \dfrac{2}{\pi}$.
Since $VO = r$, then $YV^2 = 2VO^2 = 2\left(\dfrac{2}{\pi}\right)^2 = \dfrac{8}{\pi^2}$.
This means that

$$WV^2 = WY^2 + YV^2 = 9 + \frac{8}{\pi^2} = \frac{9\pi^2+8}{\pi^2} = \frac{8+9\pi^2}{\pi^2}$$

and so $WV = \sqrt{\dfrac{8+9\pi^2}{1\cdot\pi^2}}$.
Since the coefficient of π^2 in the denominator is 1, it is not possible to "reduce" the values of a, b and c any further, and so $a = 8$, $b = 9$, and $c = 1$, which gives $a + b + c = 18$.

ANSWER: (C)

24. Starting with a list of $66 = 2 \times 33$ items, the items in the first 33 positions

$$1, 2, 3, \ldots, 31, 32, 33$$

are moved by an in-shuffle to the odd positions of the resulting list, namely to the positions

$$1, 3, 5, \ldots, 61, 63, 65$$

respectively. This means that an item in position x with $1 \leq x \leq 33$ is moved by an in-shuffle to position $2x - 1$.
We can see why this formula works by first moving the items in positions $1, 2, 3, \ldots, 31, 32, 33$ to the even positions $2, 4, 6, \ldots, 62, 64, 66$ (doubling the original position numbers) and then shifting each backwards one position to $1, 3, 5, \ldots, 61, 63, 65$.

Also, the items in the second 33 positions

$$34, 35, 36, \ldots, 64, 65, 66$$

are moved by an in-shuffle to the even positions of the resulting list, namely to the positions

$$2, 4, 6, \ldots, 62, 64, 66$$

respectively. This means that an item in position x with $34 \leq x \leq 66$ is moved by an in-shuffle to position $2(x - 33)$.
We can see why this formula works by first moving the items in positions $34, 35, 36, \ldots, 64, 65, 66$ backwards 33 positions to $1, 2, 3, \ldots, 31, 32, 33$ and then doubling their position numbers to obtain $2, 4, 6, \ldots, 62, 64, 66$.

In summary, the item in position x is moved by an in-shuffle to position

- $2x - 1$ if $1 \le x \le 33$
- $2(x - 33)$ if $34 \le x \le 66$

Therefore, the integer 47 is moved successively as follows:

List	Position
1	47
2	$2(47 - 33) = 28$
3	$2(28) - 1 = 55$
4	$2(55 - 33) = 44$
5	$2(44 - 33) = 22$
6	$2(22) - 1 = 43$
7	$2(43 - 33) = 20$
8	$2(20) - 1 = 39$
9	$2(39 - 33) = 12$
10	$2(12) - 1 = 23$
11	$2(23) - 1 = 45$
12	$2(45 - 33) = 24$
13	$2(24) - 1 = 47$

Because the integer 47 moves back to position 47 in list 13, this means that its positions continue in a cycle of length 12:

$$47, 28, 55, 44, 22, 43, 20, 39, 12, 23, 45, 24$$

This is because the position to which an integer moves is completely determined by its previous position and so the list will cycle once one position repeats.
We note that the integer 47 is thus in position 24 in every 12th list starting at the 12th list.
Since $12 \times 83 = 996$ and $12 \times 84 = 1008$, the cycle occurs a total of 83 complete times and so the integer 47 is in the 24th position in 83 lists. Even though an 84th cycle begins, it does not conclude and so 47 does not occur in the 24th position for an 84th time among the 1001 lists.

Answer: (C)

25. When Yann removes 4 of the n integers from his list, there are $n - 4$ integers left.
Suppose that the sum of the $n - 4$ integers left is T.
The average of these $n - 4$ integers is $89.5625 = 89.5 + 0.0625 = 89 + \frac{1}{2} + \frac{1}{16} = 89\frac{9}{16} = \frac{1433}{16}$.
Since the sum of the $n-4$ integers is T, then $\frac{T}{n-4} = \frac{1433}{16}$ which means that $16T = 1433(n-4)$.
Since 1433 and 16 have no common divisor larger than 1 (the positive divisors of 16 are 1, 2, 4, 8, 16, none of which other than 1 is a divisor of 1433), the value of $n - 4$ is a multiple of 16.

Since $100 < p < q < r < s$, the original list includes more than 100 numbers.
Since the original list includes consecutive integers starting at 1 and only 4 of more than 100 numbers are removed, it seems likely that the average of the original list and the average of the new list should be relatively similar.
Since the average of the new list is 89.5625 which is close to 90, it seems reasonable to say that the average of the original list is close to 90.
Since the original list is a list of consecutive positive integers starting at 1, this means that we would guess that the original list has roughly 180 integers in it.

In other words, n appears to be near 180.
We do know that $n-4$ is a multiple of 16. The closest multiples of 16 to 180 are 160, 176 and 192, which correspond to $n = 164$, $n = 180$, and $n = 196$.

Suppose that $n = 180$, which seems like the most likely possibility. We will show at the end of the solution that this is the only possible value of n.
The equation $\dfrac{T}{n-4} = 89.5625$ gives $T = 176 \times 89.5625 = 15\,763$.
The sum of the n integers in the original list is

$$1 + 2 + 3 + 4 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$$

When $n = 180$, the sum of the integers $1, 2, 3, \ldots, 178, 179, 180$ is $\frac{1}{2}(180)(181) = 16\,290$.
Since the sum of the numbers in the original list is 16 290 and the sum once the four numbers are removed is 15 763, the sum of the four numbers removed is $16\,290 - 15\,763 = 527$.
In other words, $p + q + r + s = 527$.

We now want to count the number of ways in which we can choose p, q, r, s with the conditions that $100 < p < q < r < s \leq 180$ and $p + q + r + s = 527$ with at least three of p, q, r, s consecutive.
The fourth of these integers is at least 101 and at most 180, which means that the sum of the three consecutive integers is at least $527 - 180 = 347$ and is at most $527 - 101 = 426$.
This means that the consecutive integers are at least $115, 116, 117$ (whose sum is 348) since $114 + 115 + 116 = 345$ which is too small and smaller integers will give sums that are smaller still.
If p, q, r equal 115, 116, 117, then $s = 527 - 348 = 179$.
The consecutive integers are at most $141, 142, 143$ (whose sum is 426) since $142+143+144 = 429$ which is too large and larger integers will give sums that are larger still.
If p, q, r equal 141, 142, 143, then $s = 527 - 426 = 101$.
When each of the three consecutive integers is increased by 1 and the sum is constant, the fourth integer is decreased by 3 to maintain this constant sum.
Using all of this, we obtain the following lists p, q, r, s:

$$115, 116, 117, 179 \;\; ; \;\; 116, 117, 118, 176 \;\; ; \;\; \ldots \;\; ; \;\; 130, 131, 132, 134$$

$$128, 132, 133, 134 \;\; ; \;\; 125, 133, 134, 135 \;\; ; \;\; \ldots \;\; ; \;\; 101, 141, 142, 143$$

Note that we cannot use 131, 132, 133, 131, since p, q, r, s must be distinct.
There are 26 lists of integers that can be removed (16 in the first set and 10 in the second set).
The corresponding values of s are:

$$179, 176, 173, 170, 167, 164, 161, 158, 155, 152, 149, 146, 143, 140, 137, 134$$

$$134, 135, 136, 137, 138, 139, 140, 141, 142, 143$$

There are 4 values of s that overlap between the two lists, and so there are $26 - 4 = 22$ possible values for s.

Why is $n = 180$ the only possible value of n?
To see this, we use the fact that the average of the list of consecutive integers starting at a and ending at b equals the average of a and b, or $\dfrac{a+b}{2}$. (This is true because the integers in the

list have a constant difference and are thus evenly distributed, which means that the average of the first and last integers will equal the average of all of the integers in the list.)

The original list of integers is $1, 2, \ldots, n-1, n$ which has an average of $\dfrac{n+1}{2}$.

If the four largest integers are removed from the list, the new list is $1, 2, \ldots, n-5, n-4$, which has an average of $\dfrac{n-3}{2}$.

If the four smallest integers are removed from the list, the new list is $5, 6, \ldots, n-1, n$, which has an average of $\dfrac{n+5}{2}$.

When any four integers are removed, the sum of the remaining integers is greater than or equal to the sum of $1, 2, \ldots, n-5, n-4$ and less than or equal to the sum of $5, 6, \ldots, n-1, n$. Since the denominator in the average calculation remains the same, the average of any of the lists after four numbers are removed is at least $\dfrac{n-3}{2}$ and at most $\dfrac{n+5}{2}$.

This means that the actual average (which is 89.5625) is greater than or equal to $\dfrac{n-3}{2}$ and less than or equal to $\dfrac{n+5}{2}$.

Since $89.5625 \geq \dfrac{n-3}{2}$, then $n-3 \leq 179.125$ and so $n \leq 182.125$.

Since $89.5625 \leq \dfrac{n+5}{2}$, then $n+5 \geq 179.125$ and so $n \geq 174.125$.

Since n is an integer, then $175 \leq n \leq 182$ and so $171 \leq n-4 \leq 178$.

Since $n-4$ is a multiple of 16, then $n-4 = 176$ and so $n = 180$, as required.

ANSWER: 22

(The correct answer was missing from the original version of the problem.)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2020 Pascal Contest

(Grade 9)

Tuesday, February 25, 2020
(in North America and South America)

Wednesday, February 26, 2020
(outside of North America and South America)

Solutions

1. The figure includes 5 groups of 5 squares.
Thus, there are $5 \times 5 = 25$ squares in total.

ANSWER: (E)

2. Calculating, $0.8 + 0.02 = 0.80 + 0.02 = 0.82$.

ANSWER: (C)

3. *Solution 1*
Since $2x + 6 = 16$, then $\frac{2x+6}{2} = \frac{16}{2}$ and so $x + 3 = 8$.
Since $x + 3 = 8$, then $x + 4 = (x + 3) + 1 = 8 + 1 = 9$.

Solution 2
Since $2x + 6 = 16$, then $2x = 16 - 6 = 10$.
Since $2x = 10$, then $\frac{2x}{2} = \frac{10}{2}$ and so $x = 5$.
Since $x = 5$, then $x + 4 = 5 + 4 = 9$.

ANSWER: (C)

4. The positive divisor pairs of 24 are:

1 and 24   2 and 12   3 and 8   4 and 6

Of these, the pair whose sum is 11 is 3 and 8.
The difference between these two integers is $8 - 3 = 5$.

ANSWER: (D)

5. *Solution 1*
Since the side lengths of the triangle are $x - 1$, $x + 1$ and 7, its perimeter is $(x - 1) + (x + 1) + 7$ which equals $2x + 7$.
Since $x = 10$, the perimeter equals $2 \times 10 + 7$ which is equal to 27.

Solution 2
Since $x = 10$, the side lengths of the triangle are $x - 1 = 9$ and $x + 1 = 11$ and 7.
The perimeter of the triangle is thus $9 + 11 + 7$ which equals 27.

ANSWER: (D)

6. We note that $2^3 = 2 \times 2 \times 2 = 8$ and $2^4 = 2^3 \times 2 = 16$.
Therefore, $\frac{2^4 - 2}{2^3 - 1} = \frac{16 - 2}{8 - 1} = \frac{14}{7} = 2$.
Alternatively, $\frac{2^4 - 2}{2^3 - 1} = \frac{2(2^3 - 1)}{2^3 - 1} = 2$.

ANSWER: (E)

7. Ewan's sequence starts with 3 and each following number is 11 larger than the previous number. Since every number in the sequence is some number of 11s more than 3, this means that each number in the sequence is 3 more than a multiple of 11. Furthermore, every such positive integer is in Ewan's sequence.
Since $110 = 11 \times 10$ is a multiple of 11, then $113 = 110 + 3$ is 3 more than a multiple of 11, and so is in Ewan's sequence.
Alternatively, we could write Ewan's sequence out until we get into the correct range:

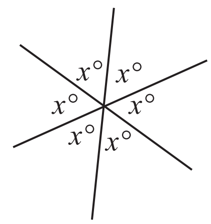
$$3, 14, 25, 36, 47, 58, 69, 80, 91, 102, 113, 124, \ldots$$

ANSWER: (A)

8. From the bar graph, Matilda saw 6 goldfinches, 9 sparrows, and 5 grackles.
   In total, she saw $6 + 9 + 5 = 20$ birds.
   This means that the percentage of birds that were goldfinches is $\frac{6}{20} \times 100\% = \frac{3}{10} \times 100\% = 30\%$.

   Answer: (C)

9. Since opposite angles are equal, then the three unmarked angles around the central point each has measure x°.

   Since the angles around a point add to 360°, then $6 \times x^\circ = 360^\circ$.
   From this, $6x = 360$ and so $x = 60$.

   Answer: (C)

10. Starting at 1:00 p.m., Jorge watches a movie that is 2 hours and 20 minutes long.
    This first movie ends at 3:20 p.m.
    Then, Jorge takes a 20 minute break.
    This break ends at 3:40 p.m.
    Then, Jorge watches a movie that is 1 hour and 45 minutes long.
    After 20 minutes of this movie, it is 4:00 p.m. and there is still 1 hour and 25 minutes left in the movie. This second movie thus ends at 5:25 p.m.
    Then, Jorge takes a 20 minute break which ends at 5:45 p.m.
    Finally, Jorge watches a movie that is 2 hours and 10 minutes long.
    This final movie ends at 7:55 p.m.

    Answer: (D)

11. 12 and 21 are multiples of 3 ($12 = 4 \times 3$ and $21 = 7 \times 3$) so the answer is not (A) or (D).
    16 is a perfect square ($16 = 4 \times 4$) so the answer is not (C).
    The sum of the digits of 26 is 8, which is not a prime number, so the answer is not (E).
    Since 14 is not a multiple of a three, 14 is not a perfect square, and the sum of the digits of 14 is $1 + 4 = 5$ which is prime, then the answer is 14, which is choice (B).

    Answer: (B)

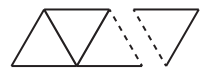
12. Since the average of three heights is 171 cm, then the sum of these three heights is 3×171 cm or 513 cm.
    Since Jiayin's height is 161 cm, then the sum of Natalie's and Harpreet's heights must equal 513 cm $-$ 161 cm $=$ 352 cm.
    Since Harpreet and Natalie are the same height, this height is $\frac{352 \text{ cm}}{2} = 176$ cm.
    Therefore, Natalie's height is 176 cm.

    Answer: (C)

13. Since the ratio of apples to bananas is $3 : 2$, then we can let the numbers of apples and bananas equal $3n$ and $2n$, respectively, for some positive integer n.
Therefore, the total number of apples and bananas is $3n + 2n = 5n$, which is a multiple of 5.
Of the given choices, only (E) 72 is not a multiple of 5 and so cannot be the total.
(Each of the other choices can be the total by picking an appropriate value of n.)

Answer: (E)

14. The first figure consists of one tile with perimeter 3×7 cm $= 21$ cm.
Each time an additional tile is added, the perimeter of the figure increases by 7 cm (one side length of a tile), because one side length of the previous figure is "covered up" and two new side lengths of a tile are added to the perimeter for a net increase of one side length (or 7 cm).



Since the first figure has perimeter 21 cm and we are looking for the figure with perimeter 91 cm, then the perimeter must increase by 91 cm $-$ 21 cm $=$ 70 cm.
Since the perimeter increases by 7 cm when each tile is added, then $\dfrac{70\text{ cm}}{7\text{ cm/tile}} = 10$ tiles need to be added to reach a perimeter of 91 cm.
In total, this figure will have $1 + 10 = 11$ tiles.

Answer: (B)

15. The total area of the shaded region equals the area of the small square (9) plus the area between the large square and medium square.
Since the area of the large square is 49 and the area of the medium square is 25, then the area of the region between these squares is $49 - 25 = 24$.
Therefore, the total area of the shaded region is $9 + 24 = 33$.

Answer: (A)

16. We look at each of the five choices:

(A) $3(x + 2) = 3x + 6$

(B) $\dfrac{-9x - 18}{-3} = \dfrac{-9x}{-3} + \dfrac{-18}{-3} = 3x + 6$

(C) $\dfrac{1}{3}(3x) + \dfrac{2}{3}(9) = x + 6$

(D) $\dfrac{1}{3}(9x + 18) = 3x + 6$

(E) $3x - 2(-3) = 3x + (-2)(-3) = 3x + 6$

The expression that is not equivalent to $3x + 6$ is the expression from (C).

Answer: (C)

17. Since there are two possible prizes that Jamie can win and each is equally likely, then the probability that Jamie wins \$30 is $\frac{1}{2}$ and the probability that Jamie wins \$40 is $\frac{1}{2}$.
If Jamie wins \$30, then for the total value of the prizes to be \$50, Ben must win \$20. The probability that Ben wins \$20 is $\frac{1}{3}$, since there are three equally likely outcomes for Ben.
If Jamie wins \$40, then for the total value of the prizes to be \$50, Ben must win \$10. The probability that Ben wins \$10 is $\frac{1}{3}$.
Since Ben's and Jamie's prizes come from different draws, we can assume that the results are independent, and so the probability that Jamie wins \$30 and Ben wins \$20 is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
Similarly, the probability that Jamie wins \$40 and Ben wins \$10 is $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.
Therefore, the probability that the total value of their prizes is \$50 is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.
ANSWER: (B)

18. Since the square root of n is between 17 and 18, then n is between $17^2 = 289$ and $18^2 = 324$.
Since n is a multiple of 7, we need to count the number of multiples of 7 between 289 and 324.
Since $41 \times 7 = 287$ and $42 \times 7 = 294$, then 294 is the smallest multiple of 7 larger than 289.
Since $46 \times 7 = 322$ and $47 \times 7 = 329$, then 322 is the largest multiple of 7 smaller than 324.
This means that the multiples of 7 between 289 and 324 are $42 \times 7 = 294$, $43 \times 7 = 301$, $44 \times 7 = 308$, $45 \times 7 = 315$, and $46 \times 7 = 322$, of which there are 5.
(We note that we could have determined that there were 5 such multiples by calculating $46 - 42 + 1$ which equals 5.)
Therefore, there are 5 possible values for n.
ANSWER: (D)

19. Each card fits into exactly one of the following categories:

(A) lower case letter on one side, even integer on the other side
(B) lower case letter on one side, odd integer on the other side
(C) upper case letter on one side, even integer on the other side
(D) upper case letter on one side, odd integer on the other side

The given statement is

"If a card has a lower case letter on one side, then it has an odd integer on the other."

If a card fits into category (B), (C) or (D), it does not violate the given statement, and so the given statement is true. If a card fits into category (A), it does violate the given statement. Therefore, we need to turn over any card that might be in category (A). Of the given cards,

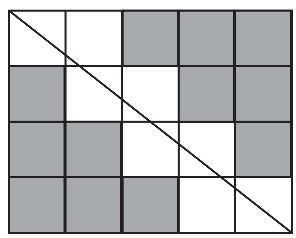
(i) 1 card shows a lower case letter and might be in (A),
(ii) 4 cards show an upper case letter and is not in (A),
(iii) 2 cards show an even integer and might be in (A), and
(iv) 8 cards show an odd integer and is not in (A).

In order to check if this statement is true, we must turn over the cards in (i) and (iii), of which there are 3.
ANSWER: (E)

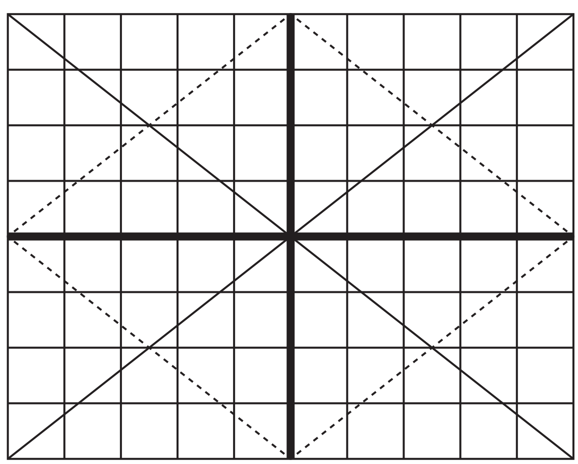
20. The original $5 \times 5 \times 5$ cube has 6 faces, each of which is 5×5.
When the three central columns of cubes is removed, one of the "1×1 squares" on each face is removed.
This means that the surface area of each face is reduced by 1 to $5 \times 5 - 1 = 24$. This means that the total exterior surface area of the cube is $6 \times 24 = 144$.
When each of the central columns is removed, it creates a "tube" that is 5 unit cubes long. Each of these tubes is $5 \times 1 \times 1$.
Since the centre cube of the original $5 \times 5 \times 5$ cube is removed when each of the three central columns is removed, this means that each of the three $5 \times 1 \times 1$ tubes is split into two $2 \times 1 \times 1$ tubes.
The interior surface area of each of these tubes consists of four faces, each of which is 2×1. (We could instead think about the exterior surface area of a $2 \times 1 \times 1$ rectangular prism, ignoring its square ends.) Thus, the interior surface area from 6 tubes each with 4 faces measuring 2×1 gives a total area of $6 \times 4 \times 2 \times 1 = 48$.
In total, the surface area of the resulting solid is $144 + 48 = 192$.

ANSWER: (E)

21. If we remove one diagonal from the given 4×5 grid, we see that 8 squares are intersected by the remaining diagonal and 12 squares are not intersected.



The result is the same whichever of the two diagonals is removed.
We can construct an 8×10 grid with two diagonals by combining four such 4×5 grids:



When these four pieces are joined together, the diagonals of the pieces join to form the diagonal of the large rectangle because their slopes are the same.
In each of the four pieces, 12 of the 1×1 squares are not intersected by either diagonal.
Overall, this means that $4 \times 12 = 48$ of the 1×1 squares are not intersected by either diagonal.

ANSWER: (D)

22. Since point R is on PQ, then $PQ = PR + QR = 6 + 4 = 10$.
Semi-circles with diameters of 10, 6 and 4 have radii of 5, 3 and 2, respectively.
A semi-circle with diameter PQ has area $\frac{1}{2} \times \pi \times 5^2 = \frac{25}{2}\pi$.
A semi-circle with diameter PR has area $\frac{1}{2} \times \pi \times 3^2 = \frac{9}{2}\pi$.
A semi-circle with diameter QR has area $\frac{1}{2} \times \pi \times 2^2 = 2\pi$.
The shaded region consists of a section to the left of PQ and a section to the right of PQ.
The area of the section to the left of PQ equals the area of the semi-circle with diameter PQ minus the area of the semi-circle with diameter PR.
This area is thus $\frac{25}{2}\pi - \frac{9}{2}\pi = 8\pi$.
The area of the section to the right of PQ equals the area of the semi-circle with diameter QR.
This area is thus 2π.
Therefore, the area of the entire shaded region is $8\pi + 2\pi = 10\pi$.
Since the large circle has radius 5, its area is $\pi \times 5^2 = 25\pi$.
Since the area of the shaded region is 10π, then the area of the unshaded region is equal to $25\pi - 10\pi = 15\pi$.
The ratio of the area of the shaded region to the area of the unshaded region is $10\pi : 15\pi$ which is equivalent to $2 : 3$.

Answer: (B)

23. Since there are 4 players in the tournament and each player plays each other player once, then each player plays 3 games.
Since each win earns 5 points and each tie earns 2 points, the possible results for an individual player are:

- 3 wins, 0 losses, 0 ties: 15 points
- 2 wins, 0 losses, 1 tie: 12 points
- 2 wins, 1 loss, 0 ties: 10 points
- 1 win, 0 losses, 2 ties: 9 points
- 1 win, 1 loss, 1 tie: 7 points
- 1 win, 2 losses, 0 ties: 5 points
- 0 wins, 0 losses, 3 ties: 6 points
- 0 wins, 1 loss, 2 ties: 4 points
- 0 wins, 2 losses, 1 tie: 2 points
- 0 wins, 3 losses, 0 ties: 0 points

In the third table given, Deb has 2 points which means that Deb had 1 tie. If one player has a tie, then another player must also have a tie. But neither 15 points nor 5 points is a possible total to obtain with a tie. Therefore, the third table is not possible.

Similarly, in the fourth table, Ali with 12 points must have had a tie, but none of the other players' scores allow for have a tie, so the fourth table is not possible.

In the second table, each of Che and Deb must have 2 ties and neither Ali nor Bea can have a tie because of their totals of 10 points each. Since Che and Deb only played each other once, then each of them must have a tie against another player, which is not possible. Therefore, the second table is not possible.

The first table is possible:

Result	Ali	Bea	Che	Deb
Ali wins against Bea	5 points	0 points		
Ali wins against Che	5 points		0 points	
Ali wins against Deb	5 points			0 points
Bea ties Che		2 points	2 points	
Bea wins against Deb		5 points		0 points
Che ties Deb			2 points	2 points
TOTAL	15 points	7 points	4 points	2 points

Therefore, exactly one of the four given final point distributions is possible.

Answer: (B)

24. If Lucas chooses 1 number only, there are 8 possibilities for the sum, namely the 8 numbers themselves: 2, 5, 7, 12, 19, 31, 50, 81.

To count the number of additional sums to be included when Lucas chooses two numbers, we make a table, adding the number on left to the number on top when it is less than the number on top (we don't need to add the numbers in both directions or a number to itself):

+	2	5	7	12	19	31	50	81
2		7	9	14	21	33	52	83
5			12	17	24	36	55	86
7				19	26	38	57	88
12					31	43	62	93
19						50	69	100
31							81	112
50								131
81								

We note two things from this table. First, any time we add two consecutive numbers from the original list who sum is not too large, we obtain another number in the list. We do not include this as a new sum as these are already accounted for as sums of 1 number only. Second, the remaining sums are all distinct and there are 20 additional sums that are less than or equal to 100.

Lastly, we consider sums formed by three numbers from the list.
The fact that the sum of any two consecutive numbers from the list equals the next number in the list becomes very important in this case.
If the three numbers chosen are three consecutive numbers in the list and their sum is not too large, then their sum is actually equal to the sum of two numbers from the list. This is because the largest two of the three numbers can be combined into one yet larger number from the list. For example, $5 + 7 + 12 = 5 + (7 + 12) = 5 + 19$, which is already counted above.
If any two of the three numbers chosen are consecutive in the list, the same thing happens. For example, $12 + 19 + 50 = (12 + 19) + 50 = 31 + 50$ and $2 + 31 + 50 = 2 + (31 + 50) = 2 + 81$.
Therefore, any additional sums that are created must come from three numbers, no two of which are consecutive.
We count these cases individually and sequentially, knowing that we are only interested in the sums less than 100 and remembering that we cannot include consecutive numbers from the list:

- $2 + 7 + 19 = 28$; $2 + 7 + 31 = 40$; $2 + 7 + 50 = 59$; $2 + 7 + 81 = 90$
- $2 + 12 + 31 = 45$; $2 + 12 + 50 = 64$; $2 + 12 + 81 = 95$
- $2 + 19 + 50 = 71$
- $5 + 12 + 31 = 48$; $5 + 12 + 50 = 67$; $5 + 12 + 81 = 98$
- $5 + 19 + 50 = 74$
- $7 + 19 + 50 = 76$

Every other combination of 3 integers from the list either includes 2 consecutive numbers (and so has been counted already) or includes both 81 and one of 31 and 19 (and so is too large).
In this case, there are 13 additional sums.

Putting the three cases together, there are $8 + 20 + 13 = 41$ different sums less than or equal to 100.

ANSWER: (E)

25. Before beginning our solution, we need several facts about prime numbers and prime factorizations:

F1. Every positive integer greater than 1 can be written as a product of prime numbers in a unique way. (If the positive integer is itself prime, this product consists of only the prime number.) This fact is called the "Fundamental Theorem of Arithmetic". This fact is often seen implicitly in finding a "factor tree" for a given integer. For example, 1500 is equal to $2^2 \times 3^1 \times 5^3$ and there is no other way of writing 1500 as a product of prime numbers. Note that rearranging the same prime factors in a different order does not count as a different factorization.

F2. If n is a positive integer and d is a positive integer that is a divisor of n, then the only possible prime factors of d are the prime factors of n. For example, if d is a positive divisor of $n = 1500$, then the only possible prime factors of d are 2, 3 and 5. This means, for example, that d cannot be divisible by 7 or by 11 or by any other prime number not equal to 2, 3 or 5. d might or might not be divisible by each of 2, 3 or 5.

F3. If n is a positive integer, d is a positive integer that is a divisor of n, and p is a prime factor of both n and d, then p cannot divide d "more times" than it divides n. For example, if d is a positive divisor of $n = 1500 = 2^2 \times 3^1 \times 5^3$ that is divisible by 5, then d can be divisible by 5 or by 5^2 or by 5^3 but cannot be divisible by 5^4 or by 5^5 or by any larger power of 5.

F4. A positive integer m greater than 1 is a perfect square exactly when each prime power in its prime factorization has an even exponent. For example, $n = 1500 = 2^2 \times 3^1 \times 5^3$ is not a perfect square but $m = 22500 = 2^2 \times 3^2 \times 5^4$ is a perfect square. This is true because if m is a perfect square, then $m = r^2$ for some positive integer r and so we can find the prime factorization of m by writing down the prime factorization of r twice. For example, if $r = 150 = 2^1 \times 3^1 \times 5^2$, then $m = 2^1 \times 3^1 \times 5^2 \times 2^1 \times 3^1 \times 5^2 = 2^2 \times 3^2 \times 5^4$. Further, if the prime factorization of m includes a prime power with an odd exponent, then the copies of this prime number cannot be equally distributed into two copies of the square root of m, which means that $\sqrt{m}$ is not an integer.

F5. One method to find the greatest common divisor (gcd) of two positive integers n and t is to write out the prime factorization of n and t and form a new integer d (the gcd) that is the product, for each common prime divisor, of the largest common prime power that divides both n and t. For example, if $n = 1500 = 2^2 \times 3^1 \times 5^3$ and $t = 7000 = 2^3 \times 5^3 \times 7^1$, then the greatest common divisor of n and t equals $2^2 \times 5^3 = 500$. The justification of this method is based on F2 (since d is a divisor of n and t it can only include prime factors common to both lists) and F3 (since d cannot include a prime power that is too large if it is to be a divisor of each of n and t).

We can now begin our solution.

Suppose that $(205\,800, 35k)$ is a happy pair.
We find the prime factorization of $205\,800$:

$$\begin{aligned} 205\,800 &= 2058 \times 100 \\ &= 2 \times 1029 \times (2 \times 5)^2 \\ &= 2 \times 3 \times 343 \times 2^2 \times 5^2 \\ &= 2 \times 3 \times 7^3 \times 2^2 \times 5^2 \\ &= 2^3 \times 3^1 \times 5^2 \times 7^3 \end{aligned}$$

Note also that $35k = 5^1 \times 7^1 \times k$.
Let d be the greatest common divisor of 205 800 and $35k$.
We want to find the number of possible values of $k \leq 2940$ for which d is a perfect square.
Since both 5 and 7 are prime divisors of 205 800 and $35k$, then 5 and 7 are both prime divisors of d (F5).
For d to be a perfect square, 5 and 7 must both divide d an even number of times (F4).
Since the prime powers of 5 and 7 in the prime factorization of 205 800 are 5^2 and 7^3, respectively, then for d to be a perfect square, it must be the case that 5^2 and 7^2 are factors of d.
Since $d = 5 \times 7 \times k$, then $k = 5 \times 7 \times j = 35j$ for some positive integer j.
Since $k \leq 2940$, then $35j \leq 2940$ which gives $j \leq 84$.
We now know that d is the gcd of $2^3 \times 3^1 \times 5^2 \times 7^3$ and $5^2 \times 7^2 \times j$.
What further information does this give us about j?

- j cannot be divisible by 3, otherwise d would have a factor of 3^1 (since both 205 800 and $35k$ would be divisible by 3) and cannot have a factor of 3^2 (since 205 800 does not) which would mean that d is not a perfect square.
- j cannot be divisible by 7, otherwise d has a factor of 7^3 and no larger power of 7, in which case d would not be a perfect square.
- If j is divisible by 2, then the prime factorization of j must include 2^2. In other words, the prime factorization of j cannot include 2^1 or 2^3.
- j can be divisible by 5 since even if j is divisible by 5, the power of 5 in d is already limited by the power of 5 in 205 800.
- j can be divisible by prime numbers other than 2, 3, 5 or 7 since 205 800 is not and so the gcd will not be affected.

Finally, we consider two cases: j is divisible by 2^2 but not by a larger power of 2, and j is not divisible by 2.

<u>Case 1: j is divisible by 2^2 but not by a larger power of 2</u>
Here, $j = 2^2h = 4h$ for some odd positive integer h.
Since $j \leq 84$, then $4h \leq 84$ which means that $h \leq 21$.
Knowing that j cannot be divisible by 3 or by 7, this means that the possible values of h are $1, 5, 11, 13, 17, 19$.
Each of these values of h produces a value of j that satisfies the conditions in the five bullets above.
There are thus 6 values of j in this case.

<u>Case 2: j is not divisible by 2</u>
Here, j is odd.
Knowing that j cannot be divisible by 3 or by 7 and that $j \leq 84$, this means that the possible values of j are:

$$1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, 53, 55, 59, 61, 65, 67, 71, 73, 79, 83$$

There are thus 24 values of j in this case.

In total, there are 30 values of j and so there are 30 possible values of $k \leq 2940$ for which $(205\,800, 35k)$ is a happy pair.

Answer: (D)

The CENTRE for EDUCATION
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2019 Pascal Contest

(Grade 9)

Tuesday, February 26, 2019
(in North America and South America)

Wednesday, February 27, 2019
(outside of North America and South America)

Solutions

1. Evaluating, $2 \times 3 + 2 \times 3 = 6 + 6 = 12$.

Answer: (D)

2. Since a square has four equal sides, the side length of a square equals one-quarter of the perimeter of the square.
Thus, the side length of a square with perimeter 28 is $28 \div 4 = 7$.

Answer: (E)

3. In the diagram, there are 9 hexagons of which 5 are shaded.
Therefore, the fraction of all of the hexagons that are shaded is $\frac{5}{9}$.

Answer: (B)

4. Since 38% of students received a muffin, then $100\% - 38\% = 62\%$ of students did not receive a muffin.
Alternatively, using the percentages of students who received yogurt, fruit or a granola bar, we see that $10\% + 27\% + 25\% = 62\%$ did not receive a muffin.

Answer: (D)

5. We know that $\frac{1}{2} = 0.5$. Since $\frac{4}{9} \approx 0.44$ is less than $\frac{1}{2} = 0.5$, then 4 cannot be placed in the box. (No integer smaller than 4 can be placed in the box either.) Since $\frac{5}{9} \approx 0.56$ is greater than $\frac{1}{2} = 0.5$, then the smallest integer that can be placed in the box is 5.

Answer: (D)

6. Since $4x + 14 = 8x - 48$, then $14 + 48 = 8x - 4x$ or $62 = 4x$.
Dividing both sides of this equation by 2, we obtain $\frac{4x}{2} = \frac{62}{2}$ which gives $2x = 31$.

Answer: (B)

7. The segment of the number line between 3 and 33 has length $33 - 3 = 30$.
Since this segment is divided into six equal parts, then each part has length $30 \div 6 = 5$.
The segment PS is made up of 3 of these equal parts, and so has length $3 \times 5 = 15$.
The segment TV is made up of 2 of these equal parts, and so has length $2 \times 5 = 10$.
Thus, the sum of the lengths of PS and TV is $15 + 10$ or 25.

Answer: (A)

8. Since $\frac{20}{19}$ is larger than 1 and smaller than 2, and $20 \times 19 = 380$, then $\frac{20}{19} < 20 \times 19 < 2019$.
We note that $19^{20} > 10^{20} > 10\,000$ and $20^{19} > 10^{19} > 10\,000$.
This means that both 19^{20} and 20^{19} are greater than 2019.
In other words, of the five numbers $19^{20}, \frac{20}{19}, 20^{19}, 2019, 20 \times 19$, the third largest is 2019.
Since the list contains 5 numbers, then its median is the third largest number, which is 2019.
(Note that it does not matter whether 19^{20} is greater than or less than 20^{19}.)

Answer: (D)

9. Since the complete angle at the centre of each circle is 360° and the unshaded sector of each circle has central angle 90°, then the unshaded sector of each circle represents $\frac{90^\circ}{360^\circ} = \frac{1}{4}$ of its area.
In other words, each of the circles is $\frac{3}{4}$ shaded.
There are 12 circles in the diagram.
Since the radius of each circle is 1, then the area of each circle is $\pi \times 1^2$ or π.
Therefore, the total shaded area is $\frac{3}{4} \times 12 \times \pi$ or 9π.

ANSWER: (D)

10. Suppose that sixty $1 \times 1 \times 1$ cubes are joined face to face in a single row on a table.
Each of the 60 cubes has its front, top and back faces exposed.
The leftmost and rightmost cubes also have their left and right faces, respectively, exposed.
No other faces are exposed.
Therefore, the number of 1×1 faces that are exposed is $60 \times 3 + 2$ which equals 182.

ANSWER: (C)

11. Using the second row, we see that the sum of the numbers in each row, column and diagonal must be $3.6 + 3 + 2.4 = 9$.
Since the sum of the numbers in the first column must be 9, then the bottom left number must be $9 - 2.3 - 3.6 = 9 - 5.9 = 3.1$
Since the sum of the numbers in the top left to bottom right diagonal must be 9, then the bottom right number must be $9 - 2.3 - 3 = 9 - 5.3 = 3.7$
Since the sum of the numbers in the bottom row must be 9, then $3.1 + x + 3.7 = 9$ and so $x + 6.8 = 9$ or $x = 9 - 6.8 = 2.2$.
We can complete the magic square as shown:

2.3	3.8	2.9
3.6	3	2.4
3.1	2.2	3.7

ANSWER: (E)

12. Since $\triangle PQX$ is right-angled at Q, then

$$\angle PXQ = 90^\circ - \angle QPX = 90^\circ - 62^\circ = 28^\circ$$

Since $\angle PXQ$ and $\angle SXR$ are opposite, then $\angle SXR = \angle PXQ = 28^\circ$.
Since $\triangle RXS$ is isosceles with $RX = SX$, then $\angle XRS = \angle XSR = y^\circ$.
Since the angles in any triangle have a sum of 180°,

$$\begin{aligned} \angle XRS + \angle XSR + \angle SXR &= 180^\circ \\ y^\circ + y^\circ + 28^\circ &= 180^\circ \\ 2y + 28 &= 180 \\ 2y &= 152 \end{aligned}$$

and so $y = 76$.

ANSWER: (C)

13. *Solution 1*
Since p, q, r, s is a list of consecutive integers in increasing order, then q is 1 more than p and r is 1 less than s.
This means that $q + r = (p + 1) + (s - 1) = p + s = 109$.

Solution 2
Since p, q, r, s is a list of consecutive integers in increasing order, then $q = p + 1$, $r = p + 2$, and $s = p + 3$.
Since $p + s = 109$, then $p + p + 3 = 109$ and so $2p = 106$ or $p = 53$.
This means that $q = 54$ and $r = 55$. Thus, $q + r = 109$.

Answer: (B)

14. Since the ratio of the number of skateboards to the number of bicycles was $7 : 4$, then the numbers of skateboards and bicycles can be written in the form $7k$ and $4k$ for some positive integer k.
Since the difference between the numbers of skateboards and bicycles is 12, then $7k - 4k = 12$ and so $3k = 12$ or $k = 4$.
Therefore, the total number of skateboards and bicycles is $7k + 4k = 11k = 11 \times 4 = 44$.

Answer: (A)

15. For Sophie's average over 5 tests to be 80%, the sum of her marks on the 5 tests must be $5 \times 80\% = 400\%$.
After the first 3 tests, the sum of her marks is $73\% + 82\% + 85\% = 240\%$.
Therefore, she will reach her goal as long as the sum of her marks on the two remaining tests is *at least* $400\% - 240\% = 160\%$.
The sums of the pairs of marks given are (A) 161%, (B) 161%, (C) 162%, (D) 156%, (E) 160%.
Thus, the pair with which Sophie would not meet her goal is (D).

Answer: (D)

16. *Solution 1*
Since the result must be the same for any real number x less than -2, we substitute $x = -4$ into each of the five expressions:

(A) $x = -4$ (B) $x + 2 = -2$ (C) $\frac{1}{2}x = -2$ (D) $x - 2 = -6$ (E) $2x = -8$

Therefore, $2x$ is the expression with the least value when $x = -4$ and thus must always be the expression with the least value.

Solution 2
For any real number x, we know that $x - 2$ is less than x which is less than $x + 2$.
Therefore, neither x nor $x + 2$ can be the least of the five values.
For any negative real number x, the value of $2x$ will be less than the value of $\frac{1}{2}x$.
Therefore, $\frac{1}{2}x$ cannot be the least of the five values.
Thus, the least of the five values is either $x - 2$ or $2x$.
When $x < -2$, we know that $2x - (x - 2) = x + 2 < 0$.
Since the difference between $2x$ and $x - 2$ is negative, then $2x$ has the smaller value and so is the least of all five values.

Answer: (E)

17. Each of the animals is either striped or spotted, but not both.
Since there are 100 animals and 62 are spotted, then there are $100 - 62 = 38$ striped animals.
Each striped animal must have wings or a horn, but not both.
Since there are 28 striped animals with wings, then there are $38 - 28 = 10$ striped animals with horns.
Each animal with a horn must be either striped or spotted.
Since there are 36 animals with horns, then there are $36 - 10 = 26$ spotted animals with horns.

Answer: (E)

18. By the Pythagorean Theorem,

$$QT^2 = QP^2 + PT^2 = k^2 + k^2 = 2k^2$$

Since $QT > 0$, then $QT = \sqrt{2}k$.
Since $\triangle QTS$ is isosceles, then $TS = QT = \sqrt{2}k$.
By the Pythagorean Theorem,

$$QS^2 = QT^2 + TS^2 = (\sqrt{2}k)^2 + (\sqrt{2}k)^2 = 2k^2 + 2k^2 = 4k^2$$

Since $QS > 0$, then $QS = 2k$.
Since $\triangle QSR$ is isosceles, then $SR = QS = 2k$.
Since $\triangle QPT$ is right-angled at P, its area is $\frac{1}{2}(QP)(PT) = \frac{1}{2}k^2$.
Since $\triangle QTS$ is right-angled at T, its area is $\frac{1}{2}(QT)(TS) = \frac{1}{2}(\sqrt{2}k)(\sqrt{2}k) = \frac{1}{2}(2k^2) = k^2$.
Since $\triangle QSR$ is right-angled at S, its area is $\frac{1}{2}(QS)(SR) = \frac{1}{2}(2k)(2k) = 2k^2$.
Since the sum of the three areas is 56, then $\frac{1}{2}k^2 + k^2 + 2k^2 = 56$ or $\frac{7}{2}k^2 = 56$ which gives $k^2 = 16$.
Since $k > 0$, then $k = 4$.

Answer: (C)

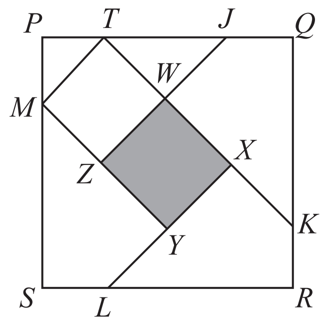
19. Since 4 balls are chosen from 6 red balls and 3 green balls, then the 4 balls could include:

- 4 red balls, or
- 3 red balls and 1 green ball, or
- 2 red balls and 2 green balls, or
- 1 red ball and 3 green balls.

There is only 1 different-looking way to arrange 4 red balls.
There are 4 different-looking ways to arrange 3 red balls and 1 green ball: the green ball can be in the 1st, 2nd, 3rd, or 4th position.
There are 6 different-looking ways to arrange 2 red balls and 2 green balls: the red balls can be in the 1st/2nd, 1st/3rd, 1st/4th, 2nd/3rd, 2nd/4th, or 3rd/4th positions.
There are 4 different-looking ways to arrange 1 red ball and 3 green balls: the red ball can be in the 1st, 2nd, 3rd, or 4th position.
In total, there are $1 + 4 + 6 + 4 = 15$ different-looking arrangements.

Answer: (A)

20. Since the sides of quadrilateral $WXYZ$ are parallel to the diagonals of square $PQRS$ and the diagonals of a square are perpendicular, then the sides of $WXYZ$ are themselves perpendicular. This means that quadrilateral $WXYZ$ has four right angles and is thus a rectangle.
Since the diagram does not change when rotated by $90°$, $180°$ or $270°$, then it must be the case that $WX = XY = YZ = YW$ which means that $WXYZ$ is a square.
We calculate the area of $WXYZ$ by first calculating the length of WZ.
Extend KW to meet PQ at T. Join M to T.



Since TK is parallel to diagonal PR, then $\angle QTK = \angle QKT = 45°$, which means that $\triangle TQK$ is isosceles with $QT = QK$.
Since $QR = 40$ and $KR = 10$, then $QK = QR - KR = 30$ and so $QT = 30$.
Since $PQ = 40$, then $PT = PQ - QT = 10$.
Since $PM = PT = 10$, then $\triangle MPT$ is right-angled and isosceles as well, which means that MT is actually parallel to diagonal SQ. (We did not construct MT with this property, but it turns out to be true.)
Since the sides of $MTWZ$ are parallel to the diagonals of the square, then $MTWZ$ is also a rectangle, which means that $WZ = MT$.
Since $PM = PT$, then

$$MT = \sqrt{PM^2 + PT^2} = \sqrt{10^2 + 10^2} = \sqrt{200}$$

by the Pythagorean Theorem.
Thus, $WZ = MT = \sqrt{200}$ and so the area of square $WXYZ$ equals WZ^2 or 200.

Answer: (B)

21. The units digit of 5^{2019} is 5.
This is because the units digit of any power of 5 is 5.
To see this, we note that the first few powers of 5 are

$$5^1 = 5 \quad 5^2 = 25 \quad 5^3 = 125 \quad 5^4 = 625 \quad 5^5 = 3125 \quad 5^6 = 15\,625$$

The units digit of a product of integers depends only on the units digits of the integers being multiplied. Since $5 \times 5 = 25$ which has a units digit of 5, the units digits remains 5 for every power of 5.

The units digit of 3^{2019} is 7.
This is because the units digits of powers of 3 cycle $3, 9, 7, 1, 3, 9, 7, 1, \ldots$.
To see this, we note that the first few powers of 3 are

$$3^1 = 3 \quad 3^2 = 9 \quad 3^3 = 27 \quad 3^4 = 81 \quad 3^5 = 243 \quad 3^6 = 729$$

Since the units digit of a product of integers depends only on the units digits of the integers being multiplied and we multiply by 3 to get from one power to the next, then once a units digit recurs in the sequence of units digits, the following units digits will follow the same pattern.
Since the units digits of powers of 3 cycle in groups of 4 and 2016 is a multiple of 4, then 3^{2016} has a units digit of 1.
Moving three additional positions along the sequence, the units digit of 3^{2019} will be 7.

Since the units digit of 5^{2019} is 5 and the units digit of 3^{2019} is 7, then the units digit of the difference will be 8. (This is the units digit of the difference whenever a smaller integer with units digit 7 is substracted from a larger integer with units digit 5.)

Answer: (E)

22. *Solution 1*
The smallest integer greater than 2019 that can be formed in this way is formed using the next two largest consecutive integers 20 and 21, giving the four-digit integer 2120.
The largest such integer is 9998.
The list of such integers is

$$2120, 2221, 2322, \ldots, 9796, 9897, 9998$$

Each pair of consecutive numbers in this list differs by 101 since the number of hundreds increases by 1 and the number of ones increases by 1 between each pair.
Since the numbers in the list are equally spaced, then their sum will equal the number of numbers in the list times the average number in the list.
The average number in the list is $\dfrac{2120 + 9998}{2} = \dfrac{12118}{2} = 6059$.
Since each number in the list is 101 greater than the previous number, then the number of increments of 101 from the first number to the last is $\dfrac{9998 - 2120}{101} = \dfrac{7878}{101} = 78$.
Since the number of increments is 78, then the number of numbers is 79.
This means that the sum of the numbers in the list is $79 \times 6059 = 478\,661$.

Solution 2
As in Solution 1, the list of such integers is $2120, 2221, 2322, \ldots, 9796, 9897, 9998$.
If the sum of these integers is S, then

$$\begin{aligned} S &= 2120 + 2221 + 2322 + \ldots + 9796 + 9897 + 9998 \\ &= (2100 + 2200 + 2300 + \cdots + 9700 + 9800 + 9900) \\ &\qquad + (20 + 21 + 22 + \cdots + 96 + 97 + 98) \\ &= 100(21 + 22 + 23 + \cdots + 96 + 97 + 98 + 99) \\ &\qquad + (20 + 21 + 22 + \cdots + 96 + 97 + 98) \\ &= 100(21 + 22 + 23 + \cdots + 96 + 97 + 98 + 99) \\ &\qquad + (21 + 22 + \cdots + 96 + 97 + 98 + 99) + 20 - 99 \\ &= 101(21 + 22 + 23 + \cdots + 96 + 97 + 98 + 99) - 79 \end{aligned}$$

There are 79 numbers in the list of consecutive integers from 21 to 99, inclusive, and the middle number in this list is 60.
Therefore, $S = 101 \times 79 \times 60 - 79 = 478\,661$.

Answer: (C)

23. Since the wheel turns at a constant speed, then the percentage of time when a shaded part of the wheel touches a shaded part of the path will equal the percentange of the total length of the path where there is "shaded on shaded" contact.
Since the wheel has radius 2 m, then its circumference is $2\pi \times 2$ m which equals 4π m.
Since the wheel is divided into four quarters, then the portion of the circumference taken by each quarter is π m.
We label the left-hand end of the path 0 m.
As the wheel rotates once, the first shaded section of the wheel touches the path between 0 m and π m ≈ 3.14 m.
As the wheel continues to rotate, the second shaded section of the wheel touches the path between 2π m ≈ 6.28 m and 3π m ≈ 9.42 m.
The path is shaded for 1 m starting at each odd multiple of 1 m, and unshaded for 1 m starting at each even multiple of 1 m.
Therefore, the first shaded section touches shaded stripes between 1 m and 2 m, and between 3 m and π m.
The second shaded section touches shaded stripes between 7 m and 8 m, and between 9 m and 3π m.
Therefore, the total length of "shaded on shaded" is $1\text{ m} + (\pi - 3)\text{ m} + 1\text{ m} + (3\pi - 9)\text{ m}$ or $(4\pi - 10)$ m.
The total length of the path along which the wheel rolls is 4π m.
This means that the required percentage of time equals $\dfrac{(4\pi - 10)\text{ m}}{4\pi\text{ m}} \times 100\% \approx 20.4\%$.
Of the given choices, this is closest to 20%, or choice (A).

ANSWER: (A)

24. First, we note that 88 663 311 000 is divisible by 792. (We can check this by division.)
Therefore, 88 663 311 000 000 is also divisible by 792.
Since 88 663 311 000 000 is divisible by 792, then 88 663 311 $pqr\,s48$ is divisible by 792 exactly when $pqr\,s48$ is divisible by 792. (This comes from the fact that if the difference between two integers is divisible by d, then either both are divisible by d or neither is divisible by d.)
The smallest integer of the form $pqr\,s48$ is 48 (which is "000 048") and the largest integer of the form $pqr\,s48$ is 999 948.
Since $999\,948 \div 792 \approx 1262.6$, then the multiples of 792 in between 48 and 999 948 are the integers of the form $792 \times n$ where $1 \leq n \leq 1262$.
Suppose that $792 \times n = pqr\,s48$ for some integer n.
Comparing units digits, we see that the units digit of n must be 4 or 9.
This means that $n = 10c + 4$ or $n = 10c + 9$ for some integer $c \geq 0$.

In the first case, $792(10c + 4) = 7920c + 3168$.
This integer has a units digit of 8.
For this integer to have a tens digit of 4, we need $2c+6$ to have a units digit of 4, which happens exactly when c has units digit 4 or 9.
This means that c can be $4, 9, 14, 19, 24, \ldots$.
This means that n can be $44, 94, 144, 194, 244, \ldots$.
Since $1 \leq n \leq 1262$, then there are 25 possible values of n with units digit 4, because there are 2 values of n between 0 and 100, 2 between 100 and 200, and so on up to 1200, with an additional 1 (namely, 1244) between 1200 and 1262.

In the second case, $792(10c + 9) = 7920c + 7128$.
This integer has a units digit of 8.

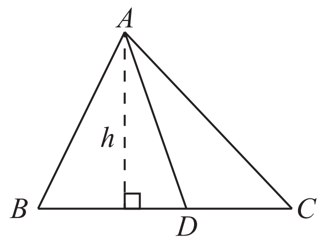
For this integer to have a tens digit of 4, we need $2c + 2$ to have units digit 4, which happens exactly when c has units digit 1 or 6.
This means that c can be $1, 6, 11, 16, 21, \ldots$.
This means that n can be $19, 69, 119, 169, 219, \ldots$.
Since $1 \leq n \leq 1262$, then there are again 25 possible values of n with units digit 9.

In total, there are $25 + 25 = 50$ 14-digit positive integers of the desired form that are divisible by 792.

Answer: (E)

25. In $\triangle ABC$, if D is on BC, then

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} = \frac{BD}{CD}$$



This is because $\triangle ABD$ and $\triangle ACD$ have a common height of h and so

$$\frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} == \frac{\frac{1}{2} \times BD \times h}{\frac{1}{2} \times CD \times h} = \frac{BD}{CD}$$

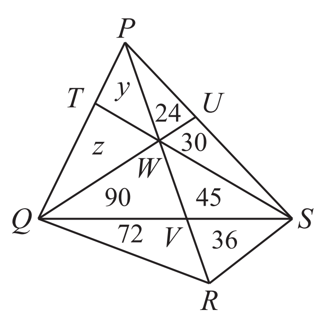
Using this in the given diagram with point V lying on side QS of $\triangle WQS$ and on side QS of $\triangle RQS$, we see that

$$\frac{\text{Area of } \triangle QVW}{\text{Area of } \triangle SVW} = \frac{QV}{SV} = \frac{\text{Area of } \triangle QVR}{\text{Area of } \triangle SVR}$$

Combining the first and third parts of this equality, we obtain the equivalent equations

$$\begin{aligned}
\frac{8x+50}{5x+20} &= \frac{8x+32}{5x+11} \\
(8x+50)(5x+11) &= (8x+32)(5x+20) \\
40x^2 + 88x + 250x + 550 &= 40x^2 + 160x + 160x + 640 \\
338x + 550 &= 320x + 640 \\
18x &= 90 \\
x &= 5
\end{aligned}$$

Since $x = 5$, then we can calculate and fill in the area of most of the pieces of the diagram:



Let the area of $\triangle PTW$ equal y and the area of $\triangle QTW$ equal z.
We know that
$$\frac{\text{Area of }\triangle QVW}{\text{Area of }\triangle SVW} = \frac{QV}{SV}$$
and so $\dfrac{QV}{SV} = \dfrac{90}{45} = 2$.
Since V lies on side QS of $\triangle PQS$, then
$$\frac{\text{Area of }\triangle PQV}{\text{Area of }\triangle PSV} = \frac{QV}{SV} = 2$$
and so $\dfrac{y+z+90}{99} = 2$ which gives $y + z + 90 = 198$ and so $y + z = 108$.
Finally, looking at points W on side TS of $\triangle PTS$ and so side TS of $\triangle QTS$, we get
$$\frac{y}{54} = \frac{TW}{WS} = \frac{z}{135}$$
and so $135y = 54z$ or $5y = 2z$ or $z = \frac{5}{2}y$.
Therefore, $y + \frac{5}{2}y = 108$ or $\frac{7}{2}y = 108$ and so $y = \frac{216}{7} = 30\frac{6}{7}$.
Of the given choices, the area of $\triangle PTW$ is closest to 31.

Answer: (E)

The CENTRE for EDUCATION
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2018 Pascal Contest

(Grade 9)

Tuesday, February 27, 2018
(in North America and South America)

Wednesday, February 28, 2018
(outside of North America and South America)

Solutions

1. When we arrange the five choices from smallest to largest, we obtain $1.2, 1.4, 1.5, 2.0, 2.1$. Thus, 1.2 is the smallest.

ANSWER: (B)

2. Evaluating, $\dfrac{2018 - 18 + 20}{2} = \dfrac{2000 + 20}{2} = \dfrac{2020}{2} = 1010$.

ANSWER: (A)

3. July 14 is 11 days after July 3 of the same year.
Since there are 7 days in a week, then July 10 and July 3 occur on the same day of the week, namely Wednesday.
July 14 is 4 days after July 10, and so is a Sunday.

ANSWER: (C)

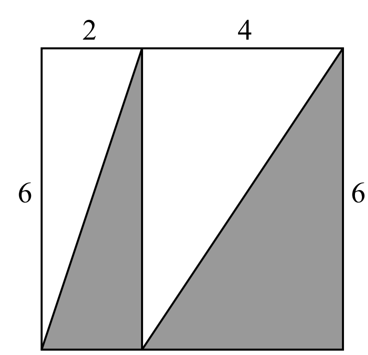
4. Since the car is charged 3 times per week for 52 weeks, it is charged $3 \times 52 = 156$ times.
Since the cost per charge is \$0.78, then the total cost is $156 \times \$0.78 = \$121.68$.

ANSWER: (E)

5. Since
$$3 \times 3 \times 5 \times 5 \times 7 \times 9 = 3 \times 3 \times 7 \times n \times n$$
then
$$n \times n = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 9}{3 \times 3 \times 7} = 5 \times 5 \times 9 = 5 \times 5 \times 3 \times 3$$
Since $n \times n = 5 \times 5 \times 3 \times 3$, then a possible value for n is $n = 5 \times 3 = 15$.

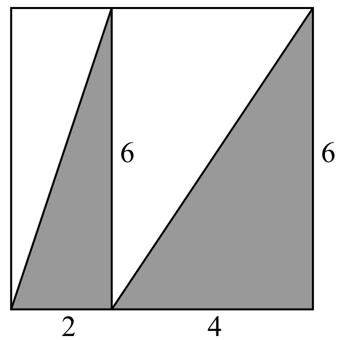
ANSWER: (A)

6. *Solution 1*
Consider the 6×6 square as being made up of a 2×6 rectangle on the left and a 4×6 rectangle on the right.



Each of these rectangles is divided in half by its diagonal, and so is half shaded.
Therefore, 50% of the total area is shaded.

Solution 2
The entire 6×6 square has area $6^2 = 36$.
Both shaded triangles have height 6 (the height of the square).
The bases of the triangles have lengths 2 and 4:



The left-hand triangle has area $\frac{1}{2} \times 2 \times 6 = 6$.
The right-hand triangle has area $\frac{1}{2} \times 4 \times 6 = 12$.
Therefore, the total shaded area is $6 + 12 = 18$, which is one-half (or 50%) of the area of the entire square.

Answer: (A)

7. There are $5 + 7 + 8 = 20$ ties in the box, 8 of which are pink.
When Stephen removes a tie at random, the probability of choosing a pink tie is $\frac{8}{20}$ which is equivalent to $\frac{2}{5}$.

Answer: (C)

8. The section of the number line between 0 and 5 has length $5 - 0 = 5$.
Since this section is divided into 20 equal parts, the width of each part is $\frac{5}{20} = \frac{1}{4} = 0.25$.
Since S is 5 of these equal parts to the right of 0, then $S = 0 + 5 \times 0.25 = 1.25$.
Since T is 5 of these equal parts to the left of 5, then $T = 5 - 5 \times 0.25 = 5 - 1.25 = 3.75$.
Therefore, $S + T = 1.25 + 3.75 = 5$.

Answer: (E)

9. If $\heartsuit = 1$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 1 \times 1 \times 1 = 1$, which is not possible since ∇ and $\heartsuit$ must be different positive integers.
If $\heartsuit = 2$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 2 \times 2 \times 2 = 8$, which is possible.
If $\heartsuit = 3$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 3 \times 3 \times 3 = 27$, which is not possible since ∇ is less than 20.
If $\heartsuit$ is greater than 3, then ∇ will be greater than 27 and so $\heartsuit$ cannot be greater than 3.
Thus, $\heartsuit = 2$ and so $\nabla = 8$.
This means that $\nabla \times \nabla = 8 \times 8 = 64$.

Answer: (D)

10. The line that passes through $(-2, 1)$ and $(2, 5)$ has slope $\dfrac{5-1}{2-(-2)} = \dfrac{4}{4} = 1$.
This means that, for every 1 unit to the right, the line moves 1 unit up.
Thus, moving 2 units to the right from the starting point $(-2, 1)$ to $x = 0$ will give a rise of 2.
Therefore, the line passes through $(-2 + 2, 1 + 2) = (0, 3)$.

Answer: (C)

11. The complete central angle of a circle measures 360°.
This means that the angle in the circle graph associated with Playing is $360^\circ - 130^\circ - 110^\circ$ or 120°.
A central angle of 120° represents $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the total central angle.
This means that the baby polar bear plays for $\frac{1}{3}$ of the day, which is $\frac{1}{3} \times 24 = 8$ hours.
Answer: (C)

12. Of the given uniform numbers,

- 11 and 13 are prime numbers
- 16 is a perfect square
- 12, 14 and 16 are even

Since Karl's and Liu's numbers were prime numbers, then their numbers were 11 and 13 in some order.
Since Glenda's number was a perfect square, then her number was 16.
Since Helga's and Julia's numbers were even, then their numbers were 12 and 14 in some order. (The number 16 is already taken.)
Thus, Ioana's number is the remaining number, which is 15.
Answer: (D)

13. Since the given equilateral triangle has side length 10, its perimeter is $3 \times 10 = 30$.
In terms of x, the perimeter of the given rectangle is $x + 2x + x + 2x = 6x$.
Since the two perimeters are equal, then $6x = 30$ which means that $x = 5$.
Since the rectangle is x by $2x$, its area is $x(2x) = 2x^2$.
Since $x = 5$, its area is $2(5^2) = 50$.
Answer: (B)

14. The average of the numbers $7, 9, 10, 11$ is $\frac{7+9+10+11}{4} = \frac{37}{4} = 9.25$, which is not equal to 18, which is the fifth number.
The average of the numbers $7, 9, 10, 18$ is $\frac{7+9+10+18}{4} = \frac{44}{4} = 11$, which is equal to 11, the remaining fifth number.
We can check that the averages of the remaining three combinations of four numbers is not equal to the fifth number.
Therefore, the answer is 11.
(We note that in fact the average of the original five numbers is $\frac{7+9+10+11+18}{5} = \frac{55}{5} = 11$, and when we remove a number that is the average of a set, the average does not change. Can you see why?)
Answer: (D)

15. We would like to find the first time after 4:56 where the digits are consecutive digits in increasing order.
It would make sense to try 5:67, but this is not a valid time.
Similarly, the time cannot start with 6, 7, 8 or 9.
No time starting with 10 or 11 starts with consecutive increasing digits.
Starting with 12, we obtain the time 12:34. This is the first such time.

We need to determine the length of time between 4:56 and 12:34.
From 4:56 to 11:56 is 7 hours, or $7 \times 60 = 420$ minutes.
From 11:56 to 12:00 is 4 minutes.
From 12:00 to 12:34 is 34 minutes.
Therefore, from 4:56 to 12:34 is $420 + 4 + 34 = 458$ minutes.

Answer: (A)

16. First, we note that we cannot have $n \leq 6$, since the first 6 letters are X's.
After 6 X's and 3 Y's, there are twice as many X's as Y's. In this case, $n = 6 + 3 = 9$.
After 6 X's and 12 Y's, there are twice as many Y's as X's. In this case, $n = 6 + 12 = 18$.
The next letters are all Y's (with 24 Y's in total), so there are no additional values of n with $n \leq 6 + 24 = 30$.
At this point, there are 6 X's and 24 Y's.
After 24 Y's and 12 X's (that is, 6 additional X's), there are twice as many Y's as X's. In this case, $n = 24 + 12 = 36$.
After 24 Y's and 48 X's (that is, 42 additional X's), there are twice as many X's as Y's. In this case, $n = 24 + 48 = 72$.
Since we are told that there are four values of n, then we have found them all, and their sum is $9 + 18 + 36 + 72 = 135$.

Answer: (C)

17. We note that $n = p^2q^2 = (pq)^2$.
Since $n < 1000$, then $(pq)^2 < 1000$ and so $pq < \sqrt{1000} \approx 31.6$.
Finding the number of possible values of n is thus equivalent to finding the number of positive integers m with $1 \leq m \leq 31 < \sqrt{1000}$ that are the product of two prime numbers.
The prime numbers that are at most 31 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31.
The distinct products of pairs of these that are at most 31 are:

$$2 \times 3 = 6 \quad 2 \times 5 = 10 \quad 2 \times 7 = 14 \quad 2 \times 11 = 22 \quad 2 \times 13 = 26$$

$$3 \times 5 = 15 \quad 3 \times 7 = 21$$

Any other product either duplicates one that we have counted already, or is larger than 31.
Therefore, there are 7 such values of n.

Answer: (E)

18. Consider $\triangle PQR$.
Since the sum of the angles in a triangle is 180°, then

$$\angle QPR + \angle QRP = 180^\circ - \angle PQR = 180^\circ - 120^\circ = 60^\circ$$

Since $\angle QPS = \angle RPS$, then $\angle RPS = \frac{1}{2}\angle QPR$.
Since $\angle QRS = \angle PRS$, then $\angle PRS = \frac{1}{2}\angle QRP$.
Therefore,

$$\begin{aligned} \angle RPS + \angle PRS &= \tfrac{1}{2}\angle QPR + \tfrac{1}{2}\angle QRP \\ &= \tfrac{1}{2}(\angle QPR + \angle QRP) \\ &= \tfrac{1}{2} \times 60^\circ \\ &= 30^\circ \end{aligned}$$

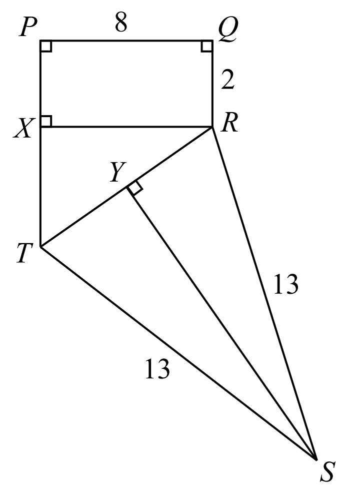
Finally, $\angle PSR = 180^\circ - (\angle RPS + \angle PRS) = 180^\circ - 30^\circ = 150^\circ$.

Answer: (E)

19. We recall that time $= \dfrac{\text{distance}}{\text{speed}}$. Travelling x km at 90 km/h takes $\dfrac{x}{90}$ hours.
Travelling x km at 120 km/h takes $\dfrac{x}{120}$ hours.
We are told that the difference between these lengths of time is 16 minutes.
Since there are 60 minutes in an hour, then 16 minutes is equivalent to $\dfrac{16}{60}$ hours.
Since the time at 120 km/h is 16 minutes less than the time at 90 km/h, then $\dfrac{x}{90} - \dfrac{x}{120} = \dfrac{16}{60}$.
Combining the fractions on the left side using a common denominator of $360 = 4 \times 90 = 3 \times 120$, we obtain $\dfrac{x}{90} - \dfrac{x}{120} = \dfrac{4x}{360} - \dfrac{3x}{360} = \dfrac{x}{360}$.
Thus, $\dfrac{x}{360} = \dfrac{16}{60}$.
Since $360 = 6 \times 60$, then $\dfrac{16}{60} = \dfrac{16 \times 6}{360} = \dfrac{96}{360}$. Thus, $\dfrac{x}{360} = \dfrac{96}{360}$ which means that $x = 96$.

ANSWER: (D)

20. We drop a perpendicular from R to X on PT, join R to T, and drop a perpendicular from S to Y on RT.



Since quadrilateral $PQRX$ has three right angles (at P, Q and X), then it must have four right angles and so is a rectangle. Its area is $8 \times 2 = 16$.
Next, $\triangle RXT$ is right-angled at X.
Since $PQRX$ is a rectangle, then $XR = PQ = 8$ and $PX = QR = 2$.
Since $PX = 2$, then $XT = PT - PX = 8 - 2 = 6$.
Thus, the area of $\triangle RXT$ is $\frac{1}{2} \times XT \times XR = \frac{1}{2} \times 6 \times 8 = 24$.
By the Pythagorean Theorem, $TR = \sqrt{XT^2 + XR^2} = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$, since $TR > 0$.
Since $\triangle TSR$ is isosceles with $ST = SR$ and SY is perpendicular to TR, then Y is the midpoint of TR.
Since $TY = YR = \frac{1}{2}TR$, then $TY = YR = 5$.
By the Pythagorean Theorem, $SY = \sqrt{ST^2 - TY^2} = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$, since $SY > 0$.
Therefore, the area of $\triangle STR$ is $\frac{1}{2} \times TR \times SY = \frac{1}{2} \times 10 \times 12 = 60$.
Finally, the area of pentagon $PQRST$ is the sum of the areas of the pieces, or $60 + 24 + 16 = 100$.

ANSWER: (D)

21. We determine the number of ways to get to each unshaded square in the grid obeying the given rules.
In the first row, there is 1 way to get to each unshaded square: by starting at that square.
In each row below the first, the number of ways to get to an unshaded square equals the sum of the number of ways to get to each of the unshaded squares diagonally up and to the left and up and to the right from the given square. This is because any path passing through any unshaded square needs to come from exactly one of these unshaded squares in the row above.
In the second row, there are 2 ways to get to each unshaded square: 1 way from each of two squares in the row above.
In the third row, there are 2 ways to get to each of the outside unshaded squares and 4 ways to get to the middle unshaded square.
Continuing in this way, we obtain the following number of ways to get to each unshaded square:

1		1		1
	2		2	
2		4		2
	6		6	
6		12		6

Since there are 6, 12 and 6 ways to get to the unshaded squares in the bottom row, then there are $6 + 12 + 6 = 24$ paths through the grid that obey the given rules.

Answer: (D)

22. Each wire has 2 ends.
Thus, 13 788 wires have $13\,788 \times 2 = 27\,576$ ends.
In a Miniou circuit, there are 3 wires connected to each node.
This means that 3 wire ends arrive at each node, and so there are $27\,576 \div 3 = 9192$ nodes.

Answer: (B)

23. The circle with centre P has radius 1 and passes through Q.
This means that $PQ = 1$.
Therefore, the circle with diameter PQ has radius $\frac{1}{2}$ and so has area $\pi\left(\frac{1}{2}\right)^2 = \frac{1}{4}\pi$.
To find the area of the shaded region, we calculate the area of the region common to both circles and subtract the area of the circle with diameter PQ.
Suppose that the two circles intersect at X and Y.
Join X to Y, P to Q, P to X, P to Y, Q to X, and Q to Y (Figure 1).
By symmetry, the area of the shaded region on each side of XY will be the same.
The area of the shaded region on the right side of XY equals the area of sector $PXQY$ of the left circle minus the area of $\triangle PXY$ (Figure 2).

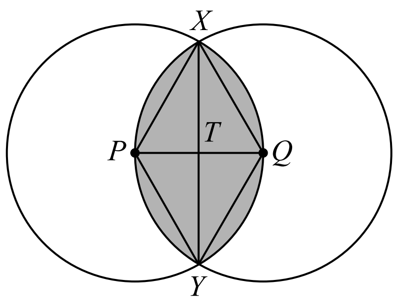


Figure 1

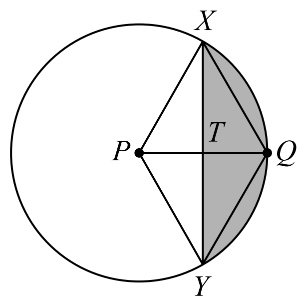


Figure 2

Since each of the large circles has radius 1, then $PQ = PX = PY = QX = QY = 1$.
This means that each of $\triangle XPQ$ and $\triangle YPQ$ is equilateral, and so $\angle XPQ = \angle YPQ = 60^\circ$.
Therefore, $\angle XPY = 120^\circ$, which means that sector $PXQY$ is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the full circle, and so has area $\frac{1}{3}\pi 1^2 = \frac{1}{3}\pi$.
Lastly, consider $\triangle PXY$.
Note that $PX = PY = 1$ and that $\angle XPQ = \angle YPQ = 60^\circ$.
Since $\triangle PXY$ is isosceles and PQ bisects $\angle XPY$, then PQ is perpendicular to XY at T and $XT = TY$.
By symmetry, $PT = TQ$. Since $PQ = 1$, then $PT = \frac{1}{2}$.
By the Pythagorean Theorem in $\triangle PTX$ (which is right-angled at T),

$$XT = \sqrt{PX^2 - PT^2} = \sqrt{1^2 - \left(\tfrac{1}{2}\right)^2} = \sqrt{1 - \tfrac{1}{4}} = \sqrt{\tfrac{3}{4}} = \tfrac{\sqrt{3}}{2}$$

since $XT > 0$.
Therefore, $XY = 2XT = \sqrt{3}$.
The area of $\triangle PXY$ equals $\frac{1}{2}(XY)(PT) = \frac{1}{2}(\sqrt{3})\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{4}$.
Now, we can calculate the area of the shaded region to the right of XY to be $\frac{1}{3}\pi - \frac{\sqrt{3}}{4}$, the difference between the area of sector $PXQY$ and the area of $\triangle PXT$.
Therefore, the area of the shaded region with the circle with diameter PQ removed is

$$2\left(\tfrac{1}{3}\pi - \tfrac{\sqrt{3}}{4}\right) - \tfrac{1}{4}\pi = \tfrac{2}{3}\pi - \tfrac{\sqrt{3}}{2} - \tfrac{1}{4}\pi = \tfrac{5}{12}\pi - \tfrac{\sqrt{3}}{2} \approx 0.443$$

Of the given choices, this is closest to 0.44.

Answer: (E)

24. We refer to the students with height 1.60 m as "taller" students and to those with height 1.22 m as "shorter" students.
For the average of four consecutive heights to be greater than 1.50 m, the sum of these four heights must be greater than 4×1.50 m $= 6.00$ m.
If there are 2 taller and 2 shorter students, then the sum of their heights is 2×1.60 m$+2 \times 1.22$ m or 5.64 m, which is not large enough.
Therefore, there must be more taller and fewer shorter students in a given group of 4 consecutive students.
If there are 3 taller students and 1 shorter student, then the sum of their heights is equal to 3×1.60 m $+ 1 \times 1.22$ m or 6.02 m, which is large enough.
Thus, in Mrs. Warner's line-up, any group of 4 consecutive students must include at least 3 taller students and at most 1 shorter student. (4 taller and 0 shorter students also give an average height that is greater than 1.50 m.)
For the average of seven consecutive heights to be less than 1.50 m, the sum of these seven heights must be less than 7×1.50 m $= 10.50$ m.
Note that 6 taller students and 1 shorter student have total height 6×1.60 m $+ 1 \times 1.22$ m or 10.82 m and that 5 taller students and 2 shorter students have total height equal to 5×1.60 m $+ 2 \times 1.22$ m or 10.44 m.
Thus, in Mrs. Warner's line-up, any group of 7 consecutive students must include at most 5 taller students and at least 2 shorter students.
Now, we determine the maximum possible length for a line-up. We use "T" to represent a taller student and "S" to represent a shorter student.
After some fiddling, we discover the line-up TTSTTTSTT.
The line-up TTSTTTSTT has length 9 and has the property that each group of 4 consecutive students includes exactly 3 T's and each group of 7 consecutive students includes exactly 5 Ts and so has the desired average height properties.
We claim that this is the longest such line-up, which makes the final answer 9 or (D).
Suppose that there was a line-up of length at least 10, and the first 10 heights in the line-up were $abcdefghjk$. Consider the following table of heights:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & h \\ c & d & e & f & g & h & j \\ d & e & f & g & h & j & k \end{array}$$

We explain why this table demonstrates that it is not possible to have at least 10 in the line-up.
Each row in this table is a list of 7 consecutive people from the line-up $abcdefghjk$ and so its sum is less than 10.50 m.
Each column in this table is a list of 4 consecutive people from the line-up $abcdefghjk$ and so its sum is greater than 6.00 m.
The sum of the numbers in this table equals the sum of the sums of the 4 rows, which must be less than 4×10.50 m or 42.00 m.
The sum of the numbers in this table equals the sum of the sums of the 7 columns, which must be greater than 7×6.00 m or 42.00 m.
The sum cannot be both less than and greater than 42.00 m.
This means that our assumption is incorrect, and so it must be impossible to have a line-up of length 10 or greater.
(There are a number of other ways to convince yourself that there cannot be more than 9 students in the line-up.)

Answer: (D)

25. Suppose that $m = 500$ and $1 \leq n \leq 499$ and $1 \leq r \leq 15$ and $2 \leq s \leq 9$ and $t = 0$.
Since $s > 0$, then the algorithm says that t is the remainder when r is divided by s.
Since $t = 0$, then r is a multiple of s. Thus, $r = as$ for some positive integer a.
Since $r > 0$, then the algorithm says that s is the remainder when n is divided by r.
In other words, $n = br + s$ for some positive integer b.
But $r = as$, so $n = bas + s = (ba + 1)s$.
In other words, n is a multiple of s, say $n = cs$ for some positive integer c.
Since $n > 0$, then r is the remainder when m is divided by n.
In other words, $m = dn + r$ for some positive integer d.
But $r = as$ and $n = cs$ so $m = dcs + as = (dc + a)s$.
In other words, m is a multiple of s, say $m = es$ for some positive integer e.
But $m = 500$ and $2 \leq s \leq 9$.
Since m is a multiple of s, then s is a divisor of 500 and so the possible values of s are $s = 2, 4, 5$.
(None of $1, 3, 6, 7, 8, 9$ is a divisor of 500.)
We know that r is a multiple of s, that $r > s$ (because s is the remainder when n is divided by r), and that $1 \leq r \leq 15$.
If $s = 5$, then $r = 10$ or $r = 15$.
If $s = 4$, then $r = 8$ or $r = 12$.
If $s = 2$, then $r = 4, 6, 8, 10, 12, 14$.

Suppose that $s = 5$ and $r = 10$.
Since $m = dn + r$, then $500 = dn + 10$ and so $dn = 490$.
Therefore, n is a divisor of 490, is a multiple of 5 (because $n = cs$), must be greater than $r = 10$, and must be 5 more than a multiple of 10 (because the remainder when n is divided by r is s).
Since $490 = 5 \times 2 \times 7^2$, then the divisors of 490 that are multiples of 5 are $5, 10, 35, 70, 245, 490$ (these are 5 times the divisors of 2×7^2). Among these, those greater than $r = 10$ having remainder 5 when divided by 10 are 35 and 245, and so the possible values of n in this case are 35 and 245.

For each possible pair s and r, we determine the values of n that satisfy the following conditions:

- n is a divsior of $500 - r$,
- n is a multiple of s,
- n is greater than r, and
- the remainder when n is divided by r is s.

We make a table:

s	r	$500 - r$	Divisors of $500 - r$ that are multiples of s	Possible n
5	10	$490 = 5 \times 2 \times 7^2$	$5, 10, 35, 70, 245, 490$	$35, 245$
5	15	$485 = 5 \times 97$	$5, 485$	485
4	8	$492 = 4 \times 3 \times 41$	$4, 12, 164, 492$	$12, 164, 492$
4	12	$488 = 4 \times 2 \times 61$	$4, 8, 244, 488$	244
2	4	$496 = 2 \times 2^3 \times 31$	$2, 4, 8, 16, 62, 124, 248, 496$	62
2	6	$494 = 2 \times 13 \times 19$	$2, 26, 38, 494$	$26, 38, 494$
2	8	$492 = 2 \times 2 \times 3 \times 41$	$2, 4, 6, 12, 82, 164, 246, 492$	82
2	10	$490 = 2 \times 5 \times 7^2$	$2, 10, 14, 70, 98, 490$	None
2	12	$488 = 2 \times 2^2 \times 61$	$2, 4, 8, 122, 244, 488$	122
2	14	$486 = 2 \times 3^5$	$2, 6, 18, 54, 162, 486$	None

In each row, we calculate the prime factorization in $500 - r$ in the third column, list the divisors of $500 - r$ that are multiples of s in the fourth column, and determine which of these are greater than r and give a remainder of s when divided by r in the fifth column.

Therefore, the possible values of n are $35, 245, 485, 12, 164, 492, 244, 62, 26, 38, 494, 82, 122,$ of which there are 13.

Answer: (E)

The CENTRE for EDUCATION
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2017 Pascal Contest

(Grade 9)

Tuesday, February 28, 2017
(in North America and South America)

Wednesday, March 1, 2017
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{4 \times 3}{2+1} = \dfrac{12}{3} = 4$.

   Answer: (A)

2. In each of the 6 rows, there is 1 unshaded square, and so there are $6 - 1 = 5$ shaded squares per row.
   Since there are 6 rows, then there are $6 \times 5 = 30$ shaded squares.
   Alternatively, we note that there are $6 \times 6 = 36$ squares in the grid and 6 unshaded squares, which means there are $36 - 6 = 30$ shaded squares in the grid.

   Answer: (B)

3. In the diagram, there are 5 shaded and 3 unshaded triangles, and so the ratio of the number of shaded triangles to the number of unshaded triangles is $5 : 3$.

   Answer: (B)

4. We note that $7 = \sqrt{49}$ and that $\sqrt{40} < \sqrt{49} < \sqrt{50} < \sqrt{60} < \sqrt{70} < \sqrt{80}$.
   This means that $\sqrt{40}$ or $\sqrt{50}$ is the closest to 7 of the given choices.
   Since $\sqrt{40} \approx 6.32$ and $\sqrt{50} \approx 7.07$, then $\sqrt{50}$ is closest to 7.

   Answer: (C)

5. We need to determine the time that is 30 hours after 2 p.m. on Friday.
   The time that is 24 hours after 2 p.m. on Friday is 2 p.m. on Saturday.
   The time that is 30 hours after 2 p.m. on Friday is an additional 6 hours later.
   This time is 8 p.m. on Saturday.

   Answer: (E)

6. The time period in which the number of people at the zoo had the largest increase is the time period over which the height of the bars in the graph increases by the largest amount.
   Looking at the graph, this is between 11:00 a.m. and 12:00 p.m.
   (We note that the first three bars represent numbers between 200 and 400, and the last three bars represent numbers between 600 and 800, so the time period between 11:00 a.m. and 12:00 p.m. is the only one in which the increase was larger than 200.)

   Answer: (C)

7. Since $2x - 3 = 10$, then $2x = 13$ and so $4x = 2(2x) = 2(13) = 26$.
   (We did not have to determine the value of x.)

   Answer: (D)

8. The three integers from the list whose product is 80 are 1, 4 and 20, since $1 \times 4 \times 20 = 80$.
   The sum of these integers is $1 + 4 + 20 = 25$.
   (Since 80 is a multiple of 5 and 20 is the only integer in the list that is a multiple of 5, then 20 must be included in the product. This leaves two integers to choose, and their product must be $\dfrac{80}{20} = 4$. From the given list, these integers must be 1 and 4.)

   Answer: (C)

9. Since Jovin, Anna and Olivia take $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{4}$ of the pizza, respectively, then the fraction of the pizza with which Wally is left is

   $$1 - \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{4} = \tfrac{12}{12} - \tfrac{4}{12} - \tfrac{2}{12} - \tfrac{3}{12} = \tfrac{3}{12} = \tfrac{1}{4}\,.$$

   Answer: (B)

10. When $n = 1$, the values of the five expressions are $2014, 2018, 2017, 2018, 2019$.
When $n = 2$, the values of the five expressions are $2011, 2019, 4034, 2021, 2021$.
Only the fifth expression $(2017 + 2n)$ is odd for both of these choices of n, so this must be the correct answer.
We note further that since 2017 is an odd integer and $2n$ is always an even integer, then $2017 + 2n$ is always an odd integer, as required.

Answer: (E)

11. When Ursula runs 30 km at 10 km/h, it takes her $\dfrac{30\text{ km}}{10\text{ km/h}} = 3$ h.
This means that Jeff completes the same distance in $3\text{ h} - 1\text{ h} = 2$ h.
Therefore, Jeff's constant speed is $\dfrac{30\text{ km}}{2\text{ h}} = 15$ km/h.

Answer: (D)

12. Since the area of the larger square equals the sum of the areas of the shaded and unshaded regions inside, then the area of the larger square equals $2 \times 18\text{ cm}^2 = 36\text{ cm}^2$.
Since the larger square has an area of 36 cm^2, then its side length is $\sqrt{36\text{ cm}^2} = 6$ cm.

Answer: (C)

13. *Solution 1*
We undo Janet's steps to find the initial number.
To do this, we start with 28, add 4 (to get 32), then divide the sum by 2 (to get 16), then subtract 7 (to get 9).
Thus, Janet's initial number was 9.

Solution 2
Let Janet's initial number be x.
When she added 7 to her initial number, she obtained $x + 7$.
When she multiplied this sum by 2, she obtained $2(x + 7)$ which equals $2x + 14$.
When she subtracted 4 from this result, she obtained $(2x + 14) - 4$ which equals $2x + 10$.
Since her final result was 28, then $2x + 10 = 28$ or $2x = 18$ and so $x = 9$.

Answer: (A)

14. Since the tax rate is 10%, then the tax on each \$2.00 app is $\$2.00 \times \dfrac{10}{100} = \$0.20$.
Therefore, including tax, each app costs $\$2.00 + \$0.20 = \$2.20$.
Since Tobias spends \$52.80 on apps, he downloads $\dfrac{\$52.80}{\$2.20} = 24$ apps.
Therefore, $m = 24$.

Answer: (D)

15. Let s be the side length of the square with area k.
The sum of the heights of the squares on the right side is $3 + 8 = 11$.
The sum of the heights of the squares on the left side is $1 + s + 4 = s + 5$.
Since the two sums are equal, then $s + 5 = 11$, and so $s = 6$.
Therefore, the square with area k has side length 6, and so its area is $6^2 = 36$.
In other words, $k = 36$.

Answer: (C)

16. The six angles around the centre of the spinner add to 360°.
Thus, $140^\circ + 20^\circ + 4x^\circ = 360^\circ$ or $4x = 360 - 140 - 20 = 200$, and so $x = 50$.
Therefore, the sum of the central angles of the shaded regions is $140^\circ + 50^\circ + 50^\circ = 240^\circ$.
The probability that the spinner lands on a shaded region is the fraction of the entire central angle that is shaded, which equals the sum of the central angles of the shaded regions divided by the total central angle (360°), or $\frac{240^\circ}{360^\circ} = \frac{2}{3}$. (We can ignore the possibility that the spinner lands exactly on one of the dividing lines, since we assume that they are infinitesimally thin.)

Answer: (A)

17. Since Igor is shorter than Jie, then Igor cannot be the tallest.
Since Faye is taller than Goa, then Goa cannot be the tallest.
Since Jie is taller than Faye, then Faye cannot be the tallest.
Since Han is shorter than Goa, then Han cannot be the tallest.
The only person of the five who has not been eliminated is Jie, who must thus be the tallest.

Answer: (E)

18. From the number line shown, we see that $x < x^3 < x^2$.
If $x > 1$, then successive powers of x are increasing (that is, $x < x^2 < x^3$).
Since this is not the case, then it is not true that $x > 1$.
If $x = 0$ or $x = 1$, then successive powers of x are equal. This is not the case either.
If $0 < x < 1$, then successive powers of x are decreasing (that is, $x^3 < x^2 < x$). This is not the case either.
Therefore, it must be the case that $x < 0$.
If $x < -1$, we would have $x^3 < x < 0 < x^2$. This is because when $x < -1$, then x is negative and we have $x^2 > 1$ which gives $x^3 = x^2 \times x < 1 \times x$. This is not the case here either.
Therefore, it must be the case that $-1 < x < 0$.
From the given possibilities, this means that $-\frac{2}{5}$ is the only possible value of x.
We can check that if $x = -\frac{2}{5} = -0.4$, then $x^2 = 0.16$ and $x^3 = -0.064$, and so we have $x < x^3 < x^2$. We can also check by substitution that none of the other possible answers gives the correct ordering of x, x^2 and x^3.

Answer: (C)

19. Since $\angle XMZ = 30^\circ$, then $\angle XMY = 180^\circ - \angle XMZ = 180^\circ - 30^\circ = 150^\circ$.
Since the angles in $\triangle XMY$ add to 180°, then

$$\angle YXM = 180^\circ - \angle XYZ - \angle XMY = 180^\circ - 15^\circ - 150^\circ = 15^\circ$$

(Alternatively, since $\angle XMZ$ is an exterior angle of $\triangle XMY$, then $\angle XMZ = \angle YXM + \angle XYM$ which also gives $\angle YXM = 15^\circ$.)
Since $\angle XYM = \angle YXM$, then $\triangle XMY$ is isosceles with $MX = MY$.
But M is the midpoint of YZ, and so $MY = MZ$.
Since $MX = MY$ and $MY = MZ$, then $MX = MZ$.
This means that $\triangle XMZ$ is isosceles with $\angle XZM = \angle ZXM$.
Therefore, $\angle XZY = \angle XZM = \frac{1}{2}(180^\circ - \angle XMZ) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.

Answer: (A)

20. We call the $n \times n \times n$ cube the "large cube", and we call the $1 \times 1 \times 1$ cubes "unit cubes". The unit cubes that have exactly 0 gold faces are those unit cubes that are on the "inside" of the large cube.
In other words, these are the unit cubes none of whose faces form a part of any of the faces of the large cube.
These unit cubes form a cube that is $(n-2) \times (n-2) \times (n-2)$.
To see why this is true, imagine placing the original painted large cube on a table.
Each unit cube with at least one face that forms part of one of the outer faces (or outer layers) has paint on at least one face.
First, we remove the top and bottom layers of unit cubes. This creates a rectangular prism that is $n-2$ cubes high and still has a base that is $n \times n$.
Next, we can remove the left, right, front, and back faces.
This leaves a cube that is $(n-2) \times (n-2) \times (n-2)$.
Therefore, $(n-2)^3$ unit cubes have 0 gold faces.

The unit cubes that have exactly 1 gold face are those unit cubes that are on the outer faces of the large cube but do not touch the edges of the large cube.
Consider each of the six $n \times n$ faces of the large cube. Each is made up of n^2 unit cubes.
The unit cubes that have 1 gold face are those with at least one face that forms part of a face of the large cube, but do not share any edges with the edges of the large cube. Using a similar argument to above, we can see that these unit cubes form a $(n-2) \times (n-2)$ square.
There are thus $(n-2)^2$ cubes on each of the 6 faces that have 1 painted face, and so $6(n-2)^2$ cubes with 1 painted face.
We calculate the values of $(n-2)^3$ and $6(n-2)^2$ for each of the possible choices for n:

Choice	n	$(n-2)^3$	$6(n-2)^2$
(A)	7	125	150
(B)	8	216	216
(C)	9	343	294
(D)	10	512	384
(E)	4	8	24

From this information, the smallest possible value of n when $(n-2)^3$ is larger than $6(n-2)^2$ must be $n = 9$.

To see this in another way, we can ask the question "When is $(n-2)^3$ greater than $6(n-2)^2$?". Note that $(n-2)^3 = (n-2) \times (n-2)^2$ and $6(n-2)^2 = 6 \times (n-2)^2$, and so $(n-2)^3$ is greater than $6(n-2)^2$ when $(n-2)$ is greater than 6, which is when n is greater than 8.
The smallest positive integer value of n for which this is true is $n = 9$.

ANSWER: (C)

21. The averages of groups of three numbers are equal if the sums of the numbers in each group are equal, because in each case the average equals the sum of the three numbers divided by 3. Therefore, the averages of three groups of three numbers are equal if the sum of each of the three groups are equal.
The original nine numbers have a sum of

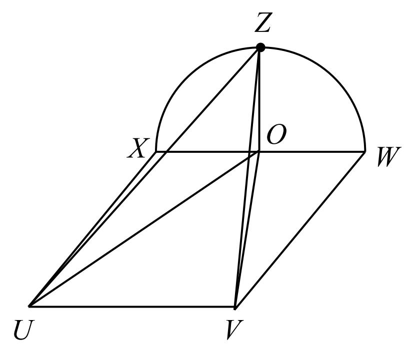
$$1 + 5 + 6 + 7 + 13 + 14 + 17 + 22 + 26 = 111$$

and so if these are divided into three groups of equal sum, the sum of each group is $\frac{111}{3} = 37$.
Consider the middle three numbers. Since two of the numbers are 13 and 17, then the third

number must be $37 - 13 - 17 = 7$. We note that the remaining six numbers can be split into the groups $5, 6, 26$ and $1, 14, 22$, each of which also has a sum of 37.
Therefore, the number that is placed in the shaded circle is 7.

Answer: (D)

22. The perimeter of $\triangle UVZ$ equals $UV + UZ + VZ$.
We know that $UV = 20$. We need to calculate UZ and VZ.
Let O be the point on XW directly underneath Z.
Since Z is the highest point on the semi-circle and XW is the diameter, then O is the centre of the semi-circle.
We join UO, VO, UZ, and VZ.



Since $UVWX$ is a rectangle, then $XW = UV = 20$ and $UX = VW = 30$.
Since XW is a diameter of the semi-circle and O is the centre, then O is the midpoint of XW and so $XO = WO = 10$.
This means that the radius of the semi-circle is 10, and so $OZ = 10$ as well.
Now $\triangle UXO$ and $\triangle VWO$ are both right-angled, since $UVWX$ is a rectangle.
By the Pythagorean Theorem, $UO^2 = UX^2 + XO^2 = 30^2 + 10^2 = 900 + 100 = 1000$ and $VO^2 = VW^2 + WO^2 = 30^2 + 10^2 = 1000$.
Each of $\triangle UOZ$ and $\triangle VOZ$ is right-angled at O, since the semi-circle is vertical and the rectangle is horizontal.
Therefore, we can apply the Pythagorean Theorem again to obtain $UZ^2 = UO^2 + OZ^2$ and $VZ^2 = VO^2 + OZ^2$.
Since $UO^2 = VO^2 = 1000$, then $UZ^2 = VZ^2 = 1000 + 10^2 = 1100$ or $UZ = VZ = \sqrt{1100}$.
Therefore, the perimeter of $\triangle UVZ$ is $20 + 2\sqrt{1100} \approx 86.332$.
Of the given choices, this is closest to 86.

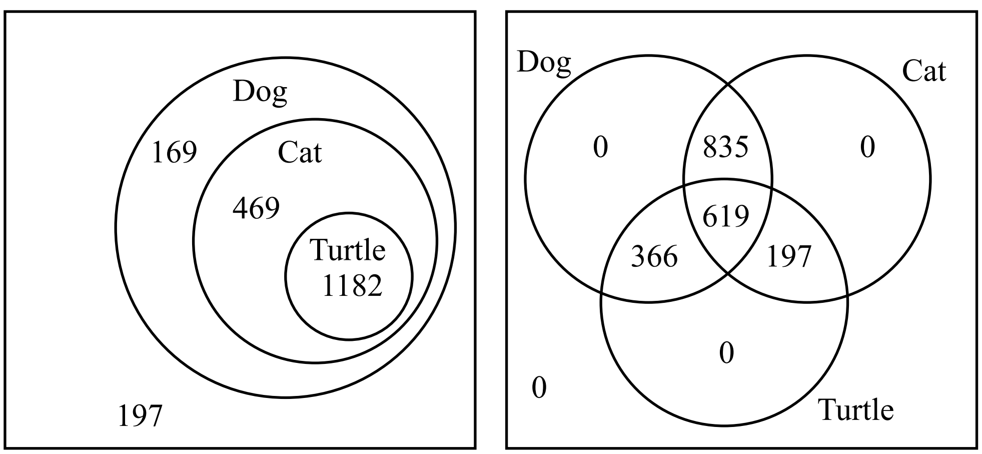
Answer: (B)

23. The squares of the one-digit positive integers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are $1, 4, 9, 16, 25, 36, 49, 64, 81$, respectively.
Of these, the squares $1, 25, 36$ end with the digit of their square root.
In other words, $k = 1, 5, 6$ are Anderson numbers.
Thus, $k = 6$ is the only even one-digit Anderson number.
To find all even two-digit Anderson numbers, we note that any two-digit even Anderson number k must have a units (ones) digit of 6. This is because the units digit of k and the units digit of k^2 must match (by the definition of an Anderson number) and because the units digit of k completely determines the units digit of k^2. (We can see this by doing "long multiplication".)
So we need to look for two-digit Anderson numbers k with digits $c6$.
Another way of writing the number $c6$ is $k = 10c+6$. (This form uses the place values associated with the digits.)
In this case, $k^2 = (10c+6)^2 = (10c+6)(10c+6) = (10c)^2+6(10c)+10c(6)+6^2 = 100c^2+120c+36$.
Note that $k^2 = 100(c^2 + c) + 10(2c + 3) + 6$ and so the units digit of k^2 is 6.
For k to be an Anderson number, we need the tens digit of k^2 to be c, in which case the final two digits of k^2 will be $c6$.
Thus, the tens digit of k^2 is equal to the units digit of $2c + 3$.
This means that $k = 10c + 6$ is an Anderson number exactly when the units digit of $2c + 3$ is equal to the digit c.
When we check the nine possible values for c, we find that the only possibility is that $c = 7$.
This means that $k = 76$ is the only two-digit even Anderson number.
Note that $76^2 = 5776$, which ends with the digits 76.
Next, we look for three-digit even Anderson numbers k.
Using a similar argument to above, we see that k must have digits $b76$.
In other words, $k = 100b + 76$ for some digit b.
In this case, $k^2 = (100b + 76)^2 = 10000b^2 + 15200b + 5776$.
We note that the tens and units digits of k^2 are 76, which means that, for k to be an Anderson number, the hundreds digit of k^2 must be b.
Now $k^2 = 1000(10b^2 + 15b + 5) + 100(2b + 7) + 76$.
Thus, k is an Anderson number exactly when the units digit of $2b + 7$ is equal to the digit b.
Again, checking the nine possible values for b shows us that $b = 3$ is the only possibility.
This means that $k = 376$ is the only three-digit even Anderson number.
Note that $376^2 = 141\,376$, which ends with the digits 376.
Since Anderson numbers are less than 10 000, then we still need to look for four-digit even Anderson numbers.
Again, using a similar argument, we see that k must have digits $a376$.
In other words, $k = 1000a + 376$ for some digit a.
In this case, $k^2 = (1000a + 376)^2 = 1\,000\,000a^2 + 752000a + 141\,376$.
We note that the hundreds, tens and units digits of k^2 are 376, which means that, for k to be an Anderson number, the thousands digit of k^2 must be a.
Now $k^2 = 10000(100a^2 + 75a + 14) + 1000(2a + 1) + 376$.
Thus, k is an Anderson number exactly when the units digit of $2a + 1$ is equal to the digit a.
Again, checking the nine possible values for a shows us that $a = 9$ is the only possibility.
This means that $k = 9376$ is the only four-digit even Anderson number.
Note that $9376^2 = 87\,909\,376$, which ends with the digits 9376.
Thus, S, the sum of the even Anderson numbers, equals $6 + 76 + 376 + 9376 = 9834$.
The sum of the digits of S is $9 + 8 + 3 + 4 = 24$.

Answer: (E)

24. Since there are 1182 houses that have a turtle, then there cannot be more than 1182 houses that have a dog, a cat, and a turtle.
Since there are more houses with dogs and more houses with cats than there are with turtles, it is possible that all 1182 houses that have a turtle also have a dog and a cat.
Therefore, the maximum possible number of houses that have all three animals is 1182, and so $x = 1182$.

Since there are 1182 houses that have a turtle and there are 2017 houses in total, then there are $2017 - 1182 = 835$ houses that do not have a turtle.
Now, there are 1651 houses that have a cat.
Since there are 835 houses that do not have a turtle, then there are at most 835 houses that have a cat and do not have a turtle. In other words, not all of the houses that do not have a turtle necessarily have a cat.
This means that there are at least $1651 - 835 = 816$ houses that have both a cat and a turtle.
Lastly, there are 1820 houses that have a dog.
Since there are at least 816 houses that have both a cat and a turtle, then there are at most $2017 - 816 = 1201$ houses that either do not have a cat or do not have a turtle (or both).
Since there are 1820 houses that do have a dog, then there are at least $1820 - 1201 = 619$ houses that have a dog and have both a cat and a turtle as well.
In other words, the minimum possible number of houses that have all three animals is 619, and so $y = 619$.
The two Venn diagrams below show that each of these situations is actually possible:



Since $x = 1182$ and $y = 619$, then $x - y = 563$.

Answer: (C)

25. We label the digits of the unknown number as $vwxyz$.
Since $vwxyz$ and 71794 have 0 matching digits, then $v \neq 7$ and $w \neq 1$ and $x \neq 7$ and $y \neq 9$ and $z \neq 4$.
Since $vwxyz$ and 71744 have 1 matching digit, then the preceding information tells us that $y = 4$.
Since $vwx4z$ and 51545 have 2 matching digits and $w \neq 1$, then $vwxyz$ is of one of the following three forms: $5wx4z$ or $vw54z$ or $vwx45$.

Case 1: $vwxyz = 5wx4z$
Since $5wx4z$ and 21531 have 1 matching digit and $w \neq 1$, then either $x = 5$ or $z = 1$.
If $x = 5$, then $5wx4z$ and 51545 would have 3 matching digits, which violates the given condition. Thus, $z = 1$.

Thus, $vwxyz = 5wx41$ and we know that $w \neq 1$ and $x \neq 5, 7$.
To this point, this form is consistent with the 1st, 2nd, 3rd and 7th rows of the table.
Since $5wx41$ and 59135 have 1 matching digit, this is taken care of by the fact that $v = 5$ and we note that $w \neq 9$ and $x \neq 1$.
Since $5wx41$ and 58342 have 2 matching digits, this is taken care of by the fact that $v = 5$ and $y = 4$, and we note that $w \neq 8$ and $x \neq 3$.
Since $5wx41$ and 37348 have 2 matching digits and $y = 4$, then either $w = 7$ or $x = 3$.
But we already know that $x \neq 3$, and so $w = 7$.
Therefore, $vwxyz = 57x41$ with the restrictions that $x \neq 1, 3, 5, 7$.
We note that the integers $57041, 57241, 57441, 57641, 57841, 57941$ satisfy the requirements, so are all possibilities for Sam's numbers.

Case 2: $vwxyz = vw54z$
Since $vw54z$ and 51545 have only 2 matching digits, so $v \neq 5$ and $z \neq 5$.
Since $vw54z$ and 21531 have 1 matching digit, then this is taken care of by the fact that $x = 5$, and we note that $v \neq 2$ and $z \neq 1$. (We already know that $w \neq 1$.)
Since $vw54z$ and 59135 have 1 matching digit, then $v = 5$ or $w = 9$ or $z = 5$.
This means that we must have $w = 9$.
Thus, $vwxyz = v954z$ and we know that $v \neq 2, 7, 5$ and $z \neq 1, 4, 5$.
To this point, this form is consistent with the 1st, 2nd, 3rd, 4th, and 7th rows of the table.
Since $v954z$ and 58342 have 2 matching digits and $v \neq 5$, then $z = 2$.
Since $v9542$ and 37348 have 2 matching digits, then $v = 3$.
In this case, the integer 39542 is the only possibility, and it satisfies all of the requirements.

Case 3: $vwxyz = vwx45$
Since $vwx45$ and 21531 have 1 matching digit and we know that $w \neq 1$, then $v = 2$ or $x = 5$.
But if $x = 5$, then $vw545$ and 51545 would have 3 matching digits, so $x \neq 5$ and $v = 2$.
Thus, $vwxyz = 2wx45$ and we know that $w \neq 1$ and $x \neq 5, 7$.
To this point, this form is consistent with the 1st, 2nd, 3rd and 7th rows of the table.
Since $2wx45$ and 59135 have 1 matching digit, this is taken care of by the fact that $z = 5$ and we note that $w \neq 9$ and $x \neq 1$.
Since $2wx45$ and 58342 have 2 matching digits, then $w = 8$ or $x = 3$, but not both.
Since $2wx45$ and 37348 have 2 matching digits, then $w = 7$ or $x = 3$, but not both.
If $w = 8$, then we have to have $x \neq 3$, and so neither $w = 7$ nor $x = 3$ is true.
Thus, it must be the case that $x = 3$ and $w \neq 7, 8$.
Therefore, $vwxyz = 2w345$ with the restrictions that $w \neq 1, 7, 8, 9$.
We note that the integers $20345, 22345, 23345, 24345, 25345, 26345$ satisfy the requirements, so are all possibilities for Sam's numbers.

Thus, there are 13 possibilities for Sam's numbers and the sum of these is 526758.

Answer: (E)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2016 Pascal Contest

(Grade 9)

Wednesday, February 24, 2016
(in North America and South America)

Thursday, February 25, 2016
(outside of North America and South America)

Solutions

1. Evaluating, $300 + 2020 + 10001 = 12321$.

Answer: (E)

2. We evaluate each of the five choices:

$$4^2 = 16 \qquad 4 \times 2 = 8 \qquad 4 - 2 = 2 \qquad \frac{4}{2} = 2 \qquad 4 + 2 = 6$$

Of these, the largest is $4^2 = 16$.

Answer: (A)

3. Since the grid is made up of 1×1 squares, then the lengths of the solid line segments, from top to bottom, are 5, 1, 4, 2, 3, and 3.
The sum of these lengths is $5 + 1 + 4 + 2 + 3 + 3 = 18$.
Alternatively, we could note that the first and second solid line segments can be combined to form a solid segment of length 6. The same is true with the third and fourth segments, and with the fifth and sixth segments. Thus, the total length is $6 \times 3 = 18$.

Answer: (D)

4. Since each of the five 1×1 squares has area 1, then the shaded area is 2.
Since the total area is 5, the percentage that is shaded is $\frac{2}{5} = 0.4 = 40\%$.

Answer: (D)

5. On a number line, the markings are evenly spaced.
Since there are 6 spaces between 0 and 30, each space represents a change of $\frac{30}{6} = 5$.
Since n is 2 spaces to the right of 60, then $n = 60 + 2 \times 5 = 70$.
Since m is 3 spaces to the left of 30, then $m = 30 - 3 \times 5 = 15$.
Therefore, $n - m = 70 - 15 = 55$.

Answer: (C)

6. From the definition, $\begin{array}{c|c} 4 & 5 \\ \hline 2 & 3 \end{array} = 4 \times 3 - 5 \times 2 = 12 - 10 = 2$.

Answer: (C)

7. Since there are 100 cm in 1 m, then 1 cm is 0.01 m. Thus, 3 cm equals 0.03 m.
Since there are 1000 mm in 1 m, then 1 mm is 0.001 m. Thus, 5 mm equals 0.005 m.
Therefore, 2 m plus 3 cm plus 5 mm equals $2 + 0.03 + 0.005 = 2.035$ m.

Answer: (A)

8. Since $x = 3$ and $y = 2x$, then $y = 2 \times 3 = 6$.
Since $y = 6$ and $z = 3y$, then $z = 3 \times 6 = 18$.
Therefore, the average of x, y and z is $\dfrac{x + y + z}{3} = \dfrac{3 + 6 + 18}{3} = 9$.

Answer: (D)

9. When Team A played Team B, if Team B won, then Team B scored more goals than Team A, and if the game ended in a tie, then Team A and Team B scored the same number of goals.
Therefore, if a team has 0 wins, 1 loss, and 2 ties, then it scored fewer goals than its opponent once (the 1 loss) and the same number of goals as its oppponent twice (the 2 ties).
Combining this information, we see that the team must have scored fewer goals than were scored against them.
In other words, it is not possible for a team to have 0 wins, 1 loss, and 2 ties, and to have scored more goals than were scored against them.

   We can also examine choices (A), (B), (D), (E) to see that, in each case, it is possible that the team scored more goals than it allowed.
This will eliminate each of these choices, and allow us to conclude that (C) must be correct.
(A): If the team won 2-0 and 3-0 and tied 1-1, then it scored 6 goals and allowed 1 goal.
(B): If the team won 4-0 and lost 1-2 and 2-3, then it scored 7 goals and allowed 5 goals.
(D): If the team won 4-0, lost 1-2, and tied 1-1, then it scored 6 goals and allowed 3 goals.
(E): If the team won 2-0, and tied 1-1 and 2-2, then it scored 5 goals and allowed 3 goals.
Therefore, it is only the case of 0 wins, 1 loss, and 2 ties where it is not possible for the team to score more goals than it allows.

   ANSWER: (C)

10. *Solution 1*
In the given diagram, we can see 3 of the 6 faces, or $\frac{1}{2}$ of the cube.
The remaining 3 faces (also $\frac{1}{2}$ of the cube) is unshaded.
Of the visible faces, $\frac{1}{2}$ of the area is shaded.
Therefore, the fraction of the total surface area that is shaded is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

   Solution 2
Since the cube is $2 \times 2 \times 2$, the area of each face is $2 \times 2 = 4$.
Since a cube has six faces, the total surface area of the cube is $6 \times 4 = 24$.
Each of the three faces that is partially shaded is one-half shaded, since each face is cut into two identical pieces by its diagonal.
Thus, the shaded area on each of these three faces is 2, and so the total shaded area is $3 \times 2 = 6$.
Therefore, the fraction of the total surface area that is shaded is $\frac{6}{24} = \frac{1}{4}$.

   ANSWER: (B)

11. The 7th oblong number is the number of dots in retangular grid of dots with 7 columns and 8 rows.
Thus, the 7th oblong number is $7 \times 8 = 56$.

   ANSWER: (C)

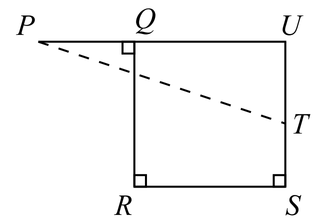
12. Since square $QRST$ has area 36, then its side length is $\sqrt{36} = 6$.
Therefore, $QR = 6$ and $RS = 6$.
Since $PQ = \frac{1}{2}QR$, then $PQ = 3$.
Now rectangle $PRSU$ has height $RS = 6$ and width $PR = PQ + QR = 3 + 6 = 9$.
Therefore, the perimeter of $PRSU$ is $2(6) + 2(9) = 12 + 18 = 30$.

   ANSWER: (B)

13. From the given information, $10x = x + 20$.
Therefore, $9x = 20$ and so $x = \frac{20}{9}$.

   ANSWER: (B)

14. Extend PQ and ST to meet at U.



Since $QUSR$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $\triangle PUT$ is right-angled at U.
By the Pythagorean Theorem, $PT^2 = PU^2 + UT^2$.
Now $PU = PQ + QU$ and $QU = RS$ so $PU = 4 + 8 = 12$.
Also, $UT = US - ST$ and $US = QR$ so $UT = 8 - 3 = 5$.
Therefore, $PT^2 = 12^2 + 5^2 = 144 + 25 = 169$.
Since $PT > 0$, then $PT = \sqrt{169} = 13$.

Answer: (E)

15. Since $75 = 3 \times 5 \times 5$, we can factor 75 in three different ways:

$$75 = 1 \times 75 = 3 \times 25 = 5 \times 15$$

If $pq = 75$ with p and q integers, then the possible values of p are thus $1, 3, 5, 15, 25, 75$.
The sum of these values is $1 + 3 + 5 + 15 + 25 + 75 = 124$.

Answer: (E)

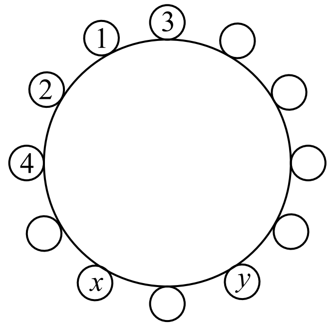
16. From 10 to 99 inclusive, there is a total of 90 integers. (Note that $90 = 99 - 10 + 1$.)
If an integer in this range includes the digit 6, this digit is either the ones (units) digit or the tens digit.
The integers in this range with a ones (units) digit of 6 are 16, 26, 36, 46, 56, 66, 76, 86, 96.
The integers in this range with a tens digit of 6 are 60, 61, 62, 63, 64, 65, 66, 67, 68, 69.
In total, there are 18 such integers. (Notice that 66 is in both lists and $9 + 10 - 1 = 18$.)
Therefore, the probability that a randomly chosen integer from 10 to 99 inclusive includes the digit 6 is $\frac{18}{90} = \frac{1}{5}$.

Answer: (A)

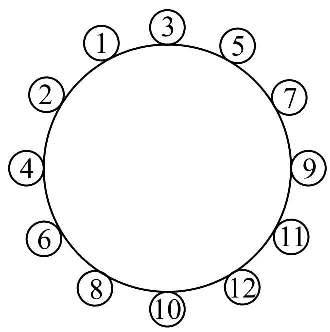
17. Among the list 10, 11, 12, 13, 14, 15, the integers 11 and 13 are prime.
Also, $10 = 2 \times 5$ and $12 = 2 \times 2 \times 3$ and $14 = 2 \times 7$ and $15 = 3 \times 5$.
For an integer N to be divisible by each of these six integers, N must include at least two factors of 2 and one factor each of 3, 5, 7, 11, 13.
Note that $2^2 \times 3 \times 5 \times 7 \times 11 \times 13 = 60\,060$.
(This is the least common multiple of 10, 11, 12, 13, 14, 15.)
To find the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, 15, we can find the smallest six-digit positive integer that is a multiple of 60 060.
Note that $1 \times 60\,060 = 60\,060$ and that $2 \times 60\,060 = 120\,120$.
Therefore, the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, 15 is 120 120.
The tens digit of this number is 2.

Answer: (C)

18. Because two integers that are placed next to each other must have a difference of at most 2, then the possible neighbours of 1 are 2 and 3.
Since 1 has exactly two neighbours, then 1 must be between 2 and 3.
Next, consider 2. Its possible neighbours are 1, 3 and 4. The number 2 is already a neighbour of 1 and cannot be a neighbour of 3 (since 3 is on the other side of 1). Therefore, 2 is between 1 and 4.
This allows us to update the diagram as follows:



Continuing in this way, the possible neighbours of 3 are 1, 2, 4, 5. The number 1 is already next to 3. Numbers 2 and 4 cannot be next to 3. So 5 must be next to 3.
The possible neighbours of 4 are 2, 3, 5, 6. The number 2 is already next 4. Numbers 3 and 5 cannot be next to 4. So 6 must be next to 4.
Continuing to complete the circle in this way, we obtain:



Note that when the even numbers and odd numbers meet (with 12 and 11) the conditions are still satisfied.
Therefore, $x = 8$ and $y = 12$ and so $x + y = 8 + 12 = 20$.

Answer: (D)

19. Suppose that there were n questions on the test.
Since Chris received a mark of 50% on the test, then he answered $\frac{1}{2}n$ of the questions correctly.
We know that Chris answered 13 of the first 20 questions correctly and then 25% of the remaining questions.
Since the test has n questions, then after the first 20 questions, there are $n - 20$ questions.
Since Chris answered 25% of these $n - 20$ questions correctly, then Chris answered $\frac{1}{4}(n - 20)$ of these questions correctly.
The total number of questions that Chris answered correctly can be expressed as $\frac{1}{2}n$ and also as $13 + \frac{1}{4}(n - 20)$.
Therefore, $\frac{1}{2}n = 13 + \frac{1}{4}(n - 20)$ and so $2n = 52 + (n - 20)$, which gives $n = 32$.
(We can check that if $n = 32$, then Chris answers 13 of the first 20 and 3 of the remaining 12 questions correctly, for a total of 16 correct out of 32.)

Answer: (C)

20. Since $\angle TQP$ and $\angle RQU$ are opposite angles, then $\angle RQU = \angle TQP = x^\circ$.
Similarly, $\angle QRU = \angle VRS = y^\circ$.
Since the angles in a triangle add to 180°, then

$$\angle QUR = 180^\circ - \angle RQU - \angle QRU = 180^\circ - x^\circ - y^\circ$$

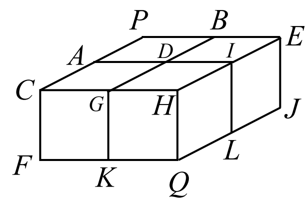
Now $\angle WQP$ and $\angle WQR$ are supplementary, as they lie along a line.
Thus, $\angle WQR = 180^\circ - \angle WQP = 180^\circ - 2x^\circ$.
Similarly, $\angle WRQ = 180^\circ - \angle WRS = 180^\circ - 2y^\circ$.
Since the angles in $\triangle WQR$ add to 180°, then

$$\begin{aligned} 38^\circ + (180^\circ - 2x^\circ) + (180^\circ - 2y^\circ) &= 180^\circ \\ 218^\circ &= 2x^\circ + 2y^\circ \\ x^\circ + y^\circ &= 109^\circ \end{aligned}$$

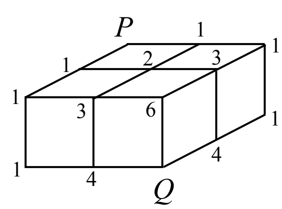
Finally, $\angle QUR = 180^\circ - x^\circ - y^\circ = 180^\circ - (x^\circ + y^\circ) = 180^\circ - 109^\circ = 71^\circ$.

Answer: (A)

21. We label the remaining points on the diagram as shown



There is exactly one path that the squirrel can take to get to each of A, C, F, B, E, and J.
For example, to get to F the squirrel must walk from P to A to C to F.
The number of paths that the squirrel can take to point D is 2, since there is 1 path to each of A and B, and to get to D, the squirrel must go through exactly one of A or B.
Similarly, the number of paths to G is the sum of the number of paths to C and to D (that is, $1 + 2 = 3$), because for the squirrel to get to G, it must walk through exactly one of C or D.
Using this process, we add to the diagram the number of paths to reach each of H, I, K, and L.



Finally, to get to Q, the squirrel must go through exactly one of H, K, or L, so the number of paths to Q is $6 + 4 + 4 = 14$.

Answer: (A)

22. *Solution 1*

Suppose that, when the n students are put in groups of 2, there are g complete groups and 1 incomplete group.
Since the students are being put in groups of 2, an incomplete group must have exactly 1 student in it.
Therefore, $n = 2g + 1$.
Since the number of complete groups of 2 is 5 more than the number of complete groups of 3, then there were $g - 5$ complete groups of 3.
Since there was still an incomplete group, this incomplete group must have had exactly 1 or 2 students in it.
Therefore, $n = 3(g - 5) + 1$ or $n = 3(g - 5) + 2$.
If $n = 2g+1$ and $n = 3(g-5)+1$, then $2g+1 = 3(g-5)+1$ or $2g+1 = 3g-14$ and so $g = 15$.
In this case, $n = 2g + 1 = 31$ and there were 15 complete groups of 2 and 10 complete groups of 3.
If $n = 2g+1$ and $n = 3(g-5)+2$, then $2g+1 = 3(g-5)+2$ or $2g+1 = 3g-13$ and so $g = 14$.
In this case, $n = 2g + 1 = 29$ and there were 14 complete groups of 2 and 9 complete groups of 3.
If $n = 31$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
If $n = 29$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
Since the difference between the number of complete groups of 3 and the number of complete groups of 4 is given to be 3, then it must be the case that $n = 31$.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

Solution 2

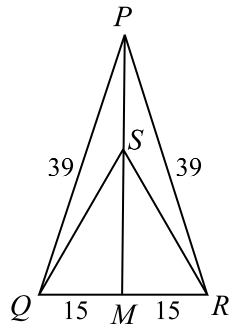
Since the n students cannot be divided exactly into groups of 2, 3 or 4, then n is not a multiple of 2, 3 or 4.
The first few integers larger than 1 that are not divisible by 2, 3 or 4 are 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35.
In each case, we determine the number of complete groups of each size:

n	5	7	11	13	17	19	23	25	29	31	35
# of complete groups of 2	2	3	5	6	8	9	11	12	14	15	17
# of complete groups of 3	1	2	3	4	5	6	7	8	9	10	11
# of complete groups of 4	1	1	2	3	4	4	5	6	7	7	8

Since the number of complete groups of 2 is 5 more than the number of complete groups of 3 which is 3 more than the number of complete groups of 4, then of these possibilities, $n = 31$ works.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.
(Since the problem is a multiple choice problem and we have found a value of n that satisfies the given conditions and for which an answer is present, then this answer must be correct. Solution 1 shows why $n = 31$ is the only value of n that satisfies the given conditions.)

Answer: (B)

23. Join S to the midpoint M of QR.
Since $\triangle SQR$ is equilateral with side length 30, then $QM = MR = \frac{1}{2}QR = 15$.



Since $\triangle SQR$ is equilateral, then SM is perpendicular to QR.
Since $\triangle PQR$ is isosceles with $PQ = PR$, then PM is also perpendicular to QR.
Since PM is perpendicular to QR and SM is perpendicular to QR, then PM and SM overlap, which means that S lies on PM.
By the Pythagorean Theorem,

$$PM = \sqrt{PQ^2 - QM^2} = \sqrt{39^2 - 15^2} = \sqrt{1521 - 225} = \sqrt{1296} = 36$$

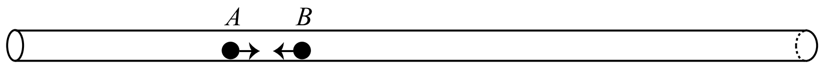
By the Pythagorean Theorem,

$$SM = \sqrt{SQ^2 - QM^2} = \sqrt{30^2 - 15^2} = \sqrt{900 - 225} = \sqrt{675} = 15\sqrt{3}$$

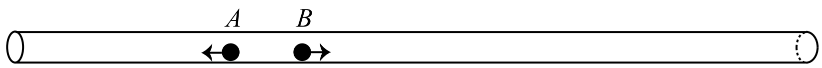
Therefore, $PS = PM - SM = 36 - 15\sqrt{3}$.
Since QM is perpendicular to PS extended, then the area of $\triangle PQS$ is equal to $\frac{1}{2}(PS)(QM)$.
(We can think of PS as the base and QM as the perpendicular height.)
Therefore, the area of $\triangle PQS$ equals $\frac{1}{2}(36 - 15\sqrt{3})(15) \approx 75.14$.
Of the given answers, this is closest to 75.

Answer: (B)

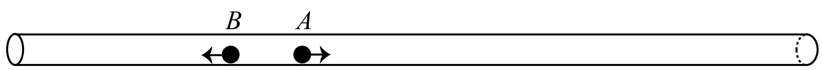
24. Since the rubber balls are very small and the tube is very long (55 m), we treat the balls as points with negligible width.
Since the 10 balls begin equally spaced along the tube with equal spaces before the first ball and after the last ball, then the 10 balls form 11 spaces in the tube, each of which is $\frac{55}{11} = 5$ m long.
When two balls meet and collide, they instantly reverse directions. Before a collision, suppose that ball A is travelling to the right and ball B is travelling to the left.



After this collision, ball A is travelling to the left and ball B is travelling to the right.



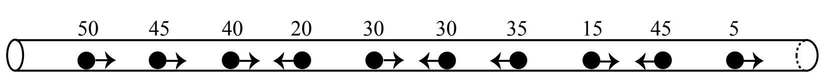
Because the balls have negligible size we can instead pretend that balls A and B have passed through each other and that now ball A is still travelling to the right and ball B is travelling to the left. The negligible size of the balls is important here as it means that we can ignore the fact that the balls will travel slightly further by passing through each other than they would by colliding.



In other words, since one ball is travelling to the left and one is travelling to the right, it actually does not matter how we label them.
This means that we can effectively treat each of the 10 balls as travelling in separate tubes and determine the amount of time each ball would take to fall out of the tube if it travelled in its original direction.
In (A),

- the first ball is 50 m from the right end of the tube, so will take 50 s to fall out
- the second ball is 45 m from the right end of the tube, so will take 45 s to fall out
- the third ball is 40 m from the right end of the tube, so will take 40 s to fall out
- the fourth ball is 20 m from the left end of the tube, so will take 20 s to fall out (note that this ball is travelling to the left)

and so on.
For configuration (A), we can follow the method above and label the amount of time each ball would take to fall out:



We can then make a table that lists, for each of the five configuration, the amount of time, in seconds that each ball, counted from left to right, will take to fall out:

Configuration	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
(A)	50	45	40	20	30	30	35	15	45	5
(B)	5	45	15	35	30	25	20	40	45	50
(C)	50	10	15	35	30	30	35	15	45	5
(D)	5	45	40	20	30	30	35	40	45	5
(E)	50	10	40	20	30	30	35	15	45	50

Since there are 10 balls, then more than half of the balls will have fallen out when 6 balls have fallen out.
In (A), the balls fall out after 5, 15, 20, 30, 30, 35, 40, 45, 45, and 50 seconds, so 6 balls have fallen out after 35 seconds.
The corresponding times for (B), (C), (D), and (E) are 35, 30, 35, and 35 seconds.
Therefore, the configuration for which it takes the least time for more than half of the balls to fall out is (C).

ANSWER: (C)

25. Since each row in the grid must contain at least one 1, then there must be at least three 1s in the grid.
Since each row in the grid must contain at least one 0, then there must be at least three 0s in the grid. Since there are nine entries in the grid, then there must be at most six 1s in the grid.
Thus, there are three 1s and six 0s, or four 1s and five 0s, or five 1s and four 0s, or six 1s and three 0s.
The number of grids with three 1s and six 0s must be equal to the number of grids with six 1s and three 0s. This is because each grid of one kind can be changed into a grid of the other kind by replacing all of the 0s with 1s and all of the 1s with 0s.
Similarly, the number of grids with four 1s and five 0s will be equal to the number of grids with five 1s and four 0s.
Therefore, we count the number of grids that contain three 1s and the number of grids that contain four 1s, and double our total to get the final answer.

Counting grids that contain three 1s
Since each row must contain at least one 1 and there are only three 1s to use, then there must be exactly one 1 in each row.
Since each column must also contain a 1, then the three rows must be $\boxed{1\ 0\ 0}$, $\boxed{0\ 1\ 0}$, and $\boxed{0\ 0\ 1}$ in some order.
There are thus 3 choices for the first row.
For each of these choices, there are 2 choices for the second row. The first and second rows completely determine the third row.
Therefore, there are $3 \times 2 = 6$ (or $3 \times 2 \times 1 = 6$) configurations for the grid.
We note that each of these also includes at least one 0 in each row and in each column, as desired.

Counting grids that contain four 1s
Since each row must contain at least one 1 and there are four 1s to use, then there must be two 1s in one row and one 1 in each of the other two rows. This guarantees that there is at least one 0 in each row.
Suppose that the row containing two 1s is $\boxed{1\ 1\ 0}$.
One of the remaining rows must have a 1 in the third column, so must be $\boxed{0\ 0\ 1}$.
The remaining row could be any of $\boxed{1\ 0\ 0}$, $\boxed{0\ 1\ 0}$, and $\boxed{0\ 0\ 1}$.
We note that in any combination of these rows, each column will contain at least one 0 as well.
With the rows $\boxed{1\ 1\ 0}$, $\boxed{0\ 0\ 1}$, and $\boxed{0\ 0\ 1}$, there are 3 arrangements.
This is because there are 3 choices of where to put the row $\boxed{1\ 1\ 0}$, and then the remaining two rows are the same and so no further choice is possible.
With the rows $\boxed{1\ 1\ 0}$, $\boxed{0\ 0\ 1}$, and $\boxed{0\ 1\ 0}$, there are 6 arrangements, using a similar argument to the counting in the "three 1s" case above.
Similarly, with rows $\boxed{1\ 1\ 0}$, $\boxed{0\ 0\ 1}$, and $\boxed{1\ 0\ 0}$, there are 6 arrangements.
So there are $3 + 6 + 6 = 15$ configurations that include the row $\boxed{1\ 1\ 0}$.

Using similar arguments, we can find that there are 15 configurations that include the row $\boxed{\texttt{1 0 1}}$ and 15 configurations that include the row $\boxed{\texttt{0 1 1}}$.
Therefore, there are $3 \cdot 15 = 45$ configurations that contain four 1s.

Finally, by the initial comment, this means that there are $2(6 + 45) = 102$ configurations.

ANSWER: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015 Pascal Contest

(Grade 9)

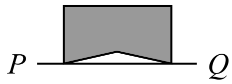
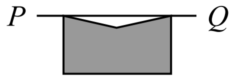
Tuesday, February 24, 2015
(in North America and South America)

Wednesday, February 25, 2015
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{20+15}{30-25} = \dfrac{35}{5} = 7$.

   Answer: (D)

2. When the shaded figure in  is reflected about the line segment PQ, the resulting figure is 

   This is because the figure started above the line and so finished below the line, and the figure initially touched the line at two points, so finishes still touching the line at two points.

   Answer: (A)

3. Since $8+6=n+8$, then subtracting 8 from both sides, we obtain $6=n$ and so n equals 6.
   Alternatively, since the order in which we add two numbers does not change the sum, then $8+6=6+8$ and so $n=6$.

   Answer: (C)

4. Each of 0.07, -0.41, 0.35, and -0.9 is less than 0.7 (that is, each is to the left of 0.7 on the number line).
   The number 0.8 is greater than 0.7.

   Answer: (C)

5. Converting from fractions to decimals, $4 + \frac{3}{10} + \frac{9}{1000} = 4 + 0.3 + 0.009 = 4.309$.

   Answer: (B)

6. Since the average of the three ages is 22, then the sum of the three ages is $3 \cdot 22 = 66$.
   Since Andras' age is 23 and Frances' age is 24, then Gerta's age is $66 - 23 - 24 = 19$.

   Answer: (A)

7. When $n=7$, we have

   $$9n = 63 \qquad n+8=15 \qquad n^2 = 49 \qquad n(n-2) = 7(5) = 35 \qquad 8n = 56$$

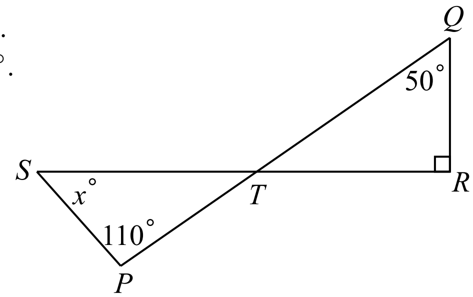
   Therefore, $8n$ is even.
   We note that for every integer n, the expression $8n$ is equal to an even integer, since 8 is even and the product of an even integer with any integer is even.
   If n were even, then in fact all five choices would be even. If n is odd, only $8n$ is even.

   Answer: (E)

8. After Jitka hiked 60% of the trail, $100\% - 60\% = 40\%$ of the trail was left.
   From the given information, 40% of the length of the trail corresponds to 8 km.
   This means that 10% of the trail corresponds to one-quarter of 8 km, or 2 km.
   Since 10% of the trail has length 2 km, then the total length of the trail is $10 \cdot 2 = 20$ km.

   Answer: (E)

9. Since the sum of the angles in a triangle is 180°, then $\angle QTR = 180^\circ - \angle TQR - \angle QRT = 180^\circ - 50^\circ - 90^\circ = 40^\circ$.
Since opposite angles are equal, then $\angle STP = \angle QTR = 40^\circ$.
Since the sum of the angles in $\triangle STP$ is 180°, then



$$\begin{aligned} \angle PST + \angle SPT + \angle STP &= 180^\circ \\ x^\circ + 110^\circ + 40^\circ &= 180^\circ \\ x + 150 &= 180 \\ x &= 30 \end{aligned}$$

Therefore, the value of x is 30.

Answer: (A)

10. Evaluating,

$$\sqrt{16 \times \sqrt{16}} = \sqrt{16 \times 4} = \sqrt{64} = 8$$

Since $8 = 2 \times 2 \times 2 = 2^3$, then $\sqrt{16 \times \sqrt{16}} = 2^3$.

Answer: (C)

11. *Solution 1*
The sequence of symbols includes 5 $\heartsuit$'s and 2 $\spadesuit$'s.
This means that, each time the sequence is written, there are $5 - 2 = 3$ more $\heartsuit$'s written than $\spadesuit$'s.
When the sequence is written 50 times, in total there are $50 \cdot 3 = 150$ more $\heartsuit$'s written than $\spadesuit$'s.

Solution 2
The sequence of symbols includes 5 $\heartsuit$'s and 2 $\spadesuit$'s.
When the sequence is written 50 times, there will be a total of $50 \cdot 5 = 250$ $\heartsuit$'s written and a total of $50 \cdot 2 = 100$ $\spadesuit$'s written.
This means that there are $250 - 100 = 150$ more $\heartsuit$'s written than $\spadesuit$'s.

Answer: (B)

12. Since 9 is a multiple of 3, then every positive integer that is a multiple of 9 is also a multiple of 3.
Therefore, we can simplify the problem to find the smallest positive integer that is a multiple of each of 5, 7 and 9.
The smallest positive integer that is a multiple of each of 7 and 9 is $7 \cdot 9 = 63$, since 7 and 9 have no common divisor larger than 1. (We could also list the positive multiples of 9 until we found the first one that is also a multiple of 7.)
Thus, the positive integers that are multiples of 7 and 9 are those which are multiples of 63.
We list the multiples of 63 until we find the first one that is divisible by 5 (that is, that ends in a 0 or in a 5):

$$63 \cdot 1 = 63 \qquad 63 \cdot 2 = 126 \qquad 63 \cdot 3 = 189 \qquad 63 \cdot 4 = 252 \qquad 63 \cdot 5 = 315$$

Therefore, the smallest positive integer that is a multiple of each of 3, 5, 7, and 9 is 315.

Answer: (D)

13. Each of the squares that has area 400 m^2 has side length $\sqrt{400} = 20$ m.
Anna's path walks along exactly 20 side lengths of squares, so has length $20 \cdot 20 = 400$ m.
Aaron's path walks along exactly 12 side lengths of squares, so has length $12 \cdot 20 = 240$ m.
Therefore, the total distance that they walk is $400 + 240 = 640$ m.

Answer: (E)

14. From the given definition,

$$4 \otimes 8 = \frac{4}{8} + \frac{8}{4} = \frac{1}{2} + 2 = \frac{5}{2}$$

Answer: (E)

15. We make a table of the total amount of money that each of Steve and Wayne have at the end of each year. After the year 2000, each entry in Steve's column is found by doubling the previous entry and each entry in Wayne's column is found by dividing the previous entry by 2. We stop when the entry in Steve's column is larger than that in Wayne's column:

Year	Steve	Wayne
2000	\$100	\$10 000
2001	\$200	\$5000
2002	\$400	\$2500
2003	\$800	\$1250
2004	\$1600	\$625

Therefore, the end of 2004 is the first time at which Steve has more money than Wayne at the end of the year.

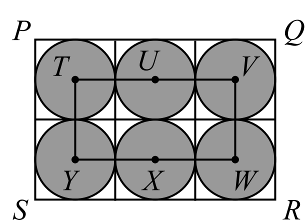
Answer: (C)

16. Since Bruce drove 200 km at a speed of 50 km/h, this took him $\frac{200}{50} = 4$ hours.
Anca drove the same 200 km at a speed of 60 km/h with a stop somewhere along the way.
Since Anca drove 200 km at a speed of 60 km/h, the time that the driving portion of her trip took was $\frac{200}{60} = 3\frac{1}{3}$ hours.
The length of Anca's stop is the difference in driving times, or $4 - 3\frac{1}{3} = \frac{2}{3}$ hours.
Since $\frac{2}{3}$ hours equals 40 minutes, then Anca stopped for 40 minutes.

Answer: (A)

17. Let r be the radius of each of the six circles.
Then $TY = TU = UV = YX = XW = VW = 2r$ and $PQ = SR = 6r$ and $PS = QR = 4r$:

> Since each circle has radius r, then each circle has diameter $2r$, and so can be enclosed in a square with side length $2r$ whose sides are parallel to the sides of rectangle $PQRS$.
> Each circle touches each of the four sides of its enclosing square.
> Because each of the circles touches one or two sides of rectangle $PQRS$ and each of the circles touches one or two of the other circles, then these six squares will fit together without overlapping to completely cover rectangle $PQRS$.



Therefore, $PQ = SR = 6r$ and $PS = QR = 4r$ since rectangle $PQRS$ is three squares wide and two squares tall.
Finally, since the centre of each circle is the centre of its square, then the distance between the centres of each pair of horizontally or vertically neighbouring squares is $2r$ (which is two times half the side length of one of the squares).

Therefore, the perimeter of rectangle $TVWY$ is

$$TV + TY + YW + VW = 2r + 4r + 4r + 2r = 12r$$

Since the perimeter of rectangle $TVWY$ is 60, then $12r = 60$ or $r = 5$.
Since $r = 5$, then in the larger rectangle, we have $PQ = SR = 30$ and $PS = QR = 20$.
Therefore, the area of rectangle $PQRS$ is $PQ \cdot PS = 30 \cdot 20 = 600$.

Answer: (A)

18. In a magic square, the numbers in each row, the numbers in each column, and numbers on each diagonal have the same sum.
Since the sum of the numbers in the first row equals the sum of the numbers in the first column, then $a + 13 + b = a + 19 + 12$ or $b = 19 + 12 - 13 = 18$.
Therefore, the sum of the numbers in any row, in any column, or along either diagonal equals the sum of the numbers in the third column, which is $18 + 11 + 16 = 45$.
Using the first column, $a + 19 + 12 = 45$ or $a = 14$.
Using the second row, $19 + c + 11 = 45$ or $c = 15$.
Thus, $a + b + c = 14 + 18 + 15 = 47$.

Answer: (C)

19. *Solution 1*
We work backwards from the last piece of information given.
Krystyna has 16 raisins left after giving one-half of her remaining raisins to Anna.
This means that she had $2 \cdot 16 = 32$ raisins immediately before giving raisins to Anna.
Immediately before giving raisins to Anna, she ate 4 raisins, which means that she had $32 + 4 = 36$ raisins immediately before eating 4 raisins.
Immediately before eating these raisins, she gave one-third of her raisins to Mike, which would have left her with two-thirds of her original amount.
Since two-thirds of her original amount equals 36 raisins, then one-third equals $\frac{36}{2} = 18$ raisins.
Thus, she gave 18 raisins to Mike and so started with $36 + 18 = 54$ raisins.

Solution 2
Suppose Krystyna starts with x raisins.
She gives $\frac{1}{3}x$ raisins to Mike, leaving her with $x - \frac{1}{3}x = \frac{2}{3}x$ raisins.
She then eats 4 raisins, leaving her with $\frac{2}{3}x - 4$ raisins.
Finally, she gives away one-half of what she has left to Anna, which means that she keeps one-half of what she has left, and so she keeps $\frac{1}{2}\left(\frac{2}{3}x - 4\right)$ raisins.
Simplifying this expression, we obtain $\frac{2}{6}x - \frac{4}{2} = \frac{1}{3}x - 2$ raisins.
Since she has 16 raisins left, then $\frac{1}{3}x - 2 = 16$ and so $\frac{1}{3}x = 18$ or $x = 54$.
Therefore, Krystyna began with 54 raisins.

Answer: (B)

20. Since $\$10 = 2 \cdot \$5$, then \$10 can be formed using 0, 1 or 2 \$5 bills and cannot be formed using more than 2 \$5 bills.

Using 2 \$5 bills, we obtain \$10 exactly. There is no choice in this case, and so this gives exactly 1 way to make \$10.

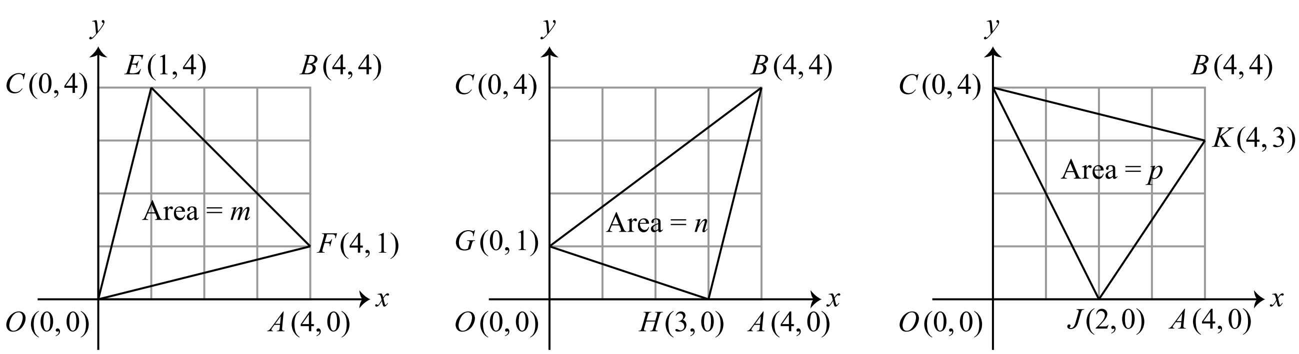
Using 1 \$5 bill, we need an additional \$5.
Since 5 is odd and any amount of dollars made up using \$2 coins will be even, we need an odd number of \$1 coins to make up the difference.
We can use 5 \$1 coins and 0 \$2 coins, or 3 \$1 coins and 1 \$2 coin, or 1 \$1 coin and 2 \$2 coins to obtain \$5.
This is 3 more ways.

Using 0 \$5 bills, we need an additional \$10.
Since 10 is even and any amount of dollars made up using \$2 coins will be even, we need an even number of \$1 coins to make up the difference.
The numbers of \$1 coins and \$2 coins that we can use are 10 and 0, 8 and 1, 6 and 2, 4 and 3, 2 and 4, or 0 and 5.
This is 6 more ways.

In total, there are $1 + 3 + 6 = 10$ ways in which André can make \$10.

ANSWER: (A)

21. In each diagram, we label the origin $(0, 0)$ as O, the point $(4, 0)$ as A, the point $(4, 4)$ as B, and the point $(0, 4)$ as C.
Thus, in each diagram, square $OABC$ is 4 by 4 and so has area 16.
In the first diagram, we label $(1, 4)$ as E and $(4, 1)$ as F.
In the second diagram, we label $(0, 1)$ as G and $(3, 0)$ as H.
In the third diagram, we label $(2, 0)$ as J and $(4, 3)$ as K.



In the first diagram, the area of $\triangle OEF$ equals the area of square $OABC$ minus the areas of $\triangle OCE$, $\triangle EBF$ and $\triangle FAO$.
Each of these three triangles is right-angled at a corner of the square.
Since $OC = 4$ and $CE = 1$, the area of $\triangle OCE$ is $\frac{1}{2}(4)(1) = 2$.
Since $EB = 3$ and $BF = 3$, the area of $\triangle EBF$ is $\frac{1}{2}(3)(3) = \frac{9}{2}$.
Since $FA = 1$ and $AO = 4$, the area of $\triangle FAO$ is $\frac{1}{2}(1)(4) = 2$.
Therefore, the area of $\triangle OEF$ equals $16 - 2 - \frac{9}{2} - 2 = \frac{15}{2}$, or $m = \frac{15}{2}$.

In the second diagram, the area of $\triangle GBH$ equals the area of square $OABC$ minus the areas of $\triangle GCB$, $\triangle BAH$ and $\triangle HOG$.

Each of these three triangles is right-angled at a corner of the square.
Since $GC = 3$ and $CB = 4$, the area of $\triangle GCB$ is $\frac{1}{2}(3)(4) = 6$.
Since $BA = 4$ and $AH = 1$, the area of $\triangle BAH$ is $\frac{1}{2}(4)(1) = 2$.
Since $HO = 3$ and $OG = 1$, the area of $\triangle HOG$ is $\frac{1}{2}(1)(3) = \frac{3}{2}$.
Therefore, the area of $\triangle HOG$ equals $16 - 6 - 2 - \frac{3}{2} = \frac{13}{2}$, or $n = \frac{13}{2}$.

In the third diagram, the area of $\triangle CKJ$ equals the area of square $OABC$ minus the areas of $\triangle CBK$, $\triangle KAJ$ and $\triangle JOC$.
Each of these three triangles is right-angled at a corner of the square.
Since $CB = 4$ and $BK = 1$, the area of $\triangle CBK$ is $\frac{1}{2}(4)(1) = 2$.
Since $KA = 3$ and $AJ = 2$, the area of $\triangle KAJ$ is $\frac{1}{2}(3)(2) = 3$.
Since $JO = 2$ and $OC = 4$, the area of $\triangle JOC$ is $\frac{1}{2}(2)(4) = 4$.
Therefore, the area of $\triangle CKJ$ equals $16 - 2 - 3 - 4 = 7$, or $p = 7$.

Since $m = \frac{15}{2} = 7\frac{1}{2}$, and $n = \frac{13}{2} = 6\frac{1}{2}$, and $p = 7$, then $n < p < m$.

ANSWER: (D)

22. *Solution 1*
Let $\$c$ be the cost per square metre of installing carpeting.
Then in each situation, the area of the room times the cost per square metre equals the total price.
From the top left entry in the table, $15 \cdot 10 \cdot \$c = \$397.50$.
From the top right entry in the table, $15 \cdot y \cdot \$c = \$675.75$.
From the bottom left entry in the table, $x \cdot 10 \cdot \$c = \$742.00$.
From the bottom right entry in the table, $x \cdot y \cdot \$c = \$z$.
Now,

$$z = x \cdot y \cdot c = x \cdot y \cdot c \cdot \frac{10 \cdot 15 \cdot c}{10 \cdot 15 \cdot c} = \frac{(x \cdot 10 \cdot c) \cdot (15 \cdot y \cdot c)}{15 \cdot 10 \cdot c} = \frac{(742.00) \cdot (675.75)}{397.50} = 1261.40$$

Therefore, $z = 1261.40$.

Solution 2
Let $\$c$ be the cost per square metre of installing carpeting.
Then in each situation, the area of the room times the cost per square metre equals the total price.
From the top left entry in the table, $15 \cdot 10 \cdot \$c = \$397.50$.
Thus, $c = \dfrac{397.50}{15 \cdot 10} = 2.65$.
From the top right entry in the table, $15 \cdot y \cdot \$c = \$675.75$.
Thus, $y = \dfrac{675.75}{15 \cdot 2.65} = 17$.
From the bottom left entry in the table, $x \cdot 10 \cdot \$c = \$742.00$.
Thus, $x = \dfrac{742.00}{10 \cdot 2.65} = 28$.
From the bottom right entry in the table, $x \cdot y \cdot \$c = \$z$.
Thus, $z = 28 \cdot 17 \cdot 2.65 = 1261.40$.
Therefore, $z = 1261.40$.

ANSWER: (E)

23. Since the left side of the given equation is a multiple of 6, then the right side, c^2, is also a multiple of 6.
Since c^2 is a multiple of 6, then c^2 is a multiple of 2 and a multiple of 3.
Since 2 and 3 are different prime numbers, then the positive integer c itself must be a multiple of 2 and a multiple of 3. This is because if c is not a multiple of 3, then c^2 cannot be a multiple of 3, and if c is not even, then c^2 cannot be even.
Therefore, c is a multiple of each of 2 and 3, and so is a multiple of 6.
Thus, there are five possible values for c in the given range: 6, 12, 18, 24, 30.
If $c = 6$, then $6ab = 36$ and so $ab = 6$.
Since $1 \leq a < b < 6$ (because $c = 6$), then $a = 2$ and $b = 3$.
If $c = 12$, then $6ab = 144$ and so $ab = 24$.
Since $1 \leq a < b < 12$, then $a = 3$ and $b = 8$ or $a = 4$ and $b = 6$.
(The divisor pairs of 24 are $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. Only the pairs $24 = 3 \cdot 8 = 4 \cdot 6$ give solutions that obey the given restrictions, since in the other two pairs, the larger divisor does not satisfy the restriction of being less than 12.)
If $c = 18$, then $6ab = 324$ and so $ab = 54$.
Since $1 \leq a < b < 18$, then $a = 6$ and $b = 9$.
(The divisor pairs of 54 are $54 = 1 \cdot 54 = 2 \cdot 27 = 3 \cdot 18 = 6 \cdot 9$.)
If $c = 24$, then $6ab = 576$ and so $ab = 96$.
Since $1 \leq a < b < 24$, then $a = 6$ and $b = 16$ or $a = 8$ and $b = 12$.
(The divisor pairs of 96 are $96 = 1 \cdot 96 = 2 \cdot 48 = 3 \cdot 32 = 4 \cdot 24 = 6 \cdot 16 = 8 \cdot 12$.)
If $c = 30$, then $6ab = 900$ and so $ab = 150$.
Since $1 \leq a < b < 30$, then $a = 6$ and $b = 25$ or $a = 10$ and $b = 15$.
(The divisor pairs of 150 are $150 = 1 \cdot 150 = 2 \cdot 75 = 3 \cdot 50 = 5 \cdot 30 = 6 \cdot 25 = 10 \cdot 150$.)
Therefore, the triples (a, b, c) of positive integers that are solutions to the equation $6ab = c^2$ and that satisfy $a < b < c \leq 35$ are

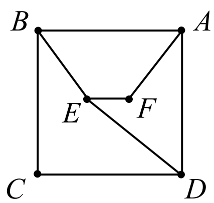
$$(a, b, c) = (2, 3, 6), (3, 8, 12), (4, 6, 12), (6, 9, 18), (6, 16, 24), (8, 12, 24), (6, 25, 30), (10, 15, 30)$$

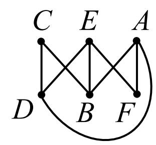
There are 8 such triplets.

Answer: (B)

24. We show that two of the five drawings can represent the given information.
In each drawing, we call each point a *vertex* (points are *vertices*), each line or curve joining two vertices an *edge*, and the number of edges meeting at each vertex the *degree* of the vertex.
We use the labels A, B, C, D, E, and F to represent Ali, Bob, Cai, Dee, Eve, and Fay, respectively.

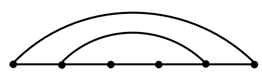
The second and fourth drawings can represent the data using the following labelling:





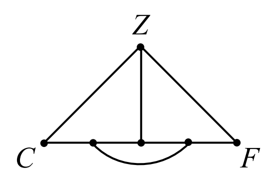
In each case, the 8 links $AB, BC, CD, DE, EF, FA, AD, BE$ are shown by edges and no additional edges are present.
Therefore, these drawings represent the given information.

The first drawing cannot represent the given information as it only includes 7 edges which cannot represent the 8 given links.

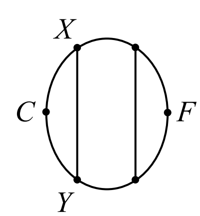


To analyze the third and fifth drawings, we note that there are exactly two suspects (Cai and Fay) who are part of only two links (Cai: BC and CD; Fay: EF and FA).

In the fifth drawing, there are two vertices of degree 2 (that is, at which exactly two edges meet). If this drawing is to represent the given information, these vertices must be C and F in some order. Because the diagram is symmetrical, we can label them as shown without loss of generality.
Consider the vertex labelled Z.
Since it is linked to C, it must be B or D.
But Z is also linked to F, and neither B nor D is linked to F.
Therefore, this drawing cannot represent the given information.



In the third drawing, there are two vertices of degree 2. If this drawing is to represent the given information, these vertices must be C and F in some order. Because the diagram is symmetrical, we can label them as shown without loss of generality.
Consider the vertices labelled X and Y.
Since they are linked to C, they must be B and D in some order.
But X and Y are joined by an edge, while B and D are not linked by the given information.
This means that this drawing cannot represent the given information.



Therefore, two of the five drawings can be labelled to represent the given data.

Answer: (B)

25. We begin by noting that an integer either appears in column V or it appears in one or more of the columns W, X, Y, Z:

    If an integer v appears in column V, then v cannot have appeared in an earlier row in the table. In particular, v cannot have appeared in any of columns W, X, Y, Z earlier in the table.
    Furthermore, v cannot appear again later in the table, since each entry in its row is larger than it and each entry in later rows is again larger (since the entries in V in those later rows will be larger).

    If an integer a appears in one or more of the columns W, X, Y, Z, then it will not appear in column V later in the table, since every entry in V is one that has not yet appeared in the table.
    Furthermore, a cannot have appeared in V earlier in the table since every entry in V before a appears in W, X, Y, or Z will be smaller than a.

    Therefore, an integer that appears in the table either appears in column V or it appears in one or more of the columns W, X, Y, Z.

    If an integer v has not appeared in the table, then it will eventually be the smallest positive integer that has not appeared in the table so far, and so v will appear in column V in the next row.

    We check to see if 2731 appears in column Z.
    Since $2731 = 7(390) + 1$, then 2731 appears in Z if 390 appears in V.
    Since 389 is not a multiple of 2, 3, 5 or 7 (we can check by dividing by each of these), then 390 cannot appear in W, X, Y, Z (because 390 cannot be written in the form $2n+1$, $3n+1$, $5n+1$, or $7n+1$ for some positive integer n).
    This means that 390 appears in V and so $2731 = 7(390) + 1$ appears in Z in this row.
    This eliminates answer (A).
    We note also that, since 2731 appears in Z, it cannot appear in V.

    Next, we show that 2731 appears in column Y.
    Since $2731 = 5(546) + 1$, then 2731 appears in Y if 546 appears in V.
    We now show that 546 does appear in V.
    Since 545 is a multiple of 5, but not of 2, 3 or 7, then either 546 appears in column V or in column Y, as explained above.
    We will show that 546 does not appear in Y.
    Since $546 = 5(109) + 1$, then 546 appears in Y if 109 appears in V.
    We will show that 109 does not appear in V.
    Since 108 is a multiple of 2 and 3, but not of 5 or 7, then 109 could appear in V, W or X.
    We will show that 109 appears in X.
    Since $109 = 3(36) + 1$, then 109 appears in X if 36 appears in V.
    We will show that 36 appears in V.
    Since 35 is a multiple of 5 and 7, but not of 2 or 3, then 36 could appear in V, Y or Z.
    If 36 appears in Z, then the V entry in this row would be 5.
    If 36 appears in Y, then the V entry in this row would be 7.
    Neither 5 nor 7 is in column V. (Each appears in the second row.)
    Therefore, 36 appears in V, which means that 109 appears in X.
    This means that 109 does not appear in V and so 546 does not appear in Y.
    Since 546 is in V or Y, then 546 is in V, which means that 2731 is in Y.
    This eliminates answers (C) and (E).

Lastly, we show that 2731 does not appear in column X.
Since 29 is not a multiple of 2, 3, 5 or 7, then 30 must appear in V.
Since 30 appears in V, then $151 = 5(30) + 1$ appears in Y.
Since 151 appears in Y, it does not appear in V.
Since 151 does not appear in V, then $303 = 2(151) + 1$ does not appear in W.
Since 303 does not appear in W, then 303 appears in V. (This is because 302 is not a multiple of 3, 5 or 7 and so 303 cannot appear in X, Y or Z.)
Since 303 appears in V, then $910 = 3(303) + 1$ appears in X.
Since 910 appears in X, then 910 does not appear in V.
Since 910 does not appear in V, then $2731 = 3(910) + 1$ does not appear in X.

Therefore, 2731 appears in W, Y and Z.

ANSWER: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Pascal Contest

(Grade 9)

Thursday, February 20, 2014
(in North America and South America)

Friday, February 21, 2014
(outside of North America and South America)

Solutions

1. We evaluate the expression by first evaluating the expressions in brackets:

$$(8 \times 6) - (4 \div 2) = 48 - 2 = 46$$

Answer: (C)

2. Since the sum of the angles in a triangle is 180°, then $50^\circ + x^\circ + 45^\circ = 180^\circ$ or $x + 95 = 180$. Therefore, $x = 85$.

Answer: (C)

3. 30% of 200 equals $\frac{30}{100} \times 200 = 60$.
Alternatively, we could note that 30% of 100 is 30 and $200 = 2 \times 100$, so 30% of 200 is 30×2 which equals 60.

Answer: (D)

4. Since $x = 3$, the side lengths of the figure are 4, 3, 6, and 10.
Thus, the perimeter of the figure is $4 + 3 + 6 + 10 = 23$.
(Alternatively, the perimeter is $x+6+10+(x+1) = 2x+17$. When $x = 3$, this equals $2(3)+17$ or 23.)

Answer: (A)

5. The team earns 2 points for each win, so 9 wins earns $2 \times 9 = 18$ points.
The team earns 0 points for each loss, so 3 losses earns 0 points.
The team earns 1 point for each tie, so 4 ties earns 4 points.
In total, the team earns $18 + 0 + 4 = 22$ points.

Answer: (E)

6. The line representing a temperature of 3° is the horizontal line passing halfway between 2° and 4° on the vertical axis.
There are two data points on this line: one at 2 p.m. and one at 9 p.m.
The required time is 9 p.m.

Answer: (A)

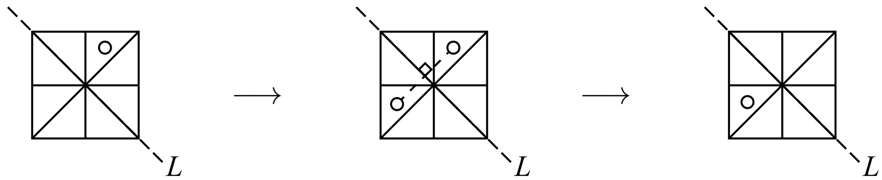
7. *Solution 1*
We rewrite the left side of the given equation as $5 \times 6 \times (2 \times 3) \times (2 \times 3)$.
Since $5 \times 6 \times (2 \times 3) \times (2 \times 3) = 5 \times 6 \times n \times n$, then a possible value of n is 2×3 or 6.

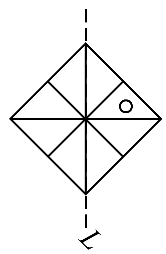
Solution 2
Since $2 \times 2 \times 3 \times 3 \times 5 \times 6 = 5 \times 6 \times n \times n$, then $1080 = 30n^2$ or $n^2 = 36$.
Thus, a possible value of n is 6. (The second possible value for n is -6.)

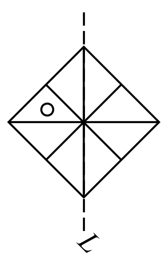
Solution 3
Dividing both sides by 5×6, we obtain $2 \times 2 \times 3 \times 3 = n \times n$, which is equivalent to $n^2 = 36$.
Thus, a possible value of n is 6. (The second possible value for n is -6.)

Answer: (E)

8. *Solution 1*
The square, its 8 pieces and 2 diagonals are symmetrical about line L.
Upon reflection, the circle moves from the triangular above L which is adjacent to and above the second diagonal to the triangular region below L which is adjacent to and above the second diagonal.



Solution 2
We can see that the final position of the figure is given in (D) by first rotating the original figure by 45° clockwise to obtain 

and then reflecting the figure through the now vertical line to obtain 

When we reposition the figure and line back to the original position (by rotating counterclockwise by 45°), we obtain the figure in (D).

ANSWER: (D)

9. We note that $2^2 = 2 \times 2 = 4$, $2^3 = 2^2 \times 2 = 4 \times 2 = 8$, and $2^4 = 2^2 \times 2^2 = 4 \times 4 = 16$.
Therefore, $2^4 - 2^3 = 16 - 8 = 8 = 2^3$.

ANSWER: (D)

10. For $\frac{3}{4} + \frac{4}{\square} = 1$ to be true, we must have $\frac{4}{\square} = 1 - \frac{3}{4} = \frac{1}{4}$.
Since $\frac{1}{4} = \frac{4}{16}$, we rewrite the right side using the same numerator to obtain $\frac{4}{\square} = \frac{4}{16}$.
Therefore, $\square = 16$ makes the equation true.
(We can check that $\frac{3}{4} + \frac{4}{16} = 1$, as required.)

ANSWER: (E)

11. *Solution 1*
The faces not visible on the top cube are labelled with 2, 3 and 6 dots.
The faces not visible on the bottom cube are labelled with 1, 3, 4, and 5 dots.
Thus, the total number of dots on these other seven faces is $2 + 3 + 6 + 1 + 3 + 4 + 5 = 24$.

Solution 2
Since each of the two cubes has faces labelled with 1, 2, 3, 4, 5, and 6 dots, then the total number of dots on the two cubes is $2 \times (1 + 2 + 3 + 4 + 5 + 6) = 2 \times 21 = 42$.
The five visible faces have a total of $4 + 1 + 5 + 6 + 2 = 18$ dots.
Therefore, the seven other faces have a total of $42 - 18 = 24$ dots.

ANSWER: (E)

12. Since each  has length $\frac{2}{3}$, then a strip of three  will have length $3 \times \frac{2}{3} = 2$.
We need two strips of length 2 to get a strip of length 4.
Since 3  make a strip of length 2, then 6  make a strip of length 4.

ANSWER: (A)

13. We rearrange the given subtraction to create the addition statement $45 + 8Y = 1X2$.
Next, we consider the units digits.
From the statement, the sum $5 + Y$ has a units digit of 2. This means that $Y = 7$. This is the only possibility. Therefore, we have $45 + 87 = 1X2$.
But $45 + 87 = 132$, so $X = 3$.
Therefore, $X + Y = 3 + 7 = 10$.
(We can check that $132 - 87 = 45$, as required.)

ANSWER: (C)

14. We simplify first, then substitute $x = 2y$:

$$(x + 2y) - (2x + y) = x + 2y - 2x - y = y - x = y - 2y = -y$$

Alternatively, we could substitute first, then simplify:

$$(x + 2y) - (2x + y) = (2y + 2y) - (2(2y) + y) = 4y - 5y = -y$$

ANSWER: (B)

15. *Solution 1*
Since $\triangle RPS$ is right-angled at P, then by the Pythagorean Theorem, $PR^2 + PS^2 = RS^2$ or $PR^2 + 18^2 = 30^2$.
This gives $PR^2 = 30^2 - 18^2 = 900 - 324 = 576$, from which $PR = 24$, since $PR > 0$.
Since P, S and Q lie on a straight line and RP is perpendicular to this line, then RP is actually a height for $\triangle QRS$ corresponding to base SQ.
Thus, the area of $\triangle QRS$ is $\frac{1}{2}(24)(14) = 168$.

Solution 2
Since $\triangle RPS$ is right-angled at P, then by the Pythagorean Theorem, $PR^2 + PS^2 = RS^2$ or $PR^2 + 18^2 = 30^2$.
This gives $PR^2 = 30^2 - 18^2 = 900 - 324 = 576$, from which $PR = 24$, since $PR > 0$.
The area of $\triangle QRS$ equals the area of $\triangle RPQ$ minus the area of $\triangle RPS$.
Since $\triangle RPQ$ is right-angled at P, its area is $\frac{1}{2}(PR)(PQ) = \frac{1}{2}(24)(18 + 14) = 12(32) = 384$.
Since $\triangle RPS$ is right-angled at P, its area is $\frac{1}{2}(PR)(PS) = \frac{1}{2}(24)(18) = 12(18) = 216$.
Therefore, the area of $\triangle QRS$ is $384 - 216 = 168$.

ANSWER: (B)

16. From the second row, $\triangle + \triangle + \triangle + \triangle = 24$ or $4\triangle = 24$, and so $\triangle = 6$.
From the first row, $\heartsuit + \triangle + \triangle + \heartsuit = 26$ or $2\heartsuit + 2\triangle = 26$.
Since $\triangle = 6$, then $2\heartsuit = 26 - 12 = 14$, and so $\heartsuit = 7$.
From the fourth row, $\square + \heartsuit + \square + \triangle = 33$.
Since $\triangle = 6$ and $\heartsuit = 7$, then $2\square + 7 + 6 = 33$, and so $2\square = 20$ or $\square = 10$.
Finally, from the third row, $\square + \blacklozenge + \heartsuit + \blacklozenge = 27$.
Since $\square = 10$ and $\heartsuit = 7$, then $2\blacklozenge = 27 - 10 - 7 = 10$.
Thus, $\blacklozenge = 5$.

ANSWER: (A)

17. The cube has six identical square faces, each of which is 30 by 30.
Therefore, the surface area of the cube is $6(30^2) = 5400$.
The rectangular solid has two faces that are 20 by 30, two faces that are 20 by L, and two faces that are 30 by L.
Thus, the surface area of the rectangular solid is $2(20)(30) + 2(20L) + 2(30L) = 100L + 1200$.
Since the surface areas of the two solids are equal, then $100L + 1200 = 5400$ or $100L = 4200$, and so $L = 42$.

Answer: (C)

18. The equality of the ratios $x : 4$ and $9 : y$ is equivalent to the equation $\frac{x}{4} = \frac{9}{y}$. (Note that x and y are both positive so we are not dividing by 0.)
This equation is equivalent to the equation $xy = 4(9) = 36$.
Thus, we want to determine the number of pairs of integers (x, y) for which $xy = 36$.
Since the positive divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36, then the desired pairs are

$$(x, y) = (1, 36), (2, 18), (3, 12), (4, 9), (6, 6), (9, 4), (12, 3), (18, 2), (36, 1)$$

There are 9 such pairs.

Answer: (D)

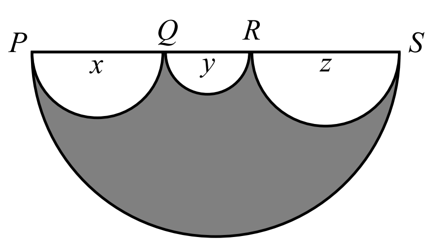
19. We make a chart that lists the possible results for the first spin down the left side, the possible results for the second spin across the top, and the product of the two results in the corresponding cells:

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Since each spin is equally likely to stop on 1, 2, 3, or 4, then each of the 16 products shown in the chart is equally likely.
Since the product 4 appears three times in the table and this is more than any of the other numbers, then it is the product that is most likely to occur.

Answer: (B)

20. The perimeter of the shaded region consists of four pieces: a semi-circle with diameter PS, a semi-circle with diameter PQ, a semi-circle with diameter QR, and a semi-circle with diameter RS.
We note that since a circle with diameter d has circumference equal to πd, then, not including the diameter itself, the length of a semi-circle with diameter d is $\frac{1}{2}\pi d$.
Suppose that the diameter of semi-circle PQ is x, the diameter of semi-circle QR is y, and the diameter of semi-circle RS is z.



We are given that the diameter of the semi-circle PS is 4.
Since $PQ + QR + RS = PS$, then $x + y + z = 4$.
Thus, the perimeter of the shaded region is

$$\tfrac{1}{2}\pi(PQ)+\tfrac{1}{2}\pi(QR)+\tfrac{1}{2}\pi(RS)+\tfrac{1}{2}\pi(PS) = \tfrac{1}{2}\pi x+\tfrac{1}{2}\pi y+\tfrac{1}{2}\pi z+\tfrac{1}{2}\pi(4) = \tfrac{1}{2}\pi(x+y+z+4) = \tfrac{1}{2}\pi(4+4) = 4\pi$$

We want to determine the area of the square whose perimeter equals this perimeter (that is, whose perimeter is 4π).
If a square has perimeter 4π, then its side length is $\frac{1}{4}(4\pi) = \pi$, and so its area is π^2.

Because this problem is multiple choice, then we should get the same answer regardless of the actual lengths of PQ, QR and RS, since we are not told what these lengths are. Therefore, we can assign to these lengths arbitrary values that satisfy the condition $PQ + QR + RS = 4$. For example, if $PQ = QR = 1$ and $RS = 2$, then we can calculate the perimeter of the shaded region to be

$$\tfrac{1}{2}\pi(PQ)+\tfrac{1}{2}\pi(QR)+\tfrac{1}{2}\pi(RS)+\tfrac{1}{2}\pi(PS) = \tfrac{1}{2}\pi(1)+\tfrac{1}{2}\pi(1)+\tfrac{1}{2}\pi(2)+\tfrac{1}{2}\pi(4) = \tfrac{1}{2}\pi(1+1+2+4) = 4\pi$$

and obtain the same answer as above.

Answer: (C)

21. *Solution 1*
The ratio of side lengths in the given 4×6 grid is $4 : 6$ which is equivalent to $2 : 3$.
The ratio of side lengths in the desired 30×45 grid is $30 : 45$ which is also equivalent to $2 : 3$.
Therefore, the 30×45 grid can be built using 2×3 blocks; the resulting grid can be seen as a 15×15 array of 2×3 blocks.
The diagonal line of the 30×45 grid has slope $\frac{30}{45} = \frac{2}{3}$ and so passes through the lower left and upper right corners of each of the diagonal blocks of the 15×15 array of 2×3 blocks. Each of these corners is a lattice point. The diagonal line does not pass through any other lattice point within each of these diagonal blocks, as can be seen in the 4×6 grid.
We note that the upper right corner of a diagonal block is the same point as the lower left corner of the next such block. This means that we have to be careful with our counting.
There are 15 diagonal blocks. The diagonal line passes through the bottom left corner of the grid and passes through the upper right corner of each of the diagonal blocks.
This means that the diagonal line passes through $1 + 15 = 16$ lattice points.

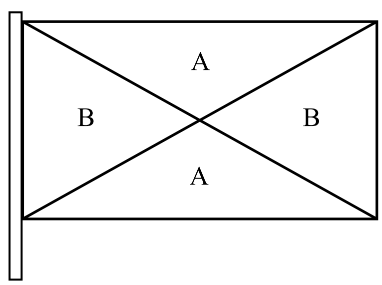
Solution 2
We assign coordinates to the desired 30×45 grid, with the bottom left corner at the origin $(0, 0)$, the vertical side lying along the positive y-axis from $(0, 0)$ to $(0, 30)$ and the horizontal side lying along the positive x-axis from $(0, 0)$ to $(45, 0)$.
The upper right corner of the grid has coordinates $(45, 30)$. The grid lines are the horizontal lines $y = 0$, $y = 1$, …, $y = 29$, $y = 30$ and the vertical lines $x = 0$, $x = 1$, …, $x = 44$, $x = 45$.
The lattice points in the grid are the points with integer coordinates.
Consider the diagonal that joins $(0, 0)$ to $(45, 30)$.
The slope of this line is $\frac{30-0}{45-0} = \frac{2}{3}$.
Since the diagonal passes through the origin, its equation is $y = \frac{2}{3}x$.
Thus, we must determine the number of lattice points that lie on the line $y = \frac{2}{3}x$ with x-coordinates between $x = 0$ and $x = 45$, inclusive.
Suppose that (a, b) is a lattice point on the line; that is, a and b are both integers.
Since $b = \frac{2}{3}a$, then for b to be an integer, it must be the case that a is a multiple of 3.

The multiples of 3 between 0 and 45, inclusive, are $0, 3, 6, \ldots, 42, 45$.
Since $45 = 15(3)$, then there are 16 numbers in the list.
Each of these values for a gives an integer for b and so gives a lattice point on the line.
Thus, there are 16 lattice points on the diagonal.

Answer: (B)

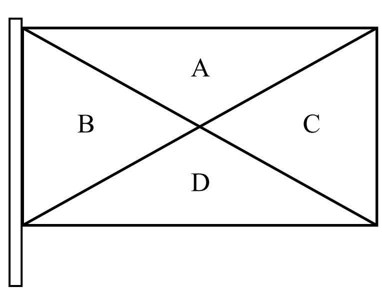
22. Since there are four triangular sections in each flag, then at most four colours can be used in a single flag.
Since no two adjacent triangles are the same colour, then at least two colours must be used. (For example, the sections Top and Left must be different colours.)
Therefore, the number of colours used is 2, 3 or 4.
We count the number of possible flags in each case.

Case 1: 2 colours
We call the colours A and B.
Assign the colour A to Top.
Since Left and Right cannot be coloured A and there is only one other colour, then Left and Right are both coloured B.
Bottom cannot be coloured B (since it shares an edge with Left and Right) so must be coloured A.
This gives us:



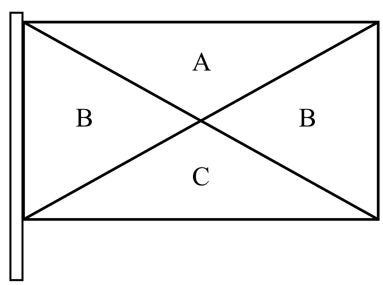
This configuration does not violate the given rule.
There are 5 possible colours for A (red, white, blue, green, purple).
For each of these 5 choices, there are 4 possible colours for B (any of the remaining 4 colours).
Therefore, there are $5(4) = 20$ possible flags in this case.

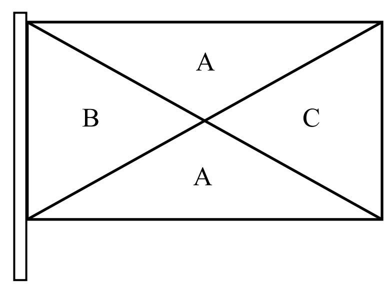
Case 2: 4 colours
We call the colours A, B, C, and D.
Since there are 4 sections and 4 colours used, then each section is a different colour.
We label them as shown:



This configuration does not violate the given rule.
There are 5 possible colours for A. For each of these 5 choices, there are 4 possible colours for B. For each of these combinations, there are 3 possible colours for C and 2 possible colours for D.
Therefore, there are $5(4)(3)(2) = 120$ possible flags in this case.

<u>Case 3: 3 colours</u>
We call the colours A, B and C.
Assign the colour A to Top.
Since Left cannot be coloured A, we assign it the colour B.
Section Right cannot be coloured A, so could be B or C.
If Right is coloured B, then in order to use all three colours, Bottom must be coloured C.
If Right is coloured C, then Bottom (which shares an edge with each of Left and Right) must be coloured A.
This gives two possible configurations:



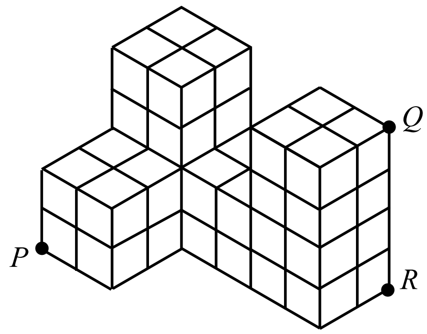


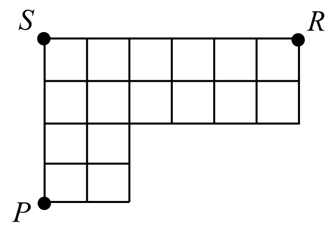
Neither configuration violates the given rule.
In each configuration, there are 5 possible colours for A. For each of these 5 choices, there are 4 possible colours for B. For each of these combinations, there are 3 possible colours for C.
Since there are two such configurations, then there are $2(5)(4)(3) = 120$ possible flags in this case.

In total, there are $20 + 120 + 120 = 260$ possible flags.

ANSWER: (E)

23. First, we calculate the distance PQ in terms of n.
Suppose that R is the point at the bottom of the solid directly under Q and S is the back left bottom corner of the figure (unseen in the problem's diagram).
Since QR is perpendicular to the bottom surface of the solid, then $\triangle PRQ$ is right-angled at R and so $PQ^2 = PR^2 + RQ^2$.
We note also that $\triangle PSR$ is right-angled at S, since the solid is made up of cubes.
Therefore, $PR^2 = PS^2 + SR^2$.





This tells us that $PQ^2 = PS^2 + SR^2 + RQ^2$.
Note that the distance PS equals four times the edge length of one of the small cubes, SR is six times the edge length, and RQ is four times the edge length.
Since the edge length is $\sqrt{n}$, then $PS = 4\sqrt{n}$, $SR = 6\sqrt{n}$, and $RQ = 4\sqrt{n}$.
Thus, $PQ^2 = (4\sqrt{n})^2 + (6\sqrt{n})^2 + (4\sqrt{n})^2 = 16n + 36n + 16n = 68n$.
Therefore, $PQ = \sqrt{68n}$.
We need to determine the smallest positive integer n for which $\sqrt{68n}$ is an integer.
For $\sqrt{68n}$ to be an integer, $68n$ must be a perfect square.
Note that $68n = 4(17n) = 2(2)(17)(n)$.
A positive integer is a perfect square whenever each prime factor occurs an even number of times.
Thus, for $68n$ to be a perfect square, n must include a factor of 17.
The smallest possible such n is in fact $n = 17$.
When $n = 17$, we have $68n = 2(2)(17)(17) = (2 \times 17)^2$ which is a perfect square.
Therefore, the smallest positive integer n for which the distance PQ is an integer is $n = 17$.
(Once we determined that $PQ = \sqrt{68n}$, we could have tried the five possible answers from smallest to largest until we obtained an integer value for PQ.)

ANSWER: (A)

24. As Nadia walks from N to G, suppose that she walks x km uphill and y km downhill. We are told that she walks 2.5 km on flat ground.
This means that when she walks from G to N, she will walk x km downhill, y km uphill, and again 2.5 km on flat ground. This is because downhill portions become uphill portions on the return trip, while uphill portions become downhill portions on the return trip.
We are told that Nadia walks at 5 km/h on flat ground, 4 km/h uphill, and 6 km/h downhill.
Since speed $= \dfrac{\text{distance}}{\text{time}}$, then distance $=$ speed $\times$ time and time $= \dfrac{\text{distance}}{\text{speed}}$.
Thus, on her trip from N to G, her time walking uphill is $\dfrac{x}{4}$ hours, her time walking downhill is $\dfrac{y}{6}$ hours, and her time walking on flat ground is $\dfrac{2.5}{5}$ hours.
Since it takes her 1 hour and 36 minutes (which is 96 minutes or $\dfrac{96}{60}$ hours), then

$$\frac{x}{4}+\frac{y}{6}+\frac{2.5}{5}=\frac{96}{60}$$

A similar analysis of the return trip gives

$$\frac{x}{6}+\frac{y}{4}+\frac{2.5}{5}=\frac{99}{60}$$

We are asked for the total distance from N to G, which equals $x+y+2.5$ km. Therefore, we need to determine $x+y$.
We add the two equations above and simplify to obtain

$$\begin{aligned}
\frac{x}{4}+\frac{x}{6}+\frac{y}{6}+\frac{y}{4}+1 &= \frac{195}{60}\\
x\left(\frac{1}{4}+\frac{1}{6}\right)+y\left(\frac{1}{4}+\frac{1}{6}\right) &= \frac{135}{60}\\
\frac{5}{12}x+\frac{5}{12}y &= \frac{9}{4}\\
x+y &= \frac{12}{5}\left(\frac{9}{4}\right)
\end{aligned}$$

Thus, $x+y=\dfrac{108}{20}=\dfrac{27}{5}=5.4$ km.
Finally, the distance from N to G is $5.4+2.5=7.9$ km.

Answer: (E)

25. First, we simplify $\dfrac{2009}{2014}+\dfrac{2019}{n}$ to obtain $\dfrac{2009n+2014(2019)}{2014n}$ or $\dfrac{2009n+4\,066\,266}{2014n}$.

Since $\dfrac{2009n+4\,066\,266}{2014n}=\dfrac{a}{b}$ and $\dfrac{a}{b}$ is in lowest terms, then $2009n+4\,066\,266=ka$ and $2014n=kb$ for some positive integer k.

Since $2009n+4\,066\,266=ka$, then if a is a multiple of 1004, we must have that $2009n+4\,066\,266$ is a multiple of 1004 as well.

Therefore, we determine the values of n for which $2009n+4\,066\,266$ is divisible by 1004 and from this list find the smallest such n that makes a divisible by 1004. (Note that even if $2009n+4\,066\,266$ is divisible by 1004, it might not be the case that a is divisible by 1004, since reducing the fraction $\dfrac{2009n+4\,066\,266}{2014n}$ might eliminate some or all of the prime factors of 1004 in the numerator.)

We note that $2008=2\times 1004$ and $4\,066\,200=4050\times 1004$, so we write

$$2009n+4\,066\,266=(2008n+4\,066\,200)+(n+66)=1004(2n+4050)+(n+66)$$

(2008 and 4 066 200 are the largest multiples of 1004 less than 2009 and 4 066 266, respectively.)

Since $1004(2n+4050)$ is a multiple of 1004, then $2009n+4\,066\,266$ is a multiple of 1004 whenever $n+66$ is a multiple of 1004, say $1004m$ (that is, $n+66=1004m$).

Thus, $2009n+4\,066\,266$ is a multiple of 1004 whenever $n=1004m-66$ for some positive integer m.

These are the values of n for which the expression $2009n+4\,066\,266$ is divisible by 1004. Now we need to determine the smallest of these n for which a is divisible by 1004.

When $m=1$, we have $n=1004-66=938$. In this case,

$$\frac{2009n+4\,066\,266}{2014n}=\frac{2009(938)+4\,066\,2006}{2014(938)}=\frac{5\,950\,708}{1\,889\,132}=\frac{1\,487\,677}{472\,283}$$

where in the last step we divided a common factor of 4 out of the numerator and denominator. Regardless of whether this last fraction is in lowest terms, its numerator is odd and so $\dfrac{a}{b}$ (the equivalent lowest terms fraction) will also have a odd, so a cannot be divisible by 1004. So, we try the next value of m.

When $m=2$, we have $n=2008-66=1942$. In this case,

$$\frac{2009n+4\,066\,266}{2014n}=\frac{2009(1942)+4\,066\,2006}{2014(1942)}=\frac{7\,967\,744}{3\,911\,188}=\frac{1\,991\,936}{977\,797}$$

where in the last step we divided a common factor of 4 out of the numerator and denominator. Now $1004=4\times 251$ and 251 is a prime number. (251 is a prime number because it is not divisible by any of the primes 2, 3, 5, 7, 11, and 13, which are all of the primes less than $\sqrt{251}$.)

Now $1\,991\,936=1984\times 1004$ so is a multiple of 1004, and 977 797 is not divisible by 4 or by 251. This means that when $\dfrac{1\,991\,936}{977\,797}$ is written in lowest terms as $\dfrac{a}{b}$, then a will be divisible by 1004.

Therefore, the smallest value of n with the desired property is $n=1942$, which has a sum of digits equal to $1+9+4+2=16$.

Answer: (A)

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2013 Pascal Contest

(Grade 9)

Thursday, February 21, 2013
(in North America and South America)

Friday, February 22, 2013
(outside of North America and South America)

Solutions

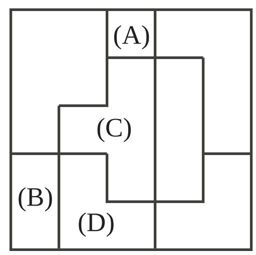
1. Simplifying first inside the brackets, $(4 + 44 + 444) \div 4 = 492 \div 4 = 123$.

   Answer: (B)

2. Since Jing purchased 8 identical items and the total cost was \$26, then to obtain the cost per item, she divides the total cost by the number of items.
   Thus, the answer is $26 \div 8$, or (A).

   Answer: (A)

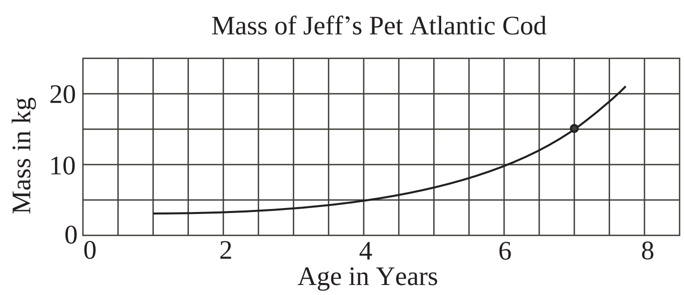
3. Each of pieces (A), (B), (C), and (D) occurs in the diagram.



   Piece (E) does not occur in the diagram.

   Answer: (E)

4. A mass of 15 kg is halfway between 10 kg and 20 kg on the vertical axis.
   The point where the graph reaches 15 kg is halfway between 6 and 8 on the horizontal axis.



   Therefore, the cod is 7 years old when its mass is 15 kg.

   Answer: (B)

5. Expanding and simplifying,

$$1^3 + 2^3 + 3^3 + 4^3 = 1 \times 1 \times 1 + 2 \times 2 \times 2 + 3 \times 3 \times 3 + 4 \times 4 \times 4 = 1 + 8 + 27 + 64 = 100$$

   Since $100 = 10^2$, then $1^3 + 2^3 + 3^3 + 4^3 = 10^2$.

   Answer: (C)

6. Since Erin walks $\frac{3}{5}$ of the way home in 30 minutes, then she walks $\frac{1}{5}$ of the way at the same rate in 10 minutes.
   She has $1 - \frac{3}{5} = \frac{2}{5}$ of the way left to walk. This is twice as far as $\frac{1}{5}$ of the way.
   Since she continues to walk at the same rate and $\frac{1}{5}$ of the way takes her 10 minutes, then it takes her $2 \times 10 = 20$ minutes to walk the rest of the way home.

   Answer: (B)

7. Simplifying, $(\sqrt{100}+\sqrt{9}) \times (\sqrt{100}-\sqrt{9}) = (10+3) \times (10-3) = 13 \times 7 = 91$.

   Answer: (A)

8. Since $PQRS$ is a rectangle, then $QR = PS = 6$.
   Therefore, $UR = QR - QU = 6 - 2 = 4$.
   Since $PQRS$ is a rectangle and TU is perpendicular to QR, then TU is parallel to and equal to SR, so $TU = 3$.
   By the Pythagorean Theorem, since $TR > 0$, then

   $$TR = \sqrt{TU^2 + UR^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

   Thus, $TR = 5$. (We could also notice the 3-4-5 right-angled triangle.)

   Answer: (C)

9. Since Owen uses an average of 1 L to drive 12.5 km, then it costs Owen \$1.20 in gas to drive 12.5 km.
   To drive 50 km, he drives 12.5 km a total of $50 \div 12.5 = 4$ times.
   Therefore, it costs him $4 \times \$1.20 = \$4.80$ in gas to drive 50 km.

   Answer: (A)

10. There are six times that can be made using each of the digits 2, 3 and 5 exactly once: 2:35, 2:53, 3:25, 3:52, 5:23, and 5:32.
    The first of these that occurs after 3:52 is 5:23.
    From 3:52 to 4:00, 8 minutes pass.
    From 4:00 to 5:00, 60 minutes pass.
    From 5:00 to 5:23, 23 minutes pass.
    Therefore, from 3:52 to 5:23, which is the next time that uses the digits 2, 3, and 5 each exactly once, a total of $8 + 60 + 23 = 91$ minutes pass.

    Answer: (D)

11. Since the sequence repeats every 4 symbols and since $13 \times 4 = 52$, then the 52nd symbol is the last symbol in a sequence of 4 symbols.
    Also, the first 52 symbols represent 13 sequences of these 4 symbols.
    Each sequence of 4 symbols includes 2 $\heartsuit$s, so the first 52 symbols include $13 \times 2 = 26$ $\heartsuit$s.
    Finally, the 53rd symbol in the pattern is the first of a sequence of 4, so is also $\heartsuit$.
    Therefore, the first 53 symbols include $26 + 1 = 27$ $\heartsuit$s.

    Answer: (C)

12. Since $x = 11$, $y = -8$ and $2x - 3z = 5y$, then $2 \times 11 - 3z = 5 \times (-8)$ or $22 - 3z = -40$.
    Therefore, $3z = 22 + 40 = 62$ and so $z = \frac{62}{3}$.

    Answer: (D)

13. The original set contains 11 elements whose sum is 66.
    When one number is removed, there will be 10 elements in the set.
    For the average of these elements to be 6.1, their sum must be $10 \times 6.1 = 61$.
    Since the sum of the original 11 elements is 66 and the sum of the remaining 10 elements is 61, then the element that must be removed is $66 - 61 = 5$.

    Answer: (B)

14. Since $\angle QTS = 76^\circ$ and $\triangle QTS$ has $QS = QT$, then $\angle QST = \angle QTS = 76^\circ$.
Since the angles in $\triangle QTS$ add to 180°, then

$$\angle SQT = 180^\circ - \angle QTS - \angle QST = 180^\circ - 76^\circ - 76^\circ = 28^\circ$$

Since PQR is a straight line segment, then $\angle PQS + \angle SQT + \angle TQR = 180^\circ$.
Thus, $x^\circ + 28^\circ + 3x^\circ = 180^\circ$.
This gives $4x + 28 = 180$ or $4x = 152$ and so $x = 38$.

Answer: (B)

15. We note that $64 = 4 \times 4 \times 4$.
Thus, $64^2 = 64 \times 64 = 4 \times 4 \times 4 \times 4 \times 4 \times 4$.
Since $4^n = 64^2$, then $4^n = 4 \times 4 \times 4 \times 4 \times 4 \times 4$ and so $n = 6$.

Answer: (C)

16. *Solution 1*
If $x = 1$, then $3x + 1 = 4$, which is an even integer.
In this case, the five given choices are

(A) $x + 3 = 4$ (B) $x - 3 = -2$ (C) $2x = 2$ (D) $7x + 4 = 11$ (E) $5x + 3 = 8$

Of these, the only odd integer is (D). Therefore, since $x = 1$ satisfies the initial criteria, then (D) must be the correct answer as the result must be true no matter what integer value of x is chosen that makes $3x + 1$ even.

Solution 2
If x is an integer for which $3x+1$ is even, then $3x$ is odd, since it is 1 less than an even integer.
If $3x$ is odd, then x must be odd (since if x is even, then $3x$ would be even).
If x is odd, then $x + 3$ is even (odd plus odd equals even), so (A) cannot be correct.
If x is odd, then $x - 3$ is even (odd minus odd equals even), so (B) cannot be correct.
If x is odd, then $2x$ is even (even times odd equals even), so (C) cannot be correct.
If x is odd, then $7x$ is odd (odd times odd equals odd) and so $7x + 4$ is odd (odd plus even equals odd).
If x is odd, then $5x$ is odd (odd times odd equals odd) and so $5x + 3$ is even (odd plus odd equals even), so (E) cannot be correct.
Therefore, the one expression which must be odd is $7x + 4$.

Answer: (D)

17. Since 40% of the songs on the updated playlist are Country, then the remaining $100\% - 40\%$ or 60% must be Hip Hop or Pop songs.
Since the ratio of Hip Hop songs to Pop songs does not change, then 65% of this remaining 60% must be Hip Hop songs.
Overall, this is $65\% \times 60\% = 0.65 \times 0.6 = 0.39 = 39\%$ of the total number of songs on the playlist.

Answer: (E)

18. The area of the shaded region is equal to the area of square $PQRS$ minus the combined areas of the four unshaded regions inside the square.
Since square $PQRS$ has side length 2, its area is $2^2 = 4$.
Since $PQRS$ is a square, then the angle at each of P, Q, R, and S is 90°.

Since each of P, Q, R, and S is the centre of a circle with radius 1, then each of the four unshaded regions inside the square is a quarter circle of radius 1. (A central angle of 90° gives a quarter of a circle.)
Thus, the combined areas of the four unshaded regions inside the square equals four quarters of a circle of radius 1, or the area of a whole circle of radius 1.
This area equals $\pi(1)^2 = \pi$.
Therefore, the shaded region equals $4 - \pi$.

ANSWER: (E)

19. Since the flag shown is rectangular, then its total area is its height multiplied by its width, or $h \times 2h = 2h^2$.
Since the flag is divided into seven stripes of equal height and each stripe has equal width, then the area of each stripe is the same.
Since the four shaded strips have total area 1400 cm^2, then the area of each strip is $1400 \div 4 = 350\text{ cm}^2$.
Since the flag consists of 7 strips, then the total area of the flag is $350\text{ cm}^2 \times 7 = 2450\text{ cm}^2$.
Since the flag is h by $2h$, then $2h^2 = 2450\text{ cm}^2$ or $h^2 = 1225\text{ cm}^2$.
Therefore, $h = \sqrt{1225\text{ cm}^2} = 35$ cm (since $h > 0$).
The height of the flag is 35 cm.

ANSWER: (C)

20. We make a chart that shows the possible combinations of the number that Sam rolls and the number that Tyler rolls. Since Sam rolls a fair four-sided die and Tyler rolls a fair six-sided die, then there are 4 possible numbers that Sam can roll and 6 possible numbers that Tyler can roll and so there are $4 \times 6 = 24$ equally likely combinations in total. In the chart, we put a Y when Sam's roll is larger than Tyler's and an N otherwise.

Tyler's roll

		1	2	3	4	5	6
Sam's roll	1	N	N	N	N	N	N
	2	Y	N	N	N	N	N
	3	Y	Y	N	N	N	N
	4	Y	Y	Y	N	N	N

Since there are 24 equally likely possibilities and Sam's roll is larger in 6 of these, then the probability that Sam's roll is larger than Tyler's is $\frac{6}{24} = \frac{1}{4}$.

ANSWER: (E)

21. We begin by factoring the given integer into prime factors.
Since 636 405 ends in a 5, it is divisible by 5, so

$$636\,405 = 5 \times 127\,281$$

Since the sum of the digits of 127 281 is a multiple of 3, then it is a multiple of 3, so

$$636\,405 = 5 \times 3 \times 42\,427$$

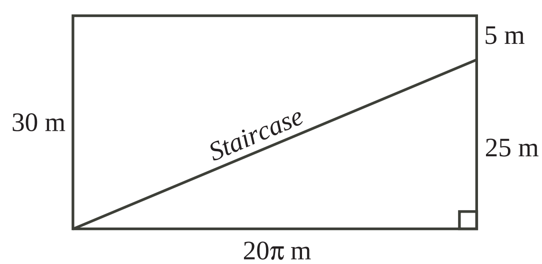
The new quotient (42 427) is divisible by 7 (can you see this without using a calculator?), which gives

$$636\,405 = 5 \times 3 \times 7 \times 6061$$

We can proceed by systematic trial and error to see if 6061 is divisible by 11, 13, 17, 19, and so on. After some work, we can see that $6061 = 11 \times 551 = 11 \times 19 \times 29$.
Therefore, $636\,405 = 3 \times 5 \times 7 \times 11 \times 19 \times 29$.
We want to rewrite this as the product of three 2-digit numbers.
Since $3 \times 5 \times 7 = 105$ which has three digits, and the product of any three of the six prime factors of $636\,405$ is at least as large as this, then we cannot take the product of three of these prime factors to form a two-digit number.
Thus, we have to combine the six prime factors in pairs.
The prime factor 29 cannot be multiplied by any prime factor larger than 3, since $29 \times 3 = 87$ which has two digits, but $29 \times 5 = 145$, which has too many digits.
This gives us $636\,405 = 87 \times 5 \times 7 \times 11 \times 19$.
The prime factor 19 can be multiplied by 5 (since $19 \times 5 = 95$ which has two digits) but cannot be multiplied by any prime factor larger than 5, since $19 \times 7 = 133$, which has too many digits.
This gives us $636\,405 = 87 \times 95 \times 7 \times 11 = 87 \times 95 \times 77$.
The sum of these three 2-digit divisors is $87 + 95 + 77 = 259$.

Answer: (A)

22. To calculate the total distance, we add the length of the vertical ladder (5 m) to the length of the spiral staircase.
The spiral staircase wraps once around the tower.
Since the tower has radius 10 m, then its circumference is $2 \times \pi \times 10 = 20\pi$ m.
We can thus "unwrap" the outside of the tower to form a rectangle of width 20π m and height 30 m.
Since the staircase has a constant slope, then the staircase becomes a straight line on the unwrapped tower.
Since the ladder accounts for the final 5 m of the height of the tower and the tower has total height 30 m, then the top of the spiral staircase is $30 - 5 = 25$ m above its base.



We can thus calculate the length of the staircase (which is positive), using the Pythagorean Theorem, to be

$$\sqrt{(20\pi \text{ m})^2 + (25 \text{ m})^2} \approx 67.62 \text{ m}$$

The total distance along the staircase and up the ladder is approximately $5 + 67.62 \approx 72.62$ m.
Of the given choices, this is closest to 72.6 m.

Answer: (A)

23. Suppose that the five distinct numbers that Joshua chooses are V, W, X, Y, Z, and that $V < W < X < Y < Z$.
We want to assign these to p, q, r, s, t so that $p < s$ and $q < s$ and $r < t$ and $s < t$.
First, we note that t must be the largest of p, q, r, s, t. This is because $r < t$ and $s < t$, and because $p < s$ and $q < s$, we get $p < s < t$ and $q < s < t$, so $p < t$ and $q < t$.

Since t is the largest, then Z must be t.

Now neither p nor q can be the second largest of the numbers (which is Y), since p and q are both smaller than s and t.
Therefore, there are two cases: $Y = r$ or $Y = s$.

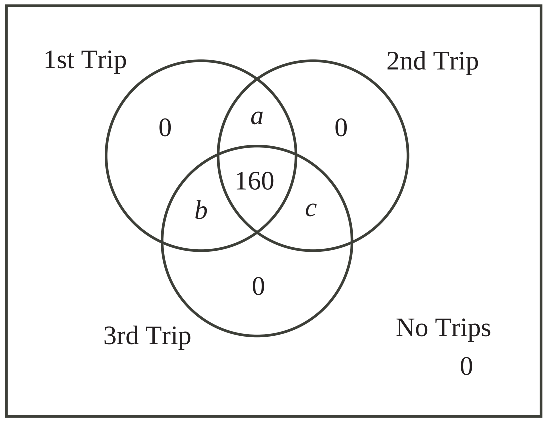
Case 1: $Y = r$
We have $Y = r$ and $Z = t$.
This leaves V, W, X (which satisfy $V < W < X$) to be assigned to p, q, s (which satisfy $p < s$ and $q < s$).
Since X is the largest of V, W, X and s is the largest of p, q, s, then $X = s$.
This leaves V, W to be assigned to p, q.
Since there is no known relationship between p and q, then there are 2 possibilities: either $V = p$ and $W = q$, or $V = q$ and $W = p$.
Therefore, if $Y = r$, there are 2 possible ways to assign the numbers.

Case 2: $Y = s$
We have $Y = s$ and $Z = t$.
This leaves V, W, X (which satisfy $V < W < X$) to be assigned to p, q, r.
There is no known relationship between p, q, r.
Therefore, there are 3 ways to assign one of V, W, X to p.
For each of these 3 ways, there are 2 ways of assigning one of the two remaining numbers to q.
For each of these 3×2 ways, there is only 1 choice for the number assigned to r.
Overall, this gives $3 \times 2 \times 1 = 6$ ways to do this assignment. (The 6 ways to assign V, W, X to p, q, r, respectively, are VWX, VXW, WVX, WXV, XVW, XWV.)
Therefore, if $Y = s$, there are 6 possible ways to assign the numbers.

Having examined the two possibilities, there are $2+6 = 8$ different ways to assign the numbers.

ANSWER: (D)

24. Let x be the total number of students at Pascal H.S.
Let a be the total number of students who went on both the first trip and the second trip, but did not go on the third trip.
Let b be the total number of students who went on both the first trip and the third trip, but did not go on the second trip.
Let c be the total number of students who went on both the second trip and the third trip, but did not go on the first trip.
We note that no student went on one trip only, and that 160 students went on all three trips.
We draw a Venn diagram:



Since the total number of students at the school is x and each region in the diagram is labelled separately, then

$$x = a + b + c + 160$$

From the given information:

– 50% of the students in the school went on the first trip, so $0.5x = a + b + 160$
– 80% of the students in the school went on the second trip, so $0.8x = a + c + 160$
– 90% of the students in the school went on the third trip, so $0.9x = b + c + 160$

Combining all of this information,

$$\begin{aligned} x &= a + b + c + 160 \\ 2x &= 2a + 2b + 2c + 160 + 160 \\ 2x &= (a + b + 160) + (a + c + 160) + (b + c) \\ 2x &= 0.5x + 0.8x + (0.9x - 160) \\ 2x &= 2.2x - 160 \\ 160 &= 0.2x \\ x &= 800 \end{aligned}$$

Therefore, there are 800 students at Pascal High School.

Answer: (D)

25. We refer to the two sequences as the GEB sequence and the difference sequence.
Since the GEB sequence is increasing and since each positive integer that does not occur in the GEB sequence must occur in the difference sequence, then each positive integer less than 12 except 1, 3, 7 (a total of 8 positive integers) must occur in the difference sequence.
Since the difference sequence is increasing, then these 8 positive integers occur in increasing order.
Therefore, the difference sequence begins $2, 4, 5, 6, 8, 9, 10, 11, \ldots$.
This allows us to continue the GEB sequence using the integers in the difference sequence as the new differences between consecutive terms. For example, since the fourth term in the GEB sequence is 12 and the fourth difference from the difference sequence is 6, then the fifth term in the GEB sequence is $12 + 6 = 18$.
Continuing in this way, we can write out more terms in the GEB sequence:

$$1, 3, 7, 12, 18, 26, 35, 45, 56, \ldots$$

In a similar way, each positive integer less than 26 except 1, 3, 7, 12, 18 (a total of 20 positive integers) must occur in the difference sequence.
Since the difference sequence is increasing, then these 20 positive integers occur in increasing order.
Therefore, the difference sequence begins

$$2, 4, 5, 6, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, \ldots$$

As above, we can write out more terms in the GEB sequence:

$$1, 3, 7, 12, 18, 26, 35, 45, 56, 69, 83, 98, 114, 131, \ldots$$

Again, every positive integer less than 114, with the exception of the 12 integers before 114 in the GEB sequence, must occur in the difference sequence, and these integers ($113 - 12 = 101$ of them in all) must occur in increasing order.
We need to determine the 100th term in the GEB sequence. We can do this by taking the first term in the GEB sequence (that is, 1) and adding to it the first 99 terms in the difference sequence. This is because the terms in the difference sequence are the differences between consecutive terms in the GEB sequence, so adding these to the first term allows us to move along the sequence.
From above, we see that 113 is 101st term in the difference sequence, so 112 is the 100th term, and 111 is the 99th term.
Since the first 99 terms in the difference sequence consist of most of the integers from 2 to 111, with the exception of a few (those in the GEB sequence), we can find the sum of these terms by adding all of the integers from 2 to 111 and subtracting the relevant integers.
Therefore, the 100th term in the GEB sequence equals

$$\begin{aligned}
&1 + (2 + 4 + 5 + 6 + 8 + \cdots + 109 + 110 + 111) \\
&\quad = 1 + (1 + 2 + 3 + 4 + \cdots + 109 + 110 + 111) \\
&\qquad\qquad -(1 + 3 + 7 + 12 + 18 + 26 + 35 + 45 + 56 + 69 + 83 + 98) \\
&\quad = 1 + \tfrac{1}{2}(111)(112) - (453) \\
&\qquad\qquad \text{(using the fact that the sum of the first } n \text{ positive integers is } \tfrac{1}{2}n(n+1)) \\
&\quad = 1 + 111(56) - 453 \\
&\quad = 5764
\end{aligned}$$

Thus, the 100th term in the GEB sequence is 5764.

Answer: (E)

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2012 Pascal Contest

(Grade 9)

Thursday, February 23, 2012
(in North America and South America)

Friday, February 24, 2012
(outside of North America and South America)

Solutions

1. Using the correct order of operations, $\dfrac{1+(3\times 5)}{2}=\dfrac{1+15}{2}=\dfrac{16}{2}=8$.

   ANSWER: (D)

2. Of the 200 students, 50% (or one-half) of the students chose Greek food.
   Since one-half of 200 is 100, then 100 students chose Greek food.

   ANSWER: (E)

3. Since $\frac{60}{8}=60\div 8=7.5$, then this choice is not equal to a whole number.
   Note as well that $\frac{60}{12}=5$, $\frac{60}{5}=12$, $\frac{60}{4}=15$, and $\frac{60}{3}=20$ are all whole numbers.

   ANSWER: (B)

4. If 7:30 a.m. was 16 minutes ago, then it is currently $30+16=46$ minutes after 7:00 a.m., or 7:46 a.m.
   Since 8:00 a.m. is 60 minutes after 7:00 a.m., then it will be 8:00 a.m. in $60-46=14$ minutes.

   ANSWER: (B)

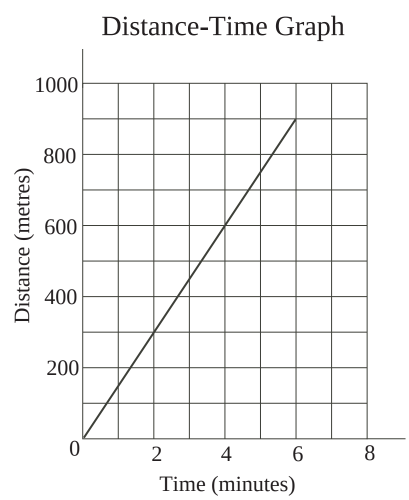
5. First, we write out the powers of 10 in full to obtain $8\times 100\,000+4\times 1000+9\times 10+5$.
   Simplifying, we obtain $800\,000+4000+90+5$ or $804\,095$.

   ANSWER: (A)

6. We write the list in increasing order: $0.023, 0.032, 0.203, 0.302, 0.320$.
   The difference between the largest and smallest of these numbers is $0.320-0.023=0.297$.

   ANSWER: (E)

7. If Anna walked 600 metres in 4 minutes, then she walked $\frac{600}{4}=150$ metres each minute.
   Therefore, in 6 minutes, she walked $6\times 150=900$ metres.

   ANSWER: (D)

8. *Solution 1*
   The segment of the ruler between each integer marking is divided into 4 equal pieces (that is, is divided into quarters).
   Therefore, the point Q is at $2+\frac{3}{4}=2\frac{3}{4}$, and P is at $\frac{2}{4}=\frac{1}{2}$.
   Thus, the length of PQ is $2\frac{3}{4}-\frac{1}{2}=\frac{11}{4}-\frac{2}{4}=\frac{9}{4}=2\frac{1}{4}$.
   Writing this as a decimal, we obtain 2.25.

   Solution 2
   The segment of the ruler between each integer marking is divided into 4 equal pieces (that is, is divided into quarters).
   There are 9 of these pieces between P and Q.
   Therefore, the length of PQ is $9\times\frac{1}{4}=\frac{9}{4}=2.25$.

   ANSWER: (A)

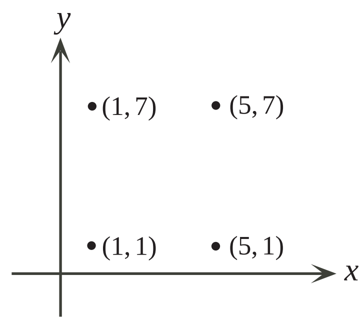
9. Substituting $y = 1$ into the second equation, we obtain $4x - 2(1) + 3 = 3x + 3(1)$.
Simplifying, we obtain $4x - 2 + 3 = 3x + 3$ or $4x + 1 = 3x + 3$.
Therefore, $4x - 3x = 3 - 1$ or $x = 2$.

Answer: (C)

10. Since Emily is a doctor and there are 5 doctors and 3 nurses aside from Emily at the hospital, then there are 6 doctors and 3 nurses in total.
Since Robert is a nurse, then aside from Robert, there are 6 doctors and 2 nurses.
Therefore, $d = 6$ and $n = 2$, so $dn = 12$.

Answer: (B)

11. Since the given three points already form a right angle, then the fourth vertex of the rectangle must be vertically above the point $(5, 1)$ and horizontally to the right of $(1, 7)$.
Therefore, the x-coordinate of the fourth vertex is 5 and the y-coordinate is 7.
Thus, the coordinates of the fourth vertex are $(5, 7)$.



Answer: (C)

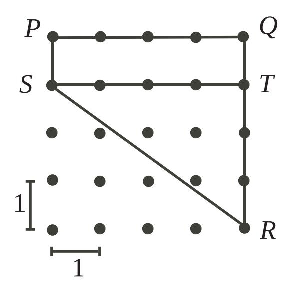
12. We can rephrase the given information by saying that each of the seven students paid \$3.71 and some of the students paid an extra \$0.01.
Since $7 \times \$3.71 = \$25.97$ and the pizza cost \$26.00 in total, then the students who paid the extra \$0.01 each must make up the final \$0.03 of the cost of the pizza.
Therefore, 3 students each paid an additional \$0.01 and so paid \$3.72 in total.

Answer: (B)

13. Using the definition of the operation, $g\nabla 6 = 45$ gives $g^2 - 6^2 = 45$.
Thus, $g^2 = 45 + 36 = 81$.
Since $g > 0$, then $g = \sqrt{81} = 9$.

Answer: (E)

14. The perimeter of quadrilateral $PQRS$ equals $PQ + QR + RS + SP$.
Since the dots are spaced 1 unit apart horizontally and vertically, then $PQ = 4$, $QR = 4$, and $PS = 1$.
Thus, the perimeter equals $4 + 4 + RS + 1$ which equals $RS + 9$.
We need to determine the length of RS.
If we draw a horizontal line from S to point T on QR, we create a right-angled triangle STR with $ST = 4$ and $TR = 3$.
By the Pythagorean Theorem, $RS^2 = ST^2 + TR^2 = 4^2 + 3^2 = 25$.
Since $RS > 0$, then $RS = \sqrt{25} = 5$.
Thus, the perimeter of quadrilateral $PQRS$ is $5 + 9 = 14$.



Answer: (C)

15. *Solution 1*
Suppose that the team has r red helmets.
Since the team has 6 more red helmets than blue helmets, then the team has $r - 6$ blue helmets.
Since the ratio of the number of red helmets to the number of blue helmets is $5 : 3$, then $\frac{r}{r-6} = \frac{5}{3}$ and so $3r = 5(r - 6)$ or $3r = 5r - 30$.
Therefore, $2r = 30$ or $r = 15$.
Thus, the team has 15 red helmets, 9 blue helmets, and $15 + 9 = 24$ helmets in total.

Solution 2
Since the ratio of the number of red helmets to the number of blue helmets equals $5 : 3$, then we can try multiplying both parts of this ratio by small numbers to see if we can obtain an equivalent ratio where the two parts differ by 6.
Multiplying by 2, we obtain $5 : 3 = 10 : 6$, which doesn't have the desired property.
Multiplying by 3, we obtain $5 : 3 = 15 : 9$. Since $15 - 9 = 6$, then we have found the correct number of red and blue helmets.
Therefore, the team has 15 red helmets, 9 blue helmets, and $15 + 9 = 24$ helmets in total.
(If we continue to multiply this ratio by larger numbers, the difference between the two parts gets bigger, so cannot equal 6 in a different case. In other words, the answer is unique.)

Answer: (C)

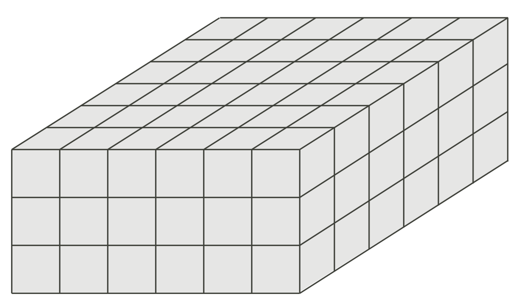
16. The quilt consists of 25 identical squares.
Of the 25 squares, 4 are entirely shaded, 8 are shaded with a single triangle that covers half of the square, and 4 are shaded with two triangles that each cover a quarter of the square.
Therefore, the shading is equivalent to the area of $4 + 8 \times \frac{1}{2} + 4 \times 2 \times \frac{1}{4} = 10$ squares.
As a percentage, the shading is $\frac{10}{25} \times 100\% = 40\%$ of the total area of the quilt.

Answer: (B)

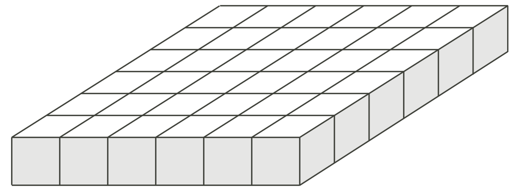
17. Since $\triangle PRS$ is isosceles with $PR = PS$, then $\angle PRS = \angle PSR$.
Since the angles in $\triangle PRS$ add to 180°, then $\angle PRS + \angle PSR + \angle RPS = 180^\circ$.
Therefore, $2(\angle PRS) + 34^\circ = 180^\circ$ or $2(\angle PRS) = 146^\circ$ or $\angle PRS = 73^\circ$.
Since $\triangle PQT$ is isosceles with $PQ = PT$, then $\angle PQT = \angle PTQ = 62^\circ$.
Since $\angle PRS$ is an exterior angle to $\triangle PQR$, then $\angle PRS = \angle PQR + \angle QPR$ or $73^\circ = 62^\circ + x^\circ$.
Therefore, $x = 73 - 62 = 11$.
(Instead, we could have determined that $\angle PRQ = 180^\circ - \angle PRS = 180^\circ - 73^\circ = 107^\circ$, and then looked at the sum of the angles in $\triangle PQR$ to get $62^\circ + x^\circ + 107^\circ = 180^\circ$ or $x = 180 - 169 = 11$.)

Answer: (A)

18. We visualize the solid as a rectangular prism with length 6, width 6 and height 3. In other words, we can picture the solid as three 6×6 squares stacked on top of each other.
Since the entire exterior of the solid is painted, then each cube in the top layer and each cube in the bottom layer has paint on it, so we can remove these.



This leaves the middle 6×6 layer of cubes.
Each cube around the perimeter of this square has paint on it, so it is only the "middle" cubes from this layer that have no paint on them.
These middle cubes form a 4×4 square, and so there are 16 cubes with no paint on them.



Answer: (A)

19. Let $PQ = a$, $QR = b$, $PW = c$, and $WV = d$.
Since the large rectangle is divided into four smaller rectangles, then $PQ = WX = VU = a$, $QR = XS = UT = b$, $PW = QX = RS = c$, and $WV = XU = ST = d$.
Since the area of rectangle $PQXW$ is 9, then $ac = 9$.
Since the area of rectangle $QRSX$ is 10, then $bc = 10$.
Since the area of rectangle $XSTU$ is 15, then $bd = 15$.
The area of rectangle $WXUV$ is ad. We want to determine the value of ad.
If we multiply the equations $ac = 9$ and $bd = 15$, we obtain $(ac)(bd) = 9 \times 15$, or $abcd = 135$.
We divided this equation by the equation $bc = 10$.
This gives $\dfrac{abcd}{bc} = \dfrac{135}{10} = \dfrac{27}{2}$, from which $ad = \dfrac{27}{2}$.

Answer: (B)

20. *Solution 1*
When N is divided by 10, 11 or 12, the remainder is 7.
This means that $M = N - 7$ is divisible by each of 10, 11 and 12.
Since M is divisible by each of 10, 11 and 12, then M is divisible by the least common multiple of 10, 11 and 12.
Since $10 = 2 \times 5$, $12 = 2 \times 2 \times 3$, and 11 is prime, then the least common multiple of 10, 11 and 12 is $2 \times 2 \times 3 \times 5 \times 11 = 660$. (To find the least common multiple, we compute the product of the highest powers of each of the prime factors that occur in the given numbers.)
Since M is divisible by 660 and $N = M + 7$ is a three-digit positive integer, then M must equal 660. (The next largest multiple of 660 is 1320.)
Therefore, $N = M + 7 = 667$, and so the sum of the digits of N is $6 + 6 + 7 = 19$.

Solution 2
When N is divided by 10, 11 or 12, the remainder is 7.
This means that $M = N - 7$ is divisible by each of 10, 11 and 12.
Since M is divisible by each of 10 and 11, then M must be divisible by 110.
We test the first few multiples of 110 until we obtain one that is divisible by 12.
The integers 110, 220, 330, 440, and 550 are not divisible by 12, but 660 is.
Therefore, M could be 660. (This means that M must be 660.)
Finally, $N = M + 7 = 667$, and so the sum of the digits of N is $6 + 6 + 7 = 19$.

Answer: (E)

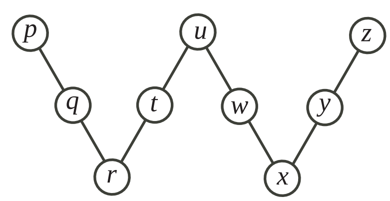
21. Let L be the length of the string.
If x is the length of the shortest piece, then since each of the other pieces is twice the length of the next smaller piece, then the lengths of the remaining pieces are $2x$, $4x$, and $8x$.
Since these four pieces make up the full length of the string, then $x + 2x + 4x + 8x = L$ or $15x = L$ and so $x = \frac{1}{15}L$.
Thus, the longest piece has length $8x = \frac{8}{15}L$, which is $\frac{8}{15}$ of the length of the string.

Answer: (A)

22. Suppose that the radius of each of the circles is r.
Since the two circles are identical, then the two circles have equal area.
Since the shaded area is common to the two circles, then the unshaded pieces of each circle have equal areas.
Since the combined area of the unshaded regions equals that of the shaded region, or 216π, then each of the unshaded regions have area $\frac{1}{2} \times 216\pi = 108\pi$.
The total area of one of the circles equals the sum of the areas of the shaded region and one unshaded region, or $216\pi + 108\pi = 324\pi$.
Since the radius of the circle is r, then $\pi r^2 = 324\pi$ or $r^2 = 324$.
Since $r > 0$, then $r = \sqrt{324} = 18$.
Therefore, the circumference of each circle is $2\pi r = 2\pi(18) = 36\pi$.
ANSWER: (C)

23. Suppose that the height of the water in each container is h cm.
Since the first container is a rectangular prism with a base that is 2 cm by 4 cm, then the volume of the water that it contains, in cm^3, is $2 \times 4 \times h = 8h$.
Since the second container is a right cylinder with a radius of 1 cm, then the volume of the water that it contains, in cm^3, is $\pi \times 1^2 \times h = \pi h$.
Since the combined volume of the water is 80 cm^3, then $8h + \pi h = 80$.
Thus, $h(8 + \pi) = 80$ or $h = \dfrac{80}{8 + \pi} \approx 7.18$.
Of the given answers, this is closest to 7.2 (that is, the height of the water is closest to 7.2 cm).
ANSWER: (B)

24. If we have a configuration of the numbers that has the required property, then we can add or subtract the same number from each of the numbers in the circles and maintain the property. (This is because there are the same number of circles in each line.)
Therefore, we can subtract 2012 from all of the numbers and try to complete the diagram using the integers from 0 to 8.
We label the circles as shown in the diagram, and call S the sum of the three integers along any one of the lines.

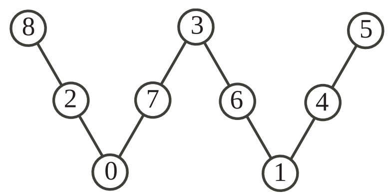


Since $p, q, r, t, u, w, x, y, z$ are 0 through 8 in some order, then

$$p + q + r + t + u + w + x + y + z = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$$

From the desired property, we want $S = p + q + r = r + t + u = u + w + x = x + y + z$.
Therefore, $(p + q + r) + (r + t + u) + (u + w + x) + (x + y + z) = 4S$.
From this, $(p+q+r+t+u+w+x+y+z)+r+u+x = 4S$ or $r+u+x = 4S-36 = 4(S-9)$.
We note that the right side is an integer that is divisible by 4.
Also, we want S to be as small as possible so we want the sum $r + u + x$ to be as small as possible.
Since $r + u + x$ is a positive integer that is divisible by 4, then the smallest that it can be is $r + u + x = 4$.

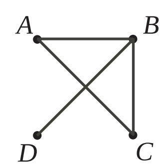
If $r+u+x=4$, then r, u and x must be 0, 1 and 3 in some order since each of r, u and x is a different integer between 0 and 8.
In this case, $4=4S-36$ and so $S=10$.
Since $S=10$, then we cannot have r and u or u and x equal to 0 and 1 in some order, or else the third number in the line would have to be 9, which is not possible.
This tells us that u must be 3, and r and x are 0 and 1 in some order.
Here is a configuration that works:



Therefore, the value of u in the original configuration is $3+2012=2015$.

Answer: (D)

25. We label the four people in the room A, B, C, and D. We represent each person by a point. There are six possible pairs of friends: AB, AC, AD, BC, BD, and CD. We represent a friendship by joining the corresponding pair of points and a non-friendship by not joining the pair of points.
Since each pair of points is either joined or not joined and there are 6 possible pairs, then there are $2\times2\times2\times2\times2\times2=2^6=64$ possible configurations of line segments in the diagram.
For example, if A and B, B and C, A and C, and B and D are friends, then we obtain the configuration to the right.
Since each pair of points has equal probability of being joined or not joined, then each of the 64 possible configurations is equally likely, so has probability $\frac{1}{64}$.

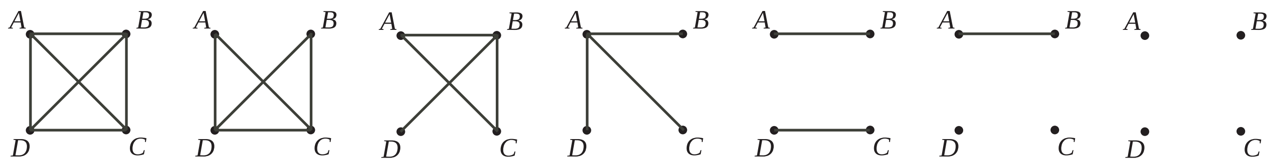


Points A and B, for example, are "connected" according to the definition, if they are joined, or if they are both joined to C, or if they are both joined to D, or if one is joined to C, the other to D and C and D are joined. In other words, points A and B are connected if we can pass from A to B along line segments, possibly through one or both of C and D.
If each pair of points is connected, we call the configuration *fully connected*.
Thus, we need to count the number of configurations that are fully connected.
First, we count the number of configurations including each possible number of line segments (0 through 6). After this, we will analyze each case to see which configurations are fully connected.

- 0 line segments: There is 1 such configuration.
- 6 line segments: There is 1 such configuration.
- 1 line segment: There are 6 such configurations, because we can choose any one of the 6 possible segments.
- 5 line segments: There are 6 such configurations, because we can start with all 6 line segments and remove any one of the 6 possible segments.
- 2 line segments: There are 15 such configurations. This is because there are 6 possible choices for the first line segment to add, and then 5 possible choices for the second line segment. This looks like 6×5 configurations, but each configuration is counted twice here (as we could choose AB and then BD or BD and then AB). Therefore, there are $\frac{1}{2}\times6\times5=15$ configurations.

- 4 line segments: There are 15 such configurations, because we can start with all 6 line segments and remove any of the 15 possible pairs of 2 line segments.
- 3 line segments: There are 20 configurations, because there are 64 configurations in total and we have already accounted for $1+1+6+6+15+15=44$ configurations.

Here is an example of each of these types of configurations:



6 segments 5 segments 4 segments 3 segments 2 segments 1 segment 0 segments

Now we analyze each of the possible classes of configurations to see if the relevant configurations are fully connected or not:

- 0 line segments
  This configuration is not fully connected.
- 6 line segments
  This configuration is fully connected.
- 1 line segment
  Each of these 6 configurations is not fully connected, since it can only have 2 points joined.
- 5 line segments
  Each of these 6 configurations is fully connected, since only one pair of points is not joined. If this pair is AB, for instance, then A and B are each joined to C, so the configuration is fully connected. (See the diagram labelled "5 segments" above.) The same is true no matter which pair is not joined.
- 2 line segments
  Each of these 15 configurations is not fully connected, since the two line segments can only include 3 points with the fourth point not connected to any other point (for example, B connected to each of A and C) or two pairs of connected points (as in the diagram labelled "2 segments" above).
- 4 line segments
  Each of these 15 configurations is fully connected.
  Consider starting with all 6 pairs of points joined, and then remove two line segments.
  There are two possibilities: either the two line segments share an endpoint, or they do not.
  An example of the first possibility is removing AB and BC to get Figure 1. This configuration is fully connected, and is representative of this subcase.
  An example of the second possibility is removing AB and CD to get Figure 2. This configuration is fully connected, and is representative of this subcase.

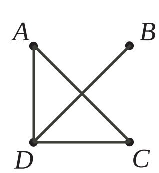


Figure 1

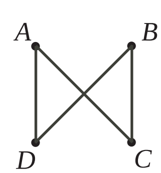
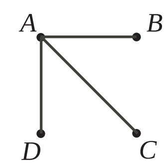


Figure 2

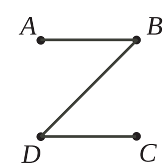
Therefore, all 15 of these configurations are fully connected.

- 3 line segments
  There are 20 such configurations.
  Each of these configurations involve joining more than 2 points, since 2 points can only be joined by 1 line segment.
  There are several possibilities:

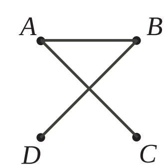
  * Some of these 20 configurations involve only 3 points. Since there are three line segments that join 3 points, these configurations look like a triangle and these are not fully connected. There are 4 such configurations, which we can see by choosing 1 of 4 points to not include.
  * Some of the remaining 16 configurations involve connecting 1 point to each of the other 3 points. Without loss of generality, suppose that A is connected to each of the other points.
    This configuration is fully connected since B and C are connected through A, as are B and D, C and D, and is representative of this subcase.

    There are 4 such configurations, which we can see by choosing 1 of 4 points to be connected to each of the other points.
  * Consider the remaining $20 - 4 - 4 = 12$ configurations. Each of these involves all 4 points and cannot have 1 point connected to the other 3 points.
    Without loss of generality, we start by joining AB.
    If CD is joined, then one of AC, AD, BC, and BD is joined, which means that each of A and B is connected to each of C and D.
    This type of configuration is fully connected.

    If CD is not joined, then two of AC, AD, BC, and BD are joined. We cannot have AC and AD both joined, or BC and BD both joined, since we cannot have 3 segments coming from the same point.
    Also, we cannot have AC and BC both joined, or AD and BD both joined, since we cannot include just 3 points.
    Therefore, we must have AC and BD joined, or AD and BC joined.
    This type of configuration is fully connected.

Therefore, of the 64 possible configurations, $1 + 6 + 15 + 4 + 12 = 38$ are fully connected.
Therefore, the probability that every pair of people in this room is connected is $\frac{38}{64} = \frac{19}{32}$.

Answer: (D)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Pascal Contest

(Grade 9)

Thursday, February 24, 2011

Solutions

1. Calculating, $6 \times (5-2) + 4 = 6 \times 3 + 4 = 18 + 4 = 22$.

   Answer: (B)

2. Converting to a numerical expression, we obtain $943 - 87$ which equals 856.

   Answer: (E)

3. Since $2011^2 = 4044121$ and $\sqrt{2011} \approx 44.8$, then the list of numbers in increasing order is $\sqrt{2011}, 2011, 2011^2$.
   (If n is a positive integer with $n > 1$, then $n^2 > n$ and $\sqrt{n} < n$, so the list $\sqrt{n}, n, n^2$ is always in increasing order.)

   Answer: (C)

4. From the graph, the mass of fats is 32 g and the mass of carbohydrates is 48 g.
   Therefore, the ratio of the mass of fats to the mass of carbohydrates is $32 : 48$.
   Since each of 32 and 48 is divisible by 16, we can reduce the ratio by dividing both parts by 16 to obtain the simplified ratio $2 : 3$.

   Answer: (B)

5. When $x = -2$, we have $(x+1)^3 = (-2+1)^3 = (-1)^3 = -1$.

   Answer: (A)

6. After Peyton has added 15 L of oil, the new mixture contains $30 + 15 = 45$ L of oil and 15 L of vinegar.
   Thus, the total volume of the new mixture is $45 + 15 = 60$ L.
   Of this, the percentage that is oil is $\frac{45}{60} \times 100\% = \frac{3}{4} \times 100\% = 75\%$.

   Answer: (A)

7. When three $1 \times 1 \times 1$ cubes are joined together as in the diagram, the resulting prism is $3 \times 1 \times 1$.
   This prism has four rectangular faces that are 3×1 and two rectangular faces that are 1×1.
   Therefore, the surface area is $4 \times (3 \times 1) + 2 \times (1 \times 1) = 4 \times 3 + 2 \times 1 = 12 + 2 = 14$.

   Answer: (B)

8. Since the 17th day of the month is a Saturday and there are 7 days in a week, then the previous Saturday was the $17 - 7 = 10$th day of the month and the Saturday before that was the $10 - 7 = 3$rd day of the month.
   Since the 3rd day of the month was a Saturday, then the 2nd day was a Friday and the 1st day of the month was a Thursday.

   Answer: (D)

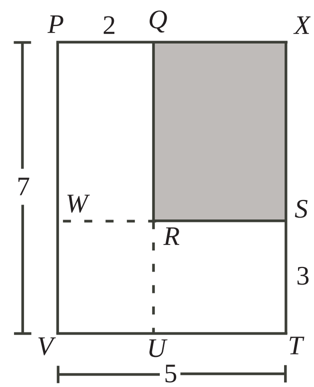
9. *Solution 1*
   Since $PQUV$ and $WSTV$ are rectangles that share a common right angle at V, then PQ, WS and VT are parallel, as are PV, QU, and ST. This tells us that all of the angles in the diagram are right angles.
   Since $PQUV$ is a rectangle, then $VU = PQ = 2$.
   Since $VT = 5$ and $VU = 2$, then $UT = VT - VU = 5 - 2 = 3$.
   Note that $RSTU$ is a rectangle, since it has four right angles.
   Therefore, the area of $PQRSTV$ equals the sum of the areas of rectangles $PQUV$ and $RSTU$, or $2 \times 7 + 3 \times 3 = 23$.

(We could also consider the area of $PQRSTV$ to be the sum of the areas of rectangle $PQRW$ and rectangle $WSTV$.)

Solution 2

Since $PQUV$ and $WSTV$ are rectangles that share a common right angle at V, then PQ, WS and VT are parallel, as are PV, QU, and ST. This tells us that all of the angles in the diagram are right angles.

We can consider $PQRSTV$ to be a large rectangle $PXTV$ with a smaller rectangle $QXSR$ removed.



The area of rectangle $PXTV$ is $7 \times 5 = 35$.
Since $PQUV$ is a rectangle, then $QU = PV = 7$.
Since PV is parallel to QU and ST, then $RU = ST = 3$.
Thus, $QR = QU - RU = 7 - 3 = 4$.
Since $WSTV$ is a rectangle, then $WS = VT = 5$.
Since VT is parallel to WS and PQ, then $WR = PQ = 2$.
Thus, $RS = WS - WR = 5 - 2 = 3$.
Therefore, rectangle $QXSR$ is 4 by 3, and so has area 12.
Therefore, the area of $PQRSTV$ is $35 - 12 = 23$.

Solution 3

Since $PQUV$ and $WSTV$ are rectangles that share a common right angle at V, then PQ, WS and VT are parallel, as are PV, QU, and ST. This tells us that all of the angles in the diagram are right angles.

If we add up the areas of rectangle $PQUV$ and $WSTV$, we get exactly the region $PQRSTV$, but have added the area of $WRUV$ twice. Thus, the area of $PQRSTV$ equals the area of $PQUV$ plus the area of $WSTV$ minus the area of $WRUV$.

We note that rectangle $PQUV$ is 2 by 7, rectangle $WSTV$ is 3 by 5, and rectangle $WRUV$ is 2 by 3 (since $WR = PQ = 2$ and $RU = ST = 3$).

Therefore, the area of $PQRSTV$ equals $2 \times 7 + 3 \times 5 - 2 \times 3 = 14 + 15 - 6 = 23$.

Answer: (E)

10. John first writes the integers from 1 to 20 in increasing order.
When he erases the first half of the numbers, he erases the numbers from 1 to 10 and rewrites these at the end of the original list.
Therefore, the number 1 has 10 numbers to its left. (These numbers are $11, 12, \ldots, 20$.)
Thus, the number 2 has 11 numbers to its left, and so the number 3 has 12 numbers to its left.
(We could write out the new list to verify this.)

Answer: (C)

11. When we convert each of the possible answers to a decimal, we obtain 1.1, 1.11, 1.101, 1.111, and 1.011.
Since the last of these is the only one greater than 1 and less than 1.1, it is closest to 1.

ANSWER: (E)

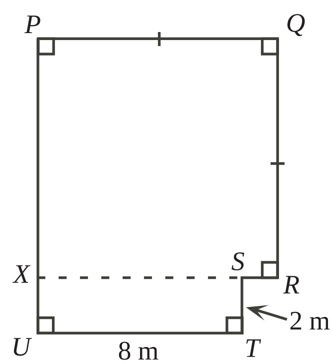
12. We note that $\frac{17}{4} = 4\frac{1}{4}$ and $\frac{35}{2} = 17\frac{1}{2}$.
Therefore, the integers between these two numbers are the integers from 5 to 17, inclusive.
The odd integers in this range are $5, 7, 9, 11, 13, 15$, and 17, of which there are 7.

ANSWER: (D)

13. The first four terms of the sequence are $1, 4, 2, 3$.
Since each term starting with the fifth is the sum of the previous four terms, then the fifth term is $1 + 4 + 2 + 3 = 10$.
Also, the sixth term is $4 + 2 + 3 + 10 = 19$, the seventh term is $2 + 3 + 10 + 19 = 34$, and the eighth term is $3 + 10 + 19 + 34 = 66$.

ANSWER: (A)

14. We extend the short horizontal side, RS, to the left until it reaches the long vertical side.



Since $PQRX$ has three right angles, then it must have a fourth right angle and so must be a rectangle.
Since $PQ = QR$, then $PQRX$ is in fact a square.
Since the exterior angle at S is a right angle, then $XSTU$ is also a rectangle.
Since $XSTU$ is 2 m by 8 m, then its area is $2 \times 8 = 16$ m^2.
Since the area of the whole garden is 97 m^2, then the area of $PQRX$ is $97 - 16 = 81$ m^2.
Since $PQRX$ is a square, then its side length is $\sqrt{81} = 9$ m.
Therefore, $PQ = QR = RX = XP = 9$ m.
Since $XSTU$ is a rectangle, then $XS = UT = 8$ m and $XU = ST = 2$ m.
Therefore, $PU = PX + XU = 9 + 2 = 11$ m and $SR = XR - XS = 9 - 8 = 1$ m.
Finally, we determine the perimeter of the garden by starting at P and proceeding clockwise.
The perimeter is $9 + 9 + 1 + 2 + 8 + 11 = 40$ m.

ANSWER: (C)

15. Since each of five friends paid an extra \$3 to cover Luxmi's portion of the bill, then Luxmi's share was $5 \times \$3 = \$15$.
Since each of the six friends had an equal share, then the total bill is $6 \times \$15 = \$90$.

ANSWER: (A)

16. The set S contains 25 multiples of 2 (that is, even numbers).
When these are removed, the set S is left with only the odd integers from 1 to 49.
At this point, there are $50 - 25 = 25$ integers in S.
We still need to remove the multiples of 3 from S.
Since S only contains odd integers at this point, then we must remove the odd multiples of 3 between 1 and 49.
These are 3, 9, 15, 21, 27, 33, 39, 45, of which there are 8.
Therefore, the number of integers remaining in the set S is $25 - 8 = 17$.

ANSWER: (D)

17. *Solution 1*
We work from right to left as we would if doing this calculation by hand.
In the units column, we have $L - 4$ giving 1. Thus, $L = 5$. (There is no borrowing required.)
This gives

$$\begin{array}{rcccc} & 6 & K & 0 & 5 \\ - & M & 9 & N & 4 \\ \hline & 2 & 0 & 1 & 1 \end{array}$$

In the tens column, we have $0 - N$ giving 1.
Since 1 is larger than 0, we must borrow from the hundreds column. Thus, $10 - N$ gives 1, which means $N = 9$. This gives

$$\begin{array}{rcccc} & & {\scriptstyle K-1} & {\scriptstyle 10} & \\ & 6 & \cancel{K} & \cancel{0} & 5 \\ - & M & 9 & 9 & 4 \\ \hline & 2 & 0 & 1 & 1 \end{array}$$

In the hundreds column, we have $K - 9$ but we have already borrowed 1 from K, so we have $(K - 1) - 9$ giving 0.
Therefore, we must be subtracting 9 from 9, which means that K should be 10, which is not possible.
We can conclude, though, that $K = 0$ and that we have borrowed from the 6. This gives

$$\begin{array}{rcccc} & {\scriptstyle 5} & & & \\ & \cancel{6} & 0 & 0 & 5 \\ - & M & 9 & 9 & 4 \\ \hline & 2 & 0 & 1 & 1 \end{array}$$

In the thousands column, we have $5 - M = 2$ or $M = 3$.
This gives $6005 - 3994 = 2011$, which is correct.
Finally, $K + L + M + N = 0 + 5 + 3 + 9 = 17$.

Solution 2
Since $6K0L - M9N4 = 2011$, then $M9N4 + 2011 = 6K0L$.
We start from the units column and work towards the left.
Considering the units column, the sum $4 + 1$ has a units digit of L. Thus, $L = 5$. (There is no carry to the tens column.) This gives

$$\begin{array}{ccccc} & M & 9 & N & 4 \\ + & 2 & 0 & 1 & 1 \\ \hline & 6 & K & 0 & 5 \end{array}$$

Considering the tens column, the sum $N + 1$ has a units digit of 0. Thus, $N = 9$. (There is a carry of 1 to the hundreds column.) This gives

$$\begin{array}{ccccc} & & {\scriptstyle 1} & & \\ & M & 9 & 9 & 4 \\ + & 2 & 0 & 1 & 1 \\ \hline & 6 & K & 0 & 5 \end{array}$$

Considering the hundreds column, the sum $9 + 0$ plus the carry of 1 from the tens column has a units digit of K. Since $9 + 0 + 1 = 10$, then $K = 0$. There is a carry of 1 from the hundreds column to the thousands column. This gives

$$\begin{array}{ccccc} & {\scriptstyle 1} & & & \\ & M & 9 & 9 & 4 \\ + & 2 & 0 & 1 & 1 \\ \hline & 6 & 0 & 0 & 5 \end{array}$$

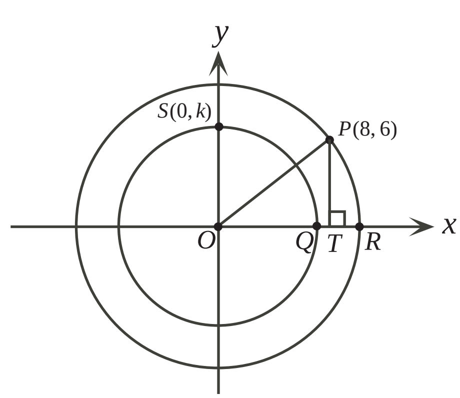
Considering the thousands column, the sum $M+2$ plus the carry of 1 from the hundreds column equals 6. Therefore, $M + 2 + 1 = 6$ or $M = 3$.
This gives $3994 + 2011 = 6005$ or $6005 - 3994 = 2011$, which is correct.
Finally, $K + L + M + N = 0 + 5 + 3 + 9 = 17$.

Answer: (A)

18. The difference between $\frac{1}{6}$ and $\frac{1}{12}$ is $\frac{1}{6} - \frac{1}{12} = \frac{2}{12} - \frac{1}{12} = \frac{1}{12}$, so $LP = \frac{1}{12}$.
Since LP is divided into three equal parts, then this distance is divided into three equal parts, each equal to $\frac{1}{12} \div 3 = \frac{1}{12} \times \frac{1}{3} = \frac{1}{36}$.
Therefore, M is located $\frac{1}{36}$ to the right of L.
Thus, the value at M is $\frac{1}{12} + \frac{1}{36} = \frac{3}{36} + \frac{1}{36} = \frac{4}{36} = \frac{1}{9}$.

Answer: (C)

19. We can determine the distance from O to P by dropping a perpendicular from P to T on the x-axis.



We have $OT = 8$ and $PT = 6$, so by the Pythagorean Theorem,

$$OP^2 = OT^2 + PT^2 = 8^2 + 6^2 = 64 + 36 = 100$$

Since $OP > 0$, then $OP = \sqrt{100} = 10$.
Therefore, the radius of the larger circle is 10.
Thus, $OR = 10$.
Since $QR = 3$, then $OQ = OR - QR = 10 - 3 = 7$.
Therefore, the radius of the smaller circle is 7.
Since S is on the positive y-axis and is 7 units from the origin, then the coordinates of S are $(0, 7)$, which means that $k = 7$.

ANSWER: (E)

20. *Solution 1*
Consider $\triangle UPV$.
Since $PU = PV$, then $\triangle UPV$ is isosceles, with

$$\angle PUV = \angle PVU = \tfrac{1}{2}(180^\circ - \angle UPV) = \tfrac{1}{2}(180^\circ - 24^\circ) = \tfrac{1}{2}(156^\circ) = 78^\circ$$

Since PVS is a straight line, then $\angle QVS = 180^\circ - \angle PVU = 180^\circ - 78^\circ = 102^\circ$.
Consider $\triangle QVS$.
The sum of the angles in this triangle is 180°, and so $102^\circ + x^\circ + y^\circ = 180^\circ$.
Therefore, $x + y = 180 - 102 = 78$.

Solution 2
Consider $\triangle UPV$.
Since $PU = PV$, then $\triangle UPV$ is isosceles, with

$$\angle PUV = \angle PVU = \tfrac{1}{2}(180^\circ - \angle UPV) = \tfrac{1}{2}(180^\circ - 24^\circ) = \tfrac{1}{2}(156^\circ) = 78^\circ$$

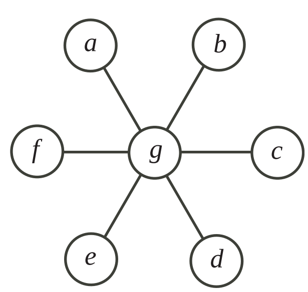
Since $\angle PVU$ is an exterior angle to $\triangle QVS$, then $\angle PVU = \angle VQS + \angle VSQ$.
Therefore, $78^\circ = y^\circ + x^\circ$ or $x + y = 78$.

ANSWER: (D)

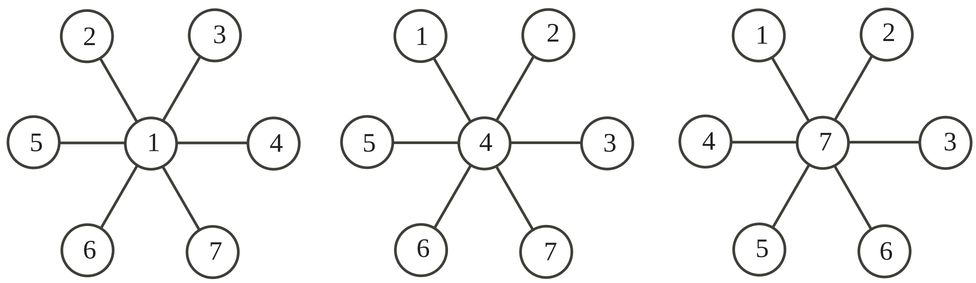
21. Since level C contains the same number of dots as level B and level D contains twice as many dots as level C, then level D contains twice as many dots as level B.
Similarly, level F contains twice as many dots as level D, level H contains twice as many dots as level F, and so on.
Put another way, the number of dots doubles from level B to level D, from level D to level F, from level F to level H, and so on.
Since there are 26 levels, then there are 24 levels after level B.
Thus, the number of dots doubles $24 \div 2 = 12$ times from level B to level Z.
Therefore, the number of dots on level Z is $2 \times 2^{12} = 2^{13} = 8192$.

ANSWER: (D)

22. We label the circles from a to g, as shown:



Let S be sum of the integers in any straight line.
Therefore, $S = a + g + d = b + g + e = c + g + f$.
Thus, $3S = (a + g + d) + (b + g + e) + (c + g + f) = a + b + c + d + e + f + 3g$.
Since the variables a to g are to be replaced by the integers from 1 to 7, in some order, then $a + b + c + d + e + f + g = 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$.
Thus, $3S = (a + b + c + d + e + f + g) + 2g = 28 + 2g$, and so $3S = 28 + 2g$.
Since $3S$ is an integer divisible by 3, then $28 + 2g$ should also be divisible by 3.
Since g must be an integer between 1 and 7, we can try the seven possibilities and see that the only values of g for which $28 + 2g$ is divisible by 3 are 1, 4 and 7.
We must verify that we can actually complete the diagram for each of these values:



Therefore, there are 3 possibilities for the number in the centre circle.

Answer: (C)

23. First, we count the number of quadruples (p, q, r, s) of non-negative integer solutions to the equation $2p+q+r+s = 4$. Then, we determine which of these satisfies $p+q+r+s = 3$. This will allow us to calculate the desired probability.
Since each of p, q, r, and s is a non-negative integer and $2p + q + r + s = 4$, then there are three possible values for p: $p = 2$, $p = 1$, and $p = 0$.
Note that, in each case, $q + r + s = 4 - 2p$.

Case 1: $p = 2$
Here, $q + r + s = 4 - 2(2) = 0$.
Since each of q, r and s is non-negative, then $q = r = s = 0$, so $(p, q, r, s) = (2, 0, 0, 0)$.
There is 1 solution in this case.

Case 2: $p = 1$
Here, $q + r + s = 4 - 2(1) = 2$.
Since each of q, r and s is non-negative, then the three numbers q, r and s must be 0, 0 and 2 in some order, or 1, 1 and 0 in some order.
There are three ways to arrange a list of three numbers, two of which are the same. (With a, a, b, the arrangements are aab and aba and baa.)
Therefore, the possible quadruples here are

$$(p, q, r, s) = (1, 2, 0, 0), (1, 0, 2, 0), (1, 0, 0, 2), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1)$$

There are 6 solutions in this case.

Case 3: $p = 0$
Here, $q + r + s = 4$.
We will look for non-negative integer solutions to this equation with $q \geq r \geq s$. Once we have found these solutions, all solutions can be found be re-arranging these initial solutions.

If $q = 4$, then $r + s = 0$, so $r = s = 0$.
If $q = 3$, then $r + s = 1$, so $r = 1$ and $s = 0$.
If $q = 2$, then $r + s = 2$, so $r = 2$ and $s = 0$, or $r = s = 1$.
The value of q cannot be 1 or 0, because if it was, then $r + s$ would be at least 3 and so r or s would be at least 2. (We are assuming that $r \leq q$ so this cannot be the case.)
Therefore, the solutions to $q + r + s = 4$ must be the three numbers 4, 0 and 0 in some order, 3, 1 and 0 in some order, 2, 2 and 0 in some order, or 2, 1 and 1 in some order.
In Case 2, we saw that there are three ways to arrange three numbers, two of which are equal. In addition, there are six ways to arrange a list of three different numbers. (With a, b, c, the arrangements are abc, acb, bac, bca, cab, cba.)
The solution $(p, q, r, s) = (0, 4, 0, 0)$ has 3 arrangements.
The solution $(p, q, r, s) = (0, 3, 1, 0)$ has 6 arrangements.
The solution $(p, q, r, s) = (0, 2, 2, 0)$ has 3 arrangements.
The solution $(p, q, r, s) = (0, 2, 1, 1)$ has 3 arrangements.
(In each of these cases, we know that $p = 0$ so the different arrangements come from switching q, r and s.)
There are 15 solutions in this case.

Overall, there are $1 + 6 + 15 = 22$ solutions to $2p + q + r + s = 4$.
We can go through each of these quadruples to check which satisfy $p + q + r + s = 3$.
The quadruples that satisfy this equation are exactly those from Case 2.
We could also note that $2p + q + r + s = 4$ and $p + q + r + s = 3$ means that

$$p = (2p + q + r + s) - (p + q + r + s) = 4 - 3 = 1$$

Therefore, of the 22 solutions to $2p + q + r + s = 4$, there are 6 that satisfy $p + q + r + s = 3$, so the desired probability is $\frac{6}{22} = \frac{3}{11}$.

Answer: (B)

24. The largest integer with exactly 100 digits is the integer that consists of 100 copies of the digit 9.
This integer is equal to $10^{100} - 1$.
Therefore, we want to determine the largest integer n for which $14n \leq 10^{100} - 1$.
This is the same as trying to determine the largest integer n for which $14n < 10^{100}$, since $14n$ is an integer.
We want to find the largest integer n for which $n < \dfrac{10^{100}}{14} = \dfrac{10}{14} \times 10^{99} = \dfrac{5}{7} \times 10^{99}$.
This is equivalent to calculating the number $\dfrac{5}{7} \times 10^{99}$ and rounding down to the nearest integer.
Put another way, this is the same as calculating $\dfrac{5}{7} \times 10^{99}$ and truncating the number at the decimal point.
The decimal expansion of $\dfrac{5}{7}$ is $0.\overline{714285}$. (We can see this either using a calculator or by doing long division.)
Therefore, the integer that we are looking for is the integer obtained by multiplying $0.\overline{714285}$ by 10^{99} and truncating at the decimal point.
In other words, we are looking for the integer obtained by shifting the decimal point in $0.\overline{714285}$ by 99 places to the right, and then ignoring everything after the new decimal point.
Since the digits in the decimal expansion repeat with period 6, then the integer consists of 16 copies of the digits 714285 followed by 714. (This has $16 \times 6 + 3 = 99$ digits.)
In other words, the integer looks like $714285\ 714285\ \cdots\ 714285\ 714$.

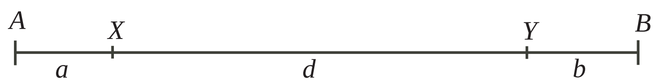
We must determine the digit that is the 68th digit from the right.
If we start listing groups from the right, we first have 714 (3 digits) followed by 11 copies of 714285 (66 more digits). This is 69 digits in total.
Therefore, the "7" that we have arrived at is the 69th digit from the right.
Moving one digit back towards the right tells us that the 68th digit from the right is 1.

Answer: (A)

25. First, we note that the three people are interchangeable in this problem, so it does not matter who rides and who walks at any given moment. We abbreviate the three people as D, M and P.
We call their starting point A and their ending point B.
Here is a strategy where all three people are moving at all times and all three arrive at B at the same time:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to a point Y before B.
D drops off M and rides back while P and M walk toward B.
D meets P at point X.
D picks up P and they drive back to B meeting M at B.
Point Y is chosen so that D, M and P arrive at B at the same time.

Suppose that the distance from A to X is a km, from X to Y is d km, and the distance from Y to B is b km.



In the time that it takes P to walk from A to X at 6 km/h, D rides from A to Y and back to X at 90 km/h.
The distance from A to X is a km.
The distance from A to Y and back to X is $a + d + d = a + 2d$ km.
Since the time taken by P and by D is equal, then $\dfrac{a}{6} = \dfrac{a+2d}{90}$ or $15a = a + 2d$ or $7a = d$.

In the time that it takes M to walk from Y to B at 6 km/h, D rides from Y to X and back to B at 90 km/h.
The distance from Y to B is b km, and the distance from Y to X and back to B is $d+d+b = b+2d$ km.
Since the time taken by M and by D is equal, then $\dfrac{b}{6} = \dfrac{b+2d}{90}$ or $15b = b + 2d$ or $7b = d$.

Therefore, $d = 7a = 7b$, and so we can write $d = 7a$ and $b = a$.
Thus, the total distance from A to B is $a + d + b = a + 7a + a = 9a$ km.
However, we know that this total distance is 135 km, so $9a = 135$ or $a = 15$.
Finally, D rides from A to Y to X to B, a total distance of $(a + 7a) + 7a + (7a + a) = 23a$ km.
Since $a = 15$ km and D rides at 90 km/h, then the total time taken for this strategy is $\frac{23\times 15}{90} = \frac{23}{6} \approx 3.83$ h.
Since we have a strategy that takes 3.83 h, then the smallest possible time is no more than 3.83 h. Can you explain why this is actually the smallest possible time?

If we didn't think of this strategy, another strategy that we might try would be:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to B.
D drops off M at B and rides back to meet P, who is still walking.
D picks up P and they drive back to B. (M rests at B.)

This strategy actually takes 4.125 h, which is longer than the strategy shown above, since M is actually sitting still for some of the time.

ANSWER: (A)

Canadian
Mathematics
Competition
An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2010 Pascal Contest

(Grade 9)

Thursday, February 25, 2010

Solutions

1. In cents, the five given choices are 50, 90, 95, 101, and 115 cents.
The differences between each of these and \$1.00 (or 100 cents), in cents, are

$$100-50=50 \quad 100-90=10 \quad 100-95=5 \quad 101-100=1 \quad 115-100=15$$

The difference between \$1.01 and \$1.00 is the smallest (1 cent), so \$1.01 is closest to \$1.00.

Answer: (D)

2. Using the correct order of operations,

$$\frac{(20-16)\times(12+8)}{4}=\frac{4\times 20}{4}=\frac{80}{4}=20$$

Answer: (C)

3. We divide the 750 mL of flour into portions of 250 mL. We do this by calculating $750 \div 250 = 3$.
Therefore, 750 mL is three portions of 250 mL.
Since 50 mL of milk is required for each 250 mL of flour, then $3 \times 50 = 150$ mL of milk is required in total.

Answer: (C)

4. There are 8 figures in total. Of these, 3 are triangles.
Therefore, the probability is $\frac{3}{8}$.

Answer: (A)

5. We simplify the left side and express it as a fraction with numerator 1:

$$\frac{1}{9}+\frac{1}{18}=\frac{2}{18}+\frac{1}{18}=\frac{3}{18}=\frac{1}{6}$$

Therefore, the number that replaces the $\square$ is 6.

Answer: (C)

6. There are 16 horizontal segments on the perimeter. Each has length 1, so the horizontal segments contribute 16 to the perimeter.
There are 10 vertical segments on the perimeter. Each has length 1, so the vertical segments contribute 10 to the perimeter.
Therefore, the perimeter is $10+16=26$.
(We could arrive at this total instead by starting at a fixed point and travelling around the outside of the figure counting the number of segments.)

Answer: (E)

7. Since $3^3 = 3\times 3\times 3 = 3\times 9 = 27$, then

$$\sqrt{3^3+3^3+3^3}=\sqrt{27+27+27}=\sqrt{81}=9$$

Answer: (B)

8. The difference between the two given numbers is $7.62-7.46=0.16$.
This length of the number line is divided into 8 equal segments.
The length of each of these segments is thus $0.16 \div 8 = 0.02$.
Point P is three of these segments to the right of 7.46.
Thus, the number represented is $7.46+3(0.02)=7.46+0.06=7.52$.

Answer: (E)

9. A 12 by 12 grid of squares will have 11 interior vertical lines and 11 interior horizontal lines. (In the given 4 by 4 example, there are 3 interior vertical lines and 3 interior horizontal lines.)
Each of the 11 interior vertical lines intersects each of the 11 interior horizontal lines and creates an interior intersection point.
Thus, each interior vertical line accounts for 11 intersection points.
Therefore, the number of interior intersection points is $11 \times 11 = 121$.

Answer: (B)

10. Because the central angle for the interior sector "Less than 1 hour" is 90°, then the fraction of the students who do less than 1 hour of homework per day is $\dfrac{90^\circ}{360^\circ} = \dfrac{1}{4}$.
In other words, 25% of the students do less than 1 hour of homework per day.
Therefore, $100\% - 25\% = 75\%$ of the students do at least 1 hour of homework per day.

Answer: (E)

11. *Solution 1*
Since there is more than 1 four-legged table, then there are at least 2 four-legged tables.
Since there are 23 legs in total, then there must be fewer than 6 four-legged tables, since 6 four-legged tables would have $6 \times 4 = 24$ legs.
Thus, there are between 2 and 5 four-legged tables.
If there are 2 four-legged tables, then these tables account for $2 \times 4 = 8$ legs, leaving $23 - 8 = 15$ legs for the three-legged tables.
Since 15 is divisible by 3, then this must be the solution, so there are $15 \div 3 = 5$ three-legged tables.
(We can check that if there are 3 or 4 four-legged tables, then the number of remaining legs is not divisible by 3, and if there are 5 four-legged tables, then there is only 1 three-legged table, which is not allowed.)

Solution 2
Since there is more than 1 table of each type, then there are at least 2 three-legged tables and 2 four-legged tables.
These tables account for $2(3) + 2(4) = 14$ legs.
There are $23 - 14 = 9$ more legs that need to be accounted for. These must come from a combination of three-legged and four-legged tables.
The only way to make 9 from 3s and 4s is to use three 3s.
Therefore, there are $2 + 3 = 5$ three-legged tables and 2 four-legged tables.

Answer: (E)

12. *Solution 1*
The total area of the rectangle is $3 \times 4 = 12$.
The total area of the shaded regions equals the total area of the rectangle (12) minus the area of the unshaded region.
The unshaded region is a triangle with base of length 1 and height 4; the area of this region is $\frac{1}{2}(1)(4) = 2$.
Therefore, the total area of the shaded regions is $12 - 2 = 10$.

Solution 2
The shaded triangle on the left has base of length 2 and height of length 4, so has an area of $\frac{1}{2}(2)(4) = 4$.

The shaded triangle on the right has base of length 3 (at the top) and height of length 4, so has an area of $\frac{1}{2}(3)(4) = 6$.
Therefore, the total area of the shaded regions is $4 + 6 = 10$.

Answer: (C)

13. Since the ratio of boys to girls at Cayley H.S. is $3 : 2$, then $\frac{3}{3+2} = \frac{3}{5}$ of the students at Cayley H.S. are boys.
Thus, there are $\frac{3}{5}(400) = \frac{1200}{5} = 240$ boys at Cayley H.S.
Since the ratio of boys to girls at Fermat C.I. is $2 : 3$, then $\frac{2}{2+3} = \frac{2}{5}$ of the students at Fermat C.I. are boys.
Thus, there are $\frac{2}{5}(600) = \frac{1200}{5} = 240$ boys at Fermat C.I.
There are $400 + 600 = 1000$ students in total at the two schools.
Of these, $240 + 240 = 480$ are boys, and so the remaining $1000 - 480 = 520$ students are girls.
Therefore, the overall ratio of boys to girls is $480 : 520 = 48 : 52 = 12 : 13$.

Answer: (B)

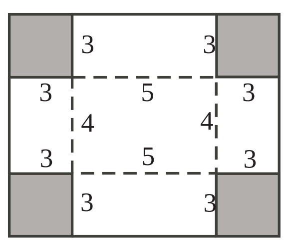
14. When the given net is folded, the face numbered 5 will be opposite the face numbered 1.
Therefore, the remaining four faces share an edge with the face numbered 1, so the product of the numbers is $2 \times 3 \times 4 \times 6 = 144$.

Answer: (B)

15. The percentage 10% is equivalent to the fraction $\frac{1}{10}$.
Therefore, $t = \frac{1}{10}s$, or $s = 10t$.

Answer: (D)

16. Since the base of the folded box measures 5 cm by 4 cm, then the area of the base of the box is $5(4) = 20\text{ cm}^2$.
Since the volume of the box is 60 cm^3 and the area of the base is 20 cm^2, then the height of the box is $\frac{60}{20} = 3$ cm.
Therefore, each of the four identical squares has side length 3 cm, because the edges of these squares form the vertical edges of the box.



Therefore, the rectangular sheet measures $3 + 5 + 3 = 11$ cm by $3 + 4 + 3 = 10$ cm, and so has area $11(10) = 110\text{ cm}^2$.

Answer: (B)

17. *Solution 1*
Since SUR is a straight line, then $\angle RUV = 180^\circ - \angle SUV = 180^\circ - 120^\circ = 60^\circ$.
Since PW and QX are parallel, then $\angle RVW = \angle VTX = 112^\circ$.
Since UVW is a straight line, then $\angle RVU = 180^\circ - \angle RVW = 180^\circ - 112^\circ = 68^\circ$.
Since the measures of the angles in a triangle add to 180°, then

$$\angle URV = 180^\circ - \angle RUV - \angle RVU = 180^\circ - 60^\circ - 68^\circ = 52^\circ$$

Solution 2
Since SUR is a straight line, then $\angle RUV = 180^\circ - \angle SUV = 180^\circ - 120^\circ = 60^\circ$.
Since PW and QX are parallel, then $\angle RST = \angle RUV = 60^\circ$.
Since STX is a straight line, then $\angle RTS = 180^\circ - \angle VTX = 180^\circ - 112^\circ = 68^\circ$.
Since the measures of the angles in a triangle add to 180°, then

$$\angle URV = \angle SRT = 180^\circ - \angle RST - \angle RTS = 180^\circ - 60^\circ - 68^\circ = 52^\circ$$

ANSWER: (A)

18. *Solution 1*
When Catherine adds 30 litres of gasoline, the tank goes from $\frac{1}{8}$ full to $\frac{3}{4}$ full.
Since $\frac{3}{4} - \frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$, then $\frac{5}{8}$ of the capacity of the tank is 30 litres.
Thus, $\frac{1}{8}$ of the capacity of the tank is $30 \div 5 = 6$ litres. Also, the full capacity of the tank is $8 \times 6 = 48$ litres.
To fill the remaining $\frac{1}{4}$ of the tank, Catherine must add an additional $\frac{1}{4} \times 48 = 12$ litres of gas.
Because each litre costs \$1.38, it will cost $12 \times \$1.38 = \$16.56$ to fill the rest of the tank.

Solution 2
Suppose that the capacity of the gas tank is x litres.
Starting with $\frac{1}{8}$ of a tank, 30 litres of gas makes the tank $\frac{3}{4}$ full, so $\frac{1}{8}x + 30 = \frac{3}{4}x$ or $\frac{5}{8}x = 30$ or $x = 48$.
The remaining capacity of the tank is $\frac{1}{4}x = \frac{1}{4}(48) = 12$ litres.
At \$1.38 per litre, it will cost Catherine $12 \times \$1.38 = \$16.56$ to fill the rest of the tank.

ANSWER: (C)

19. The area of a semi-circle with radius r is $\frac{1}{2}\pi r^2$ so the area of a semi-circle with diameter d is $\frac{1}{2}\pi(\frac{1}{2}d)^2 = \frac{1}{8}\pi d^2$.
The semicircles with diameters UV, VW, WX, XY, and YZ each have equal diameter and thus equal area. The area of each of these semicircles is $\frac{1}{8}\pi(5^2) = \frac{25}{8}\pi$.
The large semicircle has diameter $UZ = 5(5) = 25$, so has area $\frac{1}{8}\pi(25^2) = \frac{625}{8}\pi$.
The shaded area equals the area of the large semicircle, minus the area of two small semicircles, plus the area of three small semicircles, which equals the area of the large semicircle plus the area of one small semicircle.
Therefore, the shaded area equals $\frac{625}{8}\pi + \frac{25}{8}\pi = \frac{650}{8}\pi = \frac{325}{4}\pi$.

ANSWER: (A)

20. The sum of the odd numbers from 5 to 21 is

$$5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 = 117$$

Therefore, the sum of the numbers in any row is one-third of this total, or 39.
This means as well that the sum of the numbers in any column or diagonal is also 39.
Since the numbers in the middle row add to 39, then the number in the centre square is $39 - 9 - 17 = 13$.
Since the numbers in the middle column add to 39, then the number in the middle square in the bottom row is $39 - 5 - 13 = 21$.

	5	
9	13	17
x	21	

Since the numbers in the bottom row add to 39, then the number in the bottom right square is $39 - 21 - x = 18 - x$.
Since the numbers in the bottom left to top right diagonal add to 39, then the number in the top right square is $39 - 13 - x = 26 - x$.
Since the numbers in the rightmost column add to 39, then $(26 - x) + 17 + (18 - x) = 39$ or $61 - 2x = 39$ or $2x = 22$, and so $x = 11$.
We can complete the magic square as follows:

19	5	15
9	13	17
11	21	7

Answer: (B)

21. We label the numbers in the empty boxes as y and z, so the numbers in the boxes are thus $8, y, z, 26, x$.
Since the average of z and x is 26, then $x + z = 2(26) = 52$ or $z = 52 - x$.
We rewrite the list as $8, y, 52 - x, 26, x$.
Since the average of 26 and y is $52 - x$, then $26 + y = 2(52 - x)$ or $y = 104 - 26 - 2x = 78 - 2x$.
We rewrite the list as $8, 78 - 2x, 52 - x, 26, x$.
Since the average of 8 and $52 - x$ is $78 - 2x$, then

$$\begin{aligned} 8 + (52 - x) &= 2(78 - 2x) \\ 60 - x &= 156 - 4x \\ 3x &= 96 \\ x &= 32 \end{aligned}$$

Therefore, $x = 32$.

Answer: (D)

22. Since $JKLM$ is a rectangle, then the angles at J and K are each 90°, so each of $\triangle SJP$ and $\triangle QKP$ is right-angled.
By the Pythagorean Theorem in $\triangle SJP$, we have

$$SP^2 = JS^2 + JP^2 = 52^2 + 39^2 = 2704 + 1521 = 4225$$

Since $SP > 0$, then $SP = \sqrt{4225} = 65$.
Since $PQRS$ is a rhombus, then $PQ = PS = 65$.
By the Pythagorean Theorem in $\triangle QKP$, we have

$$KP^2 = PQ^2 - KQ^2 = 65^2 - 25^2 = 4225 - 625 = 3600$$

Since $KP > 0$, then $KP = \sqrt{3600} = 60$.
(Instead of using the Pythagorean Theorem, we could note instead that $\triangle SJP$ is a scaled-up version of a 3-4-5 right-angled triangle and that $\triangle QKP$ is a scaled-up version of a 5-12-13 right-angled triangle. This would allow us to use the known ratios of side lengths to calculate the missing side length.)
Since KQ and PZ are parallel and PK and WQ are parallel, then $PKQW$ is a rectangle, and

so $PW = KQ = 25$.
Similarly, $JPZS$ is a rectangle and so $PZ = JS = 52$.
Thus, $WZ = PZ - PW = 52 - 25 = 27$.
Also, $SYRM$ is a rectangle. Since JM and KL are parallel ($JKLM$ is a rectangle), JK and ML are parallel, and PQ and SR are parallel ($PQRS$ is a rhombus), then $\angle MSR = \angle KQP$ and $\angle SRM = \angle QPK$.
Since $\triangle SMR$ and $\triangle QKP$ have two equal angles, then their third angles must be equal too. Thus, the triangles have the same proportions. Since the hypotenuses of the triangles are equal, then the triangles must in fact be exactly the same size; that is, the lengths of the corresponding sides must be equal. (We say that $\triangle SMR$ is congruent to $\triangle QKP$ by "angle-side-angle".)
In particular, $MR = KP = 60$.
Thus, $ZY = SY - SZ = MR - JP = 60 - 39 = 21$.
Therefore, the perimeter of rectangle $WXYZ$ is $2(21) + 2(27) = 96$.

Answer: (D)

23. First, we note that $2010 = 10(201) = 2(5)(3)(67)$ and so $2010^2 = 2^2 3^2 5^2 67^2$.
Consider N consecutive four-digit positive integers.
For the product of these N integers to be divisible by 2010^2, it must be the case that two different integers are divisible by 67 (which would mean that there are at least 68 integers in the list) or one of the integers is divisible by 67^2.
Since we want to minimize N (and indeed because none of the answer choices is at least 68), we look for a list of integers in which one is divisible by $67^2 = 4489$.
Since the integers must all be four-digit integers, then the only multiples of 4489 the we must consider are 4489 and 8978.
First, we consider a list of N consecutive integers including 4489.
Since the product of these integers must have 2 factors of 5 and no single integer within 10 of 4489 has a factor of 25, then the list must include two integers that are multiples of 5. To minimize the number of integers in the list, we try to include 4485 and 4490.
Thus our candidate list is 4485, 4486, 4487, 4488, 4489, 4490.
The product of these integers includes 2 factors of 67 (in 4489), 2 factors of 5 (in 4485 and 4490), 2 factors of 2 (in 4486 and 4488), and 2 factors of 3 (since each of 4485 and 4488 is divisible by 3). Thus, the product of these 6 integers is divisible by 2010^2.
Therefore, the shortest possible list including 4489 has length 6.
Next, we consider a list of N consecutive integers including 8978.
Here, there is a nearby integer containing 2 factors of 5, namely 8975.
So we start with the list 8975, 8976, 8977, 8978 and check to see if it has the required property.
The product of these integers includes 2 factors of 67 (in 8978), 2 factors of 5 (in 8975), and 2 factors of 2 (in 8976). However, the only integer in this list divisible by 3 is 8976, which has only 1 factor of 3.
To include a second factor of 3, we must include a second multiple of 3 in the list. Thus, we extend the list by one number to 8979.
Therefore, the product of the numbers in the list $8975, 8976, 8977, 8978, 8979$ is a multiple of 2010^2. The length of this list is 5.
Thus, the smallest possible value of N is 5.
(Note that a quick way to test if an integer is divisible by 3 is to add its digit and see if this total is divisible by 3. For example, the sum of the digits of 8979 is 33; since 33 is a multiple of 3, then 8979 is a multiple of 3.)

Answer: (A)

24. We label the terms $x_1, x_2, x_3, \ldots, x_{2009}, x_{2010}$.
Suppose that S is the sum of the odd-numbered terms in the sequence; that is,

$$S = x_1 + x_3 + x_5 + \cdots + x_{2007} + x_{2009}$$

We know that the sum of all of the terms is 5307; that is,

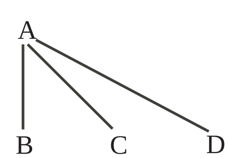
$$x_1 + x_2 + x_3 + \cdots + x_{2009} + x_{2010} = 5307$$

Next, we pair up the terms: each odd-numbered term with the following even-numbered term. That is, we pair the first term with the second, the third term with the fourth, and so on, until we pair the 2009th term with the 2010th term. There are 1005 such pairs.
In each pair, the even-numbered term is one bigger than the odd-numbered term. That is, $x_2 - x_1 = 1$, $x_4 - x_3 = 1$, and so on.
Therefore, the sum of the even-numbered terms is 1005 greater than the sum of the odd-numbered terms. Thus, the sum of the even-numbered terms is $S + 1005$.
Since the sum of all of the terms equals the sum of the odd-numbered terms plus the sum of the even-numbered terms, then $S + (S + 1005) = 5307$ or $2S = 4302$ or $S = 2151$.
Thus, the required sum is 2151.

ANSWER: (C)

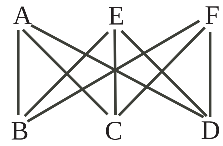
25. Before we answer the given question, we determine the number of ways of choosing 3 objects from 5 objects and the number of ways of choosing 2 objects from 5 objects.
Consider 5 objects labelled B, C, D, E, F.
The possible pairs are: BC, BD, BE, BF, CD, CE, CF, DE, DF, EF. There are 10 such pairs.
The possible triples are: DEF, CEF, CDF, CDE, BEF, BDF, BDE, BCF, BCE, BCD. There are 10 such triples.
(Can you see why there are the same number of pairs and triples?)

Label the six teams A, B, C, D, E, F.
We start by considering team A.
Team A plays 3 games, so we must choose 3 of the remaining 5 teams for A to play. As we saw above, there are 10 ways to do this.
Without loss of generality, we pick one of these sets of 3 teams for A to play, say A plays B, C and D.
We keep track of everything by drawing diagrams, joining the teams that play each other with a line.
Thus far, we have



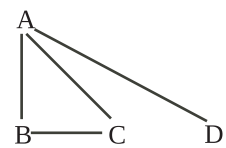
There are two possible cases now – either none of B, C and D play each other, or at least one pair of B, C, D plays each other.

Case 1: None of the teams that play A play each other
In the configuration above, each of B, C and D play two more games. They already play A and cannot play each other, so they must each play E and F.
This gives

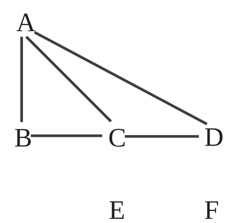


No further choices are possible.
There are 10 possible schedules in this type of configuration. These 10 combinations come from choosing the 3 teams that play A.

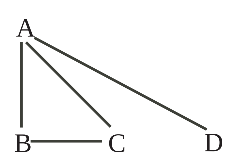
Case 2: Some of the teams that play A play each other
Here, at least one pair of the teams that play A play each other.
Given the teams B, C and D playing A, there are 3 possible pairs (BC, BD, CD).
We pick one of these pairs, say BC. (This gives $10 \times 3 = 30$ configurations so far.)



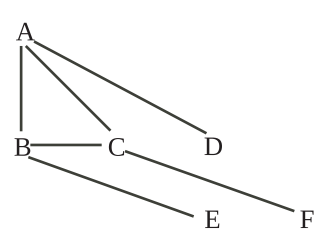
It is now not possible for B or C to also play D. If it was the case that C, say, played D, then we would have the configuration



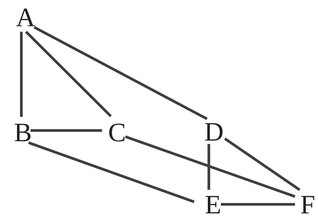
Here, A and C have each played 3 games and B and D have each played 2 games. Teams E and F are unaccounted for thus far. They cannot both play 3 games in this configuration as the possible opponents for E are B, D and F, and the possible opponents for F are B, D and E, with the "B" and "D" possibilities only to be used once.
A similar argument shows that B cannot play D.
Thus, B or C cannot also play D. So we have the configuration



Here, A has played 3 games, B and C have each played 2 games, and D has played 1 game.
B and C must play 1 more game and cannot play D or A.
They must play E and F in some order. There are 2 possible ways to assign these games (BE and CF, or BF and CE.) This gives $30 \times 2 = 60$ configurations so far.
Suppose that B plays E and C plays F.



So far, A, B and C each play 3 games and E, F and D each play 1 game.
The only way to complete the configuration is to join D, E and F.



Therefore, there are 60 possible schedules in this case.

In total, there are $10 + 60 = 70$ possible schedules.

Answer: (E)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Pascal Contest

(Grade 9)

Wednesday, February 18, 2009

Solutions

1. Calculating, $2 \times 9 - \sqrt{36} + 1 = 18 - 6 + 1 = 13$.

   Answer: (D)

2. On Saturday, Deepit worked 6 hours. On Sunday, he worked 4 hours.
   Therefore, he worked $6 + 4 = 10$ hours in total on Saturday and Sunday.

   Answer: (E)

3. Since 1 piece of gum costs 1 cent, then 1000 pieces of gum cost 1000 cents.
   Since there are 100 cents in a dollar, the total cost is \$10.00.

   Answer: (D)

4. Since each of the 18 classes has 28 students, then there are $18 \times 28 = 504$ students who attend the school.
   On Monday, there were 496 students present, so $504 - 496 = 8$ students were absent.

   Answer: (A)

5. The sum of the angles around any point is 360°.
   Therefore, $5x^\circ + 4x^\circ + x^\circ + 2x^\circ = 360^\circ$ or $12x = 360$ or $x = 30$.

   Answer: (D)

6. When -1 is raised to an even exponent, the result is 1.
   When -1 is raised to an odd exponent, the result is -1.
   Thus, $(-1)^5 - (-1)^4 = -1 - 1 = -2$.

   Answer: (A)

7. Since PQ is horizontal and the y-coordinate of P is 1, then the y-coordinate of Q is 1.
   Since QR is vertical and the x-coordinate of R is 5, then the x-coordinate of Q is 5.
   Therefore, the coordinates of Q are $(5, 1)$.

   Answer: (C)

8. When $y = 3$, we have $\dfrac{y^3 + y}{y^2 - y} = \dfrac{3^3 + 3}{3^2 - 3} = \dfrac{27 + 3}{9 - 3} = \dfrac{30}{6} = 5$.

   Answer: (D)

9. Since there are 4 $\clubsuit$'s in each of the first two columns, then at least 1 $\clubsuit$ must be moved out of each of these columns to make sure that each column contains exactly three $\clubsuit$'s.
   Therefore, we need to move at least 2 $\clubsuit$'s in total.
   If we move the $\clubsuit$ from the top left corner to the bottom right corner

	♣	♣	♣	
♣	♣	♣		♣
♣	♣			
♣	♣		♣	
		♣	♣	♣

   and the $\clubsuit$ from the second row, second column to the third row, fifth column

	♣	♣	♣	
♣		♣		♣
♣	♣			♣
♣	♣		♣	
		♣	♣	♣

   then we have exactly three $\clubsuit$'s in each row and each column.

Therefore, since we must move at least 2 ♣'s and we can achieve the configuration that we want by moving 2 ♣'s, then 2 is the smallest number.
(There are also other combinations of moves that will give the required result.)

ANSWER: (B)

10. *Solution 1*
Since $z = 4$ and $x + y = 7$, then $x + y + z = (x + y) + z = 7 + 4 = 11$.

Solution 2
Since $z = 4$ and $x + z = 8$, then $x + 4 = 8$ or $x = 4$.
Since $x = 4$ and $x + y = 7$, then $4 + y = 7$ or $y = 3$.
Therefore, $x + y + z = 4 + 3 + 4 = 11$.

ANSWER: (C)

11. We write out the five numbers to 5 decimal places each, without doing any rounding:

$$\begin{array}{rcl} 5.07\overline{6} & = & 5.07666\ldots \\ 5.0\overline{76} & = & 5.07676\ldots \\ 5.07 & = & 5.07000 \\ 5.076 & = & 5.07600 \\ 5.\overline{076} & = & 5.07607\ldots \end{array}$$

We can use these representations to order the numbers as

$$5.07000, 5.07600, 5.07607\ldots, 5.07666\ldots, 5.07676\ldots$$

so the number in the middle is $5.\overline{076}$.

ANSWER: (E)

12. *Solution 1*
Since there are 24 hours in a day, Francis spends $\frac{1}{3} \times 24 = 8$ hours sleeping.
Also, he spends $\frac{1}{4} \times 24 = 6$ hours studying, and $\frac{1}{8} \times 24 = 3$ hours eating.
The number of hours that he has left is $24 - 8 - 6 - 3 = 7$ hours.

Solution 2
Francis spends $\frac{1}{3} + \frac{1}{4} + \frac{1}{8} = \frac{8+6+3}{24} = \frac{17}{24}$ of a day either sleeping, studying or eating.
This leaves him $1 - \frac{17}{24} = \frac{7}{24}$ of his day.
Since there are 24 hours in a full day, then he has 7 hours left.

ANSWER: (D)

13. *Solution 1*
Since the sum of the angles in a triangle is 180°, then

$$\angle QPS = 180^\circ - \angle PQS - \angle PSQ = 180^\circ - 48^\circ - 38^\circ = 94^\circ$$

Therefore, $\angle RPS = \angle QPS - \angle QPR = 94^\circ - 67^\circ = 27^\circ$.

Solution 2
Since the sum of the angles in a triangle is 180°, then

$$\angle QRP = 180^\circ - \angle PQR - \angle QPR = 180^\circ - 48^\circ - 67^\circ = 65^\circ$$

Therefore, $\angle PRS = 180^\circ - \angle PRQ = 180^\circ - 65^\circ = 115^\circ$.
Using $\triangle PRS$,

$$\angle RPS = 180^\circ - \angle PRS - \angle PSR = 180^\circ - 115^\circ - 38^\circ = 27^\circ$$

ANSWER: (A)

14. The perimeter of the shaded region equals the sum of the lengths of OP and OQ plus the length of arc PQ.
Each of OP and OQ has length 5.
Arc PQ forms $\frac{3}{4}$ of the circle with centre O and radius 5, because the missing portion corresponds to a central angle of 90°, and so is $\frac{1}{4}$ of the total circle.
Thus, the length of arc PQ is $\frac{3}{4}$ of the circumference of this circle, or $\frac{3}{4}(2\pi(5)) = \frac{15}{2}\pi$.
Therefore, the perimeter is $5 + 5 + \frac{15}{2}\pi \approx 33.56$ which, of the given answers, is closest to 34.
Answer: (A)

15. After some trial and error, we obtain the two lists $\{4, 5, 7, 8, 9\}$ and $\{3, 6, 7, 8, 9\}$.
Why are these the only two?

If the largest number of the five integers was 8, then the largest that the sum could be would be $8 + 7 + 6 + 5 + 4 = 30$, which is too small. This tells us that we must include one 9 in the list. (We cannot include any number larger than 9, since each number must be a single-digit number.)
Therefore, the sum of the remaining four numbers is $33 - 9 = 24$.

If the largest of the four remaining numbers is 7, then their largest possible sum would be $7 + 6 + 5 + 4 = 22$, which is too small. Therefore, we also need to include an 8 in the list.
Thus, the sum of the remaining three numbers is $24 - 8 = 16$.

If the largest of the three remaining numbers is 6, then their largest possible sum would be $6 + 5 + 4 = 15$, which is too small. Therefore, we also need to include an 7 in the list.
Thus, the sum of the remaining two numbers is $16 - 7 = 9$.

This tells us that we need two different positive integers, each less than 7, that add to 9. These must be 3 and 6 or 4 and 5.
This gives us the two lists above, and shows that they are the only two such lists.
Answer: (B)

16. The area of the entire grid is $4 \times 9 = 36$.
The area of $\triangle PQR$ is $\frac{1}{2}(QR)(PQ) = \frac{1}{2}(3)(4) = 6$.
The area of $\triangle STU$ is $\frac{1}{2}(ST)(UT) = \frac{1}{2}(4)(3) = 6$.
The area of the rectangle with base RS is $2 \times 4 = 8$.
Therefore, the total shaded area is $6 + 6 + 8 = 20$ and so the unshaded area is $36 - 20 = 16$.
The ratio of the shaded area to the unshaded area is $20 : 16 = 5 : 4$.
Answer: (E)

17. We can suppose that each test is worth 100 marks.
Since the average of her five test marks is 73%, then the total number of marks that she received is $5 \times 73 = 365$.
Once her teacher removes a mark, her new average is 76% so the sum of the remaining four marks is $4 \times 76 = 304$.
Since $365 - 304 = 61$, then the mark removed was 61%.
Answer: (B)

18. *Solution 1*
From December 31, 1988 to December 31, 2008, a total of 20 years have elapsed.
A time period of 20 years is the same as five 4 year periods.

Thus, the population of Arloe has doubled 5 times over this period to its total of 3456.
Doubling 5 times is equivalent to multiplying by $2^5 = 32$.
Therefore, the population of Arloe on December 31, 1988 was $\dfrac{3456}{32} = 108$.

Solution 2
The population doubles every 4 years going forward, so is halved every 4 years going backwards in time.
The population on December 31, 2008 was 3456.
The population on December 31, 2004 was $3456 \div 2 = 1728$.
The population on December 31, 2000 was $1728 \div 2 = 864$.
The population on December 31, 1996 was $864 \div 2 = 432$.
The population on December 31, 1992 was $432 \div 2 = 216$.
The population on December 31, 1988 was $216 \div 2 = 108$.

Answer: (D)

19. Since Pat drives 60 km at 80 km/h, this takes him $\dfrac{60 \text{ km}}{80 \text{ km/h}} = \frac{3}{4}$ h.
Since Pat has 2 hours in total to complete the trip, then he has $2 - \frac{3}{4} = \frac{5}{4}$ hours left to complete the remaining $150 - 60 = 90$ km.
Therefore, he must travel at $\dfrac{90 \text{ km}}{\frac{5}{4} \text{ h}} = \frac{360}{5}$ km/h $= 72$ km/h.

Answer: (C)

20. Since the three numbers in each straight line must have a product of 3240 and must include 45, then the other two numbers in each line must have a product of $\frac{3240}{45} = 72$.
The possible pairs of positive integers are 1 and 72, 2 and 36, 3 and 24, 4 and 18, 6 and 12, and 8 and 9.
The sums of the numbers in these pairs are 73, 38, 27, 22, 18, and 17.
To maximize the sum of the eight numbers, we want to choose the pairs with the largest possible sums, so we choose the first four pairs.
Thus, the largest possible sum of the eight numbers is $73 + 38 + 27 + 22 = 160$.

Answer: (E)

21. Since each of Alice and Bob rolls one 6-sided die, then there are $6 \times 6 = 36$ possible combinations of rolls.
Each of these 36 possibilities is equally likely.
Alice wins when the two values rolled differ by 1. The possible combinations that differ by 1 are $(1, 2)$, $(2, 3)$, $(3, 4)$, $(4, 5)$, $(5, 6)$, $(2, 1)$, $(3, 2)$, $(4, 3)$, $(5, 4)$, and $(6, 5)$.
Therefore, there are 10 combinations when Alice wins.
Thus her probability of winning is $\frac{10}{36} = \frac{5}{18}$.

Answer: (C)

22. Diameters PQ and RS cross at the centre of the circle, which we call O.
The area of the shaded region is the sum of the areas of $\triangle POS$ and $\triangle ROQ$ plus the sum of the areas of sectors POR and SOQ.
Each of $\triangle POS$ and $\triangle ROQ$ is right-angled and has its two perpendicular sides of length 4 (the radius of the circle).
Therefore, the area of each of these triangles is $\frac{1}{2}(4)(4) = 8$.
Each of sector POR and sector SOQ has area $\frac{1}{4}$ of the total area of the circle, as each has

central angle 90° (that is, $\angle POR = \angle SOQ = 90^\circ$) and 90° is one-quarter of the total central angle.
Therefore, each sector has area $\frac{1}{4}(\pi(4^2)) = \frac{1}{4}(16\pi) = 4\pi$.
Thus, the total shaded area is $2(8) + 2(4\pi) = 16 + 8\pi$.

ANSWER: (E)

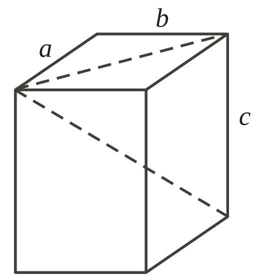
23. The maximum possible mass of a given coin is $7 \times (1 + 0.0214) = 7 \times 1.0214 = 7.1498$ g.
The minimum possible mass of a given coin is $7 \times (1 - 0.0214) = 7 \times 0.9786 = 6.8502$ g.
What are the possible numbers of coins that could make up 1000 g?
To find the largest number of coins, we want the coins to be as light as possible. If all of the coins were as light as possible, we would have $\dfrac{1000}{6.8502} \approx 145.98$ coins. Now, we cannot have a non-integer number of coins. This means that we must have at most 145 coins. (If we had 146 coins, the total mass would have to be at least $146 \times 6.8502 = 1000.1292$ g, which is too heavy.) Practically, we can get 145 coins to have a total mass of 1000 g by taking 145 coins at the minimum possible mass and making each slightly heavier.
To find the smallest number of coins, we want the coins to be as heavy as possible. If all of the coins were as heavy as possible, we would have $\dfrac{1000}{7.1498} \approx 139.86$ coins. Again, we cannot have a non-integer number of coins. This means that we must have at least 140 coins. (If we had 139 coins, the total mass would be at most $139 \times 7.1498 = 993.8222$ g, which is too light.)
Therefore, the difference between the largest possible number and smallest possible number of coins is $145 - 140 = 5$.

ANSWER: (B)

24. Divide the large cube of side length 40 into 8 smaller cubes of side length 20, by making three cuts of the large cube through its centre using planes parallel to the pairs of faces.
Each of these small cubes has the centre of the large cube as its vertex.
Each of these small cubes also just encloses one of the large spheres, in the sense that the sphere just touches each of the faces of the small cube.
We call the sphere that fits in the central space the inner sphere. To make this sphere as large possible, its centre will be at the centre of the large cube. (If this was not the case, the centre be outside one of the small cubes, and so would be farther away from one of the large spheres than from another.)
To find the radius of the inner sphere, we must find the shortest distance from the centre of the large cube (that is, the centre of the inner sphere) to one of the large spheres. (Think of starting the inner sphere as a point at the centre of the cube and inflating it until it just touches the large spheres.)

Consider one of these small cubes and the sphere inside it.
Join the centre of the small cube to one of its vertices.
Since the small cube has side length 20, then this new segment has length $\sqrt{10^2 + 10^2 + 10^2}$ or $\sqrt{300}$, since to get from the centre to a vertex, we must go over 10, down 10 and across 10. (See below for an explanation of why this distance is thus $\sqrt{300}$.)
The inner sphere will touch the large sphere along this segment.
Thus, the radius of the inner sphere will be this distance ($\sqrt{300}$) minus the radius of the large sphere (10), and so is $\sqrt{300} - 10 \approx 7.32$.
Of the given answers, this is closest to 7.3.

(We need to justify why the distance from the centre of the small cube to its vertex is $\sqrt{10^2+10^2+10^2}$.
Divide the small cube into 8 tiny cubes of side length 10 each. The distance from the centre of the small cube to its vertex is equal to the length of a diagonal of one of the tiny cubes.
Consider a rectangular prism with edge lengths a, b and c. What is the length, d, of the diagonal inside the prism?
By the Pythagorean Theorem, a face with side lengths a and b has a diagonal of length $\sqrt{a^2+b^2}$.
Consider the triangle formed by this diagonal, the diagonal of the prism and one of the vertical edges of the prism, of length c.



This triangle is right-angled since the vertical edge is perpendicular to the top face. By the Pythagorean Theorem again, $d^2 = (\sqrt{a^2+b^2})^2 + c^2$, so $d^2 = a^2+b^2+c^2$ or $d = \sqrt{a^2+b^2+c^2}$.)

ANSWER: (B)

25. The three machines operate in a way such that if the two numbers in the output have a common factor larger than 1, then the two numbers in the input would have to have a common factor larger than 1.
To see this, let us look at each machine separately. We use the fact that if two numbers are each multiples of d, then their sum and difference are also multiples of d.
Suppose that (m,n) is input into Machine A. The output is (n,m). If n and m have a common factor larger than 1, then m and n do as well.
Suppose that (m,n) is input into Machine B. The output is $(m+3n,n)$. If $m+3n$ and n have a common factor d, then $(m+3n)-n-n-n=m$ has a factor of d as each part of the subtraction is a multiple of d. Therefore, m and n have a common factor of d.
Suppose that (m,n) is input into Machine C. The output is $(m-2n,n)$. If $m-2n$ and n have a common factor d, then $(m-2n)+n+n=m$ has a factor of d as each part of the addition is a multiple of d. Therefore, m and n have a common factor of d.
In each case, any common factor that exists in the output is present in the input.

Let us look at the numbers in the five candidates.
After some work, we can find the prime factorizations of the six integers:

$$\begin{aligned}
2009 &= 7(287) = 7(7)(41)\\
1016 &= 8(127) = 2(2)(2)(127)\\
1004 &= 4(251) = 2(2)(251)\\
1002 &= 2(501) = 2(3)(167)\\
1008 &= 8(126) = 8(3)(42) = 16(3)(3)(7) = 2(2)(2)(2)(3)(3)(7)\\
1032 &= 8(129) = 8(3)(43) = 2(2)(2)(3)(43)
\end{aligned}$$

Therefore, the only one of 1002, 1004, 1008, 1016, 1032 that has a common factor larger than 1 with 2009 is 1008, which has a common factor of 7 with 2009.

How does this help? Since 2009 and 1008 have a common factor of 7, then whatever pair was input to produce $(2009, 1008)$ must have also had a common factor of 7. Also, the pair that was input to create this pair also had a common factor of 7. This can be traced back through every step to say that the initial pair that produces the eventual output of $(2009, 1008)$ must have a common factor of 7.
Thus, $(2009, 1008)$ cannot have come from $(0, 1)$.

Notes:

- This does not tell us that the other four pairs necessarily work. It does tell us, though, that $(2009, 1008)$ cannot work.
- We can trace the other four outputs back to $(0, 1)$ with some effort. (This process is easier to do than it is to describe!)
  To do this, we notice that if the output of Machine A was (a, b), then its input was (b, a), since Machine A switches the two entries.
  Also, if the output of Machine B was (a, b), then its input was $(a - 3b, b)$, since Machine B adds three times the second number to the first.
  Lastly, if the output of Machine C was (a, b), then its input was $(a + 2b, b)$, since Machine C subtracts two times the second number from the first.
  Consider $(2009, 1016)$ for example. We try to find a way from $(2009, 1016)$ back to $(0, 1)$. We only need to find one way that works, rather than looking for a specific way.

  We note before doing this that starting with an input of (m, n) and then applying Machine B then Machine C gives an output of $((m + 3n) - 2n, n) = (m + n, n)$. Thus, if applying Machine B then Machine C (we call this combination "Machine BC") gives an output of (a, b), then its input must have been $(a - b, b)$. We can use this combined machine to try to work backwards and arrive at $(0, 1)$. This will simplify the process and help us avoid negative numbers.
  We do this by making a chart and by attempting to make the larger number smaller wherever possible:

Output	Machine	Input
$(2009, 1016)$	BC	$(993, 1016)$
$(993, 1016)$	A	$(1016, 993)$
$(1016, 993)$	BC	$(23, 993)$
$(23, 993)$	A	$(993, 23)$
$(993, 23)$	BC, 43 times	$(4, 23)$
$(4, 23)$	A	$(23, 4)$
$(23, 4)$	BC, 5 times	$(3, 4)$
$(3, 4)$	A	$(4, 3)$
$(4, 3)$	BC	$(1, 3)$
$(1, 3)$	A	$(3, 1)$
$(3, 1)$	B	$(0, 1)$

  Therefore, by going up through this table, we can see a way to get from an initial input of $(0, 1)$ to a final output of $(2009, 1016)$.
  In a similar way, we can show that we can obtain final outputs of each of $(2009, 1004)$, $(2009, 1002)$, and $(2009, 1032)$.

Answer: (D)

Canadian Mathematics Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Pascal Contest

(Grade 9)

Tuesday, February 19, 2008

Solutions

1. Calculating, $\dfrac{2+3+4}{2\times3\times4} = \dfrac{9}{24} = \dfrac{3}{8}$.

Answer: (E)

2. Since $3x - 9 = 12$, then $3x = 12 + 9 = 21$.
Since $3x = 21$, then $6x = 2(3x) = 2(21) = 42$.
(Note that we did not need to determine the value of x.)

Answer: (A)

3. Calculating, $\sqrt{5^2 - 4^2} = \sqrt{25 - 16} = \sqrt{9} = 3$.

Answer: (B)

4. *Solution 1*
Since $JLMR$ is a rectangle and $JR = 2$, then $LM = 2$.
Similarly, since $JL = 8$, then $RM = 8$.
Since $RM = 8$ and $RQ = 3$, then $QM = 8 - 3 = 5$.
Since $KLMQ$ is a rectangle with $QM = 5$ and $LM = 2$, its area is $5(2) = 10$.

Solution 2
Since $JL = 8$ and $JR = 2$, then the area of rectangle $JLMR$ is $2(8) = 16$.
Since $RQ = 3$ and $JR = 2$, then the area of rectangle $JKQR$ is $2(3) = 6$.
The area of rectangle $KLMQ$ is the difference between these areas, or $16 - 6 = 10$.

Answer: (C)

5. Since $x = 12$ and $y = -6$, then

$$\frac{3x+y}{x-y} = \frac{3(12)+(-6)}{12-(-6)} = \frac{30}{18} = \frac{5}{3}$$

Answer: (C)

6. *Solution 1*
Since $\angle PQS$ is an exterior angle of $\triangle QRS$, then $\angle PQS = \angle QRS + \angle QSR$, so $136^\circ = x^\circ + 64^\circ$ or $x = 136 - 64 = 72$.

Solution 2
Since $\angle PQS = 136^\circ$, then $\angle RQS = 180^\circ - \angle PQS = 180^\circ - 136^\circ = 44^\circ$.
Since the sum of the angles in $\triangle QRS$ is 180°, then $44^\circ + 64^\circ + x^\circ = 180^\circ$ or $x = 180 - 44 - 64 = 72$.

Answer: (A)

7. In total, there are $5 + 6 + 7 + 8 = 26$ jelly beans in the bag.
Since there are 8 blue jelly beans, the probability of selecting a blue jelly bean is $\frac{8}{26} = \frac{4}{13}$.

Answer: (D)

8. Since Olave sold 108 apples in 6 hours, then she sold $108 \div 6 = 18$ apples in one hour.
A time period of 1 hour and 30 minutes is equivalent to 1.5 hours.
Therefore, Olave will sell $1.5 \times 18 = 27$ apples in 1 hour and 30 minutes.

Answer: (A)

9. Since the length of the rectangular grid is 10 and the grid is 5 squares wide, then the side length of each square in the grid is $10 \div 5 = 2$.
There are 4 horizontal wires, each of length 10, which thus have a total length of $4 \times 10 = 40$.
Since the side length of each square is 2 and the rectangular grid is 3 squares high, then the length of each vertical wire is $3 \times 2 = 6$.
Since there are 6 vertical wires, the total length of the vertical wires is $6 \times 6 = 36$.
Therefore, the total length of wire is $40 + 36 = 76$.

Answer: (E)

10. *Solution 1*
Since Q is at 46 and P is at -14, then the distance along the number line from P to Q is $46 - (-14) = 60$.
Since S is three-quarters of the way from P to Q, then S is at $-14 + \frac{3}{4}(60) = -14 + 45 = 31$.
Since T is one-third of the way from P to Q, then T is at $-14 + \frac{1}{3}(60) = -14 + 20 = 6$.
Thus, the distance along the number line from T to S is $31 - 6 = 25$.

Solution 2
Since Q is at 46 and P is at -14, then the distance along the number line from P to Q is $46 - (-14) = 60$.
Since S is three-quarters of the way from P to Q and T is one-third of the way from P to Q, then the distance from T to S is $60\left(\frac{3}{4} - \frac{1}{3}\right) = 60\left(\frac{9}{12} - \frac{4}{12}\right) = 60\left(\frac{5}{12}\right) = 25$.

Answer: (D)

11. In total, $30 + 20 = 50$ students wrote the Pascal Contest at Mathville Junior High.
Since 30% (or $\frac{3}{10}$) of the boys won certificates and 40% (or $\frac{4}{10}$) of the girls won certificates, then the total number of certificates awarded was $\frac{3}{10}(30) + \frac{4}{10}(20) = 9 + 8 = 17$.
Therefore, 17 of 50 participating students won certificates. In other words, $\frac{17}{50} \times 100\% = 34\%$ of the participating students won certificates.

Answer: (A)

12. Since the perimeter of the rectangle is 56, then

$$\begin{aligned} 2(x+4) + 2(x-2) &= 56 \\ 2x + 8 + 2x - 4 &= 56 \\ 4x + 4 &= 56 \\ 4x &= 52 \\ x &= 13 \end{aligned}$$

Therefore, the rectangle is $x + 4 = 17$ by $x - 2 = 11$, so has area $17(11) = 187$.

Answer: (B)

13. Using exponent rules, $2^3 \times 2^2 \times 3^3 \times 3^2 = 2^{3+2} \times 3^{3+2} = 2^5 \times 3^5 = (2 \times 3)^5 = 6^5$.

Answer: (A)

14. *Solution 1*
The wording of the problem tells us that $a + b + c + d + e + f$ must be the same no matter what numbers abc and def are chosen that satisfy the conditions.
An example that works is $889 + 111 = 1000$.
In this case, $a+b+c+d+e+f = 8+8+9+1+1+1 = 28$, so this must always be the value.

Solution 2
Consider performing this "long addition" by hand.
Consider first the units column.
Since $c+f$ ends in a 0, then $c+f=0$ or $c+f=10$. The value of $c+f$ cannot be 20 or more, as c and f are digits.
Since none of the digits is 0, we cannot have $c+f=0+0$ so $c+f=10$. (This means that we "carry" a 1 to the tens column.)
Since the result in the tens column is 0 and there is a 1 carried into this column, then $b+e$ ends in a 9, so we must have $b+e=9$. (Since b and e are digits, $b+e$ cannot be 19 or more.)
In the tens column, we thus have $b+e=9$ plus the carry of 1, so the resulting digit in the tens column is 0, with a 1 carried to the hundreds column.
Using a similar analysis in the hundreds column to that in the tens column, we must have $a+d=9$.
Therefore, $a+b+c+d+e+f=(a+d)+(b+e)+(c+f)=9+9+10=28$.

Answer: (D)

15. Each of $\triangle PSQ$ and $\triangle RSQ$ is right-angled at S, so we can use the Pythagorean Theorem in both triangles.
In $\triangle RSQ$, we have $QS^2=QR^2-SR^2=25^2-20^2=625-400=225$, so $QS=\sqrt{225}=15$ since $QS>0$.
In $\triangle PSQ$, we have $PQ^2=PS^2+QS^2=8^2+225=64+225=289$, so $PQ=\sqrt{289}=17$ since $PQ>0$.
Therefore, the perimeter of $\triangle PQR$ is $PQ+QR+RP=17+25+(20+8)=70$.

Answer: (E)

16. Suppose the radius of the circle is r cm.
Then the area M is πr^2 cm^2 and the circumference N is $2\pi r$ cm.
Thus, $\dfrac{\pi r^2}{2\pi r}=20$ or $\dfrac{r}{2}=20$ or $r=40$.

Answer: (C)

17. *Solution 1*
The large cube has a total surface area of 5400 cm^2 and its surface is made up of 6 identical square faces. Thus, the area of each face, in square centimetres, is $5400\div 6=900$.
Because each face is square, the side length of each face is $\sqrt{900}=30$ cm.
Therefore, each edge of the cube has length 30 cm and so the large cube has a volume of $30^3=27\,000$ cm^3.
Because the large cube is cut into small cubes each having volume 216 cm^3, then the number of small cubes equals $27\,000\div 216=125$.

Solution 2
Since the large cube has 6 square faces of equal area and the total surface area of the cube is 5400 cm^2, then the surface area of each face is $5400\div 6=900$ cm^2.
Since each face is square, then the side length of each square face of the cube is $\sqrt{900}=30$ cm, and so the edge length of the cube is 30 cm.
Since each smaller cube has a volume of 216 cm^3, then the side length of each smaller cube is $\sqrt[3]{216}=6$ cm.
Since the side length of the large cube is 30 cm and the side length of each smaller cube is 6 cm, then $30\div 6=5$ smaller cubes fit along each edge of the large cube.
Thus, the large cube is made up of $5^3=125$ smaller cubes.

Answer: (B)

18. *Solution 1*

Alex has 265 cents in total.
Since 265 is not divisible by 10, Alex cannot have only dimes, so must have at least 1 quarter.
If Alex has 1 quarter, then he has $265 - 25 = 240$ cents in dimes, so 24 dimes.
Alex cannot have 2 quarters, since $265 - 2(25) = 215$ is not divisible by 10.
If Alex has 3 quarters, then he has $265 - 3(25) = 190$ cents in dimes, so 19 dimes.
Continuing this argument, we can see that Alex cannot have an even number of quarters, since the total value in cents of these quarters would end in a 0, making the total value of the dimes end in a 5, which is not possible.
If Alex has 5 quarters, then he has $265 - 5(25) = 140$ cents in dimes, so 14 dimes.
If Alex has 7 quarters, then he has $265 - 7(25) = 90$ cents in dimes, so 9 dimes.
If Alex has 9 quarters, then he has $265 - 9(25) = 40$ cents in dimes, so 4 dimes.
If Alex has more than 9 quarters, then he will have even fewer than 4 dimes, so we do not need to investigate any more possibilities since we are told that Alex has more dimes than quarters.
So the possibilities for the total number of coins that Alex has are $1 + 24 = 25$, $3 + 19 = 22$, $5 + 14 = 19$, and $7 + 9 = 16$.
Therefore, the smallest number of coins that Alex could have is 16.
(Notice that each time we increase the number of quarters above, we are in effect exchanging 2 quarters (worth 50 cents) for 5 dimes (also worth 50 cents).)

Solution 2

Suppose that Alex has d dimes and q quarters, where d and q are non-negative integers.
Since Alex has \$2.65, then $10d + 25q = 265$ or $2d + 5q = 53$.
Since the right side is odd, then the left side must be odd, so $5q$ must be odd, so q must be odd.
If $q \geq 11$, then $5q \geq 55$, which is too large.
Therefore, $q < 11$, leaving $q = 1, 3, 5, 7, 9$ which give $d = 24, 19, 14, 9, 4$.
The solution with $d > q$ and $d + q$ smallest is $q = 7$ and $d = 9$, giving 16 coins in total.

ANSWER: (B)

19. From the definition, the first and second digits of an upright integer automatically determine the third digit, since it is the sum of the first two digits.
Consider first those upright integers beginning with 1.
These are 101, 112, 123, 134, 145, 156, 167, 178, and 189, since $1 + 0 = 1$, $1 + 1 = 2$, and so on. (The second digit cannot be 9, otherwise the last "digit" would be $1 + 9 = 10$, which is impossible.) There are 9 such numbers.
Beginning with 2, the upright integers are 202, 213, 224, 235, 246, 257, 268, and 279. There are 8 of them.
We can continue the pattern and determine the numbers of the upright integers beginning with 3, 4, 5, 6, 7, 8, and 9 to be 7, 6, 5, 4, 3, 2, and 1.
Therefore, there are $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ positive 3-digit upright integers.

ANSWER: (D)

20. The sum of the six given integers is $1867 + 1993 + 2019 + 2025 + 2109 + 2121 = 12134$.
The four of these integers that have a mean of 2008 must have a sum of $4(2008) = 8032$. (We do not know which integers they are, but we do not actually need to know.)
Thus, the sum of the remaining two integers must be $12134 - 8032 = 4102$.
Therefore, the mean of the remaining two integers is $\frac{4102}{2} = 2051$.

(We can verify that 1867, 2019, 2025 and 2121 do actually have a mean of 2008, and that 1993 and 2109 have a mean of 2051.)

Answer: (D)

21. The maximum possible value of $\dfrac{p}{q}$ is when p is as large as possible (that is, 10) and q is as small as possible (that is, 12). Thus, the maximum possible value of $\dfrac{p}{q}$ is $\dfrac{10}{12} = \dfrac{5}{6}$.
The minimum possible value of $\dfrac{p}{q}$ is when p is as small as possible (that is, 3) and q is as large as possible (that is, 21). Thus, the maximum possible value of $\dfrac{p}{q}$ is $\dfrac{3}{21} = \dfrac{1}{7}$.
The difference between these two values is $\dfrac{5}{6} - \dfrac{1}{7} = \dfrac{35}{42} - \dfrac{6}{42} = \dfrac{29}{42}$.

Answer: (A)

22. Suppose that the distance from Ginger's home to her school is d km.
Since there are 60 minutes in an hour, then $3\frac{3}{4}$ minutes (or $\frac{15}{4}$ minutes) is $\frac{15}{4} \times \frac{1}{60} = \frac{1}{16}$ of an hour.
Since Ginger walks at 4 km/h, then it takes her $\dfrac{d}{4}$ hours to walk to school.
Since Ginger runs at 6 km/h, then it takes her $\dfrac{d}{6}$ hours to run to school.
Since she saves $\frac{1}{16}$ of an hour by running, then the difference between these times is $\frac{1}{16}$ of an hour, so

$$\begin{aligned} \frac{d}{4} - \frac{d}{6} &= \frac{1}{16} \\ \frac{3d}{12} - \frac{2d}{12} &= \frac{1}{16} \\ \frac{d}{12} &= \frac{1}{16} \\ d &= \frac{12}{16} = \frac{3}{4} \end{aligned}$$

Therefore, the distance from Ginger's home to her school is $\frac{3}{4}$ km.

Answer: (E)

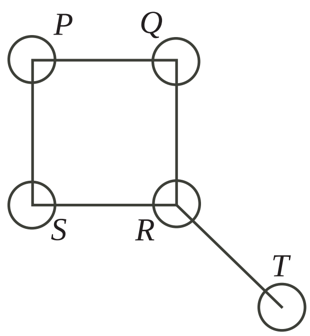
23. Suppose that the distance from line M to line L is d m.
Therefore, the total length of piece W to the left of the cut is d m.
Since piece X is 3 m from line M, then the length of piece X to the left of L is $(d - 3)$ m, because 3 of the d m to the left of L are empty.
Similarly, the lengths of pieces Y and Z to the left of line L are $(d - 2)$ m and $(d - 1.5)$ m.
Therefore, the total length of lumber to the left of line L is

$$d + (d - 3) + (d - 2) + (d - 1.5) = 4d - 6.5 \text{ m}$$

Since the total length of lumber on each side of the cut is equal, then this total length is $\frac{1}{2}(5 + 3 + 5 + 4) = 8.5$ m.
(We could instead find the lengths of lumber to the right of line L to be $5 - d$, $6 - d$, $7 - d$, and $5.5 - d$ and equate the sum of these lengths to the sum of the lengths on the left side.)
Therefore, $4d - 6.5 = 8.5$ or $4d = 15$ or $d = 3.75$, so the length of the part of piece W to the left of L is 3.75 m.

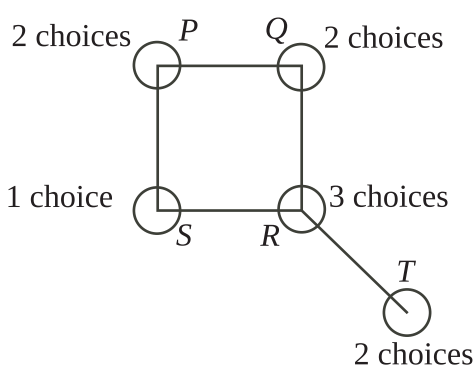
Answer: (D)

24. We label the five circles as shown in the diagram.



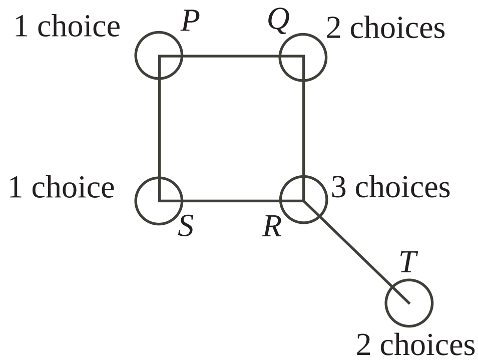
We note that there are 3 possible colours and that no two adjacent circles can be coloured the same.
Consider circle R. There are three possible colours for this circle.
For each of these colours, there are 2 possible colours for T (either of the two colours that R is not), since it cannot be the same colour as R.
Circles Q and S are then either the same colour as each other, or are different colours.

Case 1: Q and S are the same colour
In this case, there are 2 possible colours for Q (either of the colours that R is not) and 1 possibility for S (the same colour as Q).
For each of these possible colours for Q/S, there are two possible colours for P (either of the colours that Q and S are not).



In this case, there are thus $3 \times 2 \times 2 \times 1 \times 2 = 24$ possible ways of colouring the circles.

Case 2: Q and S are different colours
In this case, there are 2 possible colours for Q (either of the colours that R is not) and 1 possibility for S (since it must be different from R and different from Q).
For each of these possible colourings of Q and S, there is 1 possible colour for P (since Q and S are different colours, P is different from these, and there are only 3 colours in total).



In this case, there are thus $3 \times 2 \times 2 \times 1 \times 1 = 12$ possible ways of colouring the circles.

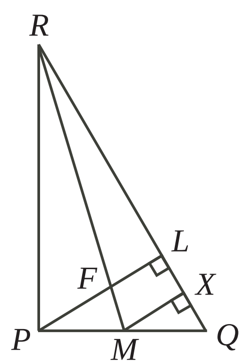
In total, there are thus $24 + 12 = 36$ possible ways to colour the circles.

Answer: (D)

25. Since $PQ = 2$ and M is the midpoint of PQ, then $PM = MQ = \frac{1}{2}(2) = 1$.
Since $\triangle PQR$ is right-angled at P, then by the Pythagorean Theorem,

$$RQ = \sqrt{PQ^2 + PR^2} = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$$

(Note that we could say that $\triangle PQR$ is a 30°-60°-90° triangle, but we do not actually need this fact.)
Since PL is an altitude, then $\angle PLR = 90°$, so $\triangle RLP$ is similar to $\triangle RPQ$ (these triangles have right angles at L and P respectively, and a common angle at R).
Therefore, $\dfrac{PL}{QP} = \dfrac{RP}{RQ}$ or $PL = \dfrac{(QP)(RP)}{RQ} = \dfrac{2(2\sqrt{3})}{4} = \sqrt{3}$.
Similarly, $\dfrac{RL}{RP} = \dfrac{RP}{RQ}$ so $RL = \dfrac{(RP)(RP)}{RQ} = \dfrac{(2\sqrt{3})(2\sqrt{3})}{4} = 3$.
Therefore, $LQ = RQ - RL = 4 - 3 = 1$ and $PF = PL - FL = \sqrt{3} - FL$.
So we need to determine the length of FL.
Drop a perpendicular from M to X on RQ.



Then $\triangle MXQ$ is similar to $\triangle PLQ$, since these triangles are each right-angled and they share a common angle at Q. Since $MQ = \frac{1}{2}PQ$, then the corresponding sides of $\triangle MXQ$ are half as long as those of $\triangle PLQ$.
Therefore, $QX = \frac{1}{2}QL = \frac{1}{2}(1) = \frac{1}{2}$ and $MX = \frac{1}{2}PL = \frac{1}{2}(\sqrt{3}) = \frac{\sqrt{3}}{2}$.
Since $QX = \frac{1}{2}$, then $RX = RQ - QX = 4 - \frac{1}{2} = \frac{7}{2}$.

Now $\triangle RLF$ is similar to $\triangle RXM$ (they are each right-angled and share a common angle at R).

Therefore, $\dfrac{FL}{MX} = \dfrac{RL}{RX}$ so $FL = \dfrac{(MX)(RL)}{RX} = \dfrac{\frac{\sqrt{3}}{2}(3)}{\frac{7}{2}} = \frac{3\sqrt{3}}{7}$.

Thus, $PF = \sqrt{3} - \frac{3\sqrt{3}}{7} = \frac{4\sqrt{3}}{7}$.

ANSWER: (C)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Pascal Contest

(Grade 9)

Tuesday, February 20, 2007

Solutions

1. Calculating, $3 \times (7 - 5) - 5 = 3 \times 2 - 5 = 6 - 5 = 1$.

   Answer: (B)

2. Since x is less than -1 and greater than -2, then the best estimate of the given choices is -1.3.

   Answer: (B)

3. The shaded square has side length 1 so has area $1^2 = 1$.
   The rectangle has dimensions 3 by 5 so has area $3 \times 5 = 15$.
   Thus, the fraction of the rectangle that is shaded is $\frac{1}{15}$.

   Answer: (A)

4. Calculating, $2^5 - 5^2 = 32 - 25 = 7$.

   Answer: (E)

5. In 3 hours, Leona earns \$24.75, so she makes $\$24.75 \div 3 = \$8.25$ per hour.
   Therefore, in a 5 hour shift, Leona earns $5 \times \$8.25 = \$41.25$.

   Answer: (E)

6. Calculating, $\dfrac{\sqrt{64}+\sqrt{36}}{\sqrt{64+36}} = \dfrac{8+6}{\sqrt{100}} = \dfrac{14}{10} = \dfrac{7}{5}$.

   Answer: (A)

7. *Solution 1*
   In total Megan and Dan inherit \$1 010 000.
   Since each donates 10%, then the total donated is 10% of \$1 010 000, or \$101 000.

   Solution 2
   Megan donates 10% of \$1 000 000, or \$100 000.
   Dan donates 10% of \$10 000, or \$1000.
   In total, they donate $\$100\,000 + \$1000 = \$101\,000$.

   Answer: (A)

8. We think of BC as the base of $\triangle ABC$. Its length is 12.
   Since the y-coordinate of A is 9, then the height of $\triangle ABC$ from base BC is 9.
   Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(12)(9) = 54$.

   Answer: (B)

9. Calculating the given difference using a common denominator, we obtain $\frac{5}{8} - \frac{1}{16} = \frac{9}{16}$.
   Since $\frac{9}{16}$ is larger than each of $\frac{1}{2} = \frac{8}{16}$ and $\frac{7}{16}$, then neither (D) nor (E) is correct.
   Since $\frac{9}{16}$ is less than each of $\frac{3}{4} = \frac{12}{16}$, then (A) is not correct.
   As a decimal, $\frac{9}{16} = 0.5625$.
   Since $\frac{3}{5} = 0.6$ and $\frac{5}{9} = 0.\overline{5}$, then $\frac{9}{16} > \frac{5}{9}$, so (C) is the correct answer.

   Answer: (C)

10. Since $M = 2007 \div 3$, then $M = 669$.
    Since $N = M \div 3$, then $N = 669 \div 3 = 223$.
    Since $X = M - N$, then $X = 669 - 223 = 446$.

    Answer: (E)

11. The mean of 6, 9 and 18 is $\dfrac{6+9+18}{3} = \dfrac{33}{3} = 11$.
Thus the mean of 12 and y is 11, so the sum of 12 and y is $2(11) = 22$, so $y = 10$.
Answer: (C)

12. In $\triangle PQR$, since $PR = RQ$, then $\angle RPQ = \angle PQR = 48^\circ$.
Since $\angle MPN$ and $\angle RPQ$ are opposite angles, then $\angle MPN = \angle RPQ = 48^\circ$.
In $\triangle PMN$, $PM = PN$, so $\angle PMN = \angle PNM$.
Therefore, $\angle PMN = \frac{1}{2}(180^\circ - \angle MPN) = \frac{1}{2}(180^\circ - 48^\circ) = \frac{1}{2}(132^\circ) = 66^\circ$.
Answer: (D)

13. The prime numbers smaller than 10 are 2, 3, 5, and 7.
The two of these numbers which are different and add to 10 are 3 and 7.
The product of 3 and 7 is $3 \times 7 = 21$.
Answer: (B)

14. Since there were 21 males writing and the ratio of males to females writing is $3 : 7$, then there are $\frac{7}{3} \times 21 = 49$ females writing.
Therefore, the total number of students writing is $49 + 21 = 70$.
Answer: (D)

15. *Solution 1*
The first stack is made up of $1 + 2 + 3 + 4 + 5 = 15$ blocks.
The second stack is made up of $1 + 2 + 3 + 4 + 5 + 6 = 21$ blocks.
There are 36 blocks in total.
We start building the new stack from the top.
Since there are more than 21 blocks, we need at least 6 rows.
For 7 rows, $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ blocks are needed.
For 8 rows, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ blocks are needed.
Therefore, Clara can build a stack with 0 blocks leftover.

Solution 2
Since the new stack will be larger than the second stack shown, let us think about adding new rows to this second stack using the blocks from the first stack.
The first stack contains $1 + 2 + 3 + 4 + 5 = 15$ blocks in total.
The first two rows that we would add to the bottom of the second stack would have 7 and 8 blocks in them, for a total of 15 blocks.
This uses all of the blocks from the first stack, with none left over, and creates a similar stack.
Therefore, there are 0 blocks left over.
Answer: (A)

16. *Solution 1*
The sum of the numbers in the second row is $10 + 16 + 22 = 48$, so the sum of the numbers in any row, column or diagonal is 48.
In the first row, $P + 4 + Q = 48$ so $P + Q = 44$.
In the third row, $R + 28 + S = 48$ so $R + S = 20$.
Therefore, $P + Q + R + S = 44 + 20 = 64$.

Solution 2
The sum of the numbers in the second row is $10 + 16 + 22 = 48$, so the sum of the numbers in any row, column or diagonal is 48.
From the first row, $P + 4 + Q = 48$ so $P + Q = 44$.
From the first column, $P + 10 + R = 48$ so $P + R = 38$.
Subtracting these two equations gives $(P + Q) - (P + R) = 44 - 38$ or $Q - R = 6$.
From one of the diagonals, $R + 16 + Q = 48$ or $Q + R = 32$.
Adding these last two equations, $2Q = 38$ or $Q = 19$, so $R = 32 - Q = 13$.
Also, $P = 44 - Q = 25$.
From the last row, $13 + 28 + S = 48$, or $S = 7$.
Thus, $P + Q + R + S = 25 + 19 + 13 + 7 = 64$.

Solution 3
The sum of all of the numbers in the grid is

$$P + Q + R + S + 10 + 16 + 22 + 28 + 4 = P + Q + R + S + 80$$

But the sum of the three numbers in the second column is $4 + 16 + 28 = 48$, so the sum of the three numbers in each column is 48.
Thus, the total of the nine numbers in the grid is $3(48) = 144$, so $P + Q + R + S + 80 = 144$ or $P + Q + R + S = 64$.

Answer: (C)

17. At present, the sum of Norine's age and the number of years that she has worked is $50+19 = 69$.
This total must increase by $85 - 69 = 16$ before she can retire.
As every year passes, this total increases by 2 (as her age increases by 1 and the number of years that she has worked increases by 1).
Thus, it takes 8 years for her total to increase from 69 to 85, so she will be $50 + 8 = 58$ when she can retire.

Answer: (C)

18. By the Pythagorean Theorem in $\triangle PQR$, $PQ^2 = PR^2 - QR^2 = 13^2 - 5^2 = 144$, so $PQ = \sqrt{144} = 12$.
By the Pythagorean Theorem in $\triangle PQS$, $QS^2 = PS^2 - PQ^2 = 37^2 - 12^2 = 1225$, so $QS = \sqrt{1225} = 35$.
Therefore, the perimeter of $\triangle PQS$ is $12 + 35 + 37 = 84$.

Answer: (D)

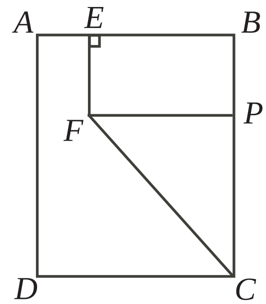
19. Since the reciprocal of $\frac{3}{10}$ is $\left(\frac{1}{x} + 1\right)$, then

$$\begin{aligned} \frac{1}{x} + 1 &= \frac{10}{3} \\ \frac{1}{x} &= \frac{7}{3} \\ x &= \frac{3}{7} \end{aligned}$$

so $x = \frac{3}{7}$.

Answer: (C)

20. Draw a line from F to BC, parallel to AB, meeting BC at P.



Since EB is parallel to FP and $\angle FEB = 90^\circ$, then $EBPF$ is a rectangle.
Since $EB = 40$, then $FP = 40$; since $EF = 30$, then $BP = 30$.
Since $AD = 80$, then $BC = 80$, so $PC = 80 - 30 = 50$.
Therefore, the area of $EBCF$ is sum of the areas of rectangle $EBPF$ (which is $30 \times 40 = 1200$) and $\triangle FPC$ (which is $\frac{1}{2}(40)(50) = 1000$), or $1200 + 1000 = 2200$.
Since the areas of $AEFCD$ and $EBCF$ are equal, then each is 2200, so the total area of rectangle $ABCD$ is 4400.
Since $AD = 80$, then $AB = 4400 \div 80 = 55$.
Therefore, $AE = AB - EB = 55 - 40 = 15$.

Answer: (D)

21. Let us first consider the possibilities for each integer separately:

- The two-digit prime numbers are 11, 13, 17, 19. The only one whose digits add up to a prime number is 11. Therefore, $P = 11$.
- Since Q is a multiple of 5 between 2 and 19, then the possible values of Q are 5, 10, 15.
- The odd numbers between 2 and 19 that are not prime are 9 and 15, so the possible values of R are 9 and 15.
- The squares between 2 and 19 are 4, 9 and 16. Only 4 and 9 are squares of prime numbers, so the possible values of S are 4 and 9.
- Since $P = 11$, the possible value of Q are 5, 10 and 15, and T is the average of P and Q, then T could be 8, 10.5 or 13. Since T is also a prime number, then T must be 13, so $Q = 15$.

We now know that $P = 11$, $Q = 15$ and $T = 13$.
Since the five numbers are all different, then R cannot be 15, so $R = 9$.
Since $R = 9$, S cannot be 9, so $S = 4$.
Therefore, the largest of the five integers is $Q = 15$.

Answer: (B)

22. By the Pythagorean Theorem, $PR = \sqrt{QR^2 + PQ^2} = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$ km.
Asafa runs a total distance of $8 + 15 + 7 = 30$ km at 21 km/h in the same time that Florence runs a total distance of $17 + 7 = 24$ km.
Therefore, Asafa's speed is $\frac{30}{24} = \frac{5}{4}$ of Florence's speed, so Florence's speed is $\frac{4}{5} \times 21 = \frac{84}{5}$ km/h.

Asafa runs the last 7 km in $\frac{7}{21} = \frac{1}{3}$ hour, or 20 minutes.

Florence runs the last 7 km in $\frac{7}{\frac{84}{5}} = \frac{35}{84} = \frac{5}{12}$ hour, or 25 minutes.
Since Asafa and Florence arrive at S together, then Florence arrived at R 5 minutes before Asafa.

ANSWER: (E)

23. The total area of the larger circle is $\pi(2^2) = 4\pi$, so the total area of the shaded regions must be $\frac{5}{12}(4\pi) = \frac{5}{3}\pi$.

Suppose that $\angle ADC = x^\circ$.
The area of the unshaded portion of the inner circle is thus $\frac{x}{360}$ of the total area of the inner circle, or $\frac{x}{360}(\pi(1^2)) = \frac{x}{360}\pi$ (since $\angle ADC$ is $\frac{x}{360}$ of the largest possible central angle (360°)).
The area of the shaded portion of the inner circle is thus $\pi - \frac{x}{360}\pi = \frac{360-x}{360}\pi$.

The total area of the outer ring is the difference of the areas of the outer and inner circles, or $\pi(2^2) - \pi(1^2) = 3\pi$.
The shaded area in the outer ring will be $\frac{x}{360}$ of this total area, since $\angle ADC$ is $\frac{x}{360}$ of the largest possible central angle (360°).
So the shaded area in the outer ring is $\frac{x}{360}(3\pi) = \frac{3x}{360}\pi$.

So the total shaded area (which must equal $\frac{5}{3}\pi$) is, in term of x, $\frac{3x}{360}\pi + \frac{360-x}{360}\pi = \frac{360+2x}{360}\pi$.
Therefore, $\frac{360+2x}{360} = \frac{5}{3} = \frac{600}{360}$, so $360 + 2x = 600$ or $x = 120$.
Thus, $\angle ADC = 120^\circ$.

ANSWER: (B)

24. First, we complete the next several spaces in the spiral to try to get a better sense of the pattern:

17	16	15	14	13	
18	5	4	3	12	
19	6	1	2	11	
20	7	8	9	10	
21	22	23	24	25	26

We notice from this extended spiral that the odd perfect squares lie on a diagonal extending down and to the right from the 1, since 1, 9, 25 and so on will complete a square of numbers when they are written. (Try blocking out the numbers larger than each of these to see this.) This pattern does continue since when each of these odd perfect squares is reached, the number of spaces up to that point in the sequence actually does form a square.
The first odd perfect square larger than 2007 is $45^2 = 2025$.
2025 will lie 18 spaces to the left of 2007 in this row. (The row with 2025 will actually be long enough to be able to move 18 spaces to the left from 2025.)
The odd perfect square before 2025 is $43^2 = 1849$, so 1850 will be the number directly above 2025, as the row containing 1849 will continue one more space to the right before turning up.

Since 1850 is directly above 2025, then 1832 is directly above 2007.
The odd perfect square after 2025 is $47^2 = 2209$, so 2208 will be the number directly below 2025, since 2209 will be one space to the right and one down.
Since 2208 is directly below 2025, then 2190 is directly below 2007.
Therefore, the sum of the numbers directly above and below 2007 is $1832 + 2190 = 4022$.

Answer: (E)

25. For x and $3x$ to each have even digits only, x must be in one of the following forms. (Here, a, b, c represent digits that can each be 0, 2 or 8, and n is a digit that can only be 2 or 8.)

- $nabc$ ($2 \times 3 \times 3 \times 3 = 54$ possibilities)
- $na68$ ($2 \times 3 = 6$ possibilities)
- $n68a$ ($2 \times 3 = 6$ possibilities)
- $68ab$ ($3 \times 3 = 9$ possibilities)
- $n668$ (2 possibilities)
- $668a$ (3 possibilities)
- 6668 (1 possibility)
- 6868 (1 possibility)

In total, there are 82 possibilities for x.

In general terms, these are the only forms that work, since digits of 0, 2 and 8 in x produce even digits with an even "carry" (0 or 2) thus keeping all digits in $3x$ even, while a 6 may be used, but must be followed by 8 or 68 or 668 in order to give a carry of 2.

More precisely, why do these forms work, and why are they the only forms that work?
First, we note that $3 \times 0 = 0$, $3 \times 2 = 6$, $3 \times 4 = 12$, $3 \times 6 = 18$ and $3 \times 8 = 24$.
Thus, each even digit of x will produce an even digit in the corresponding position of $3x$, but may affect the next digit to the left in $3x$ through its "carry".
Note that a digit of 0 or 2 in x produces no carry, while a digit of 8 in x produces an even carry.
Therefore, none of these three digits can possibly create an odd digit in $3x$ (either directly or through carrying), as they each create an even digit in the corresponding position of $3x$ and do not affect whether the next digit to the left is even or odd. (We should note that the carry into any digit in $3x$ can never be more than 2, so we do not have to worry about creating a carry of 1 from a digit in x of 0 or 2, or a carry of 3 from a digit of 8 in x through multiple carries.)
So a digit of 2 or 8 can appear in any position of x and a digit of 0 can appear in any position of x except for the first position.

A digit of 4 can never appear in x, as it will always produce a carry of 1, and so will always create an odd digit in $3x$.

A digit of 6 can appear in x, as long as the carry from the previous digit is 2 to make the carry forward from the 6 equal to 2. (The carry into the 6 cannot be larger than 2.) When this happens, we have $3 \times 6 + \text{Carry} = 20$, and so a 2 is carried forward, which does not affect whether next digit is even or odd.
A carry of 2 can occur if the digit before the 6 is an 8, or if the digit before the 6 is a 6 which is preceded by 8 or by 68.

Combining the possible uses of 0, 2, 6, and 8 gives us the list of possible forms above, and hence 82 possible values for x.

Answer: (A)

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Pascal Contest

(Grade 9)

Wednesday, February 22, 2006

Solutions

1. Calculating each of the numerator and denominator first, $\dfrac{550+50}{5^2+5} = \dfrac{600}{25+5} = \dfrac{600}{30} = 20$.

   ANSWER: (E)

2. Calculating under each square root first, $\sqrt{36+64} - \sqrt{25-16} = \sqrt{100} - \sqrt{9} = 10 - 3 = 7$.

   ANSWER: (B)

3. The positive whole numbers which divide exactly into 18 are 1, 2, 3, 6, 9, 18, of which there are 6.

   ANSWER: (D)

4. Since $A + B = 5$, then $B - 3 + A = (A + B) - 3 = 5 - 3 = 2$.

   ANSWER: (A)

5. The volume of the rectangular solid is $2 \times 4 \times 8 = 64$.
   If the length of each edge of the cube is s, then the volume of the cube is s^3, which must be equal to 64. Since $s^3 = 64$, then $s = 4$, so the length of each edge of the cube is 4.

   ANSWER: (B)

6. Since Ravindra ate $\frac{2}{5}$ of the pizza and Hongshu ate half as much as Ravindra, then Hongshu ate $\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$ of the pizza.
   After these two had eaten, there was $1 - \frac{2}{5} - \frac{1}{5} = \frac{2}{5}$ of the pizza left.
   As a percentage, $\frac{2}{5}$ is equivalent to 40%, so there was 40% of the original pizza left.

   ANSWER: (C)

7. Since 1 triangle balances 2 squares, then 2 triangles balance 4 squares.
   Since 2 triangles also balance 3 circles, then 3 circles balance 4 squares.

   ANSWER: (E)

8. Since the areas of the three squares are 16, 49 and 169, then their side lengths are $\sqrt{16} = 4$, $\sqrt{49} = 7$ and $\sqrt{169} = 13$, respectively.
   Thus, the average of their side lengths is $\dfrac{4+7+13}{3} = 8$.

   ANSWER: (A)

9. Since the rectangle has width w, length 8, and perimeter 24, then $2w+2(8) = 24$ or $2w+16 = 24$ or $2w = 8$ or $w = 4$.
   Therefore, the ratio of the width to the length is $4 : 8 = 1 : 2$.

   ANSWER: (C)

10. *Solution 1*
    Looking at the numbers in terms of their digits, then $M4 - 3N = 16$ or $M4 = 3N + 16$.
    In order to get a units digit of 4 from $3N + 16$, then N must be an 8.
    Thus, $M4 = 38 + 16 = 54$.
    Therefore, the digit M is a 5, and so $M + N = 5 + 8 = 13$.

    Solution 2
    Looking at the numbers in terms of their digits, then $M4 - 3N = 16$.
    In order to get a units digit of 6 from $M4 - 3N$, then N must be an 8.
    Thus, $M4 - 38 = 16$ or $M4 = 36 + 16 = 54$.
    Therefore, the digit M is a 5, and so $M + N = 5 + 8 = 13$.

    ANSWER: (D)

11. Evaluating each of the given choices with $x = 9$,

$$\sqrt{9} = 3 \qquad \frac{9}{2} = 4\tfrac{1}{2} \qquad 9 - 5 = 4 \qquad \frac{40}{9} = 4\tfrac{4}{9} \qquad \frac{9^2}{20} = \frac{81}{20} = 4\tfrac{1}{20}$$

Since $\frac{1}{2}$ is larger than either $\frac{4}{9}$ or $\frac{1}{20}$, then the largest of the possibilities when $x = 9$ is $\dfrac{x}{2}$.

Answer: (B)

12. Since the perimeter of the triangle is 36, then $7 + (x + 4) + (2x + 1) = 36$ or $3x + 12 = 36$ or $3x = 24$ or $x = 8$.
Thus, the lengths of the three sides of the triangle are 7, $8+4 = 12$ and $2(8)+1 = 17$, of which the longest is 17.

Answer: (C)

13. *Solution 1*
From the given information, $P + Q = 16$ and $P - Q = 4$.
Adding these two equations, we obtain $P + Q + P - Q = 16 + 4$ or $2P = 20$ or $P = 10$.

Solution 2
The value of P is increased by Q to give 16 and decreased by Q to give 4.
Thus, the difference of 12 between these two answers is twice the value of Q, so $2Q = 12$ whence $Q = 6$.
Since $P + Q = 16$, we have $P + 6 = 16$ or $P = 10$.

Answer: (D)

14. Using a common denominator of 12, we have $\dfrac{6}{12} + \dfrac{8}{12} + \dfrac{9}{12} + \dfrac{n}{12} = \dfrac{24}{12}$ or $\dfrac{23 + n}{12} = \dfrac{24}{12}$.
Comparing numerators, we obtain $23 + n = 24$ or $n = 1$.

Answer: (E)

15. *Solution 1*
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes or $1\frac{3}{4}$ hours or $\frac{7}{4}$ hours.
Since Jim drives 84 km in $\frac{7}{4}$ hours at a constant speed, then this speed is $\dfrac{84}{\frac{7}{4}} = 84 \times \dfrac{4}{7} = 48$ km/h.

Solution 2
Since Jim drives from 7:45 p.m. to 9:30 p.m., then Jim drives for 1 hour and 45 minutes, which is the same as 7 quarters of an hour.
Since he drives 84 km in 7 quarters of an hour, he drives 12 km in 1 quarter of an hour, or 48 km in one hour, so his speed is 48 km/h.

Answer: (E)

16. We make a chart to determine the sum of each possible combination of top faces. In the chart, the numbers across the top are the numbers from the first die and the numbers down the side are the numbers from the second die. For example, the number in the fourth column and fifth row is the sum of the fourth possible result from the first die and the fifth possible result from the second die, or $3 + 5 = 8$.

	2	2	3	3	5	8
2	4	4	5	5	7	10
2	4	4	5	5	7	10
3	5	5	6	6	8	11
3	5	5	6	6	8	11
5	7	7	8	8	10	13
8	10	10	11	11	13	16

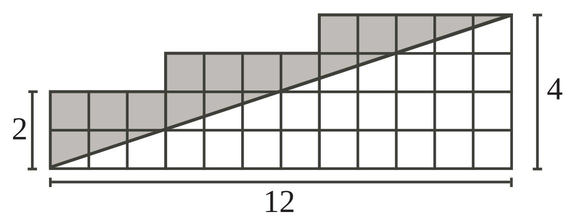
So the possibilities are 4, 5, 6, 7, 8, 10, 11, 13, 16, or nine possibilities in total.
(We could have cut down the size of our table since we didn't have to include both 2's and both 3's either across the top or down the side. As well, we could have also calculated only the numbers on the diagonal and above, since the chart is symmetric.)

Answer: (D)

17. Since $\triangle ADE$ is isosceles, then $\angle AED = \angle EAD = 70^\circ$.
Since the angles in $\triangle ADE$ add to 180°, then $\angle ADE = 180^\circ - 2(70^\circ) = 40^\circ$.
Since $\angle DEC = 2(\angle ADE)$, then $\angle DEC = 2(40^\circ) = 80^\circ$.
Since AEB is a straight line, then $\angle CEB = 180^\circ - 80^\circ - 70^\circ = 30^\circ$.
Since $\triangle EBC$ is isosceles, then $\angle ECB = \angle EBC$.
Thus, in $\triangle EBC$, $30^\circ + 2(\angle EBC) = 180^\circ$ or $2(\angle EBC) = 150^\circ$ or $\angle EBC = 75^\circ$.

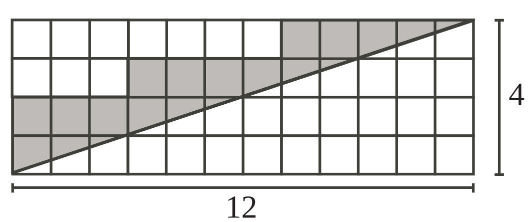
Answer: (A)

18. *Solution 1*
The area of the entire grid in the diagram is 38. (We can obtain this either by counting the individual squares, or by dividing the grid into a 2 by 3 rectangle, a 3 by 4 rectangle, and a 4 by 5 rectangle.)



The area of shaded region is equal to the area of the entire grid minus the area of the unshaded triangle, which is right-angled with a base of 12 and a height of 4.
Therefore, the area of the shaded region is $38 - \frac{1}{2}(12)(4) = 38 - 24 = 14$.

Solution 2
First, we "complete the rectangle" by adding more unshaded squares to obtain a 4 by 12 rectangle whose area is $4(12) = 48$.



Note that we added 10 unshaded squares (whose combined area is 10).
The area of the triangle under the line is half of the area of the entire rectangle, or $\frac{1}{2}(48) = 24$.
Thus, the area of the shaded region is the area of the entire rectangle minus the area of the unshaded region, or $48 - 24 - 10 = 14$.

ANSWER: (C)

19. *Solution 1*
Let the ten integers be n, $n+1$, $n+2$, $n+3$, $n+4$, $n+5$, $n+6$, $n+7$, $n+8$, and $n+9$.
Therefore, $S = n+(n+1)+(n+2)+(n+3)+(n+4)+(n+5)+(n+6)+(n+7)+(n+8)+(n+9)$ or $S = 10n + 45$ and $T = 10n$.
Thus, $S - T = (10n + 45) - 10n = 45$.

Solution 2
Since the question implies that the value of $S-T$ must be the same no matter what 10 integers we try, then we calculate $S - T$ for the integers 1 through 10.
In this case, $S = 1+2+3+4+5+6+7+8+9+10 = 55$ and $T = 10(1) = 10$ or $S-T = 45$.

ANSWER: (A)

20. Let w be the width of each of the identical rectangles.
Since $PQ = 3w$, $RS = 2x$ and $PQ = RS$ (because $PQRS$ is a rectangle), then $2x = 3w$, or $w = \frac{2}{3}x$.
Therefore, the area of each of the five identical rectangles is $x\left(\frac{2}{3}x\right) = \frac{2}{3}x^2$.
Since the area of $PQRS$ is 4000 and it is made up of five of these identical smaller rectangles, then $5\left(\frac{2}{3}x^2\right) = 4000$ or $\frac{10}{3}x^2 = 4000$ or $x^2 = 1200$ or $x \approx 34.6$, which, of the possible answers, is closest to 35.

ANSWER: (A)

21. *Solution 1*
Looking at the third row of the table, $(m+8)+(4+n) = 6$ or $m+n+12 = 6$ or $m+n = -6$.
The sum of the nine numbers in the table is
$m+4+m+4+8+n+8+n+m+8+4+n+6 = 3(m+n)+42 = 3(-6)+42 = 24$

Solution 2
Try setting $m = 0$.

Then the table becomes

0	4	4
8	n	$8+n$
8	$4+n$	6

.

From the third row, $8+(4+n) = 6$ or $n+12 = 6$ or $n = -6$.

The table thus becomes

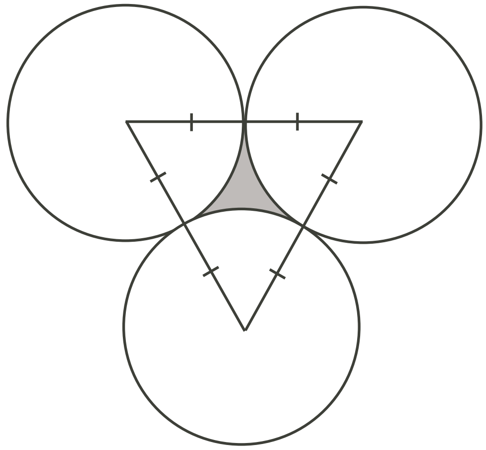
0	4	4
8	-6	2
8	-2	6

.

The sum of the nine numbers in the table is $0+4+4+8+(-6)+2+8+(-2)+6 = 24$.

ANSWER: (E)

22. Join the centre of each circle to the centre of the other two.
Since each circle touches each of the other two, then these line segments pass through the points where the circles touch, and each is of equal length (that is, is equal to twice the length of the radius of one of the circles).



Since each of these line segments have equal length, then the triangle that they form is equilateral, and so each of its angles is equal to 60°.
Now, the perimeter of the shaded region is equal to the sum of the lengths of the three circular arcs which enclose it. Each of these arcs is the arc of one of the circles between the points where this circle touches the other two circles.
Thus, each arc is a 60° arc of one of the circles (since the radii joining either end of each arc to the centre of its circle form an angle of 60°), so is $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$ of the total circumference of the circle, so has length $\frac{1}{6}(36) = 6$.
Therefore, the perimeter of the shaded region is $3(6) = 18$.

Answer: (A)

23. *Solution 1*
Let A be the number of CDs that Anna has, and let B be the number of CDs that Ben has.
If Anna gives 6 CDs to Ben, then Anna would have $A-6$ CDs and Ben would have $B+6$ CDs, so from the given information, $B+6 = 2(A-6)$.
If Anna takes 6 CDs from Ben, then Anna would have $A+6$ CDs and Ben would have $B-6$ CDs, so from the given information, $A+6 = B-6$.
From the first equation, $B = 2A-18$; from the second equation, $B = A+12$.
Therefore, $2A-18 = A+12$ or $A = 30$, and so $B = A+12 = 42$.
Thus, the total number of CDs that Ben and Anna have is $30+42 = 72$.

Solution 2
Let A be the number of CDs that Anna has.
If Anna receives 6 CDs from Ben, then the two of them would have the same number of CDs.
This tells us that Ben has 12 more CDs than Anne, or that Ben has $A+12$ CDs.
If Anna gives 6 CDs to Ben, then Anna would have $A-6$ CDs and Ben would have $A+18$ CDs.
From the given information, $A+18 = 2(A-6)$ or $A+18 = 2A-12$ or $A = 30$.
Therefore, Anna has 30 CDs and Ben has $30+12 = 42$ CDs, so they have $30+42 = 72$ CDs in total.

Answer: (C)

24. *Solution 1*
Suppose that Igor has removed some balls from the bag, and the remaining balls do not satisfy the required condition. What is the maximum number of balls that can remain? In order to

not satisfy the required condition, either there are not 4 balls of any colour (so the maximum number is 9 balls, ie. 3 of each colour) or there are at least 4 balls of one colour, but there are not 3 of either of the other colours.
In this second case, we could have 2 balls of each of two colours, and as many as possible of the third colour. The maximum number of balls of any colour that can be in the bag is 8 (the number of yellow balls with which Igor starts). So the maximum number of balls still in the bag in this case is 12.
Therefore, if Igor removes 8 or more balls, then the remaining balls might not satisfy the required condition.
However, if Igor removes 7 or fewer balls, then the remaining balls will satisfy the required condition, since the maximum number of balls in any case which does not satisfy the condition is 12.
Therefore, the maximum possible value of N is 7.

Solution 2
Since we want to determine the maximum possible value of N, we start with the largest of the answers and rule out answers until we come to the correct answer.
If Igor removed 10 marbles, he might remove 5 red and 5 black marbles, leaving 8 yellow, 2 red, and 0 black marbles, which does not meet the required condition.
Thus, 10 is not the answer.
If Igor removed 9 marbles, he might remove 5 red and 4 black marbles, leaving 8 yellow marbles, 2 red marbles, and 1 black marble, which does not meet the required condition.
Thus, 9 is not the answer.
If Igor removed 8 marbles, he might remove 5 red and 3 black marbles, leaving 8 yellow, 2 red, and 2 black marbles, which does not meet the required condition.
Thus, 8 is not the answer.

Is 7 the answer?
There are $8 + 7 + 5 = 20$ marbles to begin with. If 7 are removed, there are 13 marbles left.
Since there are 13 marbles left, then it is not possible to have 4 or fewer marbles of each of the three colours (otherwise there would be at most 12 marbles). Thus, there are at least 5 marbles of one colour.
Could there be 2 or fewer marbles of each of the other two colours? If so, then since there are 13 marbles in total, there must be at least 9 marbles of the first colour. But there cannot be 9 or more marbles of any colour, as there were at most 8 of each colour to begin with. Therefore, there must be at least 3 of one of the other two colours of marbles.
This tells us that if 7 marbles are removed, there are at least 5 marbles of one colour and 3 of another colour, so choosing $N = 7$ marbles guarantees us the required condition.
Therefore, 7 is the maximum possible value of N.

ANSWER: (B)

25. We will refer to the digits of each of John's and Judith's numbers from the left. Thus, "the first digit" will be the leftmost digit.

If the first digit of John's number is 1, then Judith's number will begin 112.
If the first digit of John's number is 2, then Judith's number will begin 111.
In either case, Judith's number begins with a 1.
Since the first 2187 digits are the same, then John's number begins with a 1.

Since John's number begins with a 1, then Judith's begins 112, so John's begins 112.
Since John's number begins 112, then Judith's begins 112112111, so John's begins 112112111.
Each time we repeat this process, the length of the string which we know will be multiplied by 3.
We continue this process to construct the $2187 = 3^7$ digits of John's number.

We make a table to keep track of this information. We notice that if at one step, the string ends in a 1, then at the next step it will end in a 2, since the 1 becomes 112. Similarly, if at one step, the string ends in a 2, then at the next step, it ends in a 1, since the 2 becomes 111. Also, since each 1 becomes 112 and each 2 becomes 111, then the number of 2's at a given step will be equal to the number of 1's at the previous step. Similarly, the number of 1's at a given step equals 2 times the number of 1's at the previous step plus 3 times the number of 2's at the previous step. (Alternatively, we could determine the total number of 1's by subtracting the number of 2's from the length of the string.)

Step #	Length	# of 1's	# of 2's	Ends in
0	1	1	0	1
1	3	2	1	2
2	9	7	2	1
3	27	20	7	2
4	81	61	20	1
5	243	182	61	2
6	729	547	182	1
7	2187	1640	547	2

How can five consecutive 1's (that is, 11111) be produced in this step 7?
There can never be two consecutive 2's at a given step, since every 2 is the end of one of the blocks and so must be followed by a 1.
Thus, there can never be two consecutive 2's which would produce 111111.
This tells us that 11111 can only be produced by 21 at the previous step.
So the number of occurrences of 11111 at step 7 is equal to the number of occurrences of 21 at step 6. But every 2 at step 6 is followed by a 1 (since the string at step 6 does not end with a 2), so this is equal to the number of 2's at step 6.
Therefore, there are 182 occurrences of 11111 in the 2187 digits of John's number.

Answer: (A)

Canadian Mathematics Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2005 Pascal Contest

(Grade 9)

Wednesday, February 23, 2005

Solutions

1. Calculating each of the numerator and denominator first, $\dfrac{200+10}{20+10} = \dfrac{210}{30} = 7$.

   Answer: (E)

2. Simplifying the terms in pairs, $6a - 5a + 4a - 3a + 2a - a = a + a + a = 3a$.

   Answer: (A)

3. When we substitute $x = 3$, the product becomes $3(2)(1)(0)(-1)$.
   Since one of the factors is 0, the entire product is equal to 0.

   Answer: (C)

4. Of the numbers 2, 3, 4, 5, 6, 7, only the numbers 2, 3, 5, and 7 are prime.
   Since 4 out of the 6 numbers are prime, then the probability of choosing a ball with a prime number is $\frac{4}{6} = \frac{2}{3}$.

   Answer: (D)

5. Calculating from the inside out, $\sqrt{36 \times \sqrt{16}} = \sqrt{36 \times 4} = \sqrt{144} = 12$.

   Answer: (A)

6. When half of the water is removed from the glass, the mass decreases by $1000 - 700 = 300$ g.
   In other words, the mass of half of the original water is 300 g.
   Therefore, the mass of the empty glass equals the mass of the half-full glass, minus the mass of half of the original water, or $700 - 300 = 400$ g.

   Answer: (D)

7. *Solution 1*
   Since $\frac{1}{3}x = 12$, then $x = 3 \times 12 = 36$, so $\frac{1}{4}x = \frac{1}{4}(36) = 9$.

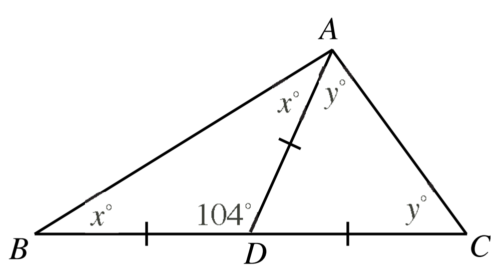
   Solution 2
   Since $\frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$, then $\frac{1}{4}x = \frac{3}{4} \times (\frac{1}{3}x) = \frac{3}{4}(12) = 9$.

   Answer: (C)

8. We calculate the square of each number: $(-5)^2 = 25$, $(\frac{3}{2})^2 = \frac{9}{4}$, $2^2 = 4$, $(\frac{3}{5})^2 = \frac{9}{25}$, and $8^2 = 64$.
   We can quickly tell that each of -5, 2 and 8 is less than its square.
   Since $\frac{3}{2} = 1\frac{1}{2}$ and $\frac{9}{4} = 2\frac{1}{4}$, then $\frac{3}{2}$ is also less than its square.
   However, $\frac{3}{5} = \frac{15}{25}$ is larger than its square $\frac{9}{25}$.

   Answer: (D)

9. *Solution 1*
   Since $\triangle BDA$ is isosceles, $\angle BAD = \angle ABD = x^\circ$.
   Since $\triangle CDA$ is isosceles, $\angle CAD = \angle ACD = y^\circ$.

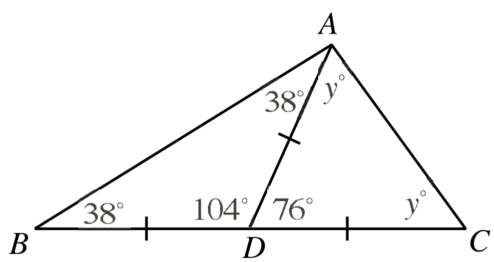


Therefore, $\angle BAC = (x+y)^\circ$.
Since the sum of the angles in $\triangle ABC$ is 180°, then

$$\begin{aligned} x^\circ + y^\circ + (x+y)^\circ &= 180^\circ \\ (2x+2y)^\circ &= 180^\circ \\ 2x+2y &= 180 \\ x+y &= 90 \end{aligned}$$

Therefore, $x+y=90$.

Solution 2
Since $\triangle BDA$ is isosceles, $\angle BAD = \angle ABD = x^\circ$.
Therefore, looking at the sum of the angles in $\triangle ABD$, we have $x^\circ + x^\circ + 104^\circ = 180^\circ$ or $2x+104=180$ or $2x=76$ or $x=38$.
Since $\angle BDA$ and $\angle ADC$ are supplementary, then $\angle ADC = 180^\circ - 104^\circ = 76^\circ$.
Since $\triangle CDA$ is isosceles, $\angle CAD = \angle ACD = y^\circ$.



Therefore, looking at the sum of the angles in $\triangle CDA$, we have $y^\circ + y^\circ + 76^\circ = 180^\circ$ or $2y+76=180$ or $2y=104$ or $y=52$.
Therefore, $x+y=38+52=90$.

Answer: (D)

10. The third term in the sequence is, by definition, the average of the first two terms, namely 32 and 8, or $\frac{1}{2}(32+8)=\frac{1}{2}(40)=20$.
The fourth term is the average of the second and third terms, or $\frac{1}{2}(8+20)=14$.
The fifth term is the average of the third and fourth terms, or $\frac{1}{2}(20+14)=17$.
Therefore, $x=17$.

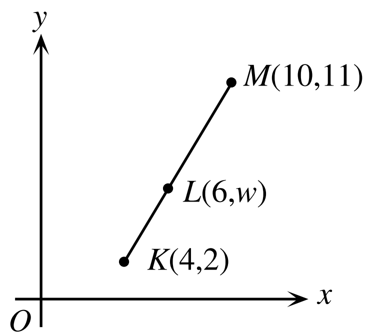
Answer: (A)

11. If a and b are positive integers with $a\times b=13$, then since 13 is a prime number, we must have a and b equal to 1 and 13, or 13 and 1, respectively.
If $b=1$, then since $b\times c=52$, we must have $c=52$. But then we could not have $c\times a=4$. So b cannot be 1.
Thus, $b=13$, so $a=1$, and $c=4$ (which we can get either from knowing $b=13$ and $b\times c=52$, or from knowing $a=1$ and $c\times a=4$.
Therefore, $a\times b\times c = 1\times 13\times 4 = 52$.

Answer: (E)

12. *Solution 1*

To get from K to M, we move 6 units to the right and 9 units up.



Since L lies on the line segment KM and to get from K to L we move $2 = \frac{1}{3} \times 6$ units to the right, then we must move $\frac{1}{3} \times 9 = 3$ units up.
Thus, $w = 2 + 3 = 5$.

Solution 2

The slope of line segment KM is $\dfrac{11-2}{10-4} = \dfrac{9}{6} = \dfrac{3}{2}$.
Since L lies on line segment KM, then the slope of line segment KL will also be $\dfrac{3}{2}$.
Therefore, $\dfrac{w-2}{6-4} = \dfrac{3}{2}$ or $\dfrac{w-2}{2} = \dfrac{3}{2}$, so $w - 2 = 3$ or $w = 5$.

Answer: (B)

13. *Solution 1*

Each unit cube has three exposed faces on the larger cube, and three faces hidden.
Therefore, when the faces of the larger cube are painted, 3 of the 6 faces (or $\frac{1}{2}$ of the surface area) of each small cube are painted, so overall exactly $\frac{1}{2}$ of the surface of area of the small cubes is painted.

Solution 2

Since the bigger cube is 2 by 2 by 2, then it has six 2 by 2 faces, for a total surface area of $6 \times 2 \times 2 = 24$ square units. All of this surface area will be painted.
Each of the unit cubes has a surface area of 6 square units (since each has six 1 by 1 faces), so the total surface area of the 8 unit cubes is $8 \times 6 = 48$ square units.
Of this 48 square units, 24 square units is red, for a fraction of $\frac{24}{48} = \frac{1}{2}$.

Answer: (C)

14. Every integer between 2005 and 3000 has four digits.
Since palindromes are integers whose digits are the same when read forwards and backwards, then a four-digit palindrome must be of the form $xyyx$ where x and y are digits.
Since 3000 is not a palindrome, then the first digit of each of these palindromes must be 2, ie. it must be of the form $2yy2$.
Since each palindrome is at least 2005 and at most 3000, then y can take any value from 1 to 9.
Therefore, there are 9 such palindromes: 2112, 2222, 2332, 2442, 2552, 2662, 2772, 2882, and 2992.

Answer: (C)

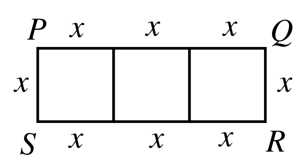
15. When 14 is divided by n, the remainder is 2, so n must divide evenly into $14 - 2 = 12$.
Therefore, n could be 1, 2, 3, 4, 6, or 12.
But the remainder has to be *less than* the number we are dividing by, so n cannot be 1 or 2.
Thus, n can be 3, 4, 6 or 12, so there are 4 possible values of n.

ANSWER: (D)

16. To get the largest possible number by using the digits 1, 2, 5, 6, and 9 exactly once, we must choose the largest digit to be the first digit, the next largest for the next digit, and so on. Thus, the largest possible number is 96 521.
Also, using this reasoning, there are only two numbers that use these digits which are at least 96 500, namely 96 521 and 96 512.
Therefore, 96 512 must be the largest even number that uses each of these digits exactly once.
Reversing our logic, the smallest possible number formed using these digits is 12 569, and the only other such number smaller than 12 600 is 12 596, so this must be the smallest even number formed using these digits.
Calculating the difference, we get $96\,512 - 12\,596 = 83\,916$.

ANSWER: (A)

17. Let the side length of each of the squares be x.



Then the perimeter of $PQRS$ equals $8x$, so $8x = 120$ cm or $x = 15$ cm.
Since $PQRS$ is made up of three squares of side length 15 cm, then its area is $3(15 \text{ cm})^2$ or $3(225)\text{ cm}^2 = 675\text{ cm}^2$.

ANSWER: (B)

18. First, $2005^2 = 4020025$, so the last two digits of 2005^2 are 25.
We need to look at 2005^5, but since we only need the final two digits, we don't actually have to calculate this number entirely.
Consider $2005^3 = 2005^2 \times 2005 = 4020025 \times 2005$. When we carry out this multiplication, the last two digits of the product will only depend on the last two digits of the each of the two numbers being multiplied (try this by hand!), so the last two digits of 2005^3 are the same as the last two digits of $25 \times 5 = 125$, ie. are 25.
Similarly, to calculate 2005^4, we multiply 2005^3 (which ends in 25) by 2005, so by the same reasoning 2005^4 ends in 25.
Similarly, 2005^5 ends in 25.
Therefore, 2005^2 and 2005^5 both end in 25.
Also, $2005^0 = 1$, so the expression overall is equal to $\ldots 25 + 1 + 1 + \ldots 25 = \ldots 52$.
Therefore, the final two digits are 52.

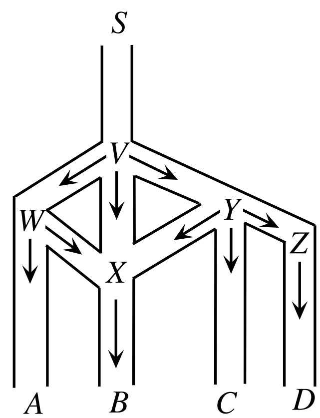
ANSWER: (A)

19. The easiest way to count these numbers is to list them by considering their hundreds digit.
Hundreds digit of 1: No such numbers, since tens digit must be 0, which leaves no option for units digit.
Hundreds digit of 2: Only one possible number, namely 210.

Hundreds digit of 3: Here the tens digit can be 2 or 1, giving numbers 321, 320, 310.
Hundreds digit of 4: Here the tens digit can be 3, 2 or 1, giving numbers 432, 431, 430, 421, 420, 410.
Therefore, there are 10 such numbers in total.

Answer: (B)

20. We label the five junctions as V, W, X, Y, and Z.



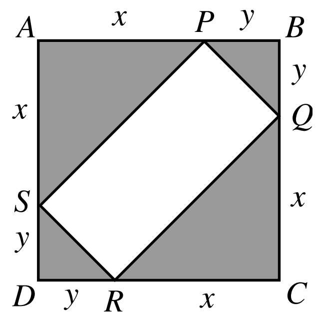
From the arrows which Harry can follow, we see that in order to get to B, he must go through X (and from X, he has to go to B). So we calculate the probability that he gets to X.
To get to X, Harry can go S to V to W to X, or S to V to Y to X, or S to V to X directly.
At V, the probability that Harry goes down any of the three paths (that is, towards W, X or Y) is $\frac{1}{3}$.
So the probability that Harry goes directly from V to X to $\frac{1}{3}$.
At W, the probability that Harry turns to X is $\frac{1}{2}$, so the probability that he goes from V to W to X is $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$.
At Y, the probability that Harry turns to X is $\frac{1}{3}$, so the probability that he goes from V to Y to X is $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.
Therefore, the probability that Harry gets to X (and thus to B) is $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{6+3+2}{18} = \frac{11}{18}$.

Answer: (C)

21. We try each of the five possibilities.
If $m : n = 9 : 1$, then we set $m = 9x$ and $n = x$, so $9x + x = 300$ or $10x = 300$ or $x = 30$, so $m = 9(30) = 270$ and $n = 30$.
If $m : n = 17 : 8$, then we set $m = 17x$ and $n = 8x$, so $17x + 8x = 300$ or $25x = 300$ or $x = 12$, so $m = 17(12) = 204$ and $n = 8(12) = 96$.
If $m : n = 5 : 3$, then we set $m = 5x$ and $n = 3x$, so $5x + 3x = 300$ or $8x = 300$ or $x = \frac{75}{2}$, so $m = 5(\frac{75}{2}) = \frac{375}{2}$ and $n = 3(\frac{75}{2}) = \frac{225}{2}$.
If $m : n = 4 : 1$, then we set $m = 4x$ and $n = x$, so $4x + x = 300$ or $5x = 300$ or $x = 60$, so $m = 4(60) = 240$ and $n = 60$.
If $m : n = 3 : 2$, then we set $m = 3x$ and $n = 2x$, so $3x + 2x = 300$ or $5x = 300$ or $x = 60$, so $m = 3(60) = 180$ and $n = 2(60) = 120$.
The only one of the possibilities for which m and n are integers, each greater than 100, is $m : n = 3 : 2$.

Answer: (E)

22. Let $AS = x$ and $SD = y$.
Since $\triangle SAP$ and $\triangle SDR$ are isosceles, then $AP = x$ and $DR = y$.
Since there are two pairs of identical triangles, then $BP = BQ = y$ and $CQ = CR = x$.



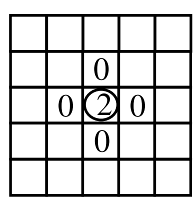
$\triangle SDR$ is right-angled (since $ABCD$ is a square) and isosceles, so its area (and hence the area of $\triangle BPQ$) is $\frac{1}{2}y^2$.
Similarly, the area of each of $\triangle SAP$ and $\triangle QCR$ is $\frac{1}{2}x^2$.
Therefore, the total area of the four triangles is $2(\frac{1}{2}x^2) + 2(\frac{1}{2}y^2) = x^2 + y^2$, so $x^2 + y^2 = 200$.
Now, by the Pythagorean Theorem, used first in $\triangle PRS$, then in $\triangle SAP$ and $\triangle SDR$,

$$\begin{aligned} PR^2 &= PS^2 + SR^2 \\ &= (SA^2 + AP^2) + (SD^2 + DR^2) \\ &= 2x^2 + 2y^2 \\ &= 2(200) \\ &= 400 \end{aligned}$$

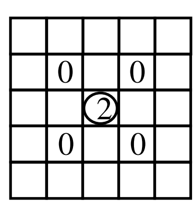
so $PR = 20$ m.

Answer: (B)

23. From the centre 2, there are 8 possible 0s to which we can move initially: 4 "side" 0s



and 4 "corner" 0s.



From a side 0, there are 4 possible 0s to which we can move: 2 side 0s and 2 corner 0s

	0	0	0	
	0		0	

From a corner 0, there are 2 possible 0s to which we can move: 2 side 0s.

	0	0		
	0			

From a side 0, there are 3 possible 5s to which we can move.

5				
5	0			
5				

From a corner 0, there are 5 possible 5s to which we can move.

5	5	5		
5	0			
5				

So the three possible combinations of 0s are "side-side", "side-corner" and "corner-side".
For the combination "side-side", there is a total of $4 \times 2 \times 3 = 24$ paths (because there are initially 4 side 0s that can be moved to, and from *each* of these, there are 2 side 0s that can be moved to, and from *each* of those, there are 3 5s that can be moved to).
For the combination "side-corner", there is a total of $4 \times 2 \times 5 = 40$ paths.
For the combination "corner-side", there is a total of $4 \times 2 \times 3 = 24$ paths.
Therefore, in total there are $24 + 40 + 24 = 88$ paths that can be followed to form 2005.

Answer: (E)

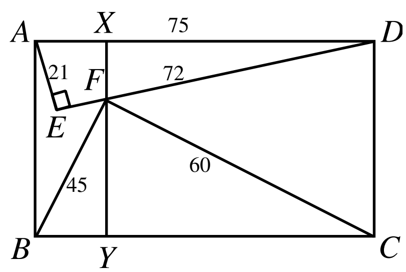
24. Since we have an increasing sequence of integers, the 1000th term will be at least 1000.
We start by determining the number of perfect powers less than or equal to 1000. (This will tell us how many integers less than 1000 are "skipped" by the sequence.)
Let us do this by making a list of all perfect powers less than 1100 (we will need to know the locations of some perfect powers bigger than 1000 anyways):

Perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089
Perfect cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000
Perfect 4th powers: 1, 16, 81, 256, 625
Perfect 5th powers: 1, 32, 243, 1024
Perfect 6th powers: 1, 64, 729
Perfect 7th powers: 1, 128
Perfect 8th powers: 1, 256
Perfect 9th powers: 1, 512
Perfect 10th powers: 1, 1024

In these lists, there are 41 distinct perfect powers less than or equal to 1000.
Thus, there are 959 positive integers less than or equal to 1000 which are not perfect powers.
Therefore, 999 will be the 959th term in the sequence. (This is because 1000 itself is actually a perfect power; if it wasn't, 1000 would be the 959th term.)
Thus, 1001 will be the 960th term.
The next perfect power larger than 1000 is 1024.
Thus, 1023 will be the 982nd term and 1025 will be the 983rd term.
The next perfect power larger than 1024 is 1089.
Therefore, the 1042 will be the 1000th term.
The sum of the squares of the digits of 1042 is $1^2 + 0^2 + 4^2 + 2^2 = 21$.

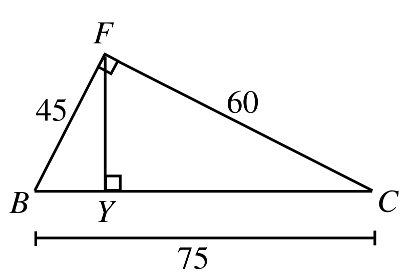
ANSWER: (E)

25. By the Pythagorean Theorem in $\triangle AED$, $AD^2 = AE^2 + ED^2 = 21^2 + 72^2 = 5625$, so $AD = 75$.
Since $ABCD$ is a rectangle, $BC = AD = 75$. Also, by the Pythagorean Theorem in $\triangle BFC$, $FC^2 = BC^2 - BF^2 = 75^2 - 45^2 = 3600$, so $FC = 60$.
Draw a line through F parallel to AB, meeting AD at X and BC at Y.
To determine the length of AB, we can find the lengths of FY and FX.



Step 1: Calculate the length of FY

The easiest method to do this is to calculate the area of $\triangle BFC$ in two different ways.
We know that $\triangle BFC$ is right-angled at F, so its area is equal to $\frac{1}{2}(BF)(FC)$ or $\frac{1}{2}(45)(60) = 1350$.
Also, we can think of FY as the height of $\triangle BFC$, so its area is equal to $\frac{1}{2}(FY)(BC)$ or $\frac{1}{2}(FY)(75)$.



Therefore, $\frac{1}{2}(FY)(75) = 1350$, so $FY = 36$.

(We could have also approached this by letting $FY = h$, $BY = x$ and so $YC = 75-x$. We could have then used the Pythagorean Theorem twice in the two little triangles to create two equations in two unknowns.)

Since $FY = 36$, then by the Pythagorean Theorem,

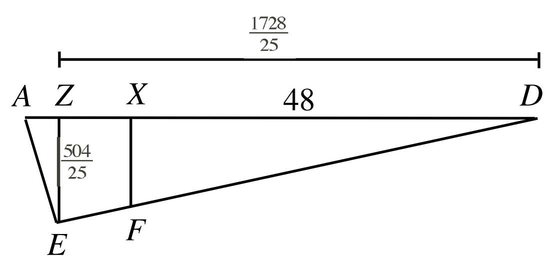
$$BY^2 = BF^2 - FY^2 = 45^2 - 36^2 = 729$$

so $BY = 27$.
Thus, $YC = BC - BY = 48$.

Step 2: Calculate the length of FX

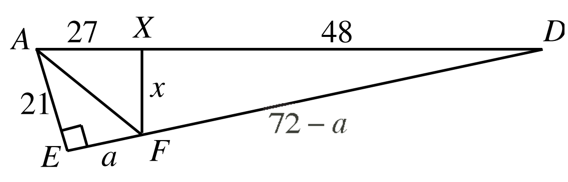
Method 1 – Similar triangles
Since $\triangle AED$ and $\triangle FXD$ are right-angled at E and X respectively and share a common angle D, then they are similar.
Since $YC = 48$, then $XD = 48$.
Since $\triangle AED$ and $\triangle FXD$ are similar, then $\dfrac{FX}{XD} = \dfrac{AE}{ED}$ or $\dfrac{FX}{48} = \dfrac{21}{72}$ so $FX = 14$.

Method 2 – Mimicking Step 1
Drop a perpendicular from E to AD, meeting AD at Z.



We can use exactly the same argument from Step 1 to calculate that $EZ = \frac{504}{25}$ and that $ZD = \frac{1728}{25}$.
Since DF is a straight line, then the ratio $FX : EZ$ equals the ratio $DX : DZ$, ie. $\dfrac{FX}{\frac{504}{25}} = \dfrac{48}{\frac{1728}{25}}$ or $\dfrac{FX}{504} = \dfrac{48}{1728}$ or $FX = 14$.

Method 3 – Areas
Join A to F. Let $FX = x$ and $EF = a$. Then $FD = 72 - a$.



Since $AE = 21$ and $ED = 72$, then the area of $\triangle AED$ is $\frac{1}{2}(21)(72) = 756$.
Now, the area of $\triangle AED$ is equal to the sum of the areas of $\triangle AEF$ and $\triangle AFD$, or

$$756 = \frac{1}{2}(21)(a) + \frac{1}{2}(75)(x)$$

so $21a + 75x = 1512$ or $a + \frac{25}{7}x = 72$.
Now in $\triangle FXD$, we have $FX = x$, $GD = 48$ and $FD = 72 - a$.

By the Pythagorean Theorem, $x^2 + 48^2 = (72 - a)^2 = (\frac{25}{7}x)^2 = \frac{625}{49}x^2$.
Therefore, $48^2 = \frac{576}{49}x^2$, or $x = 14$.

Therefore, $AB = XY = FX + FY = 36 + 14 = 50$.

Answer: (A)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
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2004 Solutions
Pascal Contest (Grade 9)

for

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2004 Pascal Contest Solutions

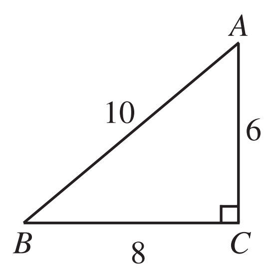
1. Using the proper order of operations,
$$5\times(10-6)\div 2=5\times 4\div 2=20\div 2=10$$
Answer: (A)

2. *Solution 1*
Since the average of 2, x and 12 is 8,
$$\frac{2+x+12}{3}=8$$
$$14+x=24$$
$$x=10$$

Solution 2
Since the average of three numbers is 8, their sum must be 24. Since the sum of 2, x and 12 is 24, then x must be 10.
Answer: (E)

3. The lowest common denominator of three fractions is the least common multiple of the three denominators. The least common multiple of 9, 4 and 18 is 36.
Answer: (D)

4. The area of triangle ABC is $\frac{1}{2}(BC)(AC)=4(AC)$, since the triangle is right-angled at C.
By the Pythagorean Theorem, $AC^2+BC^2=AB^2$ or $AC^2+8^2=10^2$ or $AC^2=36$, which tells us that $AC=6$.
Therefore, the area of the triangle is 24.



Answer: (E)

5. Calculating,
$$\frac{5-\sqrt{4}}{5+\sqrt{4}}=\frac{5-2}{5+2}=\frac{3}{7}$$
Answer: (A)

6. Calculating,
$$4^1+3^2-2^3+1^4=4+9-8+1=6$$
Answer: (C)

7. Substituting $x=-3$,

$$3x^2+2x=3(-3)^2+2(-3)=3(9)-6=21$$

Answer: (D)

8. *Solution 1*

Since 18% of 42 is equal to 27% of *x*, then

$$\frac{18}{100}(42)=\frac{27}{100}x$$
$$18(42)=27x$$
$$x=28$$

Solution 2

Since 18% is two-thirds of 27%, and 18% of 42 is equal to 27% of *x*, then *x* must be two-thirds of 42. Thus, *x* equals 28.

Answer: (A)

9. A cube has six faces, each of which is a square.
If a cube has a surface area of 96 cm^2, then each of its faces has an area of one-sixth of this total, or 16 cm^2.
Since each face has an area of 16 cm^2, then the edge length of the cube is 4 cm.
Since the edge length of the cube is 4 cm, its volume is $4^3\text{ cm}^3=64\text{ cm}^3$.

Answer: (B)

10. *Solution 1*

Since $y=3x-5$ and $z=3x+3$, then $z-y=(3x+3)-(3x-5)=8$, ie. *z* is 8 more than *y*. Since $y=1$, then $z=9$.

Solution 2

Since $y=3x-5$ and $y=1$, then $3x-5=1$ or $x=2$.
Since $x=2$, then $z=3(2)+3=9$.

Answer: (E)

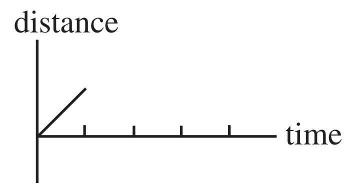
11. In the diagram, the square is divided into four rectangles, each of which has been divided in half to form two identical triangles. So in each of the four rectangles, the area of the shaded triangle equals the area of the unshaded triangle. Thus, the area of the shaded region is $\frac{1}{2}$ of the overall area of the square, or $\frac{1}{2}$ of 16 square units, or 8 square units.

Answer: (B)

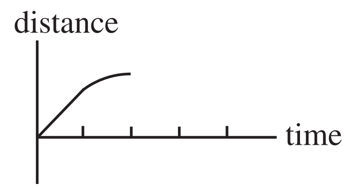
12. From the two given balances, 3 ▲'s balance 5 ●'s, and 1 ▲ balances 2 ■'s and 1 ●. Tripling the quantities on the second balance implies that 3 ▲'s will balance 6 ■'s and 3 ●'s. Therefore, 5 ●'s will balance 6 ■'s and 3 ●'s, and removing 3 ●'s from each side implies that 2 ●'s will balance 6 ■'s, or 1 ● will balance 3 ■'s.

Answer: (C)

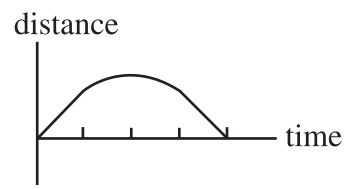
13. Suppose the length of one side of the park is 1. When Nadia is one quarter of the way around the park (ie. the top corner), she is at a distance of 1 from *S*.



From the quarter-way point to the half-way point, her distance from *S* is steadily increasing. When she is half-way around the park, her distance from *S* will be $\sqrt{2}$ (since we can form a right-angled triangle with two sides of length 1 and the hypotenuse joining *S* to the opposite corner).



As she completes her circuit of the park, the graph will be completed as follows.



Therefore, the graph must be the one from (C), since this is the only one which satisfies this condition. (The graph is indeed slightly rounded in the middle.)

Answer: (C)

14. *Solution 1*

The first figure has 8 unshaded squares. The second figure has 12 unshaded squares. The third figure has 16 unshaded squares. So the number of unshaded squares increases by 4 with each new figure. So the number of unshaded squares in the tenth figure should be $8+4(9)=44$ (we add 4 nine times to get from the first figure to the second, from the second to the third, and so on).

Solution 2

The first figure is a 3 by 3 square with a 1 by 1 square shaded in. The second figure is a 4 by 4 square with a 2 by 2 square shaded in. The third figure is a 5 by 5 square with a 3 by 3 square shaded in. Therefore, the tenth figure should be a 12 by 12 square with a 10 by 10 square shaded in. So the number of unshaded squares is $12^2-10^2=44$.

Answer: (D)

15. Since each child has at least 2 brothers, then each boy has at least 2 brothers. So there have to be at least 3 boys.
Since each child has at least 1 sister, then each girl has at least 1 sister. So there have to be at least 2 girls.
Therefore, the Pascal family has at least 5 children.

Answer: (C)

16. We try positive integer values for a to see the resulting value of b is a positive integer.
If $a=1$, then $a^2=1$, so $3b=32$, so b is not a positive integer.
If $a=2$, then $a^2=4$, so $3b=29$, so b is not a positive integer.
If $a=3$, then $a^2=9$, so $3b=24$, so $b=8$.
If $a=4$, then $a^2=16$, so $3b=17$, so b is not a positive integer.
If $a=5$, then $a^2=25$, so $3b=8$, so b is not a positive integer.
If a is at least 6, then a^2 is at least 36, so $3b$ is negative, and b cannot be a positive integer.
Therefore, there is only one possible pair of values for a and b, so $ab=3(8)=24$.

Answer: (B)

17. Since $0.\overline{12}$ has a period of length 2 and $0.\overline{123}$ has a period of length 3, we must expand each of the three given decimals to six places:

$0.\overline{1}=0.111111...$
$0.\overline{12}=0.121212...$
$0.\overline{123}=0.123123...$

When we add these three numbers as decimals, we get

$0.\overline{1}+0.\overline{12}+0.\overline{123}=0.355446...$

so the answer must be $0.\overline{1}+0.\overline{12}+0.\overline{123}=0.\overline{355446}$.

Answer: (D)

18. Using the definition of the symbol,

$$\begin{aligned}(x-1)(-5)-(2)(3)&=9\\ -5x+5-6&=9\\ -5x&=10\\ x&=-2\end{aligned}$$

Answer: (C)

19. Every week, a branch that is at least two weeks old produces a new branch. Therefore, after a given week, the total number of branches is equal to the number of branches at the beginning of the week plus the number of branches that were not new during the previous week (ie. the number of "old" branches). So we make a chart:

Week #	Branches at beginning	Old branches at beginning	Branches at end
1	0	0	1
2	1	0	1
3	1	1	2
4	2	1	3
5	3	2	5
6	5	3	8
7	8	5	13
8	13	8	21

Therefore, there are 21 branches at the end of the eighth week.

Answer: (A)

20. We start by making a chart in which we determine the position after the next turn given any current position:

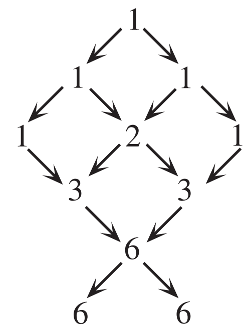
Current Position	Position after next turn
1	2
2	4
3	6
4	1
5	3
6	5
7	7

So for the arrow to point to the 6 after the 21^{st} turn, it must have pointed to the 3 after the 20^{th} turn (from the third row of the chart). For it to point to the 3 after the 20^{th} turn, it must have pointed to the 5 after the 19^{th} turn, and to the 6 after the 18^{th} turn.
This pattern now continues in a cycle, from which we can conclude the arrow pointed to the 6 after the 15^{th} turn, the 12^{th} turn, the 9^{th} turn, the 6^{th} turn and the 3^{rd} turn.
Since it pointed to the 6 after the 3^{rd} turn, it pointed to the 3 after the 2^{nd} turn and the 5 after the 1^{st} turn.

Answer: (C)

21. First we notice that any path following the arrows from the top P to one of the two bottom L's actually does spell the word "PASCAL", so we need to count the total number of paths from the top to the bottom.
We will proceed by counting the number of paths that reach each of the letters in the diagram. To do this, we see that the number of paths reaching a letter in the diagram is the sum of the number of paths reaching all of the letters which directly lead to the desired letter.
So we can fill in the number of paths leading to each letter

So in total, there are 12 paths through the diagram – 6 which lead to the left "L" and 6 which lead to the right "L".



Answer: (C)

22. Let d be the original depth of the water.
Then the total volume of water in the container initially is $20 \times 20 \times d = 400d$.
The volume of the gold cube is $15 \times 15 \times 15 = 3375$.
After the cube has been added to the water, the total volume inside the container that is filled is $20 \times 20 \times 15 = 6000$, since the container with base 20 cm by 20 cm has been filled to a depth of 15 cm.
Therefore, $400d + 3375 = 6000$ or $400d = 2625$ or $d = 6.5625$. So d is closest to 6.56 cm.

Answer: (A)

23. We start by labelling the two quarters Q_1 and Q_2, the two dimes D_1 and D_2, and the two nickels N_1 and N_2. We then make a chart where the labels on the left tell us which coin is chosen first and the labels on the top tell us which coin is chosen second. Inside the chart, we put a Y if the combination will pay the toll and an N if the combination will not pay the toll. (We put X's on the diagonal, since we cannot choose a coin first and then the same coin the second time.)

	Q_1	Q_2	D_1	D_2	N_1	N_2
Q_1	X	Y	Y	Y	Y	Y
Q_2	Y	X	Y	Y	Y	Y
D_1	Y	Y	X	N	N	N
D_2	Y	Y	N	X	N	N
N_1	Y	Y	N	N	X	N
N_2	Y	Y	N	N	N	X

For example, if Q_1 is chosen first and then N_1, the driver has chosen 30 cents, so can pay the toll.
If D_2 is chosen first and then D_1, then 20 cents has been chosen, and he cannot pay the toll.

From the chart, there are 30 possible combinations that can be chosen (since the X's on the main diagonal indicate that these combinations are not possible), with 18 of them enough to pay the toll, and 12 not enough to pay the toll.

Therefore, the probability is $\frac{18}{30} = \frac{3}{5}$.

Answer: (A)

24. In analyzing this sequence of fractions, we start by observing that this large sequence is itself made up of smaller sequences. Each of these smaller sequences is of the form

$$\frac{n}{1},\ \frac{n-1}{2},\ \frac{n-2}{3},\ \ldots,\ \frac{3}{n-2},\ \frac{2}{n-1},\ \frac{1}{n}$$

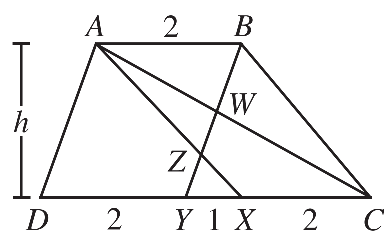
with the denominators increasing from 1 to n and the numerators decreasing from n to 1.
We observe that there is 1 term in the first of these smaller sequences, 2 terms in the second of these, and so on. This can be seen in the following grouping:

$$\left(\frac{1}{1}\right),\ \left(\frac{2}{1},\ \frac{1}{2}\right),\ \left(\frac{3}{1},\ \frac{2}{2}\ \frac{1}{3}\right),\ \left(\frac{4}{1},\ \frac{3}{2},\ \frac{2}{3},\ \frac{1}{4}\right),\ \ldots,\ \left(\frac{9}{1},\ \frac{8}{2},\ \frac{7}{3},\ \frac{6}{4},\ \frac{5}{5},\ \frac{4}{6},\ \frac{3}{7},\ \frac{2}{8},\ \frac{1}{9}\right),\ \ldots$$

If we take any fraction in any of these smaller sequences, the sum of the numerator and denominator is 1 greater than the number of terms in this smaller sequence. For example, if we take the first occurrence of $\frac{3}{7}$, it would occur in the sequence with 9 terms.
This implies that the fifth occurrence of a fraction equivalent to $\frac{3}{7}$, namely $\frac{5\times 3}{5\times 7}=\frac{15}{35}$, would occur in the sequence with 49 terms, and would be the 35th term in that sequence.
Since the smaller sequences before this particular sequence have 1, 2, 3, …, 48 terms, so the term $\frac{15}{35}$ is term number $(1+2+\cdots+47+48)+35=\frac{1}{2}(48)(49)+35=1176+35=1211$.

Answer: (E)

25. Let the height of trapezoid $ABCD$ be h.
Then its total area is $\frac{1}{2}(AB+CD)h=\frac{7}{2}h$.
Since $AB=2$ and AX is parallel to BC, then $XC=2$.
Since $AB=2$ and BY is parallel to AD, then $DY=2$.
Since $CD=5$, $XC=2$ and $DY=2$, then $YX=1$.
Now we want to determine the area of ΔAZW, so we will determine the areas of ΔAZB and ΔAWB and subtract them.



First, we calculate the area of ΔAZB. Since AB is parallel to CD, $\angle ZAB=\angle ZXY$ and $\angle ZBA=\angle ZYX$, so ΔAZB is similar to ΔXZY. Since the ratio of AB to XY is 2 to 1, then the ratio of the heights of these two triangles will also be 2 to 1, since they are similar. But the sum of their heights must be the height of the trapezoid, h, so the height of ΔAZB is $\frac{2}{3}h$. Therefore, the area of ΔAZB is $\frac{1}{2}(2)\left(\frac{2}{3}h\right)=\frac{2}{3}h$.

Next, we calculate the area of ΔAWB. Since AB is parallel to CD, $\angle WAB=\angle WCY$ and $\angle WBA=\angle WYC$, so ΔAWB is similar to ΔCWY. Since the ratio of AB to CY is 2 to 3, then the ratio of the heights of these two triangles will also be 2 to 3, since they are similar. But the sum of their heights must be the height of the trapezoid, h, so the height of ΔAWB is $\frac{2}{5}h$. Therefore, the area of ΔAWB is $\frac{1}{2}(2)\left(\frac{2}{5}h\right)=\frac{2}{5}h$.

Therefore, the area of ΔAZW is the difference between the areas of ΔAZB and ΔAWB, or $\frac{2}{3}h-\frac{2}{5}h=\frac{4}{15}h$. Thus, the ratio of the area of ΔAZW to the area of the whole trapezoid is $\frac{4}{15}h : \frac{7}{2}h=\frac{4}{15} : \frac{7}{2}=4(2) : 7(15)=8 : 105$

Answer: (B)

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2003 Solutions
Pascal Contest (Grade 9)

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Awards

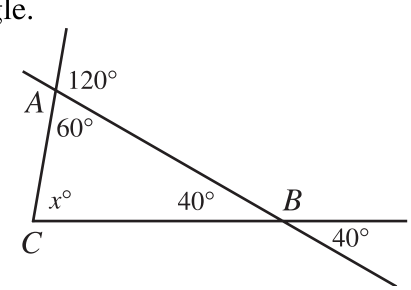
2003 Pascal Contest Solutions

1. Calculating, $\sqrt{169} - \sqrt{25} = 13 - 5 = 8$. ANSWER: (A)

2. Looking at the first few terms in the sequence, we can see that to get from one term to the next, we multiply by 3 (since $6 = 3(2)$, $18 = 3(6)$, etc.) So the missing term is $3(54) = 162$. (To check that this is correct, we can see that $486 = 3(162)$.)
[The term "geometric sequence" means that every term after the first is obtained from the previous term by multiplying by the same number. In this question, the pattern can be guessed and filled in without knowing this definition.]
ANSWER: (D)

3. Using the BEDMAS order of operations,
$\dfrac{6 + 6 \times 3 - 3}{3} = \dfrac{6 + 18 - 3}{3} = \dfrac{21}{3} = 7$. ANSWER: (B)

4. We label the three points of intersection as A, B and C, as shown.
Then $\angle ABC = 40^\circ$ since it is equal to its opposite angle.
Also, $\angle CAB = 60^\circ$ since it is the supplement of a 120° angle.
In ΔABC,
$$x^\circ + 60^\circ + 40^\circ = 180^\circ$$
$$x = 80$$



ANSWER: (E)

5. *Solution 1*
Using exponent laws,
$$\frac{2^8}{8^2} = \frac{2^8}{\left(2^3\right)^2} = \frac{2^8}{2^6} = 2^2 = 4.$$

Solution 2
Calculating directly,
$$\frac{2^8}{8^2} = \frac{256}{64} = 4.$$
ANSWER: (C)

6. Evaluating each of the 5 choices,

(A) $\frac{6^2}{10}=\frac{36}{10}=\frac{18}{5}$ (B) $\frac{1}{5}\left[6(3)\right]=\frac{1}{5}[18]=\frac{18}{5}$ (C) $\frac{18+1}{5+1}=\frac{19}{6}\neq\frac{18}{5}$

(D) $3.6=\frac{36}{10}=\frac{18}{5}$ (E) $\sqrt{\frac{324}{25}}=\sqrt{\frac{18^2}{5^2}}=\frac{18}{5}$

Therefore, the only choice not equal to $\frac{18}{5}$ is (C). ANSWER: (C)

7. Starting from the bottom of the diagram, we first determine the value of F. From the conditions given, either $F-7=3$ or $7-F=3$. Since F must be one of 1, 2, 4, 5, 6, and 8, then $F=4$, giving

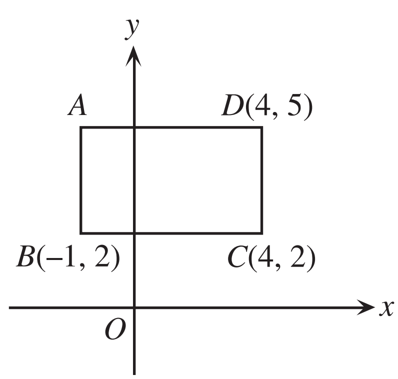
A 10 B C
D 9 E
7 4
3

Similarly, we can determine that $E=5$ and $D=2$, giving

A 10 B C
2 9 5
7 4
3

and then $A=8$, $B=1$, and $C=6$. (We notice that this does use each of the six possibilities exactly once.)
Thus, $A+C=14$. ANSWER: (E)

8. Since the sides of the rectangle are parallel to the axes, we can determine the side lengths by taking the difference of the appropriate coordinates.
The length of BC is the difference of the x-coordinates, that is $4-(-1)=5$.
The length of DC is the difference of the y-coordinates, that is $5-2=3$.
Therefore, the area of the rectangle is $3\times 5=15$.



ANSWER: (A)

9. Every prime number, with the exception of 2, is an odd number. To write an odd number as the sum of two whole numbers, one must be an even number and one must be an odd number.
So in this case, we want to write a prime as the sum of an odd prime number and an even prime number. Since the only even prime number is 2, then we want to write a prime number as 2 plus another prime number.
From highest to lowest, the prime numbers less than 30 are 29, 23, 19, 17, 13, 11, 7, 5, 3 and 2.

The largest prime less than 30 which is 2 more than another prime number is 19.

ANSWER: (C)

10. We write out each of the five choices to 8 decimal places:

(A) $3.2571 = 3.25710000...$

(B) $3.\overline{2571} = 3.25712571...$

(C) $3.2\overline{571} = 3.25715715...$

(D) $3.25\overline{71} = 3.25717171...$

(E) $3.257\overline{1} = 3.25711111...$

These five real numbers agree to four decimals, but are all different in the fifth decimal place. Therefore, $3.25\overline{71}$ is the largest.

ANSWER: (D)

11. Substituting $x = 2$ and $y = -3$, we obtain

$$\begin{aligned} 2(2)^2 + k(2)(-3) &= 4 \\ 8 - 6k &= 4 \\ 4 &= 6k \\ k &= \tfrac{2}{3} \end{aligned}$$

ANSWER: (A)

12. From the first exchange rate, 1 calculator is worth 100 rulers.
From the second exchange rate, 100 rulers are worth $\frac{100}{10}(30) = 300$ compasses.
From the third exchange rate, 300 compasses are worth $\frac{300}{25}(50) = 600$ protractors.
From this, we can see that 600 protractors are equivalent to 1 calculator.

ANSWER: (B)

13. If we look at the top left four squares, we see that they must all be coloured a different colour, since they all have a vertex in common, so no two can be the same colour.
So we need at least 4 colours. Can we colour the 15 squares with only 4 colours?
If we try to do this, we can see that this is possible by taking a block of four squares, colouring the squares different colours, and shifting this block around the grid. (In the diagram, different numbers represent the four different colours.)

1	2	1	2	1
3	4	3	4	3
1	2	1	2	1

What would happen if the grid were bigger than 3 by 5? Would 4 colours still be enough? This problem is related to a famous math problem called the "Four Colour Problem". Try looking this up on the Web!

ANSWER: (B)

14. *Solution 1*

Since x and y have to be positive integers and add up to 5, it is easy to make a chart of the possibilities:

x	y	$2x-y$
1	4	–2
2	3	1
3	2	4
4	1	7

Of the choices, only –2 is a possibility.

Solution 2

We can rewrite $2x-y=3x-x-y=3x-(x+y)=3x-5$.

Of the five possibilities, the only one that is 5 less than a multiple of 3 is –2.

ANSWER: (D)

15. *Solution 1*

We can find the area of square *KLMN* by subtracting the areas of the four triangles *KAN*, *NDM*, *MCL* and *LBK* from the area of square *ABCD*.

To do this, we note that square *ABCD* has a side length of 6, and that each of the four triangles is right-angled and has one leg of length 2 and the other of length 4. (A "leg" of right-angled triangle is a side that is not the hypotenuse.)

So the area of square *KLMN* is $6^2-4\left[\frac{1}{2}(2)(4)\right]=36-4[4]=20$ square units.

Solution 2

Since *KLMN* is a square, its area is the square of its side length; in particular, the area is equal to NM^2.

To calculate its side length, we look at right-angled triangle *DNM* and calculate NM^2 using Pythagoras:

$$NM^2=ND^2+DM^2$$

$$NM^2=2^2+4^2$$

$$NM^2=20$$

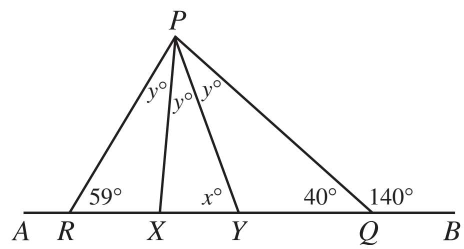
Therefore, the area of the square is 20 square units.

ANSWER: (D)

16. Since n can be any integer, let us choose $n=0$. Then the values of the 5 integers are 3, -9, -4, 6, and -1. When we arrange these from smallest to largest, we get -9, -4, -1, 3, and 6, so the number in the middle is -1, which is $n-1$.

ANSWER: (E)

17. We add the extra labels to the diagram, as shown.



Then $\angle PQR = 180^\circ - \angle PQB = 40^\circ$.
Considering ΔPQR,

$$\begin{aligned} 59^\circ + 3y^\circ + 40^\circ &= 180^\circ \\ 3y &= 81 \\ y &= 27 \end{aligned}$$

We next look at ΔPRY, where

$$\begin{aligned} 59^\circ + 2y^\circ + x^\circ &= 180^\circ \\ x &= 180 - 59 - 2(27) \\ x &= 67 \end{aligned}$$

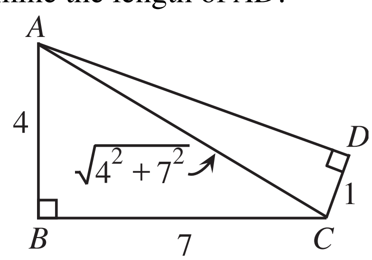
ANSWER: (A)

18. The average of a list of numbers is their sum divided by how many numbers are in the list. Thus if n numbers have an average of 7 their sum is $7n$.
When -11 is added to the list of numbers, there are then $n+1$ numbers whose sum is $7n-11$. Using this, we obtain

$$\begin{aligned} \frac{7n-11}{n+1} &= 6 \\ 7n - 11 &= 6n + 6 \\ n &= 17 \end{aligned}$$

ANSWER: (E)

19. If we join A to C, then the quadrilateral is divided into two right-angled triangles.
The area of triangle ABC is $\frac{1}{2}(4)(7) = 14$.
To find the area of triangle ADC, first we need to determine the length of AD.
Using Pythagoras in both triangles,



$$\begin{aligned} AD^2 &= AC^2 - DC^2 \\ AD^2 &= \left(AB^2 + BC^2\right) - 1^2 \\ AD^2 &= 7^2 + 4^2 - 1^2 \\ AD^2 &= 64 \\ AD &= 8 \end{aligned}$$

The area of triangle ADC is thus $\frac{1}{2}(1)(8) = 4$.
Therefore, the total area of quadrilateral $ABCD$ is the sum of the areas of the two triangles, or $14 + 4 = 18$.

ANSWER: (C)

20. *Solution 1*
Suppose instead of using the digits 0, 2, 4, 6, 8, that Evenlanders use the digits 0, 1, 2, 3, 4 (each of the previous digits divided by 2).
Then we can see that the Evenlanders' numbers system is our base 5 number system, with all of the digits doubled.

Writing 111 as a sum of powers of 5, we see that $111 = 4\left(5^2\right) + 2\left(5^1\right) + 1\left(5^0\right)$, or 111 can be written as 421 in base 5. Doubling the digits, the Evenlanders write 842 for the integer 111.

Solution 2

In order to determine the Evenlanders' version of 111, we need to find a pattern in the Evenlanders' numbers. So we write out the first several:

2, 4, 6, 8, 20, 22, 24, 26, 28, 40, 42, 44, 46, 48, 60, 62, 64, 66, 68, 80, …

which correspond to the integers 1 through 20. We can see from these that the last digit of the Evenlanders' numbers has a 5 digit cycle (2, 4, 6, 8, 0). Since we are looking for their version of 111, the last digit must be 2.

In order to figure out the other digits, we need to extend our pattern. So continuing to count, we get

80, 82, 84, 86, 88, 200, 202, 204, 206, 208, 220, …

for the integers 20 through 30.

The numbers will then continue to have three digits with first digit 2 as the last two digits go from 00 to 88. After 288 will come 400 and the cycle begins again.

How many three digit numbers beginning with 2 do the Evenlanders have? Since these are the numbers 200 through 288, it is the same number as counting 00 through 88, which is 25 numbers, since 88 corresponds to the integer 24.

So 200 represents the integer 25, which means that 400 represents the integer 50, 600 will represent the integer 75, and 800 will represent the integer 100.

To get from the integer 100 to the integer 111, we can either count as Evenlanders from 800 upwards, or we can look for the 11th number in our original list, which is 42 (which represents the integer 11), and put an 8 at the front.

Therefore, the Evenlanders' version of 111 is 842.

ANSWER: (D)

21. Since each light turns red 10 seconds after the preceding one, and the lights are all on cycles of equal time, then each light will turn green 10 seconds after the preceding one. So let's say that the first light turns green at time 0 seconds. The question to ask is, "When will the last light turn green?"

Since it is the eighth light, it will turn green at time 70 seconds (seven intervals of 10 seconds later). At this point, the first light is still green, since it remains green for 1.5 minutes or 90 seconds, and will remain green for another 20 seconds.

(This first light will then turn yellow at time 90 seconds, and then red at time 93 seconds, and then each of the other lights will turn red until the eighth light turns red at time 163 seconds, when the first light is still red.)

So the only time when all eight lights are green is the interval above, and so the longest interval of time when all the lights are green is 20 seconds.

(Note that the length of time that the light was yellow has no relevance to the situation.)

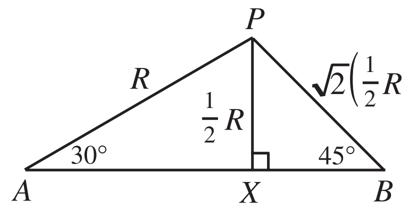
ANSWER: (C)

22. *Solution 1*

Join A to B (meeting PQ at X) and consider the triangle APB.

By symmetry, $\angle PAB = 30^\circ$, $\angle PBA = 45^\circ$ and PX is perpendicular to AB.

Since $AP = R$ and ΔAPX is a 30-60-90 triangle, then $PX = \frac{1}{2}R$.



Since ΔBPX is a 45-45-90 triangle, then $PB = \sqrt{2}\left(\frac{1}{2}R\right)$.

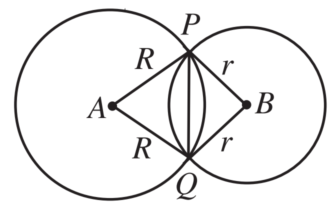
Therefore, $r = \sqrt{2}\left(\frac{1}{2}R\right)$ or $r^2 = \frac{1}{2}R^2$.

Since the two circles have areas of πr^2 and πR^2, then the ratio of their areas is 2 : 1.

Solution 2

Suppose that the radius of the circle with centre A is R, and the radius of the circle with centre B is r. Join P to Q.

Since PQ is a chord in each of the two circles, we will try to find the length of PQ in terms of R and in terms of r, in order to find a relationship between R and r.



Consider ΔAPQ. Then $AP = AQ = R$ and $\angle PAQ = 60^\circ$, so ΔAPQ is isosceles and has an angle of 60°, so must be equilateral. Therefore, $PQ = R$, or $PQ^2 = R^2$.

Now looking at ΔBPQ. Then $BP = BQ = r$, and $\angle PBQ = 90^\circ$, so $PQ^2 = r^2 + r^2 = 2r^2$, by Pythagoras.

So we know that $PQ^2 = R^2 = 2r^2$, or $\pi R^2 = 2\left(\pi r^2\right)$.

But the expression on the left side is the area of the circle with centre A and the expression in the parentheses on the right side is the area of the circle with centre B.

Thus, the required ratio is 2:1, since the area of the left-hand circle is twice the area of the right-hand circle.

ANSWER: (D)

23. *Solution 1*

Suppose that the escalator was two floors long, instead of just one, and that Jack and Jill start walking at the same time. Then Jack will reach the second floor at the same time Jill reaches the first floor (since it takes Jill twice as long to climb one floor). In that time, Jack will have climbed $2(29)$ steps and Jill will have climbed 11 steps, so there will be $47 = 2(29) - 11$ steps between them on the escalator.

These 47 steps represents the distance between two floors, or the length of the escalator.

Solution 2

Suppose that N is the total number of steps on the elevator.

As Jack walks up the escalator, he walks 29 steps, so he is carried $N - 29$ steps by the escalator.

As Jill walks up the escalator, she walks 11 steps, so is carried $N - 11$ steps by the escalator.

Since Jill's trip takes twice as long as Jack's trip, then she must be carried up twice the number of steps that Jack is carried, ie.

$$N-11=2(N-29)$$
$$N-11=2N-58$$
$$N=47$$

Therefore, the number of steps is 47.

ANSWER: (A)

24. *Solution 1*

Suppose that the artist uses squares of side length s, and that he uses M of the squares along the length, and N along the width. (Because of the constraints of the problem, M and N must both be integers.)

Then we must have $Ms = 60\frac{1}{2}$ cm and $Ns = 47\frac{2}{3}$ cm.

The total number of squares covering the rectangle is MN, and so we don't need to know s. Dividing the two equations will cancel s to obtain

$$\frac{Ms}{Ns} = \frac{60\frac{1}{2}}{47\frac{2}{3}}$$

or

$$\frac{M}{N} = \frac{\left[\frac{121}{2}\right]}{\left[\frac{143}{3}\right]} = \frac{363}{286} = \frac{11(33)}{11(26)} = \frac{33}{26}$$

So we want to determine positive integers M and N as small as possible so that $\frac{M}{N} = \frac{33}{26}$.

Since the fraction $\frac{33}{26}$ is in lowest terms, then the smallest integers M and N that will work are $M = 33$ and $N = 26$, which gives a total of $MN = (33)(26) = 858$ squares.

Solution 2

The area of the rectangle is $\frac{121}{2} \times \frac{143}{3} = \frac{11 \times 11}{2} \times \frac{11 \times 13}{3}$.

We would like to write this expression as the number of small squares times the area of the square, that is as an integer times the square of some number.

We can do this by rearranging the above to give

$$\frac{11 \times 11}{2} \times \frac{11 \times 13}{3} = \frac{11 \times 13}{6} \times 11^2$$

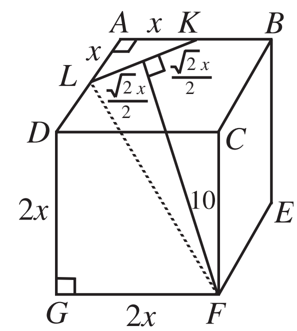
At this point, we need to make the first factor into an integer, so we want to incorporate the 6 into the "squared" part, which we can do by rewriting again as

$$\frac{11 \times 13}{6} \times 11^2 = \frac{6 \times 11 \times 13}{6^2} \times 11^2 = [6 \times 11 \times 13] \times \left(\frac{11}{6}\right)^2 = 858 \times \left(\frac{11}{6}\right)^2$$

Thus there are 858 squares which are $\frac{11}{6}$ by $\frac{11}{6}$.

ANSWER: (B)

25. We label the cube as shown in the diagram.
Notice that since $AL = AK$ that ΔALK is isosceles and right-angled. This means that $LK = \sqrt{2}x$.
Next, we draw the line from F which is perpendicular to LK and meets LK at Q. By symmetry, Q is the midpoint of LK. Thus we can label $FQ = 10$ and $LQ = \frac{\sqrt{2}x}{2}$.



Using Pythagoras in ΔDGF, $DF^2 = (2x)^2 + (2x)^2 = 8x^2$.
Also, ΔFDL is right-angled at D, so using Pythagoras again, $FL^2 = 8x^2 + x^2 = 9x^2$.
Lastly, using Pythagoras in ΔLQF,

$$LF^2 = LQ^2 + QF^2$$

$$9x^2 = \left(\frac{\sqrt{2}x}{2}\right)^2 + 10^2$$

$$9x^2 = \tfrac{1}{2}x^2 + 100$$

$$\tfrac{17}{2}x^2 = 100$$

$$x \approx 3.429$$

The volume is thus $8x^3 \approx 322.82$, which rounding off is 323.

ANSWER: (A)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2002 Solutions
Pascal Contest (Grade 9)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

1. By order of operations,
$$\frac{15+9-6}{3\times 2}=\frac{18}{6}=3$$
ANSWER: (C)

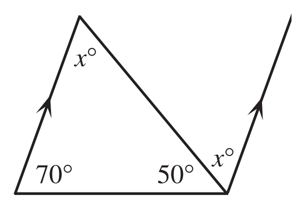
2. 50% of 2002 is $\frac{1}{2}$ of 2002, which is 1001. ANSWER: (E)

3. Since $10=x+2$, then $x=8$. Since $6=y-1$, then $y=7$. Thus, $x+y=15$.
ANSWER: (B)

4. Evaluating,
$$\left(3^2-3\right)^2=(9-3)^2=36.$$
ANSWER: (A)

5. Since Sofia goes up 7 floors, then down 6 floors, and then finally up 5 floors, the net result is that she has gone up $7-6+5=6$ floors. Since she finishes on the 20th floor, she must have started on floor number 14. ANSWER: (A)

6. Because of alternate angles, the third angle in the triangle is x°. The sum of angles in a triangle is 180°, thus
$$x^\circ+70^\circ+50^\circ=180^\circ$$
$$x^\circ=60^\circ$$
$$x=60$$



ANSWER: (B)

7. Since $n=\frac{5}{6}(240)$, then $\frac{2}{5}n=\frac{2}{5}\left(\frac{5}{6}\right)(240)=\frac{1}{3}(240)=80$. ANSWER: (B)

8. Evaluating, using the rules for negative exponents,
$$1-\left(5^{-2}\right)=1-\tfrac{1}{5^2}=1-\tfrac{1}{25}=\tfrac{24}{25}.$$
ANSWER: (A)

9. We can determine the area of the shaded region by guessing the side lengths of the various rectangles and then cross-checking our results. Let us suppose that the top left rectangle has a width of 2 and a height of 3. Then the top right rectangle has a width of 5, since its height is also 3. Thus, we can conclude that the height of the bottom right rectangle is 5. This tells us that the shaded rectangle is 2 by 5, or has an area of 10. This problem can also be solved with a more algebraic approach, but this is the most straightforward way.
ANSWER: (E)

10. For 1 square, 4 toothpicks are needed.
For 2 squares, 7 toothpicks are needed.

For 3 squares, 10 toothpicks are needed.
So when each additional square (after the first) is added in the pattern, 3 more toothpicks are added. Thus, to form 10 squares, we need to add 9 groups of 3 toothpicks to the original 4, so we need $4+9(3)=31$ toothpicks in total. ANSWER: (C)

11. Since $ABCD$ is a square, then $AB=BC$, or

$$\begin{aligned} x+16&=3x \\ 16&=2x \\ x&=8 \end{aligned}$$

Thus the side length of the square is $x+16=3x=24$, and the perimeter is $4(24)=96$.

ANSWER: (C)

12. Let the first number in the list be a. Then the second and third numbers are $2a$ and $4a$, respectively. So $2a+4a=24$, or $a=4$. Now the fourth, fifth and sixth numbers will be $8a$, $16a$ and $32a$, respectively. So the sixth number is $32(4)=128$.

ANSWER: (E)

13. Side AC of ΔABC is parallel to the y-axis and so is perpendicular to the x-axis, and crosses the x-axis at $(1,0)$. So we can think of AC as the base of the triangle (length 6) and the part of the x-axis inside ΔABC as its height (length 3). So the area of ΔABC is $\frac{1}{2}(6)(3)=9$.

ANSWER: (E)

14. To calculate the overall class average, we can assume that each of the 25 students who averaged 75% each actually got a mark of 75%. Similarly, we may assume that the other 5 students each actually got a mark of 40%. So the overall average is

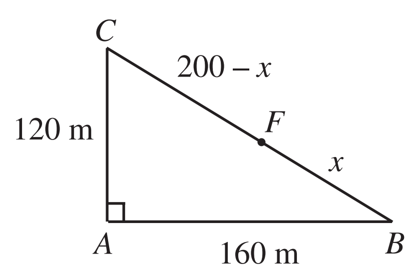
$$\frac{25(75)+5(40)}{30}=\frac{2075}{30}=69.2$$

which is closest to 69 of all of the possible choices. ANSWER: (B)

15. By Pythagoras,

$$\begin{aligned} BC^2&=120^2+160^2 \\ BC^2&=40000 \\ BC&=200 \end{aligned}$$

Let $FB=x$. Then $FC=200-x$.



So Jack jogs $AB+BF=160+x$, and Jill jogs $AC+CF=120+200-x$. Since they jog the same distance,

$$\begin{aligned} 160+x&=120+200-x \\ 2x&=160 \\ x&=80 \end{aligned}$$

Therefore, it is 80 metres from B to F. ANSWER: (D)

16. First, we see that both 5^3 and 7^{52} are odd integers, so their product is odd. Also, 5^3 is a multiple of 5, so $\left(5^3\right)\left(7^{52}\right)$ is a multiple of 5, and thus is an odd multiple of 5. Since all odd multiples of 5 end with the digit 5, then the units digit of $\left(5^3\right)\left(7^{52}\right)$ is 5.

ANSWER: (A)

17. We know that $1000 = 10 \times 10 \times 10 = 2 \times 5 \times 2 \times 5 \times 2 \times 5$. To write 1000 as the product of two integers neither of which contains a 0, we must ensure that neither has a factor of 10. This tells us that we want to separate the factors of 2 and the factors of 5, ie.
$1000 = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 8 \times 125$, and $8 + 125 = 133$. ANSWER: (C)

18. Since Akira and Jamie together weigh 101 kg, and Akira and Rabia together weigh 91 kg, then Jamie weighs 10 kg more than Rabia. So let Rabia's weight be x kg. Then Jamie's weight is $(10 + x)$ kg. So from the third piece of information given,

$$\begin{aligned} x + x + 10 &= 88 \\ 2x &= 78 \\ x &= 39 \end{aligned}$$

Since Rabia weighs 39 kg, then Akira weighs 52 kg, since their combined weight is 91 kg.

ANSWER: (D)

19. If we divide 2002 by 7, we see that $2002 = 7(286)$. Since there are 7 natural numbers in each row, and the last entry in each row is the multiple of 7 corresponding to the row number, then 2002 must lie in the 7th column of the 286th row. So $m = 7$, $n = 286$, and $m + n = 293$. ANSWER: (D)

20. For $\sqrt{25 - x^2}$ to be defined, $25 - x^2 \geq 0$, or $x^2 \leq 25$, or $-5 \leq x \leq 5$. So we make a table of x and $25 - x^2$ and check to see when $25 - x^2$ is a perfect square, ie. $\sqrt{25 - x^2}$ is an integer.

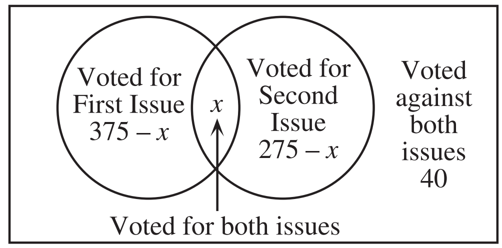
x	$25 - x^2$	Perfect Square?
0	25	Yes
± 1	24	No
± 2	21	No
± 3	16	Yes
± 4	9	Yes
± 5	0	Yes

Therefore, there are 7 integer values for x (namely $0,\ \pm 3,\ \pm 4,\ \pm 5$) that make $\sqrt{25 - x^2}$ an integer. ANSWER: (A)

21. The original rectangular block measures 5 cm by 6 cm by 4 cm, and so has volume $5 \times 6 \times 4 = 120 \text{ cm}^3$. This tells us that it is made up of 120 of the small cubes. The largest cube that can be formed by removing cubes is a 4 cm by 4 cm by 4 cm cube, since each side length can be no more longer than the side lengths of the original block. This new cube is made up of $4 \times 4 \times 4 = 64$ of the 1 cm^3 cubes, and so 56 of the original 1 cm^3 cubes have been removed.

ANSWER: (E)

22. Let x be the number of students who voted in favour of both issues. We construct a Venn diagram of the results of the vote:



Since the total number of students is 500, then

$$375 - x + x + 275 - x + 40 = 500$$
$$690 - x = 500$$
$$x = 190$$

So 190 students voted in favour of both.

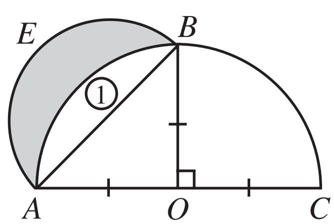
ANSWER: (C)

23. We must find all ordered pairs (a,b) of integers which satisfy $a^b = 64 = 2^6$. We can first say that we know that b must be positive, since a^b is bigger than 1. However, it is possible that a is negative.

If we want both a and b to be positive, the only ways to write 64 are $64 = 64^1 = 8^2 = 4^3 = 2^6$ (since a must be a power of 2). If b is even, then it possible for a to be negative, ie. $64 = (-8)^2 = (-2)^6$. So there are 6 ordered pairs (a,b) that satisfy the equation.

ANSWER: (D)

24. The easiest way to calculate the area of the shaded region is to take the area of the semi-circular region AEB and subtract the area of the region labelled 1. But the area of this region 1 is the area of the quarter circle ABO minus the area of the triangle ABO. We now calculate these areas.



Since $AO = BO = 1$, then the area of ΔABO is $\frac{1}{2}(1)(1) = \frac{1}{2}$.

Since $AO = BO = 1$, then radius of the quarter circle ABO is 1, and so the area of the quarter circle is $\frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$.

Lastly, since $AO = BO = 1$, then $AB = \sqrt{2}$, and so the radius of the semi-circle AEB is $\frac{\sqrt{2}}{2}$, which means that the area of the semi-circle AEB is $\frac{1}{2}\pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$.

Thus, the area of the shaded region is

(Area of semicircle AEB) – (Area of region 1)

$= (\text{Area of semicircle } AEB) - [(\text{Area of quarter circle } AOB) - (\text{Area of } \Delta AOB)]$
$= \frac{\pi}{4} - \left[\frac{\pi}{4} - \frac{1}{2}\right]$
$= \frac{1}{2}$

ANSWER: (B)

25. First, we calculate the volumes of the two cylindrical containers:
$V_{\text{large}} = \pi(6)^2(20) = 720\pi \text{ cm}^3$
$V_{\text{small}} = \pi(5)^2(18) = 450\pi \text{ cm}^3$

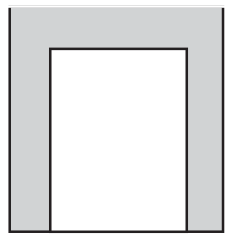


Figure 3

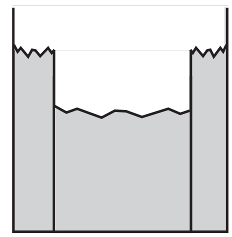


Figure 4

The volume of water initially contained in the large cylinder is
$V_{\text{water, initial}} = \pi(6)^2(17) = 612\pi \text{ cm}^3$
The easiest way to determine the final depth of water in the small cylinder is as follows. Imagine putting a lid on the smaller container and lowering it all the way to the bottom of the larger container, as shown in Figure 3. So there will be water beside and above the smaller container. Note that the larger container will be filled to the brim (since the combined volume of the small container and the initial water is greater than the volume of the large container) and some water will have spilled out of the larger container.
Now if the lid on the small container is removed, all of the water in the large container above the level of the brim of the small container will spill into the small container, as shown in Figure 4. This water occupies a cylindrical region of radius 6 cm and height 2 cm, and so has a volume of $\pi(6)^2(2) = 72\pi \text{ cm}^3$. This is the volume of water that is finally in the small container. Since the radius of the small container is 5 cm, then the depth of water is
$$\text{Depth} = \frac{72\pi \text{ cm}^3}{\pi(5 \text{ cm})^2} = \frac{72}{25} \text{ cm} = 2.88 \text{ cm}$$

ANSWER: (D)

Canadian
Mathematics
Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2001 Solutions

Pascal Contest (Grade 9)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

Part A

1. The value of $\frac{5(6)-3(4)}{6+3}$ is

(**A**) 1 (**B**) 2 (**C**) 6 (**D**) 12 (**E**) 31

Solution
By evaluating the numerator and denominator we have

$$\frac{5(6)-3(4)}{6+3}=\frac{30-12}{9}=\frac{18}{9}=2.$$

ANSWER: (B)

2. When $12\,345\,678$ is divided by 10, the remainder is

(**A**) 0 (**B**) 2 (**C**) 4 (**D**) 6 (**E**) 8

Solution
Applying the standard division algorithm we would have

$$12\,345\,678=10(1\,234\,567)+8.$$

The remainder is 8. ANSWER: (E)

3. Evaluate $\frac{2^5-2^3}{2^2}$.

(**A**) 6 (**B**) 1 (**C**) $\frac{1}{4}$ (**D**) 0 (**E**) 30

Solution
Evaluating the given expression we would have

$$\frac{2^5-2^3}{2^2}=\frac{32-8}{4}=\frac{24}{4}=6.$$

ANSWER: (A)

4. If $x=\frac{1}{4}$, which of the following has the largest value?

(**A**) x (**B**) x^2 (**C**) $\frac{1}{2}x$ (**D**) $\frac{1}{x}$ (**E**) $\sqrt{x}$

Solution
If we calculate the value of the given expressions, we get

(**A**) $\frac{1}{4}$ (**B**) $\left(\frac{1}{4}\right)^2$ (**C**) $\frac{1}{2}\left(\frac{1}{4}\right)$ (**D**) $\frac{1}{\frac{1}{4}}$ (**E**) $\sqrt{\frac{1}{4}}$

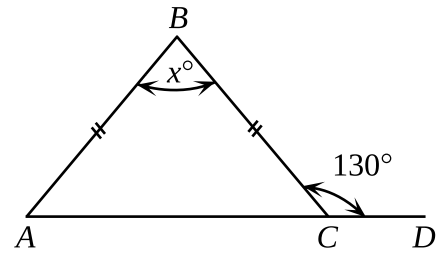
$= \frac{1}{16}$ $= \frac{1}{8}$ $= 1 \times 4$ $= \frac{1}{2}$

$= 4$

ANSWER: (D)

5. In the diagram, the value of x is

(**A**) 100 (**B**) 65 (**C**) 80
(**D**) 70 (**E**) 50



Solution

Since $\angle ACB + \angle BCD = 180°$ (Supplementary angles)

$$\angle ACB = 180° - 130°$$
$$= 50°$$

Thus, $\angle BAC = 50°$. (ΔABC is isosceles)

Therefore, $x° = 180° - (50° + 50°)$

$$x° = 80°.$$

The value of x is 80. ANSWER: (C)

6. Anna's term mark was 80%. Her exam mark was 90%. In calculating her final mark, the term mark was given a weight of 70% and the exam mark a weight of 30%. What was her final mark?

(**A**) 81% (**B**) 83% (**C**) 84% (**D**) 85% (**E**) 87%

Solution

Anna's final mark is $80(0.7) + 90(0.3)$

$$= 56 + 27$$
$$= 83.$$

ANSWER: (B)

7. The least value of x which makes $\dfrac{24}{x-4}$ an integer is

(**A**) -44 (**B**) -28 (**C**) -20 (**D**) -8 (**E**) 0

Solution

If we determine the value of the expression for each of the given values of x we would have,

(**A**) $\dfrac{24}{-44-4}$ (**B**) $\dfrac{24}{-28-4}$ (**C**) $\dfrac{24}{-20-4}$ (**D**) $\dfrac{24}{-8-4}$ (**E**) $\dfrac{24}{0-4}$

$=\frac{24}{-48}$	$=\frac{24}{-32}$	$=\frac{24}{-24}$	$=\frac{24}{-12}$	$=-6$
$=\frac{-1}{2}$	$=\frac{-3}{4}$	$=-1$	$=-2$	

The smallest value of x is thus -20. Note that the answers for (**A**) and (**B**) are non-integers.

ANSWER: (C)

8. The 50th term in the sequence 5, $6x$, $7x^2$, $8x^3$, $9x^4$, … is

(**A**) $54x^{49}$ (**B**) $54x^{50}$ (**C**) $45x^{50}$ (**D**) $55x^{49}$ (**E**) $46x^{51}$

Solution

If we start by looking at the numerical coefficient of each term we make the observation that if we add 1 to 5 to get the second term and 2 to 5 to get the third term we will then add 49 to 5 to get the fiftieth term. Thus the fiftieth term has a numerical coefficient of 54.

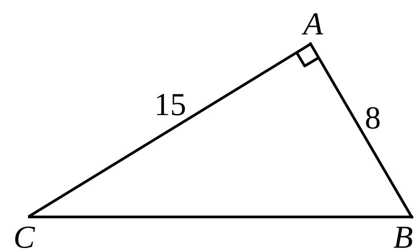
Similarly, if we observe the literal coefficient of each term, the first term has a literal coefficient of x^0 which has an exponent of 0. The second term has an exponent of 1, the third an exponent of 2 so that the exponent of the fiftieth term is 49 which gives a literal part of $54x^{49}$.

Thus the fiftieth term is $54x^{49}$.

ANSWER: (A)

9. The perimeter of ΔABC is

(**A**) 23 (**B**) 40 (**C**) 42
(**D**) 46 (**E**) 60



Solution

Since the given triangle is right-angled at A, if we apply the Pythagorean Theorem we would have

$$BC^2 = 8^2 + 15^2$$
$$= 289.$$

Therefore, $BC = 17 \quad (BC > 0)$.

The perimeter is $15 + 8 + 17 = 40$.

ANSWER: (B)

10. Dean scored a total of 252 points in 28 basketball games. Ruth played 10 fewer games than Dean. Her scoring average was 0.5 points per game higher than Dean's scoring average. How many points, in total, did Ruth score?

(**A**) 153 (**B**) 171 (**C**) 180 (**D**) 266 (**E**) 144

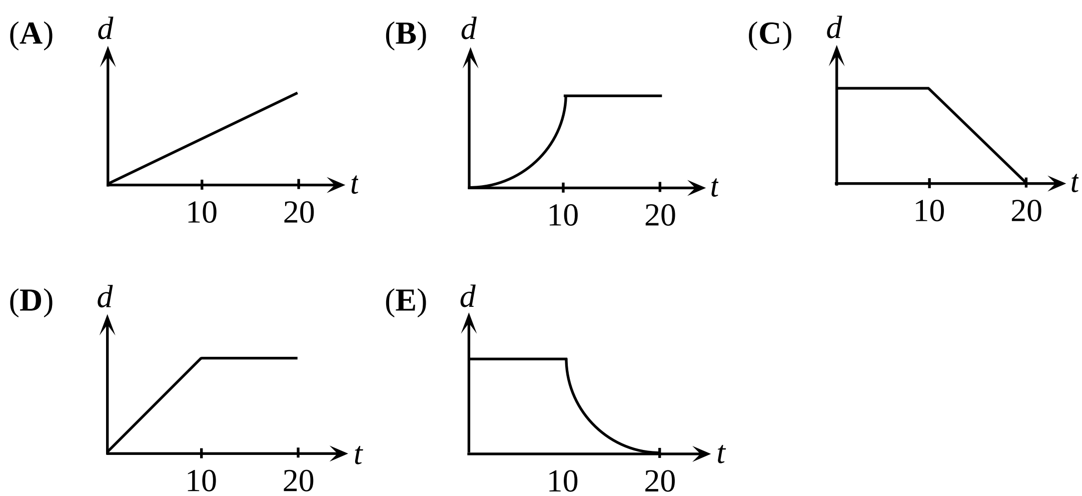
Solution

If Dean scored 252 points in 28 games this implies that he averages $\frac{252}{28}$ or 9 points per game. Ruth must then have averaged 9.5 points in each of the 18 games she played. In total she scored 9.5×18 or 171 points.

ANSWER: (B)

Part B

11. Sahar walks at a constant rate for 10 minutes and then rests for 10 minutes. Which of these distance, *d*, versus time, *t*, graphs best represents his movement during these 20 minutes?



Solution

(A) Since the line given has a constant slope over the entire 20 minute interval, this implies that Sahar would have walked at a constant rate for this length of time.

(B) Since the part of the graph from 0 to 10 minutes is non-linear, this implies that Sahar was not walking at a constant rate. The implication is that the rate at which he was walking is increasing over this interval. The graph from the 10 minute point and beyond is flat which implies that Sahar was resting for 10 minutes.

(C) Since the graph is flat on the interval from 0 to 10, this again implies that Sahar was resting. From the 10 minute point onward, Sahar was travelling at a negative but constant rate. This implies that he was returning from some point at a constant rate.

(D) This graph is the correct graph.

(E) This graph implies that Sahar was resting for 10 minutes and then walked at a gradually decreasing rate for the next 10 minutes.

ANSWER: (D)

12. A bag contains 20 candies: 4 chocolate, 6 mint and 10 butterscotch. Candies are removed randomly from the bag and eaten. What is the minimum number of candies that must be removed to be *certain* that at least two candies of each flavour have been eaten?

(**A**) 6 (**B**) 10 (**C**) 12 (**D**) 16 (**E**) 18

Solution

At most, 17 candies could be removed before the second chocolate candy is removed, that is all 10 butterscotch, all 6 mint, and 1 chocolate.

So we need to remove 18 candies to ensure that 2 of each flavour have been eaten.

ANSWER: (E)

13. Pierre celebrated his birthday on February 2, 2001. On that day, his age equalled the sum of the digits in the year in which he was born. In what year was Pierre born?

(**A**) 1987 (**B**) 1980 (**C**) 1979 (**D**) 1977 (**E**) 1971

Solution

We consider each of the possibilities in the following table:

	Year	Sum of digits	Birth year given this age
(A)	1987	$1+9+8+7=25$	$2001-25=1976$
(B)	1980	$1+9+8+0=18$	$2001-18=1983$
(C)	1979	$1+9+7+9=26$	$2001-26=1975$
(D)	1977	$1+9+7+7=24$	$2001-24=1977$
(E)	1971	$1+9+7+1=18$	$2001-18=1973$

From the chart we can see that Pierre's birth date was 1977. ANSWER: (D)

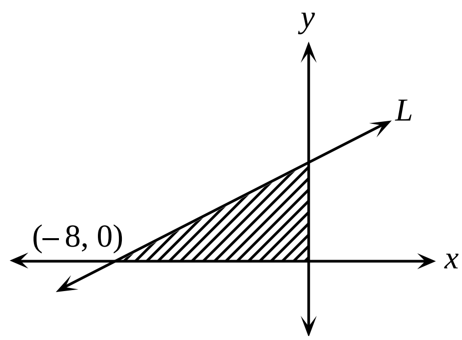
14. Twenty tickets are numbered from one to twenty. One ticket is drawn at random with each ticket having an equal chance of selection. What is the probability that the ticket shows a number that is a multiple of 3 or 5?

(**A**) $\frac{3}{10}$ (**B**) $\frac{11}{20}$ (**C**) $\frac{2}{5}$ (**D**) $\frac{9}{20}$ (**E**) $\frac{1}{2}$

Solution

The numbers that are between one and twenty and are multiples of 3 or 5 are: 3, 5, 6, 9, 10, 12, 15, 18, 20. The probability of selecting a 3 or 5 is thus $\frac{9}{20}$. ANSWER: (D)

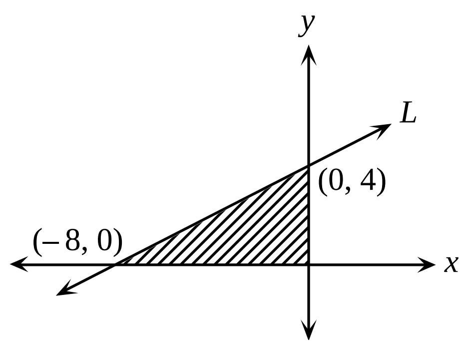
15. The line L crosses the x-axis at $(-8, 0)$. The area of the shaded region is 16. What is the slope of the line L?



(**A**) $\frac{1}{2}$ (**B**) 4 (**C**) $-\frac{1}{2}$
(**D**) 2 (**E**) -2

Solution

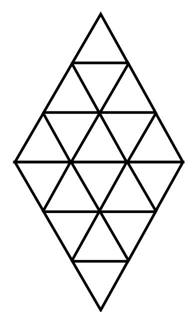
If the area of the shaded region is 16 and its base has a length of 8, its height must then be 4.
Thus we have the changes noted in the diagram.
Thus the slope is $\frac{4-0}{0-(-8)} = \frac{1}{2}$ or $\frac{1}{2}$ because the line slopes down from right to left and the line has a rise of 4 and a run of 8.



Area $= \frac{1}{2}|-8||4| = 16$

ANSWER: (A)

16. In the diagram, all triangles are equilateral. The total number of equilateral triangles of any size is

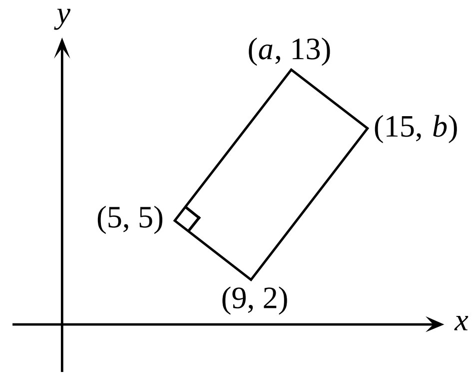


(**A**) 18 (**B**) 20 (**C**) 24
(**D**) 26 (**E**) 28

Solution

From the diagram, if we start by counting just the small triangles we would achieve a total of 18 triangles. If we count triangles that have a side length of 2, there would be 6 including 3 in each of the top and bottom halves. However, there are an additional 2 that are overlapping the two halves. If we count the 2 additional triangles of side length 3, we would have a total of 28. ANSWER: (E)

17. In the rectangle shown, the value of $a-b$ is



(**A**) -3 (**B**) -1 (**C**) 0
(**D**) 3 (**E**) 1

Solution
To go from the point $(5,5)$ to the point $(9,2)$ we must move over 4 and down 3.
Since we are dealing with a rectangle, the same must be true for $(a,13)$ and $(15,b)$.
Thus, $a+4=15$ and $13-3=b$. From this, $a=11$ and $b=10$. So $a-b=11-10=1$.

ANSWER: (E)

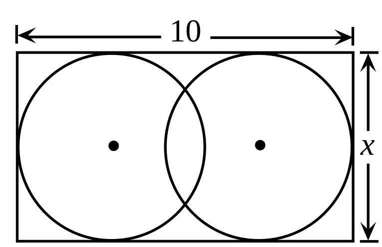
18. The largest four-digit number whose digits add to 17 is 9800. The 5th largest four-digit number whose digits have a sum of 17 is

(**A**) 9521 (**B**) 9620 (**C**) 9611 (**D**) 9602 (**E**) 9530

Solution
The largest four-digit number having a digit sum of 17 is 9800. The next 2 largest are 9710 and then 9701. The next 3 largest will then be 9620, 9611 and 9602. The 5th largest is 9611.

ANSWER: (C)

19. Two circles with equal radii are enclosed by a rectangle, as shown. The distance between their centres is $\frac{2x}{3}$. The value of x is



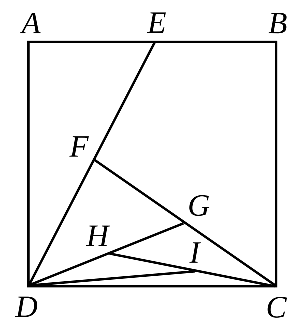
(**A**) $\frac{15}{4}$ (**B**) 5 (**C**) 6
(**D**) $\frac{60}{7}$ (**E**) $\frac{15}{2}$

Solution
We observe first of all that the width of the rectangle is x which corresponds to the diameter of a circle or $2r=x$. If the distance between the centres is $\frac{2}{3}x$ the length of the rectangle is equal to two radii plus the distance between the two centres. Or, $x+\frac{2}{3}x=10$

$$\frac{5}{3}x = 10$$
$$x = 6.$$

ANSWER: (C)

20. Square $ABCD$ has an area of 4. E is the midpoint of AB. Similarly, F, G, H, and I are the midpoints of DE, CF, DG, and CH, respectively. The area of ΔIDC is

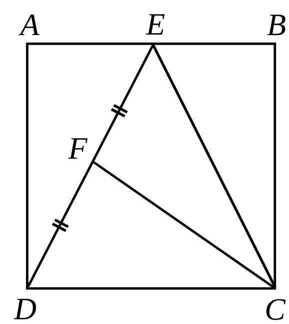


(**A**) $\frac{1}{4}$ (**B**) $\frac{1}{8}$ (**C**) $\frac{1}{16}$

(**D**) $\frac{1}{32}$ (**E**) $\frac{1}{64}$

Solution

We start by joining E to C and noting that ΔDEC has half the area of the square or an area of 2. From here, we observe ΔDEC and the dividing line, CF. Since F is the midpoint of DE, triangles DFC and EFC have the same height and the same base and hence the same area. Therefore, ΔDFC will have an area of 1 unit. If we continue using this same idea, ΔDGC has an area of $\frac{1}{2}$, ΔCHD has an area of $\frac{1}{4}$ and ΔIDC had area of $\frac{1}{8}$.



ANSWER: (B)

Part C

21. Cindy leaves school at the same time every day. If she cycles at 20 km/h, she arrives home at 4:30 in the afternoon. If she cycles at 10 km/h, she arrives home at 5:15 in the afternoon. At what speed, in km/h, must she travel to arrive home at 5:00 in the afternoon?

(**A**) $16\frac{2}{3}$ (**B**) 15 (**C**) $13\frac{1}{3}$ (**D**) 12 (**E**) $18\frac{3}{4}$

Solution

Since the distance from Cindy's home to school is unknown, represent this distance by d, in kilometres. We will consider the problem in two separate cases, the first in which she travels at 20 km/h and the second when she travels at 10 km/h.

Distance travelled at 20 km/h = Distance travelled at 10 km/h

Let the time that Cindy takes travelling home at 20 km/h be t hours.

If Cindy arrives home $\frac{3}{4}$ h later when travelling at 10 km/h, then the length of time travelling is $\left(t+\frac{3}{4}\right)$ hours. The previous equation becomes

$$\begin{aligned} 20t &= 10\left(t+\frac{3}{4}\right) \\ 20t &= 10t+\frac{30}{4} \\ 10t &= \frac{15}{2} \\ t &= \frac{15}{20} \text{ or } \frac{3}{4}. \end{aligned}$$

Therefore the distance from school to home is $d = 20 \times \frac{3}{4}$, or $d = 15$ km.

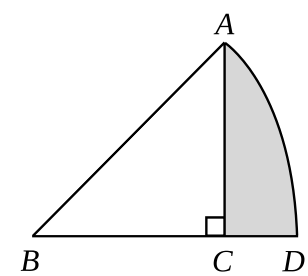
If Cindy arrives home at 5:00 in the afternoon, she would have travelled home in $\frac{3}{4}+\frac{1}{2}=\frac{5}{4}$ hours over a distance of 15 kilometres.

Therefore, $s = \frac{d}{t} = \frac{15}{\frac{5}{4}} = 15 \times \frac{4}{5} = 12$ km/h.

Therefore, Cindy would have had to travel at 12 km/h to arrive home at 5:00 p.m.

ANSWER: (D)

22. In the diagram, AB and BD are radii of a circle with centre B. The area of sector ABD is 2π, which is one-eighth of the area of the circle. The area of the shaded region is



(**A**) $2\pi - 4$ (**B**) π (**C**) $2\pi - 2$

(**D**) $2\pi - 4.5$ (**E**) $2\pi - 8$

Solution

Since the sector ABD represents one-eighth a circle that has an area of 2π, the area of the whole circle is $8 \times 2\pi$ or 16π. Also, since the whole circle has $360°$ around the centre $\angle ABD = \frac{360°}{8} = 45°$.

We find the radius of the circle by solving, $\pi r^2 = 16\pi$.

Therefore, $r = 4 \quad (r > 0)$.

Since $r = AB = 4$ and $\angle ABC = 45°$, then $\angle BAC = 45°$ and $AC = BC = 2\sqrt{2}$.

Thus the area of ΔABC is $\frac{1}{2}\left(2\sqrt{2}\right)\left(2\sqrt{2}\right) = 4$.

We observe that the area of the shaded region equals the area of the sector minus the area of ΔABC.

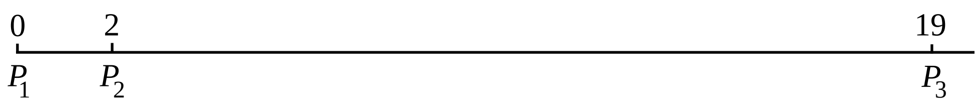
Thus, the required shaded area is $2\pi - 4$. ANSWER: (A)

23. Five points are located on a line. When the ten distances between pairs of points are listed from smallest to largest, the list reads: 2, 4, 5, 7, 8, k, 13, 15, 17, 19. What is the value of k?

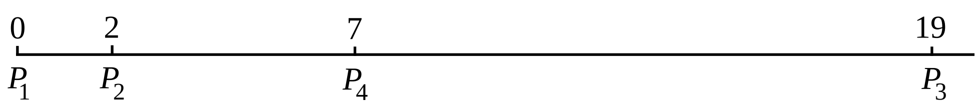
(**A**) 11 (**B**) 9 (**C**) 13 (**D**) 10 (**E**) 12

Solution

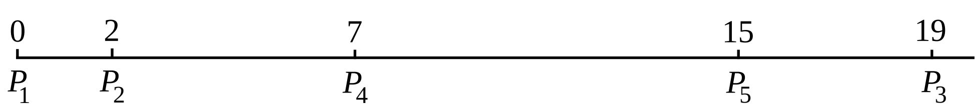
The solution to this problem is difficult to write up because it really comes down to systematic trial and error. We'll start by drawing a number line and placing two points at 0 and 19. Since the first required distance is 2, we'll place our first three points as shown.



Since we need to have a distance of 7, we'll place a point at 7 on the number line. This gives us distances which are not inconsistent with the required.



Distances so far are, 2, 7, 19, 5, 17 and 12.
And finally, if we place a point at 15 on the number line, we will have the following.



If we place the point anywhere else, we will have inconsistencies with the required. If the points are placed as shown we will have the distances, {2, 7, 15, 19, 5, 13, 17, 8, 12, 4}.
Thus the required distance is 12. ANSWER: (E)

24. A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?

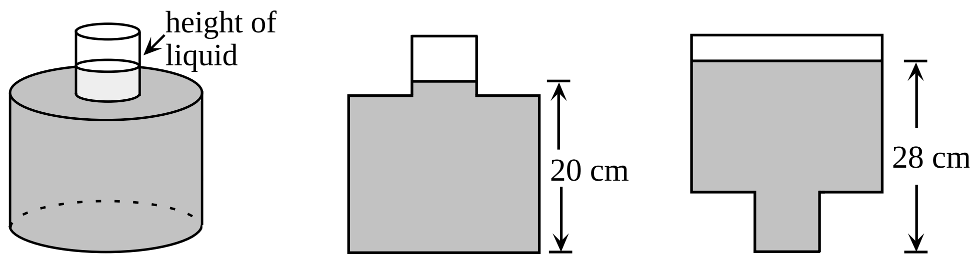


Figure A Figure B Figure C

(**A**) 29 (**B**) 30 (**C**) 31 (**D**) 32 (**E**) 48

Solution

We'll start by representing the height of the large cylinder as h_1 and the height of the small cylinder as h_2. For simplicity, we'll let $x = h_1 + h_2$.

If the bottom cylinder is completely filled and the top cylinder is only partially filled the top cylinder will have a cylindrical space that is not filled. This cylindrical space will have a height equal to $x - 20$ and a volume equal to, $\pi(1)^2(x-20)$.

Similarly, if we turn the cylinder upside down there will be a cylindrical space unfilled that will have a height equal to $x - 28$ and a volume equal to, $\pi(3)^2(x-28)$.

Since these two unoccupied spaces must be equal, we then have,

$$\begin{aligned} \pi(1)^2(x-20) &= \pi(3)^2(x-28) \\ x - 20 &= 9x - 252 \\ 8x &= 232 \\ x &= 29. \end{aligned}$$

Therefore, the total height is 29. ANSWER: (A)

25. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2882 is a four-digit palindrome and 49194 is a five-digit palindrome. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there?

(**A**) 28 (**B**) 32 (**C**) 36 (**D**) 40 (**E**) 44

Solution

Our first observation is that since we are adding two four-digit palindromes to form a five-digit palindrome then the following must be true,

$$\begin{array}{r} a\,b\,b\,a \\ c\,d\,d\,c \\ \hline 1\,e\,f\,e\,1 \end{array}.$$

(i.e. the first digit of the 5-digit palindrome is 1.)

From this, we can see that $a+c=11$ since $a+c$ has a units digit of 1 and $10<a+c<20$.
We first note that there are four possibilities for *a* and *c*. We list the possibilities:

a	2	3	4	5
c	9	8	7	6

Note that there are only four possibilities here.
(If we extended this table, we would get an additional four possibilities which would be duplicates of these four, with *a* and *c* reversed.)
Let us consider one case, say $a=2$ and $c=9$.

$$\begin{array}{r} 2\,b\,b\,2 \\ 9\,d\,d\,9 \\ \hline 1\,e\,f\,e\,1 \end{array}$$

From this, we can only get palindromes in two ways. To see this we note that *e* is either 1 or 2 depending on whether we get a carry from the previous column (we see this looking at the thousands digit of 1 *e f e* 1). If $e=1$, then $b+d$ has no carry and so looking at the tens digit $e=1$, we see that $b+d=0$ to get this digit.
If $e=2$, we do get a carry from $b+d$, so looking again at the tens digit $e=2$, we see that $b+d=11$.

Possibility 1 $b=d=0$
Since there are only four possibilities for *a* and *c* and just one way of selecting *b* and *d* so that $b+d=0$ for each possibility, there are just four possibilities.

Possibility 2 $b+d=11$
For each of the four possible ways of choosing *a* and *c*, there are eight ways of choosing *b* and *d* so that $b+d=11$ thus giving 32 possibilities.
This gives a total of $4+32=36$ possibilities.

ANSWER: (C)

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Canadian
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An activity of The Centre for Education
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Awards

Part A:

1. The value of $5^2 + 2(5-2)$ is

(**A**) 16 (**B**) 19 (**C**) 31 (**D**) 36 (**E**) 81

Solution
If we use the rules for order of operations we would have the following,
$5^2 + 2(5-2) = 25 + 2(3) = 25 + 6 = 31$.

ANSWER: (C)

2. The sum of $29 + 12 + 23$ is

(**A**) 32^2 (**B**) 2^6 (**C**) 3^4 (**D**) 1^{64} (**E**) 64^0

Solution
Evaluating each of the given choices gives the following: $32^2 = 1024$, $2^6 = 64$, $3^4 = 81$, $1^{64} = 1$, and $64^0 = 1$. Since $29 + 12 + 23 = 64$, the correct choice is B.

ANSWER: (B)

3. If $x = 4$ and $y = -3$, then the value of $\dfrac{x-2y}{x+y}$ is

(**A**) $-\frac{1}{2}$ (**B**) -2 (**C**) $\frac{10}{7}$ (**D**) $-\frac{2}{7}$ (**E**) 10

Solution
The value of the expression, $\dfrac{x-2y}{x+y}$, after making we have the substitution, $\dfrac{4-2(-3)}{4+(-3)} = \dfrac{10}{1} = 10$.

ANSWER: (E)

4. If the following sequence of five arrows repeats itself continuously, what arrow would be in the 48th position?

$\longrightarrow$, $\nearrow$, $\nwarrow$, $\longleftarrow$, $\downarrow$

(**A**) $\longrightarrow$ (**B**) $\nearrow$ (**C**) $\nwarrow$ (**D**) $\longleftarrow$ (**E**) $\downarrow$

Solution
Since this sequence repeats itself, once it has completed nine cycles it will be the same as starting at the beginning. Thus the 48th arrow will be the same as the third one.

ANSWER: (C)

5. If $y = 6 + \frac{1}{6}$, then $\frac{1}{y}$ is

(A) $\frac{6}{37}$ **(B)** $\frac{37}{6}$ **(C)** $\frac{6}{7}$ **(D)** $\frac{7}{6}$ **(E)** 1

Solution
If we perform the addition we would have, $6 + \frac{1}{6} = \frac{36}{6} + \frac{1}{6} = \frac{37}{6}$. Since $\frac{1}{y} = \frac{37}{6}$, $y = \frac{6}{37}$.

ANSWER: (A)

6. If $\frac{2}{3}$, $\frac{23}{30}$, $\frac{9}{10}$, $\frac{11}{15}$, and $\frac{4}{5}$ are written from smallest to largest then the middle fraction will be

(A) $\frac{23}{30}$ **(B)** $\frac{4}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{9}{10}$ **(E)** $\frac{11}{15}$

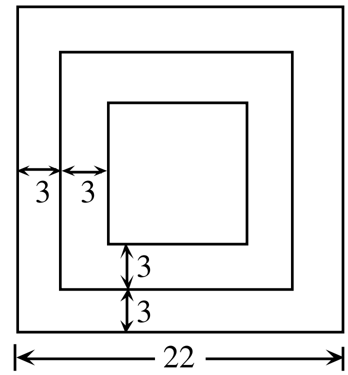
Solution
In order to compare the five fractions, we write them with the same denominator and then compare numerators. Since the lowest common multiple of 3, 30, 10, 15, and 5 is 30, we must change each of the fractions into its equivalent form with denominator 30. The fractions are shown in the following table:

Given Fraction	Equivalent Form
$\frac{2}{3}$	$\frac{2}{3} \times \frac{10}{10} = \frac{20}{30}$
$\frac{23}{30}$	Unchanged
$\frac{9}{10}$	$\frac{9}{10} \times \frac{3}{3} = \frac{27}{30}$
$\frac{11}{15}$	$\frac{11}{15} \times \frac{2}{2} = \frac{22}{30}$
$\frac{4}{5}$	$\frac{4}{5} \times \frac{6}{6} = \frac{24}{30}$

In arranging the fractions in the required order they become: $\frac{20}{30}$, $\frac{22}{30}$, $\frac{23}{30}$, $\frac{24}{30}$, and $\frac{27}{30}$.
The middle fraction is $\frac{23}{30}$.

ANSWER: (A)

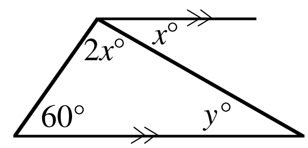
7. Three squares with the same centre and corresponding parallel sides are drawn. The distance between the sides of successive squares is 3 and the side length of the largest square is 22, as shown. What is the perimeter of the smallest square?



(**A**) 40 (**B**) 100 (**C**) 10
(**D**) 64 (**E**) 20

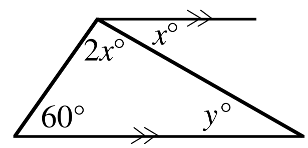
Solution
Each side of a square will decrease by 6 units each time it gets smaller. The second square will thus have a side length of $22-6=16$ and the smallest square will then have a side length of 10.
Its perimeter is 4×10 or 40. ANSWER: (A)

8. In the diagram, the value of y is



(**A**) 30 (**B**) 20 (**C**) 80
(**D**) 60 (**E**) 40

Solution
Since the two horizontal lines are parallel then $x=y$ because of alternate angles. In the triangle, this implies that $2x°+x°+60°=180°$. Solving this equation we determine that $3x°=120°$ or $x=40$. Since $x=y$, $y=40$.



ANSWER: (E)

9. The ages of three contestants in the Pascal Contest are 14 years, 9 months; 15 years, 1 month; and 14 years, 8 months. Their average (mean) age is

(**A**) 14 years, 8 months (**B**) 14 years, 9 months (**C**) 14 years, 10 months
(**D**) 14 years, 11 months (**E**) 15 years

Solution 1
Consider one of the ages, say the youngest, as a base age. The other two contestants are one month and five months older respectively. Since $\frac{0+1+5}{3}=2$, this implies that the average age is two months greater than the youngest. This gives an average age of 14 years, 10 months.

Solution 2

This second solution involves more calculation but gives the same correct answer. Since there are twelve months in a year, the age of the first contestant, in months, is $14 \times 12 + 9$ or 177 months. Similarly, the ages of the other two students would be 181 and 176 months. The average age would be $\frac{177 + 181 + 176}{3}$ or 178 months. The average age is then 14 years, 10 months because $178 = 12 \times 14 + 10$.

ANSWER: (C)

10. The number of integers between $-\sqrt{8}$ and $\sqrt{32}$ is

(**A**) 5 (**B**) 6 (**C**) 7 (**D**) 8 (**E**) 19

Solution

Since $-\sqrt{8} \doteq -2.8$ and $\sqrt{32} \doteq 5.7$, the integers between -2.8 and 5.7 would be -2, –1, 0, 1, 2, 3, 4, and 5. There are eight integers.

ANSWER: (D)

Part B:

11. A store had a sale on T-shirts. For every two T-shirts purchased at the regular price, a third T-shirt was bought for \$1.00. Twelve T-shirts were bought for \$120.00. What was the regular price for one T-shirt?

(**A**) \$10.00 (**B**) \$13.50 (**C**) \$14.00 (**D**) \$14.50 (**E**) \$15.00

Solution

We will start this question by representing the regular price of one T-shirt as x dollars. If a person bought a 'lot' of three T-shirts, they would thus pay $(2x + 1)$ dollars. Since the cost of twelve T-shirts is \$120.00, this implies that a single 'lot' would be \$30. This allows us to write the equation, $2x + 1 = 30$, $x = 14.50$. The regular price of a T-shirt is \$14.50.

ANSWER: (D)

12. In the diagram, every number beginning at 30 equals twice the sum of the two numbers to its immediate left. The value of c is

10	a	30	b	c

(**A**) 50 (**B**) 70 (**C**) 80 (**D**) 100 (**E**) 200

Solution

If we start the question by working with the first three boxes, we have:

$$30 = 2(10 + a)$$

or, $$15 = a + 10$$

$$a = 5$$

If $a = 5$ then $b = 2(30+5) = 70$ and, in turn, $c = 2(70+30) = 200$. ANSWER: (E)

13. In the expression $\frac{a}{b} + \frac{c}{d} + \frac{e}{f}$ each letter is replaced by a different digit from 1, 2, 3, 4, 5, and 6. What is the largest possible value of this expression?

(**A**) $8\frac{2}{3}$ (**B**) $9\frac{5}{6}$ (**C**) $9\frac{1}{3}$ (**D**) $9\frac{2}{3}$ (**E**) $10\frac{1}{3}$

Solution

To maximize the value of the given expression, we must make the individual fractions as large as possible. We do this by selecting the largest value possible for the numerator of a fraction and then the smallest possible value from the numbers remaining, as its denominator. Using this principle, the first 'fraction' would be $\frac{6}{1}$, the second $\frac{5}{2}$ and the third $\frac{4}{3}$. This gives, $6 + \frac{5}{2} + \frac{4}{3} = \frac{36+15+8}{6} = \frac{59}{6} = 9\frac{5}{6}$. ANSWER: (B)

14. The numbers 6, 14, *x*, 17, 9, *y*, 10 have a mean of 13. What is the value of $x + y$?

(**A**) 20 (**B**) 21 (**C**) 23 (**D**) 25 (**E**) 35

Solution

If the 7 numbers have a mean of 13, this implies that these numbers would have a sum of $7 \times 13 = 91$.
We now can calculate $x + y$ since, $6 + 14 + x + 17 + 9 + y + 10 = 91$.
Therefore, $x + y = 35$. ANSWER: (E)

15. The digits 1, 1, 2, 2, 3, and 3 are arranged to form an odd six digit integer. The 1's are separated by one digit, the 2's by two digits, and the 3's by three digits. What are the last three digits of this integer?

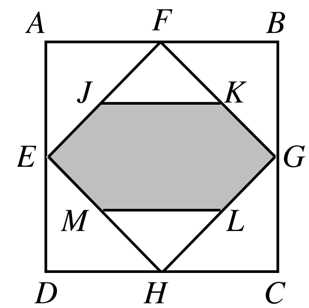
(**A**) 3 1 2 (**B**) 1 2 3 (**C**) 1 3 1 (**D**) 1 2 1 (**E**) 2 1 3

Solution

There are two numbers, either 3 1 2 1 3 2 or 2 3 1 2 1 3 that meet the given conditions. The requirement that the numbers must be odd, however, means that only the second number satisfies the requirements. The required answer is, '213'.

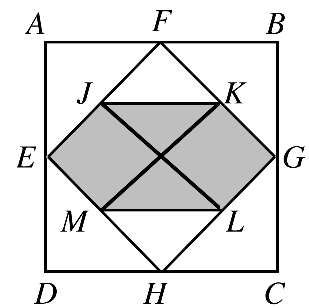
ANSWER: (E)

16. The area of square $ABCD$ is 64. The midpoints of its sides are joined to form the square $EFGH$. The midpoints of its sides are J, K, L, and M. The shaded area is



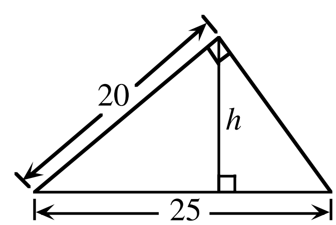
(**A**) 32 (**B**) 24 (**C**) 20
(**D**) 28 (**E**) 16

Solution
If we join vertices as illustrated in the diagram, we note that the square $FGHE$ is divided into eight equal parts, six of which are shaded. Since the square $FGHE$ is itself one half the area of the larger square $ABCD$, the shaded region is then, $\frac{6}{8} \times \left(\frac{1}{2} \times 64\right) = 24$.



ANSWER: (B)

17. In the diagram, the value of the height h is



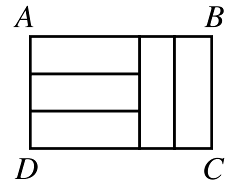
(**A**) 6 (**B**) 9 (**C**) 10
(**D**) 12 (**E**) 15

Solution
Using the fact that the given triangle is similar to any $3:4:5$ triangle we can see that the missing side is 15 since $3:4:5 = 15:20:25$. (We could also have used Pythagorus' Theorem and used the calculation $\sqrt{25^2 - 20^2} = 15$ to find the missing side.) We now calculate the area in two ways to determine h. Therefore, $\frac{1}{2}(20)(15) = \frac{1}{2}(h)(25)$.

$$300 = 25h$$
$$h = 12$$

ANSWER: (D)

18. In the diagram the five smaller rectangles are identical in size and shape. The ratio of $AB:BC$ is

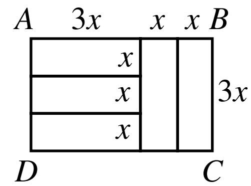


(**A**) 3:2 (**B**) 2:1 (**C**) 5:2
(**D**) 5:3 (**E**) 4:3

Solution

We let the width of each rectangle be x units and the length of each rectangle be $3x$ units. (We have illustrated this in the diagram.) The length, AB, is now $3x+x+x$ or $5x$ units and $BC=3x$. Since $AB:BC=5x:3x$

$$=5:3,\quad x\neq 0.$$



ANSWER: (D)

19. The year 2000 is a leap year. The year 2100 is not a leap year. The following are the complete rules for determining a leap year:

(i) Year Y is not a leap year if Y is not divisible by 4.
(ii) Year Y is a leap year if Y is divisible by 4 but not by 100.
(iii) Year Y is not a leap year if Y is divisible by 100 but not by 400.
(iv) Year Y is a leap year if Y is divisible by 400.

How many leap years will there be from the years 2000 to 3000 inclusive?

(**A**) 240 (**B**) 242 (**C**) 243 (**D**) 244 (**E**) 251

Solution

If we consider the stretch of 1001 years between 2000 and 3000, there are 251 'years' altogether that are divisible by 4. Every year is a leap year except 2100, 2200, 2300, 2500, 2600, 2700, 2900, and 3000. This means that there are $251-8=243$ leap years between 2000 and 3000.

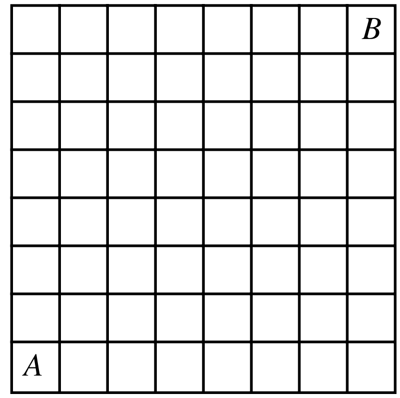
ANSWER: (C)

20. A straight line is drawn across an 8 by 8 checkerboard. What is the greatest number of 1 by 1 squares through which this line could pass?

(**A**) 12 (**B**) 14 (**C**) 16 (**D**) 11 (**E**) 15

Solution

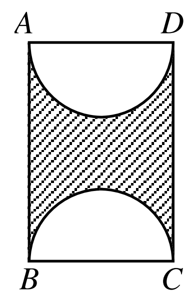
Let's suppose that we start in square *A* and end in square *B*. In order to achieve the maximum number of squares, we must cross seven horizontal and seven vertical lines. If we avoid going through the corners of the squares, we enter one new square for every line we cross. Thus we enter fourteen new squares for a total of fifteen squares.



ANSWER: (E)

Part C: Each question is worth 8 credits.

21. $ABCD$ is a rectangle with $AD = 10$. If the shaded area is 100, then the shortest distance between the semicircles is



(**A**) 2.5π (**B**) 5π (**C**) π
(**D**) $2.5\pi + 5$ (**E**) $2.5\pi - 2.5$

Solution

If AD is given as ten units then the radius of each of the two semicircles is $\frac{1}{2}(10) = 5$. Since the two semicircles make one complete circle with $r = 5$, they have a total area of $\pi\left(5^2\right) = 25\pi$. The total area of the rectangle is now $25\pi + 100$. Since the width of the rectangle is given as 10, $10(AB) = 25\pi + 100$ or $AB = 2.5\pi + 10$. The closest distance between the two circles is then $(2.5\pi + 10) - 10$ or 2.5π.

ANSWER: (A)

22. A wooden rectangular prism has dimensions 4 by 5 by 6. This solid is painted green and then cut into 1 by 1 by 1 cubes. The ratio of the number of cubes with exactly two green faces to the number of cubes with three green faces is

(**A**) 9:2 (**B**) 9:4 (**C**) 6:1 (**D**) 3:1 (**E**) 5:2

Solution

The cubes with two green faces are the cubes along the edges, not counting the corner cubes. In each dimension, we lost two cubes to the corners so we then have four edges with 4 cubes, four with 3 cubes and four with 2 cubes. The total number of cubes with paint on two edges is then

$4(4)+4(3)+4(2)=36$. The number of cubes that have paint on three sides are the corner cubes of which there are eight. The required ratio is then 36:8 or 9:2.

ANSWER: (A)

23. The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17 or 23. The 2000th digit may be either 'a' or 'b'. What is the value of $a+b$?

(**A**) 3 (**B**) 7 (**C**) 4 (**D**) 10 (**E**) 17

Solution
We start by noting that the two-digit multiples of 17 are 17, 34, 51, 68, and 85. Similarly we note that the two-digit multiples of 23 are 23, 46, 69, and 92. The first digit is 3 and since the only two-digit number in the two lists starting with 3 is 34, the second digit is 4. Similarly the third digit must be 6. The fourth digit, however, can be either 8 or 9. From here, we consider this in two cases.

Case 1
If the fourth digit is 8, the number would be 3468517 and would stop here since there isn't a number in the two lists starting with 7.

Case 2
If the fourth digit is 9, the number would be 34692 34692 34 ... and the five digits '34692' would continue repeating indefinitely as long as we choose 9 to follow 6.

If we consider a 2000 digit number, its first 1995 digits must contain 399 groups of '34692.' The last groups of five digits could be either 34692 or 34685 which means that the 2000th digit may be either 2 or 5 so that $a+b=2+5=7$.

ANSWER: (B)

24. There are seven points on a piece of paper. Exactly four of these points are on a straight line. No other line contains more than two of these points. Three of these seven points are selected to form the vertices of a triangle. How many triangles are possible?

(**A**) 18 (**B**) 28 (**C**) 30 (**D**) 31 (**E**) 33

Solution
There are three cases to consider corresponding to zero vertices, one vertex or two vertices on the given line.

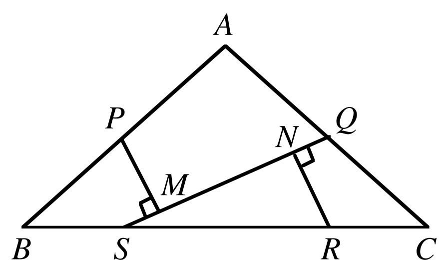
Case 1 'zero vertices on the given line'
Since there are exactly three points not on the line, there can only be one triangle formed with these three points.

Case 2 'one vertex on the given line'
There are four choices for the point on the line and for each of these four points there are three ways of selecting the pair of vertices not on the line. Thus, there are 3×4 or 12 possible triangles.

Case 3 '2 vertices on the given line'
There are six ways of choosing the pair of points on the line and for each of these six pairs there are three ways of selecting the vertex not on the line giving a total of 6×3 or 18 possibilities.

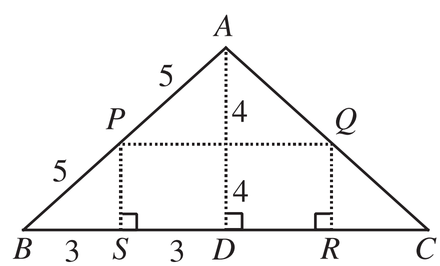
In total there are $1+12+18$ or 31 triangles. ANSWER: (D)

25. ΔABC is an isosceles triangle in which $AB = AC = 10$ and $BC = 12$. The points S and R are on BC such that $BS:SR:RC = 1:2:1$. The midpoints of AB and AC are P and Q respectively. Perpendiculars are drawn from P and R to SQ meeting at M and N respectively. The length of MN is

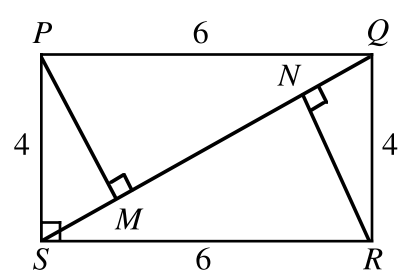


(A) $\frac{9}{\sqrt{13}}$ **(B)** $\frac{10}{\sqrt{13}}$ **(C)** $\frac{11}{\sqrt{13}}$ **(D)** $\frac{12}{\sqrt{13}}$ **(E)** $\frac{5}{2}$

Solution
The triangle ABC is isosceles and we start by drawing a perpendicular from A to BC to meet at D, the midpoint of BC. This makes $BD = 6$ and using 'Pythagorus' in ΔABD $AD = 8$. If we join P to Q, P to S and Q to R we note that because $BP = PA = 5$ and $BS = SD = 3$ we conclude that $PS \| AD$ and $\angle PSB = 90°$. This implies that $PS = 4$. We use symmetry to also conclude that $QR = 4$ and $QR \| AD$. This implies that $PS = QR = 4$ and $PS \| QR$. $PQRS$ is thus a rectangle and $PQ = 6$ since $PQ = SR = 6$.



To make the calculations easier we place all the information on the diagram.



Now, $SQ^2 = 4^2 + 6^2 = 52$ so $SQ = \sqrt{52}$. In ΔSQR we calculate the value of NR by using the fact that it has an area of 12. (ΔSQR has an area of 12 because it is one-half the area of the rectangle $PQRS$ which itself has area 24.)

Therefore, $\frac{1}{2}\left(\sqrt{52}\right)(NR) = 12$

$$NR = \frac{24}{\sqrt{52}} = \frac{24}{2\sqrt{13}} = \frac{12}{\sqrt{13}}.$$

Using 'Pythagoras', $NR^2 + NQ^2 = QR^2$.

Substituting, $\left(\frac{12}{\sqrt{13}}\right)^2 + NQ^2 = 4^2$.

Therefore, $NQ^2 = 16 - \frac{144}{13}$

$$NQ^2 = \frac{208 - 144}{13}$$

$$NQ^2 = \frac{64}{13}.$$

And, $NQ = \frac{8}{\sqrt{13}}$ and $MS = \frac{8}{\sqrt{13}}$ because $NQ = MS$.

Therefore, $MN = 2\sqrt{13} - 2\left(\frac{8}{\sqrt{13}}\right) = \frac{26}{\sqrt{13}} - \frac{16}{\sqrt{13}} = \frac{10}{\sqrt{13}}$. ANSWER: (B)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1999 Solutions
Pascal Contest (Grade 9)

for the
NATIONAL BANK OF CANADA
Awards

Part A

1. The value of $\frac{4 \times 4 + 4}{2 \times 2 - 2}$ is

(**A**) 2 (**B**) 6 (**C**) 10 (**D**) 12 (**E**) 18

Solution
$\frac{4 \times 4 + 4}{2 \times 2 - 2} = \frac{16 + 4}{4 - 2} = \frac{20}{2} = 10$ ANSWER: (C)

2. If $k = 2$, then $(k^3 - 8)(k + 1)$ equals

(**A**) 0 (**B**) 3 (**C**) 6 (**D**) 8 (**E**) -6

Solution
For $k = 2$, $(k^3 - 8)(k + 1)$

$$= (2^3 - 8)(2 + 1)$$
$$= 0(3)$$
$$= 0$$

ANSWER: (A)

3. If $4(\heartsuit)^2 = 144$, then a value of $\heartsuit$ is

(**A**) 3 (**B**) 6 (**C**) 9 (**D**) 12 (**E**) 18

Solution
$$4(\heartsuit)^2 = 144$$
$$\heartsuit^2 = 36$$
$$\heartsuit = \pm 6$$

ANSWER: (B)

4. Which of the following numbers divide exactly into $(15 + \sqrt{49})$?

(**A**) 3 (**B**) 4 (**C**) 5 (**D**) 7 (**E**) 11

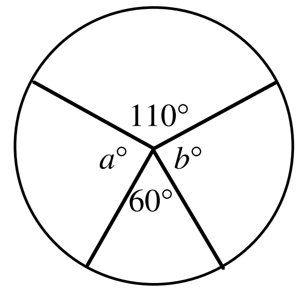
Solution
$15 + \sqrt{49} = 15 + 7 = 22$
The only integer listed that divides 22 evenly is 11. ANSWER: (E)

5. If 10% of 400 is decreased by 25, the result is

(**A**) 15 (**B**) 37.5 (**C**) 65 (**D**) 260 (**E**) 3975

Solution
(10% of 400) $- 25 = 40 - 25 = 15$. ANSWER: (A)

6. In the diagram, $a+b$ equals



(**A**) 10 (**B**) 85 (**C**) 110
(**D**) 170 (**E**) 190

Solution
The number of degrees at the centre of a circle is 360.
Thus, $a+b+110+60=360$ (measured in degrees).
Therefore $a+b=190$. ANSWER: (E)

7. If $2x-1=5$ and $3y+2=17$, then the value of $2x+3y$ is

(**A**) 8 (**B**) 19 (**C**) 21 (**D**) 23 (**E**) 25

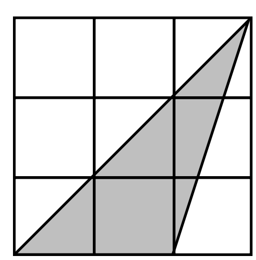
Solution
$2x-1=5$, $3y+2=17$
$2x=6$ $3y=15$
Thus, $2x+3y=6+15=21$. ANSWER: (C)
Note: It is not necessary to solve the equations to find actual values for x and y although this would of course lead to the correct answer. It is, however, a little more efficient to solve for $2x$ and $3y$.

8. The average of four test marks was 60. The first three marks were 30, 55 and 65. What was the fourth mark?

(**A**) 40 (**B**) 55 (**C**) 60 (**D**) 70 (**E**) 90

Solution
The total number of marks scored on the four tests was 4×60 or 240. The total number of marks scored on the first three tests was 150. The fourth mark was $240-150=90$. ANSWER: (E)

9. In the diagram, each small square is 1 cm by 1 cm. The area of the shaded region, in square centimetres, is



(**A**) 2.75 (**B**) 3 (**C**) 3.25
(**D**) 4.5 (**E**) 6

Solution
The shaded triangle has a base of 2 cm and a height of 3 cm.

Its area is $\dfrac{2\times 3}{2}=3$ (sq. cm). ANSWER: (B)

10. $10+10^3$ equals

(**A**) 2.0×10^3 (**B**) 8.0×10^3 (**C**) 4.0×10^1 (**D**) 1.0×10^4 (**E**) 1.01×10^3

Solution
$10+10^3=10+1000=1010=1.01\times 10^3$ ANSWER: (E)

Part B

11. Today is Wednesday. What day of the week will it be 100 days from now?

(**A**) Monday (**B**) Tuesday (**C**) Thursday (**D**) Friday (**E**) Saturday

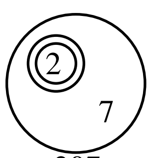
Solution
Since there are 7 days in a week it will be Wednesday in 98 days.
In 100 days it will thus be Friday. ANSWER: (D)

12. The time on a digital clock is 5:55. How many minutes will pass before the clock next shows a time with all digits identical?

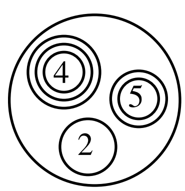
(**A**) 71 (**B**) 72 (**C**) 255 (**D**) 316 (**E**) 436

Solution
The digits on the clock will next be identical at 11:11. This represents a time difference of 316 minutes. (Notice that times like 6:66, 7:77 etc. are not possible.) ANSWER: (D)

13. In *Circle Land*, the numbers 207 and 4520 are shown in the following way:

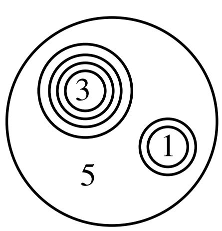


207



4520

In *Circle Land*, what number does the following diagram represent?



(**A**) 30 105 (**B**) 30 150 (**C**) 3105 (**D**) 3015 (**E**) 315

Solution 1

$= 3 \times 10^4 = 30\,000$

$= 1 \times 10^2 = 100$

5 $= 5 \times 10^0 = 5$

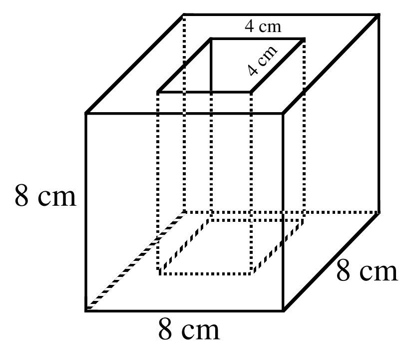
The required number is $30\,000 + 100 + 5 = 30\,105$.

Solution 2

Since there are four circles around the '3' this corresponds to $3 \times 10^4 = 30\,000$.
The '5' corresponds to a 5 in the units digit which leads to $30\,105$ as the only correct possibility.

ANSWER: (A)

14. An 8 cm cube has a 4 cm square hole cut through its centre, as shown. What is the remaining volume, in cm^3?



(**A**) 64 (**B**) 128 (**C**) 256
(**D**) 384 (**E**) 448

Solution

Remaining volume $= 8 \times 8 \times 8 - 8 \times 4 \times 4$ (in cm^3)
$= 8(64 - 16)$
$= 8 \times 48$
$= 384$

ANSWER: (D)

15. For how many different values of k is the 4-digit number $7k52$ divisible by 12?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

Solution

Since $12 = 4 \times 3$ the number $7k52$ must be divisible by both 4 and 3. Since 52 is the number formed by the last two digits divisible by 4 then we need only ask, 'for what values of k is $7k52$ divisible by 3?' If a number is divisible by 3 the sum of its digits must be a multiple of 3. Thus $7 + k + 5 + 2$ or $14 + k$ must be a multiple of 3. The only acceptable values for k are 1, 4 or 7.
Thus, are three values.

ANSWER: (D)

16. In an election, Harold received 60% of the votes and Jacquie received all the rest. If Harold won by 24 votes, how many people voted?

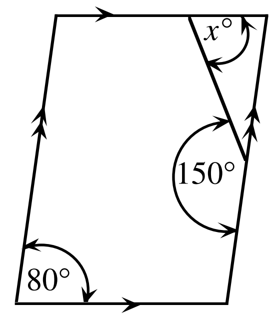
(**A**) 40 (**B**) 60 (**C**) 72 (**D**) 100 (**E**) 120

Solution

If Harold received 60% of the votes this implies that Jacquie received 40% of the total number of votes. The difference between them, 20%, represents 24 votes.
Therefore, the total number of votes cast was $5 \times 24 = 120$. ANSWER: (E)

17. In the parallelogram, the value of x is

(**A**) 30 (**B**) 50 (**C**) 70
(**D**) 80 (**E**) 150

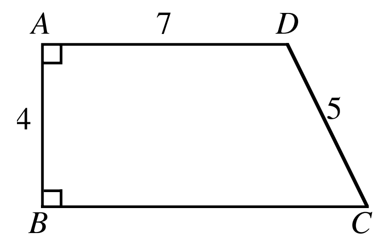


Solution

The angle in the parallelogram opposite the angle measuring 80° is also 80°. The angle supplementary to 150° is 30°.
In the given triangle we now have, $x° + 80° + 30° = 180°$.
Therefore $x = 70$. ANSWER: (C)

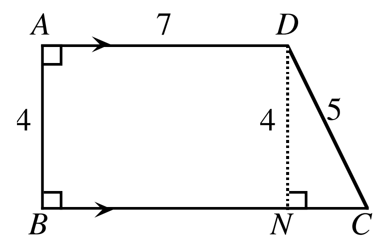
18. In the diagram, $AD < BC$. What is the perimeter of $ABCD$?

(**A**) 23 (**B**) 26 (**C**) 27
(**D**) 28 (**E**) 30



Solution

From D we draw a line perpendicular to BC that meets BC at N. Since $ADNB$ is a rectangle and $AD \parallel BC$, $DN = 4$. We use Pythagoras to find $NC = 3$. We now know that $BC = BN + NC = 7 + 3 = 10$.
The required perimeter is $7 + 5 + 10 + 4 = 26$.



ANSWER: (B)

19. The numbers 49, 29, 9, 40, 22, 15, 53, 33, 13, 47 are grouped in pairs so that the sum of each pair is the same. Which number is paired with 15?

(**A**) 33 (**B**) 40 (**C**) 47 (**D**) 49 (**E**) 53

Solution
If we arrange the numbers in ascending order we would have: 9, 13, 15, 22, 29, 33, 40, 47, 49, 53. If the sum of each pair is equal they would be paired as: $9 \leftrightarrow 53$, $13 \leftrightarrow 49$, $15 \leftrightarrow 47$, $22 \leftrightarrow 40$, $29 \leftrightarrow 33$.
ANSWER: (C)

20. The units (ones) digit in the product $(5+1)\left(5^3+1\right)\left(5^6+1\right)\left(5^{12}+1\right)$ is

(**A**) 6 (**B**) 5 (**C**) 2 (**D**) 1 (**E**) 0

Solution
We start by observing that each of 5^3, 5^6 and 5^{12} have a units digit of 5. This implies that each of $5+1$, 5^3+1, 5^6+1 and $5^{12}+1$ will then have a units digit of 6.
If we multiply any two numbers having a units digit of 6, their product will also have a units digit of 6. Applying this to the product of four numbers, we see that the final units digit must be a 6.
ANSWER: (A)

Part C

21. A number is *Beprisque* if it is the only natural number between a prime number and a perfect square (e.g. 10 is Beprisque but 12 is not). The number of *two-digit* Beprisque numbers (including 10) is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution
We start with the observation that it is necessary to consider only the odd perfect squares and the integers adjacent to them. It is not necessary to consider the even perfect squares because if we add 2 or subtract 2 from an even number the result is even and it is required by the conditions set out in the question that this number be prime. Considering then the odd perfect squares we have: $\{9, 10, 11\}$, $\{23, 24, 25, 26, 27\}$, $\{47, 48, 49, 50, 51\}$, $\{79, 80, 81, 82, 83\}$.
The Beprisque numbers are those that are circled.
ANSWER: (E)

22. If $w = 2^{129} \times 3^{81} \times 5^{128}$, $x = 2^{127} \times 3^{81} \times 5^{128}$, $y = 2^{126} \times 3^{82} \times 5^{128}$, and $z = 2^{125} \times 3^{82} \times 5^{129}$, then the order from smallest to largest is

(**A**) *w, x, y, z* (**B**) *x, w, y, z* (**C**) *x, y, z, w* (**D**) *z, y, x, w* (**E**) *x, w, z, y*

Solution
We start with the observation that $2^{125} \times 3^{81} \times 5^{128}$ is a common factor to each of the given numbers.

For the basis of comparison, we remove the common factor and write the numbers as follows:

$$w = 2^4 \cdot 5^3\left(2^{125} \times 3^{81} \times 5^{128}\right) = 2000k$$

$$x = 2^2 \cdot 5^3\left(2^{125} \times 3^{81} \times 5^{128}\right) = 500k$$

$$y = 2 \cdot 3 \cdot 5^3\left(2^{125} \times 3^{81} \times 5^{128}\right) = 750k$$

$$z = 3 \cdot 5^4\left(2^{125} \times 3^{81} \times 5^{128}\right) = 1875k, \text{ where } k = 2^{125} \times 3^{81} \times 5^{128}.$$

Thus, $x < y < z < w$. ANSWER: (C)

23. Al and Bert must arrive at a town 22.5 km away. They have one bicycle between them and must arrive at the same time. Bert sets out riding at 8 km/h, leaves the bicycle and then walks at 5 km/h. Al walks at 4 km/h, reaches the bicycle and rides at 10 km/h. For how many minutes was the bicycle not in motion?

(**A**) 60 (**B**) 75 (**C**) 84 (**D**) 94 (**E**) 109

Solution
Let x represent the distance that Bert rides his bicycle.
Therefore, he walks for $(22.5 - x)$ km.
Bert's total time for the trip is $\left(\frac{x}{8} + \frac{22.5 - x}{5}\right)$ hours and Al's is $\left(\frac{x}{4} + \frac{22.5 - x}{10}\right)$ hours.
Since their times are equal,

$$\frac{x}{8} + \frac{22.5 - x}{5} = \frac{x}{4} + \frac{22.5 - x}{10}$$

$$\frac{22.5 - x}{5} - \frac{22.5 - x}{10} = \frac{x}{4} - \frac{x}{8}$$

$$\frac{2(22.5 - x)}{10} - \frac{22.5 - x}{10} = \frac{2x}{8} - \frac{x}{8}$$

$$\frac{22.5 - x}{10} = \frac{x}{8}$$

$$10x = 180 - 8x$$

$$18x = 180$$

$$x = 10.$$

This means that Bert rode for 1.25 h before he left the bicycle and Al walked for 2.5 h before he picked it up. The bicycle was thus not in motion for 1.25 h or 75 minutes. ANSWER: (B)

24. A number is formed using the digits 1, 2, ..., 9. Any digit can be used more than once, but adjacent digits cannot be the same. Once a pair of adjacent digits has occurred, that pair, in that order, cannot be used again. How many digits are in the largest such number?

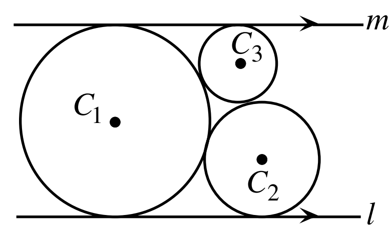
(**A**) 72 (**B**) 73 (**C**) 144 (**D**) 145 (**E**) 91

Solution
Since there are $9(8)=72$ ordered pairs of consecutive digits, and since the final digit has no successor, we can construct a 73 digit number by adding a 9. The question is, of course, can we actually construct this number? The answer is 'yes' and the largest such number is,

98 97 96 95 94 93 92 91 87 86 85 84 83 82 81 76 75 74 73 72 71 65 64 63 62 61
54 53 52 51 43 42 41 32 31 21 9.

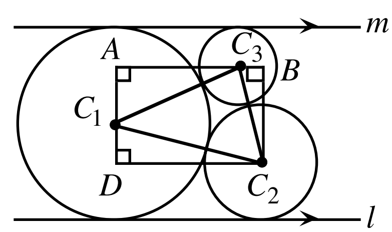
If we count the numbers in the string we can see that there are actually 73 numbers contained within it. ANSWER: (B)

25. Two circles C_1 and C_2 touch each other externally and the line l is a common tangent. The line m is parallel to l and touches the two circles C_1 and C_3. The three circles are mutually tangent. If the radius of C_2 is 9 and the radius of C_3 is 4, what is the radius of C_1?

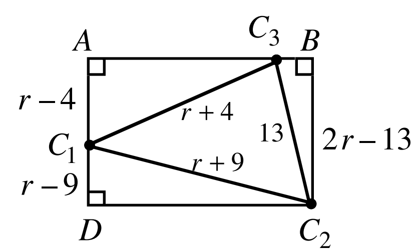


(**A**) 10.4 (**B**) 11 (**C**) $8\sqrt{2}$
(**D**) 12 (**E**) $7\sqrt{3}$

Solution
We start by joining the centres of the circles to form $\Delta C_1C_2C_3$. (The lines joining the centres pass through the corresponding points of tangency.)
Secondly, we construct the rectangle ABC_2D as shown in the diagram. If the radius of the circle with centre C_1 is r we see that: $C_1C_2=r+9$, $C_1C_3=r+4$ and $C_2C_3=13$.



We now label lengths on the rectangle in the way noted in the diagram.



To understand this labelling, look for example at C_1D. The radius of the large circle is r and the radius of the circle with centre C_2 is 9. The length C_1D is then $r-9$.
This same kind of reasoning can be applied to both C_1A and BC_2.

Using Pythagoras we can now derive the following:
In ΔAC_3C_1, $C_3A^2=(r+4)^2-(r-4)^2$
$=16r$.

Therefore $C_3A=4\sqrt{r}$.

In ΔDC_1C_2, $\quad (DC_2)^2 = (r+9)^2 - (r-9)^2$
$= 36r.$

Therefore $DC_2 = 6\sqrt{r}$.

In ΔBC_3C_2, $\quad (C_3B)^2 = 13^2 - (2r-13)^2$
$= -4r^2 + 52r.$

Therefore $C_3B = \sqrt{-4r^2 + 52r}$.

In a rectangle opposite sides are equal, so:

$DC_2 = C_3A + C_3B$

or, $\quad 6\sqrt{r} = 4\sqrt{r} + \sqrt{-4r^2 + 52r}$

$2\sqrt{r} = \sqrt{-4r^2 + 52r}$.

Squaring gives, $4r = -4r^2 + 52r$

$4r^2 - 48r = 0$

$4r(r-12) = 0$

Therefore $r = 0$ or $r = 12$.

Since $r > 0$, $r = 12$. ANSWER: (D)

Canadian
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1998 Solutions
Pascal Contest (Grade 9)

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Awards

PART A:

1. The value of $\frac{1+3+5}{10+6+2}$ is

(**A**) $\frac{1}{6}$ (**B**) 2 (**C**) $\frac{1}{2}$ (**D**) $1\frac{1}{2}$ (**E**) $3\frac{1}{10}$

Solution 1

$$\frac{1+3+5}{10+6+2} = \frac{9}{18} = \frac{1}{2}$$

Solution 2

$$\frac{1+3+5}{10+6+2} = \frac{(1+3+5)}{2(5+3+1)} = \frac{1}{2}$$

ANSWER: (C)

2. If $3(x-5)=3(18-5)$, then x is

(**A**) $\frac{44}{3}$ (**B**) $\frac{32}{3}$ (**C**) 9 (**D**) 18 (**E**) 81

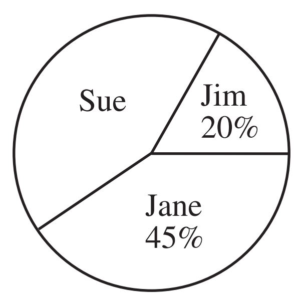
Solution

Since $3(x-5)=3(18-5)$, divide both sides by 3 to get $(x-5)=(18-5)$.

Therefore, $x=18$. ANSWER: (D)

3. The pie chart shows a percentage breakdown of 1000 votes in a student election. How many votes did Sue receive?

(**A**) 550 (**B**) 350 (**C**) 330
(**D**) 450 (**E**) 935



Solution

Sue received $100-(20+45)=35$ percent of the total number of votes. Since there was a total of 1000 votes, Sue received $0.35(1000)=350$ votes. ANSWER: (B)

4. The value of $\left(\sqrt{169}-\sqrt{25}\right)^2$ is

(**A**) 64 (**B**) 8 (**C**) 16 (**D**) 144 (**E**) 12

Solution

$$\left(\sqrt{169}-\sqrt{25}\right)^2 = (13-5)^2 = 8^2 = 64$$

ANSWER: (A)

5. The value of $\dfrac{5^6 \times 5^9 \times 5}{5^3}$ is

(**A**) 5^{18} (**B**) 25^{18} (**C**) 5^{13} (**D**) 25^{13} (**E**) 5^{51}

Solution

$$\frac{5^6 \times 5^9 \times 5}{5^3} = \frac{5^{16}}{5^3} = 5^{13}$$

ANSWER: (C)

6. If $x = 3$, which of the following expressions is even?

(**A**) $9x$ (**B**) x^3 (**C**) $2\left(x^2+9\right)$ (**D**) $2x^2+9$ (**E**) $3x^2$

Solution 1

Since the expression $2\left(x^2+9\right)$ contains a factor of 2, it must be even regardless of the value of *x*.

Solution 2

If $x = 3$, then $9x = 27$, $x^3 = 27$, $2\left(x^2+9\right) = 36$, $2x^2+9 = 27$, and $3x^2 = 27$.

Therefore, $2\left(x^2+9\right)$ is the only expression that is an even number. ANSWER: (C)

7. The value of $490-491+492-493+494-495+\ldots-509+510$ is

(**A**) 500 (**B**) -10 (**C**) -11 (**D**) 499 (**E**) 510

Solution

The value of $490-491+492-493+494-495+\ldots-509+510$ is

$$(490-491)+(492-493)+(494-495)+\ldots+(508-509)+510 = \underbrace{(-1)+(-1)+(-1)+\ldots+(-1)}_{\text{10 pairs, each add to } -1}+510$$

$$= -10+510$$

$$= 500$$

ANSWER: (A)

8. The average (mean) of a list of 10 numbers is 0. If 72 and -12 are added to the list, the new average will be
(**A**) 30 (**B**) 6 (**C**) 0 (**D**) 60 (**E**) 5

Solution
If the average (mean) of a list of 10 numbers is 0, then the sum of the numbers is 10(0) = 0. When 72 and –12 are added to the list, the sum of these 12 numbers is $0+72-12=60$.
Thus, the average of the 12 numbers is $60 \div 12 = 5$. ANSWER: (E)

9. What is one-half of 1.2×10^{30}?
(**A**) 6.0×10^{30} (**B**) 6.0×10^{29} (**C**) 0.6×5^{30} (**D**) 1.2×10^{15} (**E**) 1.2×5^{30}

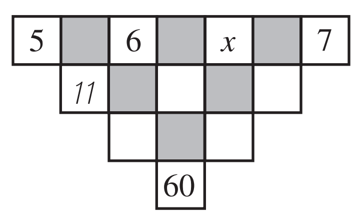
Solution
$$\begin{aligned}\frac{1}{2}\left(1.2 \times 10^{30}\right) &= 0.6 \times 10^{30} \\ &= 0.6 \times 10 \times 10^{29} \\ &= 6.0 \times 10^{29}\end{aligned}$$
ANSWER: (B)

10. If $x+y+z=25$ and $y+z=14$, then x is
(**A**) 8 (**B**) 11 (**C**) 6 (**D**) -6 (**E**) 31

Solution
We are given that $x+y+z=25$ (1)
and $y+z=14$ (2).
Subtract equation (2) from equation (1) to get $x=11$. ANSWER: (B)

PART B:

11. The number in an unshaded square is obtained by adding the numbers connected to it from the row above. (The '*11*' is one such number.) The value of x is
(**A**) 4 (**B**) 6 (**C**) 9
(**D**) 15 (**E**) 10



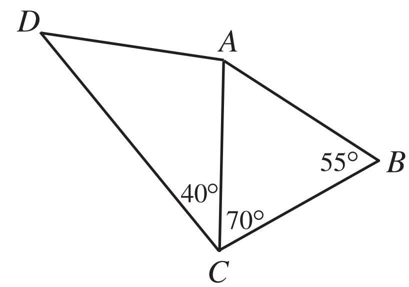
Solution
The three entries in row two, from left to right, are 11, $6+x$, and $x+7$. The two entries in row three, from left to right, are $11+(6+x)=17+x$ and $(6+x)+(x+7)=2x+13$. The single entry in row four is $(17+x)+(2x+13)=3x+30$.

$$\begin{aligned} \text{Thus, } 3x + 30 &= 60 \\ 3x &= 30 \\ x &= 10 \end{aligned}$$

ANSWER: (E)

12. In the diagram, $DA = CB$. What is the measure of $\angle DAC$?

(**A**) 70° (**B**) 100° (**C**) 95° (**D**) 125° (**E**) 110°



Solution

$$\begin{aligned} \text{In } \Delta ABC, \angle BAC &= 180^\circ - \left(70^\circ + 55^\circ\right) \\ &= 55^\circ \end{aligned}$$

Since $\angle BAC = \angle ABC$, then ΔABC is isosceles with $AC = CB$.

We are given that $DA = CB$, so $DA = AC$ and ΔADC is also isosceles.

Thus, $\angle ADC = \angle ACD = 40^\circ$

$$\begin{aligned} \text{and } \angle DAC &= 180^\circ - \left(40^\circ + 40^\circ\right) \\ &= 100^\circ \end{aligned}$$

ANSWER: (B)

13. A three-wheeled vehicle travels 100 km. Two spare wheels are available. Each of the five wheels is used for the same distance during the trip. For how many kilometres is each wheel used?

(**A**) 20 (**B**) 25 (**C**) $33\frac{1}{3}$ (**D**) 50 (**E**) 60

Solution

Since only three of the five wheels are in use at any time, the total distance travelled by all the wheels is 3(100) = 300 kilometres. However, each of the five wheels is used for the same distance during the trip. Thus, each wheel is used for $300 \div 5 = 60$ kilometres. ANSWER: (E)

14. The sum of the digits of a five-digit positive integer is 2. (A five-digit integer cannot start with zero.) The number of such integers is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

If the sum of the digits of a five-digit positive integer is 2, then the only possible integers are 20 000, 11 000, 10 100, 10 010, and 10 001. There are 5 such integers. ANSWER: (E)

15. Four points are on a line segment, as shown.
If $AB : BC = 1:2$ and $BC : CD = 8:5$, then $AB : BD$ equals

(**A**) 4:13 (**B**) 1:13 (**C**) 1:7
(**D**) 3:13 (**E**) 4:17

Solution

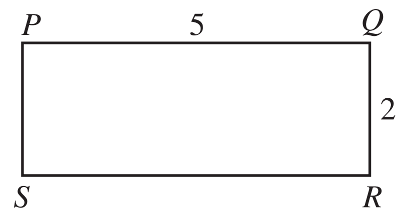
In order to compare the given ratios, we must rewrite the ratio $AB : BC = 1:2$ as $AB : BC = 4:8$. Now both ratios express BC as 8 units and we can write $AB : BC : CD = 4:8:5$.

Thus, $AB : BD = 4:(8+5)$

$= 4:13$

ANSWER: (A)

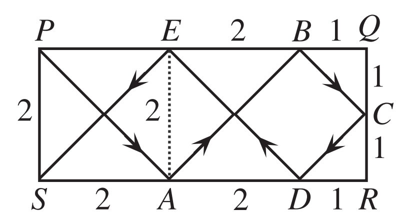
16. On a rectangular table 5 units long and 2 units wide, a ball is rolled from point P at an angle of $45°$ to PQ and bounces off SR. The ball continues to bounce off the sides at $45°$ until it reaches S. How many bounces of the ball are required?



(**A**) 9 (**B**) 8 (**C**) 7
(**D**) 5 (**E**) 4

Solution

Since the ball bounces off the sides of the rectangular table at $45°$, right-angled isosceles triangles are created as shown. The ball begins at point P then bounces at points $A, B, C, D,$ and E before reaching S, for a total of 5 bounces.



ANSWER: (D)

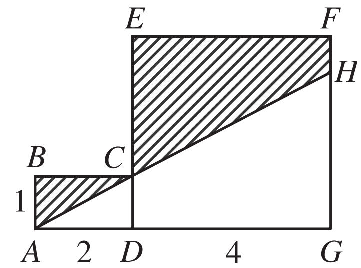
17. If $1998 = p^s q^t r^u$, where p, q and r are prime numbers, what is the value of $p + q + r$?

(**A**) 222 (**B**) 48 (**C**) 42 (**D**) 66 (**E**) 122

Solution

The prime factorization of 1998 is $2 \times 3^3 \times 37$. Thus, $p, q,$ and r have values 2, 3, and 37 (in any order), and $p + q + r = 42$. ANSWER: (C)

18. In the diagram, $DEFG$ is a square and $ABCD$ is a rectangle. A straight line is drawn from A, passes through C and meets FG at H. The area of the shaded region is



(**A**) 8 (**B**) 8.5 (**C**) 10
(**D**) 9 (**E**) 10.5

Solution

We are given that AH is a straight line segment, and C is a point on AH. Since $AD : DC = 2:1$, then $AG : GH = 2:1$. Since the length of AG is 6, the length of GH is 3.

The area of rectangle $ABCD$ is $1 \times 2 = 2$. The area of square $DEFG$ is $4^2 = 16$. The area of ΔAHG is $\frac{1}{2}(6)(3) = 9\cdot$

Therefore, the area of the shaded region is $2 + 16 - 9 = 9$. ANSWER: (D)

19. Using only digits 1, 2, 3, 4, and 5, a sequence is created as follows: one 1, two 2's, three 3's, four 4's, five 5's, six 1's, seven 2's, and so on.

The sequence appears as: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, ...

The 100th digit in the sequence is

(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

Solution

The total number of digits in n groups of the sequence is given by $1 + 2 + 3 + ... + n$. In order to determine the group containing the 100th digit in the sequence, we must find the positive integer n such that $1 + 2 + 3 + ... + (n-1) < 100$ and $1 + 2 + 3 + ... + n > 100$. By examining a few of these sums we find that $1 + 2 + 3 + ... + 13 = 91$ and $1 + 2 + 3 + ... + 13 + 14 = 105$. Thus the 100th digit in the sequence is in the 14th group. The 100th digit is a 4. ANSWER: (D)

20. Driving between two towns at 110 km/h instead of 100 km/h saves 9 minutes. The distance in kilometres between the two towns is

(**A**) 210 (**B**) 99 (**C**) 165 (**D**) 9900 (**E**) 150

Solution

Let x represent the distance, in kilometres, between the two towns. Driving at 100 km/h, it takes $\frac{x}{100}$ hours to travel between the towns. Driving at 110 km/h, it takes $\frac{x}{110}$ hours. We know that these two times differ by 9 minutes, or $\frac{9}{60}$ hours.

Thus, $\frac{x}{110} + \frac{9}{60} = \frac{x}{100}$

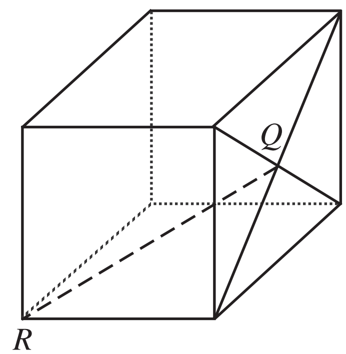
$$\frac{x}{11}+\frac{3}{2}=\frac{x}{10}$$
$$\frac{3}{2}=\frac{11x}{110}-\frac{10x}{110}$$
$$\frac{3}{2}=\frac{x}{110}$$
$$165=x$$

The two towns are 165 km apart. ANSWER: (C)

PART C:

21. Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of QR is

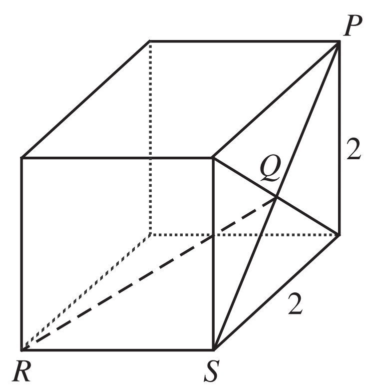
(**A**) 2 (**B**) $\sqrt{8}$ (**C**) $\sqrt{5}$
(**D**) $\sqrt{12}$ (**E**) $\sqrt{6}$



Solution

Label points P and S as shown. Since each face of the cube is a square with sides of length 2, use the Pythagorean Theorem to find the length of diagonal PS.

$$PS^2=2^2+2^2$$
$$=8$$
$$PS=2\sqrt{2}$$



Then QS has length $\sqrt{2}$, as Q is the midpoint of diagonal PS.
Because we are working with a cube, $\angle QSR=90^\circ$ and ΔQRS is a right-angled triangle. Use the Pythagorean Theorem in ΔQRS to get

$$QR^2=2^2+\left(\sqrt{2}\right)^2$$
$$=6$$
$$QR=\sqrt{6}$$

ANSWER: (E)

22. A deck of 100 cards is numbered from 1 to 100. Each card has the same number printed on both sides. One side of each card is red and the other side is yellow. Barsby places all the cards, red side up, on a table. He first turns over every card that has a number divisible by 2. He then examines all the cards, and turns over every card that has a number divisible by 3. How many cards have the red side up when Barsby is finished?

(**A**) 83 (**B**) 17 (**C**) 66 (**D**) 50 (**E**) 49

Solution

Initially, all 100 cards have the red side up. After Barsby's first pass only the 50 odd-numbered cards have the red side up, since he has just turned all the even-numbered cards from red to yellow.

During Barsby's second pass he turns over all cards whose number is divisible by 3. On this pass Barsby will turn any odd-numbered card divisible by 3 from red to yellow. Between 1 and 100, there are 17 odd numbers that are divisible by 3, namely 3, 9, 15, 21, ... , 93, and 99. Also on this pass, Barsby will turn any even-numbered card divisible by 3 from yellow to red. Between 1 and 100, there are 16 even numbers that are divisible by 3, namely 6, 12, 18, 24, ... , 90, and 96.

When Barsby is finished, the cards that have the red side up are the 50 odd-numbered cards from the first pass, minus the 17 odd-numbered cards divisible by 3 from the second pass, plus the 16 even-numbered cards divisible by 3, also from the second pass.

Thus, $50-17+16=49$ cards have the red side up. ANSWER: (E)

23. The numbers $123\,456\,789$ and $999\,999\,999$ are multiplied. How many of the digits in the final result are 9's?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 17

Solution

Rewrite the product as follows:

$$\begin{aligned}(123\ \ 456\ \ 789)(999\ \ 999\ \ 999) &= (123\ \ 456\ \ 789)\left(10^9-1\right)\\ &= (123\ \ 456\ \ 789)\times 10^9-(123\ \ 456\ \ 789)\end{aligned}$$

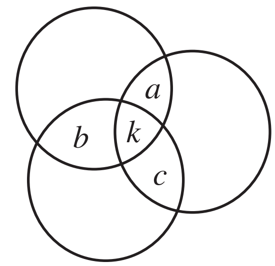
When 123 456 789 is subtracted from $(123\ \ 456\ \ 789)\times 10^9$ the result is 123 456 788 876 543 211. None of the digits are 9's. ANSWER: (A)

24. Three rugs have a combined area of 200 m^2. By overlapping the rugs to cover a floor area of 140 m^2, the area which is covered by exactly two layers of rug is 24 m^2. What area of floor is covered by three layers of rug?

(**A**) 12 m^2 (**B**) 18 m^2 (**C**) 24 m^2 (**D**) 36 m^2 (**E**) 42 m^2

Solution

Draw the rugs in the following manner, where $a+b+c$ represents the amount of floor covered by exactly two rugs and k represents the amount of floor covered by exactly three rugs. We are told that $a+b+c=24$ (1).



Since the total amount of floor covered when the rugs do not overlap is 200 m^2 and the total covered when they do overlap is 140 m^2, then 60 m^2 of rug is wasted on double or triple layers. Thus, $a+b+c+2k=60$ (2). Subtract equation (1) from equation (2) to get $2k=36$ and solve for $k=18$. Thus, the area of floor covered by exactly three layers of rug is 18 m^2. ANSWER: (B)

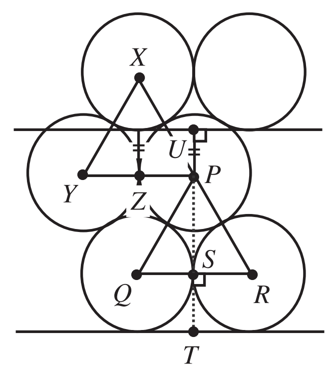
25. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

(**A**) 647 (**B**) 1442 (**C**) 1343 (**D**) 1443 (**E**) 1344

Solution

Remove one circle from every second row and shift to form the given configuration. Label the diagram as shown. Since each circle has diameter 1, ΔPQR and ΔPXY are equilateral triangles with sides of length 1.

In ΔPQR, altitude PS bisects side QR. Use the Pythagorean Theorem to find PS.



$$PS^2 = (1)^2 - \left(\tfrac{1}{2}\right)^2$$
$$= \tfrac{3}{4}$$
$$PS = \tfrac{\sqrt{3}}{2}$$

Also, $XZ = \tfrac{\sqrt{3}}{2}$

Since all radii have length $\frac{1}{2}$, then $PU = \frac{\sqrt{3}}{2} - \frac{1}{2}$ and $TU = \frac{1}{2} + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = \sqrt{3}$. This tells us that two rows of circles require a height of $\sqrt{3}$ before a third row begins.

Since $\dfrac{100}{\sqrt{3}} = 57.7$, we can pack 57 double rows, each containing 100+99=199 circles.

Can we pack one final row of 100 circles? Yes. The square has sides of length 100 and our configuration of 57 double rows requires a height of $57\sqrt{3}$ before the next row begins. Since $100 - 57\sqrt{3} > 1$, and since the circles each have diameter 1, there is room for one final row of 100 circles.

The number of circles used in this new packing is $57(199)+100=11\ 443$
Thus, the maximum number of extra circles that can be packed into the square is
$11\ 443-10\ 000=1443$. ANSWER: (D)

Canadian
Mathematics
Competition

An activity of The Centre for Education
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1997 Solutions
Pascal Contest (Grade 9)

for the
NATIONAL BANK OF CANADA
Awards

PART A:

1. ***Solution***

$$\frac{4+35}{8-5} = \frac{39}{3}$$
$$= 13$$

ANSWER: (E)

2. ***Solution***

Since $268 + 189 = 457$, the ♥ represents the digit 5. ANSWER: (D)

3. ***Solution***

$$2\frac{1}{10} + 3\frac{11}{100} + 4\frac{111}{1000} = 2.1 + 3.11 + 4.111$$
$$= 9.321$$

ANSWER: (A)

4. ***Solution***

$$(1)^{10} + (-1)^8 + (-1)^7 + (1)^5 = 1 + 1 - 1 + 1$$
$$= 2$$

ANSWER: (C)

5. ***Solution 1***

Since 60% of the number equals 42, then 1% of the number equals $\frac{42}{60}$.

Therefore 50% of the number equals $50 \times \frac{42}{60}$ or 35. ANSWER: (D)

Solution 2

Let x represent the number.

Then $0.6x = 42$

$x = 70$

Thus, 50% of the number is $0.5(70) = 35$. ANSWER: (D)

6. ***Solution***

First, simplify the expression.

$$(x)\left(x^2\right)\left(\frac{1}{x}\right) = x^2$$

If $x = -2$, the value of the expression is $(-2)^2 = 4$. ANSWER: (A)

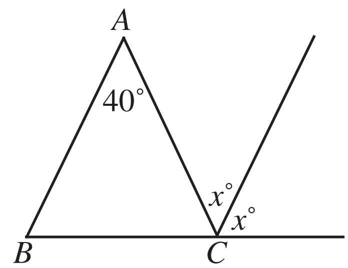
7. ***Solution***

Since ΔABC is isosceles, $\angle ABC = \angle ACB$.

The value of each of these angles is $\dfrac{(180° - 40°)}{2} = 70°$.

Since the points B, C, and D lie on a straight line,

$$70 + 2x = 180$$
$$2x = 110$$
$$x = 55$$



ANSWER: (B)

8. ***Solution***

If the first day is a Monday, then every seventh day is also a Monday, and Monday falls on the following numbered days: 1, 8, 15, 22, 29, 36, 43.

In the first 45 days of the year, the maximum number of Mondays is seven.

ANSWER: (C)

9. ***Solution***

The number is $9 \times 6 + 4 = 58$. ANSWER: (A)

10. ***Solution***

Since the sum of the nine integers is 99, their average is $\frac{99}{9}$ or 11.

Because there is an odd number of consecutive integers, 11 is the middle integer and 15 is the largest. ANSWER: (E)

PART B:

11. ***Solution***

The first ten balloons are popped in the following order: C, F, I, L, D, H, A, G, B, and K. The remaining two balloons are E and J. ANSWER: (D)

12. ***Solution***

The total number of students answering the question was $300 + 1100 + 100 + 600 + 400 = 2500$.

Thus, the percentage of students who selected the correct response was $\left(\frac{1100}{2500}\right) \times 100$ or 44%.

ANSWER: (E)

13. ***Solution 1***

Since Janet has 10 coins, seven of which are dimes and quarters, then three coins are nickels.
Since eight coins are nickels or dimes, then five are dimes.
Thus Janet has 5 dimes. ANSWER: (D)

Solution 2
If Janet has n nickels, d dimes and q quarters, we can write

$$n + d + q = 10 \qquad (1)$$
$$d + q = 7 \qquad (2)$$
$$n + d = 8 \qquad (3)$$

Subtracting equation (2) from equation (1), we get $n = 3$. Substituting this in (3), we get $d = 5$.
Thus, Janet has 5 dimes. ANSWER: (D)

14. ***Solution***
In the diagram, there are four small and two large triangles, for a total of 18 points. As well, there are four small and one large square, for a total of 20 points. Altogether, 38 points can be achieved. ANSWER: (A)

15. ***Solution***
The greatest possible value of p^q is 3^4 or 81.
The greatest possible value of $p^q + r^s$ is $3^4 + 2^1$ or 83. ANSWER: (D)

16. ***Solution***
In the diagram, there are 27 black triangles. If the entire diagram was divided into the smallest size equilateral triangles, there would be $8 + 2(7) + 2(6) + 2(5) + 2(4) + 2(3) + 2(2) + 2(1) = 64$ (counting by rows). Thus, $\frac{27}{64}$ of ΔABC is coloured black. ANSWER: (E)

17. ***Solution***
The first twelve numbers in the list begin with either the digit 1 or 2. The next six begin with the digit 3. In order, these six numbers are 3124, 3142, 3214, 3241, 3412, 3421.
We see that the number 3142 is in the fourteenth position. ANSWER: (B)

18. ***Solution***

Since each factor of 10 produces a zero at the end of the integer we want to know how many 10's occur in the product.
The product of 20^{50} and 50^{20} can be rewritten as follows:

$$\begin{aligned}(20^{50})(50^{20}) &= (2^2 \cdot 5)^{50}(5^2 \cdot 2)^{20} \\ &= 2^{100} \cdot 5^{50} \cdot 5^{40} \cdot 2^{20} \\ &= 2^{120} \cdot 5^{90} \\ &= 2^{30}(2^{90} \cdot 5^{90}) \\ &= 2^{30} \cdot 10^{90}\end{aligned}$$

From this, we see that there are 90 zeros at the end of the resulting integer.

ANSWER: (C)

19. ***Solution***

Each time a ball is drawn from the bag, there are three possible outcomes. Since a ball is drawn three different times, there are $3^3 = 27$ possible outcomes for the sum, although not all 27 are unique.
To determine how many of these outcomes give a sum that is less than eight, we first determine how many give a sum of eight or nine.
The only one way the sum of the three recorded numbers could be nine, is if three 3's are drawn.
To yield a sum of 8, the following three combinations are possible: 3, 3, 2 or 3, 2, 3 or 2, 3, 3.
Thus, of the 27 outcomes, $27 - 1 - 3 = 23$ give a sum less than 8, so the probability of obtaining the required sum is $\frac{23}{27}$.

ANSWER: (A)

20. ***Solution***

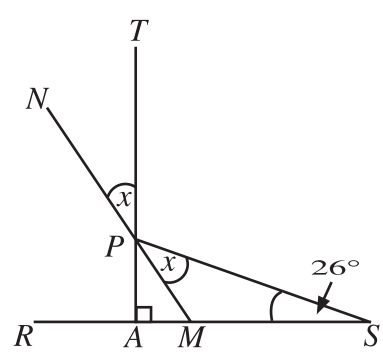
In the diagram, extend TP to meet RS at A. Since $AT \perp RS$, then $\angle SPA = 180° - 90° - 26°$

$$= 64°$$

Label points M and N. Since $\angle TPN$ and $\angle MPA$ are vertically opposite angles, they are equal, so $\angle MPA = x$.
Since $\angle SPA = 2x$, $\quad 2x = 64°$

$$x = 32°$$

Thus, the value of x is $32°$.

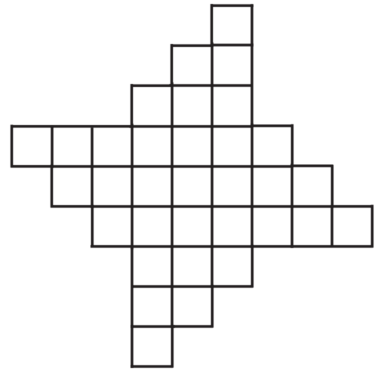


ANSWER: (B)

PART C:

21. ***Solution***

Since all of the shorter edges are equal in length, the diagram can be subdivided into 33 small squares, as shown. Each of these squares has area $\frac{528}{33} = 16$ and the length of each side is $\sqrt{16} = 4$.
By counting, we find 36 sides and a perimeter of 144.



ANSWER: (C)

22. ***Solution***

Rewrite $\frac{97}{19}$ as $5 + \frac{2}{19} = 5 + \frac{1}{\left(\frac{19}{2}\right)}$

$$= 5 + \frac{1}{9 + \frac{1}{2}}$$

By comparison, $w = 5$, $x = 9$ and $y = 2$.
Thus, $w + x + y = 16$. ANSWER: (A)

23. ***Solution***

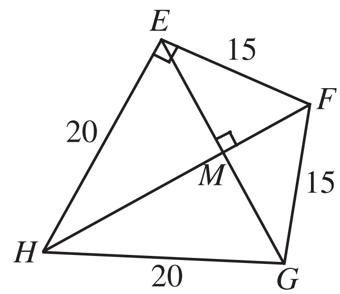
Since 25 is the sum of two squares, the only possible values for x are 0, 3, 4 and 5. Substituting each value of x into the equation and finding the corresponding value of y gives five different values for y.

24. ***Solution***

Let the speed of the faster ship be x metres per second and the speed of the slower ship be y metres per second.
When the ships are travelling in opposite directions, their relative speed is $(x + y)$ m/s and the distance required to pass is 300 m, giving $10(x + y) = 300$.
When the ships are travelling in the same direction, their relative speed is $(x - y)$ m/s and the distance required to pass is still 300 m, giving $25(x - y) = 300$.
Solving for x gives $x = 21$.
Thus, the speed of the faster ship is 21 metres per second. ANSWER: (D)

25. ***Solution***

In order to calculate AG, we must first calculate EG. Consider the base quadrilateral $EFGH$ and join E to G and F to H. Where the diagonals meet, call the point M.



In ΔHGF apply Pythagorus to find $FH = 25$.

Notice that the triangles EFG and EHG are both isosceles so the diagonals of quadrilateral $EFGH$ meet at right angles.

ΔFMG and ΔFGH are similar triangles because they each contain a right angle and $\angle MFG$ is common to both. Using this similarity, $\frac{MG}{15} = \frac{20}{25}$, $MG = 12$ and thus, $EG = 24$.

Since $EG = 24$ and $AE = 32$, apply Pythagorus in ΔAEG to find $AG = 40$.

ANSWER: (B)