The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2024 Hypatia Contest

Thursday, April 4, 2024
(in North America and South America)

Friday, April 5, 2024
(outside of North America and South America)

Solutions

1. (a) Of the 4050 trucks sold, 32% were white or $\dfrac{32}{100} \cdot 4050 = 1296$ were white.

(b) *Solution 1*

Of the 4050 trucks sold, 24% were grey or $\dfrac{24}{100} \cdot 4050 = 972$ were grey.

Since $\dfrac{1}{4}$ of the grey trucks sold were electric, then $\dfrac{1}{4} \cdot 972 = 243$ trucks sold were both grey and electric.

Solution 2

Since 24% of the trucks sold were grey, and $\dfrac{1}{4}$ of those were electric, then $\dfrac{24}{100} \cdot \dfrac{1}{4} = \dfrac{6}{100}$ (or 6%) were both grey and electric.

Thus, of the 4050 trucks sold, $\dfrac{6}{100} \cdot 4050 = 243$ were both grey and electric.

(c) *Solution 1*

Of the 4050 trucks sold, 44% were black or $\dfrac{44}{100} \cdot 4050 = 1782$ were black.

Thus, the total number of black trucks, sold and unsold, was $1782 + k$, and the total number of trucks, sold and unsold, was $4050 + k$.

Since 46% of all trucks, sold and unsold, were black, then $\dfrac{1782 + k}{4050 + k} = \dfrac{46}{100}$.

Solving, we get

$$\begin{aligned}
\frac{1782 + k}{4050 + k} &= \frac{46}{100} \\
\frac{1782 + k}{4050 + k} &= \frac{23}{50} \\
50(1782 + k) &= 23(4050 + k) \\
89\,100 + 50k &= 93\,150 + 23k \\
27k &= 4050 \\
k &= 150
\end{aligned}$$

and so there were 150 unsold trucks, all of which were black.

Solution 2

Of the 4050 trucks sold, 44% were black and so $100\% - 44\% = 56\%$ were not black.

Therefore, $\dfrac{56}{100} \cdot 4050 = 2268$ trucks sold were not black.

Since all unsold trucks were black, then there were 2268 trucks, sold and unsold, that were not black.

Since 46% of all trucks, sold and unsold, were black, then $100\% - 46\% = 54\%$ of all trucks, sold and unsold, were not black.

The total number of trucks, sold and unsold, was $4050 + k$ and 54% of these trucks were not black, thus $\dfrac{2268}{4050 + k} = \dfrac{54}{100}$.

Solving, we get

$$\begin{aligned}\frac{2268}{4050+k} &= \frac{54}{100}\\ \frac{2268}{4050+k} &= \frac{27}{50}\\ 50(2268) &= 27(4050+k)\\ 113\,400 &= 109\,350+27k\\ 4050 &= 27k\\ k &= 150\end{aligned}$$

and so there were 150 unsold trucks, all of which were black.

2. (a) Evaluating, we get $f(132) = 132 + 1 + 3 + 2 = 138$.

(b) Suppose that n is equal to the 3-digit positive integer abc.
Then $f(n) = f(abc) = 100a + 10b + c + a + b + c = 101a + 11b + 2c$.
Since $f(n) = 175$, then $101a + 11b + 2c = 175$.
It cannot be the case that $a \geq 2$, since if we had $a \geq 2$, then $101a \geq 202$ which is too large, noting that $11b + 2c$ is always at least 0.
Therefore, $a < 2$ which means that $a = 1$.
When $a = 1$, we get $101 + 11b + 2c = 175$ or $11b + 2c = 74$.
It cannot be the case that $b \geq 7$, since if we had $b \geq 7$, then $11b \geq 77$ which is too large, noting that $2c$ is always at least 0.
Therefore, $b < 7$. If $b = 6$, then $66 + 2c = 74$ or $2c = 8$, and so $c = 4$.
If $b \leq 5$, then $11b \leq 55$, and so $2c \geq 74 - 55 = 19$, which is not possible since $c \leq 9$.
We can confirm that $f(164) = 164 + 1 + 6 + 4 = 175$, and so $n = 164$.

(c) Suppose that n is equal to the 3-digit positive integer pqr.
Then $f(pqr) = 100p + 10q + r + p + q + r$, and so $101p + 11q + 2r = 204$.
If $p \geq 3$, then $101p \geq 303$, and so $p = 1$ or $p = 2$.

If $p = 1$, then $101 + 11q + 2r = 204$ or $11q + 2r = 103$.
Since $r \leq 9$, then $2r \leq 18$ and so $11q \geq 103 - 18 = 85$.
Therefore, $q = 8$ or $q = 9$.
If $q = 8$, then $88 + 2r = 103$ or $2r = 15$, which is not possible since r is an integer.
If $q = 9$, then $99 + 2r = 103$ or $2r = 4$, and so $r = 2$.
In this case, $n = 192$ and we can confirm that $f(192) = 192 + 1 + 9 + 2 = 204$.

If $p = 2$, then $202 + 11q + 2r = 204$ or $11q + 2r = 2$.
The only possible solution to $11q + 2r = 2$ is $q = 0$ and $r = 1$.
In this case, $n = 201$ and we can confirm that $f(201) = 201 + 2 + 0 + 1 = 204$.
Therefore, if $f(n) = 204$, then the possible values of n are 192 and 201.

3. (a) To determine the coordinates of F, we find the point of intersection of the line through A and C and the line through B and E.
The line through $A(0, 0)$ and $C(12, 12)$ has slope $\frac{12-0}{12-0} = 1$.
Since it passes through $(0, 0)$, this line has equation $y = x$.
The line through $B(12, 0)$ and $E(0, 6)$ has slope $\frac{6-0}{0-12} = -\frac{1}{2}$.
Since it passes through $(0, 6)$, this line has equation $y = -\frac{1}{2}x + 6$.

To determine the x-coordinate of the point of intersection, F, we solve $x = -\frac{1}{2}x + 6$, which gives $\frac{3}{2}x = 6$ or $3x = 12$, and so $x = 4$.

Since F lies on the line with equation $y = x$, then the coordinates of F are $(4, 4)$.

(b) *Solution 1*

Consider $\triangle AEF$ as having base $AE = 6$.

Then $\triangle AEF$ has height equal to the perpendicular distance from F to AE, which is 4, the x-coordinate of F.

The area of $\triangle AEF$ is thus $\frac{1}{2} \cdot 6 \cdot 4 = 12$.

Solution 2

We can determine the area of $\triangle AEF$ by subtracting the area of $\triangle AFB$ from the area of $\triangle AEB$.

Consider $\triangle AFB$ as having base $AB = 12$.

Then $\triangle AFB$ has height equal to the perpendicular distance from F to AB, which is 4, the y-coordinate of F.

The area of $\triangle AFB$ is thus $\frac{1}{2} \cdot 12 \cdot 4 = 24$.

The area of $\triangle AEB$ is $\frac{1}{2} \cdot 12 \cdot 6 = 36$, and so the area of $\triangle AEF$ is $36 - 24 = 12$.

(c) To determine the area of quadrilateral $GDEF$, our strategy will be to subtract the area of $\triangle AEF$ and the area of $\triangle CDG$ from the area of $\triangle ACD$.

We need to find the area of $\triangle CDG$ still, which means finding the coordinates of G.

We can find the coordinates of G by determining the intersection of the line through A and C with the given circle.

Thus, we proceed by finding the equation of the circle.

Since the circle has diameter EB, then its centre is the midpoint of EB, which is $\left(\frac{0+12}{2}, \frac{6+0}{2}\right)$ or $(6, 3)$.

The diameter has length $EB = \sqrt{(12-0)^2 + (0-6)^2}$ or $EB = \sqrt{180}$, which simplifies to $EB = 6\sqrt{5}$.

Thus the radius of the circle is $r = \frac{1}{2} \cdot 6\sqrt{5} = 3\sqrt{5}$, and so the circle has equation $(x-6)^2 + (y-3)^2 = (3\sqrt{5})^2$ or $(x-6)^2 + (y-3)^2 = 45$.

Suppose the x-coordinate of G is g.

Since G lies on the line with equation $y = x$, then the coordinates of G are (g, g).

The point G also lies on the circle, and thus the coordinates of G satisfy the equation of the circle.

That is, $(g-6)^2 + (g-3)^2 = 45$, and solving for g, we get

$$\begin{aligned} g^2 - 12g + 36 + g^2 - 6g + 9 &= 45 \\ 2g^2 - 18g + 45 &= 45 \\ 2g^2 - 18g &= 0 \\ 2g(g-9) &= 0 \end{aligned}$$

and so $g = 0$ or $g = 9$.

Since G is distinct from A, then $g = 9$ and G has coordinates $(9, 9)$.

We may now determine the area of $\triangle CDG$.

Consider $\triangle CDG$ as having base $CD = 12$.

Then $\triangle CDG$ has height equal to the perpendicular distance from G to CD, which is $12 - 9 = 3$, since CD lies along the line $y = 12$ and the y-coordinate of G is 9.

The area of $\triangle CDG$ is thus $\frac{1}{2} \cdot 12 \cdot 3 = 18$.

The area of $\triangle ACD$ is half the area of square $ABCD$ or $\frac{1}{2} \cdot 12^2 = 72$.

From part (b), the area of $\triangle AEF$ is 12, and so the area of $GDEF$ is $72 - 18 - 12 = 42$.

4. (a) Each Hewitt number, H, can be written as $H = (n-1)^3 + n^3 + (n+1)^3$ where n is an integer and $n \geq 2$.
(We chose $H = (n-1)^3 + n^3 + (n+1)^3$ instead of $H = n^3 + (n+1)^3 + (n+2)^3$, since the quadratic term and constant term subtract out when simplified, as shown below.)
Expanding and simplifying, we get

$$\begin{aligned} H &= (n-1)^3 + n^3 + (n+1)^3 \\ &= n^3 - 3n^2 + 3n - 1 + n^3 + n^3 + 3n^2 + 3n + 1 \\ &= 3n^3 + 6n \\ &= 3n(n^2+2) \end{aligned}$$

A Hewitt number is divisible by 10 exactly when its units digit is equal to 0.
If for example the units digit of n is 9, then the units digit of $3n$ is 7, the units digit of n^2+2 is 3, and so the units digit of $H = 3n(n^2+2)$ is 1 (since 7×3 has units digit 1).
For each possible units digit of n, we determine the units digit of H in the table below.

Units digit of n	0	1	2	3	4	5	6	7	8	9
Units digit of $3n$	0	3	6	9	2	5	8	1	4	7
Units digit of n^2+2	2	3	6	1	8	7	8	1	6	3
Units digit of $H = 3n(n^2+2)$	0	9	6	9	6	5	4	1	4	1

Thus, for a Hewitt number to be divisible by 10, the units digit of n must be 0, and so n must be divisible by 10.
When $n = 10$, $H = 3(10)(10^2+2) = 3060$ which is less than 10 000.
When $n = 20$, $H = 3(20)(20^2+2) = 24\,120$ which lies between 10 000 and 100 000.
When $n = 30$, $H = 3(30)(30^2+2) = 81\,180$ which lies between 10 000 and 100 000.
When $n \geq 40$, $H \geq 3(40)(40^2+2) = 192\,240$ which is greater than 100 000.
Thus, there are 2 Hewitt numbers between 10 000 and 100 000 that are divisible by 10.

(b) From part (a), each Hewitt number can be written as $H = 3n(n^2+2)$ where n is an integer and $n \geq 2$.
Since $216 = 2^3 \cdot 3^3$, then a Hewitt number is divisible by 216 exactly when $3n(n^2+2)$ is divisible by $2^3 \cdot 3^3$, or exactly when $n(n^2+2)$ is divisible by $2^3 \cdot 3^2$.
That is, we need $n(n^2+2)$ to be divisible by $2^3 = 8$ and by $3^2 = 9$.

We begin by considering what is required for $n(n^2+2)$ to be divisible by 8.
If $n(n^2+2)$ is divisible by 8, then $n(n^2+2)$ is divisible by 2 and thus is even.
If n is odd, then $n(n^2+2)$ is odd, and so n must be even.
Since n is even, then $n = 2a$ for some positive integer a, and so $n^2+2 = 4a^2+2$ which is 2 more than a multiple of 4 and so n^2+2 is not divisible by 4 (but it is divisible by 2).
Since $n(n^2+2)$ is divisible by 8 and n^2+2 contains exactly one factor of 2, then n must be divisible by 4.

Next, we consider what is required for $n(n^2+2)$ to be divisible by 9.
If $n(n^2+2)$ is divisible by 9, then at least one of the following must be true:

(i) n is divisible by 3 and n^2+2 is divisible by 3, or
(ii) n is divisible by 9, or
(iii) n^2+2 is divisible by 9.

Assume that n is divisible by 9, and thus divisible by 3.
Then $n = 3b$ for some positive integer b, and so $n^2+2 = 9b^2+2 = 3(3b^2)+2$ which is 2 more than a multiple of 3 and so n^2+2 is not divisible by 3, and thus not divisible by 9.

This tells us that if n is divisible by 3, then n^2+2 is not divisible by 3, and so (i) cannot be true.
Further, if n is divisible by 9, then n^2+2 is not divisible by 9, and so exactly one of (ii) or (iii) is true.

Summarizing, we get $n(n^2+2)$ is divisible by 8 and by 9 (and thus a Hewitt number is divisible by 216) exactly when n is divisible by 4 and by 9, or when n is divisible by 4 and n^2+2 is divisible by 9.

Case 1: n is divisible by 4 and by 9
Since 4 and 9 share no common divisor larger than 1, then n is divisible by 4 and by 9 exactly when n is divisible by $4 \cdot 9 = 36$.
In this case, $n = 36k$ for positive integers k.
The first Hewitt number occurs when $n = 2$, and so the 2024th Hewitt number occurs when $n = 2025$.
That is, $2 \leq n \leq 2025$ or $2 \leq 36k \leq 2025$, and so $\frac{2}{36} \leq k \leq \frac{2025}{36}$.
Since k is an integer and $\frac{2025}{36} = 56.25$, then $1 \leq k \leq 56$.
Thus in this case, 56 of the smallest 2024 Hewitt numbers are divisible by 216.

Case 2: n is divisible by 4 and n^2+2 is divisible by 9
Since n is divisible by 4, then $n = 4m$ for some positive integer m, and so $n^2+2 = 16m^2+2$ is divisible by 9.
For some non-negative integers q and r, where $0 \leq r \leq 8$, every positive integer m can be written as $m = 9q + r$, depending on its remainder, r, when divided by 9.
Since $16m^2+2$ must be divisible by 9, then each of the following equivalent expressions must also be divisible by 9:

$$\begin{aligned} 16m^2+2 &= 16(9q+r)^2+2 \\ &= 16(9^2q^2+2\cdot 9qr+r^2)+2 \\ &= 16(9^2q^2+2\cdot 9qr)+16r^2+2 \\ &= 9\cdot 16(9q^2+2qr)+16r^2+2 \end{aligned}$$

which is divisible by 9 exactly when $16r^2+2$ is divisible by 9.
That is, n is divisible by 4 and $n^2+2 = 16m^2+2$ is divisible by 9 exactly when $16r^2+2$ is divisible by 9, where r is the remainder when m is divided by 9.
For each of the possible remainders $0 \leq r \leq 8$, we may determine the remainder when $16r^2+2$ is divided by 9.
For example, when $r = 2$, $16r^2+2 = 16(2)^2+2 = 66$ leaves remainder 3 when divided by 9.
Similarly, we determine the remainder when $16r^2+2$ is divided by 9 for each of the possible values of r:

Value of r	0	1	2	3	4	5	6	7	8
Remainder when $16r^2+2$ is divided by 9	2	0	3	2	6	6	2	3	0

Thus, $16r^2+2$ is divisible by 9 when $r = 1$ or when $r = 8$.
Summarizing, we get that n is divisible by 4 and n^2+2 is divisible by 9 exactly when $n = 4m = 4(9q+1)$ or when $n = 4(9q+8)$ for non-negative integers q.
For the smallest 2024 Hewitt numbers, $2 \leq n \leq 2025$ or $2 \leq 4(9q+1) \leq 2025$, and so $0 \leq q \leq 56$ (since q is a non-negative integer).
In this case, 57 of the smallest 2024 Hewitt numbers are divisible by 216.
Similarly, $2 \leq 4(9q+8) \leq 2025$, and so $0 \leq q \leq 55$.

In this case, 56 of the smallest 2024 Hewitt numbers are divisible by 216.

Of the smallest 2024 Hewitt numbers, $56 + 57 + 56 = 169$ are divisible by 216.

(c) From part (a), each Hewitt number is given by $H = 3n(n^2 + 2)$ where n is an integer and $n \geq 2$.
If S is the sum of two distinct Hewitt numbers, then $S = 3n(n^2 + 2) + 3m(m^2 + 2)$ for some integers m and n and for which we may assume that $2 \leq m < n$.
If there are two distinct Hewitt numbers whose sum is equal to $9 \cdot 2^k$ for some positive integer k, then we get the following equivalent equations

$$\begin{aligned} S &= 3n(n^2+2) + 3m(m^2+2) \\ 9 \cdot 2^k &= 3n(n^2+2) + 3m(m^2+2) \\ 3 \cdot 2^k &= n(n^2+2) + m(m^2+2) \\ 3 \cdot 2^k &= n^3 + m^3 + 2n + 2m \\ 3 \cdot 2^k &= (n+m)(n^2 - nm + m^2) + 2(n+m) \\ 3 \cdot 2^k &= (n+m)(n^2 - nm + m^2 + 2) \end{aligned}$$

and so if there are two distinct Hewitt numbers whose sum is equal to $9 \cdot 2^k$, then $(n+m)(n^2 - nm + m^2 + 2) = 3 \cdot 2^k$ for some positive integers k, m, n where $2 \leq m < n$.
If m and n have different parity (one is even and the other is odd), then $n + m$ is odd.
Also, $n^2 - nm + m^2 + 2$ is odd since exactly one of n^2 or m^2 is odd and the remaining three terms in the sum are even.
In this case, $n + m$ and $n^2 - nm + m^2 + 2$ are both odd, and so their product is odd.
However, $3 \cdot 2^k$ is even for all positive integers k. Therefore, m and n must have the same parity (they must both be odd or they must both be even).

<u>Case 1: m and n are both odd</u>
If m and n are both odd, then $n^2 - nm + m^2 + 2$ is odd.
Since the only odd factors of $3 \cdot 2^k$ are 1 and 3, then $n^2 - nm + m^2 + 2$ must equal 1 or 3.
However, if m and n are both odd with $2 \leq m < n$, then $m \geq 3$ and $n \geq 5$ and $n - m \geq 2$.
Therefore,

$$\begin{aligned} n^2 - nm + m^2 + 2 &= n(n-m) + m^2 + 2 \\ &\geq 5(2) + 3^2 + 2 \\ &= 21 \end{aligned}$$

and so $n^2 - nm + m^2 + 2$ cannot equal 1 or 3.
Therefore, m and n cannot both be odd.

<u>Case 2: m and n are both even</u>
If m and n are both even, then $m = 2a$ and $n = 2b$ for some integers a and b with $1 \leq a < b$.
Substituting and simplifying, we get

$$\begin{aligned} 3 \cdot 2^k &= (n+m)(n^2 - nm + m^2 + 2) \\ 3 \cdot 2^k &= (2b+2a)(4b^2 - 4ab + 4a^2 + 2) \\ 3 \cdot 2^k &= 4(b+a)(2b^2 - 2ab + 2a^2 + 1) \\ 3 \cdot 2^{k-2} &= (b+a)(2b^2 - 2ab + 2a^2 + 1) \end{aligned}$$

Since the right side is the product of two integers, then $k \geq 2$.

The factor $2b^2 - 2ab + 2a^2 + 1$ is one more than a multiple of 2, and thus is odd and so it must be equal to 1 or 3.
Since $1 \leq a < b$, then $a \geq 1$, $b \geq 2$ and $b - a \geq 1$, and so

$$\begin{aligned} 2b^2 - 2ab + 2a^2 + 1 &= 2b(b-a) + 2a^2 + 1 \\ &\geq 4(1) + 2(1^2) + 1 \\ &= 7 \end{aligned}$$

and so $2b^2 - 2ab + 2a^2 + 1$ cannot equal 1 or 3.
Therefore, m and n cannot both be even.

We can conclude that there cannot be two distinct Hewitt numbers whose sum is equal to $9 \cdot 2^k$ for some positive integer k.

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2023 Hypatia Contest

Wednesday, April 5, 2023
(in North America and South America)

Thursday, April 6, 2023
(outside of North America and South America)

Solutions

1. (a) Jasmin's total score was $3 \cdot 2 + 4 \cdot 5 = 6 + 20$ or 26 points.

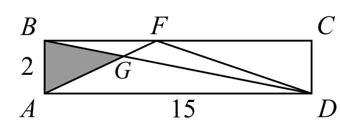
   (b) Suppose that Sam had n throws that each scored 5 points; then Sam had $2n$ throws that each scored 2 points.
   Since Sam's total score was 36 points, then $5 \cdot n + 2 \cdot 2n = 36$ or $9n = 36$, and so $n = 4$.
   In total, Sam took $n + 2n = 3n$ throws, which is $3 \cdot 4 = 12$ throws.

   (c) Since Tia's total score was 37 points, then $2t + 5f = 37$.
   Since $2t$ is even for all integer values of t, then $5f$ must be odd since their sum is 37 (which is odd).
   The value of $5f$ is odd exactly when f is odd.
   When $f = 1$, we get $2t + 5 = 37$ or $2t = 32$, and so $t = 16$.
   When $f = 3$, we get $2t + 15 = 37$ or $2t = 22$, and so $t = 11$.
   When $f = 5$, we get $2t + 25 = 37$ or $2t = 12$, and so $t = 6$.
   When $f = 7$, we get $2t + 35 = 37$ or $2t = 2$, and so $t = 1$.
   When $f \geq 9$, $5f \geq 45$ and so $2t + 5f > 37$.
   The possible ordered pairs (t, f) are $(16, 1)$, $(11, 3)$, $(6, 5)$, and $(1, 7)$.

   (d) If a throws each score 6 points and b throws each score 21 points, then $6a+21b$ or $3(2a+7b)$ points are scored.
   Since a and b are non-negative integers, then $2a + 7b$ is a non-negative integer, and so the total number of points scored, $3(2a + 7b)$, is a multiple of 3.
   Since 182 is not a multiple of 3, then it is not possible to have a total score of 182 points.

2. (a) The total area of the shaded regions can be determined by subtracting the area of $\triangle AED$ from the area of rectangle $ABCD$.
   The area of rectangle $ABCD$ is $2 \cdot 15 = 30$.
   The base length of $\triangle AED$ is $AD = 15$, its height is equal to $AB = 2$, and so its area is $\frac{1}{2} \cdot 15 \cdot 2 = 15$.
   Thus, the total area of the shaded regions is $30 - 15 = 15$.

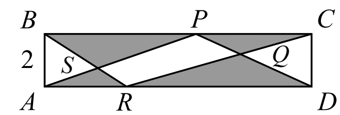
   (b) *Solution 1*
   Since $\triangle AFD$ has base length $AD = 15$ and height equal to $AB = 2$, then its area is $\frac{1}{2} \cdot 15 \cdot 2 = 15$.
   The area of $\triangle AFD$ is equal to the sum of the areas of $\triangle AGD$ and $\triangle FGD$.



   Since the area of $\triangle AFD$ is 15, and the area of $\triangle FGD$ is 5, then the area of $\triangle AGD$ is $15 - 5 = 10$.
   The area of $\triangle ABD$ is $\frac{1}{2} \cdot 15 \cdot 2 = 15$.
   The area of $\triangle ABD$ is equal to the sum of the areas of $\triangle AGD$ and $\triangle ABG$.
   Since the area of $\triangle ABD$ is 15, and the area of $\triangle AGD$ is 10, then the area of $\triangle ABG$ (the shaded area) is $15 - 10 = 5$.

   Solution 2
   Consider $\triangle AFD$ and $\triangle ABD$. Each has base AD and height AB and thus the two triangles have equal areas.
   The area of $\triangle AFD$ is equal to the sum of the areas of $\triangle AGD$ and $\triangle FGD$.
   The area of $\triangle ABD$ is equal to the sum of the areas of $\triangle AGD$ and $\triangle ABG$.
   Thus, the area of $\triangle ABG$ (the shaded area) is equal to the area of $\triangle FGD$, which is 5.

(c) The total area of the bottom two shaded regions ($\triangle ASR$ and $\triangle RQD$) can be determined by subtracting the area of $PQRS$ from the area of $\triangle APD$.
The base length of $\triangle APD$ is $AD = 15$, its height is equal to $AB = 2$, and so its area is $\frac{1}{2} \cdot 15 \cdot 2 = 15$.
Since the area of $PQRS$ is 6, the total area of the two bottom shaded regions is $15 - 6 = 9$.



We can similarly show that the total area of the two top shaded regions ($\triangle BSP$ and $\triangle PQC$) can be determined by subtracting the area of $PQRS$ from the area of $\triangle BRC$.
Since $\triangle BRC$ also has area 15, then the total area of the two top shaded regions is also $15 - 6 = 9$.
Thus, the total area of the shaded regions is $9 + 9 = 18$.

3. (a) Switching the second and fourth digits of 6238 gives the cousin 6832.
Since 6832 is not in the list, then it is the missing cousin.

(b) Of the 16 six-digit integers in the given list, 15 are cousins of the original integer.
If the original six-digit integer is $abcdef$, then there are exactly 5 cousins for which a is not the 1st digit.
These 5 cousins are a result of switching the 1st digit, a, with each of the other 5 digits, b, c, d, e, f, and thus are $bacdef$, $cbadef$, $dbcaef$, $ebcdaf$, and $fbcdea$.
Each of the remaining $15 - 5 = 10$ cousins of $abcdef$ have first digit a.
Since the digits are distinct, then the 1st digit of the original integer is the integer that occurs most frequently as the 1st digit in the given list, which is 7.
A similar argument can be made for each of the other digits.
That is, the nth digit of the original integer is the integer that occurs most frequently as the nth digit in the given list.
Thus, the original integer has 2nd digit 2, 3rd digit 6, 4th digit 4, 5th digit 9, 6th digit 1, and so the original integer is 726 491.
We may check that if the original integer is 726 491, then the cousins are indeed the remaining 15 integers in the given list.

(c) The cousins of the three-digit integer $cd3$ are $dc3$, $3dc$ and $c3d$.
The difference between the two integers $cd3$ and $dc3$ could be negative, in which case it's not the right difference. If it is positive, then $cd3$ minus $dc3$ has ones digit 0, and thus cannot equal $d95$.
Similarly, the difference between the two integers $cd3$ and $c3d$ could be negative, in which case it's not the right difference. Since $cd3$ and $c3d$ each have hundreds digit c, then $cd3$ minus $c3d$ has hundreds digit 0, and thus cannot equal $d95$ (since d is a non-zero digit).
Therefore, $cd3$ minus $3dc$ is equal to $d95$.
Since the difference, $d95$, has ones digit 5, then $c = 8$. (You should confirm for yourself that this is the only possible value for c before moving on.)
With $c = 8$, we get $8d3$ minus $3d8$ is equal to $d95$.
The three-digit integer $8d3$ is equal to $800 + 10d + 3$.
The three-digit integer $3d8$ is equal to $300 + 10d + 8$.
Thus, $8d3$ minus $3d8$ is equal to $(800 + 10d + 3) - (300 + 10d + 8) = 500 - 5 = 495$, which is equal to $d95$, and so $d = 4$.
(We may confirm that $843 - 348 = 495$.)
The value of c is 8 and the value of d is 4, and no other values are possible.

(d) *Solution 1*

The six cousins of the four-digit integer $mn97$ are $nm97$, $9nm7$, $7n9m$, $m9n7$, $m79n$, and $mn79$.

The cousin $nm97$ is equal to $1000n + 100m + 90 + 7$.

The cousin $9nm7$ is equal to $9000 + 100n + 10m + 7$.

The cousin $7n9m$ is equal to $7000 + 100n + 90 + m$.

The cousin $m9n7$ is equal to $1000m + 900 + 10n + 7$.

The cousin $m79n$ is equal to $1000m + 700 + 90 + n$.

The cousin $mn79$ is equal to $1000m + 100n + 70 + 9$.

The sum, S, of the six cousins is thus

$$\begin{aligned} S &= 1000(n+9+7+m+m+m) + 100(m+n+n+9+7+n) + 10(9+m+9+n+9+7) \\ &\quad + (7+7+m+7+n+9) \\ &= 1000(3m+n+16) + 100(m+3n+16) + 10(m+n+34) + (m+n+30) \end{aligned}$$

The sum, S, must be equal to the five-digit integer $nmnm7$, which has ones digit 7.

The ones digit of $S = 1000(3m+n+16)+100(m+3n+16)+10(m+n+34)+(m+n+30)$ is equal to the ones digit of $m+n+30$, which is equal to the ones digit of $m+n$ (since the ones digit of 30 is 0).

If the ones digit of $m+n$ is 7, then $m+n=7$ or $m+n=17$.

Since m and n are distinct non-zero digits, then the possible pairs (m,n) are $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, $(6,1)$, $(8,9)$, and $(9,8)$.

If $(m,n)=(1,6)$, then the five-digit integer $nmnm7$ is 61 617, and

$$\begin{aligned} S &= 1000(3m+n+16) + 100(m+3n+16) + 10(m+n+34) + (m+n+30) \\ &= 1000(3(1)+6+16) + 100(1+3(6)+16) + 10(1+6+34) + (1+6+30) \\ &= 1000(25) + 100(35) + 10(41) + 37 \\ &= 28\,947 \end{aligned}$$

and so $(m,n)=(1,6)$ is not a possible pair.

We continue to check the remaining possible pairs (m,n) in the table below.

(m,n)	$nmnm7$	S
$(2,5)$	52 527	30 747
$(3,4)$	43 437	32 547
$(4,3)$	34 347	34 347
$(5,2)$	25 257	36 147
$(6,1)$	16 167	37 947
$(8,9)$	98 987	54 657
$(9,8)$	89 897	56 457

The only pair of distinct, non-zero digits (m,n) is $(4,3)$.

Solution 2

The six cousins of the four-digit integer $mn97$ are $nm97$, $9nm7$, $7n9m$, $m9n7$, $m79n$, and $mn79$. As in Solution 1, the sum, S, of the six cousins is

$$\begin{aligned} S &= 1000(n+9+7+m+m+m) + 100(m+n+n+9+7+n) + 10(9+m+9+n+9+7) \\ &\quad + (7+7+m+7+n+9) \\ &= 1000(3m+n+16) + 100(m+3n+16) + 10(m+n+34) + (m+n+30) \\ &= 3000m + 1000n + 16\,000 + 100m + 300n + 1600 + 10m + 10n + 340 + m + n + 30 \\ &= 3111m + 1311n + 17\,970 \end{aligned}$$

The sum, S, must be equal to the five-digit integer $nmnm7$, which is equal to $10\,000n + 1000m + 100n + 10m + 7$ or $10\,100n + 1010m + 7$.
Setting these two expressions equal to one another and simplifying, we get

$$\begin{aligned} 3111m + 1311n + 17\,970 &= 10\,100n + 1010m + 7 \\ 8789n - 2101m &= 17\,963 \\ 799n - 191m &= 1633 \quad \text{(after division by 11)} \end{aligned}$$

Since $799n = 1633 + 191m$ and $m \geq 1$, then $799n \geq 1633 + 191(1)$ or $799n \geq 1824$, and so $n \geq \frac{1824}{799} \approx 2.3$.
Since $m \leq 9$, then $799n \leq 1633 + 191(9)$ or $799n \leq 3352$, and so $n \leq \frac{3352}{799} \approx 4.2$.
Since $2.3 \leq n \leq 4.2$ and n is an integer, then the possible values of n are 3 and 4.
Substituting $n = 4$, we get $799(4) = 1633 + 191m$ or $191m = 1563$, and since $\frac{1563}{191}$ is not an integer, then $n \neq 4$.
Substituting $n = 3$, we get $799(3) = 1633 + 191m$ or $191m = 764$, and so $m = 4$.
When $m = 4$ and $n = 3$, the five-digit integer $nmnm7$ is $34\,347$ and $S = 3111(4) + 1311(3) + 17\,970$ or $S = 34\,347$, and so the only pair of distinct, non-zero digits (m, n) is $(4, 3)$.

4. (a) Since there are 9 ways to generate each of the three integers, there are a total of $9 \cdot 9 \cdot 9 = 729$ ways to generate all three integers.
Next, we consider the different cases for which the product of the three integers is a prime number.
The integers from 1 to 9 include the composite numbers 4, 6, 8, and 9.
If any one of the three integers generated is composite, then the product of the three integers is composite.
Thus, the integers must be chosen from 1, 2, 3, 5, and 7.
Of these five integers, 2, 3, 5, and 7 are prime numbers.
If two or more of the three integers generated are prime numbers, then the product of the three integers is composite.
Thus, at most one integer is a prime number.
If no integer generated is a prime number (all three are equal to 1), then the product is 1, which is not prime.
Thus, Amarpreet's product is a prime number exactly when one of the integers is 2, 3, 5, or 7, and each of the other two integers is equal to 1.
There are 4 ways to choose one of the prime numbers, p, and 3 ways to arrange the integers 1, 1, p, and so there are $4 \cdot 3 = 12$ ways to generate three integers whose product is a prime number.
Therefore, the probability that the product is a prime number is $\frac{12}{729} = \frac{4}{243}$.

(b) *Solution 1*
We begin by considering the different cases for which the product of the four integers is divisible by 5, but not divisible by 7.
The only integer from 1 to 9 that is divisible by 5 is 5, and so the product of the four integers is divisible by 5 exactly when at least one of the four integers is a 5.
Similarly, the only integer from 1 to 9 that is divisible by 7 is 7, and so the product of the four integers is not divisible by 7 exactly when a 7 is not among the four integers

generated.
We proceed to count the number of ways to generate such arrangements of four integers by considering the number of times that 5 appears in the arrangement.

<u>Case 1: there are four 5s</u>
Since each of the four integers must be 5, there is 1 such possibility in this case.

<u>Case 2: there are exactly three 5s</u>
The three 5s can be arranged in 4 different ways: 555_, 55_5, 5_55, _555.
There are 7 choices for the integer that is not a 5 since it can be any of the nine integers, except 5 and 7.
Thus, there are $4 \cdot 7 = 28$ possibilities in this case.

<u>Case 3: there are exactly two 5s</u>
The two 5s can be arranged in 6 different ways: 55__, 5_5_, 5__5, _55_, _5_5, __55.
Once the two 5s have been placed, there are 7 choices for each of the remaining two integers (since it is possible that these integers are equal to one another).
Thus, there are $6 \cdot 7 \cdot 7 = 294$ possibilities in this case.

<u>Case 4: there is exactly one 5</u>
The 5 can be placed in 4 different ways.
Once the 5 has been placed, there are 7 choices for each of the remaining three integers (since it is possible that these integers are equal to one another).
Thus, there are $4 \cdot 7 \cdot 7 \cdot 7 = 1372$ possibilities in this case.

In total, there are $1 + 28 + 294 + 1372 = 1695$ ways to select the four integers.
Since there are 9 ways to generate each of the four integers, there are a total of $9^4 = 6561$ ways to generate all four integers.
Therefore, the probability that Braxton's product is divisible by 5, but not divisible by 7 is $\dfrac{1695}{6561} = \dfrac{565}{2187}$.

Solution 2
Let A be the event that the product generated by Braxton is divisible by 5, and $\overline{A}$ be the event that the product is *not* divisible by 5.
Let B be the event that the product generated by Braxton is divisible by 7, and $\overline{B}$ be the event that the product is *not* divisible by 7.
We are asked to find the probability of A and $\overline{B}$, which we write as $P(A \text{ and } \overline{B})$.
Consider the Venn diagram shown in Figure 1.
The shaded region of this diagram represents $P(A \text{ and } \overline{B})$.

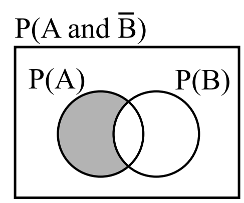


Figure 1

We wish to determine a more efficient method for determining $P(A \text{ and } \overline{B})$ than that which was shown in Solution 1.
To do so, we proceed to use Venn diagrams to help express $P(A \text{ and } \overline{B})$ in an equivalent form.

Consider the next two Venn diagrams shown below.
In Figure 2, the shaded region represents $P(\overline{B})$.
In Figure 3, the shaded region represents $P(\overline{A} \text{ and } \overline{B})$.

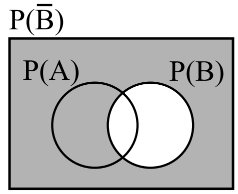


Figure 2

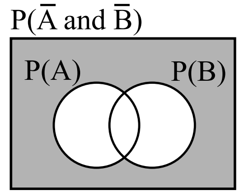


Figure 3

Notice that if the region that is shaded in Figure 3 is removed (unshaded) in Figure 2, then the resulting Venn diagram is equivalent to that in Figure 1.
That is, mathematically, $P(A \text{ and } \overline{B}) = P(\overline{B}) - P(\overline{A} \text{ and } \overline{B})$.

(Without the use of the Venn diagrams, we may have similarly noticed that since exactly one of A and $\overline{A}$ occurs, then $P(\overline{B}) = P(A \text{ and } \overline{B}) + P(\overline{A} \text{ and } \overline{B})$, and so $P(\overline{B}) - P(\overline{A} \text{ and } \overline{B}) = P(A \text{ and } \overline{B}) + P(\overline{A} \text{ and } \overline{B}) - P(\overline{A} \text{ and } \overline{B}) = P(A \text{ and } \overline{B})$.)

Thus, the probability that the product is divisible by 5, but not divisible by 7, is equal to the probability that the product is not divisible by 7, minus the probability that the product is both not divisible by 5 and not divisible by 7.
The only integer from 1 to 9 that is divisible by 7 is 7, and so the product of the four integers is not divisible by 7 exactly when a 7 is not among the four integers generated.
The probability that the integer generated is not 7 is $\frac{8}{9}$, and so the probability that the product of the four integers is not divisible by 7 is $P(\overline{B}) = \left(\frac{8}{9}\right)^4$.

The only integer from 1 to 9 that is divisible by 5 is 5, and so the product of the four integers is not divisible by 5 exactly when a 5 is not among the four integers generated.
The probability that the integer generated is not divisible by both 5 and 7 is thus $\frac{7}{9}$, and so the probability that the product of the four integers is not divisible 5 and not divisible by 7 is $P(\overline{A} \text{ and } \overline{B}) = \left(\frac{7}{9}\right)^4$.

Therefore, the probability that the product is divisible by 5, but not divisible by 7, is
$\left(\frac{8}{9}\right)^4 - \left(\frac{7}{9}\right)^4 = \frac{8^4 - 7^4}{9^4} = \frac{4096 - 2401}{6561} = \frac{1695}{6561} = \frac{565}{2187}$.

Solution 3
Suppose that the four numbers generated are a, b, c, d.
When there are no restrictions, there are 9 choices for each of a, b, c, d, and so a total of 9^4 possible combinations of 4 digits that can be generated.
If $abcd$ is not divisible by 7, then none of a, b, c, d can be 7; thus, there are 8 possibilities for each of a, b, c, d, and so 8^4 of these 9^4 combinations give products that are not divisible by 7.
If $abcd$ is not divisible by 7 and not divisible by 5, then none of a, b, c, d can be 5; thus, there are 7 possibilities for each of a, b, c, d, and so 7^4 of these 8^4 combinations whose products are not divisible by 7 give products that are also not divisible by 5.
This means that there are $8^4 - 7^4$ combinations whose product is not divisible by 7 but

whose product is divisible by 5.
Therefore, the probability that a random combination gives a product that is not divisible by 7 but is divisible by 5 is $\dfrac{8^4-7^4}{9^4}=\dfrac{4096-2401}{6561}=\dfrac{1695}{6561}=\dfrac{565}{2187}$.

(c) If any one of the 2023 integers generated is a 6, then the product is divisible by 6.
Thus, we exclude 6 from the choice of possible integers.
Further, an integer is divisible by 6 exactly when it is divisible by both 2 and 3.
With 6 excluded, the remaining integers that are divisible by 2 are 2, 4 and 8, and the remaining integers that are divisible by 3 are 3 and 9.
Therefore, the product is also divisible by 6 when at least one of the integers generated is 2 or 4 or 8, and at least one of the integers generated is 3 or 9.
To summarize, the product is not divisible by 6 exactly when

- 6 is not among the integers generated, and
- there are integers from at most one of the two lists 2, 4, 8, and 3, 9.

Thus, there are exactly 3 cases which produce a product that is not divisible by 6, as follows.

Case	Is there a 6?	Is there a 2, 4 or 8?	Is there a 3 or 9?
1	no	no	no
2	no	no	yes
3	no	yes	no

For each of these 3 cases, we count the number of ways that the product generated is not divisible by 6.
Case 1: there is no 6, 2, 4, 8, 3, or 9
In this case, there are 3 integers that may be chosen (1, 5 and 7), and so there are 3^{2023} ways to generate this product.

Case 2: there is no 6, 2, 4, or 8
In this case, there are 5 integers that may be chosen (1, 3, 5, 7, and 9), and so there are 5^{2023} ways to generate this product.

However, this count includes the cases in which only 1, 5 and 7 are chosen (3 and 9 are not), which were counted in Case 1.
Thus, the number of ways to generate the products in Case 2, different from those in Case 1, is $5^{2023}-3^{2023}$.

Case 3: there is no 6, 3, or 9
In this case, there are 6 integers that may be chosen (1, 2, 4, 5, 7, and 8), and there are 6^{2023} ways to generate this product.

However, this count includes the cases in which only 1, 5 and 7 are chosen (2, 4 and 8 are not), which were counted in Case 1.
Thus, the number of ways to generate the products in Case 3, different from those in Case 1, is $6^{2023}-3^{2023}$.

Thus, the total number of ways to generate all products that are not divisible by 6 is

$$3^{2023}+5^{2023}-3^{2023}+6^{2023}-3^{2023}=5^{2023}+6^{2023}-3^{2023}$$

When there are no restrictions, there are 9 choices for each integer generated, and so there is a total of 9^{2023} possible ways to generate the product.

Thus, the probability, p, that the product is not divisible by 6 is

$$p = \frac{5^{2023} + 6^{2023} - 3^{2023}}{9^{2023}}$$

and so $p \cdot 9^{2023} = 5^{2023} + 6^{2023} - 3^{2023}$, which is equal to the sum and difference of integers and thus an integer.
To determine the ones digit of the integer equal to $p \cdot 9^{2023}$, we determine the ones digit of the integer equal to $5^{2023} + 6^{2023} - 3^{2023}$.
The ones digit of 5^a is equal to 5 for all positive integers a, and thus the ones digit of 5^{2023} is 5.
The ones digit of 6^b is equal to 6 for all positive integers b, and thus the ones digit of 6^{2023} is 6.
Consider the ones digit of successive integer powers of 3 beginning at 3^1:

- 3^1 has ones digit 3
- 3^2 has ones digit 9
- 3^3 has ones digit 7
- 3^4 has ones digit 1
- 3^5 has ones digit 3

Since 3^1 and 3^5 have the same ones digit and we are performing the same action to get from one step to the next, the results begin to repeat. Thus the block of 4 ones digits (3, 9, 7, 1) must form a cycle.
Since $2023 = 4 \cdot 505 + 3$, then the ones digit of 3^{2023} is the third number in the repeating block, which is 7.
Finally, the ones digit of the integer equal to $p \cdot 9^{2023}$ is the ones digit of $5 + 6 - 7$, which is equal to 4.

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2022 Hypatia Contest

Tuesday, April 12, 2022
(in North America and South America)

Wednesday, April 13, 2022
(outside of North America and South America)

Solutions

1. (a) Regular hexagon $ABCDEF$ has side length $2x$, and so $AB = 2x$.
Since $\triangle OAB$ is equilateral, then $OA = OB = AB = 2x$.
The radius of the circle is equal to OA and thus is $2x$.

(b) Since M is the midpoint of AB and $OA = OB$, then OM is perpendicular to AB.
Since M is the midpoint of AB, then $AM = \frac{1}{2}AB = x$.
Using the Pythagorean Theorem in right-angled $\triangle OAM$, we get $OA^2 = OM^2 + AM^2$ or $(2x)^2 = OM^2 + x^2$, and so $OM^2 = 3x^2$ or $OM = \sqrt{3}x$ (since $OM > 0$).
Alternatively, notice that $\triangle OAM$ is a 30°-60°-90° triangle, and so $AM : OA : OM = 1 : 2 : \sqrt{3} = x : 2x : \sqrt{3}x$.

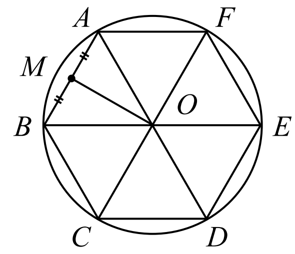

Figure 2

(c) The diagonals AD, BE and CF divide $ABCDEF$ into six congruent equilateral triangles. Thus the area of $ABCDEF$ is six times the area of $\triangle OAB$ (having base AB and height OM), or

$$6 \times \tfrac{1}{2} \times AB \times OM = 3 \times 2x \times \sqrt{3}x = 6\sqrt{3}x^2$$

(d) The area of the shaded region is determined by subtracting the area of $ABCDEF$ from the area of the circle with centre O and radius $2x$.
Thus, the area of the shaded region is

$$\pi(2x)^2 - 6\sqrt{3}x^2 = 4\pi x^2 - 6\sqrt{3}x^2 = (4\pi - 6\sqrt{3})x^2$$

The area of this shaded region is 123, and so $(4\pi - 6\sqrt{3})x^2 = 123$ or $x^2 = \dfrac{123}{4\pi - 6\sqrt{3}}$.

Since $x > 0$, we get $x = \sqrt{\dfrac{123}{4\pi - 6\sqrt{3}}}$ and so $x = 7.5$ when rounded to the nearest tenth.

2. (a) With 1 kg of muffin batter, exactly 24 mini muffins and 2 large muffins can be made. Thus with 2 kg of muffin batter, exactly $2 \times 24 = 48$ mini muffins and $2 \times 2 = 4$ large muffins can be made, and so $n = 4$.

(b) *Solution 1*
With 2 kg of muffin batter, exactly 36 mini muffins and 6 large muffins can be made. With 2 kg of muffin batter, exactly 48 mini muffins and 4 large muffins can be made. Adding these, we get that with 2 kg + 2 kg = 4 kg of muffin batter, exactly $36 + 48 = 84$ mini muffins and $6 + 4 = 10$ large muffins can be made, and so $x = 4$.

Solution 2
Let m be the number of kilograms of muffin batter needed to make 1 mini muffin. Let ℓ be the number of kilograms of muffin batter needed to make 1 large muffin. Since 1 kg of muffin batter makes exactly 24 mini muffins and 2 large muffins, then $1 = 24m + 2\ell$.
Since 2 kg of muffin batter makes exactly 36 mini muffins and 6 large muffins, then $2 = 36m + 6\ell$.
Subtracting the second equation from 3 times the first equation, we get $3 \times 1 - 2 = 3 \times (24m + 2\ell) - (36m + 6\ell)$ or $1 = 36m$, and so $m = \dfrac{1}{36}$.

Substituting $m = \dfrac{1}{36}$ into the second equation, we get $2 = 36\left(\dfrac{1}{36}\right) + 6\ell$ or $1 = 6\ell$, and so $\ell = \dfrac{1}{6}$.

Since $\dfrac{1}{36}$ kg of muffin batter is needed to make 1 mini muffin, then $84 \cdot \dfrac{1}{36} = \dfrac{7}{3}$ kg of muffin batter is needed to make 84 mini muffins.

Since $\dfrac{1}{6}$ kg of muffin batter is needed to make 1 large muffin, then $10 \cdot \dfrac{1}{6} = \dfrac{5}{3}$ kg of muffin batter is needed to make 10 large muffins.

Thus, exactly 84 mini muffins and 10 large muffins can be made with $\dfrac{7}{3} + \dfrac{5}{3} = \dfrac{12}{3} = 4$ kg of muffin batter, and so $x = 4$.

(c) In part (b) *Solution 2*, it was determined that $\frac{1}{36}$ kg of muffin batter is needed to make 1 mini muffin, and $\frac{1}{6}$ kg of muffin batter is needed to make 1 large muffin.
Therefore, the number of kilograms of muffin batter needed to make 1 large muffin is 6 times the number of kilograms of batter needed to make 1 mini muffin $\left(6 \times \frac{1}{36} = \frac{1}{6}\right)$.
In other words, the batter needed to make 1 large muffin can make 6 mini muffins.
Then, using the amount of batter that makes 7 large muffins, $6 \times 7 = 42$ mini muffins can be made.

3. (a) If the first number in a sequence is 3 and the sequence is generated by the function $x^2 - 3x + 1$, then the second number in the sequence is

$$3^2 - 3(3) + 1 = 1,$$

and the third number in the sequence is

$$1^2 - 3(1) + 1 = -1,$$

and the fourth number in the sequence is

$$(-1)^2 - 3(-1) + 1 = 5.$$

The first four numbers in the sequence are $3, 1, -1, 5$.

(b) Let the first and second numbers in the sequence generated by the function $x^2 - 4x + 7$ be f and s, respectively.
Then, the first three numbers in the sequence are $f, s, 7$.
Since the third number in the sequence is 7, then $s^2 - 4s + 7 = 7$.
Solving this equation, we get $s^2 - 4s = 0$ or $s(s - 4) = 0$, which has solutions $s = 0$ and $s = 4$, and so the first three numbers in the sequence could be $f, 0, 7$ or $f, 4, 7$.
If the second number in the sequence is 0, then $f^2 - 4f + 7 = 0$.
The discriminant of this equation is $(-4)^2 - 4(1)(7) = -12$ (less than zero) and so there are no real solutions.
Thus, there is no first number in this sequence for which the second number is 0.

If the second number in the sequence is 4, then $f^2 - 4f + 7 = 4$, or $f^2 - 4f + 3 = 0$ and so $(f - 1)(f - 3) = 0$, which has solutions $f = 1$ and $f = 3$.
Therefore, if 7 is the third number in a sequence generated by the function $x^2 - 4x + 7$, then the first three numbers in the sequence could be $1, 4, 7$ or $3, 4, 7$, and so the possible first numbers in the sequence are 1 and 3.

(c) The first two numbers in the sequence are c, c, and so $c^2 - 7c - 48 = c$.
Solving this equation, we get $c^2 - 8c - 48 = 0$ or $(c + 4)(c - 12) = 0$, which has solutions $c = -4$ and $c = 12$.

(d) The first number in the sequence is a and the second number is b, and so

$$a^2 - 12a + 39 = b.$$

The second number in the sequence is b and the third number is a, and so

$$b^2 - 12b + 39 = a.$$

Subtracting the second equation from the first and simplifying, we get

$$\begin{aligned}
(a^2 - 12a + 39) - (b^2 - 12b + 39) &= b - a \\
a^2 - b^2 - 12a + 12b &= b - a \\
a^2 - b^2 - 11a + 11b &= 0 \\
(a-b)(a+b) - 11(a-b) &= 0 \\
(a-b)(a+b-11) &= 0
\end{aligned}$$

Since $a \neq b$, then $a - b \neq 0$ and so $a + b - 11 = 0$ or $b = 11 - a$.
Substituting into the first equation, we get $a^2 - 12a + 39 = 11 - a$ or $a^2 - 11a + 28 = 0$.
Factoring gives $(a-4)(a-7) = 0$ and so the possible values of a are 4 and 7.
(Note that the two possible sequences are $4, 7, 4, \ldots$ and $7, 4, 7, \ldots$.)

4. (a) Written as a product of its prime factors, $240 = 2^4 3^1 5^1$ and so

$$f(240) = (1+4)(1+1)(1+1) = (5)(2)(2) = 20$$

(b) Suppose $f(N) = 6$ and N is refactorable. Then 6 divides N.
N is divisible by 6 exactly when its prime factors include at least one 2 and at least one 3 (since $6 = 2 \times 3$).
Assume that N contains an additional prime factor p, distinct from 2 and 3.
In this case, the divisors of N are $1, 2, 3, 6, p, 2p, 3p$, and $6p$, which is too many divisors.
Thus, it must be that N is a positive integer of the form $N = 2^u 3^v$, where u and v are positive integers, and so $f(N) = (1+u)(1+v) = 6$.
Since u and v are positive integers, then $1 + u \geq 2$ and $1 + v \geq 2$ and so there are exactly two possibilities: $u = 1$ and $v = 2$ or $u = 2$ and $v = 1$.
When $u = 1$ and $v = 2$, $N = 2^1 3^2 = 18$ and when $u = 2$ and $v = 1$, $N = 2^2 3^1 = 12$.
The refactorable numbers N with $f(N) = 6$ are 12 and 18.

(c) Since N is refactorable, then $f(N) = 256 = 2^8$ divides N.
Thus for some integer $w \geq 8$, N is of the form $N = 2^w p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $2 < p_1 < p_2 < \cdots < p_k$ are prime numbers, and $a_1, a_2, \cdots, a_k$ are positive integers.
In this case, $f(N) = (1+w)(1+a_1)(1+a_2)\cdots(1+a_k) = 2^8$, and so each of the factors $1+w, 1+a_1, 1+a_2, \cdots, 1+a_k$ is a power of 2 (since their product is 2^8).
This means that each of the exponents $w, a_1, a_2, \cdots, a_k$ is one less than a power of two.
Since $w \geq 8$, then $1 + w \geq 9$ and so $1 + w \geq 2^4$ (the smallest power of 2 greater than 8).
Since $f(N) = (1+w)(1+a_1)(1+a_2)\cdots(1+a_k) = 2^8$, then $2^4 \leq 1 + w \leq 2^8$ and $1 \leq (1+a_1)(1+a_2)\cdots(1+a_k) \leq 2^4$, and so w must be equal to $15, 31, 63, 127$, or 255, and each of $a_1, a_2, \cdots, a_k$ must be equal to $1, 3, 7$, or 15.
For example, if $1 + w = 2^8$, then $w = 256 - 1 = 255$ and $N = 2^{255}$.
If $1 + w = 2^4$, then $w = 15$ and N is of the form $N = 2^{15} p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$.
Thus, the smallest N having the greatest number of distinct prime factors is $N = 2^{15} 3^1 5^1 7^1 11^1$, and we may confirm that $f(N) = (1+15)(1+1)(1+1)(1+1)(1+1) = 2^4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^8$,

as required.
Comparing these first two values of N, we recognize that $2^{15}3^1 5^1 7^1 11^1$ is significantly less than 2^{255}.
Further, we recognize that

- all remaining possible values of N have exactly 2, 3 or 4 distinct prime factors,
- where exponents are equal, smaller prime factors give smaller values of N,
- the greatest exponents must occur on the smallest prime factors, and
- we recall that each exponent is one less than a power of 2.

In the table below, we use the above information to determine the smallest possible values of N having 1, 2, 3, 4, and 5 distinct prime factors.
Further, we compare the size of each of these values of N within each of the 5 groups.

Number of distinct prime factors of N	Values of N
1	2^{255}
2	$2^{15}3^{15} < 2^{31}3^7 < 2^{63}3^3 < 2^{127}3^1$
3	$2^{15}3^3 5^3 < 2^{15}3^7 5^1 < 2^{31}3^3 5^1 < 2^{63}3^1 5^1$
4	$2^{15}3^3 5^1 7^1 < 2^{31}3^1 5^1 7^1$
5	$2^{15}3^1 5^1 7^1 11^1$

Finally, we compare the smallest values of N from each of the 5 rows in the table above.
Since $2^{15}3^3 5^1 7^1 < 2^{15}3^1 5^1 7^1 11^1 < 2^{15}3^3 5^3 < 2^{15}3^{15} < 2^{255}$, then the smallest refactorable number N with $f(N) = 256$, is $N = 2^{15}3^3 5^1 7^1 = 30\,965\,760$.

(d) Let m be a positive integer of the form $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ are prime numbers, and $a_1, a_2, \cdots, a_k$ are positive integers.
We begin by stating a value of N and then proceed to show that it works.

For each $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, choose $N = p_1^{\left(p_1^{a_1}-1\right)} p_2^{\left(p_2^{a_2}-1\right)} \cdots p_k^{\left(p_k^{a_k}-1\right)}$.
On the contest paper, the *useful fact* states that $2^n \geq n + 1$ for all positive integers n.
Since $p \geq 2$ for all prime numbers p, then $p^n \geq 2^n \geq n + 1$.
Since $p^n \geq n + 1$, then $p^n - 1 \geq n$, and so $p_i^{a_i} - 1 \geq a_i$ for all integers i where $1 \leq i \leq k$.
Thus, $p_1^{a_1}$ divides $p_1^{\left(p_1^{a_1}-1\right)}$ since each is a power with base p_1 and $p_1^{a_1} - 1 \geq a_1$.
Similarly, $p_2^{a_2}$ divides $p_2^{\left(p_2^{a_2}-1\right)}$, and so on.
Therefore, $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ divides $N = p_1^{\left(p_1^{a_1}-1\right)} p_2^{\left(p_2^{a_2}-1\right)} \cdots p_k^{\left(p_k^{a_k}-1\right)}$.
Further, $f(N) = \left(1 + p_1^{a_1} - 1\right)\left(1 + p_2^{a_2} - 1\right) \cdots \left(1 + p_k^{a_k} - 1\right) = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} = m$.
Thus for every integer $m > 1$, there exists a refactorable number N such that $f(N) = m$.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Hypatia Contest

April 2021
(in North America and South America)

April 2021
(outside of North America and South America)

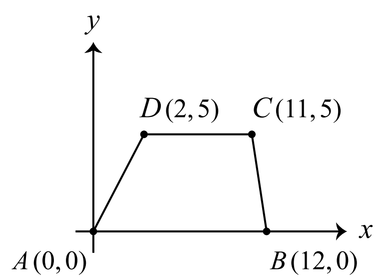
Solutions

1. (a) The total cost to rent a car is \$180.00.
If 4 people rent a car, the cost per person is $\frac{\$180.00}{4} = \$45.00$.

(b) Since the members of the group equally share the total cost to rent the vehicle, the smaller the group, the greater the cost per person.
To rent an SUV, the smallest group size required is 5 passengers and the total cost is \$200.00.
Thus, the maximum possible cost per person to rent an SUV is $\frac{\$200.00}{5} = \$40.00$.

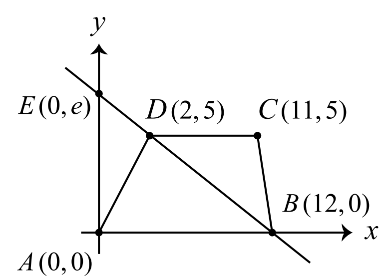
(c) Let the total cost to rent a van be v.
When renting a van, the maximum possible cost per person occurs when the number of passengers is 9 (the fewest number possible), and so this maximum cost is $\frac{v}{9}$.
The minimum possible cost per person occurs when the number of passengers is 12 (the greatest number possible), and so this minimum cost is $\frac{v}{12}$.
Then, $\frac{v}{9} - \frac{v}{12} = \$6.00$ or $\frac{4v-3v}{36} = \$6.00$, and so $v = \$6.00 \times 36$.
Thus, the total cost to rent a van is \$216.00.

2. (a) Trapezoid $ABCD$ is drawn, as shown.
The slope of line segments AB and CD are each zero and thus they are parallel.
The length of AB is the difference between the x-coordinates of A and B, or 12.
The length of CD is the difference between the x-coordinates of C and D, or $11 - 2 = 9$.



The height of the trapezoid is equal to the vertical distance between AB and CD, which is 5.
The area of trapezoid $ABCD$ is $\frac{5}{2}(AB + CD)$ or $\frac{5}{2}(21) = \frac{105}{2}$.

(b) The line passing through B and D intersects the y-axis at E. Let the coordinates of E be $(0, e)$, as shown.
The slope of the line through B and D is $\frac{5-0}{2-12} = -\frac{1}{2}$.



Solution 1
Since E, D and B lie on the same line, then the slope of ED is equal to the slope of BD.
Equating slopes, we get $\frac{e-5}{0-2} = -\frac{1}{2}$ or $\frac{e-5}{2} = \frac{1}{2}$, and so $e - 5 = 1$ or $e = 6$.
Thus, point E has coordinates $(0, 6)$.

Solution 2
The line passing through $B(12, 0)$ and $D(2, 5)$ has slope $-\frac{1}{2}$, and thus has equation $y - 5 = -\frac{1}{2}(x - 2)$.
Rearranging, we get $y - 5 = -\frac{1}{2}x + 1$ or $y = -\frac{1}{2}x + 6$.
Since this line has y-intercept 6, then point E has coordinates $(0, 6)$.

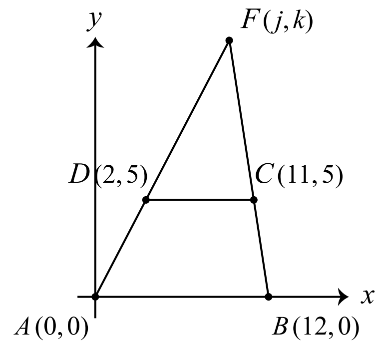
Solution 3

The line passing through $B(12,0)$ and $D(2,5)$ has slope $-\frac{1}{2}$, and thus has equation $y-5=-\frac{1}{2}(x-2)$.

This line passes through $E(0,e)$, and so $e-5=-\frac{1}{2}(0-2)$ or $e=1+5=6$.

Thus, point E has coordinates $(0,6)$.

(c) Sides AD and BC are extended to intersect at F, as shown.



Solution 1

Let the coordinates of F be (j,k).
Since A, D and F lie on the same line, then the slope of AD is equal to the slope of AF.
Equating slopes, we get $\frac{5}{2}=\frac{k}{j}$ or $k=\frac{5}{2}j$.
Since B, C and F lie on the same line, then the slope of BC is equal to the slope of BF.
Equating slopes, we get $\frac{5-0}{11-12}=\frac{k-0}{j-12}$ or $-5=\frac{k}{j-12}$, and so $k=-5(j-12)$.
Substituting $k=\frac{5}{2}j$, we get $\frac{5}{2}j=-5(j-12)$ or $j=-2(j-12)$, and so $3j=24$ or $j=8$.
When $j=8$, $k=\frac{5}{2}(8)=20$, and so F has coordinates $(8,20)$.

Solution 2

The line passing through $A(0,0)$ and $D(2,5)$ has slope $\frac{5}{2}$ and y-intercept 0, and thus has equation $y=\frac{5}{2}x$.
The line passing through $B(12,0)$ and $C(11,5)$ has slope -5 and thus has equation $y=-5(x-12)$.
These two lines intersect at F, and so the coordinates of F can be determined by solving the equation $\frac{5}{2}x=-5(x-12)$. Solving, we get $x=-2(x-12)$ or $3x=24$, and so $x=8$.
When $x=8$, $y=\frac{5}{2}(8)=20$, and so F has coordinates $(8,20)$.

(d) Let P have coordinates (r,s).
Assume AB is the base of $\triangle PAB$.
In this case, if the height of $\triangle PAB$ is h, then the area of $\triangle PAB$ is $\frac{1}{2}(AB)h=6h$.
The area of $\triangle PAB$ is 42, and so $6h=42$ or $h=7$.
That is, $P(r,s)$ is located a vertical distance of 7 units from the line through A and B, or 7 units from the x-axis.
There are two possibilities: $P(r,s)$ is located 7 units above the x-axis, and thus lies on the horizontal line $y=7$, or $P(r,s)$ is located 7 units below the x-axis, and thus lies on the horizontal line $y=-7$.
In the first case, P has coordinates $(r,7)$ and in the second case, P has coordinates $(r,-7)$.
Recall that P lies on the line passing through B and D.
The line passing through $B(12,0)$ and $D(2,5)$ has slope $-\frac{1}{2}$, and thus has

equation $y - 5 = -\dfrac{1}{2}(x-2)$.
If $P(r,7)$ lies on this line, then $7 - 5 = -\dfrac{1}{2}(r-2)$ or $-4 = r - 2$, and so $r = -2$.
Similarly, if $P(r,-7)$ lies on this line, then $-7 - 5 = -\dfrac{1}{2}(r-2)$ or $24 = r - 2$, and so in this case, $r = 26$.
The points P that lie on the line passing through B and D, so that the area of $\triangle PAB$ is 42, are $(-2,7)$ and $(26,-7)$.

3. (a) Since $a_n = 2^n$ for $n \geq 1$, then $a_5 = 2^5 = 32$.
Since $b_2 = 1, b_3 = 3$, and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 3$, then $b_4 = b_3 + 2b_2 = 3 + 2(1) = 5$ and $b_5 = b_4 + 2b_3 = 5 + 2(3) = 11$.
Therefore, $a_5 = 32$ and $b_5 = 11$.

(b) Since $b_1 = p \cdot (a_1) + q \cdot (-1)^1$ and $a_1 = 2$, then $b_1 = 2p - q$.
From the definition of sequence B, we know $b_1 = 1$, and so $2p - q = 1$.
Since $b_2 = p \cdot (a_2) + q \cdot (-1)^2$ and $a_2 = 2^2 = 4$, then $b_2 = 4p + q$.
From the definition of sequence B, we know $b_2 = 1$, and so $4p + q = 1$.
This gives two equations in two unknowns, p and q.
Adding these two equations, we get $6p = 2$, and so $p = \frac{1}{3}$.
Substituting, we get $2\left(\frac{1}{3}\right) - q = 1$ or $q = \frac{2}{3} - 1 = -\frac{1}{3}$.
Thus, the real numbers p and q for which $b_n = p \cdot (a_n) + q \cdot (-1)^n$ for all $n \geq 1$ are $p = \frac{1}{3}$ and $q = -\frac{1}{3}$.

(c) Using algebraic manipulation, and the definitions $a_n = 2^n$ and $b_n = \frac{1}{3}(a_n) - \frac{1}{3}(-1)^n$, each for $n \geq 1$, we obtain the following equivalent equations,

$$\begin{aligned}
S_n &= b_1 + b_2 + b_3 + \cdots + b_n \\
&= \left(\tfrac{1}{3}(a_1) - \tfrac{1}{3}(-1)\right) + \left(\tfrac{1}{3}(a_2) - \tfrac{1}{3}(-1)^2\right) + \left(\tfrac{1}{3}(a_3) - \tfrac{1}{3}(-1)^3\right) + \cdots + \left(\tfrac{1}{3}(a_n) - \tfrac{1}{3}(-1)^n\right) \\
&= \tfrac{1}{3}(a_1 + a_2 + a_3 + \cdots + a_n) - \tfrac{1}{3}((-1) + (-1)^2 + (-1)^3 + \cdots + (-1)^n) \\
&= \tfrac{1}{3}(2 + 2^2 + 2^3 + \cdots + 2^n) - \tfrac{1}{3}(-1 + 1 - 1 + \cdots + (-1)^n)
\end{aligned}$$

Next, we consider each of the two expressions within parentheses, separately.
The expression $2 + 2^2 + 2^3 + \cdots + 2^n$ is the sum of n terms of a geometric sequence with first term $a = 2$ and common ratio $r = 2$.
Thus, $2 + 2^2 + 2^3 + \cdots + 2^n = 2\left(\dfrac{1-2^n}{1-2}\right) = -2(1-2^n)$.
The expression $-1 + 1 - 1 + \cdots + (-1)^n$ is an alternating sum of the terms -1 and 1.
This simplifies to 0 if there are an even number of terms, that is, if n is even, and simplifies to -1 if n is odd.
Summarizing, we have

$$S_n = \begin{cases} \frac{1}{3}(-2(1-2^n)) \text{ , if } n \text{ is even} \\ \frac{1}{3}(-2(1-2^n)) + \frac{1}{3} \text{ , if } n \text{ is odd} \end{cases}$$

and simplifying, we get

$$S_n = \begin{cases} \frac{2}{3}(2^n - 1) \text{ , if } n \text{ is even} \\ \frac{2}{3}(2^n - 1) + \frac{1}{3} \text{ , if } n \text{ is odd} \end{cases}$$

We want the smallest positive integer n that satisfies $S_n \geq 16^{2021}$ and note that the value of S_n increases as n increases.
Since $16 = 2^4$, then $16^{2021} = \left(2^4\right)^{2021} = 2^{8084}$ and so we want the smallest positive integer n that satisfies $S_n \geq 2^{8084}$.

When n is even, we get

$$\begin{aligned} \tfrac{2}{3}(2^n - 1) &\geq 2^{8084} \\ 2^n - 1 &\geq 3 \cdot 2^{8083} \\ 2^n &\geq 3 \cdot 2^{8083} + 1 \end{aligned}$$

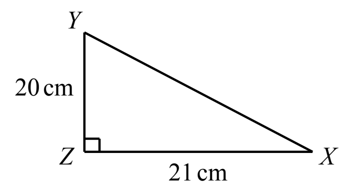
When n is odd, we get

$$\begin{aligned} \tfrac{2}{3}(2^n - 1) + \tfrac{1}{3} &\geq 2^{8084} \\ \tfrac{2}{3}(2^n - 1) &\geq 2^{8084} - \tfrac{1}{3} \\ 2^n - 1 &\geq 3 \cdot 2^{8083} - \tfrac{1}{2} \\ 2^n &\geq 3 \cdot 2^{8083} + \tfrac{1}{2} \end{aligned}$$

Since 2^n is an even integer, $3 \cdot 2^{8083} + 1$ is an odd integer and $3 \cdot 2^{8083} + \frac{1}{2}$ is between an even integer and an odd integer, and thus the inequalities $2^n \geq 3 \cdot 2^{8083} + 1$ and $2^n \geq 3 \cdot 2^{8083} + \frac{1}{2}$ are both equivalent to saying $2^n > 3 \cdot 2^{8083}$.

Since $3 \cdot 2^{8083} > 2 \cdot 2^{8083}$, then simplifying, we get $3 \cdot 2^{8083} > 2^{8084}$.
Thus, we want the smallest positive integer n that satisfies $2^n > 3 \cdot 2^{8083} > 2^{8084}$.
When $n \leq 8084$, this inequality is not true.
When $n = 8085$, we get $2^{8085} = 2^2 \cdot 2^{8083} = 4 \cdot 2^{8083}$ which is greater than $3 \cdot 2^{8083}$, as required.
Thus, the smallest positive integer n that satisfies $S_n \geq 16^{2021}$ is $n = 8085$.

4. (a) In $\triangle XYZ$, $x = 20$, $y = 21$, and $\angle XZY = 90^\circ$, as shown. Using the Pythagorean Theorem, we get $z = \sqrt{20^2 + 21^2} = 29$ (since $z > 0$).
The value of A is



$$A = \frac{1}{2}(y)(x) = \frac{1}{2}(21)(20) = 210$$

The value of P is

$$P = z + x + y = 29 + 20 + 21 = 70$$

(b) When $A = 336$, we get $\frac{1}{2}xy = 336$, and so $xy = 672$.
By the Pythagorean Theorem, $x^2 + y^2 = 50^2$, which when manipulated algebraically gives the following equivalent equations

$$\begin{aligned} x^2 + y^2 &= 2500 \\ (x+y)^2 - 2xy &= 2500 \\ (x+y)^2 &= 2500 + 2xy \\ (x+y)^2 &= 2500 + 2(672) \\ (x+y)^2 &= 3844 \end{aligned}$$

Since $x + y > 0$, then $x + y = \sqrt{3844} = 62$.
Thus, we get $P = x + y + z = 62 + 50 = 112$.
(The triangle satisfying these conditions has side lengths 14 cm, 48 cm and 50 cm.)

(c) Since $A = 3P$, we get $\frac{1}{2}xy = 3(x + y + z)$, and so $xy = 6(x + y + z)$.
When $xy = 6(x + y + z)$ is manipulated algebraically, we get the following equivalent equations

$$\begin{aligned}
xy &= 6(x+y+z) \\
xy - 6x - 6y &= 6z \\
(xy - 6x - 6y)^2 &= (6z)^2 \\
(xy)^2 - 12x(xy) - 12y(xy) + 72xy + 36x^2 + 36y^2 &= 36z^2 \\
(xy)^2 - 12x(xy) - 12y(xy) + 72xy + 36x^2 + 36y^2 &= 36(x^2+y^2) \quad (\text{since } x^2+y^2=z^2) \\
xy(xy - 12x - 12y + 72) &= 0 \\
xy - 12x - 12y + 72 &= 0 \quad (\text{since } xy \neq 0) \\
x(y-12) - 12y &= -72 \\
x(y-12) - 12y + 144 &= -72 + 144 \\
x(y-12) - 12(y-12) &= 72 \\
(x-12)(y-12) &= 72
\end{aligned}$$

Since x and y are positive integers, then $x - 12$ and $y - 12$ are a factor pair of 72.
The product $(x-12)(y-12)$ is positive (since $72 > 0$), and thus $x-12 < 0$ and $y-12 < 0$ or $x - 12 > 0$ and $y - 12 > 0$.
If $x - 12 < 0$ and $y - 12 < 0$, then $x < 12$ and $y < 12$.
There are exactly two Pythagorean triples (x, y, z) in which $x < 12$ and $y < 12$.
In the first case, $(x, y, z) = (3, 4, 5)$, which gives $A = \frac{1}{2}(3)(4) = 6$, $P = 3+4+5 = 12$, and so $A \neq 3P$.
In the second case, $(x, y, z) = (6, 8, 10)$, which gives $A = \frac{1}{2}(6)(8) = 24$, $P = 6+8+10 = 24$, and so $A \neq 3P$.
Therefore, $x - 12 > 0$ and $y - 12 > 0$, and so x and y are each greater than 12.
In the table below, we use the positive factor pairs of 72 to determine all possible integer values of x, y and z.
Further, we initially make the assumption that $x \leq y$, recognizing that by the symmetry of the equation, the values of x and y may be interchanged with one another and doing so gives the same value for z and the same triangle.

Factor pair	$x-12$	$y-12$	x	y	$z=\sqrt{x^2+y^2}$	(x,y,z)
1 and 72	1	72	13	84	85	$(13, 84, 85)$ or $(84, 13, 85)$
2 and 36	2	36	14	48	50	$(14, 48, 50)$ or $(48, 14, 50)$
3 and 24	3	24	15	36	39	$(15, 36, 39)$ or $(36, 15, 39)$
4 and 18	4	18	16	30	34	$(16, 30, 34)$ or $(30, 16, 34)$
6 and 12	6	12	18	24	30	$(18, 24, 30)$ or $(24, 18, 30)$
8 and 9	8	9	20	21	29	$(20, 21, 29)$ or $(21, 20, 29)$

It is worth noting that instead of factoring the equation $xy - 12x - 12y + 72 = 0$ as we did, we could have chosen to rewrite it as

$$\begin{aligned} x(y-12) &= 12y-72 \\ x &= \frac{12y-72}{y-12} \\ x &= \frac{12(y-12)+144-72}{y-12} \\ x &= 12+\frac{72}{y-12} \end{aligned}$$

and considered that since x is a positive integer, then $y-12$ is a divisor of 72.

(d) Since $A = kP$, we get $\frac{1}{2}xy = k(x+y+z)$, and so $xy = 2k(x+y+z)$.
When $xy = 2k(x+y+z)$ is manipulated algebraically, we get the following equivalent equations

$$\begin{aligned} xy &= 2k(x+y+z) \\ xy-2kx-2ky &= 2kz \\ (xy-2kx-2ky)^2 &= (2kz)^2 \\ (xy)^2-4kx(xy)-4ky(xy)+8k^2xy+4k^2x^2+4k^2y^2 &= 4k^2z^2 \\ (xy)^2-4kx(xy)-4ky(xy)+8k^2xy+4k^2x^2+4k^2y^2 &= 4k^2(x^2+y^2) \quad (\because x^2+y^2=z^2) \\ xy(xy-4kx-4ky+8k^2) &= 0 \\ xy-4kx-4ky+8k^2 &= 0 \quad (\text{since } xy \neq 0) \\ x(y-4k)-4ky &= -8k^2 \\ x(y-4k)-4ky+16k^2 &= -8k^2+16k^2 \\ x(y-4k)-4k(y-4k) &= 8k^2 \\ (x-4k)(y-4k) &= 8k^2 \end{aligned}$$

Since x, y and k are positive integers, then $x-4k$ and $y-4k$ are a factor pair of $8k^2$.
We begin by assuming that $k = 2$.
Substituting, we get $(x-8)(y-8) = 32$.
The product $(x-8)(y-8)$ is positive (since $32 > 0$), and thus $x-8 < 0$ and $y-8 < 0$ or $x-8 > 0$ and $y-8 > 0$.
If $x-8 < 0$ and $y-8 < 0$, then $x < 8$ and $y < 8$ which is not possible since $P = 510$, and so $x-8 > 0$ and $y-8 > 0$.
If $(x-8)(y-8) = 32$ and $x \leq y$, then $x-8$ is equal to $1, 2$ or 4, which gives $x = 9, 10, 12$ and $y-8$ is equal to $32, 16, 8$, and so $y = 40, 24, 16$, respectively.
Using the Pythagorean Theorem, we get $z = 41, 26, 20$, respectively.
For each of the three possibilities, $P = x+y+z \neq 510$ and so we conclude $k \neq 2$.
It can similarly be shown that k cannot equal 3, and thus $k \geq 5$ (since k is a prime number).
Since k is a prime number and $k \geq 5$, the positive factor pairs of $8k^2$ are

$$(1, 8k^2), (2, 4k^2), (4, 2k^2), (8, k^2), (k, 8k), (2k, 4k)$$

and the negative factor pairs of $8k^2$ are

$$(-1, -8k^2), (-2, -4k^2), (-4, -2k^2), (-8, -k^2), (-k, -8k), (-2k, -4k)$$

If for example $x - 4k = -1$ and $y - 4k = -8k^2$, then $y = 4k - 8k^2$ which is less than zero for all $k \geq 5$.
This is not possible since $y > 0$.
Assuming $x \leq y$, it can similarly be shown that when $x - 4k$ and $y - 4k$ are equal to a negative factor pair of $8k^2$, then $y \leq 0$ for all values of $k \geq 5$.
Thus, $x - 4k$ and $y - 4k$ must equal a positive factor pair of $8k^2$.

Beginning with the fact that the perimeter of the triangle is 510 cm, we get the following equivalent equations

$$\begin{aligned}
x + y + z &= 510 \\
x + y &= 510 - z \\
x^2 + 2xy + y^2 &= 510^2 - 1020z + z^2 \quad \text{(squaring both sides)} \\
2xy &= 510^2 - 1020z \quad (\text{since } x^2 + y^2 = z^2) \\
4A &= 510^2 - 1020z \quad (A = \tfrac{1}{2}xy \text{ and so } 4A = 2xy) \\
4(510k) &= 510^2 - 1020z \quad (\text{since } A = kP \text{ and } P = 510) \\
2k &= 255 - z \\
2k &= 255 - (510 - x - y) \\
x + y - 2k &= 255
\end{aligned}$$

From the first factor pair, we get $x - 4k = 1$ and $y - 4k = 8k^2$, and so $(x, y) = (1 + 4k, 8k^2 + 4k)$ (assuming $x \leq y$).
Substituting $x = 1 + 4k$ and $y = 8k^2 + 4k$ into $x + y - 2k = 255$ and simplifying, we get $8k^2 + 6k = 254$ or $k(4k + 3) = 127$, which has no solutions since 127 is a prime number.

We continue our analysis of the remaining 5 factor pairs in the table below.
As before, we make the assumption that $x \leq y$, recognizing that the values of x and y may be interchanged with one another and doing so gives the same value(s) for k.

Factor pair	x	y	$x + y - 2k = 255$ simplified	Value(s) of k
$2, 4k^2$	$2 + 4k$	$4k^2 + 4k$	$4k^2 + 6k = 253$	No k (LS is even and the RS is odd)
$4, 2k^2$	$4 + 4k$	$2k^2 + 4k$	$2k^2 + 6k = 251$	No k (LS is even and the RS is odd)
$8, k^2$	$8 + 4k$	$k^2 + 4k$	$k(k + 6) = 247$	$k = 13$
$k, 8k$	$5k$	$12k$	$15k = 255$	$k = 17$
$2k, 4k$	$6k$	$8k$	$12k = 255$	No k (LS is even and the RS is odd)

Therefore, the values of k which satisfy the given conditions are $k = 13$ and $k = 17$.

The CENTRE for EDUCATION
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2020 Hypatia Contest

Wednesday, April 15, 2020
(in North America and South America)

Thursday, April 16, 2020
(outside of North America and South America)

Solutions

1. (a) The cost of 12 bags of avocados is $\$5.00 \times 12 = \$60.00$.
Thus, the chef spent $\$135.00 - \$60.00 = \$75.00$ on mangoes.
The cost of each box of mangoes is \$12.50, and so the chef purchased $\dfrac{\$75.00}{\$12.50} = 6$ boxes of mangoes.

(b) *Solution 1*
A bag of avocados sells for \$5.00, and so a 10% discount represents a savings of $\frac{10}{100} \times \$5.00$ or $0.10 \times \$5.00$ which equals \$0.50.
Thus, the discounted price for a bag of avocados is $\$5.00 - \$0.50 = \$4.50$.
A box of mangoes sells for \$12.50, and so a 20% discount represents a savings of $\frac{20}{100} \times \$12.50$ or $0.20 \times \$12.50$ which equals \$2.50.
Thus, the discounted price for a box of mangoes is $\$12.50 - \$2.50 = \$10.00$.
On Saturdays, the total cost for 8 bags of avocados and 4 boxes of mangoes is

$$\$4.50 \times 8 + \$10.00 \times 4 = \$36.00 + \$40.00 = \$76.00$$

Solution 2
A bag of avocados sells for \$5.00, and so a 10% discount is equivalent to paying $100\% - 10\% = 90\%$ of the regular price.
Thus, the discounted price for a bag of avocados is $\frac{90}{100} \times \$5.00 = 0.90 \times \$5.00 = \$4.50$.
A box of mangoes sells for \$12.50, and so a 20% discount is equivalent to paying $100\% - 20\% = 80\%$ of the regular price.
Thus, the discounted price for a box mangoes is $\frac{80}{100} \times \$12.50 = 0.80 \times \$12.50 = \$10.00$.
On Saturdays, the total cost for 8 bags of avocados and 4 boxes of mangoes is

$$\$4.50 \times 8 + \$10.00 \times 4 = \$36.00 + \$40.00 = \$76.00$$

(c) Avocados are sold in bags of 6 and the chef needs 100 avocados.
Since $6 \times 16 = 96$ and $6 \times 17 = 102$, the chef will need to purchase 17 bags of avocados (16 bags is not enough).
Mangoes are sold in boxes of 15 and the chef needs 70 mangoes.
Since $15 \times 4 = 60$ and $15 \times 5 = 75$, the chef will need to purchase 5 boxes of mangoes.
The total cost of the purchase was $\$5.00 \times 17 + \$12.50 \times 5 = \$85.00 + \$62.50 = \$147.50$.

(d) Avocados are sold for \$5.00 per bag, and so the cost to purchase any number of bags is a whole number.
Mangoes are sold for \$12.50 per box, and so the cost to purchase mangoes is a whole number only when an even number of boxes are bought (1 box costs \$12.50, 2 boxes cost \$25.00, 3 boxes cost \$37.50, and so on).
Since the chef spends exactly \$75.00 (a whole number), then the chef must purchase an even number of boxes of mangoes.

If the chef purchases 2 boxes of mangoes, the cost is \$25.00, which leaves $\$75.00 - \$25.00 = \$50.00$ to be used to purchase $\$50.00 \div \$5.00 = 10$ bags of avocados.
In this case, the chef has $10 \times 6 = 60$ avocados and $2 \times 15 = 30$ mangoes.
Each tart requires 1 avocado and 2 mangoes and so the chef can make $30 \div 2 = 15$ tarts (he has more than 15 avocados but only 30 mangoes).

If the chef purchases 4 boxes of mangoes, the cost is $4 \times \$12.50 = \$50.00$, which leaves $\$75.00 - \$50.00 = \$25.00$ to be used to purchase $\$25.00 \div \$5.00 = 5$ bags of avocados.

In this case, the chef has $5 \times 6 = 30$ avocados and $4 \times 15 = 60$ mangoes.
Each tart requires 1 avocado and 2 mangoes and so the chef can make 30 tarts.

If the chef purchases 6 boxes of mangoes, the cost is $6 \times \$12.50 = \$75.00$, which leaves no money to purchase avocados.
Purchasing more than 6 boxes of mangoes will cost the chef more than \$75.00.

Therefore, if the chef purchases 30 avocados (5 bags) and 60 mangoes (4 boxes), she will have spent exactly \$75.00, have twice as many mangoes as avocados, and be able to make the greatest number of tarts, 30.

2. (a) The parabola $y = \frac{1}{4}x^2$ and the parabolic rectangle are each symmetrical about the y-axis, and thus a second vertex of the rectangle lies on the parabola and has coordinates $(-6, 9)$.
A third vertex of the parabolic rectangle lies on the x-axis vertically below $(6, 9)$, and thus has coordinates $(6, 0)$.
Similarly, the fourth vertex also lies on the x-axis vertically below $(-6, 9)$, and thus has coordinates $(-6, 0)$.

(b) If one vertex of a parabolic rectangle is $(-3, 0)$, then a second vertex has coordinates $(3, 0)$, and so the rectangle has length 6.
The vertex that lies vertically above $(3, 0)$ has x-coordinate 3.
This vertex lies on the parabola $y = \frac{1}{4}x^2$ and thus has y-coordinate equal to $\frac{1}{4}(3)^2 = \frac{9}{4}$.
The width of the rectangle is equal to this y-coordinate $\frac{9}{4}$, and so the area of the parabolic rectangle having one vertex at $(-3, 0)$ is $6 \times \frac{9}{4} = \frac{54}{4} = \frac{27}{2}$.

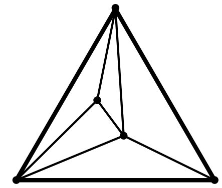
(c) Let a vertex of the parabolic rectangle be the point $(p, 0)$, with $p > 0$.
A second vertex (also on the x-axis) is thus $(-p, 0)$, and so the rectangle has length $2p$.
The width of this rectangle is given by the y-coordinate of the point that lies on the parabola vertically above $(p, 0)$, and so the width is $\frac{1}{4}p^2$.
The area of a parabolic rectangle having length $2p$ and width $\frac{1}{4}p^2$ is $2p \times \frac{1}{4}p^2 = \frac{1}{2}p^3$.
If such a parabolic rectangle has length 36, then $2p = 36$, and so $p = 18$.
The area of this rectangle is $\frac{1}{2}(18)^3 = 2916$.
If such a parabolic rectangle has width 36, then $\frac{1}{4}p^2 = 36$ or $p^2 = 144$, and so $p = 12$ (since $p > 0$).
The area of this rectangle is $\frac{1}{2}(12)^3 = 864$.
The areas of the two parabolic rectangles that have side length 36 are 2916 and 864.

(d) Let a vertex of the parabolic rectangle be the point $(m, 0)$, with $m > 0$.
A second vertex (also on the x-axis) is thus $(-m, 0)$, and so the rectangle has length $2m$.
The width of this rectangle is given by the y-coordinate of the point that lies on the parabola vertically above $(m, 0)$, and so the width is $\frac{1}{4}m^2$.
The area of a parabolic rectangle having length $2m$ and width $\frac{1}{4}m^2$ is $2m \times \frac{1}{4}m^2 = \frac{1}{2}m^3$.
If the length and width of such a parabolic rectangle are equal, then

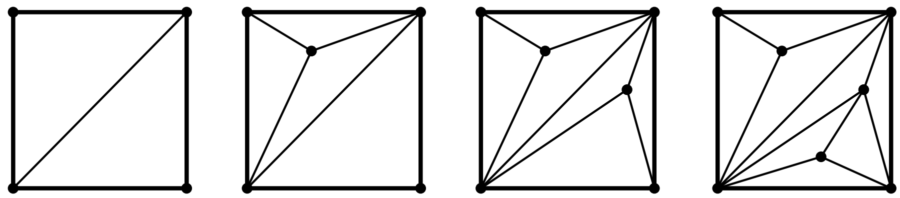
$$\begin{aligned} \tfrac{1}{4}m^2 &= 2m \\ m^2 &= 8m \\ m^2 - 8m &= 0 \\ m(m-8) &= 0 \end{aligned}$$

Thus $m = 8$ (since $m > 0$), and so the area of the parabolic rectangle whose length and width are equal is $\frac{1}{2}(8)^3 = 256$.

3. (a) For $n \geq 3$ and $k \geq 0$, the value of $T(n,k)$ is constant for all possible locations of the k interior points and all possible triangulations.
Thus, we may use the triangulation shown to determine that $T(3,2) = 5$.

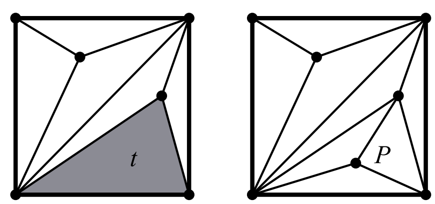


(b) We begin by drawing triangulations to determine the values of $T(4,k)$ for $k = 0, 1, 2, 3$.



$T(4,0)=2$ $T(4,1)=4$ $T(4,2)=6$ $T(4,3)=8$

Although we would obtain these same four answers by positioning the interior points in different locations, or by completing the triangulations in different ways, the diagrams above were created to help visualize a pattern.
From the answers shown, we see that $T(4,k+1) = T(4,k) + 2$, for $k = 0, 1, 2$.
We must justify why this observation is true for all $k \geq 0$ so that we may use the result to determine the value of $T(4,100)$.
Notice that each triangulation (after the first) was created by placing a new interior point inside the previous triangulation.
Further, each square is divided into triangles, and so each new interior point is placed *inside* a triangle of the previous triangulation (since no 3 points may lie on the same line).
For example, in the diagrams shown to the right, we observe that P lies in triangle t of the previous triangulation.



$T(4,2)=6$ $T(4,3)=8$

Also, each of the triangles outside of t is untouched by the addition of P, and thus they continue to contribute the same number of triangles (5) to the value of $T(4,3)$ as they did to the value of $T(4,2)$.
Triangle t contributes 1 to the value of $T(4,2)$.
To triangulate the region defined by triangle t, P must be joined to each of the 3 vertices of triangle t (no other triangulation of this region is possible).
Thus, the placement of P divides triangle t into 3 triangles for every possible location of P inside triangle t.
That is, t contributes 1 to the value of $T(4,2)$, but the region defined by t contributes 3 to the value of $T(4,3)$ after the placement of P.
To summarize, the value of $T(4,k+1)$ is 2 more than the value of $T(4,k)$ for all $k \geq 0$ since:

- the $(k+1)^{st}$ interior point may be placed anywhere inside the triangulation for $T(4,k)$ (provided it is not on an edge)
- specifically, the $(k+1)^{st}$ interior point lies inside a triangle of the triangulation which gives $T(4,k)$
- this triangle contributed 1 to the value of $T(4,k)$
- after the $(k+1)^{st}$ interior point is placed inside this triangle and joined to each of the 3 vertices of the triangle, this area contributes 3 to the value of $T(4,k+1)$
- this is a net increase of 2 triangles, and thus $T(4,k+1) = T(4,k) + 2$, for all $k \geq 0$.

$T(4,0) = 2$ and each additional interior point increases the number of triangles by 2.
Thus, k additional interior points increases the number of triangles by $2k$, and so $T(4,k) = T(4,0) + 2k = 2 + 2k$ for all $k \geq 0$.
Using this formula, we get $T(4,100) = 2 + 2(100) = 202$.

(c) In the triangulation of a regular n-gon with no interior points, we may choose any one of the n vertices and join this vertex to each of the remaining $n - 3$ non-adjacent vertices.
All such triangulations of a regular n-gon with no interior points creates $n - 2$ triangles, and so $T(n,0) = n - 2$ for all $n \geq 3$ (since $T(n,0)$ is constant).
The reasoning used in part (b) extends to any regular polygon having $n \geq 3$ vertices.
That is, each additional interior point that is added to the triangulation for $n \geq 3$ vertices and $k \geq 0$ interior points gives a net increase of 2 triangles.
Thus, $T(n,k+1) = T(n,k) + 2$ for all regular polygons having $n \geq 3$ vertices and $k \geq 0$ interior points.
So then k additional interior points increases the number of triangles by $2k$, and so $T(n,k) = T(n,0) + 2k = (n-2) + 2k$ for all $k \geq 0$.
Using this formula $T(n,k) = (n-2) + 2k$, we get $T(n,n) = (n-2) + 2n = 3n - 2$ and $3n - 2 = 2020$ when $n = \frac{2022}{3} = 674$.

4. (a) *Solution 1*
If x_0 is even, then x_0^2 is even, and so $x_1 = x_0^2 + 1$ is odd.
If x_1 is odd, then x_1^2 is odd, and so $x_2 = x_1^2 + 1$ is even.
Thus if x_0 is even, then x_2 is even and so $x_2 - x_0$ is even.

If x_0 is odd, then x_0^2 is odd, and so $x_1 = x_0^2 + 1$ is even.
If x_1 is even, then x_1^2 is even, and so $x_2 = x_1^2 + 1$ is odd.
Thus if x_0 is odd, then x_2 is odd and so $x_2 - x_0$ is even.

Therefore, for all possible values of x_0, $x_2 - x_0$ is even.

Solution 2
Using the definition twice and simplifying, we get

$$\begin{aligned}
x_2 &= (x_1)^2 + 1 \\
x_2 &= ((x_0)^2 + 1)^2 + 1 \\
x_2 &= (x_0)^4 + 2(x_0)^2 + 2 \\
x_2 &= (x_0)^4 + 2((x_0)^2 + 1) \\
x_2 - x_0 &= (x_0)^4 + 2((x_0)^2 + 1) - x_0
\end{aligned}$$

To show that $x_2 - x_0$ is even, we may show that $(x_0)^4 + 2((x_0)^2 + 1) - x_0$ is even (since the expressions are equal).
Since $2((x_0)^2 + 1)$ is the product of some integer and 2, this term is even for all possible values of x_0.
If x_0 is even, then $(x_0)^4$ is even, and so $(x_0)^4 + 2((x_0)^2 + 1) - x_0$ is the sum and difference of three even terms and thus is even.
If x_0 is odd, then $(x_0)^4$ is odd, $(x_0)^4 - x_0$ is even, and so $(x_0)^4 + 2((x_0)^2 + 1) - x_0$ is even.
Thus, $x_2 - x_0$ is even for all possible values of x_0.

Solution 3

Using the definition twice and simplifying, we get

$$\begin{aligned}
x_2 &= (x_1)^2 + 1 \\
x_2 &= ((x_0)^2 + 1)^2 + 1 \\
x_2 &= (x_0)^4 + 2(x_0)^2 + 2 \\
x_2 - x_0 &= (x_0)^4 + 2(x_0)^2 - x_0 + 2 \\
x_2 - x_0 &= (x_0)^4 + (x_0)^2 + (x_0)^2 - x_0 + 2 \\
x_2 - x_0 &= (x_0)^2((x_0)^2 + 1) + x_0(x_0 - 1) + 2
\end{aligned}$$

To show that $x_2 - x_0$ is even, we may show that $(x_0)^2((x_0)^2 + 1) + x_0(x_0 - 1) + 2$ is even (since they are equal).
Since $x_0 - 1$ is one less than x_0, then x_0 and $x_0 - 1$ are consecutive integers and so one of them is even.
Thus, the product $x_0(x_0 - 1)$ is even.
Similarly, $(x_0)^2$ is one less than $(x_0)^2 + 1$, and thus these are consecutive integers and so one of them is even.
Therefore, the product $(x_0)^2((x_0)^2 + 1)$ is even.
Since $(x_0)^2((x_0)^2 + 1) + x_0(x_0 - 1) + 2$ is the sum of three even integers, $x_2 - x_0$ is even for all possible values of x_0.

(b) An integer is divisible by 10 exactly when its units (ones) digit is 0.
The difference $x_{2026} - x_{2020}$ has units digit 0 exactly when x_{2026} and x_{2020} have equal units digits.
Thus, we will show that for all possible values of x_0, the units digit of x_{2026} is equal to the units digit of x_{2020}, and so $x_{2026} - x_{2020}$ is divisible by 10.
When a non-negative integer is divided by 10, the remainder is one of the integers from 0 through 9, inclusive.
Thus for every possible choice for x_0, there exists some non-negative integer k, so that x_0 can be expressed in exactly one of the following ways: $10k$, $10k + 1$, $10k + 2$, $\ldots$, $10k + 8$, $10k + 9$.
If for example x_0 has units digit 4, then $x_0 = 10k + 4$ for some non-negative integer k, and so $x_1 = (10k + 4)^2 + 1 = 100k^2 + 80k + 17 = 10(10k^2 + 8k + 1) + 7$, and thus has units digit 7.
Since x_1 is determined by x_0 only ($x_1 = (x_0)^2 + 1$), the units digit of x_1 is uniquely determined by the units digit of x_0.
For example, if the units digit of x_0 is 4, then the units digit of x_1 is equal to the units digit of $(4)^2 + 1$, which equals 7.
More generally, if the units digit of x_i (for a non-negative integer i) is equal to u, then the units digit of x_{i+1} is equal to the units digit of $u^2 + 1$.
(Can you explain why this is true?)
For example, if we know that $x_{15} = 29$, then the units digit of x_{16} is equal to the units digit of $9^2 + 1 = 82$, which is 2.
Given that we know all possible units digits of x_0, this provides an efficient method for determining the units digits of x_1, x_2, x_3, and so on.
In the table below, we list the units digits for the terms x_1 through x_7 for each of the 10 possible units digits of x_0, 0 through 9 inclusive.

x_0	0	1	2	3	4	5	6	7	8	9
x_1	1	2	5	0	7	6	7	0	5	2
x_2	2	5	6	1	0	7	0	1	6	5
x_3	5	6	7	2	1	0	1	2	7	6
x_4	6	7	0	5	2	1	2	5	0	7
x_5	7	0	1	6	5	2	5	6	1	0
x_6	0	1	2	7	6	5	6	7	2	1
x_7	1	2	5	0	7	6	7	0	5	2

Looking at the table, we see that for each of the possible units digits for x_0, the units digit of x_1 is equal to the units digit of x_7.
Thus beginning at x_1, each column in the table will repeat every 6 terms, and so independent of the starting value x_0, x_{i+6} and x_i have equal units digits for all integers $i \geq 1$.
Since $2026 - 2020 = 6$, then x_{2026} and x_{2020} have equal units digits, and so $x_{2026} - x_{2020}$ has units digit 0, and thus is divisible by 10.

(c) Since $x_{115} - 110 = (x_{115} - 5) - 105$, then $x_{115} - 110$ is divisible by 105 exactly when $x_{115} - 5$ is divisible by 105.
Further, $105 = 3 \times 5 \times 7$ and each of $3, 5, 7$ is a prime number, and so $x_{115} - 5$ is divisible by 105 exactly when it is divisible by 3, 5 and 7.
Every x_i is a multiple of 3, 1 more than a multiple of 3, or 2 more than a multiple of 3.
Suppose that x_i is a multiple of 3. Then $x_i = 3k$ for some non-negative integer k, and so

$$x_{i+1} = (x_i)^2 + 1 = (3k)^2 + 1 = 3(3k^2) + 1$$

which is 1 more than a multiple of 3.
If x_i is 1 more than a multiple of 3, then $x_i = 3k + 1$ for some non-negative integer k, and so

$$x_{i+1} = (x_i)^2 + 1 = (3k + 1)^2 + 1 = 9k^2 + 6k + 2 = 3(3k^2 + 2k) + 2$$

which is 2 more than a multiple of 3.
If x_i is 2 more than a multiple of 3, then $x_i = 3k + 2$ for some non-negative integer k, and so

$$x_{i+1} = (x_i)^2 + 1 = (3k + 2)^2 + 1 = 9k^2 + 12k + 5 = 3(3k^2 + 4k + 1) + 2$$

which is 2 more than a multiple of 3.
Each possible choice of x_0 is a multiple of 3, 1 more than a multiple of 3, or 2 more than a multiple of 3.
If x_0 is a multiple of 3, then x_1 is 1 more than a multiple of 3 and $x_2, x_3, x_4, \ldots$ and so on are each 2 more than a multiple of 3.
If x_0 is 1 or 2 more than a multiple of 3, then $x_1, x_2, x_3, x_4, \ldots$ and so on are each 2 more than a multiple of 3.
Therefore, $x_2, x_3, x_4, \ldots$ and so on are all 2 more than a multiple of 3 (independent of x_0), and so for all $i \geq 2$, x_i is a number of the form $3k + 2$ for some non-negative integer k.
Therefore, $x_{115} - 5 = 3k + 2 - 5 = 3(k - 1)$ is divisible by 3 for all possible choices of $x_0 = n$.
Thus, we need to only determine when $x_{115} - 5$ is divisible by 5 and 7.

For which of the possible values of x_0 is $x_{115} - 5$ divisible by 5?
$x_{115} - 5$ is divisible by 5 exactly when x_{115} is divisible by 5.
Every x_i is either a multiple of 5, 1 more than a multiple of 5, 2 more than a multiple of 5, 3 more than a multiple of 5, or 4 more than a multiple of 5.

With respect to division by 5, the table below gives the remainders of x_{i+1} given each of the 5 possible remainders of x_i, 0 through 4 inclusive.

x_i	$x_{i+1} = (x_i)^2 + 1$
$5k$	$25k^2 + 1 = 5(5k^2) + 1$
$5k+1$	$25k^2 + 10k + 2 = 5(5k^2 + 2k) + 2$
$5k+2$	$25k^2 + 20k + 5 = 5(5k^2 + 4k + 1)$
$5k+3$	$25k^2 + 30k + 10 = 5(5k^2 + 6k + 2)$
$5k+4$	$25k^2 + 40k + 17 = 5(5k^2 + 8k + 3) + 2$

From this table, we make the following observations:

- if x_i is a multiple of 5, then x_{i+1} is 1 more than a multiple of 5
- if x_i is 1 more than a multiple of 5, then x_{i+1} is 2 more than a multiple of 5
- if x_i is 2 more than a multiple of 5, then x_{i+1} is a multiple of 5
- if x_i is 3 more than a multiple of 5, then x_{i+1} is a multiple of 5
- if x_i is 4 more than a multiple of 5, then x_{i+1} is 2 more than a multiple of 5

Using these observations, we summarize the remainders of x_1, x_2, x_3, x_4 when dividing by 5 for each of the possible remainders for x_0.

x_0	0	1	2	3	4
x_1	1	2	0	0	2
x_2	2	0	1	1	0
x_3	0	1	2	2	1
x_4	1	2	0	0	2

With respect to division by 5, we see in the table above that for each of the possible remainders for x_0, the remainder for x_1 is equal to that of x_4.
Thus beginning at x_1, each column in the table repeats every 3 terms, and so independent of the starting value x_0, x_{i+3} and x_i have equal remainders after division by 5 for all integers $i \geq 1$.
Since $115 = 3(37) + 4$, then x_{115} and x_4 have the same remainders after division by 5, and so x_{115} is divisible by 5 for all choices of $x_0 = n$ which are either 2 more than a multiple of 5 or 3 more than a multiple of 5.

Finally, we want to determine for which of the possible values of x_0 is $x_{115} - 5$ divisible by 7.
$x_{115} - 5$ divisible by 7 exactly when x_{115} is 5 more than a multiple of 7.
Every x_i is exactly one of: a multiple of 7, 1 more than a multiple of 7, 2 more than a multiple of 7, and so on up to 6 more than a multiple of 7.
With respect to division by 7, the table below gives the remainders of x_{i+1} given each of the 7 possible remainders of x_i, 0 through 6 inclusive.

x_i	$x_{i+1} = (x_i)^2 + 1$
$7k$	$49k^2 + 1 = 7(7k^2) + 1$
$7k+1$	$49k^2 + 14k + 2 = 7(7k^2 + 2k) + 2$
$7k+2$	$49k^2 + 28k + 5 = 7(7k^2 + 4k) + 5$
$7k+3$	$49k^2 + 42k + 10 = 7(7k^2 + 6k + 1) + 3$
$7k+4$	$49k^2 + 56k + 17 = 7(7k^2 + 8k + 2) + 3$
$7k+5$	$49k^2 + 70k + 26 = 7(7k^2 + 10k + 3) + 5$
$7k+6$	$49k^2 + 84k + 37 = 7(7k^2 + 12k + 5) + 2$

From this table, we make the following observations:

- if x_i is a multiple of 7, then x_{i+1} is 1 more than a multiple of 7
- if x_i is 1 more than a multiple of 7, then x_{i+1} is 2 more than a multiple of 7
- if x_i is 2 more than a multiple of 7, then x_{i+1} is 5 more than a multiple of 7
- if x_i is 3 more than a multiple of 7, then x_{i+1} is 3 more than a multiple of 7
- if x_i is 4 more than a multiple of 7, then x_{i+1} is 3 more than a multiple of 7
- if x_i is 5 more than a multiple of 7, then x_{i+1} is 5 more than a multiple of 7
- if x_i is 6 more than a multiple of 7, then x_{i+1} is 2 more than a multiple of 7

Using these observations, we summarize the remainders of x_1, x_2, x_3, x_4 when dividing by 7 for each of the possible remainders for x_0.

x_0	0	1	2	3	4	5	6
x_1	1	2	5	3	3	5	2
x_2	2	5	5	3	3	5	5
x_3	5	5	5	3	3	5	5
x_4	5	5	5	3	3	5	5

Looking at the table, we see that if x_0 is $0, 1, 2, 5$, or 6 more than a multiple of 7, then x_i is 5 more than a multiple of 7 for all $i \geq 3$.
Also, if x_0 is 3 or 4 more than a multiple of 7, then x_i is 3 more than a multiple of 7 for all $i \geq 1$ (and thus never 5 more than a multiple of 7).
Therefore, $x_{115} - 5$ is a multiple of 7 exactly when x_0 is not 3 or 4 more than a multiple of 7.

<u>Summary:</u>

$x_{115} - 110$ is divisible by 105 exactly when

- x_0 is 2 or 3 more than a multiple of 5, and
- x_0 is a multiple of 7 or $1, 2, 5$, or 6 more than a multiple of 7.

The values of x_0 in the range $1 \leq x_0 \leq 35$ satisfying these properties are:

$$2, 7, 8, 12, 13, 22, 23, 27, 28, 33$$

The values of x_0 in the range $36 \leq x_0 \leq 100$ satisfying these properties must each be a mutiple of $5 \times 7 = 35$ greater than one the numbers in the above list.
Thus, there are 10 possible values for x_0 in the original list, 10 more from $2 + 35 = 37$ to $33 + 35 = 68$, and 9 more from $2 + 2(35) = 72$ to $28 + 2(35) = 98$, so 29 in total (note that $33 + 2(35) > 100$).
If Parsa chooses an integer n with $1 \leq n \leq 100$ at random and sets $x_0 = n$, the probability that $x_{115} - 110$ is divisible by 105 is $\frac{29}{100}$.

The CENTRE for EDUCATION
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2019 Hypatia Contest

Wednesday, April 10, 2019
(in North America and South America)

Thursday, April 11, 2019
(outside of North America and South America)

Solutions

1. (a) The radius of each hole is 2 cm, and so the diameter of each hole is 4 cm.
Since there are 4 holes, then their diameters combine for a total distance of $4 \times 4 = 16$ cm along the 91 cm midline.
Thus, the 5 equal spaces along the midline combine for a total distance of $91 - 16 = 75$ cm, and so the distance along the midline between adjacent holes is $\frac{75}{5} = 15$ cm.

(b) Let the radius of each hole be r cm, and so the diameter of each hole is $2r$ cm.
Since there are 4 holes, then their diameters combine for a total distance of $4 \times 2r = 8r$ cm along the midline.
The distance along the midline between adjacent holes is equal to the radius, r cm, and so these 5 equal spaces along the midline combine for a total distance of $5r$ cm.
The total length of the midline is 91 cm, and so $8r + 5r = 91$ or $13r = 91$, and so the radius of each hole is $\frac{91}{13} = 7$ cm.

(c) *Solution 1*
As in part (b), if the diameter of each hole is $2r$ cm, then the 4 diameters combine for a total distance of $4 \times 2r = 8r$ cm along the midline.
If the distance along the midline between adjacent holes is 5 cm, then the 5 equal spaces along the midline combine for a total distance of 25 cm.
The total length of the midline is 91 cm, and so we get $8r + 25 = 91$ or $8r = 66$, and so the radius of each hole is $\frac{66}{8} = 8.25$ cm.
However, the vertical distance from the midline to each edge of the metal is 8 cm, and since the holes must be circles, the radius of each hole cannot be 8.25 cm, and so the distance between adjacent holes cannot be 5 cm.

Solution 2
The minimum possible distance between adjacent holes is determined by maximizing the radius of each of the circles.
Since the holes must be circles, and the vertical distance from the midline to each edge of the metal is 8 cm, the maximum radius is 8 cm.
Since there are 4 holes, then their diameters combine for a maximum total distance of $4 \times 16 = 64$ cm along the midline.
The total length of the midline is 91 cm, and so the 5 equal spaces combine for a minimum total length of $91 - 64 = 27$ cm.
Thus, the minimum distance between adjacent holes is $\frac{27}{5} = 5.4$ cm, which is greater than the required 5 cm.

2. (a) To add a bump, the line segment of length 21 is first broken into three segments, each having length $\frac{21}{3} = 7$.
The middle segment of these three segments is removed, and two new segments each having length 7 are added.
Thus, after a bump is added to a segment of length 21, the new path will have length $4 \times 7 = 28$.

(b) A path with exactly one bump has four line segments of equal length.
If such a path has length 240, then each of the four line segments has length $\frac{240}{4} = 60$.
Thus, the original line segment had three line segments each of length 60, and so the length of the original line segment was $3 \times 60 = 180$.

(c) To add the first bump, the line segment of length 36 is broken into three segments, each having length $\frac{36}{3} = 12$.
The middle segment of these three segments is removed, and two new segments each having length 12 are added.
Thus, after the first bump is added to a segment of length 36, the new path will have length $4 \times 12 = 48$.

Next, a bump is added to each of the four segments having length 12.
Consider adding a bump to one of these four segments.
The segment of length 12 is broken into three segments, each having length $\frac{12}{3} = 4$.
The middle segment is removed, two new segments each having length 4 are added, and so the new length is $4 \times 4 = 16$.
There are four such segments to which this process happens, and so the total path length of the resulting figure is $4 \times 16 = 64$.

(d) To add a bump, the line segment of length n is first broken into three segments, each having length $\frac{n}{3}$.
The middle segment of these three segments is removed, and two new segments each having length $\frac{n}{3}$ are added.
Thus, after a bump is added to a segment of length n, Path 1 will have length $4 \times \frac{n}{3} = \frac{4}{3}n$.

To create Path 2, a bump is added to each of the four segments having length $\frac{n}{3}$.
Consider adding a bump to one of these four segments.
The segment of length $\frac{n}{3}$ is broken into three segments, each having length $\frac{1}{3} \times \frac{n}{3}$.
The middle segment is removed, two new segments each having length $\frac{1}{3} \times \frac{n}{3}$ are added, and so the new length is $4 \times \frac{1}{3} \times \frac{n}{3} = \frac{4}{3^2}n$.
There are four such segments to which this process happens, and so the total length of Path 2 is $4 \times \frac{4}{3^2}n = \frac{4^2}{3^2}n$ or $\left(\frac{4}{3}\right)^2 n$.

To summarize, when a bump is added to a line segment of length n, the length of the resulting path, Path 1, is $\frac{4}{3}n$.

When bumps are then added to Path 1, the length of the resulting Path 2 is $\left(\frac{4}{3}\right)^2 n$.

This process will continue with each new path having a total length that is $\frac{4}{3}$ of the previous path length.
That is, Path 3 will have length $\left(\frac{4}{3}\right)^3 n$, Path 4 will have length $\left(\frac{4}{3}\right)^4 n$, and Path 5 will have length $\left(\frac{4}{3}\right)^5 n$.
If the length of Path 5 is an integer, then n must be divisible by 3^5 (since there are no factors of 3 in 4^5).
The smallest integer n which is divisible by 3^5 is $3^5 = 243$.
The smallest possible integer n for which the length of Path 5 is an integer is 243.

3. (a) The arithmetic mean of 36 and 64 is $\dfrac{36+64}{2} = \dfrac{100}{2} = 50$.

The geometric mean of 36 and 64 is $\sqrt{36 \cdot 64} = \sqrt{6^2 \cdot 8^2} = 6 \cdot 8 = 48$.

(b) If the arithmetic mean of two positive real numbers x and y is 13, then $\dfrac{x+y}{2} = 13$.
If the geometric mean of two positive real numbers x and y is 12, then $\sqrt{xy} = 12$.
Multiplying the first equation by 2 gives $x + y = 26$ and so $x = 26 - y$.
Substituting $x = 26 - y$ into the second equation and squaring both sides gives $(26 - y)y = 12^2$ or $y^2 - 26y + 144 = 0$.
Factoring the left side of this equation, we get $(y - 8)(y - 18) = 0$ and so $y = 8$ or $y = 18$.
When $y = 8$, $x = 26 - 8 = 18$, and when $y = 18$, $x = 8$.
That is, the numbers 8 and 18 have an arithmetic mean of 13 and a geometric mean of 12.

(c) We are required to solve the equation $\dfrac{x+y}{2} - \sqrt{xy} = 1$, for positive integers x and y with $x < y \leq 50$.
Simplifying first, we get

$$\begin{aligned}
\frac{x+y}{2} - \sqrt{xy} &= 1 \\
x - 2\sqrt{xy} + y &= 2 \\
(\sqrt{x})^2 - 2\sqrt{xy} + (\sqrt{y})^2 &= 2 \quad \text{(since } x, y > 0\text{)} \\
(\sqrt{x} - \sqrt{y})^2 &= 2 \\
\sqrt{x} - \sqrt{y} &= \pm\sqrt{2} \\
\sqrt{y} - \sqrt{x} &= \sqrt{2} \quad \text{(since } y > x\text{)} \\
\sqrt{y} &= \sqrt{x} + \sqrt{2} \\
y &= x + 2 + 2\sqrt{2x}
\end{aligned}$$

Since x is a positive integer, then $y = 2 + x + 2\sqrt{2x}$ is a positive integer exactly when $\sqrt{2x}$ is a positive integer.
This occurs exactly when $x = 2m^2$ for integer values of m.
Since $x < y \leq 50$, then $2m^2 < 50$ or $m^2 < 25$, and so m is any integer from 1 to 4 inclusive.
We determine the corresponding values of x and y in the table below.

m	$x = 2m^2$	$y = 2 + x + 2\sqrt{2x}$
1	2	8
2	8	18
3	18	32
4	32	50

The pairs (x, y) satisfying the required conditions are $(2, 8), (8, 18), (18, 32)$, and $(32, 50)$.

4. (a) We begin by multiplying the first equation by 5 and the second equation by 3 to get

$$\begin{aligned}
15x + 20y &= 50 \\
15x + 18y &= 3c
\end{aligned}$$

By subtracting the second equation from the first, we get

$$(15x + 20y) - (15x + 18y) = 50 - 3c$$

or $2y = 50 - 3c$. Solving for y, we get

$$y = 25 - \frac{3}{2}c$$

Substituting this into the first of the original two equations, we get

$$3x + 4\left(25 - \frac{3}{2}c\right) = 10$$

Multiplying the 4 through the parentheses gives $3x + 100 - 6c = 10$ which simplifies to $3x = 6c - 90$ or $x = 2c - 30$. Therefore, in terms of c, we have

$$(x, y) = \left(2c - 30, 25 - \frac{3}{2}c\right)$$

(b) Similar to part (a), we will first solve for x and y in terms of d.
This will give us enough information to determine for which d these values of x and y are integers.
Multiplying the first equation by 4 we get $4x + 8y = 12$.
We can subtract $4x + dy = 6$ from $4x + 8y = 12$ to get $8y - dy = (8 - d)y = 6$.
This means $y = \dfrac{6}{8 - d}$.
From the first equation, we get $x = 3 - 2y$.
Substituting the expression for y into this equation and simplifying, we get

$$x = 3 - 2\left(\frac{6}{8 - d}\right) = 3 - \frac{12}{8 - d}$$

We need to find values of d so that $y = \dfrac{6}{8 - d}$ and $x = 3 - \dfrac{12}{8 - d}$ are both integers.
Since 3 is an integer, x will be an integer exactly when $\dfrac{12}{8 - d}$ is an integer.
For this to happen, we need $8 - d$ to be a divisor of 12.
However, for $y = \dfrac{6}{8 - d}$ to be an integer, we need $8 - d$ to be a divisor of 6.
Since any divisor of 6 is also a divisor of 12, this means that if y is an integer, then x is an integer.
Therefore, we need only find integers d so that y is an integer.
The divisors of 6, and hence, the possible values of $8 - d$, are $-6, -3, -2, -1, 1, 2, 3$, and 6.
Therefore, the possible values of d are $2, 5, 6, 7, 9, 10, 11$, and 14.
(We may check that each of these values of d give integer values for x and y.)

(c) In order to simplify things, we will first show that, regardless of the values of x, y and k, if n is an integer, then y must be equal to -1.
To see this, we begin by multiplying the first equation by -2 and the second equation by 3 to get

$$\begin{aligned} -(18n + 12)x + (6n + 4)y &= -6n^2 - 12n - (6k + 10) \\ (18n + 12)x + (9n^2 + 6n)y &= -3n^2 + (6k + 6) \end{aligned}$$

We now add the two equations.
When doing this, the $-(18n + 12)$ cancels with the $(18n + 12)$ to give

$$(9n^2 + 12n + 4)y = -9n^2 - 12n - 4$$

Observe that $(3n+2)^2 = 9n^2 + 12n + 4$, so we can rewrite this equation as

$$y(3n+2)^2 = -(3n+2)^2$$

Suppose $(3n+2)^2 = 0$. Then $3n+2 = 0$ so $n = -\frac{2}{3}$, which is not an integer.
Therefore, if we assume n is an integer, we can divide the above equation by $(3n+2)^2$ to get

$$y = \frac{-(3n+2)^2}{(3n+2)^2} = -1$$

We are interested in finding positive integer values of k for which there exist integers n so that the system of equations has a solution (x, y) where x and y are integers.
Thus, going forward, we assume n is an integer which we have shown means $y = -1$.
The first equation simplifies to

$$(9n+6)x - (3n+2)(-1) = 3n^2 + 6n + (3k+5)$$

which can be rearranged to get

$$(9n+6)x = 3n^2 + 3n + 3k + 3 \tag{1}$$

The second equation becomes

$$(6n+4)x + (3n^2+2n)(-1) = -n^2 + (2k+2)$$

which rearranges to

$$(6n+4)x = 2n^2 + 2n + 2k + 2 \tag{2}$$

Dividing equation (1) by 3 or equation (2) by 2 gives

$$(3n+2)x = n^2 + n + 1 + k$$

We want n, k, and x to all be integers, which means we need integers n and k so that $n^2 + n + 1 + k$ is a multiple of $3n+2$.
To simplify this, we will show that $n^2 + n + 1 + k$ is a multiple of $3n+2$ if and only if $3(n^2 + n + 1 + k)$ is a multiple of $3n+2$.
Assuming $n^2 + n + 1 + k$ is a multiple of $3n+2$, it is certainly true that $3(n^2 + n + 1 + k)$ is a multiple of $3n+2$.
Notice that $3n+2$ is 2 more than a multiple of 3, so $3n+2$ is not a multiple of 3.
This means $3n+2$ does not have a prime factor of 3, so $3(n^2 + n + k + 1)$ being a multiple of $3n+2$ means that $n^2 + n + 1 + k$ is a multiple of $3n+2$.
To summarize, we want to understand integer pairs (n, k) for which $n^2 + n + 1 + k$ is a multiple of $3n+2$.
We have shown that finding such integer pairs is the same as finding integer pairs (n, k) for which $3n^2 + 3n + 3 + 3k$ is a multiple of $3n+2$.
By rearranging and factoring, this is the same as finding integer pairs (n, k) so that $n(3n+2) + n + 3 + 3k$ is a multiple of $3n+2$.
Noticing that

$$\frac{n(3n+2) + n + 3 + 3k}{3n+2} = n + \frac{n + 3 + 3k}{3n+2}$$

the expression on the right side is an integer if and only if $n+3+3k$ is a multiple of $3n+2$.
Therefore, we need to find integer pairs (n,k) such that $n+3+3k$ is a multiple of $3n+2$.
Using the same reasoning as before, since $3n+2$ does not have a prime factor of 3, we have that $n+3+3k$ is a multiple of $3n+2$ if and only if $3(n+3+3k) = 3n+9+9k$ is a multiple of $3n+2$.
Notice that
$$\frac{3n+9+9k}{3n+2} = \frac{(3n+2)+(7+9k)}{3n+2} = 1 + \frac{7+9k}{3n+2}$$
so $3n+9+9k$ is a multiple of $3n+2$ if and only if $\dfrac{7+9k}{3n+2}$ is an integer, or $7+9k$ is a multiple of $3n+2$.
Putting all of this together, given that n and k are integers, the system of equations has a solution (x,y) where x and y are both integers precisely when $7+9k$ is a multiple of $3n+2$.
Phrasing the question in these terms, we are looking for a positive integer k for which there are exactly eight integers n so that $3n+2$ is a factor of $9k+7$.
This means we need a positive integer k so that exactly eight of the factors of $9k+7$ are two more than a multiple of 3.
Beginning with $k=1$, we get $9k+7=16$.
The factors of 16 are $-16,-8,-4,-2,-1,1,2,4,8,$ and 16, of which only $-16,-4,-1,2,$ and 8 are 2 more than a multiple of 3.
When $k=2$, $9k+7=25$.
The factors of 25 are $-25,-5,-1,1,5,$ and 25.
There are fewer than eight factors in total, so $k=2$ does not work.
When $k=3$, $9k+7=34$.
The factors of 34 are $-34,-17,-2,-1,1,2,17,34$, of which only $-34,-1,2,$ and 17 are 2 more than a multiple of 3.
When $k=4$, $9k+7=43$ which is prime, so it only has four factors in total which means $k=4$ does not work.
When $k=5$, $9k+7=52$.
The factors of 52 are $-52,-26,-13,-4,-2,-1,1,2,4,13,26,$ and 52, of which only $-52,-13,-4,-1,2,$ and 26 are 2 more than a multiple of 3.
When $k=6$, $9k+7=61$ which is prime, so it only has four factors in total which means $k=6$ does not work.
When $k=7$, $9k+7=70$.
The factors of 70 are
$$-70,-35,-14,-10,-7,-5,-2,-1,1,2,5,7,10,14,35, \text{ and } 70$$
Of these, the numbers $-70,-10,-7,-1,2,5,14$ and 35 are two more than a multiple of 3.
There are exactly 8 numbers in this list.
Therefore, if $k=7$, there are exactly eight integers n $(-70,-10,-7,-1,2,5,14,$ and $35)$, with the property that the system of equations has a solution (x,y) where x and y are integers.
(It is worth noting that there are other values of k which satisfy the given properties.)

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2018 Hypatia Contest

Thursday, April 12, 2018
(in North America and South America)

Friday, April 13, 2018
(outside of North America and South America)

Solutions

1. (a) The average of Aneesh's first six test scores was $\dfrac{17+13+20+12+18+10}{6} = \dfrac{90}{6} = 15$.

   (b) After Jon's third test, his average score was 14, and so the sum of the scores on his first three tests was $14 \times 3 = 42$.
   The sum of his scores on his first two tests was $17+12 = 29$, and so the score on his third test was $42 - 29 = 13$.
   (We may check that the average of $17, 12$ and 13 is $\dfrac{17+12+13}{3} = 14$.)

   (c) Dina wrote six tests followed by n more tests, for a total of $n+6$ tests.
   After Dina's first 6 tests, her average score was 14, and so the sum of the scores on her first 6 tests was $14 \times 6 = 84$.
   Dina scored 20 on each of her next n tests, and so the sum of the scores on her next n tests was $20n$.
   Therefore, the sum of the scores on these $n+6$ tests was $84 + 20n$.
   After Dina's $n+6$ tests, her average score was 18, and so the sum of the scores on her $n+6$ tests was $18(n+6)$.
   Thus, $84 + 20n = 18(n+6)$ or $84 + 20n = 18n + 108$ or $2n = 24$, and so $n = 12$.

2. (a) The distance from Botown to Aville is 120 km.
   Jessica drove this distance at a speed of 90 km/h, and so it took Jessica $\dfrac{120}{90} = \dfrac{4}{3}$ hours or $\dfrac{4}{3} \times 60 = 80$ minutes.

   (b) The distance from Botown to Aville is 120 km.
   The car predicted that Jessica would drive this distance at a speed of 80 km/h, and so it predicted that it would take Jessica $\dfrac{120}{80} = \dfrac{3}{2}$ hours or $\dfrac{3}{2} \times 60 = 90$ minutes.
   The ETA displayed by her car at 7:00 a.m. was 8:30 a.m..

   (c) Jessica drove from 7:00 a.m. to 7:16 a.m. (for 16 minutes) at a speed of 90 km/h, and so she travelled a distance of $\dfrac{16}{60} \times 90 = 24$ km.
   At 7:16 a.m., Jessica had a distance of 120 km $-$ 24 km $=$ 96 km left to travel.
   The car predicted that Jessica would drive this distance at a speed of 80 km/h, and so it predicted that it would take Jessica $\dfrac{96}{80} = \dfrac{6}{5}$ hours or $\dfrac{6}{5} \times 60 = 72$ minutes to complete the trip.
   The ETA displayed by her car at 7:16 a.m. was 72 minutes later or 8:28 a.m..

   (d) As in part (b), the car predicted that it would take Jessica 90 minutes or 1.5 hours to travel from Botown to Aville.
   Let the distance that Jessica travelled at 100 km/h be d km, and so the distance that Jessica travelled at 50 km/h was $(120 - d)$ km.
   The time that Jessica drove at 100 km/h was $\dfrac{d}{100}$ hours.
   The time that Jessica drove at 50 km/h was $\dfrac{120-d}{50}$ hours.
   Since the time predicted by her car is equal to the actual time that it took Jessica to travel from Botown to Aville, then $\dfrac{d}{100} + \dfrac{120-d}{50} = 1.5$.
   Solving for d, we get $d + 2(120 - d) = 1.5 \times 100$ or $-d + 240 = 150$, and so $d = 90$ km.
   Therefore, Jessica drove a distance of 90 km at a speed of 100 km/h.

3. (a) We are given that $T_1 = 1, T_2 = 2$ and $T_3 = 3$.
Evaluating, we get

$$T_4 = 1 + T_1T_2T_3 = 1 + (1)(2)(3) = 7, \text{ and}$$
$$T_5 = 1 + T_1T_2T_3T_4 = 1 + (1)(2)(3)(7) = 43.$$

(b) *Solution 1*
Each term after the second is equal to 1 more than the product of all previous terms in the sequence. Thus, $T_n = 1 + T_1T_2T_3 \cdots T_{n-1}$.
For all integers $n \geq 2$, we use the fact that $T_n = 1 + T_1T_2T_3 \cdots T_{n-1}$ to get

$$\begin{aligned} RS &= T_n^2 - T_n + 1 \\ &= T_n(T_n - 1) + 1 \\ &= T_n(1 + T_1T_2T_3 \cdots T_{n-1} - 1) + 1 \\ &= T_n(T_1T_2T_3 \cdots T_{n-1}) + 1 \\ &= T_1T_2T_3 \cdots T_{n-1}T_n + 1 \\ &= T_{n+1} \\ &= LS \end{aligned}$$

Solution 2
For all integers $n \geq 2$, we use the fact that $T_n = 1 + T_1T_2T_3 \cdots T_{n-1}$ to get

$$\begin{aligned} LS &= T_{n+1} \\ &= 1 + T_1T_2T_3 \cdots T_{n-1}T_n \\ &= 1 + (T_1T_2T_3 \cdots T_{n-1})T_n \\ &= 1 + (T_n - 1)T_n \\ &= T_n^2 - T_n + 1 \\ &= RS \end{aligned}$$

(c) Using the result from part (b), we get $T_n + T_{n+1} = T_n + T_n^2 - T_n + 1 = T_n^2 + 1$, for all integers $n \geq 2$.
Similarly,

$$\begin{aligned} T_nT_{n+1} - 1 &= T_n(T_n^2 - T_n + 1) - 1 \\ &= T_n^3 - T_n^2 + T_n - 1 \\ &= T_n^2(T_n - 1) + T_n - 1 \\ &= (T_n - 1)(T_n^2 + 1) \end{aligned}$$

Since $T_n + T_{n+1} = T_n^2 + 1$ and $T_n^2 + 1$ is a factor of $T_nT_{n+1} - 1$, then $T_n + T_{n+1}$ is a factor of $T_nT_{n+1} - 1$ for all integers $n \geq 2$.

(d) Using the result from part (b), we get $T_{2018} = T_{2017}^2 - T_{2017} + 1$.
Since T_{2017} is a positive integer greater than 1, then $T_{2017}^2 - T_{2017} + 1 > T_{2017}^2 - 2T_{2017} + 1$ and $T_{2017}^2 - T_{2017} + 1 < T_{2017}^2$.
That is, $T_{2017}^2 - 2T_{2017} + 1 < T_{2017}^2 - T_{2017} + 1 < T_{2017}^2$, and so $(T_{2017} - 1)^2 < T_{2018} < T_{2017}^2$.
Since $T_{2017} - 1$ and T_{2017} are two consecutive positive integers, then $(T_{2017} - 1)^2$ and T_{2017}^2 are two consecutive perfect squares, and so T_{2018} lies between two consecutive perfect squares and thus is not a perfect square.

4.(a)(i) By completing the square, the equations defining the two parabolas become

$$y = x^2 - 8x + 17 = x^2 - 8x + 16 + 1 = (x-4)^2 + 1, \text{ and}$$

$$y = -x^2 + 4x + 7 = -(x^2 - 4x + 4) + 11 = -(x-2)^2 + 11.$$

Thus, the parabola defined by the equation $y = x^2 - 8x + 17$ has vertex $V_1(4,1)$, and the parabola defined by the equation $y = -x^2 + 4x + 7$ has vertex $V_2(2,11)$.

(a)(ii) First, we determine the coordinates of the points of intersection P and Q.
When the two parabolas intersect,

$$\begin{aligned} x^2 - 8x + 17 &= -x^2 + 4x + 7 \\ 2x^2 - 12x + 10 &= 0 \\ x^2 - 6x + 5 &= 0 \\ (x-5)(x-1) &= 0, \end{aligned}$$

and so the two parabolas intersect at $P(5,2)$ and $Q(1,10)$.
Next, we want to show why quadrilateral V_1PV_2Q is a parallelogram.
To do this, we will use the property that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
The midpoint of diagonal V_1V_2 is $\left(\frac{4+2}{2}, \frac{1+11}{2}\right)$ or $(3,6)$, and the midpoint of diagonal PQ is $\left(\frac{5+1}{2}, \frac{2+10}{2}\right)$ or $(3,6)$.
Since the midpoint of each diagonal is the same point, $(3,6)$, then the diagonals bisect each other and so quadrilateral V_1PV_2Q is a parallelogram.
(Note that we could have also shown that each pair of opposite sides of V_1PV_2Q is parallel.)

(b)(i) By completing the square, the equation defining the parabola $y = -x^2 + bx + c$ becomes

$$\begin{aligned} y &= -x^2 + bx + c \\ &= -(x^2 - bx) + c \\ &= -(x^2 - bx + \frac{b^2}{4} - \frac{b^2}{4}) + c \\ &= -(x^2 - bx + \frac{b^2}{4}) + \frac{b^2}{4} + c \\ &= -(x - \frac{b}{2})^2 + \frac{b^2}{4} + c. \end{aligned}$$

The vertex of this parabola is $V_3\left(\frac{b}{2}, \frac{b^2}{4} + c\right)$ and the vertex of the parabola defined by the equation $y = x^2$ is $V_4(0,0)$.

First, we determine the conditions on b and c so that the points of intersection R and S exist and are distinct from one another.
When the two parabolas intersect, $-x^2 + bx + c = x^2$ or $2x^2 - bx - c = 0$.
This equation has two distinct real roots when its discriminant is greater than 0, or when $b^2 - 4(2)(-c) > 0$.
The points of intersection, R and S, exist and are distinct from one another when $c > \frac{-b^2}{8}$.
Next, we determine conditions on b and c so that each of R and S are distinct from both

V_3 and V_4.
The roots of the equation $2x^2 - bx - c = 0$ are given by the quadratic formula, and so $x = \dfrac{b \pm \sqrt{b^2+8c}}{4}$.
We let the x-coordinate of R be $x_1 = \dfrac{b+\sqrt{b^2+8c}}{4}$ and the x-coordinate of S be $x_2 = \dfrac{b-\sqrt{b^2+8c}}{4}$.
Each of the points R and S is not distinct from V_4 when $\dfrac{b \pm \sqrt{b^2+8c}}{4} = 0$ or $b = \mp\sqrt{b^2+8c}$ or $b^2 = b^2 + 8c$, and so $c = 0$.
Thus, we require that $c \neq 0$.
Similarly, each of the points R and S is not distinct from V_3 when $\dfrac{b \pm \sqrt{b^2+8c}}{4} = \dfrac{b}{2}$ or $b \pm \sqrt{b^2+8c} = 2b$ or $\pm\sqrt{b^2+8c} = b$ or $b^2 + 8c = b^2$, and so $c = 0$.
As before, we require that $c \neq 0$.
(Note that since R and V_4 lie on the same parabola, then if their x-coordinates are not equal, then they are distinct points – that is, we need not consider their y-coordinates. The same is true for points S and V_4, R and V_3, and S and V_3.)

Finally, we require that the vertices of the parabolas, $V_3\left(\dfrac{b}{2}, \dfrac{b^2}{4} + c\right)$ and $V_4(0,0)$, be distinct from one another.
Vertices V_3 and V_4 are distinct provided that if their x-coordinates are equal, then their y-coordinates are not equal (V_3 and V_4 lie on different parabolas and so we must consider both x- and y-coordinates).
If $\dfrac{b}{2} = 0$ or $b = 0$, then $\dfrac{b^2}{4} + c = \dfrac{0^2}{4} + c = c$, and since we have the requirement (from earlier) that $c \neq 0$, then vertices V_3 and V_4 are certainly distinct when $c \neq 0$.

If the two conditions $c > \dfrac{-b^2}{8}$ and $c \neq 0$ are satisfied, then for all pairs (b, c), the points R and S exist, and the points V_3, V_4, R, S are distinct.

(b)(ii) We begin by assuming that the conditions on b and c from part (b)(i) above are satisfied.
Thus, the points R and S exist, and the points V_3, V_4, R, S are distinct.
For quadrilateral V_3RV_4S to be a rectangle, it is sufficient to require that it be a parallelogram that has at least one pair of adjacent sides that are perpendicular to each other.
From (b)(i), the parabolas intersect at $R(x_1, x_1^2)$ and $S(x_2, x_2^2)$ (R and S each lie on the parabola $y = x^2$, and thus the y-coordinates are x_1^2 and x_2^2, respectively).
Recall that x_1 and x_2 are the distinct real roots of the quadratic equation $2x^2 - bx - c = 0$.
The sum of the roots of the general quadratic equation $Ax^2 + Bx + C = 0$ is equal to $\dfrac{-B}{A}$, and so $x_1 + x_2 = \dfrac{b}{2}$.
The product of the roots of the general quadratic equation $Ax^2 + Bx + C = 0$ is equal to $\dfrac{C}{A}$, and so $x_1x_2 = \dfrac{-c}{2}$.

First, we will show that quadrilateral V_3RV_4S is a parallelogram since its diagonals bisect each other.

The midpoint of diagonal V_3V_4 is $\left(\dfrac{\frac{b}{2}+0}{2}, \dfrac{\frac{b^2}{4}+c+0}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2}{8}+\dfrac{c}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2+4c}{8}\right)$.
The midpoint of diagonal RS is $\left(\dfrac{x_1+x_2}{2}, \dfrac{x_1^2+x_2^2}{2}\right)$.
However, $x_1+x_2=\dfrac{b}{2}$ and $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2=\left(\dfrac{b}{2}\right)^2-2\left(\dfrac{-c}{2}\right)$, and so the midpoint of RS is $\left(\dfrac{\frac{b}{2}}{2}, \dfrac{\left(\frac{b}{2}\right)^2+c}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2}{8}+\dfrac{c}{2}\right)$ or $\left(\dfrac{b}{4}, \dfrac{b^2+4c}{8}\right)$.
Since the midpoint of diagonal V_3V_4 is equal to the midpoint of diagonal RS, then the diagonals bisect each other, and so V_3RV_4S is a parallelogram.

Next, we require that any one pair of adjacent sides of quadrilateral V_3RV_4S be perpendicular to each other. (This will mean that all pairs of adjacent sides are perpendicular.)
The slope of V_4S is $\dfrac{x_2^2-0}{x_2-0}=x_2$ since $x_2\neq 0$ ($S(x_2,x_2^2)$ and $V_4(0,0)$ are distinct points).
Similarly, the slope of V_4R is $\dfrac{x_1^2-0}{x_1-0}=x_1$ since $x_1\neq 0$ ($R(x_1,x_1^2)$ and $V_4(0,0)$ are distinct points).
Sides V_4S and V_4R are perpendicular to each other if the product of their slopes, x_1x_2, is equal to -1.
Since $x_1x_2=\dfrac{-c}{2}$, then $\dfrac{-c}{2}=-1$, and so $c=2$.

In addition to the condition that $c=2$, the two conditions from part (b)(i), $c>\dfrac{-b^2}{8}$ and $c\neq 0$, must also be satisfied.
Clearly if $c=2$, then $c\neq 0$.
Further, when $c=2$, $c>\dfrac{-b^2}{8}$ becomes $2>\dfrac{-b^2}{8}$ or $b^2>-16$ which is true for all real values of b.
The points R and S exist, the points V_3, V_4, R, S are distinct, and quadrilateral V_3RV_4S is a rectangle for all pairs (b,c) where $c=2$ and b is any real number.

The CENTRE for EDUCATION
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2017 Hypatia Contest

Wednesday, April 12, 2017
(in North America and South America)

Thursday, April 13, 2017
(outside of North America and South America)

Solutions

1. (a) Since $ABCD$ is a cyclic quadrilateral, $\angle DCB + \angle DAB = 180^\circ$ or $(2u)^\circ + 88^\circ = 180^\circ$ or $2u = 92$ and so $u = 46$.

   (b) Since $STQR$ is a cyclic quadrilateral, $\angle SRQ + \angle STQ = 180^\circ$ or $x^\circ + 58^\circ = 180^\circ$ and so $x = 122$.
   Since $PQRS$ is a cyclic quadrilateral, $\angle SPQ + \angle SRQ = 180^\circ$ or $y^\circ + x^\circ = 180^\circ$ or $y + 122 = 180$ and so $y = 58$.

   (c) In $\triangle JKL$, $KJ = KL$ and so $\angle KLJ = \angle KJL = 35^\circ$ ($\triangle JKL$ is isosceles).
   In $\triangle JKL$, $\angle JKL = 180^\circ - 2(35^\circ) = 110^\circ$.
   Since $JKLM$ is a cyclic quadrilateral, $\angle JML + \angle JKL = 180^\circ$ or $\angle JML + 110^\circ = 180^\circ$, and so $\angle JML = 70^\circ$.
   In $\triangle JLM$, $LJ = LM$ and so $\angle MJL = \angle JML = 70^\circ$ ($\triangle JLM$ is isosceles).
   In $\triangle JLM$, $\angle JLM = 180^\circ - 2(70^\circ) = 40^\circ$, and so $w = 40$.

   (d) *Solution 1*
   Since $DEFG$ is a cyclic quadrilateral, $\angle DGF + \angle DEF = 180^\circ$ or $\angle DGF + z^\circ = 180^\circ$ and so $\angle DGF = 180^\circ - z^\circ$.
   Since FGH is a straight angle, then $\angle DGF + \angle DGH = 180^\circ$ or $(180^\circ - z^\circ) + \angle DGH = 180^\circ$ or $\angle DGH = 180^\circ - 180^\circ + z^\circ = z^\circ$.

   Solution 2
   Since $DEFG$ is a cyclic quadrilateral, $\angle DGF + \angle DEF = 180^\circ$.
   Since FGH is a straight angle, then $\angle DGF + \angle DGH = 180^\circ$.
   Therefore, $\angle DGF + \angle DGH = \angle DGF + \angle DEF$, and so $\angle DGH = \angle DEF = z^\circ$.

2. (a) We complete Row 5 of the table, as shown.
   After having completed n rows, n^2 integers have been written.
   Therefore, after having completed 5 rows, $5^2 = 25$ integers have been written.
   The 25^{th} integer written in the table is the last integer in Row 5, which is 9.

Row 1	1								
Row 2	1	2	3						
Row 3	1	2	3	4	5				
Row 4	1	2	3	4	5	6	7		
Row 5	1	2	3	4	5	6	7	8	9
$\vdots$									

   (b) After having completed 10 rows, $10^2 = 100$ integers have been written in the table.
   Therefore, the 100^{th} integer written in the table is the last integer in Row 10.
   Row n ends at the n^{th} odd integer.
   Beginning at 1, the first odd integer is $1 = 2(1) - 1$, the second odd integer is $3 = 2(2) - 1$, the third odd integer is $5 = 2(3) - 1$.
   Beginning at 1, the n^{th} odd integer is $2n - 1$ (1 less than the n^{th} even integer $2n$).
   Row 10 ends at the 10^{th} odd integer, which is $2(10) - 1 = 19$.
   Therefore, the 100^{th} integer written in the table is 19.

   (c) After completing 44 rows, $44^2 = 1936$ integers have been written in the table.
   After completing 45 rows, $45^2 = 2025$ integers have been written in the table.
   Therefore, the 2017^{th} integer written in the table is in Row 45.
   The final integer written in Row 45 is the 45^{th} odd integer, which is $2(45) - 1 = 89$.
   The 2025^{th} integer written in the table is 89, and so the 2017^{th} integer written in the table is $2025 - 2017 = 8$ less than 89, or $89 - 8 = 81$.

   (d) Each row, after the first, contains all of the integers that were written in the previous row, followed by the next two consecutive integers.
   For example, Row 3 contains all integers that were written in Row 2 (that is, $1, 2, 3$), followed by the next two consecutive integers, 4 and 5.

Therefore, the first time an integer is written in the table, it appears as the last integer in a row, or as the second last integer in a row.
Since each row ends at the n^{th} odd integer, then the last integer in each row is odd, and the second last integer in each row is even.
The first time the integer 96 appears in the table, it is the second last integer written in a row (since 96 is even), and 97 is the last integer written in that same row.
Since Row n ends with the n^{th} odd integer, $2n-1$, then when $2n-1=97$, we get $2n=98$ and so $n=49$.
That is, Row 49 ends with the integer 97 and so the second last integer written in Row 49 is 96.
Since 96 first appears in Row 49 of the table, then 96 appears in every row following Row 49 and does not appear in any row before Row 49.
Therefore, the integer 96 appears in $200-48=152$ of the first 200 rows of the table.

3. (a) When the line $y=-15$ intersects the parabola with equation $y=-x^2+2x$, the x-coordinates of the two points of intersection satisfy the equation $-15=-x^2+2x$.
Solving this equation, we get $x^2-2x-15=0$ or $(x+3)(x-5)=0$, and so $x=-3$ or $x=5$.
Since both points of intersection lie on the line $y=-15$, then the coordinates of the two points of intersection are $(-3,-15)$ and $(5,-15)$.

(b) The point with x-coordinate 4, on the parabola with equation $y=-x^2-3x$, has y-coordinate $-4^2-3(4)=-28$.
Therefore, the line intersects the parabola at the point $(4,-28)$.
The line passes through the point $(0,8)$, and so the line has slope $\dfrac{-28-8}{4-0}=\dfrac{-36}{4}=-9$.
The line has slope -9 and y-intercept 8, and so the equation of the line is $y=-9x+8$.
When the line $y=-9x+8$ intersects the parabola with equation $y=-x^2-3x$, the x-coordinates of the two points of intersection satisfy the equation $-9x+8=-x^2-3x$.
Solving this equation, we get $x^2-6x+8=0$ or $(x-2)(x-4)=0$, and so $x=2$ or $x=4$.
Therefore, the line intersects the parabola at $x=4$ and at $x=2$, and so $a=2$.

(c) The point with x-coordinate p, on the parabola with equation $y=-x^2+kx$, has y-coordinate $-p^2+kp$.
Therefore, the line intersects the parabola at the point $(p,-p^2+kp)$.
Similarly, the line also intersects the parabola at the point $(q,-q^2+kq)$.
The slope of the line passing through the points $(p,-p^2+kp)$ and $(q,-q^2+kq)$ is $\dfrac{(-p^2+kp)-(-q^2+kq)}{p-q}$, where $p\neq q$ and so $p-q\neq 0$.

Simplifying this slope, we get

$$\begin{aligned}\frac{(-p^2+kp)-(-q^2+kq)}{p-q} &= \frac{q^2-p^2+kp-kq}{p-q}\\ &= \frac{(q-p)(q+p)+k(p-q)}{p-q}\\ &= \frac{(q-p)(q+p)}{p-q}+\frac{k(p-q)}{p-q}\\ &= \frac{-(p-q)(q+p)}{p-q}+\frac{k(p-q)}{p-q}\\ &= -(q+p)+k\\ &= k-q-p\end{aligned}$$

The line has slope $k-q-p$ and passes through the point $(p,-p^2+kp)$.
Therefore, the equation of the line is $y-(-p^2+kp)=(k-q-p)(x-p)$.
(The equation of a line having slope m and passing through the point (x_1,y_1) is $y-y_1=m(x-x_1)$. This is called the point-slope form of a line.)
Finally, we determine the y-intercept of the line by substituting $x=0$ into the equation of the line $y-(-p^2+kp)=(k-q-p)(x-p)$ and solving for y.

$$\begin{aligned}y-(-p^2+kp) &= (k-q-p)(x-p)\\ y-(-p^2+kp) &= (k-q-p)(0-p)\\ y+p^2-kp &= -kp+pq+p^2\\ y &= -p^2+kp-kp+pq+p^2\\ y &= pq\end{aligned}$$

The y-intercept of the line that intersects the parabola with equation $y=-x^2+kx$ at $x=p$ and at $x=q$ with $p\neq q$, is pq.

(d) When the curve $x=\frac{1}{k^3}y^2+\frac{1}{k}y$ intersects the parabola with equation $y=-x^2+kx$, the x-coordinates of the two points of intersection ($(0,0)$ and T) satisfy the equation $x=\frac{1}{k^3}(-x^2+kx)^2+\frac{1}{k}(-x^2+kx)$, where $k\neq 0$.
Simplifying this equation, we get

$$\begin{aligned}x &= \frac{1}{k^3}(-x^2+kx)^2+\frac{1}{k}(-x^2+kx)\\ k^3x &= (-x^2+kx)^2+k^2(-x^2+kx)\\ k^3x &= x^4-2kx^3+k^2x^2-k^2x^2+k^3x\\ 0 &= x^4-2kx^3\\ 0 &= x^3(x-2k)\end{aligned}$$

Since $x^3(x-2k)=0$, then the x-coordinates of the points of intersection of the curve and the parabola are $x=0$ and $x=2k$.
Therefore, the x-coordinate of point T is $x=2k$, and the y-coordinate of T is $-(2k)^2+k(2k)=-4k^2+2k^2=-2k^2$.
Since the y-coordinate of point T does not contain a linear term in the variable k and does not contain a constant term, then the equation of the parabola on which all such points

T lie, contains a quadratic term only.
That is, all points $T(2k, -2k^2)$ lie on a parabola with an equation of the form $y = ax^2$.
Substituting, we get $-2k^2 = a(2k)^2$ or $-2k^2 = 4ak^2$ or $-2 = 4a$ (since $k \neq 0$), and so $a = -\frac{1}{2}$.
(We may verify that $x = 2k$ and $y = -2k^2$ satisfies $y = ax^2 + bx + c$ only if $a = -\frac{1}{2}$ and $b = c = 0$.)
Therefore, the equation of the required parabola is $y = -\frac{1}{2}x^2$.

4. (a) Let $N = abcdefghi$ be the largest 9-digit zigzag number.
We will determine N by assigning each of the digits $1, 2, 3, 4, 5, 6, 7, 8,$ and 9 to $a, b, c, d, e, f, g, h,$ and i.
We begin by trying to construct a zigzag number with $a = 9$, since any other choice will give us a smaller zigzag number.
It cannot be that $b = 8$ since any of the remaining possible choices for c gives $a > b > c$.
That is, the middle of the first three adjacent digits would be less than the first digit and greater than the third digit.
In our attempt to find the maximum possible zigzag number, we choose b to equal the next largest possible number, 7.
Since $a > b$, then it must be that $b < c$, which means that $c = 8$ (since 9 has already been assigned).
At this point, we have

$$N = 978defghi.$$

If the largest 9-digit zigzag number starts with 9, then it must begin with 978.
We continue to assign values to digits, ensuring that for each group of three adjacent digits, either the middle digit is greater than each of the other two digits or the middle digit is less than each of the other two digits.
Since $b < c$, then $c > d$.
This tells us that $d < e$ and since N is as large as possible, we choose $d = 5$ and $e = 6$ to give

$$N = 97856fghi.$$

Since $d < e$, then $e > f$ and thus $f < g$.
Of the remaining digits, we choose $f = 3$ and $g = 4$ to make N as large as possible.
Finally, since $f < g$, then $g > h$ and thus $h < i$ so we choose $h = 1$ and $i = 2$.
Therefore, the largest 9-digit zigzag number is $N = 978563412$.

(b) We proceed by proving several facts.

<u>Fact 1: $G(6, 2) = L(6, 5)$</u>

Consider a 6-digit zigzag number counted by $G(6, 2)$, say $n = 251634$.
We form a new 6-digit number by subtracting each of the digits of n from 7 to obtain $N = 526143$. Note that N is a 6-digit zigzag number and one that is counted by $L(6, 5)$.
Consider now an arbitrary 6-digit zigzag number counted by $G(6, 2)$, say $n = 2bcdef$, where b, c, d, e, f are the digits $1, 3, 4, 5, 6$ in some order so that $2 < b$, $b > c$, $c < d$, $d > e$, and $e < f$.
We form a new 6-digit number by subtracting each of the digits of n from 7 to obtain $N = 5(7 - b)(7 - c)(7 - d)(7 - e)(7 - f)$.
Since b, c, d, e, f are the digits $1, 3, 4, 5, 6$ in some order, then $7 - b, 7 - c, 7 - d, 7 - e, 7 - f$

are the digits $6, 4, 3, 2, 1$ in some order.
Since $2 < b$, then $-2 > -b$ and so $7 - 2 > 7 - b$ or $5 > 7 - b$.
Since $b > c$, then $-b < -c$ and so $7 - b < 7 - c$.
Similarly, $7 - c > 7 - d$ and $7 - d < 7 - e$ and $7 - e > 7 - f$.
Therefore, N is a 6-digit zigzag number and one that is counted by $L(6, 5)$.
Also, if $2bcdef$ and $2BCDEF$ are two different zigzag numbers counted by $G(6, 2)$, then one of the following must be true: $b \neq B$ or $c \neq C$ or $d \neq D$ or $e \neq E$ or $f \neq F$.
This means that $5(7-b)(7-c)(7-d)(7-e)(7-f)$ and $5(7-B)(7-C)(7-D)(7-E)(7-F)$ will be different numbers counted by $G(6, 2)$ as at least one pair of corresponding digits will be unequal.
In other words, each zigzag number counted by $G(6, 2)$ corresponds to a different zigzag number counted by $L(6, 5)$, and so $G(6, 2) \leq L(6, 5)$. (There could be zigzag numbers counted by $L(6, 5)$ that are not achieved by using this process.)
However, we can apply the same process to an arbitrary zigzag number counted by $L(6, 5)$ to obtain one counted by $G(6, 2)$, and thus show that $L(6, 5) \leq G(6, 2)$.
In other words, $G(6, 5) = L(6, 2)$.

Fact 2: $G(6, a) = L(6, 7 - a)$ for $a = 1, 2, 3, 4, 5, 6$

Using a similar argument to that in Fact #1, we can show that $G(6, a) = L(6, 7 - a)$ for each of $a = 1, 2, 3, 4, 5, 6$.

We can now prove the statement from (ii):

$$\begin{aligned} & G(6,1) + G(6,2) + G(6,3) + G(6,4) + G(6,5) + G(6,6) \\ = \; & L(6,6) + L(6,5) + L(6,4) + L(6,3) + L(6,2) + L(6,1) \\ = \; & L(6,1) + L(6,2) + L(6,3) + L(6,4) + L(6,5) + L(6,6) \end{aligned}$$

as required.

We note further that we can generalize Fact 2 in a way that will be useful in (c).

Fact 3: $G(n, a) = L(n, n + 1 - a)$ for each pair of integers a and n with $1 \leq a \leq n \leq 9$

To see this, we use a similar argument to that from Fact 1, instead subtracting each digit of a zigzag number counted by $G(n, a)$ from $n+1$ to obtain a corresponding zigzag number counted by $L(n, (n + 1) - a)$.

We now prove that $G(6, 3) = L(5, 3) + L(5, 4) + L(5, 5)$.

Fact 4: The number of zigzag numbers of the form $35cdef$ equals $L(5, 4)$

We use $G(6, 35)$ to represent the number of zigzag numbers of the form $35cdef$. (We will use analogous notation later as well.)
Consider a 6-digit zigzag number counted by $G(6, 35)$, say $n = 351624$.
We form a 5-digit number by deleting the first digit of n to obtain 51624 and then replacing the 4, 5 and 6 with 3, 4 and 5, respectively, to obtain $N = 41523$.
Note that N is a 5-digit zigzag number and one that is counted by $L(5, 4)$.
Consider now an arbitrary 6-digit zigzag number counted by $G(6, 35)$, say $n = 35cdef$, where c, d, e, f are the digits $1, 2, 4, 6$ in some order so that $3 < 5$, $5 > c$, $c < d$, $d > e$, and $e < f$.
We form a new 6-digit number by deleting the first digit of n to obtain $5cdef$ (whose digits

are $1, 2, 4, 5, 6$ in some order).
Note that we have $5 > c$ and $c < d$ and $d > e$ and $e < f$.
We then replace the digits $4, 5, 6$ with $3, 4, 5$ respectively to obtain $4c'd'e'f'$.
Since the digits of the $5cdef$ are $1, 2, 4, 5, 6$ and the digits of $4c'd'e'f'$ are $1, 2, 3, 4, 5$ (arranged in the same order), then we have not changed the relative ordering of the digits of the number, so we will still have $5 > c'$ and $c' < d'$ and $d' > e'$ and $e' < f'$.
Therefore, N is a 5-digit zigzag number and one that is counted by $L(5, 4)$.
Also, if $35cdef$ and $35CDEF$ are two different zigzag numbers counted by $G(6, 3)$, then one of the following must be true: $b \neq B$ or $c \neq C$ or $d \neq D$ or $e \neq E$ or $f \neq F$.
This means that $4c'd'e'f'$ and $4C'D'E'F'$ will be different numbers counted by $L(5, 4)$. (For example, if $c' = C'$, then we must have had $c = C$.)
In other words, each zigzag number counted by $G(6, 35)$ corresponds to a different zigzag number counted by $L(5, 4)$, and so $G(6, 35) \leq L(5, 4)$.
However, we can apply the reverse process to an arbitrary zigzag number counted by $L(5, 4)$ (change $3, 4, 5$ to $4, 5, 6$ and put a 3 on the front) to obtain one counted by $G(6, 35)$, and thus show that $L(5, 4) \leq G(6, 35)$.
In other words, $G(6, 35) = L(5, 4)$.

Fact 5: $G(6, 3a) = L(5, a - 1)$ for each of $a = 4, 5, 6$

Using a similar argument to that in Fact #4, we can show that $G(6, 3a) = L(5, a - 1)$ for each of $a = 4, 5, 6$.

We can now prove the statement from (i), noting that the second digit of a zigzag number counted by $G(6, 3)$ must be 4, 5 or 6 and so each zigzag number counted by $G(6, 3)$ must be of the form $34cdef$ or $35cdef$ or $36cdef$:

$$G(6, 3) = G(6, 34) + G(6, 35) + G(6, 36) = L(5, 3) + L(5, 4) + L(5, 5)$$

as required.

Before concluding (b), we note the following generalization of Fact 5 that will be useful in (c):

Fact 6: $G(n, ab) = L(n - 1, b - 1)$ for all integers n, a, b with $1 \leq a < b \leq n \leq 9$

To see this, we use a similar argument to that from Fact 5, again removing the first digit a and reducing all digits from $a + 1$ to n, inclusive, by 1 to obtain digits a to $n - 1$.

(c) Let T be the total number of 8-digit zigzag numbers. An 8-digit zigzag number starts with one of $1, 2, 3, 4, 5, 6, 7, 8$ and its second digit is either less than or greater than its first digit.
Therefore, T equals the sum of

$$\ell = L(8, 1) + L(8, 2) + L(8, 3) + L(8, 4) + L(8, 5) + L(8, 6) + L(8, 7) + L(8, 8)$$

and

$$g = G(8, 1) + G(8, 2) + G(8, 3) + G(8, 4) + G(8, 5) + G(8, 6) + G(8, 7) + G(8, 8)$$

From Fact 3, $L(8, 1) = G(8, 8)$ and $L(8, 2) = G(8, 7)$ and $L(8, 3) = G(8, 6)$ and so on.
Therefore,

$$\ell = G(8, 8) + G(8, 7) + G(8, 6) + G(8, 5) + G(8, 4) + G(8, 3) + G(8, 2) + G(8, 1) = g$$

and so $T = 2(G(8,1) + G(8,2) + G(8,3) + G(8,4) + G(8,5) + G(8,6) + G(8,7) + G(8,8))$.
We calculate the value of each of the terms on the right side by building a table of values of $G(n,k)$ for $3 \leq n \leq 8$ and $1 \leq k \leq 8$.
Note that if $k \geq n$, then $G(n,k) = 0$ as no n-digit zigzag number can begin with $k > n$ and if $k = n$, an n-digit zigzag number starting with $n = k$ cannot have its second digit greater than its first.
Also, the only 3-digit zigzag numbers are 132, 231, 213 and 312 because the two other ways of arranging the digits 1, 2 and 3 (123 and 321) do not satisfy the zigzag property.
Thus $G(3,1) = 1$, $G(3,2) = 1$ and $G(3,3) = 0$.
This gives us the following start to the table:

n \ k	1	2	3	4	5	6	7	8
3	1	1	0	0	0	0	0	0
4				0	0	0	0	0
5					0	0	0	0
6						0	0	0
7							0	0
8								0

To complete the table, we build a relationship between the values of G in one row and the values of G in the previous row.
For a positive digit n with $4 \leq n \leq 8$ and a positive digit $k < n$, we have:

$$\begin{aligned} G(n,k) &= G(n,k(k+1)) + G(n,k(k+2)) + \cdots + G(n,k(n-1)) + G(n,kn) \\ &\qquad \text{(since the second digit must be larger than } k\text{)} \\ &= L(n-1,k) + L(n-1,k+1) + \cdots + L(n-1,n-2) + L(n-1,n-1) \quad \text{(by Fact 6)} \\ &= G(n-1,n-k) + G(n-1,n-k-1) + \cdots + G(n-1,2) + G(n-1,1) \quad \text{(by Fact 3)} \end{aligned}$$

Using this formula,

$$\begin{aligned} G(4,1) &= G(3,3) + G(3,2) + G(3,1) = 2 \\ G(4,2) &= G(3,2) + G(3,1) = 2 \\ G(4,3) &= G(3,1) = 1 \\ G(5,1) &= G(4,4) + G(4,3) + G(4,2) + G(4,1) = 5 \\ G(5,2) &= G(4,3) + G(4,2) + G(4,1) = 5 \\ G(5,3) &= G(4,2) + G(4,1) = 4 \\ &\vdots \end{aligned}$$

Proceeding in this way, we complete the table row by row and obtain:

n \ k	1	2	3	4	5	6	7	8
3	1	1	0	0	0	0	0	0
4	2	2	1	0	0	0	0	0
5	5	5	4	2	0	0	0	0
6	16	16	14	10	5	0	0	0
7	61	61	56	46	32	16	0	0
8	272	272	256	224	178	122	61	0

Finally, $T = 2(272 + 272 + 256 + 224 + 178 + 122 + 61) = 2770$, and so the number of 8-digit zigzag numbers is 2770.

The CENTRE for EDUCATION
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2016 Hypatia Contest

Wednesday, April 13, 2016
(in North America and South America)

Thursday, April 14, 2016
(outside of North America and South America)

Solutions

1. (a) Since 5 baskets of raisins fill 2 tubs, then $5 \times 6 = 30$ baskets of raisins fill $2 \times 6 = 12$ tubs. Therefore, 12 tubs of raisins fill 30 baskets.

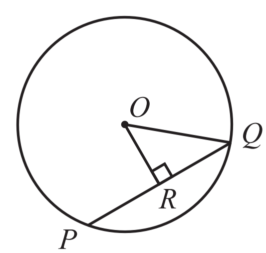
   (b) Since 5 scoops of raisins fill 1 jar, then $5 \times 6 = 30$ scoops of raisins fill $1 \times 6 = 6$ jars.
   Since 3 scoops of raisins fill 1 cup, then $3 \times 10 = 30$ scoops of raisins fill $1 \times 10 = 10$ cups.
   Since 30 scoops fill 6 jars, and 30 scoops fill 10 cups, then 10 cups of raisins fill 6 jars.

   (c) *Solution 1*
   From part (b), we know that 10 cups of raisins fill 6 jars.
   Thus, $10 \times 5 = 50$ cups of raisins fill $6 \times 5 = 30$ jars.
   Since 30 jars of raisins fill 1 tub, then 50 cups of raisins fill 1 tub, or $50 \times 2 = 100$ cups of raisins fill $1 \times 2 = 2$ tubs.
   Since 2 tubs of raisins fill 5 baskets, then 100 cups of raisins fill 5 baskets.
   This tells us that $100 \div 5 = 20$ cups of raisins fill $5 \div 5 = 1$ basket.

   Solution 2
   Since 5 baskets fill 2 tubs, then $\frac{2}{5}$ tubs fill 1 basket.
   Since 30 jars of raisins fill 1 tub, then $\frac{2}{5} \times 30 = 12$ jars of raisins fill $\frac{2}{5}$ tubs and so fill 1 basket.
   Since 5 scoops of raisins fill 1 jar, then $12 \times 5 = 60$ scoops of raisins fill 12 jars and so fill 1 basket.
   Since 3 scoops of raisins fill 1 cup, then $20 \times 1 = 20$ cups fill $20 \times 3 = 60$ scoops and so fill 1 basket.
   Therefore, 20 cups of raisins fill 1 basket.

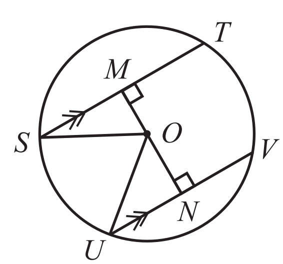
2. (a) Since M is the midpoint of chord AB, then $AM = \frac{1}{2}(AB) = 5$.
   Also, since M is the midpoint of chord AB, then OM is perpendicular to AB.
   Using the Pythagorean Theorem in $\triangle OMA$, we get $OM^2 = OA^2 - AM^2$ or $OM^2 = 13^2 - 5^2 = 169 - 25 = 144$, and so $OM = \sqrt{144} = 12$ (since $OM > 0$).

   (b) Let the circle have centre O and chord PQ, as shown.
   Since the radius is 25, then $OQ = 25$.
   The perpendicular distance from O to the chord is given by OR, and so $OR = 7$.
   In $\triangle ORQ$, the Pythagorean Theorem gives $RQ^2 = OQ^2 - OR^2$ or $RQ^2 = 25^2 - 7^2 = 625 - 49 = 576$, and so $RQ = \sqrt{576} = 24$ (since $RQ > 0$).



   Since OR is perpendicular to the chord PQ, then R is the midpoint of PQ, and so $PQ = 2(RQ) = 2(24) = 48$.
   Therefore, the length of the chord is 48.

   (c) Join O to S and O to U, as shown.
   The radius of the circle is 65, and so $OS = OU = 65$.
   Since OM is perpendicular to chord ST, then M is the midpoint of the chord and so $MS = \frac{1}{2}(ST) = \frac{1}{2}(112) = 56$.
   In $\triangle OMS$, the Pythagorean Theorem gives $OM^2 = OS^2 - MS^2$ or $OM^2 = 65^2 - 56^2 = 4225 - 3136 = 1089$, and so $OM = \sqrt{1089} = 33$ (since $OM > 0$).
   Since $MN = OM + ON = 72$, then $ON = 72 - OM = 72 - 33 = 39$.



   In $\triangle ONU$, the Pythagorean Theorem gives $NU^2 = OU^2 - ON^2$ or $NU^2 = 65^2 - 39^2 = 4225 - 1521 = 2704$, and so $NU = \sqrt{2704} = 52$ (since $NU > 0$).

Finally, since ON is perpendicular to chord UV, then N is the midpoint of the chord and so $UV = 2(NU) = 2(52) = 104$.
Therefore, the length of the chord UV is 104.

3. (a) Since $405 = 3^4 \times 5$, then 405 is divisible by 3^4 but is not divisible by 3^5.
Thus, $f(405) = 4$.

(b) First, we find all factors of 3 which exist in the product $1\times2\times3\times4\times5\times6\times7\times8\times9\times10$.
The multiples of 3 are the only numbers which contain factors of 3.
The multiples of 3 in the given product are 3, 6 and 9.
Rewriting the given product, we get

$$\begin{aligned} &1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \\ &\quad = \quad 1 \times 2 \times 3 \times 4 \times 5 \times (2 \times 3) \times 7 \times 8 \times (3 \times 3) \times 10 \\ &\quad = \quad 3^4 \times (1 \times 2 \times 4 \times 5 \times 2 \times 7 \times 8 \times 10). \end{aligned}$$

Since the product in parentheses does not include any factors of 3, then the largest power of 3 which divides the given product is 3^4, and so $f(1\times2\times3\times4\times5\times6\times7\times8\times9\times10) = 4$.

(c) First, we count the number of factors of 3 included in 100!.
Every multiple of 3 includes least 1 factor of 3.
The product 100! includes 33 multiples of 3 (since $33 \times 3 = 99$).
Counting one factor of 3 from each of the multiples of 3 (these are $3, 6, 9, 12, 15, 18, \ldots, 93, 96, 99$), we see that 100! includes at least 33 factors of 3.
However, each multiple of $3^2 = 9$ includes a second factor of 3 (since $9 = 3^2, 18 = 3^2 \times 2$, etc.) which was not counted in the previous 33 factors.
The product 100! includes 11 multiples of 9 (since $11\times 9 = 99$), and thus there are at least 11 additional factors of 3 in 100!.
Similarly, 100! includes 3 multiples of $3^3 = 27$, each of which contribute an additional factor of 3 (these are $27 = 3^3, 54 = 3^3 \times 2$, and $81 = 3^4$).
Finally, there is one multiple of $3^4 = 81$ which contributes one more factor of 3.
Since $3^5 > 100$, then 100! does not include any multiples of 3^5 and so we have counted all possible factors of 3.
Thus, 100! includes exactly $33 + 11 + 3 + 1 = 48$ factors of 3, and so $100! = 3^{48} \times t$ for some positive integer t that is not divisible by 3.
Counting in a similar way, the product 50! includes 16 multiples of 3, 5 multiples of 9, and 1 multiple of 27, and thus includes $16 + 5 + 1 = 22$ factors of 3.
Therefore, $50! = 3^{22} \times r$ for some positive integer r that is not divisible by 3.
Also, 20! includes $6 + 2 = 8$ factors of 3, and thus $20! = 3^8 \times s$ for some positive integer s that is not divisible by 3.
Therefore, $N = \dfrac{100!}{50!20!} = \dfrac{3^{48} \times t}{(3^{22} \times r)(3^8 \times s)} = \dfrac{3^{48} \times t}{(3^{30} \times rs)} = \dfrac{3^{18} \times t}{rs}$.
Since we are given that N is equal to a positive integer, then $\dfrac{3^{18} \times t}{rs}$ is a positive integer.
Since r and s contain no factors of 3 and $3^{18} \times t$ is divisible by rs, then it must be the case that t is divisible by rs.
In other words, we can re-write $N = \dfrac{3^{18} \times t}{rs}$ as $N = 3^{18} \times \dfrac{t}{rs}$ where $\dfrac{t}{rs}$ is an integer.
Since each of r, s and t does not include any factors of 3, then the integer $\dfrac{t}{rs}$ is not

divisible by 3.

Therefore, the largest power of 3 which divides $\dfrac{100!}{50!20!}$ is 3^{18}, and so $f(N) = 18$.

(d) Since $f(a) = 8$, then the exponent of the largest power of 3 that divides a is 8.
That is, $a = 3^8 m$ for some positive integer m and 3 does not divide m.
Since $f(b) = 7$, then the exponent of the largest power of 3 that divides b is 7.
That is, $b = 3^7 n$ for some positive integer n and 3 does not divide n.
Substituting and simplifying, we get

$$a + b = 3^8 m + 3^7 n = 3^7(3m + n)$$

Since 3 divides $3m$ but 3 does not divide n, then 3 does not divide the sum $3m + n$.
That is, $3m+n$ is not a multiple of 3 and so the largest power of 3 that divides $a+b$ is 3^7.
Therefore, $f(a + b) = 7$.

4. (a) (i) For every 10 cents that one restaurant's price is higher than the other restaurant's price, it loses one customer to the other restaurant.
On Monday, LP charges $\$9.30 - \$7.70 = \$1.60$ more per pizza than what EP charges.
Therefore, LP loses $\frac{1.60}{0.10} = 16$ customers to EP and thus has $50 - 16 = 34$ customers.

(ii) The cost for LP to make each pizza is \$5.00, and so LP's profit is $\$9.30-\$5.00 = \$4.30$ for each pizza sold.
On Monday, LP's total profit is $\$4.30 \times 34 = \$146.20$.

(b) *Solution 1*
Let LP's price per pizza on Tuesday be $\$L$, where $L > 0$ and L is an integer multiple of 0.10.
If LP charges $\$L$ per pizza, then its profit is $\$(L - 5)$ per pizza sold.
We note that if $L < 5$, then LP's profit per pizza sold is negative (that is, LP is losing money on each pizza it sells).
Since EP charges \$7.20 per pizza, then the number of customers that LP has is $50 + \dfrac{7.20 - L}{0.10}$.
We note that if $L < 7.20$ (LP charges less per pizza than EP charges), then $\dfrac{7.20 - L}{0.10} > 0$ and LP will have more than 50 customers. In fact, LP gains $\dfrac{7.20 - L}{0.10}$ customers.
Similarly, if $L > 7.20$ (LP charges more per pizza than EP charges), then $\dfrac{7.20 - L}{0.10} < 0$ and LP will have fewer than 50 customers. In fact, LP loses $\dfrac{L - 7.20}{0.10}$ customers.
LP's profit on Tuesday is given by the product of its number of customers and its profit per pizza sold.
That is, LP's profit in dollars, P, is $P = \left(50 + \dfrac{7.20 - L}{0.10}\right) \times (L - 5)$.
Simplifying, we get $P = \left(\dfrac{5 + 7.2 - L}{0.10}\right) \times (L - 5) = 10(12.2 - L)(L - 5)$.
Therefore, P is a quadratic function of L.
The graph of this quadratic function, $P = 10(12.2 - L)(L - 5)$, is a parabola opening downward and thus the maximum profit occurs at its vertex.
The zeros of this parabola occur when $12.2 - L = 0$ (that is, when $L = 12.2$) and when

$L - 5 = 0$ (that is, when $L = 5$).
The vertex of the parabola occurs on its axis of symmetry, which is the vertical line passing through the midpoint of its zeros, $L = 12.2$ and $L = 5$.
That is, the maximum profit occurs when $L = \dfrac{12.2 + 5}{2} = \dfrac{17.2}{2} = 8.60$.
On Tuesday, LP should charge \$8.60 per pizza to maximize their profit.

Solution 2
On Tuesday, EP charges \$7.20 per pizza.
Suppose that, on Tuesday, LP charges $\$(7.20 + 0.10d)$ per pizza for some integer d. (Note that LP's price must be an integer multiple of 10 cents higher or lower than EP's price.)
If $d > 0$, then LP will lose d customers to EP.
If $d < 0$, then LP will gain $-d$ customers from EP.
In other words, on Tuesday, LP will have $50 - d$ customers.
Since it costs LP \$5.00 to make each pizza, then LP's profit per pizza is equal to $\$(7.20 + 0.10d) - \$5.00 = \$(2.20 + 0.10d)$.
Therefore, in dollars, LP's profit on Tuesday is the product of its number of customers and its profit per pizza sold, or $P = (2.20 + 0.10d)(50 - d) = 0.10(22 + d)(50 - d)$.
Therefore, P is a quadratic function of d.
The graph of this quadratic function, $P = 0.10(22 + d)(50 - d)$, is a parabola opening downward and thus the maximum profit occurs at its vertex.
The zeros of this parabola occur when $22 + d = 0$ (that is, when $d = -22$) and when $50 - d = 0$ (that is, when $d = 50$).
The vertex of the parabola occurs on its axis of symmetry, which is the vertical line passing through the midpoint of its zeros, $d = -22$ and $d = 50$.
That is, the maximum profit occurs when $d = \dfrac{(-22) + 50}{2} = 14$.
On Tuesday, LP should charge $\$(7.20 + 0.10(14)) = \$8.60$ per pizza to maximize their profit.

(c) *Solution 1*
Suppose that EP set its price per pizza at $\$E$, where $E > 0$ and E is an integer multiple of 0.20.
After EP sets its price at $\$E$, LP maximizes its profit by setting its price per pizza at $\$L$, where $L > 0$ and L is an integer multiple of 0.10.
Let EP's profit be P_E and LP's profit be P_L.
First we determine the price per pizza, $\$L$, that LP will choose in order to maximize its profit, P_L, given that LP knows that EP has set its price per pizza at $\$E$.
LP's profit per pizza sold is $\$(L - 5)$ and, using a similar method as in (b), its number of customers is $50 + \dfrac{E - L}{0.10}$.
Thus, LP's total profit, in dollars, is given by $P_L = \left(50 + \dfrac{E - L}{0.10}\right) \times (L - 5)$.
Simplifying, we get $P_L = \left(\dfrac{5 + E - L}{0.10}\right) \times (L - 5) = 10(5 + E - L)(L - 5)$.
We think about E as fixed and L as variable, making this a quadratic function in L.
The graph of this quadratic function, $P_L = 10(5 + E - L)(L - 5)$, is a parabola opening downward and thus the maximum profit occurs at its vertex.
The zeros of this parabola occur when $5 + E - L = 0$ (that is, $L = 5 + E$) and when $L - 5 = 0$ (that is, $L = 5$).

The vertex of the parabola occurs on its axis of symmetry, which is the vertical line passing through the midpoint of its zeros, $L = 5 + E$ and $L = 5$.
That is, the maximum profit for LP occurs when $L = \dfrac{5+E+5}{2} = \dfrac{10+E}{2} = 5 + \frac{1}{2}E$.
(Since E is a multiple of 0.20, then L is a multiple of 0.10.)
Thus, if EP first sets its price per pizza at $\$E$, then LP should charge $\$(5+\frac{1}{2}E)$ per pizza to maximize its profit.

Since EP realizes what LP is doing, we can assume that EP now knows that LP will set their price per pizza at $\$(5+\frac{1}{2}E)$.
Thus, EP may determine its price per pizza, $\$E$, that will maximize its profit.
EP's profit per pizza sold is $\$(E-5)$ and its number of customers is $50 + \dfrac{L-E}{0.10}$.
(Since L and E are both multiples of 0.10, then this number is an integer.)
Thus, EP's total profit is given by $P_E = \left(50 + \dfrac{L-E}{0.10}\right) \times (E-5)$.
Simplifying, we get $P_E = \left(\dfrac{5+L-E}{0.10}\right) \times (E-5) = 10(5+L-E)(E-5)$.
Since $L = 5+\frac{1}{2}E$, the quadratic function becomes $P_E = 10(5+(5+\frac{1}{2}E)-E)(E-5)$, or $P_E = 10(10-\frac{1}{2}E)(E-5)$.
This is again a parabola opening downward and so its maximum profit occurs at its vertex.
The zeros of this parabola occur when $E = 20$ and when $E = 5$.
Thus, the maximum profit for EP occurs when $E = \dfrac{20+5}{2} = 12.50$.
However, since E must equal an integer multiple of 0.20, then E cannot equal \$12.50.
Since the quadratic relation P_E is quadratic in E and the resulting parabola opens downward, then values of E closest to the vertex give the largest values corresponding values of P_E.
Therefore, to maximize EP's profit, we choose the closest values to $E = 12.50$ that are multiples of 20 cents.
These values are $E = 12.40$ (which gives $L = 11.20$), and $E = 12.60$ (which gives $L = 11.30$).
We note that $E = 12.40$ and $E = 12.60$ are symmetric about the axis of symmetry, $E = 12.50$, and thus give equal values of $P_E = 281.20$. Further, there are no values of E which satisfy the given conditions and for which P_E is greater in value, since there are no multiples of 20 cents between \$12.40 and \$12.50 or between \$12.60 and \$12.50.
When EP sets its price at $E = 12.40$, LP's profit is $P_L = 10(5+E-L)(L-5)$ or $P_L = 10(5+12.40-11.20)(11.20-5) = 10(6.20)(6.20) = 384.40$.
When EP sets its price at $E = 12.60$, LP's profit is $P_L = 10(5+E-L)(L-5)$ or $P_L = 10(5+12.60-11.30)(11.30-5) = 10(6.30)(6.30) = 396.90$.

To maximize its profit, EP could charge \$12.40 or \$12.60 per pizza, which result in profits for LP of \$384.40 and \$396.90, respectively.

Solution 2
On Wednesday, suppose that EP charges $\$2e$ per pizza, where e is a multiple of 0.10.
Based on this fixed (but unknown) price, LP chooses its price on Wednesday to maximize its profit.
Suppose that, on Wednesday, LP charges $\$(2e+0.10n)$ per pizza for some integer n. (Note that LP's price must be an integer multiple of 10 cents higher or lower than EP's price.)

As in (b), on Wednesday, LP will have $50 - n$ customers.
Since it costs LP \$5.00 to make each pizza, then LP's profit per pizza is equal to $\$(2e + 0.10n) - \$5.00 = \$(2e + 0.10n - 5)$.
Therefore, in dollars, LP's profit on Wednesday is

$$P_L = (2e+0.10n-5)(50-n) = 0.10(20e+n-50)(50-n) = -0.10n^2+(10-2e)n+(100e-250)$$

We treat e as a constant and n as a variable. Therefore, P_L is a quadratic function of n.
Since the coefficient of n^2 is negative, the graph of this quadratic function is a parabola opening downward and thus the maximum profit for LP occurs at its vertex.
The vertex occurs when $n = -\dfrac{10-2e}{2(-0.10)} = 50 - 10e$.
In this case, LP's profit, in dollars, is

$$P_L = 0.10(20e + (50 - 10e) - 50)(50 - (50 - 10e)) = 0.10(10e)(10e) = 10e^2$$

Now, on Wednesday, EP realizes what LP is doing and so sets its initial price, \$$2e$, to maximize EP's profit (knowing that LP will pick its price afterwards to optimize LP's profit).
Since EP's price is set at \$$2e$ per pizza, then its profit per pizza is $\$(2e - 5)$.
Since LP has $50 - n$ customers and there are 100 customers in total, then EP has $100 - (50 - n) = 50 + n = 50 + (50 - 10e) = 100 - 10e$ customers. (From above, we can assume that $n = 50 - 10e$.)
Therefore, in dollars, EP's total profit on Wednesday is

$$P_E = (100 - 10e)(2e - 5) = -20e^2 + 250e - 500 = -20(e^2 - 12.5e + 25)$$

Completing the square, we obtain

$$P_E = -20((e - 6.25)^2 - 6.25^2 + 25) = -20(e - 6.25)^2 + 281.25$$

This is the equation of a parabola opening downwards. Thus, the maximum value of P_E occurs when $e = 6.25$. However, we require that e be a multiple of 0.10.
To find the maximum value(s) of P_E including this constraint, we take the closest values of e to the vertex that are multiples of 0.10. These are $e = 6.20$ and $e = 6.30$.
Since $e = 6.20$ and $e = 6.30$ are symmetric about the vertex $e = 6.25$, then they give the same profit P_E, namely $P_E = 281.20$. Since we have stayed as closed to the vertex as possible, this is EP's maximum possible profit given the constraints.
When $e = 6.20$, EP's price is \$12.40 and LP's profit is $\$10e^2 = \$10(6.20)^2 = \$384.40$.
When $e = 6.30$, EP's price is \$12.60 and LP's profit is $\$10e^2 = \$10(6.30)^2 = \$396.90$.
To maximize its profit, EP should charge \$12.40 or \$12.60 per pizza, which result in profits for LP of \$384.40 and \$396.90, respectively.

The CENTRE for EDUCATION
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2015 Hypatia Contest

Thursday, April 16, 2015
(in North America and South America)

Friday, April 17, 2015
(outside of North America and South America)

Solutions

1. (a) The distance from the end of any car (boxcar or engine car) to the end of the next car is the sum of the 2 m distance between the cars and the 15 m length of a boxcar, or 17 m.
Thus the distance from the end of the engine car to the end of the 10^{th} boxcar (the end of the train) is $10 \times 17 = 170$ m.
Since the engine car has a length of 26 m, then the total length of a train with 10 boxcars is $26 + 170 = 196$ m.

   (b) The total length of a train with n boxcars is $(26 + 15n + 2n)$ m (one 26 m engine car, n 15 m boxcars, and a 2 m distance in front of each of the n boxcars).
That is, the total length of a train with n boxcars is $(26 + 17n)$ m.
If a train has a length of 2015 m, then $26 + 17n = 2015$ or $17n = 1989$ and so $n = 117$.
A train with a total length of 2015 m has 117 boxcars.

   (c) A train with 14 boxcars has length $26 + 17(14) = 264$ m.
The length of time during which a portion of the train is in Canada and a portion of the train is in the United States at the same time is equal to the total length of time it takes the train to cross the border. (When the train is crossing the border, portions of the train are in both countries at the same time.)
The train begins to cross the border when the front of the engine car reaches the border.
The train finishes crossing the border when the end of the last boxcar reaches the border and so the front is 264 m farther.
That is, the length of time required for the train to cross the border is equal to the length of time it takes the train to travel a distance equal to the length of the train, or 264 m.
Since the train is travelling at a speed of 1.6 m/s, then the time required to travel 264 m is $\frac{264}{1.6} = 165$ s.
Therefore, the length of time during which a portion of the train is in Canada and a portion of the train is in the United States at the same time is 165 s.

2. (a) The two-digit positive integers AB and BA equal $10A + B$ and $10B + A$, respectively.
Solving $AB - BA = 72$, we get $(10A + B) - (10B + A) = 72$ or $9A - 9B = 72$ and so $A - B = 8$.
Since A and B are positive digits, then the only possibility for which $A - B = 8$ occurs when $A = 9$ and $B = 1$. (Verify for yourself that this is indeed the only possibility.)
Therefore, the positive integer AB is 91.
(We may check that $AB - BA = 91 - 19 = 72$.)

   (b) The two-digit positive integers MN and NM equal $10M + N$ and $10N + M$, respectively.
Solving $MN - NM = 80$, we get $(10M + N) - (10N + M) = 80$ or $9M - 9N = 80$ and so $9(M - N) = 80$.
Since M and N are positive digits, then $M - N$ is an integer and so $9(M - N)$ is a multiple of 9.
However, 80 is not a multiple of 9 and so $9(M - N) \neq 80$.
Therefore, it is not possible that $MN - NM = 80$.

   (c) The three-digit positive integers PQR and RQP equal $100P+10Q+R$ and $100R+10Q+P$, respectively.
Simplifying $PQR - RQP$, we get $(100P + 10Q + R) - (100R + 10Q + P)$ or $99P - 99R$ or $99(P - R)$.
Since P and R are positive digits, the maximum possible value of $P - R$ is 8 (which occurs when P is as large as possible and R is as small as possible, or $P = 9$ and $R = 1$).
Since $P > R$, the minimum possible value of $P - R$ is 1 (which occurs when $P = 9$ and $R = 8$, for example).

That is, $1 \leq P - R \leq 8$ and so there are exactly 8 possible integer values of $P - R$. (Verify for yourself that there are values for P and R so that $P - R$ is equal to each of the integers from 1 to 8.)
Since $PQR - RQP = 99(P - R)$ and there are exactly 8 possible values of $P - R$, then there are exactly 8 possible values of $PQR - RQP$.
(We note that the value of $PQR - RQP$ does not depend on the value of the digit Q.)

3. (a) Diagram 1 illustrates that $T(3) = 9$.
To determine $T(4)$, add 1 line segment to Diagram 1 as shown in Diagram 2.
We are told that this new (4^{th}) line segment must intersect each of the existing 3 line segments exactly once, creating 3 new points of intersection (labelled $1, 2, 3$).
This 4^{th} line segment also adds 2 new endpoints (labelled 4 and 5) distinct from the previous 3 new points.
In addition, each of the points which exist in the illustration of $T(3)$ (Diagram 1) continue to exist in the illustration of $T(4)$ (Diagram 2) and are distinct from each of the new points which were added.
Therefore, we get

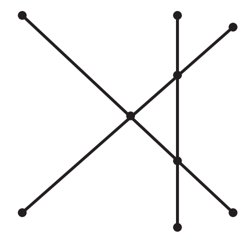

Diagram 1

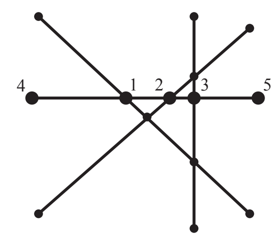

Diagram 2

$$\begin{aligned} T(4) &= T(3) + 3 + 2 \\ &= 9 + 3 + 2 \\ &= 14 \end{aligned}$$

Diagram 2 illustrates that $T(4) = 14$.
To determine $T(5)$, add 1 line segment to Diagram 2 as shown in Diagram 3.
We are told that this new (5^{th}) line segment must intersect each of the existing 4 line segments exactly once, creating 4 new points of intersection (labelled $1, 2, 3, 4$).
This 5^{th} line segment also adds 2 new endpoints (labelled 5 and 6) distinct from the previous 4 new points.
In addition, each of the points which exist in the illustration of $T(4)$ (Diagram 2) continue to exist in the illustration of $T(5)$ (Diagram 3) and are distinct from each of the new points which were added.
Therefore, we get

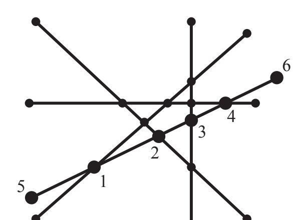

Diagram 3

$$\begin{aligned} T(5) &= T(4) + 4 + 2 \\ &= 14 + 4 + 2 \\ &= 20 \end{aligned}$$

Therefore, $T(4) = 14$ and $T(5) = 20$.

(b) As in part (a), consider finding $T(n)$ with the help of (in terms of) $T(n-1)$ for any integer $n \geq 2$.
To determine $T(n)$, add 1 line segment to any illustration of $T(n-1)$.
This new (n^{th}) line segment must intersect each of the existing $n-1$ line segments exactly once, creating $n - 1$ new points of intersection.

This n^{th} line segment also adds 2 new endpoints (distinct from the previous $n-1$ points). In addition, each of the points which exist in the illustration of $T(n-1)$ continue to exist in the illustration of $T(n)$ and are distinct from each of the new points which were added. Therefore, we get $T(n) = T(n-1) + (n-1) + 2$ or $T(n) = T(n-1) + n + 1$ and so $T(n) - T(n-1) = n + 1$ for all $n \geq 2$.

(c) From part (b), $T(n) - T(n-1) = n + 1$ and so $T(n) = T(n-1) + n + 1$.
That is, the addition of an n^{th} line segment increases $T(n-1)$ by $n+1$.
For example since $T(1) = 2$, then $T(2) = T(1) + 3 = 2 + 3$.
For small values of n, we determine $T(n)$ in the table below.

n	$T(n) = T(n-1) + n + 1, n \geq 2$
2	$T(2) = T(1) + 3 = 2 + 3$
3	$T(3) = T(2) + 4 = 2 + 3 + 4$
4	$T(4) = T(3) + 5 = 2 + 3 + 4 + 5$
5	$T(5) = T(4) + 6 = 2 + 3 + 4 + 5 + 6$
6	$T(6) = T(5) + 7 = 2 + 3 + 4 + 5 + 6 + 7$
$\vdots$	$\vdots$

We may use the pattern in the table above to establish an equation for $T(n)$.
What is the pattern?
Consider for example the row for $n = 5$.
$T(5)$ is the sum of the positive integers from 2 to $n + 1 = 5 + 1 = 6$.
This is true for each of the rows shown in the table.
That is, $T(n-1) = 2 + 3 + 4 + \cdots + n$ for any positive integer $n \geq 3$.
(Verify that this is true for each of the rows shown in the table.)
Since the addition of an n^{th} line segment increases $T(n-1)$ by $n+1$, then $T(n) = T(n-1) + n + 1$ and so $T(n) = (2 + 3 + 4 + \cdots + n) + n + 1$.
Reorganizing this equation for $T(n)$, we get

$$T(n) = 1 + 2 + 3 + 4 + \cdots + n + n.$$

Since the sum of the first n positive integers $1 + 2 + 3 + 4 + \cdots + n$ is equal to $\dfrac{n(n+1)}{2}$, then $T(n) = \dfrac{n(n+1)}{2} + n$.
Solving $T(n) = 2015$, we get

$$\begin{aligned} \frac{n(n+1)}{2} + n &= 2015 \\ n(n+1) + 2n &= 4030 \\ n^2 + 3n &= 4030 \\ n^2 + 3n - 4030 &= 0 \\ (n-62)(n+65) &= 0 \end{aligned}$$

and so $n = 62$ (since $n > 0$). (We could use the quadratic formula if we didn't see how to factor the quadratic.)
Therefore, $n = 62$ is the only value of n for which $T(n) = 2015$.

4. (a) Since $125 = 5^3$, then the positive divisors of 125 are $5^0, 5^1, 5^2, 5^3$ or $1, 5, 25, 125$.
Therefore, for each positive integer a from 1 to 125 inclusive, the possible values of $\gcd(a, 125)$ are $1, 5, 25, 125$.

The $\gcd(a, 125) = 125$ exactly when a is divisible by 125. Since $1 \le a \le 125$ and there is only one multiple of 125 in this range, then $\gcd(a, 125) = 125$ only when $a = 125$.
$\gcd(a, 125) = 25$ exactly when a is divisible by 25 and not by 125.
These a are each of the form $a = 25k$ for some positive integer k.
The possible values of a in the range $1 \le a < 125$ are 25, 50, 75, and 100.
$\gcd(a, 125) = 5$ exactly when a is divisible by 5 and not by 25.
There are 25 multiples of 5 between 1 and 125, inclusive.
5 of these have a gcd with 125 of 25 or 125, as above.
The remaining $25 - 5 = 20$ multiples of 5 must have a gcd with 125 of 5.
$\gcd(a, 125) = 1$ exactly when a is not divisible by 5.
Since there are 25 multiples of 5 between 1 and 125, inclusive, then there are $125 - 25 = 100$ integers in this range that are not multiples of 5.
In summary, when the positive integers a with $1 \le a \le 125$ are considered, there is/are

- 1 integer a for which $\gcd(a, 125) = 125$
- 4 integers a for which $\gcd(a, 125) = 25$
- 20 integers a for which $\gcd(a, 125) = 5$
- 100 integers a for which $\gcd(a, 125) = 1$

Therefore,

$$\begin{aligned} P(125) &= \gcd(1, 125) + \gcd(2, 125) + \cdots + \gcd(124, 125) + \gcd(125, 125) \\ &= 1(125) + 4(25) + 20(5) + 100(1) \\ &= 425 \end{aligned}$$

(b) Since r is a prime number, the positive divisors of r^2 are $1, r, r^2$.
Since s is a prime number, the positive divisors of s are $1, s$.
Since r and s are different prime numbers, then $\gcd(r^2, s) = 1$. (There are no common positive divisors other than 1 in these two lists.)
Therefore, $P(r^2 s) = P(r^2)P(s)$, from the given fact.

To calculate $P(s)$, we proceed as in (a).
For each positive integer a with $1 \le a \le s$, the possible values of $\gcd(a, s)$ are 1 and s.
The only multiple of s in the given range is $a = s$, so there is only one a (namely $a = s$) for which $\gcd(a, s) = s$.
There are thus $s - 1$ integers a for which $\gcd(a, s) = 1$.
Therefore,

$$\begin{aligned} P(s) &= \gcd(1, s) + \gcd(2, s) + \cdots + \gcd(s - 1, s) + \gcd(s, s) \\ &= 1(s) + (s - 1)1 \\ &= 2s - 1 \end{aligned}$$

In a similar way, there is 1 integer a with $1 \le a \le r^2$ for which $\gcd(a, r^2) = r^2$, and $r - 1$ integers a with $\gcd(a, r^2) = r$, and $r^2 - r$ integers a with $\gcd(a, r^2) = 1$.
Therefore,

$$\begin{aligned} P(r^2) &= \gcd(1, r^2) + \gcd(2, r^2) + \cdots + \gcd(r^2 - 1, r^2) + \gcd(r^2, r^2) \\ &= 1(r^2) + (r - 1)r + (r^2 - r)1 \\ &= 3r^2 - 2r \\ &= r(3r - 2) \end{aligned}$$

Therefore, $P(r^2 s) = P(r^2)P(s) = r(3r - 2)(2s - 1)$, as required.

(c) We prove that $P(r^2s)$ can never be equal to a power of a prime number by assuming that $P(r^2s)$ equals t^n for some prime number t and positive integer n, and obtaining a contradiction.
From (b), we obtain $r(3r-2)(2s-1) = t^n$.
Since the right side is a power of a prime number, then the only divisors of the right side are powers of t.
Therefore, each of the factors on the left side must be powers of t.
Since r is a factor on the left side and r is itself a prime, then $r = t$.
Therefore, $3r - 2 = 3t - 2$ must also be a power of t.
If $3t - 2 = t$, then $t = 1$, which is not prime.
If $3t - 2 = t^2$ and t is prime, then $t^2 - 3t + 2 = 0$ or $(t-2)(t-1) = 0$ and so $t = 2$ or $t = 1$. Since t is to be a prime number, then if $3t - 2 = t^2$, we must have $t = 2$.
This also tells us that if $t = 2$, then $3t - 2$ is only a power of a prime when $3t - 2 = t^2$.
Can $3t - 2 = t^u$ for some integer $u > 2$ and prime $t > 2$?
If $u > 2$, then $t^u > t^2$, since $t > 1$.
If $t > 2$, then $t^2 - 3t + 2 = (t-2)(t-1) > 0$ so $t^2 > 3t - 2$.
Thus, if $u > 2$, then $t^u > t^2 > 3t - 2$, so $3t - 2 \neq t^u$.
Therefore, if $3t - 2$ is a power of t, then $t = 2$.
If $t = 2$, then t and $3t - 2$ are both powers of $t = 2$. Furthermore, $t = 2$ is the only prime for which this can work.
But in this case, $2s - 1$ is odd and so cannot be a power of $t = 2$.
This contradicts our original assumption.
Therefore, $P(r^2s) = r(3r-2)(2s-1)$ cannot be the power of a prime number.

(d) We note that $243 = 3^5$. We try some different possible forms for m.
Suppose that $m = rs$ for some prime numbers r and s.
Then $P(m) = P(rs) = (2r-1)(2s-1)$.
Are there prime numbers r and s for which $(2r-1)(2s-1) = 243$?
Note that if p is prime, then $p \geq 2$ so $2p - 1 \geq 3$.
Therefore, we could have $2r - 1 = 3$ and $2s - 1 = 81$ (which gives $r = 2$ and $s = 41$ which are both prime) or $2r - 1 = 9$ and $2s - 1 = 27$ (which gives $r = 5$ (prime) and $s = 14$ (not prime)).
Therefore, $P(82) = P(2 \cdot 41) = 3 \cdot 81 = 243$.

Since 243 is a power of a prime, then from part (c), $P(r^2s) \neq 243$ for all primes r and s.

Let's next see if $P(r^3s)$ can equal 243.
Using a similar derivation to that in (a), we can see that

$$P(r^3) = 1(r^3) + (r-1)r^2 + (r^2 - r)r + (r^3 - r^2)(1) = 4r^3 - 3r^2$$

Therefore, $P(r^3s) = P(r^3)P(s) = (4r^3 - 3r^2)(2s-1) = r^2(4r-3)(2s-1)$.
If $r^2(4r-3)(2s-1) = 243$, then since r is prime and a factor of the left side, we must have $r = 3$.
Therefore, $3^2(4(3)-3)(2s-1) = 243$ or $81(2s-1) = 243$ or $2s - 1 = 3$, which gives $s = 2$.
Thus, $P(54) = P(3^3 \cdot 2) = 243$.

Therefore, $m = 82$ and $m = 54$ both satisfy $P(m) = 243$.

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2014 Hypatia Contest

Wednesday, April 16, 2014
(in North America and South America)

Thursday, April 17, 2014
(outside of North America and South America)

Solutions

1. (a) Using the given definition, $8 \odot 7 = \sqrt{8 + 4(7)} = \sqrt{36} = 6$.

   (b) Since $16 \odot n = 10$, then $\sqrt{16 + 4n} = 10$ or $16 + 4n = 100$ (by squaring both sides) or $4n = 84$ and so $n = 21$.
   We check that indeed $16 \odot n = 16 \odot 21 = \sqrt{16 + 4(21)} = \sqrt{100} = 10$.

   (c) We first determine the value inside the brackets: $9 \odot 18 = \sqrt{9 + 4(18)} = \sqrt{81} = 9$.
   So then $(9 \odot 18) \odot 10 = 9 \odot 10 = \sqrt{9 + 4(10)} = \sqrt{49} = 7$.

   (d) Using the definition, $k \odot k = \sqrt{k + 4k} = \sqrt{5k}$.
   So we are asked to solve the equation $\sqrt{5k} = k$.
   Squaring both sides we get, $5k = k^2$ and so $k^2 - 5k = 0$ or $k(k - 5) = 0$, and so $k = 0$ or $k = 5$.
   Checking $k = 0$, we obtain $k \odot k = 0 \odot 0 = \sqrt{0 + 4(0)} = \sqrt{0} = 0 = k$, as required.
   Checking $k = 5$, we obtain $k \odot k = 5 \odot 5 = \sqrt{5 + 4(5)} = \sqrt{25} = 5 = k$, as required.
   Thus, the only possible solutions are $k = 0$ and $k = 5$.

2. (a) The song's position on week 1 ($w = 1$), is $P(1) = 3(1)^2 - 36(1) + 110 = 77$.

   (b) The song's position is given by the quadratic function $P = 3w^2 - 36w + 110$, the graph of which is a parabola opening upward.
   The minimum value of this parabola is achieved at its vertex.
   To find the coordinates of the vertex, we may complete the square.

$$\begin{aligned} P &= 3w^2 - 36w + 110 \\ &= 3(w^2 - 12w) + 110 \\ &= 3(w^2 - 12w + 36 - 36) + 110 \\ &= 3(w^2 - 12w + 36) - 108 + 110 \\ &= 3(w - 6)^2 + 2 \end{aligned}$$

   Therefore, the vertex of the parabola occurs at $w = 6$ and $P = 2$.

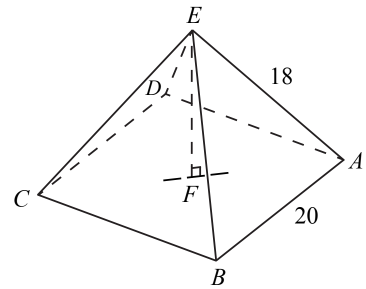
   (i) The best position that the song "Recursive Case" reaches is position #2.

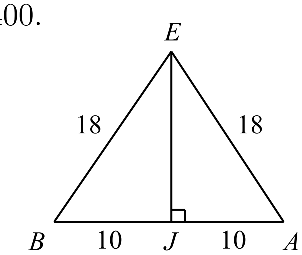
   (ii) The song reaches its best position on week 6.

   (c) To determine the last week that "Recursive Case" appears on the Top 200 list, we want to find the largest w such that $P = 3w^2 - 36w + 110 \le 200$.
   Using the vertex form from part (b), we have $3(w - 6)^2 + 2 \le 200$ or $3(w - 6)^2 \le 198$ or $(w - 6)^2 \le 66$.
   To determine the largest positive integer w such that $(w - 6)^2 \le 66$, we want to find the largest square that is less than or equal to 66.
   Since $8^2 \le 66$ and $9^2 > 66$, then the largest w satisfies $w - 6 = 8$ and so $w = 14$.
   The last week that "Recursive Case" appears on the Top 200 list is week 14.
   To check this we note that,

$$P(14) = 3(14 - 6)^2 + 2 = 194 \le 200 \text{ but } P(15) = 3(15 - 6)^2 + 2 = 245 > 200.$$

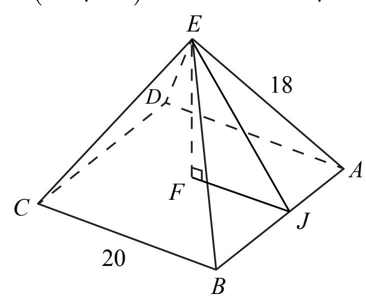
3. (a) We will denote the area of a figure using vertical bars.
For example, $|\triangle BCE|$ is the area of $\triangle BCE$.
Since $ABCD$ has equal side lengths (it is a square) and $EA = EB = EC = ED$, then the 4 triangular faces of pyramid $ABCDE$ are congruent and so all have equal area.
The surface area of pyramid $ABCDE$ is equal to the sum of the base area and the areas of the 4 triangular faces or
$|ABCD| + |\triangle EAB| + |\triangle EBC| + |\triangle ECD| + |\triangle EDA| = |ABCD| + 4|\triangle EAB|$.
Square $ABCD$ has side length 20 and so $|ABCD| = 20 \times 20 = 400$.
To determine $|\triangle EAB|$, we construct altitude EJ as shown.
$\triangle EAB$ is isosceles and so EJ bisects AB with $AJ = JB = 10$.
$\triangle EAJ$ is a right-angled triangle and so by the Pythagorean Theorem, $EA^2 = AJ^2 + EJ^2$ or $18^2 = 10^2 + EJ^2$, so then $EJ = \sqrt{224}$ or $EJ = 4\sqrt{14}$ (since $EJ > 0$).





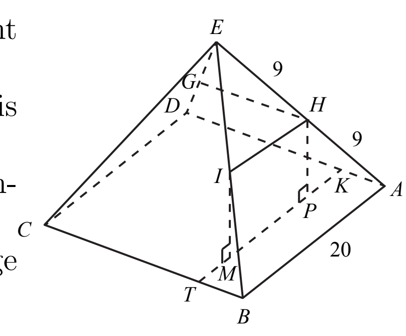
The area of $\triangle EAB$ is $\frac{1}{2}(AB)(EJ) = \frac{1}{2}(20)(4\sqrt{14}) = 40\sqrt{14}$.
Thus the surface area of $ABCDE$ is $|ABCD|+4|EAB| = 400+4(40\sqrt{14}) = 400+160\sqrt{14}$.

(b) As in part (b), J is positioned such that EJ is an altitude of $\triangle EAB$ and so $EJ = 4\sqrt{14}$. Since EF is perpendicular to the base of the pyramid, then EF is perpendicular to FJ, as shown.
Further, F is the centre of the base $ABCD$ and J is the midpoint of AB, so then FJ is parallel to CB and $FJ = \frac{1}{2} \times CB = \frac{1}{2} \times 20 = 10$.

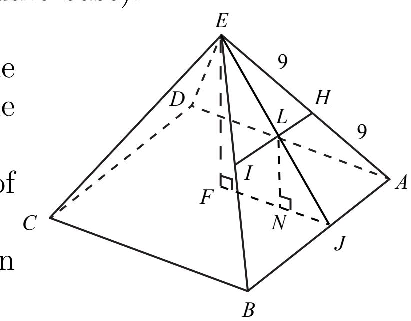


By the Pythagorean Theorem, $EJ^2 = EF^2 + FJ^2$ or $224 = EF^2 + 100$ and so $EF = \sqrt{124} = 2\sqrt{31}$ (since $EF > 0$).
Therefore, the height EF of the pyramid $ABCDE$ is $2\sqrt{31}$.

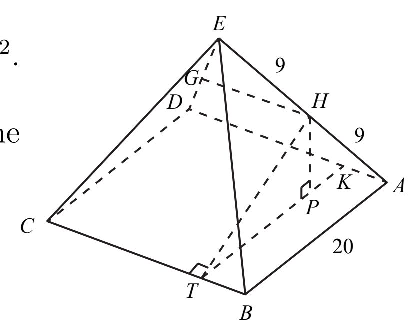
(c) Points G and H are the midpoints of ED and EA, respectively, and so $EG = GD = EH = HA = 9$.
Thus, GH is a midsegment of $\triangle EDA$ and so GH is parallel to DA and $GH = \frac{1}{2}\times DA = 10$.
(Note that this result follows from the fact that $\triangle EGH$ is similar to $\triangle EDA$. Can you prove this?)
Since GH is parallel to DA and DA is parallel to CB, then GH is parallel to CB.
That is, quadrilateral $BCGH$ (whose area we are asked to find) is a trapezoid.
To determine the area of trapezoid $BCGH$, we need the lengths of the parallel sides ($GH = 10$ and $CB = 20$) and we need the perpendicular distance between these two parallel sides.
We will proceed by showing that HT (in the diagram below) is such a perpendicular height of the trapezoid and also by determining its length.
Join H to I, the midpoint of EB, so that HI is a midsegment of $\triangle EAB$ with $HI = 10$.
Position P on the base of the pyramid such that HP is perpendicular to the base.
Similarly, position M on the base such that IM is perpendicular to the base.
Let MP extended intersect the edge BC at T and the edge AD at K, as shown.



By symmetry, $HP = IM$ and so $HPMI$ is a rectangle with $PM = HI = 10$.
Further, since AB is parallel to HI and HI is parallel to KT (both are perpendicular to HP and IM), then AB is parallel to KT. So then $ABTK$ is a rectangle and $KT = AB = 20$.
Also by symmetry, PM is centred on line segment KT such that $KP = MT = \frac{20-10}{2} = 5$ ($EA = EB$ and E lies vertically above the centre of the square base).
Therefore $PT = PM + MT = 10 + 5 = 15$.
Next, let the midpoint of HI be L and position N on the base of the pyramid such that LN is perpendicular to the base.
Since F is the centre of the square and J is the midpoint of edge AB, then FJ passes through N.
Since $\triangle EFJ$ is similar to $\triangle LNJ$ (by AA $\sim$), then $\dfrac{LN}{EF} = \dfrac{LJ}{EJ} = \dfrac{1}{2}$ (since HI is a midsegment of $\triangle EAB$).
Therefore, $LN = \frac{1}{2}(EF) = \frac{1}{2}(2\sqrt{31}) = \sqrt{31}$.



Since HI is parallel to the base of the pyramid, $ABCD$, then $HP = LN = \sqrt{31}$ (both are perpendicular to the base).
In right-angled $\triangle HPT$, $HT^2 = HP^2 + PT^2 = (\sqrt{31})^2 + 15^2$.
So $HT^2 = 256$ and $HT = 16$ (since $HT > 0$).
Since the plane containing $\triangle HPT$ is perpendicular to the base $ABCD$, then HT is perpendicular to BC.
That is, HT is the height of trapezoid $BCGH$.
Finally, $|BCGH| = \frac{HT}{2}(GH + CB) = \frac{16}{2}(10 + 20) = 240$.



4. (a) If $(4, y, z)$ is an APT, then $4^2 + y^2 = z^2 + 1$ or $z^2 - y^2 = 15$ and so $(z - y)(z + y) = 15$.
Since y and z are positive integers, then $(z + y)$ is a positive integer and thus $(z - y)$ is also a positive integer (since the product of the two factors is 15).
That is, $(z - y)$ and $(z + y)$ are the possible pairs of factors of 15, of which there are two: 1 and 15, and 3 and 5.
Since $z + y > z - y$, we have the following two systems of equations to solve:

$$\begin{aligned} z - y &= 1 \\ z + y &= 15 \end{aligned} \qquad\qquad \begin{aligned} z - y &= 3 \\ z + y &= 5 \end{aligned}$$

Adding the first pair of equations, we get $2z = 16$ or $z = 8$ and so $y = 7$.
Adding the second pair of equations, we get $2z = 8$ or $z = 4$ and so $y = 1$.
Since $y > 1$, this second solution is not an APT.
The only APT with $x = 4$ is $(4, 7, 8)$.

(b) Let positive integers u, v, w be the lengths of the sides of $\triangle UVW$ (with side length u opposite vertex U, v opposite V, and w opposite W).
Without loss of generality, assume (u, v, w) forms an APT such that $u^2 + v^2 = w^2 + 1$ with $u > 1$ and $v > 1$.
The area of $\triangle UVW$ is given by $A = \frac{1}{2}uv \sin W$ $(\star)$ (we will derive this formula at the end of the solution).
Assume that this area, A, is an integer.
In $\triangle UVW$, the cosine law gives $w^2 = u^2 + v^2 - 2uv \cos W$ or $\cos W = \dfrac{u^2 + v^2 - w^2}{2uv}$.

However, (u, v, w) is an APT and thus $u^2 + v^2 = w^2 + 1$ and so $u^2 + v^2 - w^2 = 1$.
Substituting, we get $\cos W = \dfrac{1}{2uv}$ and since $\sin^2 W = 1 - \cos^2 W$, then
$\sin^2 W = 1 - \left(\dfrac{1}{2uv}\right)^2$.

Squaring $(\star)$ and substituting for $\sin^2 W$, we get $A^2 = \frac{1}{4}u^2v^2\left(1 - \left(\dfrac{1}{2uv}\right)^2\right)$.

Simplifying, this last equation we get $A^2 = \frac{1}{4}u^2v^2\left(\dfrac{(2uv)^2 - 1}{(2uv)^2}\right)$ or $A^2 = \frac{1}{4}u^2v^2\left(\dfrac{(2uv)^2 - 1}{4u^2v^2}\right)$ or $16A^2 = (2uv)^2 - 1$ and so $4u^2v^2 - 16A^2 = 1$.
The left side of this equation has a common factor of 4 and so is an even integer for all integers u, v, A.
However, the right side of the equation is 1 (an odd integer) and so this equation has no solutions.
Our assumption that A is an integer is false and thus the area of any triangle whose sides lengths form an APT is not an integer.

<u>Derivation of $A = \frac{1}{2}uv\sin W$</u>
In $\triangle UVW$, construct the perpendicular from V meeting side UW (or side UW extended) at T.
In this case, $\triangle UVW$ has height VT and base UW.
In right-angled $\triangle VWT$, we have $\sin W = \dfrac{VT}{VW}$ and so $VT = u\sin W$.
(Note that if $\angle W$ is obtuse, then height VT lies outside the triangle and we have $\sin(180^\circ - W) = \dfrac{VT}{VW}$ and so $VT = u\sin W$ since $\sin(180^\circ - W) = \sin W$).
Thus, the area of $\triangle UVW$ is $\frac{1}{2}(UW)(VT) = \frac{1}{2}uv\sin W$.

(c) Since $(5t + p, bt + q, ct + r)$ is an APT, then $(5t + p)^2 + (bt + q)^2 = (ct + r)^2 + 1$.
Expanding, we get $25t^2 + 10pt + p^2 + b^2t^2 + 2bqt + q^2 = c^2t^2 + 2crt + r^2 + 1$, or $(25 + b^2)t^2 + (10p + 2bq)t + (p^2 + q^2) = c^2t^2 + 2crt + (r^2 + 1)$.
Each side of this equation is a quadratic polynomial in the variable t.
Since this equation is true for all positive integers t, then the corresponding coefficients on the left side and the right side are equal.
That is,

$$\begin{aligned} 25 + b^2 &= c^2 && (1)\\ 10p + 2bq &= 2cr && (2)\\ p^2 + q^2 &= r^2 + 1 && (3) \end{aligned}$$

Equation (1) is the Pythagorean relationship, and since b, c are positive integers then the Pythagorean triple $(5, 12, 13)$ satisfies (1) with $b = 12$ and $c = 13$.
(These are in fact the only values of b and c that work but we don't need to show this.)
Substituting b and c into (2) and simplifying we get $5p + 12q = 13r$ (4).
Squaring equation (4), we get $25p^2 + 120pq + 144q^2 = 169r^2$ (5).
From equation (3), we get $r^2 = p^2 + q^2 - 1$, and substituting this into (5) gives $25p^2 + 120pq + 144q^2 = 169(p^2 + q^2 - 1)$ or $144p^2 - 120pq + 25q^2 = 169$ or $(12p - 5q)^2 = 169$ and so $12p - 5q = \pm 13$.
We search for positive integers p and q with $p \geq 100$ satisfying $12p - 5q = \pm 13$ or $5q = 12p \pm 13$.

Since $12p$ is even for all integers p and 13 is odd, then $12p \pm 13$ is odd and so $5q$ must also be odd.
The units digit of $5q$ is either 0 or 5 for all positive integers q and since $5q$ is odd, then its units digit is 5.
The units digit of $5q - 13$ is 2 and the units digit of $5q + 13$ is 8, and so the units digit of $12p$ is either 2 or 8 (since $12p = 5q \pm 13$).
A value of $p \geq 100$ such that $12p$ has units digit 2 is $p = 101$.
Substituting, we get $12(101) = 5q - 13$ or $5q = 1225$ and so $q = 245$.
Substituting p and q into (4) gives $r = 265$.
A value of $p \geq 100$ such that $12p$ has units digit 8 is $p = 104$.
Substituting, we get $12(104) = 5q + 13$ or $5q = 1235$ and so $q = 247$.
Substituting p and q into (4) gives $r = 268$.
Therefore, two possible 5-tuples (b, c, p, q, r) satisfying the given conditions are $(12, 13, 101, 245, 265)$ and $(12, 13, 104, 247, 268)$.
(Note that there are an infinite number of possible solutions.)

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2013 Hypatia Contest

Thursday, April 18, 2013
(in North America and South America)

Friday, April 19, 2013
(outside of North America and South America)

Solutions

1. (a) Pulling out two of each type of bill gives Rad, $2\times(\$5+\$10+\$20+\$50) = 2\times(\$85) = \$170$. Since his total sum of money is \$175, the only other bill that Rad pulled out must have a value of $\$175 - \$170 = \$5$.
That is, Rad pulled out three \$5 bills, two \$10 bills, two \$20 bills, and two \$50 bills, for a total of $3 + 2 + 2 + 2 = 9$ bills.

(b) Sandy pulls out at least one of each type of bill and so she must have at least $\$5 + \$10 + \$20 + \$50 = \$85$.
Thus, we know four of the five bills that Sandy pulls out and that these four bills total \$85. The fifth bill that Sandy pulls could be any one of the four different types of bills.
If this fifth bill is a \$5 bill, then Sandy's total sum of money is $\$85 + \$5 = \$90$.
If this fifth bill is a \$10 bill, then Sandy's total sum of money is $\$85 + \$10 = \$95$.
If this fifth bill is a \$20 bill, then Sandy's total sum of money is $\$85 + \$20 = \$105$.
Finally, if this fifth bill is a \$50 bill, then Sandy's total sum of money is $\$85+\$50 = \$135$.
Therefore, the sums of money that Sandy could have are \$90, \$95, \$105, and \$135.

(c) Lino could have at most three \$50 bills since four \$50 bills exceeds his total sum of money ($4 \times \$50 = \$200 > \$160$).
If Lino had no \$50 bills, then the bills (6 bills at most) each have value at most \$20 and would total \$160.
However, this is not possible since $6 \times \$20 = \$120$ which is less than the required \$160.
If Lino had one \$50 bill, then the remaining bills (5 bills at most) would total $\$160 - \$50 = \$110$.
However, this is not possible since the largest denomination of the remaining bills is \$20 and $5 \times \$20 = \$100$ which is less than the required \$110.
Therefore, we proceed by considering the cases where Lino has two or three \$50 bills.
These two cases are summarized in the table below.

Number of \$50s	Money in \$50s	Money remaining	Number of \$20s	Number of \$10s	Number of \$5s	Number of bills used
3	\$150	\$10		1		4
3	\$150	\$10			2	5
2	\$100	\$60	3			5
2	\$100	\$60	2	2		6

In each case from the table above, Lino has a total sum of \$160 and has pulled out 6 or fewer bills.
Since we are given that there are only four possibilities, then we have found them all.

2. (a) The equation of a parabola, written in the form $y = (x - p)^2 + q$, has its vertex at (p, q). Thus, the parabola with equation $y = (x - 3)^2 + 1$, has its vertex at $(3, 1)$.

(b) *Solution 1*
Under a translation, the shape of a parabola remains unchanged.
That is, the new parabola is congruent to the original parabola.
After a translation of 3 units to the left and 3 units up, the original vertex $(3, 1)$ moves to the point $(3 - 3, 1 + 3)$ or $(0, 4)$.
Since this new parabola is congruent to the original, but it has its vertex at $(0, 4)$, then its equation is $y = (x - 0)^2 + 4$ or $y = x^2 + 4$.

Solution 2
Under a translation of 3 units left and 3 units up, the equation $y = (x-3)^2 + 1$ becomes

$$\begin{aligned} y-3 &= ((x+3)-3)^2+1 \\ y-3 &= x^2+1 \\ y &= x^2+4 \end{aligned}$$

(c) At the point of intersection of these two parabolas, their y values must be equal. Thus,

$$\begin{aligned} (x-3)^2+1 &= x^2+4 \\ x^2-6x+9+1 &= x^2+4 \\ -6x &= -6 \\ x &= 1 \end{aligned}$$

Substituting $x = 1$ into the equation $y = x^2 + 4$, we determine the y value of the point of intersection to be $y = 5$.
Therefore, the two parabolas intersect at the point $(1, 5)$.

(d) At the point of intersection of these two parabolas, their y values must be equal. Thus,

$$\begin{aligned} (x-3)^2+1 &= ax^2+4 \\ x^2-6x+9+1 &= ax^2+4 \\ 0 &= ax^2-x^2+6x-6 \\ 0 &= (a-1)x^2+6x-6 \end{aligned}$$

Since the two parabolas intersect at exactly one point, then the resulting equation $(a-1)x^2 + 6x - 6 = 0$ (which is quadratic since $a < 0$), has exactly one solution.
Thus, the *discriminant* of this equation must equal zero.
(Note: The discriminant of a quadratic equation of the form $ax^2 + bx + c = 0$, is $b^2 - 4ac$.)
Solving $6^2 - 4(-6)(a-1) = 0$, we get $36 + 24(a-1) = 0$ or $24a = -12$, and so $a = -\frac{1}{2}$.
That is, the parabolas with equations $y = ax^2 + 4$ and $y = (x-3)^2 + 1$ touch at exactly one point when $a = -\frac{1}{2}$.

3. (a) If the sequence begins with 3 or more P's, then it cannot be non-predictive since there are only 2 Q's.
Thus, there are two possible cases to consider; the sequence begins with exactly 1 P (that is, the sequence begins PQ since the second letter is not P), or the sequence begins with exactly 2 P's (that is, the sequence begins PPQ since the third letter is not P).

<u>Case 1: The sequence begins PQ</u>
Any sequence that begins PQ is already non-predictive, and so the remaining 7 letters can be in any order.
Therefore the number of non-predictive sequences beginning PQ, with $m = 7$ and $n = 2$, is equal to the number of ways of arranging 6 P's and 1 Q in the remaining 7 positions.
There are 7 distinct sequences having 6 P's and 1 Q since the Q can be placed in any one of the 7 remaining positions (while the P's are placed in all other positions without choice).
Thus the number of non-predictive sequences in this case is 7.

<u>Case 2: The sequence begins PPQ</u>

If the fourth letter in this sequence was P, then it would not be possible for the sequence to be non-predictive since there are only 2 Q's.
Therefore, the sequence must begin PPQQ.
Since the remaining 5 letters are all P's, then there is no choice in forming the rest of the sequence, so there is only one such sequence.

If $m = 7$ and $n = 2$, then the total number of non-predictive sequences beginning with P is $7 + 1 = 8$.

(b) Using a similar argument to that in part (a), if $m > 2$ and $n = 2$, then the non-predictive sequences beginning with P must either begin PQ or PPQQ.
In the case in which the sequence begins PQ, there are $(m-1)$ P's and 1 Q left to arrange in the remaining m positions.
Since the Q can be placed in any one of the m remaining positions, and each placement produces a distinct sequence, there are m non-predictive sequences that begin PQ.
In the case where the sequence begins PPQQ, there are only P's left to arrange.
There is only 1 way to do this.
Thus, there is only 1 non-predictive sequence that begins PPQQ.
Therefore the number of non-predictive sequences with $m > 2$ and $n = 2$, beginning with a P, is $(m + 1)$.

Any sequence that begins with a Q is a non-predictive sequence.
Therefore, the total number of non-predictive sequences that begin with Q is equal to the number of ways of arranging the remaining m P's and 1 Q in the remaining $(m + 1)$ positions.
Since the Q can be placed in any one of the $(m+1)$ positions, and each placement produces a distinct sequence, there are $(m + 1)$ non-predictive sequences that begin with a Q.

Therefore when $n = 2$, for every $m > 2$ the number of non-predictive sequences that begin with P is equal to the number of non-predictive sequences that begin with Q.

(c) If the sequence begins with 4 or more P's, then it cannot be non-predictive since there are only 3 Q's. (The number of Q's can never "catch up" to the number of P's.)
Consider the cases in which the sequence begins with $0, 1, 2$, or 3 P's.

<u>Case 1: The sequence begins with a Q (it begins with 0 P's)</u>
Any sequence that begins with a Q is a non-predictive sequence.
Therefore, the total number of non-predictive sequences that begin with Q is equal to the number of ways of arranging the remaining 10 P's and 2 Q's in 12 positions.
We count these arrangements by considering the number of distinct ways to place the 2 Q's in the 12 positions and filling in the remaining positions with the 10 P's.
There are 12 possible positions to place the first Q, followed by 11 possible positions to place the second, or 12×11 arrangements.
However, since the 2 Q's cannot be distinguished from one another (i.e. they are identical), this counts each of the possible arrangements twice.
(To see this, consider for example that placing the first Q in the 4th position and the second Q in the 7th position is the same sequence as placing the first Q in the 7th position and the second Q in the 4th position.)
Therefore in this case, the number of ways to place the two Q's is $\frac{12\times 11}{2} = 66$ and so the number of non-predictive sequences is 66.

Case 2: The sequence begins PQ

Any sequence that begins with PQ is a non-predictive sequence.
Therefore, the total number of non-predictive sequences that begin with PQ is equal to the number of ways of arranging the remaining 9 P's and 2 Q's in 11 positions.
We count these arrangements by considering the number of distinct ways to place the 2 Q's in the 11 possible positions and filling in the remaining positions with the 9 P's.
Using the same argument as was used in Case 1, this can done in $\frac{11\times 10}{2} = 55$ possible ways.

Case 3: The sequence begins PPQ

There are two possibilities for the 4th letter in the sequence beginning PPQ.
The sequence beginning PPQQ is non-predictive, however, so is the sequence that begins PPQPQQ.
(Can you verify for yourself that these are the only two ways to begin a non-predictive sequence with $m = 10$, $n = 2$ starting with PPQ?)
The total number of non-predictive sequences that begin PPQQ is equal to the number of ways of arranging the remaining 8 P's and 1 Q in 9 positions.
Since the Q can be placed in any one of the 9 positions, and each placement produces a distinct sequence, there are 9 non-predictive sequences that begin PPQQ.
There is only 1 non-predictive sequence that begins PPQPQQ since the remaining 7 letters are all P's.
Therefore in this case, the number of non-predictive sequences is $9 + 1 = 10$.

Case 4: The sequence begins PPPQ

Since there are only 2 more Q's available, this sequence must begin PPPQQQ so that it is non-predictive.
There is only 1 non-predictive sequence that begins PPPQQQ since the remaining 7 letters are all P's.
Thus in this final case, there is only 1 non-predictive sequence.

The number of non-predictive sequences with $m = 10$ and $n = 2$ is $66 + 55 + 10 + 1 = 132$.

4. (a) Since the edge length of each cube is 1 cm, then the length, width and height of the rectangular prism are 5 cm, 4 cm and 1 cm, respectively.
The top face of the prism has dimensions 5 cm by 4 cm, and so has area 20 cm^2.
Similarly, the bottom face on the opposite side of the prism has the same area, 20 cm^2.
The front and back faces of the prism each have dimensions 4 cm by 1 cm, and so have area 4 cm^2.
The right and left faces of the prism each have dimensions 5 cm by 1 cm, and so have area 5 cm^2.
Therefore, the surface area of the rectangular prism is $2 \times (20 + 4 + 5) = 2 \times 29 = 58$ cm^2.

(b) Suppose that the rectangular prism in question is l cm (cubes) in length and w cm (cubes) wide, with $l \geq w$.
Then the top surface of the rectangular prism has area $(l \times w)$ cm^2, the front surface has area $(w \times 1)$ cm^2, and the right side has area $(l \times 1)$ cm^2.
Therefore, the surface area of the entire rectangular prism is $2 \times (lw + w + l)$ cm^2.
Since the surface area is 180 cm^2, then $2 \times (lw + w + l) = 180$ or $lw + w + l = 90$.
Adding 1 to both sides of this equation, we get $lw+w+l+1 = 91$, so $w(l+1)+1(l+1) = 91$.
Factoring the left side of this equation one step further, we have $(w + 1)(l + 1) = 91$.
Since both l and w are positive integers, then $(w+1)(l+1)$ is the product of two positive integers.

The right side, 91, can be written as the product of two positive integers in exactly two different ways, 1×91 and 7×13.
Since both l and w are positive integers, then $2 \leq (w+1) \leq (l+1)$ and so neither factor can be equal to 1.
Therefore $(w+1) = 7$ and $(l+1) = 13$ (since $l \geq w$), and so $w = 6$ and $l = 12$.
Since the rectangular prism is 6 cubes wide and 12 cubes in length, then it has $6 \times 12 = 72$ cubes in total.

(c) As shown in part (b), the surface area of the original prism without the removal of the internal prism, is $2 \times (lw + w + l)$ cm^2.
To find the surface area of the frame, we must account for the area that is lost and that which is gained by removing the internal prism.

The area of the original prism that is lost is equal to the front and back rectangular faces of the internal prism.
Since the width of the original prism is w cm and the internal prism is located k cm from each side of the original prism, then the width of the internal prism is $(w - 2k)$ cm.
Similarly, the length of the internal prism is $(l - 2k)$ cm.
(Note that $w - 2k > 0$ and $l - 2k > 0$, so $w > 2k$ and $l > 2k$.)
Thus, the area from the original prism that is lost by removing the internal prism is $2 \times (w - 2k) \times (l - 2k)$ cm^2.

The area that is gained by removing the internal prism is equal to the top, bottom, left, and right rectangular surfaces of the internal prism.
Since the width of the internal prism is $(w - 2k)$ cm and its thickness is 1 cm, then the total area of the top and bottom faces is $2 \times (w - 2k) \times 1$ cm^2.
Similarly, since the length of the internal prism is $(l - 2k)$ cm and its thickness is 1 cm, then the total area of the right and left faces is $2 \times (l - 2k) \times 1$ cm^2.

Summarizing, the surface area of the original prism without the removal of the internal prism, is $2 \times (lw + w + l)$ cm^2.
The area of the original prism that is lost by removing the internal prism is, $2 \times (w - 2k) \times (l - 2k)$ cm^2.
The area that is added to the area of the original prism by removing the internal prism is, $(2 \times (w - 2k) \times 1 + 2 \times (l - 2k) \times 1)$ cm^2 or $2 \times (w - 2k + l - 2k) = 2 \times (w + l - 4k)$ cm^2.
That is, the surface area of the frame, in cm^2, is

$$2 \times (lw + w + l) - 2 \times (w - 2k) \times (l - 2k) + 2 \times (w + l - 4k).$$

Since the surface area of the frame is 532 cm^2, then we equate this with the expression for the surface area and simplify the resulting equation.

$$\begin{aligned}
2 \times (lw + w + l) - 2 \times (w - 2k) \times (l - 2k) + 2 \times (w + l - 4k) &= 532 \\
lw + w + l - (w - 2k) \times (l - 2k) + (w + l - 4k) &= \tfrac{532}{2} \\
lw + w + l - (lw - 2kw - 2kl + 4k^2) + (w + l - 4k) &= 266 \\
2w + 2l + 2kw + 2kl - 4k^2 - 4k &= 266 \\
w + l + kw + kl - 2k^2 - 2k &= \tfrac{266}{2} \\
kw + kl - 2k^2 + w + l - 2k &= 133 \\
k(w + l - 2k) + 1(w + l - 2k) &= 133 \\
(w + l - 2k)(k + 1) &= 133
\end{aligned}$$

Recall that $w > 2k$ and $l > 2k$, so then $(w + l - 2k)$ is a positive integer since w, l and k are positive integers.
Therefore, $(w + l - 2k)(k + 1)$ is the product of two positive integers.
Expressed as the product of two positive integers, 133 can only be written as 1×133 or 7×19.
Note that $(k + 1) > 1$, and $(w + l - 2k) > 2k + 2k - 2k = 2k > 1$, since k is a positive integer.
That is, $(k + 1) \neq 1$ and $(w + l - 2k) \neq 1$, and so either $k + 1 = 7$ or $k + 1 = 19$.
If $k + 1 = 7$, then $k = 6$ and $w + l - 2k = 19$ or $w + l - 12 = 19$, and so $w + l = 31$.
Since $w > 2k = 12$, we require all possible values for w and l such that $w \geq 13$, $l \geq w$, and $w + l = 31$.
Written as ordered pairs (w, l) the only possibilities are $(13, 18), (14, 17)$ and $(15, 16)$.

If $k + 1 = 19$, then $k = 18$ and $w + l - 2k = 7$ or $w + l - 36 = 19$, and so $w + l = 55$.
Since $w > 2k = 36$, then $l < 55 - 36 = 29$.
This is not possible since $l \geq w$.
Therefore, the only possible values for w and l such that the frame has surface area 532 cm^2 are 13 and 18, or 14 and 17, or 15 and 16.

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2012 Hypatia Contest

Thursday, April 12, 2012
(in North America and South America)

Friday, April 13, 2012
(outside of North America and South America)

Solutions

1. (a) In $\triangle PTQ$, $\angle PTQ = 90^\circ$.
Using the Pythagorean Theorem, $PQ^2 = 32^2 + 24^2$ or $PQ^2 = 1024 + 576 = 1600$ and so $PQ = \sqrt{1600} = 40$, since $PQ > 0$.

(b) In $\triangle QTR$, $\angle QTR = 90^\circ$.
Using the Pythagorean Theorem, $51^2 = TR^2 + 24^2$ or $TR^2 = 2601 - 576 = 2025$ and so $TR = \sqrt{2025} = 45$, since $TR > 0$.
Since $PT = 32$ and $TR = 45$, then $PR = PT + TR = 32 + 45 = 77$.
In $\triangle PQR$, QT is perpendicular to base PR and so $\triangle PQR$ has area $\frac{1}{2} \times (PR) \times (QT) = \frac{1}{2} \times 77 \times 24 = 924$.

(c) From part (b), the length of PR is 77.
Since $QS : PR = 12 : 11$, then $\dfrac{QS}{77} = \dfrac{12}{11}$ or $QS = 77 \times \dfrac{12}{11} = 84$.
So then $TS = QS - QT = 84 - 24 = 60$.
Using the Pythagorean Theorem in $\triangle PTS$, $PS = \sqrt{32^2 + 60^2}$ or $PS = \sqrt{4624} = 68$, since $PS > 0$.
Using the Pythagorean Theorem in $\triangle RTS$, $RS = \sqrt{45^2 + 60^2}$ or $RS = \sqrt{5625} = 75$, since $RS > 0$.
Thus, quadrilateral $PQRS$ has perimeter $40 + 51 + 75 + 68$ or 234.

2. (a) Expanding, $(a+b)^2 = a^2 + 2ab + b^2 = (a^2 + b^2) + 2ab$.
Since $a^2 + b^2 = 24$ and $ab = 6$, then $(a+b)^2 = 24 + 2(6) = 36$.

(b) Expanding, $(x+y)^2 = x^2 + 2xy + y^2 = (x^2 + y^2) + 2xy$.
Since $(x+y)^2 = 13$ and $x^2 + y^2 = 7$, then $13 = 7 + 2xy$ or $2xy = 6$, and so $xy = 3$.

(c) Expanding, $(j+k)^2 = j^2 + 2jk + k^2 = (j^2 + k^2) + 2jk$.
Since $j + k = 6$ and $j^2 + k^2 = 52$, then $6^2 = 52 + 2jk$ or $2jk = -16$, and so $jk = -8$.

(d) Expanding, $(m^2 + n^2)^2 = m^4 + 2m^2n^2 + n^4 = (m^4 + n^4) + 2m^2n^2$.
Since $m^2 + n^2 = 12$ and $m^4 + n^4 = 136$, then $12^2 = 136 + 2m^2n^2$ or $2m^2n^2 = 8$ or $m^2n^2 = 4$, and so $mn = \pm 2$.

3. (a) Since $\angle MON = 90^\circ$, the product of the slopes of NO and OM is -1.
The slope of NO is $\dfrac{n^2 - 0}{n - 0} = n$, since $n \neq 0$ (points N and O are distinct).
The slope of OM is $\dfrac{\frac{1}{4} - 0}{\frac{1}{2} - 0} = \dfrac{1}{2}$.
Thus, $n \times \dfrac{1}{2} = -1$ or $n = -2$.

(b) Since $\angle ABO = 90^\circ$, the product of the slopes of BA and BO is -1.
The slope of BA is $\dfrac{b^2 - 4}{b - 2} = \dfrac{(b-2)(b+2)}{b-2} = b + 2$, since $b \neq 2$ (A and B are distinct).
The slope of BO is $\dfrac{b^2 - 0}{b - 0} = b$, since $b \neq 0$ (B and O are distinct).
Thus, $(b+2) \times b = -1$ or $b^2 + 2b + 1 = 0$.
Factoring, $(b+1)(b+1) = 0$ and so $b = -1$.

(c) Since $\angle PQR = 90^\circ$, the product of the slopes of PQ and RQ is -1.
The slope of PQ is $\dfrac{p^2 - q^2}{p - q} = \dfrac{(p-q)(p+q)}{p-q} = p + q$, since $p \neq q$ (P and Q are distinct).
The slope of RQ is $\dfrac{r^2 - q^2}{r - q} = \dfrac{(r-q)(r+q)}{r-q} = r + q$, since $r \neq q$ (R and Q are distinct).

Thus, $(p+q)\times(r+q)=-1$.
Since p,q and r are integers, then $p+q$ and $r+q$ are integers.
In order that $(p+q)\times(r+q)=-1$, either $p+q=1$ and $r+q=-1$ or $p+q=-1$ and $r+q=1$ (these are the only possibilities for integers p,q,r for which $(p+q)\times(r+q)=-1$).
In the first case, we add the two equations to get $p+q+r+q=1+(-1)$ or $2q+p+r=0$.
In the second case, adding the two equations gives $p+q+r+q=-1+1$ or $2q+p+r=0$.
In either case, $2q+p+r=0$, as required.

4. (a) Since p is an odd prime integer, then $p>2$.
Since the only prime divisors of $2p^2$ are 2 and p, then the positive divisors of $2p^2$ are $1,2,p,2p,p^2$, and $2p^2$.
So then, $S(2p^2)=1+2+p+2p+p^2+2p^2=3p^2+3p+3$.
Since $S(2p^2)=2613$, then $3p^2+3p+3=2613$ or $3p^2+3p-2610=0$ or $p^2+p-870=0$.
Factoring, $(p+30)(p-29)=0$, and so $p=29$ ($p\neq-30$ since p is an odd prime).

(b) Suppose $m=2p$ and $n=9q$ for some prime numbers $p,q>3$.
The positive divisors of $2p$, thus m, are $1,2,p$, and $2p$ (since $p>3$).
Therefore, $S(m)=1+2+p+2p=3p+3$.
The positive divisors of $9q$, thus n, are $1,3,q,3q,9$, and $9q$ (since $q>3$).
Therefore, $S(n)=1+3+9+q+3q+9q=13q+13$.
Since $S(m)=S(n)$, then $3p+3=13q+13$ or $3p-13q=10$.
Also, m and n are consecutive integers and so either $m-n=1$ or $n-m=1$.

If $m-n=1$, then $2p-9q=1$.
We solve the following system of two equations and two unknowns.

$$\begin{aligned} 2p-9q &= 1 && (1)\\ 3p-13q &= 10 && (2) \end{aligned}$$

Multiplying equation (1) by 3 and equation (2) by 2 we get,

$$\begin{aligned} 6p-27q &= 3 && (3)\\ 6p-26q &= 20 && (4) \end{aligned}$$

Subtracting equation (3) from equation (4), we get $q=17$.
Substituting $q=17$ into equation (1), $2p-9(17)=1$ or $2p=154$, and so $p=77$.
However, p must be a prime and thus $p\neq 77$.
There is no solution when $m-n=1$.

If $n-m=1$, then $9q-2p=1$.
We solve the following system of two equations and two unknowns.

$$\begin{aligned} 9q-2p &= 1 && (5)\\ 3p-13q &= 10 && (6) \end{aligned}$$

Multiplying equation (5) by 3 and equation (6) by 2 we get,

$$\begin{aligned} 27q-6p &= 3 && (7)\\ 6p-26q &= 20 && (8) \end{aligned}$$

Adding equation (7) and equation (8), we get $q=23$.
Substituting $q=23$ into equation (6), $3p-13(23)=10$ or $3p=309$, and so $p=103$.
Since $q=23$ and $p=103$ are prime integers greater than 3, then $m=2(103)=206$ and $n=9(23)=207$ are the only pair of consecutive integers satisfying the given properties.

(c) Since the only prime divisors of p^3q are p and q, then the positive divisors of p^3q, are $1, p, q, pq, p^2, p^2q, p^3$, and p^3q (since p and q are distinct primes).
Therefore, $S(p^3q) = p^3q + p^3 + p^2q + p^2 + pq + p + q + 1$.
Simplifying,

$$\begin{aligned} S(p^3q) &= p^3q + p^3 + p^2q + p^2 + pq + p + q + 1 \\ &= (p^3q + p^2q + pq + q) + (p^3 + p^2 + p + 1) \\ &= q(p^3 + p^2 + p + 1) + (p^3 + p^2 + p + 1) \\ &= (q + 1)(p^3 + p^2 + p + 1) \\ &= (q + 1)(p^2(p + 1) + (p + 1)) \\ &= (q + 1)(p + 1)(p^2 + 1) \end{aligned}$$

We are to determine the number of pairs of distinct primes p and q, each less than 30, such that $(q + 1)(p + 1)(p^2 + 1)$ is not divisible by 24.
There are 10 primes less than 30. These are $2, 3, 5, 7, 11, 13, 17, 19, 23$ and 29.
Therefore, the total number of possible pairs (p, q), where $p \neq q$, is $10 \times 9 = 90$.
We will count the number of pairs (p, q) for which $(q + 1)(p + 1)(p^2 + 1)$ *is* divisible by 24 and then subtract this total from 90.

If p or q equals 23, then 24 divides $(q + 1)(p + 1)(p^2 + 1)$.
There are 9 ordered pairs of the form $(23, q)$ and 9 of the form $(p, 23)$.
Thus, we count 18 pairs and since we have exhausted all possibilities using 23, we remove it from our list of 10 primes above.

Since $24 = 2^3 \times 3$, we can determine values of q for a given value of p by recognizing that each of these prime factors (three 2s and one 3) must occur in the prime factorization of $(q + 1)(p + 1)(p^2 + 1)$.
For example if $p = 2$, then $(q + 1)(p + 1)(p^2 + 1) = (q + 1)(3)(5)$.
Therefore, for $(q + 1)(p + 1)(p^2 + 1)$ to be a multiple of 24, $q + 1$ must be a multiple of 8 (since we are missing 2^3).
Thus when $p = 2$, the only possible value of q is 7 (we get this by trying the other 8 values in the list of primes).
We organize all possibilities for p (and the resulting values of q) in the table below.

p	$(p+1)(p^2+1)$	$q+1$ must be a multiple of	q (distinct from p)	Number of ordered pairs
2	$(3)(5)$	$2^3 = 8$	$q = 7$	1
3	$(4)(10) = 2^3 \times 5$	3	$q = 2, 5, 11, 17, 29$	5
5	$(6)(26) = 2^2 \times 3 \times 13$	2	$q = 3, 7, 11, 13, 17, 19, 29$	7
7	$(8)(50) = 2^3 \times 50$	3	$q = 2, 5, 11, 17, 29$	5
11	$(12)(122) = 2^3 \times 3 \times 61$	any q will work	$q = 2, 3, 5, 7, 13, 17, 19, 29$	8
13	$(14)(170) = 2^2 \times 595$	$2 \times 3 = 6$	$q = 5, 11, 17, 29$	4
17	$(18)(290) = 2^2 \times 3 \times 435$	2	$q = 3, 5, 7, 11, 13, 19, 29$	7
19	$(20)(362) = 2^3 \times 905$	3	$q = 2, 5, 11, 17, 29$	5
29	$(30)(842) = 2^2 \times 3 \times 2105$	2	$q = 3, 5, 7, 11, 13, 17, 19$	7

The total number of pairs (p, q) for which 24 divides $S(p^3q)$ is

$$18 + 1 + 5 + 7 + 5 + 8 + 4 + 7 + 5 + 7 = 67.$$

Thus, the total number of pairs of distinct prime integers p and q, each less than 30, such that $S(p^3q)$ *is not* divisible by 24, is $90 - 67 = 23$.

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Hypatia Contest

Wednesday, April 13, 2011

Solutions

1. (a) Since D is the midpoint of AB, it has coordinates $(\frac{1}{2}(0+0), \frac{1}{2}(0+6)) = (0,3)$.
The line passing through C and D has slope $\frac{3-0}{0-8}$ or $-\frac{3}{8}$.
The y-intercept of this line is the y-coordinate of point D, or 3.
Therefore, the equation of the line passing through points C and D is $y = -\frac{3}{8}x + 3$.

(b) Since E is the midpoint of BC, it has coordinates $(\frac{1}{2}(8+0), \frac{1}{2}(0+0)) = (4,0)$.
Next, we find the equation of the line passing through the points A and E.
This line has slope $\frac{6-0}{0-4}$ or $-\frac{6}{4}$ or $-\frac{3}{2}$.
The y-intercept of this line is the y-coordinate of point A, or 6.
Therefore, the equation of the line passing through points A and E is $y = -\frac{3}{2}x + 6$.
Point F is the intersection point of the lines with equation $y = -\frac{3}{8}x+3$ and $y = -\frac{3}{2}x+6$.
To find the coordinates of point F we solve the system of equations by equating y:

$$\begin{aligned} -\tfrac{3}{8}x + 3 &= -\tfrac{3}{2}x + 6 \\ 8(-\tfrac{3}{8}x + 3) &= 8(-\tfrac{3}{2}x + 6) \\ -3x + 24 &= -12x + 48 \\ 9x &= 24 \end{aligned}$$

So the x-coordinate of point F is $\frac{24}{9}$ or $x = \frac{8}{3}$.
Substituting $x = \frac{8}{3}$ into $y = -\frac{3}{2}x + 6$, we find that $y = -\frac{3}{2} \times \frac{8}{3} + 6$ or $y = 2$.
The coordinates of point F are $(\frac{8}{3}, 2)$.

(c) Triangle DBC has base BC of length 8, and height BD of length 3.
Therefore, the area of $\triangle DBC$ is $\frac{1}{2} \times 8 \times 3$ or 12.

(d) The area of quadrilateral $DBEF$ is the area of $\triangle DBC$ minus the area of $\triangle FEC$.
Triangle FEC has base EC.
Note that $EC = BC - BE = 8 - 4$ or 4.
The height of $\triangle FEC$ is equal to the vertical distance from point F to the x-axis.
That is, the height of $\triangle FEC$ is equal to the y-coordinate of point F, or 2.
Therefore, the area of $\triangle FEC$ is $\frac{1}{2} \times 4 \times 2$ or 4.
Thus, the area of quadrilateral $DBEF$ is $12 - 4$ or 8.

2. (a) Since the tens digit to be used is 3, we must consider the number of possibilities for the ones digit.
Given that the digits 0 and 9 cannot be used, there are 8 choices remaining for the ones digit.
However if the ones digit is a 3, then the number formed, 33, is a multiple of 11 which is not permitted.
Since 33 is the only multiple of 11 with tens digit 3, each of the other 7 choices for the ones digit is possible.
There are 7 numbers in S whose tens digit is a 3.
In fact, the argument above can be repeated to show that there are 7 numbers in S for each of the possible tens digits 1 through to 8.
This will be useful information for part (d) to follow.

(b) Since the ones digit to be used is 8, we must consider the number of possibilities for the tens digit.
Given that the digits 0 and 9 cannot be used, there are 8 choices remaining for the tens digit.
However if the tens digit is 8, then the number formed, 88, is a multiple of 11 which is not

permitted.
Since 88 is the only multiple of 11 with ones digit 8, each of the other 7 choices for the tens digit is possible.
There are 7 numbers in S whose ones digit is 8.
In fact, the argument above can be repeated to show that there are 7 numbers in S for each of the possible ones digits 1 through to 8.
This will also be useful information for part (d) to follow.

(c) *Solution 1*
Ignoring the second restriction that no number in S be a multiple of 11, there are 8 choices for the ones digit and 8 choices for the tens digit.
For each of the 8 choices for the tens digit, there are 8 choices for the units digit ignoring multiples of 11.
Thus, there would be 8×8 or 64 numbers in S, ignoring the second restriction.
Included in these 64 numbers are the numbers $11, 22, 33, 44, 55, 66, 77, 88$, or 8 multiples of 11.
These are the only multiples of 11 in our 64 possibilities.
Removing these from the number of possibilities, there are $64 - 8$ or 56 numbers in S.

Solution 2
Given that the digits 0 and 9 cannot be used, there are 8 choices for the tens digit.
For each choice of a tens digit, choosing the ones digit to be equal to that tens digit gives the only number that is a multiple of 11.
That is, for each possible choice of a tens digit, the ones digit cannot equal the tens digit.
Since the ones digit cannot equal $0, 9$ or the tens digit, there are 7 possible choices of a ones digit for each choice of a tens digit.
Thus, there are $8 \times 7 = 56$ numbers in S.

(d) Our work in part (a) shows that for each of the possible tens digits, 1 through 8, there are 7 numbers in S that have that tens digit.
That is, of the 56 numbers in S there are 7 whose tens digit is 1, 7 whose tens digit is 2, and so on to include 7 whose tens digit is 8.
Similarly, our work in part (b) shows that for each of the possible ones digits, 1 through 8, there are 7 numbers in S that have that ones digit.
That is, of the 56 numbers in S there are 7 whose ones digit is 1, 7 whose ones digit is 2, and so on to include 7 whose ones digit is 8.
We may determine the sum of the 56 numbers in S by considering the sum of their tens digits separately from the sum of their ones digits.
First, consider the sum of the ones digits of all of the numbers in S.
Each of the numbers 1 through 8 appear 7 times in the ones digit.
The sum of the numbers from 1 to 8 is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = \frac{(8)(9)}{2} = 36$.
Since each of these occur 7 times, then the sum of the ones digits for all numbers in S is 7×36 or 252.
Next, consider the sum of the tens digits of all of the numbers in S.
Again, each of the numbers 1 through 8 appear 7 times in the tens digit.
The sum of the numbers from 1 to 8 is 36.
Since each of these occur 7 times, the sum of the tens digits for all numbers in S is 7×36 or 252.
Since these are tens digits, they add 10×252 or 2520 to the total sum.
Thus, the sum of all of the numbers in S is the combined sum of all 56 ones digits, 252, and all 56 tens digits, which add 2520 to the sum, for a total of 2772.

3. (a) *Solution 1*

Since $3x = 5y$, then $y = \frac{3}{5}x$.
In the given Trenti-triple, the value of x is 50.
Thus, $y = \frac{3}{5}(50)$ or $y = 30$.
Since $3x = 2z$, then $z = \frac{3}{2}x$.
Since $x = 50$, then $z = \frac{3}{2}(50)$ or $z = 75$.
The Trenti-triple is $(50, 30, 75)$.

Solution 2

Let $3x = 5y = 2z = k$.
Since x, y, z are positive integers, then k is a positive integer that is divisible by 3, 5 and 2.
Any positive integer that is divisible by 3, 5 and 2 must be divisible by their least common multiple, which is $3 \times 5 \times 2$ or 30.
Since k is divisible by 30, then $k = 30m$ for some positive integer m.
That is, $3x = 5y = 2z = 30m$ and so $x = 10m$, $y = 6m$ and $z = 15m$.
Since $x = 50$, then $50 = 10m$ or $m = 5$.
Therefore, $y = 6 \times 5$ or $y = 30$ and $z = 15 \times 5$ or $z = 75$.
The Trenti-triple is $(50, 30, 75)$.

(b) *Solution 1*

Since $3x = 5y$, then $x = \frac{5}{3}y$.
Since x is a positive integer, then $\frac{5}{3}y$ is a positive integer and so y is divisible by 3, since 5 is not.
Since $5y = 2z$, then $z = \frac{5}{2}y$.
Since z is a positive integer, then $\frac{5}{2}y$ is a positive integer and so y is divisible by 2, since 5 is not.
Thus, y is divisible by both 2 and 3 and so y is divisible by the least common multiple of 2 and 3.
Therefore, in every Trenti-triple, y is divisible by 6.

Solution 2

From our work in part (a) Solution 2, it follows that since $y = 6m$ for some positive integer m, then y is divisible by 6 for every Trenti-triple.

(c) *Solution 1*

From part (b) Solution 1, we know that y is divisible by 6 for every Trenti-triple.
We can similarly show that x is divisible by 10 for every Trenti-triple.
Since $3x = 5y$, then $y = \frac{3}{5}x$.
Since y is a positive integer, then $\frac{3}{5}x$ is a positive integer and so x is divisible by 5, since 3 is not.
Since $3x = 2z$, then $z = \frac{3}{2}x$.
Since z is a positive integer, then $\frac{3}{2}x$ is a positive integer and so x is divisible by 2, since 3 is not.
Thus, x is divisible by both 5 and 2 and so x is divisible by the least common multiple of 5 and 2.
Therefore, in every Trenti-triple, x is divisible by 10.
We can similarly show that z is divisible by 15 for every Trenti-triple.
Since $3x = 2z$, then $x = \frac{2}{3}z$.
Since x is a positive integer, then $\frac{2}{3}z$ is a positive integer and so z is divisible by 3, since 2 is not.
Since $5y = 2z$, then $y = \frac{2}{5}z$.

Since y is a positive integer, then $\frac{2}{5}z$ is a positive integer and so z is divisible by 5, since 2 is not.
Thus, z is divisible by both 3 and 5 and so z is divisible by the least common multiple of 3 and 5.
Therefore, in every Trenti-triple, z is divisible by 15.
Since in every Trenti-triple, y is divisible by 6, x is divisible by 10, and z is divisible by 15, then their product xyz is divisible by $6 \times 10 \times 15$ or 900.

Solution 2
From our work in part (a) Solution 2, we have that $x = 10m$, $y = 6m$ and $z = 15m$ for some positive integer m.
Therefore, the product xyz is $(10m)(6m)(15m)$ or $900m^3$, and thus is divisible by 900 for every Trenti-triple.

4. (a) $F(8) = 6$ since

$$\begin{aligned} 8 &= 1+1+1+1+1+1+1+1 \\ &= 1+1+1+1+1+3 \\ &= 1+1+3+3 \\ &= 1+1+1+5 \\ &= 3+5 \\ &= 1+7 \end{aligned}$$

(b) Let us first begin by defining each of the ways that a positive integer n can be written as the sum of positive odd integers as a *representation* of n.
There are $F(n)$ representations of n.
To each possible representation of n, we may add a 1 to create a representation of $n+1$.
For example, $3+3+1$ is a representation of 7; adding a 1, $3+3+1+1$, creates a representation of 8.
Since every representation of n can be used to create a representation of $n+1$ in this way, then there are at least as many representations of $n+1$ as there are of n, so $F(n+1) \geq F(n)$.
Next, we will show that in fact $F(n+1) \geq F(n)+1$ by finding one additional representation of $n+1$ not described above.
We will do this by considering the cases when $n+1$ is odd and when $n+1$ is even.

<u>Case 1: $n+1$ is odd</u>
Since $n+1$ is odd, then $n+1$ is a representation of itself.
Since this representation of $n+1$ does not include a 1 ($n > 3$), then it must be a new representation not created by adding a 1 to a representation of n as described above.
Therefore, if $n+1$ is odd then $F(n+1) \geq F(n)+1$ and so $F(n+1) > F(n)$.

<u>Case 2: $n+1$ is even</u>
Since $n+1$ is even, then n is odd.
Since n is odd and $n > 3$, then $n \geq 5$.
Since n is odd and $n \geq 5$, then $n-2$ is odd and $n-2 \geq 3$.
Since $n+1 = (n-2)+3$ and $n-2 \geq 3$, then $(n-2)+3$ is a representation of $n+1$ that does not include a 1.
Since this representation of $n+1$ does not include a 1, then it must be a new representation not created by adding a 1 to a representation of n as described above.
Therefore, if $n+1$ is even then $F(n+1) \geq F(n)+1$ and so $F(n+1) > F(n)$.

Thus for all integers $n > 3$, $F(n+1) > F(n)$.

(c) Let a_n be the representation of n as the sum of n 1s.
As an example from part (a), $a_8 = 1+1+1+1+1+1+1+1$.
Let b_n be the representation of n as $(n-1)+1$ if n is even and as $(n-2)+1+1$ if n is odd.
Therefore, $b_8 = 7+1$ since 8 is even.
Let S_n be the list of the remaining representations of n.
From part (a), list S_8 consists of the following representations:

$$\begin{aligned} 8 &= 1+1+1+1+1+3 \\ &= 1+1+3+3 \\ &= 1+1+1+5 \\ &= 3+5 \end{aligned}$$

Since each of a_n and b_n are single representations of n, there are $F(n)-2$ representations in S_n.
Note that when $n=4$, S_n has no representations for n.
Consider the representations $a_n + S_n$ of $2n$.
These representations of $2n$ are n 1s added to each of the representations of S_n.
Again using our work from part (a) as an example, the representations of 16 given by $a_8 + S_8$ are:

$$\begin{aligned} a_8 + S_8 &= (1+1+1+1+1+1+1+1)+(1+1+1+1+1+3) \\ &= (1+1+1+1+1+1+1+1)+(1+1+3+3) \\ &= (1+1+1+1+1+1+1+1)+(1+1+1+5) \\ &= (1+1+1+1+1+1+1+1)+(3+5) \end{aligned}$$

In general, consider the following representations of $2n$:

- $a_n + S_n$ (there are $F(n)-2$ representations here and when $n=4$ there are none)
- $b_n + S_n$ (there are $F(n)-2$ representations here and when $n=4$ there are none)
- $a_n + a_n$
- $a_n + b_n$
- $b_n + b_n$
- $(2n-1)+1$
- $(2n-3)+3$

There are $2 \times [F(n)-2]+5 = 2F(n)+1$ representations in this list.
If these are all distinct, then $F(2n) \geq 2F(n)+1 > 2F(n)$, as required.
Since $n > 3$, then $n-3 > 0$ or $2n-3 > n$ and thus $2n-1 > n$ also.
Since both $2n-3$ and $2n-1$ are greater than n, then there can be no overlap between the last two lists of representations and the first five lists of representations in the above list.
There is no overlap between any of the third, fourth or fifth lists of representations by the definitions of a_n and b_n.
Similarly, there can be no overlap between the first two lists of representations and the third, fourth and fifth lists of representations by the definitions of a_n, b_n and S_n.
This leaves us to consider the possibility of overlap between the first two lists of representations only.
Suppose that there is a representation of $2n$ that is included in both a_n+S_n and in b_n+S_n.
Since this representation is included in a_n+S_n, then part of it looks like a_n, so the representation includes n 1s.

Since this representation is included in $b_n + S_n$, then part of it looks like b_n, so the representation includes either $n-1$ or $n-2$ depending on whether n is even or odd. Because the representation already includes some 1s (from the a_n portion), we cannot automatically include the 1 or $1+1$ from b_n.

So this representation includes n 1s and either $n-1$ or $n-2$.

These parts add to either $2n-1$ or $2n-2$.

The only way to complete the representation is then either with a 1 or with a $1+1$.

But this means that the representation then is n 1s plus $(n-1)+1$ if n is even or is n 1s plus $(n-2)+1+1$ if n is odd.

This means that this representation must be $a_n + b_n$.

Since neither a_n or b_n is in S_n, then this representation cannot actually be in $a_n + S_n$ or in $b_n + S_n$, so there cannot be any overlap between these two collections of representations.

Therefore, $F(2n) \geq 2F(n) + 1 > 2F(n)$, as required.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Hypatia Contest

Friday, April 9, 2010

Solutions

1. (a) The cost to fly is \$0.10 per kilometre plus a \$100 booking fee.
To fly 3250 km from A to B, the cost is $3250 \times 0.10 + 100 = 325 + 100 = \$425$.

(b) Since $\triangle ABC$ is a right-angled triangle, then we may use the Pythagorean Theorem.
Thus, $AB^2 = BC^2 + CA^2$, and so $BC^2 = AB^2 - CA^2 = 3250^2 - 3000^2 = 1\,562\,500$, and $BC = 1250$ km (since $BC > 0$).
Piravena travels a distance of $3250 + 1250 + 3000 = 7500$ km for her complete trip.

(c) To fly from B to C, the cost is $1250 \times 0.10 + 100 = \$225$.
To bus from B to C, the cost is $1250 \times 0.15 = \$187.50$.
Since Piravena chooses the least expensive way to travel, she will bus from B to C.
To fly from C to A, the cost is $3000 \times 0.10 + 100 = \$400$.
To bus from C to A, the cost is $3000 \times 0.15 = \$450$.
Since Piravena chooses the least expensive way to travel, she will fly from C to A.
To check, the total cost of the trip would be $\$425 + \$187.50 + \$400 = \$1012.50$ as required.

2. (a) Substituting $x = 6$, then $f(x) - f(x-1) = 4x - 9$ becomes $f(6) - f(5) = 4 \times 6 - 9$.
Since $f(5) = 18$, then $f(6) - 18 = 24 - 9$ or $f(6) - 18 = 15$ and $f(6) = 33$.

(b) Substituting $x = 5$, then $f(x) - f(x-1) = 4x - 9$ becomes $f(5) - f(4) = 4 \times 5 - 9$.
Since $f(5) = 18$, then $18 - f(4) = 20 - 9$ or $18 - f(4) = 11$ and $f(4) = 7$.
Substituting $x = 4$, then $f(x) - f(x-1) = 4x - 9$ becomes $f(4) - f(3) = 4 \times 4 - 9$.
Since $f(4) = 7$, then $7 - f(3) = 16 - 9$ or $7 - f(3) = 7$ and $f(3) = 0$.

(c) Since $f(5) = 18$, then $2(5^2) + 5p + q = 18$, or $50 + 5p + q = 18$ and so $5p + q = -32$.
Since $f(3) = 0$, then $2(3^2) + 3p + q = 0$, or $18 + 3p + q = 0$ and so $3p + q = -18$.
We solve the system of equations:

$$\begin{aligned} 5p + q &= -32 \\ 3p + q &= -18 \end{aligned}$$

Subtracting the second equation from the first gives $2p = -14$ or $p = -7$.
Substituting $p = -7$ into the first equation gives $5(-7) + q = -32$, or $-35 + q = -32$ and $q = 3$.
Therefore if $f(x) = 2x^2 + px + q$, then $p = -7$ and $q = 3$.

3. (a) Since $\triangle ABE$ is equilateral, then $\angle ABE = 60^\circ$.
Therefore, $\angle PBC = \angle ABC - \angle ABE = 90^\circ - 60^\circ = 30^\circ$.
Since $AB = BC$, then $\triangle ABC$ is a right isosceles triangle and $\angle BAC = \angle BCA = 45^\circ$.
Then, $\angle BCP = \angle BCA = 45^\circ$ and

$$\angle BPC = 180^\circ - \angle PBC - \angle BCP = 180^\circ - 30^\circ - 45^\circ = 105^\circ.$$

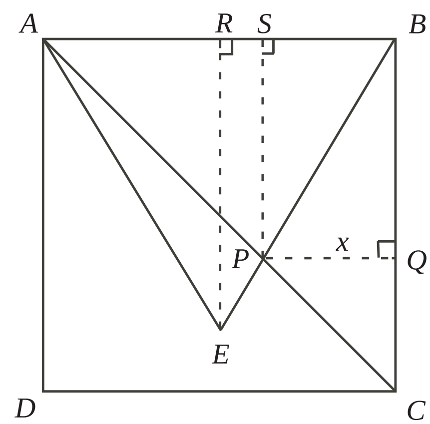
(b) *Solution 1*
In $\triangle PBQ$, $\angle PBQ = 30^\circ$ and $\angle BQP = 90^\circ$, thus $\angle BPQ = 60^\circ$.
Therefore, $\triangle PBQ$ is a 30° - 60° - 90° triangle with $PQ : PB : BQ = 1 : 2 : \sqrt{3}$.
Since $\frac{PQ}{BQ} = \frac{1}{\sqrt{3}}$, then $\frac{x}{BQ} = \frac{1}{\sqrt{3}}$ and $BQ = \sqrt{3}x$.

Solution 2
In $\triangle PQC$, $\angle QCP = 45^\circ$ and $\angle PQC = 90^\circ$, thus $\angle CPQ = 45^\circ$.
Therefore, $\triangle PQC$ is isosceles and $QC = PQ = x$.
Since $BC = 4$, then $BQ = BC - QC = 4 - x$.

(c) *Solution 1*
In $\triangle PQC$, $\angle QCP = 45^\circ$ and $\angle PQC = 90^\circ$, thus $\angle CPQ = 45^\circ$.
Therefore, $\triangle PQC$ is isosceles and $QC = PQ = x$.
Since $BC = 4$, then $BC = BQ + QC = \sqrt{3}x + x = 4$ or $x(\sqrt{3}+1) = 4$ and $x = \frac{4}{\sqrt{3}+1}$.
Rationalizing the denominator gives $x = \frac{4}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{4(\sqrt{3}-1)}{3-1} = \frac{4(\sqrt{3}-1)}{2} = 2(\sqrt{3}-1)$.

Solution 2
In $\triangle PBQ$, $\angle PBQ = 30^\circ$ and $\angle BQP = 90^\circ$, thus $\angle BPQ = 60^\circ$.
Therefore, $\triangle PBQ$ is a 30°-60°-90° triangle with $PQ : PB : BQ = 1 : 2 : \sqrt{3}$.
Since $\frac{PQ}{BQ} = \frac{1}{\sqrt{3}}$, then $\frac{x}{4-x} = \frac{1}{\sqrt{3}}$ or $\sqrt{3}x = 4 - x$ or $\sqrt{3}x + x = 4$, and $x(\sqrt{3}+1) = 4$ so $x = \frac{4}{\sqrt{3}+1}$.
Rationalizing the denominator gives $x = \frac{4}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{4(\sqrt{3}-1)}{3-1} = \frac{4(\sqrt{3}-1)}{2} = 2(\sqrt{3}-1)$.

(d) *Solution 1*
We adopt the notation $|\triangle XYZ|$ to represent the area of triangle XYZ.



Then, $|\triangle APE| = |\triangle ABE| - |\triangle ABP|$.
Since $\triangle ABE$ is equilateral, $BE = EA = AB = 4$ and the altitude from E to AB bisects side AB at R as shown.
Thus, $AR = RB = 2$ and by the Pythagorean Theorem $ER^2 = BE^2 - RB^2 = 4^2 - 2^2 = 12$ or $ER = \sqrt{12} = 2\sqrt{3}$, since $ER > 0$.
Therefore, the area of $\triangle ABE$ is $\frac{1}{2}(AB)(ER)$, or $\frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.
In $\triangle ABP$, construct the altitude from P to S on AB.
Then $PS \perp AB$ and $QB \perp AB$, so $PS \parallel QB$.
Also, $SB \perp QB$ and $PQ \perp QB$, so $SB \parallel PQ$.
Thus, $SBQP$ is a rectangle and $PS = QB$.
From (b) and (c), $QB = 4 - x = 4 - 2(\sqrt{3}-1) = 6 - 2\sqrt{3}$.
Therefore, $|\triangle ABP| = \frac{1}{2}(AB)(PS) = \frac{1}{2}(4)(6-2\sqrt{3}) = 2(6-2\sqrt{3}) = 12 - 4\sqrt{3}$.
Then, $|\triangle APE| = 4\sqrt{3} - (12 - 4\sqrt{3}) = 4\sqrt{3} - 12 + 4\sqrt{3} = 8\sqrt{3} - 12$.

Solution 2
We adopt the notation $|\triangle XYZ|$ to represent the area of triangle XYZ.
Then, $|\triangle APE| = |\triangle ABE| - |\triangle ABP|$.
However, $|\triangle ABP| = |\triangle ABC| - |\triangle BPC|$.
Thus, $|\triangle APE| = |\triangle ABE| - (|\triangle ABC| - |\triangle BPC|) = |\triangle ABE| + |\triangle BPC| - |\triangle ABC|$.
Since $\triangle ABE$ is equilateral, $BE = EA = AB = 4$ and the altitude from E to AB bisects side AB at R as shown.
Thus, $\triangle ERB$ is a 30°-60°-90° triangle with $ER : RB = \sqrt{3} : 1$ or $ER = (RB)\sqrt{3} = 2\sqrt{3}$.
Therefore, $|\triangle ABE| = \frac{1}{2}(AB)(ER) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.
Since PQ is an altitude of $\triangle BPC$, $|\triangle BPC| = \frac{1}{2}(BC)(PQ) = \frac{1}{2}(4)(2\sqrt{3}-2) = 4\sqrt{3} - 4$.
In triangle ABC, $\angle ABC = 90^\circ$.
Thus, $|\triangle ABC| = \frac{1}{2}(AB)(BC) = \frac{1}{2}(4)(4) = 8$.
Thus, $|\triangle APE| = |\triangle ABE| + |\triangle BPC| - |\triangle ABC| = (4\sqrt{3}) + (4\sqrt{3} - 4) - 8 = 8\sqrt{3} - 12$.

4. (a) We solve by factoring,

$$x^4 - 6x^2 + 8 = 0$$
$$(x^2 - 4)(x^2 - 2) = 0$$

Therefore, $x^2 = 4$ or $x^2 = 2$, and so $x = \pm 2$ or $x = \pm\sqrt{2}$.

The real values of x satisfying $x^4 - 6x^2 + 8 = 0$ are $x = -2, 2, -\sqrt{2}$, and $\sqrt{2}$.

(b) We want the smallest positive integer N for which,

$$\begin{aligned}
x^4 + 2010x^2 + N &= (x^2 + rx + s)(x^2 + tx + u) \\
x^4 + 2010x^2 + N &= x^4 + tx^3 + ux^2 + rx^3 + rtx^2 + rux + sx^2 + stx + su \\
x^4 + 2010x^2 + N &= x^4 + tx^3 + rx^3 + ux^2 + rtx^2 + sx^2 + rux + stx + su \\
x^4 + 2010x^2 + N &= x^4 + (t + r)x^3 + (u + rt + s)x^2 + (ru + st)x + su
\end{aligned}$$

Equating the coefficients from the left and right sides of this equation we have,
$t + r = 0$, $u + rt + s = 2010$, $ru + st = 0$, and $su = N$.
From the first equation we have $t = -r$.
If we substitute $t = -r$ into the third equation, then $ru - rs = 0$ or $r(u - s) = 0$.
Since $r \neq 0$, then $u - s = 0$ or $u = s$.
Thus, from the fourth equation we have $N = su = u^2$.
That is, to minimize N we need to minimize u^2.
If we substitute $t = -r$ and $s = u$ into the second equation, then $u + rt + s = 2010$ becomes $u + r(-r) + u = 2010$ or $2u - r^2 = 2010$ and so $u = \dfrac{2010 + r^2}{2}$.
Thus, $u > 0$. So to minimize u^2, we minimize u or equivalently, we minimize r.
Since u and r are integers and $r \neq 0$, u is minimized when $r = \pm 2$ (r must be even) or $u = \dfrac{2014}{2} = 1007$.
Therefore, the smallest positive integer N for which $x^4 + 2010x^2 + N$ can be factored as $(x^2 + rx + s)(x^2 + tx + u)$ with r, s, t, u integers and $r \neq 0$ is $N = u^2 = 1007^2 = 1014049$.

(c) Replacing the coefficient 2010 with M in part (b) and again equating coefficients, we have the similar four equations $t + r = 0$, $u + rt + s = M$, $ru + st = 0$, and $su = N$.
Thus we have,

$$\begin{aligned}
N - M &= su - (u + rt + s) \\
37 &= u^2 - (2u - r^2) \\
37 &= u^2 - 2u + r^2 \\
37 + 1 &= u^2 - 2u + 1 + r^2 \\
38 &= (u - 1)^2 + r^2
\end{aligned}$$

and so $r = \pm\sqrt{38 - (u - 1)^2}$.
In the table below we attempt to find integer solutions for u and r:

u	$(u-1)^2$	r
1	0	$\pm\sqrt{38}$
0 or 2	1	$\pm\sqrt{37}$
-1 or 3	4	$\pm\sqrt{34}$
-2 or 4	9	$\pm\sqrt{29}$
-3 or 5	16	$\pm\sqrt{22}$
-4 or 6	25	$\pm\sqrt{13}$
-5 or 7	36	$\pm\sqrt{2}$

We see that for all choices of u above, r is not an integer.
For any other integer choice of u not listed, $(u-1)^2 > 38$ and then $38-(u-1)^2 < 0$, so there are no real solutions for r.
Thus, when u is an integer, r cannot be, so u and r cannot both be integers. Therefore, x^4+Mx^2+N cannot be factored as in (b) for any integers M and N with $N-M=37$.
Note: Alternatively, we could have stated that $(u-1)^2+r^2$ represents the sum of two perfect squares. Since no pair of perfect squares (from the list $0, 1, 4, 9, 16, 25, 36$) sums to 38, then $(u-1)^2+r^2 \neq 38$ for any integers u and r.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2009 Hypatia Contest

Wednesday, April 8, 2009

Solutions

1. Throughout this problem, we will need to know the total number of students in the class. According to the chart, the total number is $3+2+1+2+4+2+2+3+1=20$.

   (a) There are 20 students in total.
   Of these students, 2 have both green eyes and brown hair.
   Therefore, the percentage who have both green eyes and brown hair is $\frac{2}{20} \times 100\% = 10\%$.

   (b) There are 20 students in total.
   There are five entries in the table that appear in the either the "green eyes" row or the "brown hair" column. Note that the entry that is in both the "green eyes" row and the "brown hair" column is only counted once.
   The sum of these five entries is $2+4+2+3+2=13$.
   Therefore, the percentage of the students with either green eyes or brown hair is $\frac{13}{20} \times 100\% = 65\%$.

   (c) There are $2+4+2=8$ students in total with green eyes.
   Of these students, 2 have red hair.
   Therefore, the percentage of those with green eyes who have red hair is $\frac{2}{8} \times 100\% = 25\%$.

   (d) Initially, there are 20 students in the class, of whom $1+2+1=4$ have red hair.
   Suppose that x students with red hair join the class. There will then be $20+x$ students in total, of whom $4+x$ have red hair.
   We want $\dfrac{4+x}{20+x} = \dfrac{36}{100}$.
   Reducing the fraction on the right-hand side, this equation becomes $\dfrac{4+x}{20+x} = \dfrac{9}{25}$.
   Cross-multiplying, we obtain $25(4+x) = 9(20+x)$ which gives $100+25x = 180+9x$ and so $16x = 80$ or $x = 5$.
   Therefore, 5 students with red hair must join the class.
   (We could have seen that $x = 5$ by inspection from the equation $\dfrac{4+x}{20+x} = \dfrac{9}{25}$ but this would not immediately guarantee us that this was the *only* solution to this equation.)

2. (a) *Solution 1*
   Suppose that the middle term of the sequence is x.
   If the common difference between terms in the sequence is d, then the first term is $x-d$ and the third term is $x+d$.
   Since the sum of the terms is 180, then $(x-d)+x+(x+d) = 180$ or $3x = 180$ and so $x = 60$.
   Therefore, the middle term in the sequence is 60.

   Solution 2
   Suppose that the first term in the sequence is a and the common difference between terms in the sequence is d.
   Thus, the second term is $a+d$ and the third term is $(a+d)+d = a+2d$.
   The second term is the middle term, so we need to determine the value of $a+d$.
   Since the sum of the three terms is 180, then $a+(a+d)+(a+2d) = 180$ or $3a+3d = 180$ which gives $3(a+d) = 180$.
   Thus, $a+d = 60$ and so the middle term is 60.

   (b) *Solution 1*
   We show that the middle term (which is the third term) equals 36.

Suppose that the third term of the sequence is x.
If the common difference between terms in the sequence is d, then the second term is $x-d$, the first term is $x-2d$, the fourth term is $x+d$, and the fifth term is $x+2d$.
Since the sum of the terms is 180, then

$$\begin{aligned}(x-2d)+(x-d)+x+(x+d)+(x+2d) &= 180\\ 5x &= 180\\ x &= 36\end{aligned}$$

Therefore, the middle term in the sequence is 36.
(Note that if $d=0$ then all of the terms in the sequence are equal to 36, so it is possible for more than one term to equal 36.)

Solution 2
Suppose that the first term in the sequence is a and the common difference between terms in the sequence is d.
Thus, the second term is $a+d$, the third term is $(a+d)+d=a+2d$, the fourth term is $a+3d$, and the fifth term is $a+4d$.
The third term is the middle term, so we need to determine the value of $a+2d$.
Since the sum of the five terms is 180, then

$$\begin{aligned}a+(a+d)+(a+2d)+(a+3d)+(a+4d) &= 180\\ 5a+10d &= 180\\ 5(a+2d) &= 180\\ a+2d &= 36\end{aligned}$$

Thus, $a+2d=36$ and so the middle term equals 36.
(Note that if $d=0$ then all of the terms in the sequence are equal to 36, so it is possible for more than one term to equal 36.)

(c) Suppose that the first term in the sequence is a and the common difference between terms in the sequence is d.
Thus, the second term is $a+d$, the third term is $a+2d$, the fourth term is $a+3d$, the fifth term is $a+4d$, and the sixth term is $a+5d$.
We need to determine the sum of the first and sixth terms, which equals $a+(a+5d)=2a+5d$.
Since the sum of the six terms is 180, then

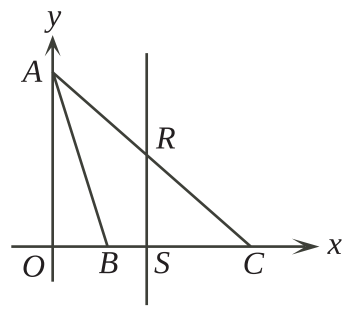
$$\begin{aligned}a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d) &= 180\\ 6a+15d &= 180\\ 3(2a+5d) &= 180\\ 2a+5d &= 60\end{aligned}$$

Thus, $2a+5d=60$ and so the sum of the first and sixth terms is 60.

3. (a) The line through B that cuts the area of $\triangle ABC$ in half is the median – that is, the line through B and the midpoint M of AC.
This line cuts the area of the triangle in half, because if we consider AC as its base, then the height of each of $\triangle AMB$ and $\triangle CMB$ is equal to the distance of point B from the

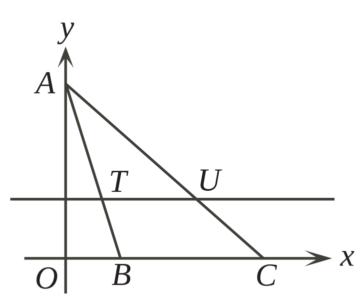
line through A and C. These two triangles also have equal bases because $AM = MC$, so their areas must be equal.
The midpoint, M, of AC has coordinates $\left(\frac{1}{2}(0+8), \frac{1}{2}(8+0)\right) = (4,4)$.
The slope of the line through $B(2,0)$ and $M(4,4)$ is $\dfrac{4-0}{4-2} = 2$.
Since this line passes through $B(2,0)$, it has equation $y - 0 = 2(x-2)$ or $y = 2x - 4$.

(b) Since line segment RS is vertical and S lies on BC, which is horizontal, then $\triangle RSC$ is right-angled at S.



Also, R lies on line segment AC, which has slope $\dfrac{0-8}{8-0} = -1$.
Since AC has a slope of -1, it makes an angle of 45° with the x-axis. In particular, the angle between RC and SC is 45°.
Since $\triangle RSC$ is right-angled at S and has a 45° angle at C, then the third-angle must be $180^\circ - 90^\circ - 45^\circ = 45^\circ$, which means that the triangle is right-angled and isosceles.
Suppose that $RS = SC = x$.
Since $\triangle RSC$ is right-angled, then the area of $\triangle RSC$ in terms of x is $\frac{1}{2}x^2$.
But we know that the area of $\triangle RSC$ is 12.5, so $\frac{1}{2}x^2 = 12.5$ or $x^2 = 25$.
Since $x > 0$, then $x = 5$.
This tells us that point S is 5 units to the left of C, so has coordinates $(8-5, 0) = (3,0)$.
Also, point R is 5 units above S, so has coordinates $(3, 0+5) = (3,5)$.

(c) *Solution 1*
Since line segment BC is horizontal and the line segment through T and U is also horizontal, then BC and TU are parallel.
Therefore $\angle ATU = \angle ABC$.



Since $\triangle ATU$ and $\triangle ABC$ also share a common angle at A, then $\triangle ATU$ is similar to $\triangle ABC$.
Since $\triangle ATU$ and $\triangle ABC$ are similar, then the ratio of their areas equals the square of the ratio of their heights.
Considering BC as the base of $\triangle ABC$, we see that its area is $\frac{1}{2}(8-2)(8) = 24$. Note that its height is 8 when considered in this direction.
Suppose that the height of $\triangle ATU$ considered from TU is h.

Then $\dfrac{13.5}{24} = \left(\dfrac{h}{8}\right)^2$ or $\dfrac{h^2}{64} = \dfrac{27}{48}$ or $h^2 = \dfrac{64(27)}{48} = 36$.
Since $h > 0$, then $h = 6$.
Therefore, the line segment TU is 6 units lower than the point $A(0,8)$, and so has equation $y = 8 - 6$ or $y = 2$.

Solution 2
Suppose that the equation of the horizontal line is $y = t$.
We find the coordinates of points T and U first.
To do this, we need to find the equation of the line through A and B and the equation of the line through A and C.
The line through A and B has slope $\dfrac{0-8}{2-0} = -4$ and passes through $(0,8)$, so has equation $y = -4x + 8$.
The line through A and C has slope $\dfrac{0-8}{8-0} = -1$ and passes through $(0,8)$, so has equation $y = -x + 8$.
The point T is the point on the line $y = -4x + 8$ with y-coordinate t.
To find the x-coordinate, we solve $t = -4x + 8$ to get $4x = 8 - t$ or $x = \frac{1}{4}(8 - t)$.
The point U is the point on the line $y = -x + 8$ with y-coordinate t.
To find the x-coordinate, we solve $t = -x + 8$ to get $x = 8 - t$.
Therefore, T has coordinates $(\frac{1}{4}(8-t), t)$, U has coordinates $(8-t, t)$, and A is at $(0,8)$.
To find the area, we remember that TU is horizontal and has length $(8-t) - \frac{1}{4}(8-t) = \frac{3}{4}(8-t)$, and the distance from TU to A is $8 - t$.
Therefore, the area in terms of t is $\frac{1}{2}\left(\frac{3}{4}(8-t)\right)(8-t) = \frac{3}{8}(8-t)^2$.
Since we know that the area equals 13.5, then $\frac{3}{8}(8-t)^2 = 13.5$ or $(8-t)^2 = \frac{8}{3}(13.5) = 36$.
Note that $t < 8$ because line segment TU is below A, so $8 - t > 0$.
Therefore, $8 - t = 6$ and so $t = 8 - 6 = 2$.
Thus, the equation of the horizontal line through T and U is $y = 2$.

4. (a) Since $\triangle ABC$ is equilateral with side length 12 and X and Y are the midpoints of CA and CB, respectively, then $CX = CY = \frac{1}{2}(12) = 6$.
Since the height of the prism is 16 and Z is the midpoint of CD, then $CZ = \frac{1}{2}(16) = 8$.
Since faces $ACDE$ and $BCDF$ are rectangles, then $\angle ACD = \angle BCD = 90^\circ$.
Thus, $\triangle XCZ$ and $\triangle YCZ$ are right-angled at C.
By the Pythagorean Theorem, $XZ = \sqrt{CX^2 + CZ^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.
Similarly, $YZ = \sqrt{CY^2 + CZ^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.
Lastly, consider $\triangle CXY$.
We know that $CX = CY = 6$ and that $\angle XCY = 60^\circ$, because $\triangle ABC$ is equilateral.
Thus, $\triangle CXY$ is isosceles with $\angle CXY = \angle CYX$.
These angles must each be equal to $\frac{1}{2}(180^\circ - \angle XCY) = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
But this means that $\triangle CXY$ is equilateral, and so $XY = CX = CY = 6$.
Therefore, $XY = 6$ and $XZ = YZ = 10$.

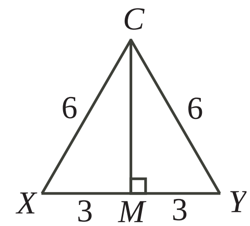
(b) To determine the surface area of solid $CXYZ$, we must determine the area of each of the four triangular faces.

Areas of $\triangle CZX$ and $\triangle CZY$
Each of these triangles is right-angled and has legs of lengths 6 and 8.
Therefore, the area of each is $\frac{1}{2}(6)(8) = 24$.

Area of $\triangle CXY$

This triangle is equilateral with side length 6.

We draw the altitude from C to M on XY. Since $\triangle CXY$ is equilateral, then M is the midpoint of XY.



Each of $\triangle CMX$ and $\triangle CMY$ is thus a 30°-60°-90° triangle, since each already has a 60° angle and a 90° angle.

Using the ratios from this special triangle, $CM = \frac{\sqrt{3}}{2}(CX) = \frac{\sqrt{3}}{2}(6) = 3\sqrt{3}$.

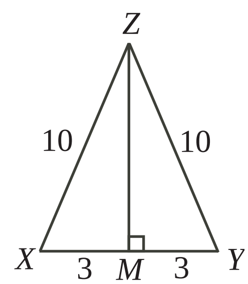
Since $XY = 6$, then the area of $\triangle CXY$ is $\frac{1}{2}(6)(3\sqrt{3}) = 9\sqrt{3}$.

Area of $\triangle XYZ$

Here, $XY = 6$ and $XZ = YZ = 10$.

Again, we drop an altitude from Z to XY.

Since $\triangle XYZ$ is isosceles, then this altitude meets XY at its midpoint, M.



Note that $XM = MY = \frac{1}{2}(XY) = 3$.

By the Pythagorean Theorem, $ZM = \sqrt{ZX^2 - XM^2} = \sqrt{10^2 - 3^2} = \sqrt{91}$.

Since $XY = 6$, then the area of $\triangle XYZ$ is $\frac{1}{2}(6)(\sqrt{91}) = 3\sqrt{91}$.

Therefore, the total surface area of solid $CXYZ$ is $24+24+9\sqrt{3}+3\sqrt{91} = 48+9\sqrt{3}+3\sqrt{91}$.

(c) Step 1: Examination of $\triangle MDN$

We know that $DM = 4$, $DN = 2$, and $\angle MDN = 60°$ (because $\triangle EDF$ is equilateral).

Since $DM : DN = 2 : 1$ and the contained angle is 60°, then $\triangle MDN$ must be a 30°-60°-90° triangle.

Therefore, MN is perpendicular to DF.

Using the ratios in the special triangle, $MN = \sqrt{3}DN = 2\sqrt{3}$.

We could have instead calculated the length of MN using the cosine law to determine this.

Step 2: Calculation of CP

We know that $QC = 8$ and $\angle QCP = 60°$.

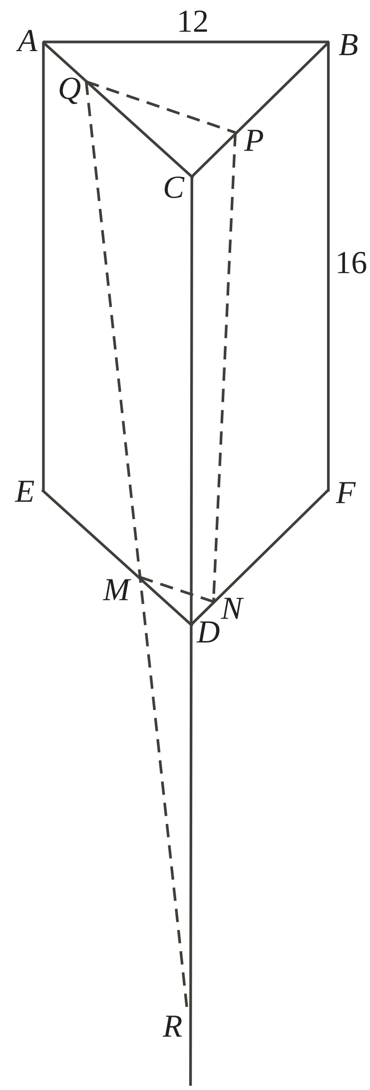
Since MN is perpendicular to DF, this tells us that the plane $MNPQ$ is perpendicular to the plane $BCDF$.

Since QP is parallel to MN (they lie in the same plane $MNPQ$ and in parallel planes ACB and DEF), then QP is perpendicular to CB.

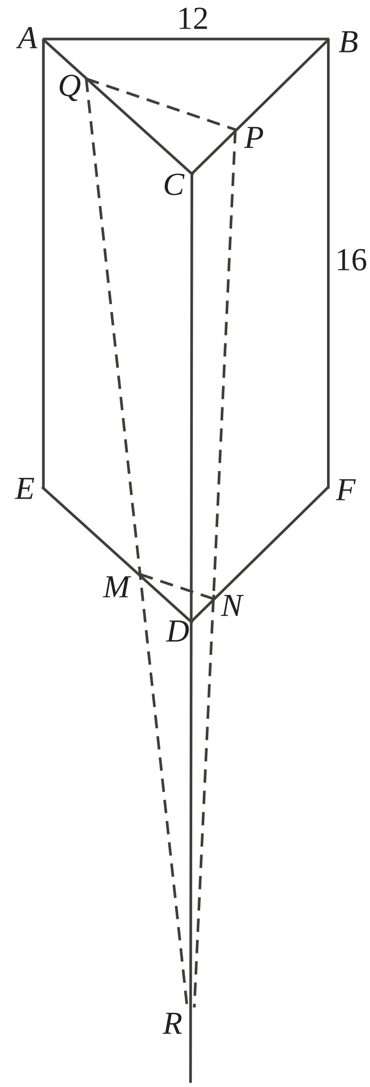
Therefore, $\triangle QCP$ is right-angled at P and contains a 60° angle, making it also a

30°-60°-90° triangle.
Thus, $CP = \frac{1}{2}(CQ) = \frac{1}{2}(8) = 4$ and $QP = \sqrt{3}CP = 4\sqrt{3}$.

Step 3: Construction
Extend CD downwards.
Next, extend QM until it intersects the extension of CD at R. (Note here that the line through QM will intersect the line through CD since they are two non-parallel lines lying in the same plane.)



Consider $\triangle RDM$ and $\triangle RCQ$.
The two triangles share a common angle at R and each is right-angled ($\triangle RDM$ at D and $\triangle RCQ$ at C), so the two triangles are similar.
Since $QC = 8$ and $MD = 4$, then their ratio of similarity is $2 : 1$.
This means that $RC = 2RD$, ie. D is the midpoint of RC.
Since $CD = 16$, then $DR = 16$.
Similarly, since $CP : DN = 2 : 1$, then when PN is extended to meet the extension of CD, it will do so at the same point R.



Step 4: Calculation of volume of $QPCDMN$

The volume of $QPCDMN$ equals the difference between the volume of the triangular based pyramid $RCQP$ and the volume of the triangular based pyramid $RDMN$.
(Another name for a triangular based pyramid is a tetrahedron.)
The volume of a tetrahedron equals one-third times the area of the base time the height.
The area of $\triangle CPQ$ is $\frac{1}{2}(CP)(QP) = \frac{1}{2}(4)(4\sqrt{3}) = 8\sqrt{3}$.
The area of $\triangle DNM$ is $\frac{1}{2}(DN)(MN) = \frac{1}{2}(2)(2\sqrt{3}) = 2\sqrt{3}$.
The length of RD is 16 and the length of RC is 32.
Therefore, the volume of $QPCDMN$ is $\frac{1}{3}(8\sqrt{3})(32) - \frac{1}{3}(2\sqrt{3})(16) = \frac{256\sqrt{3}}{3} - \frac{32\sqrt{3}}{3} = \frac{224\sqrt{3}}{3}$.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Hypatia Contest

Wednesday, April 16, 2008

Solutions

1. (a) By definition, $3\nabla 2 = 2(3) + 2^2 + 3(2) = 6 + 4 + 6 = 16$.

(b) We have

$$\begin{aligned} x\nabla(-1) &= 8 \\ 2x + (-1)^2 + x(-1) &= 8 \\ x + 1 &= 8 \\ x &= 7 \end{aligned}$$

so $x = 7$.

(c) We have

$$\begin{aligned} 4\nabla y &= 20 \\ 2(4) + y^2 + 4y &= 20 \\ y^2 + 4y + 8 &= 20 \\ y^2 + 4y - 12 &= 0 \\ (y+6)(y-2) &= 0 \end{aligned}$$

so $y = -6$ or $y = 2$.

(d) We have

$$\begin{aligned} (w-2)\nabla w &= 14 \\ 2(w-2) + w^2 + (w-2)w &= 14 \\ 2w - 4 + w^2 + w^2 - 2w &= 14 \\ 2w^2 - 4 &= 14 \\ 2w^2 &= 18 \\ w^2 &= 9 \end{aligned}$$

so $w = 3$ or $w = -3$.

2. (a) The slope of the line through $A(7,8)$ and $B(9,0)$ is $\dfrac{8-0}{7-9} = \dfrac{8}{-2} = -4$.
Therefore, the line has equation $y = -4x + b$ for some b.
Since $B(9,0)$ lies on this line, then $0 = -4(9) + b$ so $b = 36$.
Thus, the equation of the line is $y = -4x + 36$.

(b) We want to determine the point of intersection between the lines having equations $y = -4x + 36$ and $y = 2x - 10$.
Equating values of y, we obtain $-4x + 36 = 2x - 10$ or $46 = 6x$ so $x = \frac{23}{3}$.
We substitute this value of x into the equation $y = 2x - 10$ to determine the value of y, obtaining $y = 2(\frac{23}{3}) - 10 = \frac{46}{3} - \frac{30}{3} = \frac{16}{3}$.
Thus, the coordinates of P are $(\frac{23}{3}, \frac{16}{3})$.

(c) *Solution 1*
The x-coordinate of A is 7 and the x-coordinate of B is 9.
The average of these x-coordinates is $\frac{1}{2}(7 + 9) = 8$.
Since the x-coordinate of P is $\frac{23}{3} < 8$, then the x-coordinate of P is closer to that of A than that of B.
Since the points P, A and B lie on a straight line, then P is closer to A than to B.

Solution 2
The y-coordinate of A is 8 and the y-coordinate of B is 0.
The average of these y-coordinates is $\frac{1}{2}(8+0) = 4$.
Since the y-coordinate of P is $\frac{16}{3} > 4$, then the y-coordinate of P is closer to that of A than that of B.
Since the points P, A and B lie on a straight line, then P is closer to A than to B.

Solution 3
The coordinates of A are $(7,8)$, of B are $(9,0)$, and of P are $(\frac{23}{3}, \frac{16}{3})$, then

$$PA = \sqrt{(7-\tfrac{23}{3})^2 + (8-\tfrac{16}{3})^2} = \sqrt{(-\tfrac{2}{3})^2 + (\tfrac{8}{3})^2} = \sqrt{\tfrac{68}{9}}$$

and

$$PB = \sqrt{(9-\tfrac{23}{3})^2 + (0-\tfrac{16}{3})^2} = \sqrt{(\tfrac{4}{3})^2 + (-\tfrac{16}{3})^2} = \sqrt{\tfrac{272}{9}}$$

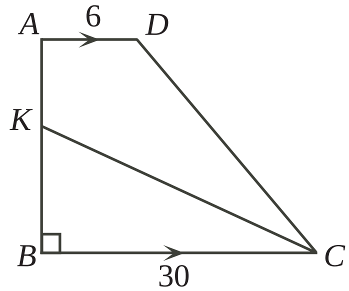
Thus, $PB > PA$, so P is closer to A than to B.

3. (a) *Solution 1*
Trapezoid $ABCD$ has bases $AD = 6$ and $BC = 30$, and height $AB = 20$. (AB is a height since it is perpendicular to BC.)
Therefore, the area of $ABCD$ is $\frac{1}{2}(6+30)(20) = 360$.

Solution 2
Join B to D.
Since AB is perpendicular to BC and AD is parallel to BC, then AB is perpendicular to AD.
Thus, $\triangle DAB$ is right-angled at A, and so has area $\frac{1}{2}(6)(20) = 60$.
Also, $\triangle BDC$ can be considered as having base $BC = 30$ and height equal to the length of BA (that is, height equal to 20), and so has area $\frac{1}{2}(30)(20) = 300$.
The area of trapezoid $ABCD$ is the sum of the areas of $\triangle DAB$ and $\triangle BDC$, so equals $60 + 300 = 360$.

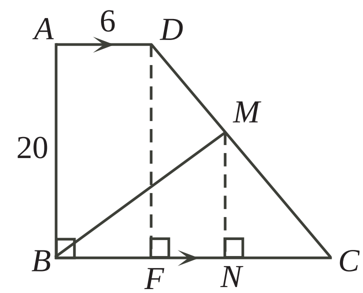
Solution 3
Since AB is perpendicular to BC and AD is parallel to BC, then AB is perpendicular to AD.
Drop a perpendicular from D to F on BC.
Then $ADFB$ is a rectangle that is 6 by 20, and so has area $6(20) = 120$.
Also, $FC = BC - BF = 30 - AD = 30 - 6 = 24$.
Thus, $\triangle DFC$ is right-angled at F, has height $DF = AB = 20$ and base $FC = 24$. Hence, the area of $\triangle DFC$ is $\frac{1}{2}(20)(24) = 240$.
The area of trapezoid $ABCD$ is the sum of the areas of rectangle $ADFB$ and $\triangle DFC$, so equals $120 + 240$ or 360.

(b) First, we note that if K is on AB, then $\triangle KBC$ and quadrilateral $KADC$ cover the entire area of trapezoid $ABCD$.



Thus, if the areas of $\triangle KBC$ and quadrilateral $KADC$ are equal, then each equals one-half of the area of trapezoid $ABCD$, or $\frac{1}{2}(360) = 180$.
Suppose that $BK = h$.
Then $\triangle KBC$ has base $BC = 30$ and height $BK = h$, and so $\frac{1}{2}(30)h = 180$, or $h = 12$.
Therefore, $BK = 12$.

(c) *Solution 1*
As in (b), we want the area of $\triangle MBC$ to equal 180. Drop a perpendicular from M to N on BC.
Again as in (b), since the base BC of $\triangle MBC$ has length 30, then the height MN of $\triangle MBC$ needs to be 12 for its area to be 180.
Drop a perpendicular from D to F on BC.
Since DF is perpendicular to BC, then $ADFB$ is a rectangle, so $BF = 6$, which gives $FC = BC - BF = 30 - 6 = 24$.
Also, $DF = AB = 20$.



By the Pythagorean Theorem, $DC = \sqrt{20^2 + 24^2} = \sqrt{400 + 576} = \sqrt{976} = 4\sqrt{61}$.
Lastly, we must calculate the length of MC.

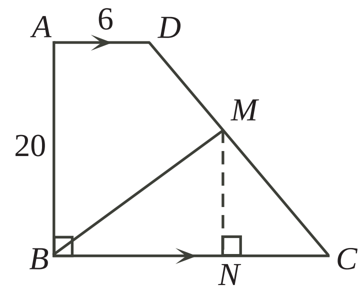
<u>Method 1</u>
We know that $\sin(\angle DCF) = \dfrac{DF}{DC} = \dfrac{20}{4\sqrt{61}} = \dfrac{5}{\sqrt{61}}$.
Since $MN = 12$, then $MC = \dfrac{MN}{\sin(\angle DCF)} = \dfrac{12}{5/\sqrt{61}}$, or $MC = \frac{12}{5}\sqrt{61}$.

<u>Method 2</u>
We know that $\triangle DFC$ is similar to $\triangle MNC$, since each is right-angled and they have a common angle at C.
Therefore, $\dfrac{MC}{MN} = \dfrac{DC}{DF}$ so $MC = \frac{12}{20}(4\sqrt{61}) = \frac{12}{5}\sqrt{61}$.

<u>Method 3</u>
Since $\triangle DFC$ is similar to $\triangle MNC$, then $\dfrac{NC}{MN} = \dfrac{FC}{DF}$ so $NC = \frac{12(24)}{20} = \frac{72}{5}$.
By the Pythagorean Theorem,

$$MC = \sqrt{MN^2 + NC^2} = \sqrt{12^2 + \left(\tfrac{72}{5}\right)^2} = \tfrac{12}{5}\sqrt{6^2 + 5^2} = \tfrac{12}{5}\sqrt{61}$$

Solution 2
As in (b), we want the area of $\triangle MBC$ to equal 180.



Drop a perpendicular from M to N on BC.
Again as in (b), since the base BC of $\triangle MBC$ has length 30, then the height MN of $\triangle MBC$ needs to be 12 for its area to be 180.
We place the diagram on a coordinate grid, with B at the origin, A on the positive y-axis, and C on the positive x-axis.
Thus, the coordinates of B are $(0,0)$, the coordinates of A are $(0,20)$, the coordinates of D are $(6,20)$, and the coordinates of C are $(30,0)$.
Since the length of MN is 12, then the coordinates of M are $(s,12)$ for some real number s.
But M lies on DC, so the slope of DC equals the slope of MC, or $\dfrac{0-20}{30-6} = \dfrac{0-12}{30-s}$ so $-20(30-s) = 24(-12)$ or $20s - 600 = -288$ or $20s = 312$ or $s = \frac{78}{5}$.
Using the coordinates of M and of C, we have

$$MC = \sqrt{\left(30 - \tfrac{78}{5}\right)^2 + (0-12)^2} = \sqrt{\left(\tfrac{72}{5}\right)^2 + 12^2} = \tfrac{12}{5}\sqrt{6^2+5^2} = \tfrac{12}{5}\sqrt{61}$$

4. (a) Taking all possible products of pairs,

$$\begin{aligned} 2(3) + 2x + 2(2x) + 3x + 3(2x) + x(2x) &= -7 \\ 6 + 15x + 2x^2 &= -7 \\ 2x^2 + 15x + 13 &= 0 \\ (2x+13)(x+1) &= 0 \end{aligned}$$

so $x = -1$ or $x = -\frac{13}{2}$.

(b) Since each term equals 1, -1 or 2, then the pairs of the terms whose products is 1 are those coming from 1×1 and $(-1) \times (-1)$.
We know that there are m terms equal to 1. How many pairs can these terms form?
To form a pair, there are m choices for the first entry and then $m-1$ choices for the first entry (all but the first term chosen). This gives $m(m-1)$ pairs.
But we have counted each pair twice here (as we have counted both a and b as well as b and a, while we only want to count one of these), so we divide by 2 to obtain $\frac{1}{2}m(m-1)$ pairs.
(Alternatively, we could have said that there were $\dbinom{m}{2} = \dfrac{m(m-1)}{2}$ pairs.)
We also know that there are n terms equal to -1.
In a similar way, these will form $\frac{1}{2}n(n-1)$ pairs.
Therefore, in total, there are $\frac{1}{2}m(m-1) + \frac{1}{2}n(n-1)$ pairs of distinct terms whose product is 1.

(c) Suppose that the sequence contains m terms equal to 2 and so $n = 100 - m$ terms equal to -1.
The terms equal to 2 form $\frac{1}{2}m(m-1)$ pairs, each contributing $2 \times 2 = 4$ to the peizi-sum.

The terms equal to -1 form $\frac{1}{2}n(n-1) = \frac{1}{2}(100-m)(99-m)$ pairs each equal to $(-1)\times(-1) = 1$.
Since there are m terms equal to 2 and $100-m$ terms equal to -1, then there are $m(100-m)$ pairs formed by choosing one 2 and one -1, and so $m(100-m)$ pairs of terms, each contributing $2\times(-1) = -2$ to the peizi-sum.
These are all of the possible types of pairs, so the peizi-sum, S, is

$$S = 4\left(\tfrac{1}{2}m(m-1)\right) + 1\left(\tfrac{1}{2}(100-m)(99-m)\right) + (-2)\left(m(100-m)\right)$$

or

$$S = 2m^2 - 2m + 50(99) - \tfrac{199}{2}m + \tfrac{1}{2}m^2 - 200m + 2m^2$$

or

$$S = \tfrac{9}{2}m^2 - \tfrac{603}{2}m + 4950$$

The equation $S = \frac{9}{2}m^2 - \frac{603}{2}m + 4950$ is a quadratic equation in m (forming a parabola opening upwards), and so is minimized at its vertex, which occurs at

$$m = -\frac{-\frac{603}{2}}{2\left(\frac{9}{2}\right)} = \tfrac{67}{2} = 33\tfrac{1}{2}$$

However, this value of m is not an integer, so is not the value of m that solves our problem.
The parabola formed by this equation is symmetric about its vertex and increases in either direction from its vertex, so the minimum value at an integer value of m occurs at both $m = 33$ and $m = 34$ (since these are the closest integers to $33\frac{1}{2}$ and are the same distance from $33\frac{1}{2}$).
We can substitute either value to determine the minimum possible peizi-sum, which is $\frac{9}{2}(33)^2 - \frac{603}{2}(33) + 4950 = -99$.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Hypatia Contest

Wednesday, April 18, 2007

Solutions

1. (a) The possible routes are:
$A \to B \to C \to D \to A$ $\quad$ $A \to B \to D \to C \to A$
$A \to C \to B \to D \to A$ $\quad$ $A \to C \to D \to B \to A$
$A \to D \to B \to C \to A$ $\quad$ $A \to D \to C \to B \to A$

(b) We list each route and its length:

$A \to B \to C \to D \to A$: Length $AB + BC + CD + DA = 80 + 120 + 90 + 40 = 330$ km
$A \to B \to D \to C \to A$: Length $AB + BD + DC + CA = 80 + 60 + 90 + 105 = 335$ km
$A \to C \to B \to D \to A$: Length $AC + CB + BD + DA = 105 + 120 + 60 + 40 = 325$ km
$A \to C \to D \to B \to A$: Length $AC + CD + DB + BA = 105 + 90 + 60 + 80 = 335$ km
$A \to D \to B \to C \to A$: Length $AD + DB + BC + CA = 40 + 60 + 120 + 105 = 325$ km
$A \to D \to C \to B \to A$: Length $AD + DC + CB + BA = 40 + 90 + 120 + 80 = 330$ km

The two routes of shortest length are $A \to C \to B \to D \to A$ and $A \to D \to B \to C \to A$, which are each of length 325 km.
The two routes of longest length are $A \to B \to D \to C \to A$ and $A \to C \to D \to B \to A$, which are each of length 335 km.

(c) *Solution 1*
We can list the possible routes:
$A \to B \to C \to E \to D \to A$ $\quad$ $A \to B \to D \to E \to C \to A$
$A \to C \to B \to E \to D \to A$ $\quad$ $A \to C \to D \to E \to B \to A$
$A \to D \to B \to E \to C \to A$ $\quad$ $A \to D \to C \to E \to B \to A$
Therefore, there are 6 possible routes.
(Note that in fact each route from (a) gives a route here in (c) by adding an E between the third and fourth stops on the original route.)

Solution 2
Consider a route $A \to x \to y \to E \to z \to A$.
There are 3 possibilities for x (B, C or D).
For each of these possibilities, there are 2 possibilities for y.
After x and y are chosen, there is only 1 possibility of z.
So there are $3 \times 2 = 6$ possible routes.

(d) From the first piece of information, $AD + DC + CE + EB + BA = 600$ km so $40 + 90 + CE + EB + 80 = 600$ km or $CE + EB = 390$ km.
From the second piece of information, $AC + CD + DE + EB + BA = 700$ km so $105 + 90 + 225 + EB + 80 = 700$ km or $EB = 200$ km.
Since $EB = 200$ km and $CE + EB = 390$ km, then $CE = 190$ km, so the distance from C to E is 190 km.

2. (a) Here is a sequence of moves that works:

Move #	P	Q	R	S	Comment
	9	9	1	5	
1	8	8	4	4	3 added to R
2	7	7	7	3	3 added to R
3	6	6	6	6	3 added to S

There are other sequences of moves that will work.

(b) i. In total, there are $31 + 27 + 27 + 7 = 92$ marbles, so if there is an equal number in each pail, there must be 23 in each pail.

Here is a sequence of moves that works:

Move #	P	Q	R	S	Comment
	31	27	27	7	
1	30	26	26	10	3 added to S
2	29	25	25	13	3 added to S
3	28	24	24	16	3 added to S
4	27	23	23	19	3 added to S
5	26	22	22	22	3 added to S
6	25	21	21	25	3 added to S
7	24	24	20	24	3 added to Q
8	23	23	23	23	3 added to R

There are other sequences of moves that will work.

ii. Initially, pail P contains 31 marbles.
We want pail P to contain 23 marbles, so we must decrease the number of marbles in pail P by 8.
In any legal move, the number of marbles in pail P decreases by at most 1 (that is, it decreases by 1 or increases by 3).
Therefore, we need at least 8 legal moves in which the number of marbles in pail P decreases (and potentially some where the number of marbles in pail P increases).
Thus, it takes at least 8 legal moves to obtain the same number of marbles in each pail.
(Note that in part (i), we showed that we could do this in 8 legal moves, so 8 is the minimum number of moves needed.)

(c) *Solution 1*
Starting with 10, 8, 11, and 7 marbles in the pails, there are $10 + 8 + 11 + 7 = 36$ marbles in total.
To have an equal number of marbles in each pail, we would need $36 \div 4 = 9$ marbles in each pail.
On any legal move, the number of marbles in any pail decreases by 1 or increases by 3.
If the pail contains an even number n of marbles before a legal move, then it will contain either $n-1$ or $n+3$ marbles after the legal move, so will contain an odd number of marbles.
Similarly, if the pail contains an odd number of marbles before a legal move, then it will contain an even number of marbles after the legal move.
But we start with two pails containing an even number of marbles and two pails containing an odd number of marbles.
After the first legal move, the pails originally containing an even number of marbles will contain an odd number of marbles and the pails originally containing an odd number of marbles will contain an even number of marbles.
This gets us back to the the same situation – two pails with an even number and two pails with an odd number of marbles.
Therefore, after any move, this situation will not change.
Therefore, it is impossible to ever have 9 marbles in each pail, as there will always be two pails containing an even number of marbles.

Solution 2
Starting with 10, 8, 11, and 7 marbles in the pails, there are $10 + 8 + 11 + 7 = 36$ marbles in total.
To have an equal number of marbles in each pail, we would need $36 \div 4 = 9$ marbles in

each pail.
On any legal move, the number of marbles in any pail decreases by 1 or increases by 3.

Assume that it is possible to end up with 9 marbles in each pail.
We show that this cannot happen by proving the following fact:

> If it is possible to end up with 9 marbles in each pail, then after each move, the difference between the number of marbles in any two pails must be a multiple of 4.

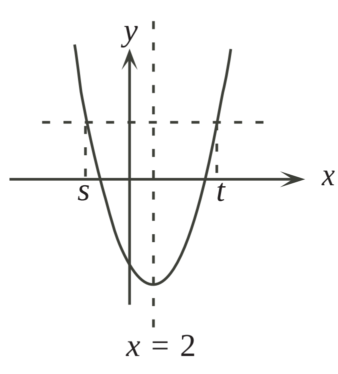
Assume that this fact was true after a certain move. (We know that it is true at the end, since the difference between the numbers in any pair of pails is 0.)
Suppose that there were a, b, c and d marbles in the pails.
Pick two of the four pails (say, the pails with a and b marbles). Before this move (that is, after the previous move), either these two pails each had 1 more marble in each (so $a+1$ and $b+1$ marbles which preserves the difference) or one pail had 1 more marble and the other had 3 fewer marbles (so $a+1$ and $b-3$ or $a-3$ and $b+1$ which changes the difference by 4).
Therefore, before this move, the differences between the number of marbles in the pails are all multiples of 4.
This tells us that, to end up with 9 marbles in each pail, the difference between the numbers of marbles in any pair of pails is always a multiple of 4.
But this is not true with our initial condition of 10, 8, 11 and 7 marbles (since, for example, $11-10$ is not a multiple of 4).
Therefore, it is impossible to end up with an equal number of marbles in each pail.

3. (a) If $f(x)=0$, then $x^2-4x-21=0$.
Factoring the left side, we obtain $(x-7)(x+3)=0$, so $x=7$ or $x=-3$.
(We could obtain the same values of x by using the quadratic formula.)

(b) *Solution 1*
Completing the square in the original function,

$$f(x)=x^2-4x-21=x^2-4x+4-4-21=(x-2)^2-25$$

so the axis of symmetry of the parabola $y=f(x)$ is the vertical line $x=2$. (The axis of symmetry could also have been found using the average of the roots from (a).)
If $f(s)=f(t)$, then s and t are symmetrically located around the axis of symmetry.



In other words, the average value of s and t is the x-coordinate of the axis of symmetry, so $\frac{1}{2}(s+t)=2$ or $s+t=4$.

(Note that this agrees with our answer from part (a), but that we needed to proceed formally here to make sure that there were no other answers.)

Solution 2
Rearranging,

$$\begin{aligned} s^2-4s-21 &= t^2-4t-21 \\ s^2-t^2-4s+4t &= 0 \\ (s+t)(s-t)-4(s-t) &= 0 \\ (s+t-4)(s-t) &= 0 \end{aligned}$$

Therefore, $s+t-4=0$ or $s-t=0$.
Since we are told that s and t are different real numbers, then $s-t\neq 0$.
Therefore, $s+t-4=0$ or $s+t=4$.

(c) *Solution 1*
Proceeding algebraically in a similar way to part (b), Solution 2,

$$\begin{aligned} (a^2-4a-21)-(b^2-4b-21) &= 4 \\ a^2-b^2-4a+4b &= 4 \\ (a+b-4)(a-b) &= 4 \end{aligned}$$

Since a and b are integers, then $a+b-4$ and $a-b$ are integers as well. In particular, they are integers whose product is 4.
We make a table to check the possibilities:

$a+b-4$	$a-b$	$2a-4$	a	b
4	1	5	$\frac{9}{2}$	$\frac{7}{2}$
2	2	4	4	2
1	4	5	$\frac{9}{2}$	$\frac{1}{2}$
-4	-1	-5	$-\frac{1}{2}$	$\frac{1}{2}$
-2	-2	-4	0	2
-1	-4	-5	$-\frac{1}{2}$	$\frac{7}{2}$

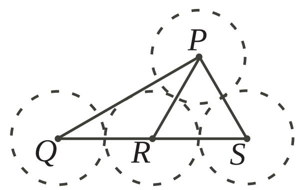
Therefore, the one pairs of positive integer values of a and b that works is $(a,b)=(4,2)$.
(Note that we could have cut down our work in this table by noticing that if $a+b-4=x$ and $a-b=y$, then $2a=x+y$, so $x+y$ (that is, the sum of the values of $a+b-4$ and $a-b$) must be even, which eliminates all but two of the rows in the table.)

Solution 2
As in part (b), the axis of symmetry of the parabola $y=f(x)$ is $x=2$.
Since the parabola has leading coefficient $+1$, then it is the same shape as the parabola $y=x^2$.
In the parabola $y=x^2$ (and so in the parabola $y=f(x)$), the lattice points moving to the right from the axis of symmetry are $(0,0)$, $(1,1)$, $(2,4)$, $(3,9)$, $(4,16)$, and so on. The vertical distances moving from one point to the next are 1, 3, 5, 7, and so on.
A similar pattern is true when we move successive units to the left from the axis of symmetry.
Starting from the left, the sequence of successive vertical differences is thus

$$\ldots,-7,-5,-3,-1,1,3,5,7,\ldots$$

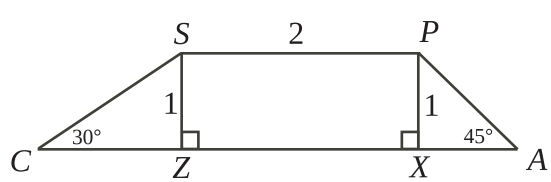
For $f(a) - f(b) = 4$ with a and b integers, we must find a sequence of consecutive differences that add to 4 or -4 (depending on whether a or b is further to the left).
We can only get 4 or -4 by using $(-3)+(-1)$ or $1+3$. The relative positions of these are starting at the axis of symmetry and moving two units to the right, or starting two units to the left of the axis of symmetry and moving two units to the right.
Since the axis of symmetry for the given parabola is $x = 2$, then the only solution is $(a, b) = (4, 2)$, since a and b must both be positive.

4. (a) Join PQ, PR, PS, RQ, and RS.
Since the circles with centre Q, R and S are all tangent to BC, then QR and RS are each parallel to BC (as the centres Q, R and S are each 1 unit above BC).
This tells us that QS passes through R.
When the centres of tangent circles are joined, the line segments formed pass through the associated point of tangency, and so have lengths equal to the sum of the radii of those circles.
Therefore, $QR = RS = PR = PS = 1 + 1 = 2$.

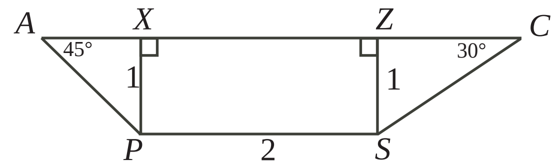


Since $PR = PS = RS$, then $\triangle PRS$ is equilateral, so $\angle PSR = \angle PRS = 60^\circ$.
Since $\angle PRS = 60^\circ$ and QRS is a straight line, then $\angle QRP = 180^\circ - 60^\circ = 120^\circ$.
Since $QR = RP$, then $\triangle QRP$ is isosceles, so $\angle PQR = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$.
Since $\angle PQS = 30^\circ$ and $\angle PSQ = 60^\circ$, then $\angle QPS = 180^\circ - 30^\circ - 60^\circ = 90^\circ$, so $\triangle PQS$ is a 30°-60°-90° triangle.

(b) In (a), we saw that QS is parallel to BC.
Similarly, since P and S are each one unit from AC, then PS is parallel to AC.
Also, since P and Q are each one unit from AB, then PQ is parallel to AB.
Therefore, the sides of $\triangle PQS$ are parallel to the corresponding sides of $\triangle ABC$.
Thus, the angles of $\triangle ABC$ are equal to the corresponding angles of $\triangle PQS$, so $\triangle ABC$ is a 30°-60°-90° triangle.
This means that if we can determine one of the side lengths of $\triangle ABC$, we can then determine the lengths of the other two sides using the side ratios in a 30°-60°-90° triangle.
Consider side AC.
Since the circle with centre P is tangent to sides AB and AC, then the line through A and P bisects $\angle BAC$. Thus, $\angle PAC = 45^\circ$.
Similarly, the line through C and S bisects $\angle ACB$. Thus, $\angle SCA = 30^\circ$.
We extract trapezoid $APSC$ from the diagram, obtaining

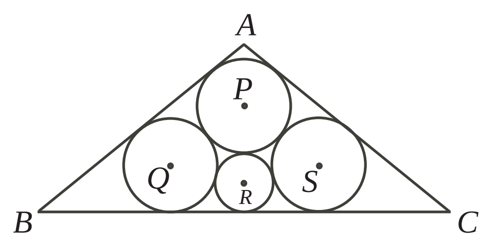


or

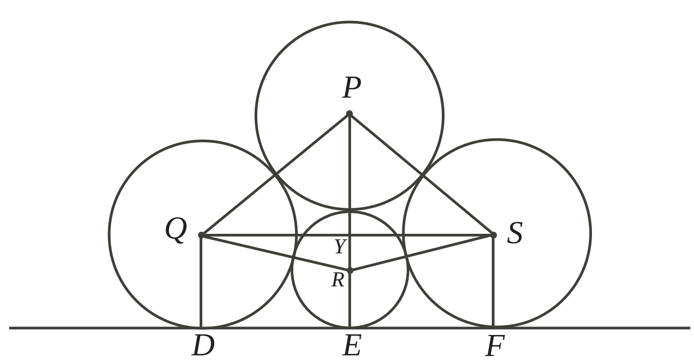


depending on your perspective. Drop perpendiculars from P and S to X and Z on side AC.
Since PS is parallel to AC and PX and SZ are perpendicular to AC, then $PXZS$ is a rectangle, so $XZ = PS = 2$.
Since $\triangle AXP$ is right-angled at X, has $PX = 1$ (the radius of the circle), and $\angle PAX = 45°$, then $AX = PX = 1$.
Since $\triangle CZS$ is right-angled at Z, has $SZ = 1$ (the radius of the circle), and $\angle SCZ = 30°$, then $CZ = \sqrt{3}SZ = \sqrt{3}$ (since $\triangle SZC$ is also a 30°-60°-90° triangle).
Thus, $AC = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$.
Since $\triangle ABC$ is a 30°-60°-90° triangle, with $\angle ACB = 60°$ and $\angle CAB = 90°$, then $BC = 2AC = 6 + 2\sqrt{3}$, and $AB = \sqrt{3}AC = \sqrt{3}(3 + \sqrt{3}) = 3\sqrt{3} + 3$.
Therefore, the side lengths of $\triangle ABC$ are $AC = 3+\sqrt{3}$, $AB = 3\sqrt{3}+3$, and $BC = 6+2\sqrt{3}$.

(c) After the described transformation, we obtain the following diagram.



Drop perpendiculars from Q, R and S to D, E and F respectively on BC. Since the circles with centres Q, R and S are tangent to BC, then D, E and F are the points of tangency of these circles to BC.
Thus, $QD = SF = 1$ and $RE = r$.
Join QR, RS, PS, PQ, and PR.
Since we are connecting centres of tangent circles, then $PQ = PS = 2$ and $QR = RS = PR = 1 + r$.
Join QS.
By symmetry, PRE is a straight line (that is, PE passes through R).
Since QS is parallel to BC as in parts (a) and (b), then QS is perpendicular to PR, meeting at Y.



Since $QD = 1$, then $YE = 1$. Since $RE = r$, then $YR = 1 - r$.
Since $QR = 1+r$, $YR = 1-r$ and $\triangle QYR$ is right-angled at Y, then, by the Pythagorean

Theorem,

$$QY^2 = QR^2 - YR^2 = (1+r)^2 - (1-r)^2 = (1+2r+r^2) - (1-2r+r^2) = 4r$$

Since $PR = 1 + r$ and $YR = 1 - r$, then $PY = PR - YR = 2r$.
Since $\triangle PYQ$ is right-angled at Y, then

$$\begin{aligned} PY^2 + YQ^2 &= PQ^2 \\ (2r)^2 + 4r &= 2^2 \\ 4r^2 + 4r &= 4 \\ r^2 + r - 1 &= 0 \end{aligned}$$

By the quadratic formula, $r = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \dfrac{-1 \pm \sqrt{5}}{2}$.
Since $r > 0$, then $r = \dfrac{-1 + \sqrt{5}}{2}$ (which is the reciprocal of the famous “golden ratio”).

Canadian
Mathematics
Competition
An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2006 Hypatia Contest

Thursday, April 20, 2006

Solutions

1. (a) *Solution 1*

The first odd positive integer is 1. The second odd positive integer is 3, which is 2 larger than the first. The third odd positive integer is 5, which is 2 larger than the second.
Therefore, the 25th odd positive integer will be $24 \times 2 = 48$ larger than the first odd positive integer, since we must add 2 to get to each successive odd number.
Thus, the 25th odd positive integer is $1 + 48 = 49$.
There are $1 + 2 + 3 + 4 + 5 + 6 = 21$ integers in the first six rows of the pattern, so 49 must appear in the 7th row.

Solution 2

The first odd positive integer is 1, which is 1 less than the first even positive integer, namely 2.
The second odd positive integer is 3, which is 1 less than the second even positive integer, namely 4.
This pattern continues, with the 25th odd positive integer being 1 less than the 25th even positive integer, which is $25 \times 2 = 50$.
Therefore, the 25th odd positive integer is 49.
There are $1 + 2 + 3 + 4 + 5 + 6 = 21$ integers in the first six rows of the pattern, so 49 must appear in the 7th row.

(b) In the triangular pattern, the first row contains 1 number, the second row 2 numbers, the third row 3 numbers, and so on.
Thus, the first twenty rows contain $1 + 2 + 3 + \cdots + 19 + 20$ numbers in total. This total equals $\dfrac{20(21)}{2} = 210$ (using the fact that $1 + 2 + \cdots + (n-1) + n = \dfrac{n(n+1)}{2}$).
Therefore, the 19th integer in the 21st row is the $210 + 19 = 229$th odd positive integer.
Using either of the methods of part (a), this integer is $1 + 228(2) = 229(2) - 1 = 457$.

(c) To get from 1 to 1001, we must add 2 a total of 500 times, so 1001 is the 501st odd positive integer.
From part (b), we know that the first 20 rows contain 210 integers, so the row number is larger than 20.
How many integers do the first 30 rows contain? They contain $1 + 2 + \cdots + 29 + 30 = \dfrac{30(31)}{2} = 465$ integers.
This tells us that the first 31 rows contain $465 + 31 = 496$ integers.
Since 1001 is the 501st odd positive integer, it must be the 5th integer in the 32nd row.

2. (a) We find CE by first finding BE.
Since $AE = 24$ and $\angle AEB = 60^\circ$, then $BE = 24\cos(60^\circ) = 24\left(\frac{1}{2}\right) = 12$.
(We could have also used the fact that $\triangle ABE$ is a 30°-60°-90° triangle and the appropriate ratios.)
Since $BE = 12$ and $\angle BEC = 60^\circ$, then $CE = 12\cos(60^\circ) = 12\left(\frac{1}{2}\right) = 6$.

(b) Using the same strategy as in (a),

$$\begin{aligned} AB &= 24\sin(60^\circ) = 24\left(\tfrac{\sqrt{3}}{2}\right) = 12\sqrt{3} \\ BC &= 12\sin(60^\circ) = 12\left(\tfrac{\sqrt{3}}{2}\right) = 6\sqrt{3} \\ CD &= 6\sin(60^\circ) = 6\left(\tfrac{\sqrt{3}}{2}\right) = 3\sqrt{3} \\ ED &= 6\cos(60^\circ) = 6\left(\tfrac{1}{2}\right) = 3 \end{aligned}$$

The perimeter of quadrilateral $ABCD$ is equal to $AB + BC + CD + DA$ and $DA = DE + EA$, so the perimeter is $12\sqrt{3} + 6\sqrt{3} + 3\sqrt{3} + 3 + 24 = 27 + 21\sqrt{3}$.

(c) The area of quadrilateral $ABCD$ is equal to the sum of the areas of triangles ABE, BCE and CDE.
Thus,

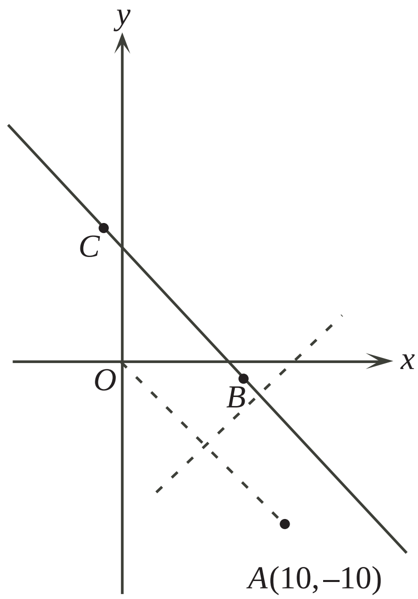
$$\begin{aligned} \text{Area} &= \tfrac{1}{2}(BE)(BA) + \tfrac{1}{2}(CE)(BC) + \tfrac{1}{2}(DE)(DC) \\ &= \tfrac{1}{2}(12)(12\sqrt{3}) + \tfrac{1}{2}(6)(6\sqrt{3}) + \tfrac{1}{2}(3)(3\sqrt{3}) \\ &= 72\sqrt{3} + 18\sqrt{3} + \tfrac{9}{2}\sqrt{3} \\ &= \tfrac{189}{2}\sqrt{3} \end{aligned}$$

3. (a) The line through points B and C has slope $\dfrac{-1-7}{7-(-1)} = -1$.
Since $(7,-1)$ lies on the line, the line has equation $y - (-1) = -1(x-7)$ or $y = -x + 6$.

(b) *Solution 1*
Suppose P has coordinates (x, y). Since P lies on ℓ, then $y = -x+6$, so P has coordinates $(x, -x+6)$.
Since A has coordinates $(10, -10)$, then $PA = \sqrt{(x-10)^2 + (-x+16)^2}$.
Since O has coordinates $(0, 0)$, then $PO = \sqrt{x^2 + (-x+6)^2}$.
For $PA = PO$, we must have $PA^2 = PO^2$ or

$$\begin{aligned} (x-10)^2 + (-x+16)^2 &= x^2 + (-x+6)^2 \\ x^2 - 20x + 100 + x^2 - 32x + 256 &= x^2 + x^2 - 12x + 36 \\ 320 &= 40x \\ x &= 8 \end{aligned}$$

so P has coordinates $(8, -2)$.



Solution 2

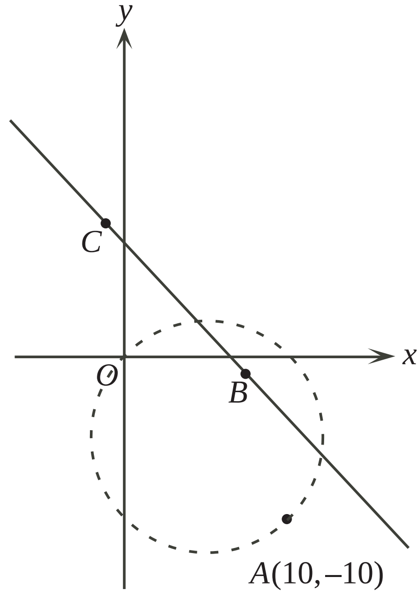
For P to be equidistant from A and O, P must lie on the perpendicular bisector of AO. Since A has coordinates $(10, -10)$ and O has coordinates $(0, 0)$, then AO has slope -1, so the perpendicular bisector has slope 1 and passes through the midpoint, $(5, -5)$, of AO. Therefore, the perpendicular bisector has equation $y - (-5) = x - 5$ or $y = x - 10$. Thus, P must be the point of intersection of the lines $y = x - 10$ and the line ℓ which has equation $y = -x + 6$, so $-x + 6 = x - 10$ or $2x = 16$ or $x = 8$.
Therefore, P has coordinates $(8, -2)$.

(c) *Solution 1*

Since Q lies on the line ℓ, then its coordinates are of the form $(q, -q + 6)$, as in (b).
For $\angle OQA = 90^\circ$, we need the slopes of OQ and QA to be negative reciprocals (that is, the product of the slopes is -1).
The slope of OQ is $\dfrac{-q+6}{q}$.
The slope of QA is $\dfrac{-q+6-(-10)}{q-10} = \dfrac{-q+16}{q-10}$.
Therefore, we must solve

$$\begin{aligned} \frac{-q+6}{q} \cdot \frac{-q+16}{q-10} &= -1 \\ (-q+6)(-q+16) &= -q(q-10) \\ q^2 - 22q + 96 &= -q^2 + 10q \\ 2q^2 - 32q + 96 &= 0 \\ q^2 - 16q + 48 &= 0 \\ (q-4)(q-12) &= 0 \end{aligned}$$

Therefore, $q = 4$ or $q = 12$, so Q has coordinates $(4, 2)$ or $(12, -6)$.



Solution 2

For $\angle OQA = 90^\circ$, then Q must lie on the circle with diameter OA.
The centre of the circle with diameter OA is the midpoint M of OA, or $(5, -5)$.
The radius of the circle with diameter OA is OM or $\sqrt{5^2 + (-5)^2} = \sqrt{50}$.
Therefore, this circle has equation $(x-5)^2 + (y+5)^2 = 50$.
The points Q that we are seeking are the points on both the circle and the line ℓ, ie. the points of intersection. Since $y = -x + 6$,

$$\begin{aligned}
(x-5)^2 + (-x+6+5)^2 &= 50 \\
x^2 - 10x + 25 + x^2 - 22x + 121 &= 50 \\
2x^2 - 32x + 96 &= 0 \\
x^2 - 16x + 48 &= 0 \\
(x-4)(x-12) &= 0
\end{aligned}$$

Therefore, $x = 4$ or $x = 12$, so Q has coordinates $(4, 2)$ or $(12, -6)$.

4. (a) Suppose p is a prime number.
The only positive divisors of p are 1 and p, so $\sigma(p) = 1 + p$.
Thus,

$$I(p) = \frac{1+p}{p} = \frac{1}{p} + 1 \leq \tfrac{1}{2} + 1 = \tfrac{3}{2}$$

since $p \geq 2$.

(b) *Solution 1*

Suppose that p is an odd prime number and k is a positive integer. Note that $p \geq 3$.
The positive divisors of p^k are $1, p, p^2, \ldots, p^{k-1}, p^k$, so

$$I(p^k) = \frac{1 + p + p^2 + \cdots + p^k}{p^k} = \frac{1}{p^k}\left(\frac{1(p^{k+1}-1)}{p-1}\right) = \frac{p^{k+1}-1}{p^{k+1}-p^k}$$

Thus,

$$\begin{array}{rrcl} & I(p^k) & < & 2 \\ \iff & \dfrac{p^{k+1}-1}{p^{k+1}-p^k} & < & 2 \\ \iff & p^{k+1}-1 & < & 2(p^{k+1}-p^k) \\ \iff & 0 & < & p^{k+1}-2p^k+1 \\ \iff & 0 & < & p^k(p-2)+1 \end{array}$$

which is true since $p \geq 3$.

Solution 2
Suppose that p is an odd prime number and k is a positive integer. Note that $p \geq 3$. The positive divisors of p^k are 1, p, p^2, …, p^{k-1}, p^k, so

$$\begin{aligned} I(p^k) &= \frac{p^k+p^{k-1}+\cdots+p+1}{p^k} \\ &= 1+\frac{1}{p}+\cdots+\frac{1}{p^{k-1}}+\frac{1}{p^k} \\ &= \frac{1\left(1-\left(\frac{1}{p}\right)^{k+1}\right)}{1-\frac{1}{p}} \qquad \text{(geometric series)} \\ &< \frac{1}{1-\frac{1}{p}} \\ &= \frac{p}{p-1} \\ &= 1+\frac{1}{p-1} \\ &\leq 1+\frac{1}{2} \\ &< 2 \end{aligned}$$

(c) Since p is a prime number, the positive divisors of p^2 are 1, p and p^2, so $I(p^2) = \dfrac{1+p+p^2}{p^2}$.
Since q is a prime number, $I(q) = \dfrac{1+q}{q}$, as in (a).
Since p and q are prime numbers, the positive divisors of p^2q are 1, p, p^2, q, pq, and p^2q,

so

$$\begin{aligned} I(p^2q) &= \frac{1+p+p^2+q+pq+p^2q}{p^2q} \\ &= \frac{(1+p+p^2)+q(1+p+p^2)}{p^2q} \\ &= \frac{(1+p+p^2)(1+q)}{p^2q} \\ &= \frac{1+p+p^2}{p^2}\cdot\frac{1+q}{q} \\ &= I(p^2)I(q) \end{aligned}$$

as required.

(d) We start by listing a number of facts:

- From part (b), if n is a prime number, then $I(n) < 2$. Thus, to obtain an n with $I(n) > 2$, we must combine multiple primes.
- If $n = {p_1}^{e_1}{p_2}^{e_2}\cdots{p_m}^{e_m}$ where the p_i's are distinct primes and the e_i's are positive integers, then by extending the ideas of (c), we can see that $I(n) = I({p_1}^{e_1})I({p_2}^{e_2})\cdots I({p_m}^{e_m})$.
- If p and q are primes with $p < q$, then $I(p^k) > I(q^k)$.
  (This is because $I(p^k) = \dfrac{p^k+p^{k-1}+\cdots+p+1}{p^k} = 1+\dfrac{1}{p}+\cdots+\dfrac{1}{p^{k-1}}+\dfrac{1}{p^k}$ and $I(q^k) = 1+\dfrac{1}{q}+\cdots+\dfrac{1}{q^{k-1}}+\dfrac{1}{q^k}$. If $p < q$, then $\dfrac{1}{p} > \dfrac{1}{q}$, $\dfrac{1}{p^2} > \dfrac{1}{q^2}$, etc., so $I(p^k) > I(q^k)$.)
- The previous point tells us that smaller primes are more efficient than larger primes at increasing $I(n)$. In other words, given an n, we can find an m with $I(m) > I(n)$ by replacing some of the prime factors of n with smaller primes to obtain m. (For example, if $n = 5^2 7^3 11$, then $m = 5^2 3^3 7$ would gives $I(m) > I(n)$.)
- From (b), $I(p^k) < \dfrac{p}{p-1}$ so in particular $I(3^a) < \frac{3}{2}$ and $I(5^b) < \frac{5}{4}$. Therefore, $I(3^a5^b) = I(3^a)I(5^b) < \frac{3}{2}\cdot\frac{5}{4} = \frac{15}{8} < 2$.
- The previous two points tell us that no odd integer n with at most two prime factors can have $I(n) > 2$. Therefore, to get $I(n) > 2$, n must have at least three prime factors.

Let us consider integers of the form $n = 3^a5^b7^c$ and try to find one with $I(n) > 2$:

a	b	c	n	$I(n)$
1	1	1	105	$I(n) = I(3)I(5)I(7) = \frac{4}{3}\cdot\frac{6}{5}\cdot\frac{8}{7} = \frac{64}{35} < 2$
2	1	1	315	$I(n) = I(3^2)I(5)I(7) = \frac{13}{9}\cdot\frac{6}{5}\cdot\frac{8}{7} = \frac{208}{105} < 2$
3	1	1	945	$I(n) = I(3^3)I(5)I(7) = \frac{40}{27}\cdot\frac{6}{5}\cdot\frac{8}{7} = \frac{128}{63} > 2$

So $I(945) > 2$.

Why is $n = 945$ the smallest odd integer with $I(n) > 2$?

- We notice that any positive odd integer with at least 4 prime factors is at least $3(5)(7)(11)$ or 1155, which is larger than 945, so we can restrict to looking at integers with at most 3 prime factors.

- By our opening remarks, we can restrict our search even further, looking at only those odd integers with 3, 5 and 7 as possible prime factors, since we can decrease the primes and make $I(n)$ larger at the same time.
- Furthermore, we only need to consider those integers $n = 3^{e_1}5^{e_2}7^{e_3}$ with $e_1 \geq e_2 \geq e_3$, since otherwise we could reassign the exponents in this order and obtain a smaller integer. (For example, $3^2 5^3 7$ is larger than $3^3 5^2 7$.)
- Since n must have at least three prime factors and we can assume that these prime factors are 3, 5 and 7 and that the exponents have the property from the previous point, then there are no integers smaller than 945 left to check other than the ones in the table above (none of which have $I(n) > 2$).

Therefore, $n = 945$ is the smallest odd positive integer with $I(n) > 2$.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2005 Hypatia Contest

Wednesday, April 20, 2005

Solutions

1. (a) By definition, $2 \diamond 3 = 2^2 - 4(3) = 4 - 12 = -8$.

   (b) By definition, $k \diamond 2 = k^2 - 4(2) = k^2 - 8$.
   By definition, $2 \diamond k = 2^2 - 4(k) = 4 - 4k$.
   So we want to solve

$$\begin{aligned} k^2 - 8 &= 4 - 4k \\ k^2 + 4k - 12 &= 0 \\ (k+6)(k-2) &= 0 \end{aligned}$$

   so $k = -6$ or $k = 2$.
   Checking, $(-6) \diamond 2 = (-6)^2 - 4(2) = 28$, $2 \diamond (-6) = 2^2 - 4(-6) = 28$, so $k = -6$ works.
   Also, if $k = 2$, then $2 \diamond 2 = 2 \diamond 2$ so $k = 2$ works as well.

   (c) Since $3 \diamond x = y$, then $3^2 - 4x = y$ or $9 - 4x = y$.
   Since $2 \diamond y = 8x$, then $2^2 - 4y = 8x$ or $4 - 4y = 8x$.
   We now have a system of two equations in two unknowns.
   Since $4 - 4y = 8x$ and $y = 9 - 4x$, then

$$\begin{aligned} 4 - 4(9 - 4x) &= 8x \\ 4 - 36 + 16x &= 8x \\ 8x &= 32 \\ x &= 4 \end{aligned}$$

   Since $x = 4$, then $y = 9 - 4(4) = -7$.
   (We could have solved this system of equations in several different ways instead.)
   Checking, $3 \diamond x = 3 \diamond 4 = 3^2 - 4(4) = -7 = y$ and $2 \diamond y = 2 \diamond (-7) = 2^2 - 4(-7) = 32 = 8x$, so $x = 4$, $y = -7$ is indeed the solution.

2. (a) Since 3, then 1, then 4 toothpicks have been removed from the initial pile of 11 toothpicks, there are now 3 toothpicks remaining.
   Since players have removed 1, 3 and 4 toothpicks on turns already, then Chris can only remove 2 or 5 toothpicks now on his turn, because of rules 2 and 3.
   Since there are only 3 toothpicks remaining, Chris must remove 2 toothpicks.
   This leaves 1 toothpick in the pile, and the only possible move that Gwen can now make is to remove 5 toothpicks, which is impossible.
   Therefore, Gwen cannot make her turn.
   Since Chris was the last player able to move, then Chris wins.

   (b) After Gwen has removed 5 toothpicks, there are 5 remaining and Chris can remove 1, 2, 3, or 4 on his turn.
   If Chris removes 1, there are 4 remaining and Gwen can remove all of them (since no one has yet removed 4 toothpicks on a turn). This empties the pile, so Gwen wins.
   If Chris removes 2, there are 3 remaining and Gwen can remove all of them (since no one has yet removed 3 toothpicks on a turn). This empties the pile, so Gwen wins.
   If Chris removes 3, there are 2 remaining and Gwen can remove all of them (since no one has yet removed 2 toothpicks on a turn). This empties the pile, so Gwen wins.
   If Chris removes 4, there is 1 remaining and Gwen can remove all of them (since no one has yet removed 1 toothpick on a turn). This empties the pile, so Gwen wins.
   Therefore, no matter what Chris removes, Gwen can always win the game.

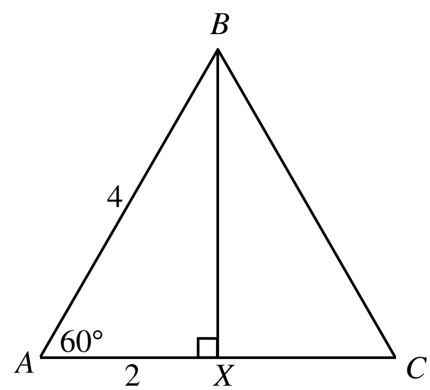
(c) After Gwen has removed 2 toothpicks, there are 7 toothpicks remaining, and Chris can take 1, 3, 4, or 5 on his turn.
If Chris removes 5 toothpicks, there are 2 remaining and Gwen can remove 1, 3 or 4. Thus, Gwen must remove 1, leaving 1 toothpick and Chris can remove 3 or 4. He is unable to make his turn, so Gwen wins. So Chris should not remove 5 toothpicks.

If Chris removes 3 or 4 toothpicks, there are 4 or 3 toothpicks remaning, and Gwen can remove all of them (since in either case that number of toothpicks hasn't yet been removed on a turn), so Gwen wins. So Chris should not remove 3 or 4 toothpicks. (If Gwen removed 1 toothpick instead of 4 or 3 toothpicks, she would still be guaranteed to win, since Chris would be unable to go again. Why?)

If Chris removes 1 toothpick, there are 6 remaining and Gwen can remove 3, 4 or 5. If Gwen now removes 5 toothpicks, there is 1 remaining, and Chris is unable to make his move, since he can now only remove 3 or 4 toothtpicks. So Gwen wins. Similarly, if Gwen had removed 4 toothpicks, there would be 2 remaining and Chris cannot remove 1 or 2 since these numbers have already been used, so Gwen wins. If Gwen had removed 3 toothpicks, there would be 3 remaining and Chris cannot remove 1, 2 or 3, since these numbers have already been used, so Gwen wins.

Thus, regardless of what Chris does on his turn, Gwen will win.

3. (a) *Solution 1*
Drop a perpendicular from B to X on AC.
Since $\triangle ABC$ is equilateral, then $AB = CB$, so X will be the midpoint of AC, so $AX = 2$.



By the Pythagorean Theorem, $BX = \sqrt{AB^2 - AX^2} = \sqrt{4^2 - 2^2} = \sqrt{12} = 2\sqrt{3}$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(BX) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.

Solution 2
Drop a perpendicular from B to X on AC.
Since $\triangle ABC$ is equilateral, then $AB = CB$, so X will be the midpoint of AC, so $AX = 2$.
Since $\angle BAX = 60^\circ$ and BX is perpendicular to AX, then $\triangle BAX$ is a 30°-60°-90° triangle, so $BX = \sqrt{3}AX = 2\sqrt{3}$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(BX) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.

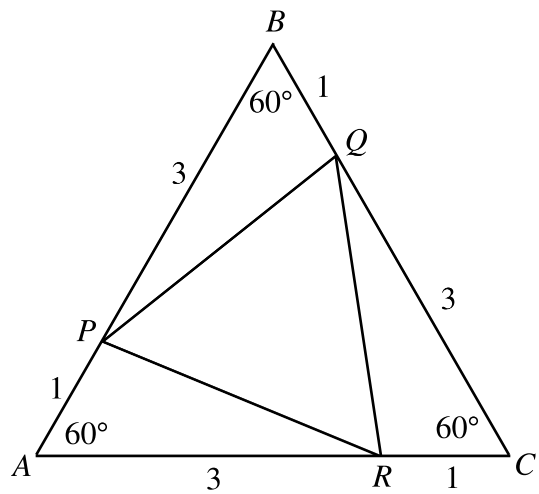
Solution 3
Drop a perpendicular from B to X on AC.
Since $\triangle ABC$ is equilateral, then $AB = CB$, so X will be the midpoint of AC, so $AX = 2$.

Since $\angle BAX = 60^\circ$, then $BX = BA\sin(60^\circ) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(AC)(BX) = \frac{1}{2}(4)(2\sqrt{3}) = 4\sqrt{3}$.

Solution 4
The area of $\triangle ABC$ is given by the formula

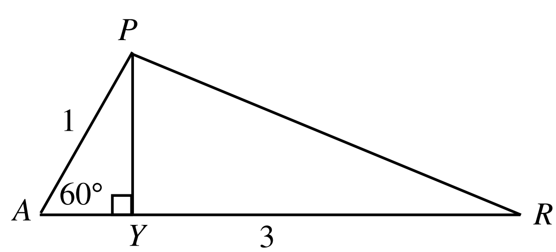
$$\frac{1}{2}(AB)(AC)\sin(\angle BAC) = \frac{1}{2}(4)(4)\sin(60^\circ) = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$$

(b) Since $AP = BQ = CR = 1$, then $AR = BP = CQ = 3$, since the side length of $\triangle ABC$ is 4.



Since $AP = BQ = CR = 1$, $PB = QC = RA = 3$ and $\angle RAP = \angle BPQ = \angle QCR = 60^\circ$, then $\triangle RAP$, $\triangle PBQ$ and $\triangle QCR$ are all congruent (by side-angle-side). Therefore, the areas of $\triangle PBQ$, $\triangle RAP$ and $\triangle QCR$ will all be equal.
Finding the area of any of these three triangles will give us the area of all three, so we determine the area of $\triangle RAP$, because this is easiest to visualize.

Method #1
Drop a perpendicular from P to Y on AR.



Then, the area of $\triangle RAP$ is equal to $\frac{1}{2}(AR)(PY) = \frac{1}{2}(3)(PY) = \frac{3}{2}(PY)$, so we need to find the length of PY.
Since $\angle RAP = 60^\circ$, then $PY = AP\sin(60^\circ) = 1\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$.
Therefore, the area of $\triangle RAP$ is $\frac{3}{2}(PY) = \frac{3\sqrt{3}}{4}$.

Method #2
The area of $\triangle RAP$ is $\frac{1}{2}(RA)(AP)\sin(\angle RAP) = \frac{1}{2}(3)(1)\sin(60^\circ) = \frac{3\sqrt{3}}{4}$.

Using either method, we obtain that the area of $\triangle PBQ$ is $\frac{3\sqrt{3}}{4}$.

Lastly, we must determine the area of $\triangle PQR$.
Method #1
To do this, we can subtract the combined areas of $\triangle PBQ$, $\triangle RAP$ and $\triangle QCR$ from the area of the large triangle, $\triangle ABC$.
But the areas of these three triangles are equal (as stated above), and we found the area of $\triangle ABC$ in part (a).
Therefore, the area of $\triangle PQR$ is $4\sqrt{3} - 3\left(\frac{3\sqrt{3}}{4}\right) = \frac{16\sqrt{3}}{4} - \frac{9\sqrt{3}}{4} = \frac{7\sqrt{3}}{4}$.

Method #2
Since $\triangle RAP$, $\triangle PBQ$ and $\triangle QCR$ are all congruent, then $PQ = QR = RP$, so $\triangle PQR$ is equilateral.
So, if we can calculate the side length of $\triangle PQR$, then we can use a similar method to any of the methods from (a) to calculate the area of $\triangle PQR$.
Using the cosine law in $\triangle RAP$, we can find PR:

$$\begin{aligned} PR^2 &= PA^2 + AR^2 - 2(PA)(AR)\cos(\angle PAR) \\ PR^2 &= 1^2 + 3^2 - 2(1)(3)\cos(60^\circ) \\ PR^2 &= 10 - 6\left(\frac{1}{2}\right) \\ PR^2 &= 7 \end{aligned}$$

so $PR = \sqrt{7}$.
We can then use any of the methods from (a) to determine that the area of $\triangle PQR$ is $\frac{7\sqrt{3}}{4}$.

4. (a) *Solution 1*
Since the middle number has to be the largest of the three numbers in the triple, then the only possibilities for b are 3, 4 and 5.
If $b = 3$, then a and c can only be 1 and 2 or 2 and 1, a total of 2 possible triples.
If $b = 4$, then a and c can be 1 and 2, 1 and 3, 2 and 3, or their reverses, a total of 6 possible triples.
If $b = 5$, then a and c can be 1 and 2, 1 and 3, 1 and 4, 2 and 3, 2 and 4, 3 and 4, or their reverses, a total of 12 possible triples.
Thus, in total there are 20 possible triples.

Solution 2
This solution uses the combinatorial notation $\binom{n}{r}$ and $n!$.
First, we choose three different numbers from the set $\{1, 2, 3, 4, 5\}$.
There are $\binom{5}{3} = 10$ ways of doing this.
From these three numbers, to form a triple (a, b, c) with $a < b$ and $b > c$, the middle number b must be the largest, so there is no choice as to what to put in the middle position.
With the two remaining numbers we can put them in the first and last position in either order (ie. two possibilities).
Therefore, each choice of 3 different numbers gives us two possible triples, so the total number of possible triples is $10 \times 2 = 20$.

(To be totally rigorous, we should also note that we can obtain every such triple in this way, and that we don't get any overlap, since we're choosing three different numbers always, and we can't have equal triples coming from two different choices of three numbers.)

(b) *Solution 1*

Each arrangement of $\{1, 2, 3, 4, 5, 6\}$ has one number in each of six positions.
If 254 occurs as a block in the arrangement, then the arrangement must be of one of the forms $254xyz$, $x254yz$ $xy254z$, or $xyz254$, where x, y and z are 1, 3 and 6 in some order.
With each of these 4 forms, there are 6 ways in which the numbers 1, 3 and 6 can fill the three remaining places – either 1,3,6 or 1,6,3 or 3,1,6 or 3,6,1 or 6,1,3 or 6,3,1.
Therefore, there are $4 \times 6 = 24$ arrangements containing 254 consecutively in that order.

Solution 2

We treat 254 as a single block and call it B, say.
Then the arrangements which we are counting correspond to the arrangements of $\{1, 2, 3, B\}$.
There are $4! = 24$ arrangements of the 4 element set $\{1, 2, 3, B\}$ (since there are four possibilities for the first element of the arrangement, and for each of these there are three possibilities for the second element, and so on).
Therefore, there are 24 arrangements containing 254 consecutively in that order.

(c) *Solution 1*

To determine the average number of local peaks in all of the arrangements, we count the total number of local peaks in all 40 320 arrangements and then divide by this total number of arrangements.
To count the total number of local peaks, instead of looking at the arrangements and counting the number of local peaks in the arrangements, we look at the possible local peaks and count the number of arrangements in which each occurs.
In an arrangement of $\{1, 2, 3, 4, 5, 6, 7, 8\}$, a local peak is a sequence of three numbers abc inside the arrangement where $a < b$ and $b > c$.
How many such sequences of three numbers are there?
This is an extension of part (a). Using either technique from (a), we can determine that the total number of such sequences is 112.
Now fix one of these 112 sequences abc. In how many of the 40 320 arrangements does this sequence occur as a block?
This is an extension of part (b). Using either technique from (b), we can determine that the total number of such arrangements is $6! = 720$.
Thus, each of the 112 possible local peak sequences occurs in 720 arrangements, so there are a total of $112 \times 720 = 80\,640$ local peaks in all possible arrangements.
(We have indeed counted all such local peaks, since every local peak occurs as a subsequence of three numbers in this way.)
Therefore, the average number of local peaks in the 40 320 arrangements is $\dfrac{80640}{40320} = 2$.

Solution 2

In an arrangement of the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, a local peak involves three consecutive positions, so can occur in one of six places – in positions 1 to 3, 2 to 4, 3 to 5, 4 to 6, 5 to 7, or 6 to 8.
Let's focus on one of these places, say positions 1 to 3. Our arguments will apply equally to all such places.
What fraction of all of the arrangements will have local peak in this position?
Choose three numbers a, b, c from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$, say with $a < b < c$.

There are six possible ways to arrange these three numbers: abc, acb, bac, bca, cab, cba. Of these six possible ways, two will give a local peak: acb and bca (from the condition that $a < b < c$). So $\frac{1}{3}$ of the possible ways to arrange a, b, c give a local peak.
Consider all of the arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose first three numbers are a, b, c in some order.
Since the same total number of arrangements begin with abc as begin with acb as begin with bac as begin with any of the six ways of ordering a, b and c, then exactly $\frac{1}{3}$ of the arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ starting with a, b, c in some order have a local peak across positions 1 to 3.
Since the number of arrangements with any fixed set of three numbers in positions 1 to 3 is the same, then we can extend our argument to say that exactly $\frac{1}{3}$ of all arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have a local peak across positions 1 to 3.
This argument applies to any of the 6 possible places in which a local peak can occur.
Therefore, the average number of local peaks in all of the arrangements of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is $6 \times \frac{1}{3} = 2$.

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2004 Solutions
Hypatia Contest (Grade 11)

1. (a) *Solution 1*

Factoring the given equation,

$$x^2 + 5x + 6 = 0$$
$$(x+2)(x+3) = 0$$

so the roots are $x = -2$ and $x = -3$.

Solution 2

Using the quadratic formula,

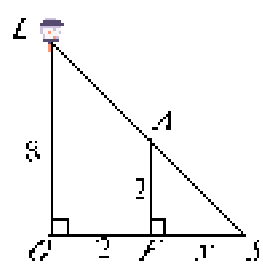
$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)} = \frac{-5 \pm \sqrt{1}}{2}$$

so $x = \frac{-5+1}{2} = -2$ or $x = \frac{-5-1}{2} = -3$.

(b) When we increase the roots (–2 and –3) from (a) by 7, we get 5 and 4.
A quadratic equation that has 5 and 4 as roots is $(x-5)(x-4) = 0$ or $x^2 - 9x + 20 = 0$.

(c) First, we need to find the roots of this equation.
Since $x - 4$ is a factor, then $x = 4$ is a root.
So we must find the roots of $3x^2 - x - 2 = 0$. We can do this either by factoring the left side or by using the quadratic formula.
The easiest way is by factoring. We get $3x^2 - x - 2 = (3x+2)(x-1)$.
Therefore, the second and third roots are $x = -\frac{2}{3}$ and $x = 1$.
When we add 1 to each of the three roots, we obtain 5, $\frac{1}{3}$ and 2, so an equation having these three roots is $(x-5)(3x-1)(x-2) = 0$ or $(x-5)(3x^2 - 7x + 2) = 0$ or $3x^3 - 22x^2 + 37x - 10 = 0$. (Of course, there are many other equations that have these three numbers as roots.)

2. (a) In the diagram, L is the top of the lamp-post, O is the base of the lamp-post, A is the top of Alan's head, F is the point on the ground where Alan is standing, and S is the tip of Alan's shadow.



We know that LO and AF are perpendicular to SO, and LAS is a straight line.
Therefore, we see that triangle LOS is similar to triangle AFS, since they have a common angle and each has a right angle.

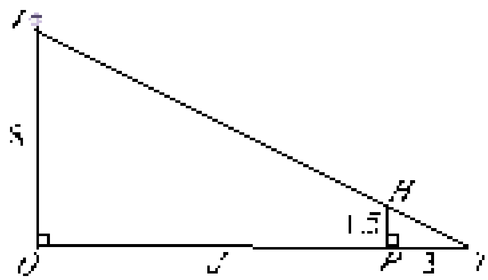
Therefore, $\dfrac{LO}{SO} = \dfrac{AF}{SF}$, or

$$\begin{aligned}
\frac{8}{2+x} &= \frac{2}{x} \\
8x &= 4 + 2x \\
6x &= 4 \\
x &= \tfrac{2}{3}
\end{aligned}$$

Therefore, Alan's shadow has length $\frac{2}{3}$ m.

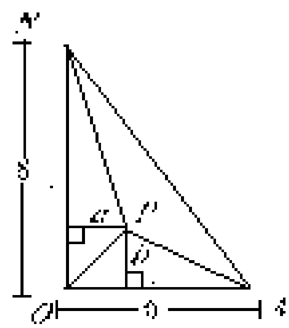
(b) In the new diagram, L and O are as before, H is the top of Bobbie's head, P is the point on the ground where Bobbie is standing, and T is the tip of Bobbie's shadow.
As in (a), we have that triangle LOT is similar to triangle HPT, and so, if d is the distance from the lamp to where Bobbie is standing (ie. the length of OP), then

$$\begin{aligned}
\frac{LO}{TO} &= \frac{HP}{TP} \\
\frac{8}{d+3} &= \frac{1.5}{3} \\
1.5d + 4.5 &= 24 \\
1.5d &= 19.5 \\
d &= 13
\end{aligned}$$



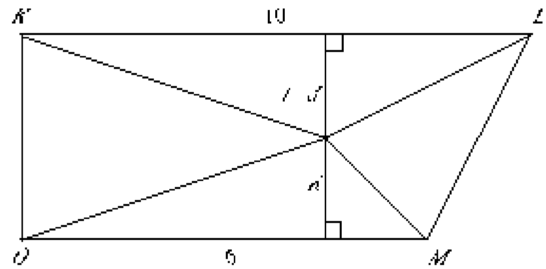
Therefore, Bobbie is standing 13 m from the lamp-post.

3\. (a) Since triangle OMN is right-angled, its area is $\frac{1}{2}(OM)(ON) = \frac{1}{2}(8)(6) = 24$.
For the areas of triangles POM, PON and PMN all to be equal, they each must equal 8, ie. one-third of the total area.
Consider triangle POM. Its base has length 6 (the length of OM) and its height has length b (the distance from P to OM), so its area is $\frac{1}{2}(6)(b) = 3b$. For the area of triangle POM to be 8, we must have $b = \frac{8}{3}$.



Consider now triangle PON. Its base has length 8 (the length of ON) and its height has length a (the distance from P to ON), so its area is $\frac{1}{2}(8)(a) = 4a$. For the area of triangle PON to be 8, we must have $a = 2$.
So if P has coordinates $\left(2, \frac{8}{3}\right)$, then each of triangles POM and PON has area 8, and so triangle PMN must also have area 8, since the area of the whole triangle OMN is 24.

(b) First, we calculate the area of quadrilateral *OMLK*.
OMLK is a trapezoid with *OM* parallel to *KL*, and so has area equal to the average of the bases times the height, ie. $\frac{1}{2}(10+6)t = 8t$.
So for the areas of the triangles *QOM*, *QML*, *QLK*, and *QKO* to be all equal, each of these four areas must be equal to $2t$, since the sum of these four areas is the area of the whole quadrilateral.



Consider triangle *QOM*. Its base has length 6 (the length of *OM*) and its height has length d (the distance from Q to *OM*), so its area is $\frac{1}{2}(6)(d) = 3d$. For the area of triangle *QOM* to be $2t$, we must have $d = \frac{2}{3}t$.
Consider next triangle *QLK*. Its base has length 10 (the length of *LK*) and its height has length $\frac{1}{3}t$ (the distance from Q to *LK*, since Q has y-coordinate $\frac{2}{3}t$). Thus, the area of triangle *QLK* is $\frac{1}{2}(10)\left(\frac{1}{3}t\right) = \frac{5}{3}t$,
which is not
equal to $2t$. (If we tried to set $\frac{5}{3}t = 2t$, we would then get $\frac{1}{3}t = 0$ or $t = 0$, which is not possible since we are told that $t > 0$.)
So it is impossible for the areas of both triangles *QOM* and *QLK* to be equal to $2t$.
Therefore, there is no point Q so that the areas of all four triangles area equal.

4. (a) We solve this problem by considering all of the possible cases. We will use the notation (G,Y,R) to denote the number of green (G), yellow (Y) and red (R) balls remaining. For example, the initial position is $(1,1,2)$.
If the green and yellow are chosen at the beginning, we then get $(0,0,3)$, so all of the remaining balls are red.
If the green and a red ball are chosen at the beginning, we then get $(0,2,1)$, so the next choice must be one yellow and one red, leaving $(1,1,0)$, and so the final two balls are chosen, leaving $(0,0,1)$, and so the final ball is red.
Similarly, the yellow and a green ball are chosen at the beginning, we then get $(2,0,1)$, so the next choice must be one green and one red, leaving $(1,1,0)$, and so the final two balls are chosen, leaving $(0,0,1)$, and so the final ball is red.
Thus, in all cases, the colour of the remaining ball or balls is always red.

(b) Here, we could again proceed by cases, but the number of cases would quickly get very large, so we should look for a better approach.

We notice that when a "move" is made (that is, two balls are removed and one is replaced), the parity of all three colours of balls changes. This is because the number of balls of each colour is being increased by 1 or decreased by 1, and so changes from odd to even or even to odd.
Therefore, since all three parities change together, the number of green balls and the number of red balls must always be both even or both odd, and the number of yellow balls is of the opposite parity. (Green and red start out both odd, after one move they will be both even, and so on; yellow starts out even, after one move it will be odd, and so on.)
When the process finally finishes, two of the numbers of balls will be 0, and so both be even. The only two colours which could have the same parity are green and red, so at the end there are 0 green and 0 red balls, so the colour of the remaining ball or balls is always yellow.
(Note that since the number of balls *decreases* on each turn, we are sure to actually end the game after at most 11 repetitions of the process.)

(c) *Solution 1*
In this version of the game, the total number of balls does not change, since the number of balls removed on each turn (2) is equal to the number of balls replaced on each turn (2).
Therefore, if the process were to end eventually with only one colour of ball remaining, then there would be 12 of that colour and 0 of each of the other two colours.
Consider $G - Y$ the difference between the number of balls that are green and the number of balls that are yellow. If the game got to 12 of one colour and 0 of the other two, then $G - Y$ would be equal to 12, 0 or –12.
We know that $G - Y$ is initially equal to –1. We will show that no matter what happens on each step of the process, $G - Y$ will always change by 0, 3 or –3 (ie. a multiple of 3). Once we have shown this, we can conclude that $G - Y$ can never be equal to 12, 0 or –12, since we cannot add and subtract 3's to –1 to get a multiple of 3. This will show that it is impossible for all of the remaining balls to be the same colour.
Let's suppose that at some point, the number of green balls is g and the number of yellow balls is y. After the next turn:

i) if a green and a yellow are replaced by two reds, there are $g-1$ green balls and $y-1$ yellow balls, so the difference is $(g-1)-(y-1)=g-y$, the same as its previous value
ii) if a green and a red are replaced by two yellows, there are $g-1$ green balls and $y+2$ yellow balls, so the difference is $(g-1)-(y+2)=g-y-3$
iii) if a yellow and a red are replaced by two greens, there are $g+2$ green balls and $y-1$ yellow balls, so the difference is $(g+2)-(y-1)=g-y+3$

So the difference always changes by 0, –3 or 3, so starting from –1, $G - Y$ can never become 0, so it is impossible for all of the balls to be the same colour.

Solution 2

Since all of the balls were eventually all of the same colour, then there would be 12 of one colour and 0 of each of the two remaining colours (since the number of balls does not change).

Suppose we were able to reach this state. In this solution, we will list the number of balls remaining by decreasing size (that is, we won't worry about which number goes with which colour). We will show by working backwards from the end, that it is impossible to get to 5,4,3.

When the process is reversed, we subtract 2 from one of the colours and add 1 to each of the others.

Thus, 12,0,0 can only come from 10,1,1.

10,1,1 can come from only 8,2,2.

8,2,2 can come from 6,3,3 or 9,3,0.

9,3,0 can come from 7,4,1 or 10,1,1.

7,4,1 can come from 5,5,2 or 8,2,2.

5,5,2 can come from 6,6,0 or 6,3,3.

6,6,0 can come from only 7,4,1.

6,3,3 can come from 4,4,4 or 7,4,1.

4,4,4 can come from only 5,5,2.

This creates a loop of possibilities: 12,0,0; 10,1,1; 9,3,0; 8,2,2; 7,4,1; 6,6,0; 6,3,3; 5,5,2; 4,4,4. It is impossible starting at 12,0,0 to get a position not in this list.

Thus, starting with 5,4,3 it is impossible to get to 12,0,0.

Canadian
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2003 Solutions
Hypatia Contest (Grade 11)

1. (a) Let N be the number of tiles in Quentin's possession.
Since he has 92 extra tiles after forming an n by n square, then
$$N = n^2 + 92$$
Since he is 100 tiles short of forming an $(n+2)$ by $(n+2)$ square, then
$$N = (n+2)^2 - 100$$
To solve for N, the easiest approach is to solve for n first. Since we have two expressions for N, then
$$\begin{aligned} (n+2)^2 - 100 &= n^2 + 92 \\ n^2 + 4n + 4 - 100 &= n^2 + 92 \\ 4n - 96 &= 92 \\ n &= 47 \end{aligned}$$
and thus
$$N = (47)^2 + 92 = 2209 + 92 = 2301$$
Therefore, Quentin has 2301 tiles.

(b) Let B be the total number of blocks in Rufus' pile.
When Quentin tries to make a cube with 8 blocks along an edge, he is 24 blocks short, and so he must have taken $8^3 - 24 = 488$ blocks.
Let r be the edge length of the cube that Rufus makes with his portion of the blocks.
Since the edge length of Rufus' original cube is r, he uses r^3 blocks.
When Quentin and Rufus combine their blocks, they can make a cube of edge length $(r+2)$, and so have $(r+2)^3$ blocks in total.
Thus,
$$\begin{aligned} (r+2)^3 &= r^3 + 488 \\ (r+2)^2(r+2) &= r^3 + 488 \\ \left(r^2 + 4r + 4\right)(r+2) &= r^3 + 488 \\ r^3 + 4r^2 + 4r + 2r^2 + 8r + 8 &= r^3 + 488 \\ 6r^2 + 12r - 480 &= 0 \\ r^2 + 2r - 80 &= 0 \\ (r+10)(r-8) &= 0 \end{aligned}$$
Since r is positive, then $r = 8$.
Therefore, they have $(r+2)^3 = 10^3 = 1000$ blocks in total.

Extension

Let N be the total number of tiles that Quentin has, let x be the side length of the first square that Quentin tries to make, and let y be the side length of the second square that Quentin tries to make.
As in (a), since Quentin has 92 extra tiles when he tries to make the first square and is 100 tiles short when he tries to make the second square, then

$$N = x^2 + 92$$
and
$$N = y^2 - 100$$
Again equating the values of *N*, we get
$$x^2 + 92 = y^2 - 106$$
$$y^2 - x^2 = 192$$
$$y^2 - 100 = x^2 + 92$$
$$y^2 - x^2 = 192$$
$$(y - x)(y + x) = 192$$
So we want to find the number of solutions to this equation where *x* and *y* are both integers with $y > x$. Thus, $y - x$ and $y + x$ are integers whose product is 192. Note also that $y + x$ is bigger than $y - x$. Now, $192 = 2(96) = 2(16)(6) = 2^6 3^1$.
The divisors of 192 are

1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, 96, 192

We should verify that each possibility gives an allowable value for *N*:

$y - x$	$y + x$	y	x	N
1	192			
2	96	49	47	2301
3	64			
4	48	26	22	584
6	32	19	13	269
8	24	16	8	164
12	16	14	2	104

(Notice that $y = \frac{1}{2}\left[(y - x) + (y + x)\right]$ so these equations are not too hard to solve!)
In the cases where $y - x$ is 1 and 3, *y* is not an integer, which is not allowed.
Therefore, there are 5 possible values for the number of tiles that Quentin has.

Comment
It is also possible to set up the equations in the following fashion
$$n - 92 = x^2$$
$$n + 100 = (x + p)^2$$
Subtracting the first equation from the second, we have
$$192 = (x + p)^2 - x^2 = 2xp + p^2 = p(2x + p)$$
The analysis continues in the same fashion as the first solution in that we must write $192 = 3 \cdot 2^6$ and consider divisors *p* and $2x + p$, where *p* is the smallest of the two divisors.

2. (a) *Solution 1*

First we ask the question: When does a player have a winning move? Since to win a player must remove the last coin, then a player has a winning move when he or she is choosing from a position with coins in only one pile (and a second empty pile).

So for Yolanda to win, she wants to ensure that she will always be passed an empty pile and a non-empty pile at some point. How can she force Xavier to pass her an empty pile? Xavier can only be forced to empty a pile if he receives two piles both of which have 1 coin (otherwise, he could reduce, but not empty, one of the piles).
So if Yolanda chooses from piles with 1 and 3 coins or 1 and 2 coins, then she can pass back piles with 1 and 1 coins, and be sure to win.

Thus, Xavier does not want to initially remove 2 coins from one pile, otherwise Yolanda can follow her strategy above. Also, Xavier does not want to remove 3 coins from one pile, or Yolanda can immediately win by removing the other 3 coins.

So Xavier should start by removing 1 coin, and passing piles with 2 and 3 coins to Yolanda. She does not want to pass an empty pile or a pile with 1 coin in it to Xavier (or he can use *her* strategy from above), so she removes 1 coin from the larger pile, and passes back 2 and 2 coins. Xavier is then forced to empty one pile, or reduce one pile to 1 coin, and so Yolanda can then guarantee that she wins.

Therefore, in all cases, Yolanda can guarantee that she wins.

Solution 2

Yolanda will always win the game if she can guarantee that at some point when it is her turn to choose that she is selecting coins from just one pile. If she is selecting coins from just one pile, she will win the game by removing all of the coins from that pile.

She can guarantee that this will happen by duplicating Xavier's move only in the other pile. Thus, if Xavier takes 1, 2 or 3 coins, then Yolanda will take the same number of coins from the other pile. This strategy, on Yolanda's part, will mean that Xavier will always empty one pile first, and thus guarantee that Yolanda will win.

Solution 3

If Yolanda can ensure that she passes two equal piles to Xavier, then Xavier can never win, because he can never empty the last pile (he'll always have two non-empty piles).
So if Xavier reduces to 2 and 3 coins, Yolanda passes back 2 and 2 coins.
If Xavier reduces to 1 and 3 coins, Yolanda passes back 1 and 1 coins.
If Xavier reduces to 0 and 3 coins, Yolanda can immediately win by removing the last 3 coins.
From 1 and 1 coins, Xavier must reduce to 1 and 0 coins, and so Yolanda wins.
From 2 and 2 coins, Xavier must reduce to 1 and 2 coins (allowing Yolanda to pass back 1 and 1 coins) or to 0 and 2 coins, allowing Yolanda to win immediately.
Therefore, Yolanda can always win by following an "equalizing" strategy.

(b) In part (a), we saw that Yolanda always won the game if she could guarantee that Xavier was choosing when there were two piles with an equal number of coins in each pile.

Starting with piles of 1, 2 and 3 coins, Yolanda can always win, because she can always after her first turn give two equal piles (and an empty third pile) back to Xavier. We see this by examining the possibilities:

Xavier's first move	Yolanda's first move
0,2,3	0,2,2
1,1,3	1,1,0
1,0,3	1,0,1
1,2,2	0,2,2
1,2,1	1,0,1
1,2,0	1,1,0

In any of these cases, Yolanda can be sure to win by following her "equalizing" strategy from part (a).
So Yolanda's strategy is to create two equal piles (and a third empty pile) after her first turn, and so force Xavier to lose, using her strategy from (a).

Extension

In part (b), we saw that if Xavier chooses first from three piles with 1, 2 and 3 coins, then Yolanda can always win.
In part (a), we saw that if Xavier chooses first from two piles with an equal number of coins, then Yolanda can again always win.
So on his first move, Xavier does not want to create two equal piles (eg. 2,4,4 or 2,2,5 etc.), otherwise Yolanda would remove the third unequal pile and Xavier would then be choosing on his second turn from two equal piles.
Similarly, Xavier does not want to create a situation where Yolanda can reduce immediately to 1,2,3, otherwise Yolanda will win by following the strategy from (b).

So we consider the possible first moves for Xavier:

Xavier's first move	Yolanda's first move	Winner
2,4,4	0,4,4	Yolanda
2,4,3	2,1,3	Yolanda
2,4,2	2,0,2	Yolanda
2,4,1	2,3,1	Yolanda
2,4,0	2,2,0	Yolanda
2,3,5	2,3,1	Yolanda
2,2,5	2,2,0	Yolanda
2,1,5	2,1,3	Yolanda
2,0,5	2,0,2	Yolanda
1,4,5	??	??
0,4,5	0,4,4	Yolanda

So if Xavier makes any move other than to 1,4,5, Yolanda will win by following the correct strategy.

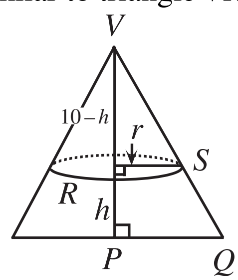
What if Xavier moves to 1,4,5? There are then 10 possible moves for Yolanda.
As above, if Yolanda makes her first move to 0,4,5 or 1,1,5 or 1,0,5 or 1,4,4 or 1,4,1 or 1,4,0, then Xavier can reduce to two equal piles.
If Yolanda reduces to 1,3,5 or 1,2,5 or 1,4,3 or 1,4,2, then Xavier can reduce to some ordering of 1,2,3, and so Xavier can win.
Therefore, Xavier can win by reducing first to 1,4,5, and then to either two equal piles or some ordering of 1,2,3, and then following Yolanda's strategy from (a) or (b).

3. We slice the two solids with a vertical plane through the vertex of the cone and highest point of the sphere. The cross-sections we get then are an isosceles triangle with a height of 10 cm and a base of 10 cm, and a circle of diameter 10 cm.

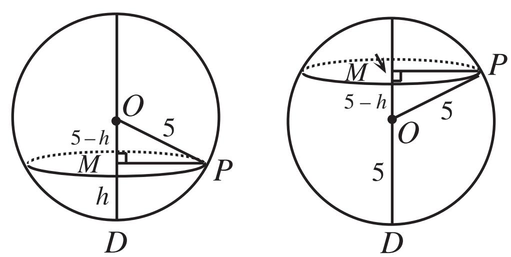
We want to find the height, h, that produces equal cross-sectional areas. Since the formula for the area of a circle is πr^2 and we want to find h, we need to find a relationship between h and r for the cone and for the sphere.

Case 1 – Cone
From V, the vertex of the cone, we draw in the principal axis of the cone as shown. It is clear by symmetry that triangle VPQ is similar to triangle VRS.
Thus,

$$\frac{10}{5} = \frac{10-h}{r}$$
$$2r = 10-h$$
$$r = \tfrac{1}{2}(10-h)$$



Case 2 – Sphere
From O, the centre of the sphere, we draw a line through the centre of the cross-sectional circle. This line is perpendicular to the circle.
We have $OP = 5$ (the radius of the circle), $DO = 5, DM = h$, and $MP = r$.
We need an expression for r – the answer depends on whether h is bigger or smaller than 5.



If h is bigger than or equal to 5, then $OM = h-5$. If h is smaller than 5, then $OM = 5-h$.

Thus, by Pythagoras,
$$r = \sqrt{OP^2 - OM^2} = \sqrt{5^2 - (5-h)^2} = \sqrt{25 - \left(h^2 - 10h + 25\right)} = \sqrt{10h - h^2}$$
(Notice that even if $OM = h-5$, then we get the same answer, since $(h-5)^2 = (5-h)^2$.)

So, we compare the areas of the cross-sectional circles to solve for h:

$$\pi\left[\tfrac{1}{2}(10-h)\right]^2 = \pi\left[\sqrt{10h-h^2}\,\right]^2$$
$$(10-h)^2 = 4\left(10h-h^2\right)$$
$$100-20h+h^2 = 40h-4h^2$$
$$5h^2-60h+100=0$$
$$h^2-12h+20=0$$
$$(h-10)(h-2)=0$$

and therefore, $h=10$ or $h=2$.
(Note that $h=10$ gives a horizontal plane that just passes through the vertex of the cone and is tangent to the top of the sphere.)

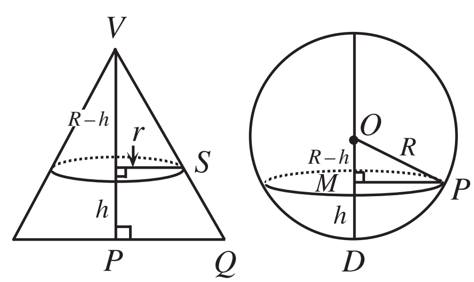
Therefore, the height is 10 cm or 2 cm.

Extension

To avoid fractions, we start by letting $d=2R$, so the cone has radius R and height R.
As above, we slice the two solids with a vertical plane.
In the cone, at a height of h, we have $\dfrac{R-h}{R}=\dfrac{r}{R}$ or $r=R-h$ as in the previous set-up.
In the sphere, at a height of h we have

$$r=\sqrt{OP^2-OM^2}=\sqrt{R^2-(R-h)^2}=\sqrt{R^2-\left(h^2-2Rh+R^2\right)}=\sqrt{2hR-h^2}$$

Note that $h\le R$ since the cone has height R.
Therefore, the sum of the areas of the two circular cross-sections is

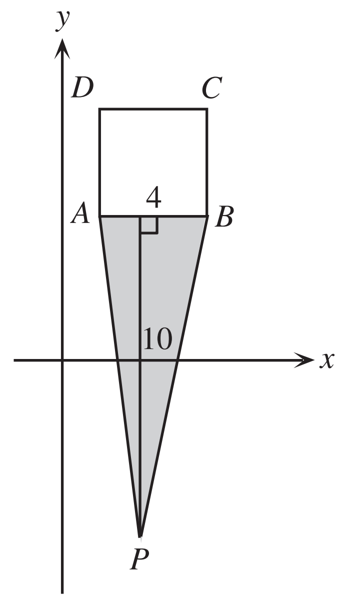


$$\pi\left(\sqrt{2hR-h^2}\right)^2+\pi(R-h)^2$$
$$=\pi\left(2hR-h^2\right)+\pi\left(R^2-2hR+h^2\right)$$
$$=\pi R^2$$

which is a constant, since R is a constant.

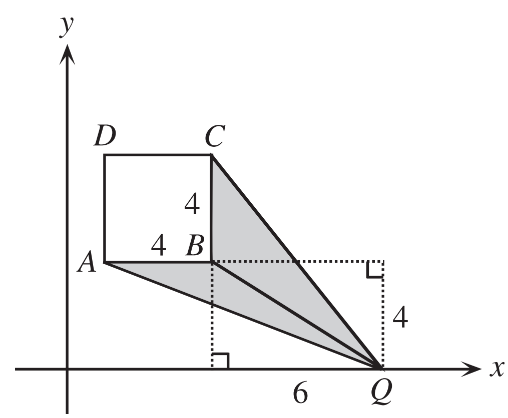
4. (a) ***Solution***

The visible area of $P(2,-6)$ is the area of ΔABP.
ΔABP has base AB of length 4, and its height is the distance from AB to the point P, which is 10, since AB is parallel to the x-axis.
Thus, the area of ΔABP is $\frac{1}{2}bh = \frac{1}{2}(4)(10) = 20$ square units, ie. the visible area of P is 20 square units.



(b) ***Solution***

The visible area of $Q(11,0)$ is the sum of the areas of ΔQBC and ΔQBA.
ΔQBC has base BC of length 4, and its height is the distance from Q to the line through B and C, which is 6.
Thus, the area of ΔQBC is $\frac{1}{2}bh = \frac{1}{2}(4)(6) = 12$ square units.
ΔQBA has base BA of length 4, and its height is the distance from Q to the line through B and A, which is also 4. Thus, the area of ΔQBA is $\frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$ square units.
So the visible area of Q is the sum of these two areas, or 20 square units.



(c) ***Solution***

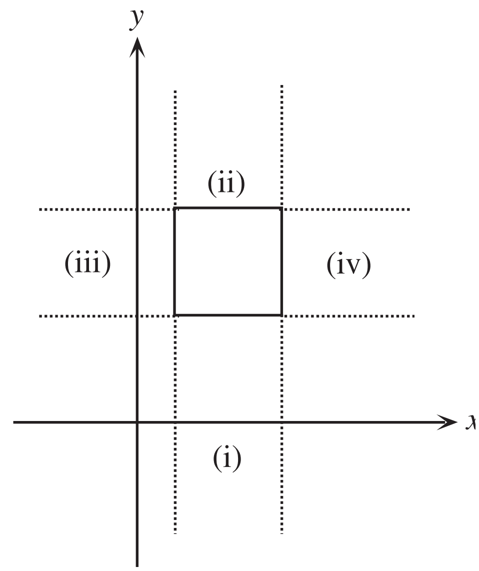
From any point P, there are either 2 or 3 visible vertices of the square.

We first consider those points P for which there are 2 visible vertices.

Geometrically, these will be points which lie "directly opposite" an edge of the square.

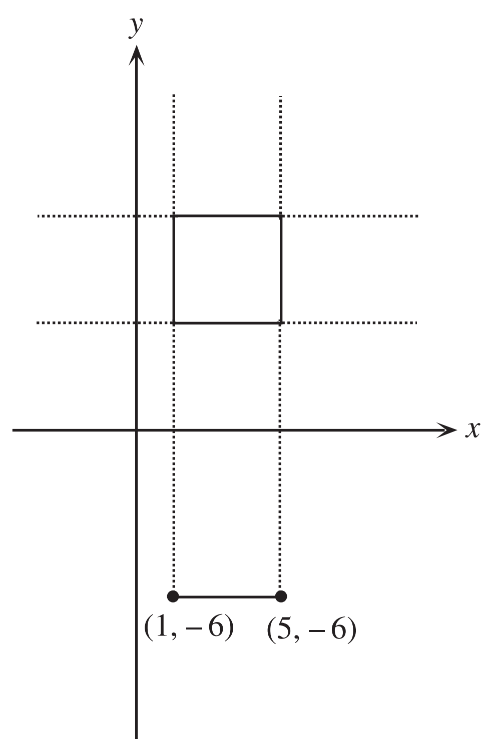
That is, they will be point P which

i) lie below the square, with x-coordinate between 1 and 5, inclusive
ii) lie above the square, with x-coordinate between 1 and 5, inclusive
iii) lie to the left of the square, with y-coordinate between 4 and 8, inclusive
iv) lie to the right of the square, with y-coordinate between 4 and 8, inclusive



In each of these four cases, the visible area will be a single triangle whose base is a side of the square (that is, of length 4). For the visible area of one of these points P to be 20, the height of this triangle must be 10.

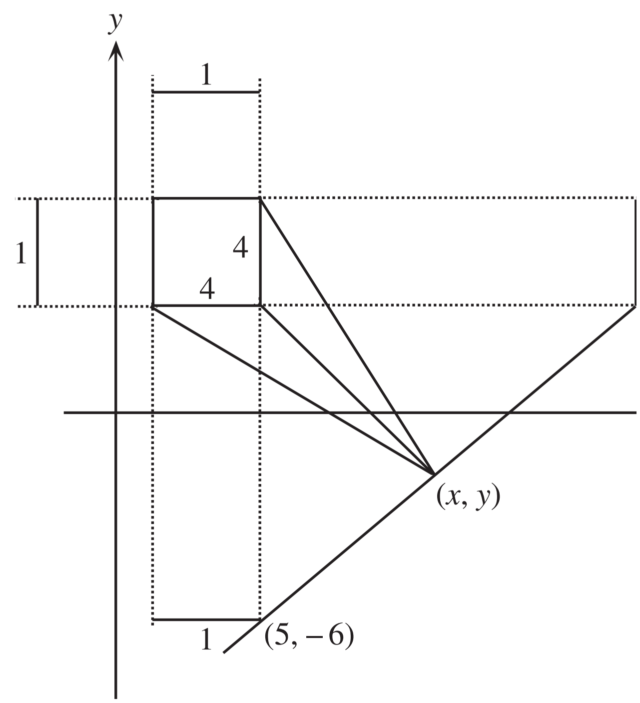
Thus, for those points in case (i), all points P which lie 10 units below the square, and have x-coordinates between 1 and 5 will be in the 20/20 set. In other words, the points P lying on the line segment joining $(1,-6)$ to $(5,-6)$ will lie in the 20/20 set. (Notice that this includes the point $(2,-6)$ from (a).) These are all of the points in this region that lie in the 20/20 set. So in this region, the 20/20 set has length 4.



The other three regions will give the same result, by symmetry, so we have four line segments, each of length 4, in the 20/20 set thus far.

There are four more regions to consider – that is, the regions that do not lie directly opposite an edge of the square (for example, the region of points $P(x,y)$ where $x \geq 5$ and $y \leq 4$). By symmetry each of these regions should give us the same result.

So suppose that $P(x,y)$ in the region $x \geq 5$ and $y \leq 4$ has a visible area of 20. Then the sum of the areas of ΔPBC and ΔPBA is 20.
ΔPBC has a base of length 4 and a height of $x-5$.
ΔPBA has a base of length 4 and a height of $4-y$.



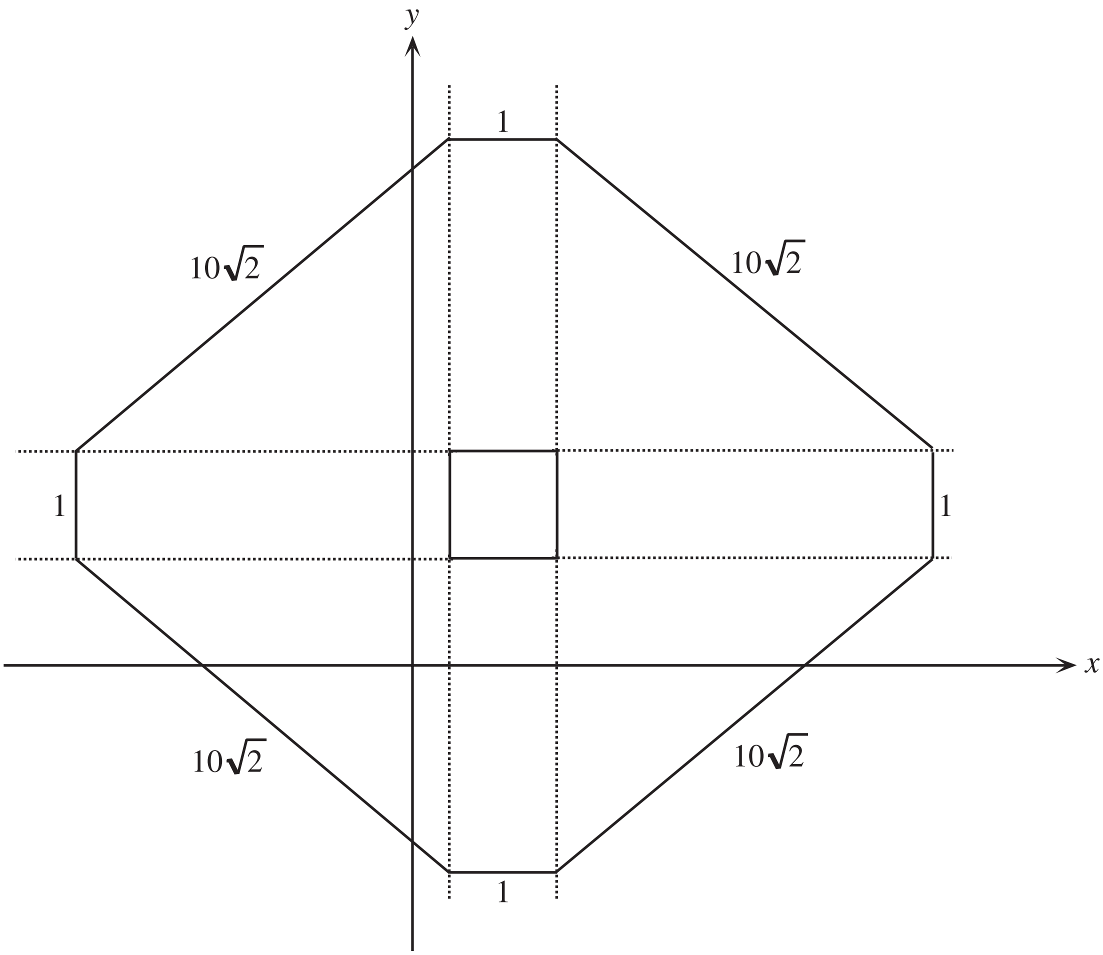
Therefore, for P to be in the 20/20 set, we have

$$\tfrac{1}{2}(4)(x-5)+\tfrac{1}{2}(4)(4-y)=20$$
$$x-5+4-y=10$$
$$y=x-11$$

ie. P lies on the straight line of slope 1, $y=x-11$. (Notice that this includes the point $(11,0)$ from (b).) So in this region, the points in the 20/20 set are the points on the line segment joining $(5,-6)$ to $(15,4)$, which has length $10\sqrt{2}$. (It is worth noting that the endpoints of this line segment are the endpoints of the line segments from regions (i) and (iv) above.)

We can argue by symmetry that the other sides in the required octagon will be as shown in the diagram.

So the 20/20 is a polygon (in fact, an octagon) with four sides of length 4 and four sides of length $10\sqrt{2}$. Therefore, the perimeter of the 20/20 set is $16+40\sqrt{2}$.



Extension

We approach this in an analogous manner to the 2-dimensional case.

Case 1: 4 visible vertices

Where will a point P lie so that it has exactly 4 visible vertices?
In this case, P will lie directly opposite one of the 6 faces of the cube.
The visible volume will be formed by one square-based pyramid.
The base of this pyramid will be a unit square, so since the volume of a pyramid is one-third times the area of the base times the height, then for a volume of 20, the height of the pyramid must be 60.
So the points P above one of the faces which have a visible volume of 20 are all those points 60 units above the faces. In other words, the points form a square (again, of side length 1) which is 60 units above the face. So the 20/20 set has six square faces each of area 1.

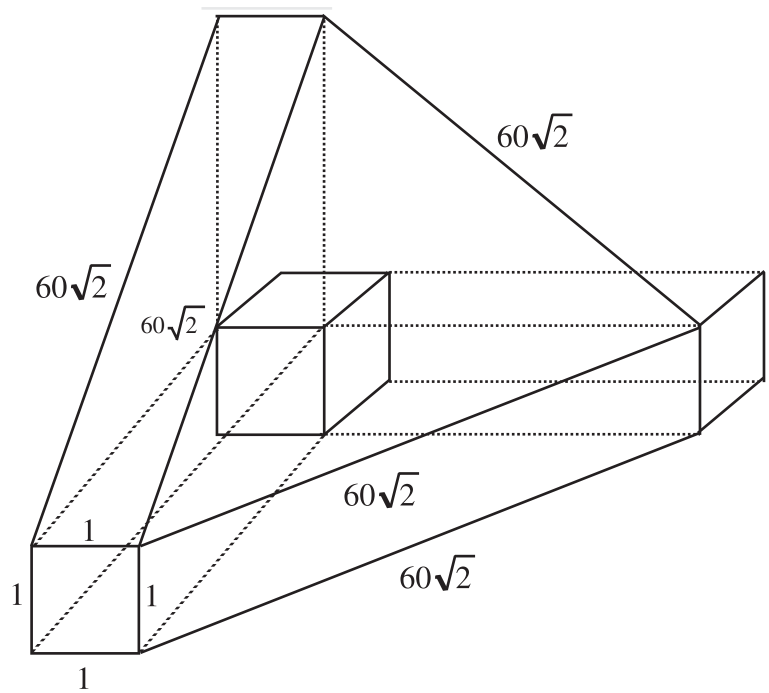
Case 2: 6 visible vertices

Where will a point P lie so that it has exactly 6 visible vertices?
In this case, the 6 visible vertices are the 6 vertices of two faces which share a common edge, and P lies in the region which is between the outer edges of these two faces, but not

directly above either face. We say that P lies above the edge. Since the cube has 12 edges, there will be 12 such regions.
Consider points P in one of these regions. The visible volume here will be formed by two square-based pyramids, each of which has a base which is a unit square. So the visible volume will be one-third times the area of the base (which is 1) times the sum of the two heights. Since the visible volume should be 20, the sum of the two heights will be 60.
The set of points above the edge which give a combined height of 60 for the two pyramids will form a rectangle which joins the edges of the two squares over the two adjacent faces. This rectangle will thus have one edge of length 1 and the other edge of length $60\sqrt{2}$ (since the long edge will form the hypotenuse of a right-angled triangle with two legs each of length 60).
(Why does this give us a portion of a plane? If we consider points P which lie in a cross-section which is perpendicular to the two adjacent faces, then we are asking for P to have sum of the heights to two perpendicular line segments to be 60, which gives a straight line as in (b). Sliding this cross-section across the faces gives us a segment of a plane.)
Thus, we have twelve rectangular faces, each of area $60\sqrt{2}$.

Case 3: 7 visible vertices
Where will a point P have 7 visible vertices?
P will have 7 visible vertices if it lies in a position that has not yet been considered. That is, P will in essence be above one of the 8 vertices of the cube. (This seems sensible since the 20/20 set currently has 8 holes in it!)
This time, the visible volume is made up of three square-based pyramids, so we want points P so the sum of the heights of these three pyramids is 60, as before.
This region will again form part of a plane, and it should connect to the three rectangles already formed over the three edges that meet at the vertex in question (since at the edge of each of these rectangles, we can consider having a third pyramid of height 0).
So this segment of the 20/20 set is an equilateral triangle of side length $60\sqrt{2}$.

To find the area of this triangle, we join the top vertex to the midpoint of the base, so each half of the triangle is a 30-60-90 right-angled with a leg of length $30\sqrt{2}$ opposite the 30° angle. Thus, the height of the triangle is $\sqrt{3}\left(30\sqrt{2}\right) = 30\sqrt{6}$, and so the area of the whole equilateral triangle is $\frac{1}{2}\left(60\sqrt{2}\right)\left(30\sqrt{6}\right) = 900\sqrt{12} = 1800\sqrt{3}$.



A partial sketch of the desired region.
Therefore, the total surface area of the 20/20 set is

$$6(1)+12\left(60\sqrt{2}\right)+8\left(1800\sqrt{3}\right)=6+720\sqrt{2}+14400\sqrt{3}$$