The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2024 Galois Contest*

**Thursday, April 4, 2024**
**(in North America and South America)**

**Friday, April 5, 2024**
**(outside of North America and South America)**

*Solutions*

1. (a) *Solution 1*

The length of the expanded garden is $(5 + 2 \times 2)$ m $= 9$ m, and the width is 4 m.
Thus, the total area of the expanded garden is 9 m $\times$ 4 m $= 36$ m$^2$.

*Solution 2*

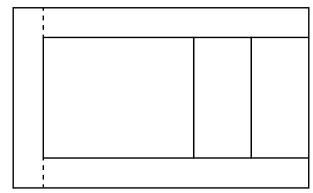
The area of the original 5 m by 4 m garden is 5 m $\times$ 4 m $= 20$ m$^2$.
Each additional 2 m by 4 m plot has area 2 m $\times$ 4 m $= 8$ m$^2$, and so the total area of the expanded garden is $(20 + 2 \times 8)$ m$^2 = 36$ m$^2$.

(b) *Solution 1*

The combined garden and path has length 9 m + 1 m = 10 m, and width $(4 + 2 \times 1)$ m $= 6$ m.
Thus, the area of the garden and the path is 10 m $\times$ 6 m $= 60$ m$^2$.

*Solution 2*

Consider splitting the path into three rectangles, as shown.



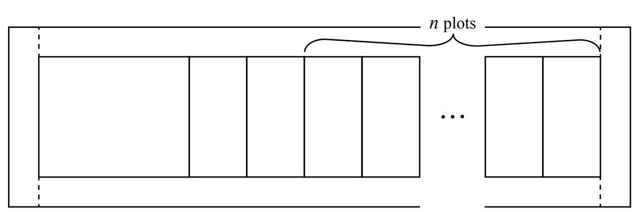
Each of the rectangles above and below the garden has dimensions 9 m by 1 m, and thus each has area 9 m $\times$ 1 m $= 9$ m$^2$.
The remaining section of the path has height $(4 + 2 \times 1)$ m $=$ 6 m and width 1 m, and thus has area 6 m $\times$ 1 m $= 6$ m$^2$.
The area of the expanded garden is 36 m$^2$, and so the total combined area of the garden and the path is $(36 + 2 \times 9 + 6)$ m$^2 = 60$ m$^2$.

(c) *Solution 1*

Each of the new plots has length 2 m, and so $n$ plots increase the 9 m length of the garden by $2n$ m.
Thus, the combined length of the garden and the path is $(9+2n+2\times 1)$ m $= (2n+11)$ m.
The combined width of the garden and the path is $(4 + 2 \times 1)$ m $= 6$ m.
Thus in m$^2$, the total combined area of the garden and the path is $6 \times (2n + 11)$.
Solving $6 \times (2n + 11) = 150$, we get $2n + 11 = \frac{150}{6} = 25$ or $2n = 14$, and so $n = 7$.

*Solution 2*

Consider splitting the combined area of the garden and path into three rectangles, as shown.




Each of the rectangles to the left and right of the garden has height $(4 + 2 \times 1)$ m $= 6$ m, width 1 m, and thus each has area 6 m $\times$ 1 m $= 6$ m$^2$.
The remaining rectangle, which combines the garden and the remaining sections of the path, also has height 6 m.
Each of the new plots has length 2 m, and so $n$ plots increase the 9 m length of the garden by $2n$ m.
Thus, the length of this remaining rectangle is $(2n + 9)$ m.
Measured in m$^2$, the total combined area of the garden and the path is $2\times 6+6\times(2n+9)$ or $12 + 6 \times (2n + 9)$.
Solving $12+6\times(2n+9) = 150$, we get $6\times(2n+9) = 138$ or $2n+9 = \frac{138}{6} = 23$ or $2n = 14$, and so $n = 7$.

2. (a) Beginning with the point $(5, 11)$, and applying $R$ then $T$, the resulting coordinates are $(11, -3)$, as shown:

$$(5, 11) \xrightarrow{R} (11, -5) \xrightarrow{T} (11, -3)$$

(b) *Solution 1*

When a point is rotated $90^\circ$ about the origin 4 times, the result is a rotation of $4 \times 90^\circ = 360^\circ$ or one full rotation about the origin.

Thus beginning with the point $(-3, 7)$, when $R$ is applied 4 times, the point returns to its original location, and so the resulting coordinates are $(-3, 7)$.

When $R$ is applied a 5th time, the resulting coordinates are $(7, 3)$.

*Solution 2*

Beginning with the point $(-3, 7)$, and applying $R$ 5 times, the resulting coordinates are $(7, 3)$, as shown:

$$(-3, 7) \xrightarrow{R} (7, 3) \xrightarrow{R} (3, -7) \xrightarrow{R} (-7, -3) \xrightarrow{R} (-3, 7) \xrightarrow{R} (7, 3)$$

(c) Beginning with the point $(9, 1)$, and applying the sequence $R$, $R$, $T$, the resulting coordinates are $(-9, 1)$, as shown:

$$(9, 1) \xrightarrow{R} (1, -9) \xrightarrow{R} (-9, -1) \xrightarrow{T} (-9, 1)$$

Continuing with the point $(-9, 1)$, and applying the sequence $R$, $R$, $T$ again, the resulting coordinates are $(9, 1)$, as shown:

$$(-9, 1) \xrightarrow{R} (1, 9) \xrightarrow{R} (9, -1) \xrightarrow{T} (9, 1)$$

Beginning with the point $(9, 1)$, and applying the sequence $R$, $R$, $T$ twice, the resulting coordinates are $(9, 1)$ (that is, the point returns to its original location).

This will continue to occur each time the sequence $R$, $R$, $T$ is applied an even number of times, and so after applying $R$, $R$, $T$ 10 times, the resulting coordinates are $(9, 1)$.

Beginning with the point $(9, 1)$, and applying the sequence $R$, $R$, $T$ an 11th time, the resulting coordinates are $(-9, 1)$, the steps to which were previously shown.

3. (a) Of the 7 balls in the hat, there are 3 balls that are even-numbered (numbered 2, 4 and 6) and so the probability that the first ball drawn is even-numbered is $\frac{3}{7}$.

(b) There are 7 possible choices for the first ball, and since a drawn ball is neither replaced nor returned to the hat, there are 6 choices for the second ball, and thus $7 \times 6 = 42$ ways that the first two balls may be drawn.

The sum of the numbers on the first two balls drawn is 5 exactly when the numbers are 1 and 4, in some order, or 2 and 3, in some order.

Thus there are 4 possible ways that the first two balls drawn have a sum of 5: 1 and 4, 4 and 1, 2 and 3, or 3 and 2.

The probability that the sum of the numbers on the first two balls drawn is 5 is $\frac{4}{42} = \frac{2}{21}$.

(c) Suppose the probability that the sum of the numbers on the first two balls drawn is greater than or equal to 6 is $p$.
Then we let $\overline{p}$ equal the probability that the sum of the numbers on the first two balls drawn is *not* greater than or equal to 6.
That is, $\overline{p}$ is equal to the probability that the sum of the numbers on the first two balls drawn is less than 6, and so $p = 1 - \overline{p}$.

If sum of the numbers on the first two balls drawn is less than 6, then this sum is either 5, 4 or 3 (since two different balls are drawn, the smallest possible sum is $1 + 2 = 3$).
From part (b), there are exactly 4 ways that the first two balls drawn have a sum of 5.
There are exactly 2 ways in which the sum is 4: 1 and 3 or 3 and 1 (2 and 2 is not possible since there is only one 2).
There are exactly 2 ways in which the sum is 3: 1 and 2 or 2 and 1.
Therefore, of the $7 \times 6 = 42$ ways that the first two balls may be drawn, there are $4 + 2 + 2 = 8$ ways that the sum is less than 6, and so $\overline{p} = \frac{8}{42} = \frac{4}{21}$.
Finally, the probability that the sum of the numbers on the first two balls drawn is greater than or equal to 6 is $p = 1 - \overline{p} = 1 - \frac{4}{21} = \frac{17}{21}$.

Note: We may have instead chosen to determine $p$ directly. That is, we may have determined the probability that the sum of the numbers on the first two balls drawn was 6, 7, 8, 9, 10, 11, 12, or 13 and then added each of these probabilities together to determine $p$. We chose to determine $\overline{p}$ since it required considering that the sum of the numbers on the first two balls drawn was 3, 4 or 5, and thus was less work than it would be to determine $p$ directly.

(d) The probability that the sum of the numbers on the first two balls drawn is greater than or equal to 7 is $q = \frac{3}{4}$.
As in part (c), we similarly define $\overline{q}$ to be the probability that the sum of the numbers on the first two balls is less than 7, and thus $q = 1 - \overline{q}$, or $\frac{3}{4} = 1 - \overline{q}$, and so $\overline{q} = \frac{1}{4}$.
There are 8 possible choices for the first ball (since an eighth ball was added to the hat) and 7 choices for the second ball, and thus $8 \times 7 = 56$ ways that the first two balls may be drawn.
Since $\overline{q} = \frac{1}{4} = \frac{14}{56}$, then there are 14 ways that the sum of the numbers on the first two balls drawn is less than 7.
Without using the new gold ball, there are 12 ways that the sum of the numbers on the first two balls drawn can be less than 7.
These are: $1 + 5$, $1 + 4$, $1 + 3$, $1 + 2$, $2 + 4$, $2 + 3$, and their reversals.
Thus, the new gold ball, numbered with the integer $k$, where $1 \leq k \leq 7$, must give 2 additional ways to produce a sum that is less than 7.
If $k = 5$, then the gold ball may be paired with the ball numbered 1 (drawn in either order) to give 2 additional ways to produce a sum that is less than 7.
Further, if $k = 5$, the gold ball cannot be paired with any other ball to give a sum that is less than 7, and so the correct value of $k$ is 5.
(You should confirm for yourself that if $k = 6$ or 7, there are no additional ways for the sum to be less than 7, and if $k = 1, 2, 3,$ or 4, then there are more than 2 additional ways for the sum to be less than 7.)

4. (a) We denote the number in row $r$, column $c$ as $[r, c]$, and so $[2, 1] = 1$, for example.
We begin by determining the numbers in column 2.
The neighbours of $[1, 1]$ are $[2, 1]$ and $[1, 2]$, and so $[1, 1] = [2, 1] \times [1, 2]$ or $-1 = 1 \times [1, 2]$, which gives $[1, 2] = -1$.
The neighbours of $[2, 1]$ are $[1, 1]$, $[3, 1]$ and $[2, 2]$, and so $[2, 1] = [1, 1] \times [3, 1] \times [2, 2]$ or $1 = (-1) \times (-1) \times [2, 2]$, which gives $[2, 2] = 1$.
The neighbours of $[3, 1]$ are $[2, 1]$ and $[3, 2]$, and so $[3, 1] = [2, 1] \times [3, 2]$ or $-1 = 1 \times [3, 2]$, which gives $[3, 2] = -1$.
It is important to note that there was no choice in determining the numbers in column 2. That is, the properties of the numbers in column 1 are satisfied only when $[1, 2] = -1$, $[2, 2] = 1$ and $[3, 2] = -1$.
Continuing in this way, the numbers in column 3 are $[1, 3] = 1$, $[2, 3] = 1$ and $[3, 3] = 1$.
The numbers in column 3 are once again necessary to satisfy the properties of the numbers in column 2.
If we were to stop here, is the $3 \times 3$ grid shown to the right a Griffin Grid?

| | | |
|---|---|---|
| −1 | −1 | 1 |
| 1 | 1 | 1 |
| −1 | −1 | 1 |

The neighbours of $[1, 3] = 1$ are $[1, 2] = -1$ and $[2, 3] = 1$, however $1 \neq (-1) \times 1$, and so it is not possible to construct a $3 \times 3$ Griffin Grid with the given first column.
Continuing, we complete columns 4 and 5, as shown below. The numbers in column 5 are chosen so that the numbers in column 4 satisfy the properties of a Griffin Grid.
We must check that the numbers in column 5 also satisfy the properties of a Griffin Grid. Since each cell in column 5 contains a $-1$ or a 1, and the number in each cell is equal to the product of the numbers in all cells that are neighbours, then the completed $3 \times 5$ grid is indeed a Griffin Grid.

| | | | | |
|---|---|---|---|---|
| −1 | −1 | 1 | −1 | −1 |
| 1 | 1 | 1 | 1 | 1 |
| −1 | −1 | 1 | −1 | −1 |

(b) In the first column of a $3 \times 5$ grid, there are two possibilities for the number in each cell, and so there are $2 \times 2 \times 2 = 8$ possible first columns.
As was demonstrated in part (a), the remainder of the grid is completely determined by the three entries in the first column, and so there are at most 8 different $3 \times 5$ Griffin Grids.
Expressed as an ordered triple, consider the two grids with first columns $(1, -1, -1)$ and $(-1, -1, 1)$.
Since each of these columns is a vertical reflection of the other, then their completed $3 \times 5$ grids will be vertical reflections of one another.
That is, the $3 \times 5$ grid with first column $(-1, -1, 1)$ is a Griffin Grid exactly when the $3 \times 5$ grid with first column $(1, -1, -1)$ is a Griffin Grid, and so we may consider these two grid types as one case.
Similarly, the grids with first columns $(1, 1, -1)$ and $(-1, 1, 1)$ are also vertical reflections of one another and so we may consider these two grid types as one case.

Each cell of a grid with first column $(1, 1, 1)$ is a 1, and thus gives a $3 \times 5$ Griffin Grid.
At this point, we are left to consider the 5 grids whose first columns are:

$$A(-1, -1, -1), B(1, -1, 1), C(-1, 1, -1), D(1, -1, -1), \text{ and } E(1, 1, -1)$$

We complete each of the 5 grids below, replacing 1 with + and $-1$ with $-$.

$A$

| | | | | |
|---|---|---|---|---|
| − | + | + | + | − |
| − | − | + | − | − |
| − | + | + | + | − |

$B$

| | | | | |
|---|---|---|---|---|
| + | − | + | − | + |
| − | − | + | − | − |
| + | − | + | − | + |

$C$

| | | | | |
|---|---|---|---|---|
| − | − | + | − | − |
| + | + | + | + | + |
| − | − | + | − | − |

$D$

| | | | | |
|---|---|---|---|---|
| + | − | − | + | − |
| − | + | + | + | − |
| − | + | − | − | + |

$E$

| | | | | |
|---|---|---|---|---|
| + | + | − | − | − |
| + | − | + | − | + |
| − | − | − | + | + |

As was demonstrated in part (a), the numbers in column 5 are chosen so that the numbers in column 4 satisfy the properties of a Griffin Grid.
We must check if the numbers in column 5 also satisfy the properties of a Griffin Grid.
Since each cell in each column 5 contains a $-1$ or a 1, and the number in each cell is equal to the product of the numbers in all cells that are neighbours, then each of the completed $3 \times 5$ grids is indeed a Griffin Grid.
Thus, each of the 8 possible first columns produces a $3 \times 5$ Griffin Grid, and so there are a total of 8 Griffin Grids of this size.

Additional note: The grid with first column $F(-1, -1, 1)$ produces a $3 \times 5$ Griffin Grid that is a vertical reflection of the grid with first column $D$.
Similarly, the grid with first column $G(-1, 1, 1)$ produces a $3 \times 5$ Griffin Grid that is a vertical reflection of the grid with first column $E$.
These two Griffin Grids, along with the grid with first column $H(1, 1, 1)$, are shown below.

$F$

| | | | | |
|---|---|---|---|---|
| − | + | − | − | + |
| − | + | + | + | − |
| + | − | − | + | − |

$G$

| | | | | |
|---|---|---|---|---|
| − | − | − | + | + |
| + | − | + | − | + |
| + | + | − | − | − |

$H$

| | | | | |
|---|---|---|---|---|
| + | + | + | + | + |
| + | + | + | + | + |
| + | + | + | + | + |

(c) Continuing our work from part (b), if we extend each $3 \times 5$ grid one additional column, we see that each has the same 6th column, $(1, 1, 1)$.
This means that for each $3 \times 7$ grid, the 7th column will match the 5th.
Can you see why this is? (For example, see the 2nd, 3rd and 4th columns of the grid with first column $C$ in part (b).)
As was demonstrated in part (b), the grid with first column $F$ is a vertical reflection of the grid with first column $D$, and so their completed $3 \times n$ grids will be vertical reflections of one another.
That is, the $3 \times n$ grid with first column $D$ is a Griffin Grid exactly when the $3 \times n$ grid with first column $F$ is a Griffin Grid, and so we may consider these two grid types as one case. The same is true for the grids with first columns $E$ and $G$.

Each cell of a grid with first column $(1, 1, 1)$ is a 1, and thus gives a $3 \times n$ Griffin Grid for all values of $n \geq 2$.
The first 7 columns of the grids with first columns $A, B, C, D, E$ are shown below.

*A*

| | | | | | | |
|---|---|---|---|---|---|---|
| − | + | + | + | − | + | − |
| − | − | + | − | − | + | − |
| − | + | + | + | − | + | − |

*B*

| | | | | | | |
|---|---|---|---|---|---|---|
| + | − | + | − | + | + | + |
| − | − | + | − | − | + | − |
| + | − | + | − | + | + | + |

*C*

| | | | | | | |
|---|---|---|---|---|---|---|
| − | − | + | − | − | + | − |
| + | + | + | + | + | + | + |
| − | − | + | − | − | + | − |

*D*

| | | | | | | |
|---|---|---|---|---|---|---|
| + | − | − | + | − | + | − |
| − | + | + | + | − | + | − |
| − | + | − | − | + | + | + |

*E*

| | | | | | | |
|---|---|---|---|---|---|---|
| + | + | − | − | − | + | − |
| + | − | + | − | + | + | + |
| − | − | − | + | + | + | + |

We will refer to each of the above grids by their first column, $A$, $B$, $C$, $D$, and $E$.
Given a first column and an integer $n \geq 2$, there either is no $3 \times n$ Griffin Grid with that first column, or there is exactly one.
For each possible first column, we are going to count the number of $n$, where $2 \leq n \leq 2024$, for which there is a $3 \times n$ Griffin Grid.
Notice that the 7th column of each of the above grids $A$, $B$ and $C$ matches the 1st column of the grid.
Further, since the 6th column in each grid is $(1, 1, 1)$, then the 8th column will match the 2nd, the 9th will match the 3rd, and in general, column $n + 6$ will match column $n$.
Each of the grids $A$, $B$ and $C$ repeats every 6 columns, and so if a $3 \times n$ grid ($n \geq 2$) is a Griffin Grid, then a $3 \times (n + 6)$ grid is also a Griffin Grid.
This means that to determine for which values of $n$ a $3 \times n$ grid is a Griffin Grid, we need only consider $2 \leq n \leq 7$ (we don't consider $n = 1$ and since the pattern repeats every 6 columns, we check $n = 7$). We then use the fact that the pattern repeats to determine the number of Griffin Grids for all values of $n$ where $2 \leq n \leq 2024$.

In grids $D$ and $E$, the 7th column is a vertical reflection of the 1st column.
Further, since the 6th column in each grid is $(1, 1, 1)$, then the 8th column will be a vertical reflection of the 2nd, and in general, column $n + 6$ is a vertical reflection of column $n$.
That is, in each of the grids $D$ and $E$, each group of 6 columns beginning with the 7th column is a vertical reflection of the previous group of 6 columns.
This tells us that if grid $D$ or $E$ (and thus $F$ or $G$) is a $3 \times n$ Griffin Grid, then the $3 \times (n + 6)$ grid is also a Griffin Grid.
Why is this? To determine if a $3 \times k$ grid is a Griffin Grid, we check that each number in column $k$ is the product of its neighbours, which are numbers in column $k$ and in the previous column, $k - 1$.
Reflecting both of these columns vertically does not change the product of the neighbouring cells, and so both the $3 \times n$ and $3 \times (n + 6)$ grids are Griffin Grids, or they both are not.

Next, we must determine for which values of $n$ with $2 \leq n \leq 7$ the grids $A$, $B$, $C$, $D$, and $E$ are Griffin Grids.
In the diagram below, we place a "Y" below column $n$ if the $3 \times n$ grid is a Griffin Grid, otherwise we leave it blank.

*A*

| | | | | | | |
|---|---|---|---|---|---|---|
| − | + | + | + | − | + | − |
| − | − | + | − | − | + | − |
| − | + | + | + | − | + | − |
| | Y | | | Y | | |

*B*

| | | | | | | |
|---|---|---|---|---|---|---|
| + | − | + | − | + | + | + |
| − | − | + | − | − | + | − |
| + | − | + | − | + | + | + |
| | Y | | | Y | | |

*C*

| | | | | | | |
|---|---|---|---|---|---|---|
| − | − | + | − | − | + | − |
| + | + | + | + | + | + | + |
| − | − | + | − | − | + | − |
| | Y | | | Y | | |

*D*

| | | | | | | |
|---|---|---|---|---|---|---|
| + | − | − | + | − | + | − |
| − | + | + | + | − | + | − |
| − | + | − | − | + | + | + |
| | | | | Y | | |

*E*

| | | | | | | |
|---|---|---|---|---|---|---|
| + | + | − | − | − | + | − |
| + | − | + | − | + | + | + |
| − | − | − | + | + | + | + |
| | | | | Y | | |

Thus, grids $A$, $B$ and $C$ are $3 \times n$ Griffin grids for $n = 2, 5, 8, 11, 14, 17, \ldots$ and so on, while grids $D$ and $E$ are $3 \times n$ Griffin grids for $n = 5, 11, 17, 23, \ldots$ and so on.
Since $2024 = 6 \times 337 + 2$, the complete pattern of 6 columns repeats 337 times for each of the grids.

In each group of 6, there are 2 values of $n$ for which grids $A$, $B$ and $C$ are Griffin Grids, and so there are $2 \times 337 = 674$ Griffin Grids for $2 \leq n \leq 2022$ for each of these 3 grid types.
However, the $3 \times 2024$ grid is also a Griffin Grid in each case, and so there are 675 Griffin Grids for each of the grids $A$, $B$ and $C$.
In each group of 6, there is 1 value of $n$ for which grids $D$, $E$, $F$, and $G$ are Griffin Grids, and so there are 337 Griffin Grids for each of these 4 grid types.
Finally, grid $H$ (the grid of all 1s) is a Griffin Grid for all values of $n$, and so there are 2023 Griffin Grids in this case.

Thus, the sum of the numbers of $3 \times n$ Griffin Grids for $2 \leq n \leq 2024$ is

$$S = (3 \times 675) + (4 \times 337) + 2023 = 5396$$

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# *2023 Galois Contest*

**Wednesday, April 5, 2023**
**(in North America and South America)**

**Thursday, April 6, 2023**
**(outside of North America and South America)**

*Solutions*

1. (a) A grid with 12 rows and 15 columns has $12 \times 15 = 180$ pieces.

   (b) *Solution 1*

   We begin by recognizing that the middle pieces in each grid form a rectangle.
   In a grid with 6 rows, the 1st row and the 6th row are each composed entirely of edge pieces, and thus the grid has $6 - 2 = 4$ rows that contain some middle pieces.
   In each of these 4 rows, the 1st column and the 4th column are each composed entirely of edge pieces, and thus the grid has $4 - 2 = 2$ columns that contain some middle pieces.
   Therefore, a grid with 6 rows and 4 columns contains a rectangular grid of middle pieces having 4 rows and 2 columns, and thus has $4 \times 2 = 8$ middle pieces.

   *Solution 2*

   A grid with 6 rows and 4 columns has $6 \times 4 = 24$ pieces.
   We proceed to find the number of edge pieces, and then subtract this number from 24 to determine the number of middle pieces.
   The first column of the grid contains 6 edge pieces (since there are 6 rows), and the fourth column of the grid also contains 6 edge pieces.
   The first row of the grid contains 4 edge pieces (since there are 4 columns).
   However, the first and last of these edge pieces (the top left and right corners of the grid) were previously included in the count of edge pieces in the first and last columns, respectively, and so there are $4 - 2 = 2$ additional edge pieces in the first row.
   Similarly, there are 2 additional edge pieces in the sixth row.
   Thus, there are $6+6+2+2 = 16$ edge pieces, and so there are $24 - 16 = 8$ middle pieces.

   (c) Since 14 has two possible factor pairs, 1 and 14 or 2 and 7, then the dimensions of the rectangular grid of middle pieces has either 1 row and 14 columns (or vice versa), or it has 2 rows and 7 columns (or vice versa).

   If the rectangular grid of middle pieces has 1 row, then the puzzle grid has $1 + 2 = 3$ rows since there is a row of edge pieces both above and below the 1 row of middle pieces.
   Similarly, if the rectangular grid of middle pieces has 14 columns, then the puzzle grid has $14 + 2 = 16$ columns since there is a column of edge pieces both to the right and to the left of the middle pieces.
   In this case, the puzzle grid has 3 rows and 16 columns (or vice versa), and thus has $3 \times 16 = 48$ pieces.
   A puzzle grid with 48 pieces, including 14 middle pieces, has $48 - 14 = 34$ edge pieces.

   If the rectangular grid of middle pieces has 2 rows, then the puzzle grid has $2 + 2 = 4$ rows, as above.
   Similarly, if the rectangular grid of middle pieces has 7 columns, then the puzzle grid has $7 + 2 = 9$ columns.
   In this case, the puzzle grid has 4 rows and 9 columns (or vice versa), and thus has $4 \times 9 = 36$ pieces.
   A puzzle grid with 36 pieces, including 14 middle pieces, has $36 - 14 = 22$ edge pieces.
   The values of $s$ and $t$ are 34 and 22.

   (d) A grid with 5 rows and $c$ columns contains $5c$ pieces.
   A grid with 5 rows and $c$ columns contains a rectangular grid of middle pieces with $5 - 2 = 3$ rows and $c - 2$ columns, and thus has $3(c - 2)$ middle pieces.
   Since the number of edge pieces is equal to the number of middle pieces, then the total number of pieces is twice the number of middle pieces.
   Thus, $5c = 2 \times 3(c - 2)$ or $5c = 6(c - 2)$. Solving, we get $5c = 6c - 12$ and so $c = 12$.

2. (a) If the first term is 7, then the second term is $7 + 3 = 10$ (since 7 is odd, we add 3).
If the second term is 10, then the third term is $10 + 4 = 14$ (since 10 is even, we add 4).
Similarly, the fourth term is $14 + 4 = 18$, and the fifth term is $18 + 4 = 22$.
If the first term in an Ing sequence is 7, then the fifth term in the sequence is 22.

(b) Suppose that a term, $x$, is odd. The next term, $x + 3$, is even since an odd integer plus an odd integer is even.
Suppose that a term, $x$, is even. The next term, $x + 4$, is even since an even integer plus an even integer is even.
This means that in an Ing sequence, each term after the first is an even integer.
Thus, if the fifth term is 62, then the fourth term cannot equal $62 - 3 = 59$ (since 59 is odd), and so it must equal $62 - 4 = 58$.
Similarly, the third term is $58 - 4 = 54$, and the second term is $54 - 4 = 50$.
If the first term is an even integer, then the first term is $50 - 4 = 46$, and if it is an odd integer, then the first term is $50 - 3 = 47$.
If the fifth term in an Ing sequence is 62, then the first term is 46 (the terms are 46, 50, 54, 58, 62) or the first term is 47 (the terms are 47, 50, 54, 58, 62).

(c) If the first term is 49, then the second term is $49 + 3 = 52$, and so each term after the second is 4 more than the previous term.
This means that for every positive integer $k$, there is a term of the form $52 + 4k$ in the sequence.
Since $52 + 4k = 4(13 + k)$, then each of the remaining terms in the sequence is a multiple of 4.
The integers that are greater than 318 and less than 330, and that are equal to a multiple of 4 are $320 = 4 \times 80$, $324 = 4 \times 81$, and $328 = 4 \times 82$.

(d) If 18 appears somewhere in an Ing sequence after the first term, then the term preceding 18 is either $18 - 3 = 15$, or it is $18 - 4 = 14$.
Each of these is a possible value of $n$, the first term of the sequence.
As was shown in part (b), each term after the first in an Ing sequence is even, and so if 15 appears in the sequence, then 15 can only be the first term of the sequence.
Since 14 is even, then it could be the first term, but it could also be a term after the first.
If 14 appears in the sequence after the first term, then the preceding term is either $14 - 3 = 11$, or it is $14 - 4 = 10$.
Each of these is a possible value of $n$.
If 11 appears in the sequence, then 11 is the first term (since 11 is odd).
Since 10 is even, then it could be the first term, but it could also be a term after the first.
If 10 appears in the sequence after the first term, then the preceding term is either $10 - 3 = 7$, or it is $10 - 4 = 6$.
Each of these is a possible value of $n$.
If 7 appears in the sequence, then 7 is the first term.
Since 6 is even, then it could be the first term, but it could also be a term after the first.
If 6 appears in the sequence after the first term, then the preceding term is either $6 - 3 = 3$, or it is $6 - 4 = 2$.
Each of these is a possible value of $n$.
If 3 appears in the sequence, then 3 is the first term.
If 2 appears in the sequence, then 2 must also be the first term of the sequence since both $2 - 3 = -1$ and $2 - 4 = -2$ are not positive integers.
Thus, if 18 appears somewhere in an Ing sequence after the first term, then the possible values of the first term $n$ are 2, 3, 6, 7, 10, 11, 14, and 15.

3. (a) The line $x = a$ intersects the line $y = x$ at the point $(a, a)$.
Thus, the length of the base and the height of the triangle are each equal to $a$, and so the area of the triangle is $\frac{1}{2} \times a \times a$.
Solving $\frac{1}{2}a^2 = 32$, we get $a^2 = 64$, and so $a = 8$ (since $a > 0$).

(b) *Solution 1*
The line $x = 10$ intersects the line $y = 2x$ at the point $(10, 20)$.
The line $x = 4$ intersects the line $y = 2x$ at the point $(4, 8)$.
Thus, the trapezoid has parallel sides of length 20 and 8, and the distance between the parallel sides is $10 - 4 = 6$.
The area of the trapezoid is $\frac{6}{2}(20 + 8) = 3(28)$ which is equal to 84.

*Solution 2*
If the area of the trapezoid is $T$, the area of the new unshaded triangle is $B$, and the area of the original triangle is $A$, then $T = A - B$.
The line $x = 4$ intersects the line $y = 2x$ at the point $(4, 8)$.
Thus, the unshaded triangle has base length 4 and height 8, and so $B = \frac{1}{2} \times 4 \times 8 = 16$.
The line $x = 10$ intersects the line $y = 2x$ at the point $(10, 20)$.
Thus, the original triangle has base length 10 and height 20, and so $A = \frac{1}{2} \times 10 \times 20 = 100$.
The area of the trapezoid is $T = A - B$ or $T = 100 - 16$ which is 84.

(c) *Solution 1*
We begin by determining the area of the trapezoid.
The line $x = 21$ intersects the line $y = 3x$ at the point $(21, 63)$.
The line $x = c$ intersects the line $y = 3x$ at the point $(c, 3c)$.
Thus, the trapezoid has parallel sides of length 63 and $3c$, and the distance between the parallel sides is $21 - c$ (since $0 < c < 21$).
The area of the trapezoid is $\frac{21 - c}{2}(63 + 3c)$.
Next, we determine the area of the new triangle.
If the length of its base is $c$, then its height is $3c$, and so the area of the new triangle is $\frac{1}{2} \times c \times 3c = \frac{1}{2} \times 3c^2$.
The area of the trapezoid is 8 times the area of the new triangle.
Solving, we get

$$\begin{aligned}
\frac{21 - c}{2}(63 + 3c) &= 8 \times \frac{1}{2} \times 3c^2 \\
(21 - c)(63 + 3c) &= 8 \times 3c^2 \\
(21 - c)(21 + c) &= 8 \times c^2 \\
441 - c^2 &= 8c^2 \\
441 &= 9c^2 \\
c^2 &= 49
\end{aligned}$$

and so $c = 7$ (since $c > 0$).

*Solution 2*
If the area of the trapezoid is $T$, the area of the new triangle is $B$, and the area of the original triangle is $A$, then $T = A - B$.
The area of the trapezoid is 8 times the area of the new triangle, or $T = 8B$.
Substituting, we get $8B = A - B$ or $9B = A$.

The line $x = 21$ intersects the line $y = 3x$ at the point $(21, 63)$.
Thus, $A = \frac{1}{2} \times 21 \times 63 = \frac{1323}{2}$.
The line $x = c$ intersects the line $y = 3x$ at the point $(c, 3c)$.
Thus, $B = \frac{1}{2} \times c \times 3c = \frac{3c^2}{2}$.
Substituting into $9B = A$ and solving, we get

$$\begin{aligned} 9 \times \frac{3c^2}{2} &= \frac{1323}{2} \\ 27c^2 &= 1323 \\ c^2 &= 49 \end{aligned}$$

and so $c = 7$ (since $c > 0$).

(d) *Solution 1*
As was shown in parts (b) and (c), the vertical line drawn at $x = p$ divides the original triangle into a trapezoid and a new triangle.
We begin by determining the area of the trapezoid.
The line $x = 1$ intersects the line $y = 4x$ at the point $(1, 4)$.
The line $x = p$ intersects the line $y = 4x$ at the point $(p, 4p)$.
Thus, the trapezoid has parallel sides of length 4 and $4p$, and the distance between the parallel sides is $1 - p$ (since $0 < p < 1$).
The area of the trapezoid is $\frac{1-p}{2}(4 + 4p)$.

Next, we determine the area of the new triangle.
If the length of its base is $p$, then its height is $4p$, and so the area of the new triangle is $\frac{1}{2} \times p \times 4p = \frac{1}{2} \times 4p^2$.
The line $x = p$ divides the area of the original triangle in half, and so the area of the trapezoid is equal to the area of the new triangle.
Solving, we get

$$\begin{aligned} \frac{1-p}{2}(4 + 4p) &= \frac{1}{2} \times 4p^2 \\ (1-p)(4+4p) &= 4p^2 \\ (1-p)(1+p) &= p^2 \\ 1 - p^2 &= p^2 \\ 1 &= 2p^2 \\ p^2 &= \frac{1}{2} \end{aligned}$$

and so $p = \frac{1}{\sqrt{2}}$ (since $p > 0$).

Ahmed repeats the process by drawing a second vertical line at $x = q$, where $0 < q < p$.

We wish to determine the value of $q$ in terms of $p$, so that we may use this relationship to determine the position of the 12th vertical line (without needing to repeat these calculations 12 times).

That is, we will repeat the above process without substituting $p = \frac{1}{\sqrt{2}}$ so that we may determine the value of $q$ in terms of $p$.

The vertical line drawn at $x = q$ divides the triangle bounded by the $x$-axis, the line $y = 4x$, and the line $x = p$ into a new trapezoid and a new triangle.
We begin by determining the area of the trapezoid.
The line $x = p$ intersects the line $y = 4x$ at the point $(p, 4p)$.
The line $x = q$ intersects the line $y = 4x$ at the point $(q, 4q)$.
Thus, the trapezoid has parallel sides of length $4p$ and $4q$, and the distance between the parallel sides is $p - q$ (since $0 < q < p$).
The area of the trapezoid is $\frac{p-q}{2}(4p + 4q)$.

Next, we determine the area of the triangle.
If the length of its base is $q$, then its height is $4q$, and so the area of the triangle is $\frac{1}{2} \times q \times 4q = \frac{1}{2} \times 4q^2$.
The line $x = q$ divides the area of the previous triangle in half, and so the area of the trapezoid is equal to the area of the new triangle.
Solving, we get

$$\begin{aligned}
\frac{p-q}{2}(4p+4q) &= \frac{1}{2} \times 4q^2 \\
(p-q)(4p+4q) &= 4q^2 \\
(p-q)(p+q) &= q^2 \\
p^2 - q^2 &= q^2 \\
p^2 &= 2q^2 \\
q^2 &= \frac{1}{2} \times p^2
\end{aligned}$$

and so $q = \frac{1}{\sqrt{2}} \times p$ (since $q > 0$).

This tells us that if Ahmed draws a vertical line at $x = n$ (where $n > 0$ and $n$ is less than the $x$-intercept of the vertical line previously drawn), then the next vertical line is drawn at $x = \frac{1}{\sqrt{2}} \times n$ (since the process repeats).
Since the original vertical line is at $x = 1$, then the 12th vertical line drawn by Ahmed is at $x = 1 \times \left(\frac{1}{\sqrt{2}}\right)^{12}$ or $x = \left(\left(\frac{1}{\sqrt{2}}\right)^2\right)^6$ or $x = \left(\frac{1}{2}\right)^6$, and so $k = \frac{1}{64}$.

*Solution 2*
As was shown in parts (b) and (c), the vertical line drawn at $x = p$ divides the original triangle into a trapezoid and a new triangle.
The line $x = 1$ intersects the line $y = 4x$ at the point $(1, 4)$, and so the area of the original triangle is $\frac{1}{2} \times 1 \times 4 = 2$.
The line $x = p$ intersects the line $y = 4x$ at the point $(p, 4p)$, and so the area of the new triangle is $\frac{1}{2} \times p \times 4p = 2p^2$.

The area of the new triangle is half of the area of the original triangle, and so $2p^2 = 1$ or $p^2 = \frac{1}{2}$, and so $p = \frac{1}{\sqrt{2}}$ (since $p > 0$).
Ahmed repeats the process by drawing a second vertical line at $x = q$, where $0 < q < p$.
We wish to determine the value of $q$ in terms of $p$, so that we may use this relationship to determine the position of the 12th vertical line (without needing to repeat these calculations 12 times).

That is, we will repeat the above process without substituting $p = \frac{1}{\sqrt{2}}$ so that we may determine the value of $q$ in terms of $p$.

The vertical line drawn at $x = q$ divides the triangle bounded by the $x$-axis, the line $y = 4x$, and the line $x = p$ into a new trapezoid and a new triangle.
As was determined above, the triangle bounded by the $x$-axis, the line $y = 4x$, and the line $x = p$ has area $2p^2$.
The line $x = q$ intersects the line $y = 4x$ at the point $(q, 4q)$, and so the area of the new triangle is $\frac{1}{2} \times q \times 4q = 2q^2$.

The area of the new triangle is half of the area of the previous triangle, and so $2q^2 = \frac{2p^2}{2}$ or $q^2 = \frac{1}{2} \times p^2$, and so $q = \frac{1}{\sqrt{2}} \times p$ (since $q > 0$).
This tells us that if Ahmed draws a vertical line at $x = n$ (where $n > 0$ and $n$ is less than the $x$-intercept of the vertical line previously drawn), then the next vertical line is drawn at $x = \frac{1}{\sqrt{2}} \times n$ (since the process repeats).
Since the original vertical line is at $x = 1$, then the 12th vertical line drawn by Ahmed is at $x = 1 \times \left(\frac{1}{\sqrt{2}}\right)^{12}$ or $x = \left(\left(\frac{1}{\sqrt{2}}\right)^2\right)^6$ or $x = \left(\frac{1}{2}\right)^6$, and so $k = \frac{1}{64}$.

*Solution 3*
Ahmed draws the 12th vertical line at $x = k$.
The line $x = k$ intersects the line $y = 4x$ at the point $(k, 4k)$, and so the area of the new triangle to the left of this line is $\frac{1}{2} \times k \times 4k = 2k^2$.
Since the area of each new triangle is half of the area of the previous triangle, then the triangle with area $2k^2$ has $\left(\frac{1}{2}\right)^{12}$ of the area of the original triangle.
The line $x = 1$ intersects the line $y = 4x$ at the point $(1, 4)$, and so the area of the original triangle is $\frac{1}{2} \times 1 \times 4 = 2$.
Equating the areas and solving for $k$, we get

$$\begin{aligned} 2k^2 &= \left(\frac{1}{2}\right)^{12} \times 2 \\ k^2 &= \left(\frac{1}{2}\right)^{12} \\ k &= \left(\frac{1}{2}\right)^6 \end{aligned}$$

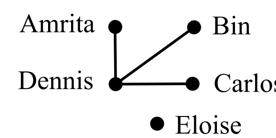
and so $k = \frac{1}{64}$.

4. (a) *Solution 1*

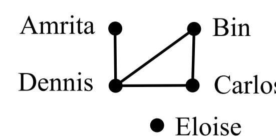
Amrita shook hands with exactly 1 person, Bin and Carlos each shook hands with exactly 2 people, and Dennis shook hands with exactly 3 people, and so this gives $1+2+2+3=8$ handshakes, except each of these handshakes is counted twice.
That is, when Person X shakes Person Y's hand, Person Y shakes Person X's hand, and so this one handshake is counted twice.
In general, if $S$ is the sum of the number of hands shaken by each person, and $N$ is the number of handshakes that occurred, then $N=S\div 2$.
Thus, the total number of handshakes that took place was $8\div 2=4$.

*Solution 2*

Dennis shook hands with exactly 3 people and Eloise did not shake hands with anyone. Therefore, Dennis must have shaken hands with Amrita, Bin and Carlos (and Amrita, Bin and Carlos each shook hands with Dennis).
If a line segment drawn between 2 people represents a handshake, then the diagram to the right shows the handshakes accounted for to this point.
The diagram shows that Amrita has 1 handshake, Dennis has 3 and Eloise has 0, and so all handshakes for Amrita, Dennis and Eloise have been accounted for.

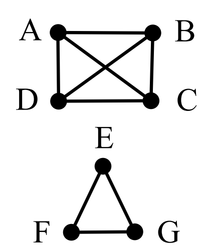



Bin and Carlos each shook hands with exactly 2 people, and so their second handshakes must be with one another (since they can't be with Amrita, Dennis or Eloise).
The diagram to the right shows all handshakes that occurred, and so there were a total of 4 handshakes.




(b) As in part (a) Solution 1, if 9 people each shook hands with exactly 3 people, then $S=9\times 3=27$, and the total number of handshakes was $N=27\div 2=13.5$.
Since the number of handshakes that took place must be an integer, then it is not possible that each of 9 people shook hands with exactly 3 others.

(c) We represent the 7 people with the letters, A, B, C, D, E, F, and G.
If each of A, B, C, and D shook hands with one another, and each of E, F, and G shook hands with one another, and no other handshakes occurred, then a total of 9 handshakes took place, as shown in the diagram.
We will show that this set of 9 handshakes satisfies the given conditions and that fewer than 9 handshakes does not, and thus $m=9$.




We begin with an explanation of why the set of 9 handshakes shown in the diagram satisfies the condition that at least one handshake occurred within each group of 3 people.
Let Group 1 be the group A, B, C, D, and Group 2 be the group E, F, G.
In any group of 3 people chosen from the 7 people, either all 3 people are from Group 1, or all 3 people are from Group 2, or 1 person is from one of the two groups, and 2 people are from the other group.
That is, at least 2 of the 3 people chosen must belong to either Group 1 or to Group 2.
Since a handshake occurs between each pair of people in Group 1 and between each pair of people in Group 2, and every group of 3 people chosen must contain at least 2 people

from the same group, then at least one handshake occurs within each group of 3 people chosen.

Next, we give an explanation of why fewer than 9 handshakes does not satisfy the given conditions.
Define $S$ and $N$ as in part (a) Solution 1.
Assume that $N \leq 8$.
Since each handshake occurs between 2 people and $N \leq 8$, then $S$ is at most $8 \times 2 = 16$.
If each of the 7 people shook 3 or more hands, then $S$ would be at least $7 \times 3 = 21$.
Since $S$ is at most 16, then at least one of the 7 people shook hands with 2 or fewer people.
Suppose that it was E who shook hands with 2 or fewer people. (It might not be E but whoever it is, the reasoning is identical.)
Then at least 4 people did not shake E's hand.
Suppose that A, B, C, D did not shake E's hand. (Again, the reasoning is the same if it is any other four people.)
In this case, each pair of people from the group A, B, C, D must have shaken hands with one another, otherwise the pair that did not shake hands, along with E, form a group of 3 people in which no handshakes occurred.
Since each pair of people in the group A, B, C, D shook hands, there are 6 handshakes within this group (A and B, A and C, A and D, B and C, B and D, C and D).
Since $N$ is at most 8, then E, F and G participate in at most $8 - 6 = 2$ handshakes in total.
That is, E, F, G could participate in 0, 1 or 2 handshakes, giving 3 cases to consider.

Case 1: No pair of people in the group E, F, G shook hands
If no pair of people in the group E, F, G shook hands, then they are a group of 3 people in which no handshakes occurred.

Case 2: Exactly one pair of people in the group E, F, G shook hands
In this case, there is at most one handshake between one of E, F, G, and one of A, B, C, D.
Suppose E and F shook hands (recognizing that the argument holds when choosing any pair from E, F, G).
There is at least one person in the group A, B, C, D who did not shake hands with F and did not shake hands with G, and so this is a group of 3 people in which no handshakes occurred.

Case 3: Exactly two pairs of people in the group E, F, G shook hands
Suppose E and F shook and E and G shook (recognizing that the argument holds when choosing any two pairs from E, F, G).
In this case, F and G and one of A, B, C, D is a group of 3 people in which no handshakes occurred.

Therefore, 8 or fewer handshakes is not possible, and so $m = 9$.

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# *2022 Galois Contest*

**Tuesday, April 12, 2022**
**(in North America and South America)**

**Wednesday, April 13, 2022**
**(outside of North America and South America)**

*Solutions*

1. (a) The ratio of Alice's contribution to Bello's contribution was $3:8$.
This means that for every $\$3 + \$8 = \$11$ contributed to starting the business, Bello contributed \$8, and so Bello contributed $\frac{8}{11}$ of the cost of starting the new business.
If the cost of starting the new business was \$9240, then Bello's contribution to this cost was $\frac{8}{11} \times \$9240 = \$6720$.

(b) The ratio of Alice's share of the first year profits to Bello's share was $3:8$.
This means that for every $\$3 + \$8 = \$11$ of profit in the first year, Alice's share was \$3.
Let the first year profit be $\$P$.
Since Alice's share of the first year profit was \$1881, then $\frac{3}{11} \times P = 1881$ or $P = \frac{1881 \times 11}{3}$, and so $P = \frac{20\,691}{3} = 6897$.
The total profit of the business for the first year was \$6897.

(c) The ratio of Alice's share of the second year profits to Bello's share was $3:(8+x)$.
This means that for every $\$3 + \$(8+x) = \$(11+x)$ of profit in the second year, Bello's share was $\$(8+x)$.
Since Bello's share of the second year profit was \$5440, and the profit that year was \$6400, then $\frac{(8+x)}{(11+x)} \times 6400 = 5440$.
Solving this equation, we get

$$\begin{aligned}
\frac{(8+x)}{(11+x)} \times 6400 &= 5440 \\
6400(8+x) &= 5440(11+x) \\
20(8+x) &= 17(11+x) \quad \text{(dividing both sides by 320)} \\
160 + 20x &= 187 + 17x \\
3x &= 27
\end{aligned}$$

and so $x = 9$.

2. (a) Line $L_1$ has equation $y = \frac{3}{2}x + k$, and thus has slope $\frac{3}{2}$.
Since $L_2$ is perpendicular to $L_1$, then its slope is $-\frac{2}{3}$.

(b) Since $L_1$ has equation $y = \frac{3}{2}x + k$, its $y$-intercept is $k$.
Line $L_2$ has the same $y$-intercept as $L_1$ (which is at $P(0,k)$).
Thus, $L_2$ has slope $-\frac{2}{3}$ and $y$-intercept $k$, and so it has equation $y = -\frac{2}{3}x + k$.
Since $L_2$ intersects the $x$-axis at $Q$, the $x$-coordinate of point $Q$ is the $x$-intercept of $L_2$.
Setting $y = 0$ in the equation for $L_2$ and solving for $x$, we get $0 = -\frac{2}{3}x + k$ or $\frac{2}{3}x = k$, and so $x = \frac{3k}{2}$.
Written in terms of $k$, the $x$-coordinate of point $Q$ is $\frac{3k}{2}$.

(c) From part (b), the coordinates of $P$ are $(0,k)$, and the coordinates of $Q$ are $\left(\frac{3k}{2}, 0\right)$.
To determine an expression for the area of $\triangle PQR$, we first need to determine the coordinates of point $R$.
$L_3$ is parallel to $L_1$ and thus has slope $\frac{3}{2}$ and equation $y = \frac{3}{2}x + b$, for some $y$-intercept $b$.
Line $L_3$ passes through point $Q\left(\frac{3k}{2}, 0\right)$, and so $0 = \frac{3}{2}\left(\frac{3k}{2}\right) + b$ or $b = -\frac{9k}{4}$.
Therefore the $y$-intercept of $L_3$ is $-\frac{9k}{4}$ and so $R$ has coordinates $\left(0, -\frac{9k}{4}\right)$.

If we call the origin $O(0,0)$, then the area of $\triangle PQR$ is given by $\frac{1}{2}\times PR\times OQ$, since height $OQ$ is perpendicular to the base $PR$.
Since $PR = k-\left(-\frac{9k}{4}\right) = \frac{13k}{4}$ and $OQ = \frac{3k}{2}$, then the area of $\triangle PQR$ is $\frac{1}{2}\times\frac{13k}{4}\times\frac{3k}{2} = \frac{39k^2}{16}$.
The area of $\triangle PQR$ is 351, and so $\frac{39k^2}{16} = 351$ or $k^2 = \frac{351\times 16}{39}$ or $k^2 = 144$, and so $k = 12$ (since $k > 0$).

3. (a) We begin by recognizing that a number is a perfect square if the exponent on each prime factor in its prime factorization is even, and conversely that the prime factorization of every perfect square has an even exponent on each prime factor.
The prime factorization of 84 is $2^2\times 3\times 7$.
For the product $84\times k = 2^2\times 3\times 7\times k$ to be a perfect square, $k$ must at least include 3 and 7 as prime factors (since the exponent on each is an odd number), and so $k$ must be divisible by 21.
If $k = 3\times 7$, then $84\times k$ is a perfect square because

$$84\times 3\times 7 = 2^2\times 3^2\times 7^2 = (2\times 3\times 7)\times(2\times 3\times 7).$$

Thus, the smallest value of the positive integer $k$ for which $84\times k$ is a perfect square is $k = 3\times 7 = 21$.

(b) The prime factorization of 572 is $2^2\times 11\times 13$, and so $\ell$ must at least include 11 and 13 as prime factors (since the exponent on each is an odd number).
Since $572\times\ell$ is a perfect square, then any factor of $\ell$ in addition to 11 and 13 must be a perfect square.
That is, $\ell$ is a number of the form $11\times 13\times n^2$ for some positive integer $n$, because

$$572\times\ell = 572\times 11\times 13\times n^2 = (2\times 11\times 13\times n)\times(2\times 11\times 13\times n).$$

Since $\ell = 11\times 13\times n^2 = 143n^2$ and $\ell < 6000$, then $143n^2 < 6000$, and so $n^2 < \frac{6000}{143}\approx 41.96$.
The greatest value of the positive integer $n$ for which $n^2\leq 41$ is $n = 6$.
Therefore, the greatest possible value of $\ell$ is $11\times 13\times 6^2 = 5148$.

(c) The prime factorization of 525 000 is $2^3\times 3\times 5^5\times 7$, and so if $525\,000\times m$ is a perfect square, then $m$ must at least include $2, 3, 5$, and 7 as prime factors (since the exponent on each is an odd number).
If $m$ includes the prime factors 2, 3, 5, and 7, then $m$ is greater than or equal to $2\times 3\times 5\times 7 = 210$.
Therefore, if $m$ is a positive integer less than 200, then $525\,000\times m$ cannot be a perfect square.

(d) Suppose that the three powers of 10 chosen from the list are $10^a, 10^b$, and $10^c$, where $a, b, c$ are odd positive integers between 1 and 99 inclusive, and $a < b < c$.
Since $a < b$ and $a < c$, the sum of these three powers, $S = 10^a + 10^b + 10^c$, can be factored so that $S = 10^a(1 + 10^{b-a} + 10^{c-a})$.
Since $b - a$ is a positive integer, then $10^{b-a}$ is an even positive integer.
Similarly, $10^{c-a}$ is an even positive integer, and so $1 + 10^{b-a} + 10^{c-a}$ is an odd positive integer.
Thus for some odd positive integers $a, b, c$, and $d$,

$$S = 10^a(1 + 10^{b-a} + 10^{c-a}) = 10^a\times d = (2\times 5)^a\times d = 2^a\times 5^a\times d$$

Since $d$ is an odd integer, then 2 is not a prime factor of $d$, and so the prime factor 2 occurs an odd number of times in the prime factorization of $S$ (since $a$ is odd).
Therefore, the sum of every choice of three different powers of 10 from the list is not a perfect square.

4. (a) *Solution 1*

For each Bauman string of length 5 in which the first and last letters are both $A$, the second and fourth letters are not $A$.
For such Bauman strings, either the third letter is $A$, or the third letter is not $A$.

Case 1: The third letter is $A$.
In this case, the string is of the form $A\_A\_A$.
There are 4 choices for the second letter ($B, C, D$, or $E$) and 4 choices for the fourth letter, and so there are $4 \times 4 = 16$ such Bauman strings.

Case 2: The third letter is not $A$.
In this case, the string is of the form $A\_x\_A$, where the third letter $x$ is $B, C, D$, or $E$, and so there are 4 choices for the third letter.
The second letter must be different than the third letter and must not be $A$, and so there are 3 choices for the second letter.
Similarly, there are 3 choices for the fourth letter, and so there are $4 \times 3 \times 3 = 36$ such Bauman strings.

In total, the number of Bauman strings of length 5 in which the first and last letters are both $A$ is $16 + 36 = 52$.

*Solution 2*

For each Bauman string of length 5 in which the first and last letters are both $A$, either the second and fourth letters are the same, or they are different.

Case 1: The second and fourth letters are the same.
In this case, the string is of the form $Ax\_xA$.
There are 4 choices for the second letter ($B, C, D$, or $E$) and 1 choice for the fourth letter since it is the same as the second letter.
The third letter must be different than the second and fourth letters (which are the same) and so there are 4 choices for the third letter.
Thus, there are $4 \times 1 \times 4 = 16$ such Bauman strings.

Case 2: The second and fourth letters are different.
In this case, the string is of the form $Ax\_yA$, where $x$ and $y$ represent different letters.
There are 4 choices for the second letter ($B, C, D$, or $E$) and 3 choices for the fourth letter (it is different than the second letter and not $A$).
The third letter must be different than the second and fourth letters (which are different) and so there are 3 choices for the third letter.
Thus, there are $4 \times 3 \times 3 = 36$ such Bauman strings.

In total, the number of Bauman strings of length 5 in which the first and last letters are both $A$ is $16 + 36 = 52$.

(b) *Solution 1*

We may determine the number of Bauman strings of length 6 that contain more than one $B$ indirectly.
That is, we may subtract the number of Bauman strings that contain 0 $B$'s and the number of Bauman strings that contain exactly 1 $B$ from the total number of Bauman strings of length 6.

For a Bauman string of length 6 (with no restrictions), there are 5 choices for the first letter and 4 choices for each of the remaining letters, and so there are a total of $5 \times 4^5 = 5120$

such strings.

For a Bauman string of length 6 that contains 0 $B$'s, there are 4 choices for the first letter and 3 choices for each of the remaining letters, and so there are a total of $4 \times 3^5 = 972$ such strings.

Next, we count the number of Bauman strings that contain exactly 1 $B$.
If the first letter of the string is a $B$, then there are 4 choices for the second letter and 3 choices for each of the remaining four letters.
Similarly, if the last letter is a $B$, then there are 4 choices for the letter adjacent to the $B$ and 3 choices for each of the remaining four letters.
Thus, there are $1 \times 4 \times 3^4 = 324$ strings that begin with a $B$ and 324 strings that end with a $B$.
If the second letter of the string is a $B$, then there are 4 choices for the first letter, 4 choices for the third letter, and 3 choices for each of the fourth, fifth and sixth letters.
Thus, there are $4 \times 1 \times 4 \times 3^3 = 432$ such strings.
Similarly, if the third, fourth or fifth letter in the string is a $B$, then there are 432 such strings.

In total, there are $5120 - 972 - 2 \times 324 - 4 \times 432 = 1772$ Bauman strings of length 6 that contain more than one $B$.

*Solution 2*
A Bauman string of length 6 cannot contain more than 3 $B$'s (confirm for yourself why this is true before proceeding).
We may determine the number of Bauman strings of length 6 that contain more than one $B$ directly.
That is, we may count the number of strings that contain exactly 3 $B$'s and the number of strings that contain exactly 2 $B$'s.
Thus, there are two cases to consider.
Case 1: The Bauman string has exactly 3 $B$'s
In this case, the string must take one of four possible forms:

$$B\_B\_B\_\,, \quad B\_B\_\_B, \quad B\_\_B\_B, \quad \_B\_B\_B.$$

We begin by counting the number of strings of the form $B\_B\_B\_$.
There are 4 choices for the second letter ($A, C, D$, or $E$), 4 choices for the fourth letter, and 4 choices for the sixth letter, and so there are $4^3 = 64$ such strings.
Similarly, there are $4^3 = 64$ strings of the form $\_B\_B\_B$.
It is worth noting that $B\_B\_B\_$ and $\_B\_B\_B$ have identical forms when one is read left to right and the other right to left.
We may call such pairs of forms *symmetric*, and recognize that under the same restrictions, symmetric forms have an equal number of Bauman strings.

For strings of the form $B\_B\_\_B$, there are 4 choices for the second letter, 4 choices for the fourth letter, and 3 choices for the fifth letter (since the fifth letter must be different than $B$ and different than the fourth letter, which is not $B$), and so $4^2 \times 3 = 48$ such strings.
Since $B\_\_B\_B$ is symmetric to $B\_B\_\_B$, there are similarly $4^2 \times 3 = 48$ strings of this form.
Thus, there are $2 \times 64 + 2 \times 48 = 224$ Bauman strings with exactly 3 $B$'s.

<u>Case 2</u>: The Bauman string has exactly 2 $B$'s
In this case, the string must take one of ten possible forms.
Eight of these occur in one of the following four symmetric pairs

$$B\_B\_\_\_ \text{ and } \_\_\_B\_B, \qquad B\_\_B\_\_ \text{ and } \_\_B\_\_B,$$

$$B\_\_\_B\_ \text{ and } \_B\_\_\_B, \qquad \_B\_B\_\_ \text{ and } \_\_B\_B\_$$

and the final two forms (which are not a symmetric pair) are

$$B\_\_\_\_B \text{ and}$$

$$\_B\_\_B\_.$$

We begin by counting the number of strings of the form $B\_B\_\_\_$.
There are 4 choices for the second letter ($A, C, D$, or $E$), 4 choices for the fourth letter, 3 choices for each of the fifth and sixth letters, and so there are $4^2 \times 3^2 = 144$ such strings.
Similarly, there are $4^2 \times 3^2 = 144$ strings for each of the next five forms listed above, $\_\_\_B\_B$, $B\_\_B\_\_$, $\_\_B\_\_B$, $B\_\_\_B\_$, and $\_B\_\_\_B$.

For strings of the form $\_B\_B\_\_$, there are 4 choices for the first letter, 4 choices for the third letter, 4 choices for the fifth letter, and 3 choices for the sixth letter, and so there are $4^3 \times 3 = 192$ such strings.
Since $\_\_B\_B\_$ is symmetric to $\_B\_B\_\_$, there are similarly $4^3 \times 3 = 192$ strings of this form.

Finally, there are $4 \times 3^3 = 108$ strings of the form $B\_\_\_\_B$, and $4^3 \times 3 = 192$ strings of the form $\_B\_\_B\_$.
Thus, there are $6 \times 144 + 2 \times 192 + 108 + 192 = 1548$ Bauman strings with exactly 2 $B$'s.

In total, there are $224 + 1548 = 1772$ Bauman strings of length 6 that contain more than one $B$.

(c) *Solution 1*
Consider all Bauman strings of length $n$ that begin with $C$.
There are 4 choices for each of the remaining $n - 1$ letters, and so there are $4^{n-1}$ such Bauman strings.
Each of these strings either ends with $D$, or it does not end with $D$.
We call those that end with $D$, $d_n$, and we define $|d_n|$ to be the number of such strings.
Similarly, we call those that do not end in $D$, $x_n$, and we define $|x_n|$ to be the number of such strings.
For example, $d_1$ represents the Bauman strings of length 1 that begin with $C$ and end with $D$, and since no such string exists, then $|d_1| = 0$.
Similarly, $x_1$ represents the Bauman strings of length 1 that begin with $C$ and do not end with $D$, and so $|x_1| = 1$ (the string is $C$).
We may confirm that $4^{n-1} = |d_n| + |x_n|$ when $n = 1$, since $4^{1-1} = |d_1| + |x_1| = 0 + 1$.
Further, we know that $|d_2| = 1$ (the string is $CD$), and $|x_2| = 3$ (the strings are $CA$, $CB$, $CE$), and again confirm that $4^{2-1} = 1 + 3$.

Next, consider the Bauman strings of length $n$ that begin with $C$ and do not end with $D$, that is, $x_n$.
Each of these strings could have a $D$ added to its end to form a Bauman string of length $n + 1$ that begins with $C$ and ends with $D$, or $d_{n+1}$.
Since adding a $D$ to the end of every string $x_n$ gives all possible strings $d_{n+1}$,

then $|d_{n+1}| = |x_n|$.
From our earlier work, we may confirm that $|d_2| = |x_1| = 1$.

Further, $|x_{n+1}| = 4|d_n| + 3|x_n|$. Why is this true?
Every Bauman string of length $n+1$ that begins with $C$ and does not end with $D$, that is $x_{n+1}$, is either a string $d_n$ with an $A$, $B$, $C$, or $E$ added to its end, or it is a string $x_n$ that has a choice of 3 letters added to its end (the letter added can not be $D$ and it can not be the last letter of $x_n$, leaving 3 possibilities).
The number of strings $d_n$ that have $A$, $B$, $C$, or $E$ added to its end is $4|d_n|$.
The number of strings $x_n$ that have a choice of 3 letters added to its end is $3|x_n|$.
Therefore, we conclude that $|x_{n+1}| = 4|d_n| + 3|x_n|$.
From our earlier work, we may confirm that $|x_2| = 4|d_1| + 3|x_1| = 4(0) + 3(1) = 3$.
We use these two formulas $|d_{n+1}| = |x_n|$ and $|x_{n+1}| = 4|d_n| + 3|x_n|$, which are equivalent to $|d_n| = |x_{n-1}|$ and $|x_n| = 4|d_{n-1}| + 3|x_{n-1}|$, to build the table below.

| $n$ | $\lvert d_n\rvert = \lvert x_{n-1}\rvert$ | $\lvert x_n\rvert = 4\lvert d_{n-1}\rvert + 3\lvert x_{n-1}\rvert$ |
|---|---|---|
| 1 | $\lvert d_1\rvert = 0$ | $\lvert x_1\rvert = 1$ |
| 2 | $\lvert d_2\rvert = 1$ | $\lvert x_2\rvert = 3$ |
| 3 | $\lvert d_3\rvert = \lvert x_2\rvert = 3$ | $\lvert x_3\rvert = 4\lvert d_2\rvert + 3\lvert x_2\rvert = 4(1) + 3(3) = 13$ |
| 4 | $\lvert d_4\rvert = \lvert x_3\rvert = 13$ | $\lvert x_4\rvert = 4\lvert d_3\rvert + 3\lvert x_3\rvert = 4(3) + 3(13) = 51$ |
| 5 | $\lvert d_5\rvert = 51$ | $\lvert x_5\rvert = 4(13) + 3(51) = 205$ |
| 6 | $\lvert d_6\rvert = 205$ | $\lvert x_6\rvert = 4(51) + 3(205) = 819$ |
| 7 | $\lvert d_7\rvert = 819$ | $\lvert x_7\rvert = 4(205) + 3(819) = 3277$ |
| 8 | $\lvert d_8\rvert = 3277$ | $\lvert x_8\rvert = 4(819) + 3(3277) = 13\,107$ |
| 9 | $\lvert d_9\rvert = 13\,107$ | $\lvert x_9\rvert = 4(3277) + 3(13\,107) = 52\,429$ |
| 10 | $\lvert d_{10}\rvert = 52\,429$ | not needed |

Therefore, the number of Bauman strings of length 10 in which the first letter is $C$ and the last letter is $D$ is 52 429.

*Solution 2*
Let $S_n$ be the set of Bauman strings of length $n$ in which the first letter is $C$ and the last letter is $D$.
Further, we define $|S_n|$ to be the number of such strings.
For example, $S_2 = \{CD\}$ and so $|S_2| = 1$, and $S_3 = \{CAD, CBD, CED\}$ and so $|S_3| = 3$.
Each string in $S_{10}$ is of the form $C\_\_\_\_\_\_\_\_D$ or $CT_8D$ where $T_8$ is a Bauman

string of length 8 that does not begin with $C$ and does not end with $D$.
Therefore, $|S_{10}| = |T_8|$. The number of such strings, $|T_8|$, is equal to

the total number of Bauman strings of length 8
$-$ the number of Bauman strings of length 8 that begin with $C$
$-$ the number of Bauman strings of length 8 that end with $D$
$+$ the number of Bauman strings of length 8 that begin with $C$ and end with $D$

In the above, we note that strings beginning with $C$ include those that end with $D$, as well as others.
Similarly, strings that end with $D$ include those that begin with $C$, as well as others.
Since we have subtracted the number of strings that begin with $C$ and end with $D$ twice from the total, we conclude by adding the number of such strings.

For a Bauman string of length 8, there are 5 choices for the first letter and 4 choices for each of the remaining 7 letters.
Thus, the total number of Bauman strings of length 8 is equal to $5 \times 4^7$.
For a Bauman string of length 8 that begins with $C$, there is 1 choice for the first letter and 4 choices for each of the remaining 7 letters.
Thus, the total number of Bauman strings of length 8 that begin with $C$ is equal to $1 \times 4^7$.
Also, the total number of Bauman strings of length 8 that end with $D$ is equal to $1 \times 4^7$.
The number of Bauman strings of length 8 that begin with $C$ and end with $D$ is equal to $|S_8|$.
Therefore, we get

$$|S_{10}| = |T_8| = 5 \times 4^7 - 1 \times 4^7 - 1 \times 4^7 + |S_8| = 3 \times 4^7 + |S_8|$$

and so the number of Bauman strings of length 10 that begin with $C$ and end with $D$ is dependent on the number of Bauman strings of length 8 that begin with $C$ and end with $D$.
At this point, we could repeat the process above to determine $|S_8|$, however it may be more efficient to generalize the work above to determine a formula for $|S_n|$.

For integers $n \geq 3$, each string in $S_n$ is of the form $CT_{n-2}D$ where $T_{n-2}$ is a Bauman string of length $n-2$ that does not begin with $C$ and does not end with $D$.
The number of such strings, $|T_{n-2}|$, is equal to

the total number of Bauman strings of length $n-2$
$-$ the number of Bauman strings of length $n-2$ that begin with $C$
$-$ the number of Bauman strings of length $n-2$ that end with $D$
$+$ the number of Bauman strings of length $n-2$ that begin with $C$ and end with $D$

For a Bauman string of length $n-2$, there are 5 choices for the first letter and 4 choices for each of the remaining $n-3$ letters.
Thus, the total number of Bauman strings of length $n-2$ is equal to $5 \times 4^{n-3}$.
For a Bauman string of length $n-2$ that begins with $C$, there is 1 choice for the first letter and 4 choices for each of the remaining $n-3$ letters.
Thus, the total number of Bauman strings of length $n-2$ that begin with $C$ is equal to $1 \times 4^{n-3}$.
Also, the total number of Bauman strings of length $n-2$ that end with $D$ is equal to $1 \times 4^{n-3}$.
The number of Bauman strings of length $n-2$ that begin with $C$ and end with $D$ is equal

to $|S_{n-2}|$.
Therefore, we get

$$|S_n| = |T_{n-2}| = 5\times 4^{n-3} - 1\times 4^{n-3} - 1\times 4^{n-3} + |S_{n-2}| = 3\times 4^{n-3} + |S_{n-2}|, \text{ for integers } n \geq 3.$$

Using this recursive formula and the fact that $|S_2| = 1$, we get

$$\begin{array}{rclclcl} |S_4| &=& 3\times 4^{4-3} + |S_{4-2}| &=& 3\times 4 + |S_2| &=& 13 \\ |S_6| &=& 3\times 4^{6-3} + |S_{6-2}| &=& 3\times 4^3 + |S_4| &=& 205 \\ |S_8| &=& 3\times 4^{8-3} + |S_{8-2}| &=& 3\times 4^5 + |S_6| &=& 3277 \\ |S_{10}| &=& 3\times 4^{10-3} + |S_{10-2}| &=& 3\times 4^7 + |S_8| &=& 52\,429 \end{array}$$

The number of Bauman strings of length 10 in which the first letter is $C$ and the last letter is $D$ is 52 429.

The CENTRE for EDUCATION
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# *2021 Galois Contest*

**April 2021**
**(in North America and South America)**

**April 2021**
**(outside of North America and South America)**

*Solutions*

1. (a) Substituting $a = 5$ and $b = 1$, we get $5\triangle 1 = 5(2 \times 1 + 4) = 5(6) = 30$.

   (b) If $k\triangle 2 = 24$, then $k(2 \times 2 + 4) = 24$ or $8k = 24$, and so $k = 3$.

   (c) Solving the given equation for $p$, we get

$$\begin{aligned} p\triangle 3 &= 3\triangle p \\ p(2 \times 3 + 4) &= 3(2p + 4) \\ p(10) &= 6p + 12 \\ 10p - 6p &= 12 \\ 4p &= 12 \\ p &= 3 \end{aligned}$$

   The only value of $p$ for which $p\triangle 3 = 3\triangle p$ is $p = 3$.

   (d) Simplifying the given equation, we get

$$\begin{aligned} m\triangle(m + 1) &= 0 \\ m(2(m + 1) + 4) &= 0 \\ m(2m + 2 + 4) &= 0 \\ m(2m + 6) &= 0 \end{aligned}$$

   Thus, $m = 0$ or $2m + 6 = 0$ which gives $m = -3$.
   The values of $m$ for which $m\triangle(m + 1) = 0$ are $m = 0$ and $m = -3$.
   (Substituting each of these values of $m$, we may check that $0\triangle 1 = 0(2\times 1+4) = 0(6) = 0$, and that $(-3)\triangle(-2) = -3(2 \times (-2) + 4) = -3(0) = 0$.)

2. (a) Team $P$ played 27 games which included 10 wins and 14 losses.
   Thus, Team $P$ had $27 - 10 - 14 = 3$ ties at the end of the season.

   (b) Team $Q$ had 2 more wins than Team $P$, or $10 + 2 = 12$ wins.
   Team $Q$ had 4 fewer losses than Team $P$, or $14 - 4 = 10$ losses.
   Since Team $Q$ played 27 games, they had $27 - 12 - 10 = 5$ ties.
   At the end of the season, Team $Q$ had a total of $(2 \times 12) + (0 \times 10) + (1 \times 5)$ or 29 points.

   (c) *Solution 1*
   Assume that Team $R$ finished the season with exactly 6 ties.
   Since 6 ties contribute 6 points to their points total, then Team $R$ earned the remaining $25 - 6 = 19$ points as a result of their wins.
   However, each win contributes 2 points to the total, and thus it is not possible to earn an odd number of points from wins.
   Therefore, Team $R$ could not have finished the season with exactly 6 ties.

   *Solution 2*
   Assume that Team $R$ finished the season with exactly $w$ wins.
   If Team $R$ finished with exactly 6 ties, then they finished the season with a total of $(2 \times w) + (1 \times 6)$ or $2w + 6 = 2(w + 3)$ points (they earn 0 points for losses).
   Since $w$ is an integer, then $w + 3$ is an integer and so $2(w + 3)$ is an even integer.
   However, this is not possible since Team $R$ finished the season with 25 points, an odd number of points.
   Therefore, Team $R$ could not have finished the season with exactly 6 ties.

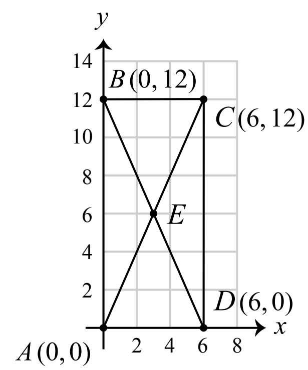
(d) *Solution 1*

Let the number of losses that Team $S$ had at the end of the season be $\ell$.
Team $S$ had 4 more wins than losses and thus finished the season with $\ell + 4$ wins.
Since Team $S$ played 27 games, then each of their remaining $27 - \ell - (\ell + 4) = 23 - 2\ell$ games resulted in a tie.
Therefore, Team $S$ finished the season with a total of $(2 \times (\ell + 4)) + (0 \times \ell) + (1 \times (23 - 2\ell))$ or $2\ell + 8 + 23 - 2\ell = 31$ points.

*Solution 2*

Each of the 4 teams played 27 games, 2 teams played in each game, and so the season finished with a total of $\frac{4\times 27}{2} = 54$ games played.
Each of the 54 games resulted in a total of 2 points being awarded (either 2 points to a winning team and 0 to the losing team or 1 point to each of the two teams that tied).
Thus, the total points earned by all 4 teams at the end of the season was $2 \times 54 = 108$.
The table shows that Team $P$ finished with 23 points, Team $R$ had 25 points, and in part (b) we determined that Team $Q$ had 29 points at the end of the season.
Therefore, Team $S$ finished the season with $108 - 23 - 25 - 29 = 31$ points.

3. (a) *Solution 1*

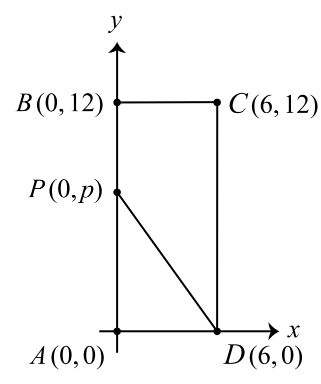



We begin by drawing and labelling a diagram, as shown.
The diagonals of a rectangle intersect at the centre of the rectangle. That is, $E$ is the midpoint of $AC$. Thus, the $x$-coordinate of $E$ is the average of the $x$-coordinates of $A$ and $C$, or $\frac{0+6}{2} = 3$.
The $y$-coordinate of $E$ is the average of the $y$-coordinates of $A$ and $C$, or $\frac{0+12}{2} = 6$, and so the coordinates of $E$ are $(3, 6)$.
Consider base $AD = 6$ of $\triangle ADE$, then its height is equal to the distance from $E$ to the $x$-axis, which is 6.
The area of $\triangle ADE$ is $\frac{1}{2}(6)(6) = 18$.

*Solution 2*

The diagonals of a rectangle divide the rectangle into 4 non-overlapping triangles having equal area. (You should consider why this is true before reading on.)
Thus, the area of $\triangle ADE$ is equal to $\frac{1}{4}$ of the area of rectangle $ABCD$ or $\frac{1}{4}(6)(12) = 18$.

(b) *Solution 1*




We begin by drawing and labelling a diagram, as shown.
The area of rectangle $ABCD$ is equal to the area of trapezoid $BCDP$ plus the area of $\triangle PAD$.
Since the area of trapezoid $BCDP$ is twice the area of $\triangle PAD$, then the area of $\triangle PAD$ is $\frac{1}{3}$ the area of $ABCD$ (and the area of trapezoid $BCDP$ is $\frac{2}{3}$ the area of $ABCD$).
The area of rectangle $ABCD$ is $6 \times 12 = 72$, and so the area of $\triangle PAD$ is $\frac{1}{3} \times 72 = 24$.
The area of $\triangle PAD$ is $\frac{1}{2}(AD)(AP) = \frac{1}{2}(6)(p) = 3p$, and so $3p = 24$ or $p = 8$.

*Solution 2*

Point $P$ has coordinates $(0, p)$ and so $AP = p$ and $BP = 12 - p$.
The area of $\triangle PAD$ is $\frac{1}{2}(AD)(AP) = \frac{1}{2}(6)(p) = 3p$.
The area of trapezoid $BCDP$ is $\frac{1}{2}(BC)(BP + CD) = \frac{1}{2}(6)(12 - p + 12) = 3(24 - p)$.

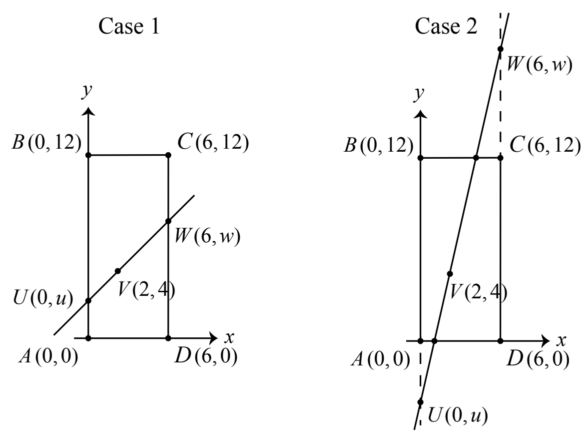
The area of trapezoid $BCDP$ is twice the area of $\triangle PAD$, and so $3(24-p) = 2(3p)$ or $24 - p = 2p$, and so $3p = 24$ or $p = 8$.

(c) The area of rectangle $ABCD$ is $6 \times 12 = 72$.
The sum of the areas of the two trapezoids is equal to the area of rectangle $ABCD$.
Since the ratio of the areas of these two trapezoids is $5 : 3$, then the areas of the two trapezoids are $\frac{5}{8} \times 72 = 45$ and $\frac{3}{8} \times 72 = 27$.
(We may check that $45 : 27 = 5 : 3$ and $45 + 27 = 72$.)

Let $\ell$ be the line that passes through $U$, $V$ and $W$.
Begin by assuming $\ell$ does not pass through a vertex of $ABCD$. In this case, $\ell$ either intersects opposite sides of $ABCD$, or it intersects adjacent sides of $ABCD$.
If $\ell$ intersects opposite sides of $ABCD$, then $\ell$ divides $ABCD$ into two trapezoids, as required.
If $\ell$ intersects adjacent sides of $ABCD$, then $\ell$ divides $ABCD$ into a triangle and a pentagon. This is not possible.

Assume $\ell$ passes through at least one vertex of $ABCD$.
In this case, $\ell$ divides $ABCD$ into two figures, at least one of which is a triangle. This is not also possible.
Thus, $\ell$ intersects opposite sides of $ABCD$ and does not pass through $A$, $B$, $C$, or $D$.

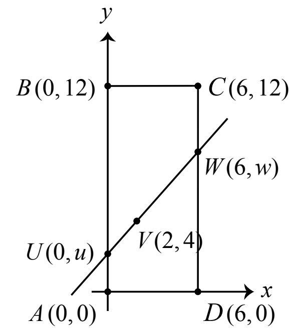
That is, line $\ell$ can intersect opposite sides of $ABCD$ in the two different ways shown below.




In each case, since $\ell$ is a straight line passing through $U$, $V$ and $W$, then the slope of $UV$ is equal to the slope of $VW$.
That is,

$$\begin{aligned} \frac{4-u}{2-0} &= \frac{w-4}{6-2} \\ 4(4-u) &= 2(w-4) \\ 2(4-u) &= w-4 \\ 8-2u &= w-4 \\ w &= 12-2u \end{aligned}$$

Case 1: Line $\ell$ intersects sides $AB$ and $CD$.
That is, $U$ lies between $A$ and $B$, and $W$ lies between $C$ and $D$.




In this case, $0 < u < 12$, $0 < w < 12$, $AU = u$, and $DW = w$.
The area of trapezoid $ADWU$ is

$$\tfrac{1}{2}(AD)(DW + AU) = \tfrac{1}{2}(6)(w + u) = 3(w + u).$$

Since $w = 12 - 2u$, the area of trapezoid $ADWU$ becomes $3(12 - u)$.
We consider each of two possibilities: the area of trapezoid $ADWU$ is equal to 27, or the area is equal to 45.

If the area of trapezoid $ADWU$ is equal to 27, then

$$\begin{aligned} 3(12 - u) &= 27 \\ 12 - u &= 9 \\ u &= 3 \end{aligned}$$

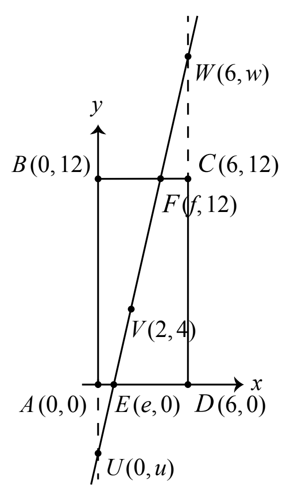
Substituting $u = 3$ into $w = 12 - 2u$, we get $w = 12 - 6 = 6$.
The Case 1 conditions that $0 < u < 12$ and $0 < w < 12$ are satisfied and thus the ratio of the areas of the two trapezoids is $5 : 3$ for the pair of points $U(0, 3)$ and $W(6, 6)$.

If the area of trapezoid $ADWU$ is equal to 45, then

$$\begin{aligned} 3(12 - u) &= 45 \\ 12 - u &= 15 \\ u &= -3 \end{aligned}$$

Here, the condition that $0 < u < 12$ is not satisfied and so there is no pair of points $U$ and $W$ for which the ratio of the areas of the two trapezoids is $5 : 3$.

Case 2: Line $\ell$ intersects sides $AD$ and $BC$.
That is, $U$ lies on $AB$ extended, outside of side $AB$, and $W$ lies on $CD$ extended, outside of side $CD$.




We begin by drawing and labelling a diagram, including $E(e, 0)$ and $F(f, 12)$, the points where $\ell$ intersects sides $AD$ and $BC$ respectively, as shown.
In this case, $u < 0$ and $w > 12$ (as in the diagram shown), or $u > 12$ and $w < 0$ (when $U$ lies above $B$ and $W$ lies below $D$).
We note that what follows is true for each of these two cases, and thus we need not consider them separately.
In this case, we require that $0 < e < 6$, $0 < f < 6$, and so we get $BF = f$ and $AE = e$.
The area of trapezoid $BFEA$ is

$$\tfrac{1}{2}(AB)(BF + AE) = \tfrac{1}{2}(12)(f + e) = 6(f + e).$$

Further, since $\ell$ is a straight line passing through $E$, $V$ and $F$, then the slope of $EV$ is equal to the slope of $FV$.

That is,

$$\begin{aligned}
\frac{4-0}{2-e} &= \frac{12-4}{f-2} \\
\frac{4}{2-e} &= \frac{8}{f-2} \\
4(f-2) &= 8(2-e) \\
f-2 &= 2(2-e) \\
f &= 6-2e
\end{aligned}$$

Since $f = 6-2e$, the area of trapezoid $BFEA$ becomes $6(6-e)$.
We consider each of two possibilities: the area of trapezoid $BFEA$ is equal to 27, or the area is equal to 45.

If the area of trapezoid $BFEA$ is equal to 27, then

$$\begin{aligned}
6(6-e) &= 27 \\
6-e &= \tfrac{9}{2} \\
e &= \tfrac{3}{2}
\end{aligned}$$

Substituting $e = \frac{3}{2}$ into $f = 6-2e$, we get $f = 3$, and these values satisfy the Case 2 conditions $0 < e < 6$ and $0 < f < 6$.
Here, we get $E(\frac{3}{2}, 0)$ and $F(3, 12)$ and use these points to determine $U$ and $W$.
The slope of $FV$ is $\dfrac{12-4}{3-2} = 8$ and so the slope of $WV$ is also 8, which gives $\dfrac{w-4}{4} = 8$, and solving we get $w = 36$.
Similarly, the slope of $VU$ is also 8, which gives $\dfrac{4-u}{2} = 8$, and solving we get $u = -12$.
We note that $w = 36$ and $u = -12$ satisfy the conditions $w > 12$ and $u < 0$ and so the ratio of the areas of the two trapezoids is $5:3$ for the points $U(0, -12)$ and $W(6, 36)$.

If the area of trapezoid $BFEA$ is equal to 45, then

$$\begin{aligned}
6(6-e) &= 45 \\
6-e &= \tfrac{15}{2} \\
e &= -\tfrac{3}{2}
\end{aligned}$$

Here, the condition that $0 < e < 6$ is not satisfied and so there is no pair of points $E$ and $F$ and thus no pair of points $U$ and $W$ for which the ratio of the areas of the two trapezoids is $5:3$.

Thus, there are two pairs of points $U$ and $W$ for which the ratio of the areas of the two trapezoids is $5:3$. These are $U(0, 3)$, $W(6, 6)$, and $U(0, -12)$, $W(6, 36)$.

4. (a) When $x = 6$, $\dfrac{5}{x} + \dfrac{14}{y} = 2$ becomes $\dfrac{5}{6} + \dfrac{14}{y} = 2$ and so $\dfrac{14}{y} = 2 - \dfrac{5}{6}$ or $\dfrac{14}{y} = \dfrac{7}{6}$, which gives $y = 12$.

(b) *Solution 1*

Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

$$\begin{aligned} \frac{4}{x} + \frac{5}{y} &= 1 \\ \frac{4}{x}(xy) + \frac{5}{y}(xy) &= 1(xy) \qquad \text{(since } xy \neq 0\text{)} \\ 4y + 5x &= xy \\ xy - 5x - 4y &= 0 \\ x(y-5) - 4y &= 0 \\ x(y-5) - 4y + 20 &= 20 \\ x(y-5) - 4(y-5) &= 20 \\ (x-4)(y-5) &= 20 \end{aligned}$$

Since $x$ and $y$ are positive integers, then $x - 4$ and $y - 5$ are integers and thus are a factor pair of 20.
Since $y > 0$, then $y - 5 > -5$.
The factors of 20 which are greater than $-5$ are: $-4, -2, -1, 1, 2, 4, 5, 10$, and 20.
If $y - 5$ is equal to $-4$, then $x - 4 = -5$ (since $(-5)(-4) = 20$), and so $x = -1$.
This is not possible since $x$ is a positive integer.
Similarly, $y - 5$ cannot equal $-2$ or $-1$ (since each gives $x < 0$), and so $y - 5$ is a positive factor of 20.
In the table below, we determine the values of $x$ and $y$ corresponding to each of the positive factor pairs of 20.

| Factor Pair | $x-4$ | $y-5$ | $x$ | $y$ |
|---|---|---|---|---|
| 1 and 20 | 1 | 20 | 5 | 25 |
| 20 and 1 | 20 | 1 | 24 | 6 |
| 2 and 10 | 2 | 10 | 6 | 15 |
| 10 and 2 | 10 | 2 | 14 | 7 |
| 4 and 5 | 4 | 5 | 8 | 10 |
| 5 and 4 | 5 | 4 | 9 | 9 |

Thus, the ordered pairs of positive integers $(x, y)$ that are solutions to the given equation are $(5, 25)$, $(24, 6)$, $(6, 15)$, $(14, 7)$, $(8, 10)$, and $(9, 9)$.

*Solution 2*

Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

$$\begin{aligned}
\frac{4}{x} + \frac{5}{y} &= 1 \\
\frac{4}{x}(xy) + \frac{5}{y}(xy) &= 1(xy) \qquad \text{(since } xy \neq 0\text{)} \\
4y + 5x &= xy \\
xy - 5x &= 4y \\
x(y-5) &= 4y \\
x &= \frac{4y}{y-5} \quad (y \neq 5) \\
x &= \frac{4y - 20 + 20}{y-5} \\
x &= \frac{4(y-5) + 20}{y-5} \\
x &= 4 + \frac{20}{y-5}
\end{aligned}$$

Since $x$ and $y$ are positive integers, then $y - 5$ is a divisor of 20.
Since $y > 0$, then $y - 5 > -5$.
The divisors of 20 which are greater than $-5$ are: $-4, -2, -1, 1, 2, 4, 5, 10$, and 20.
If $y - 5$ is equal to $-4$, then $x = 4 + \frac{20}{-4} = -1$, which is not possible since $x$ is a positive integer.
Similarly, $y - 5$ cannot equal $-2$ or $-1$ (since each gives $x < 0$), and so $y - 5$ is a positive divisor of 20.
In the table below, we determine the values of $y$ and $x$ corresponding to each of the positive divisors of 20.

| $y-5$ | 1 | 2 | 4 | 5 | 10 | 20 |
|---|---|---|---|---|---|---|
| $y$ | 6 | 7 | 9 | 10 | 15 | 25 |
| $x$ | 24 | 14 | 9 | 8 | 6 | 5 |

Thus, the ordered pairs of positive integers $(x, y)$ that are solutions to the given equation are $(24, 6), (14, 7), (9, 9), (8, 10), (6, 15)$, and $(5, 25)$.

(c) *Solution 1*

Since $x \geq 1$ and $y \geq 1$, then $\dfrac{16}{x} + \dfrac{25}{y} \leq 16 + 25 = 41$, and so $5 \leq p \leq 41$. That is, the possible prime numbers $p$ come from the list $5, 7, 11, 13, 17, 19, 23, 29, 31, 37$, and 41.

Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

$$\begin{aligned}
\frac{16}{x} + \frac{25}{y} &= p \\
\frac{16}{x}(xy) + \frac{25}{y}(xy) &= p(xy) \qquad \text{(since } xy \neq 0\text{)} \\
16y + 25x &= pxy \\
pxy - 25x - 16y &= 0 \\
p^2xy - 25px - 16py &= 0 \qquad \text{(since } p > 0\text{)} \\
px(py - 25) - 16py &= 0 \\
px(py - 25) - 16py + 400 &= 400 \\
px(py - 25) - 16(py - 25) &= 400 \\
(px - 16)(py - 25) &= 400
\end{aligned}$$

Since $p$, $x$ and $y$ are positive integers, then $px - 16$ and $py - 25$ are integers and thus are a factor pair of 400.

Since $p \geq 5$ and $x \geq 1$, then $px \geq 5$, and so $px - 16 \geq 5 - 16$ or $px - 16 \geq -11$.

The factors of 400 which are greater than or equal to $-11$, and are less than 0, are: $-1, -2, -4, -5, -8$, and $-10$.

If $px - 16 = -1$, then $py - 25 = -400$.

In this case, we get $py = -375$ which is not possible since both $p$ and $y$ are positive.

We can similarly show that $px - 16$ cannot equal $-2, -4, -5, -8$, and $-10$ (since each gives $py < 0$) and so $px - 16$ is a positive factor of 400 and thus $py - 25$ is also.

In the table below, we determine possible values of $p$ corresponding to each of the positive factor pairs of 400.

Recall from earlier that we only need to consider possible values of $p$ for which $5 \leq p \leq 41$.

| $px - 16$ | $py - 25$ | $px$ | $py$ | New common prime factor of the integers $px$ and $py$ |
|---|---|---|---|---|
| 1 | 400 | 17 | $425 = 17 \times 25$ | 17 |
| 2 | 200 | 18 | 225 | |
| 4 | 100 | $20 = 5 \times 4$ | $125 = 5 \times 25$ | 5 |
| 5 | 80 | $21 = 7 \times 3$ | $105 = 7 \times 15$ | 7 |
| 8 | 50 | 24 | 75 | |
| 10 | 40 | $26 = 13 \times 2$ | $65 = 13 \times 5$ | 13 |
| 16 | 25 | 32 | 50 | |
| 20 | 20 | 36 | 45 | |
| 25 | 16 | 41 | 41 | 41 |
| 40 | 10 | 56 | 35 | |
| 50 | 8 | $66 = 11 \times 6$ | $33 = 11 \times 3$ | 11 |
| 80 | 5 | 96 | 30 | |
| 100 | 4 | $116 = 29 \times 4$ | 29 | 29 |
| 200 | 2 | 216 | 27 | |
| 400 | 1 | 416 | 26 | |

The values of $p$ for which there is at least one ordered pair of positive integers $(x, y)$ that is a solution to the given equation are $5, 7, 11, 13, 17, 29$, and $41$.
We may check, for example, that when $(x, y) = (6, 3)$ we get,

$$\frac{16}{x} + \frac{25}{y} = \frac{16}{6} + \frac{25}{3} = \frac{16}{6} + \frac{50}{6} = \frac{66}{6} = 11$$

as given in the table above.

*Solution 2*
Since $x \geq 1$ and $y \geq 1$, then $\frac{16}{x} + \frac{25}{y} \leq 16 + 25 = 41$, and so $5 \leq p \leq 41$. That is, the possible prime numbers $p$ come from the list $5, 7, 11, 13, 17, 19, 23, 29, 31, 37$, and $41$.
When $x$ is a positive divisor of 16, $\frac{16}{x}$ is a positive integer.
Specifically, when $x = 1, 2, 4, 8, 16$, the values of $\frac{16}{x}$ are $16, 8, 4, 2, 1$, respectively.
Similarly, when $y$ is a positive divisor of 25, $\frac{25}{y}$ is a positive integer.
Specifically, when $y = 1, 5, 25$, the values of $\frac{25}{y}$ are $25, 5, 1$, respectively.
We may use this observation to determine some values of $p$ for which there is at least one ordered pair of positive integers $(x, y)$ that is a solution to the equation.
We summarize these solutions in the table below.

| $p$ | $x$ | $y$ | $\frac{16}{x} + \frac{25}{y}$ |
|---|---|---|---|
| 5 | 4 | 25 | $\frac{16}{4} + \frac{25}{25} = 4 + 1$ |
| 7 | 8 | 5 | $\frac{16}{8} + \frac{25}{5} = 2 + 5$ |
| 13 | 2 | 5 | $\frac{16}{2} + \frac{25}{5} = 8 + 5$ |
| 17 | 1 | 25 | $\frac{16}{1} + \frac{25}{25} = 16 + 1$ |
| 29 | 4 | 1 | $\frac{16}{4} + \frac{25}{1} = 4 + 25$ |
| 41 | 1 | 1 | $\frac{16}{1} + \frac{25}{1} = 16 + 25$ |

From our previous list of possible values of $p$, we have only 11, 19, 23, 31, and 37 remaining to consider.

Since $x$ and $y$ are positive integers, we obtain the following equivalent equations,

$$\begin{aligned} \frac{16}{x}+\frac{25}{y} &= p \\ \frac{16}{x}(xy)+\frac{25}{y}(xy) &= p(xy) \qquad \text{(since } xy \neq 0\text{)} \\ 16y+25x &= pxy \\ pxy-25x &= 16y \\ x(py-25) &= 16y \\ x &= \frac{16y}{py-25} \qquad (p \geq 11 \text{ and so no multiple of } p \text{ can equal } 25) \end{aligned}$$

Since $x > 0$ and $16y > 0$ and $x = \dfrac{16y}{py-25}$, then $py - 25 > 0$ and so $py > 25$.

Further, $x$ is an integer and so $x \geq 1$, which gives $\dfrac{16y}{py-25} \geq 1$.

Simplifying, we get $16y \geq py - 25$ or $py - 16y \leq 25$, and so $y \leq \dfrac{25}{p-16}$ when $p > 16$.

We may use this inequality to determine restrictions on $y$ given each of the remaining possible values of $p$ which are greater than 16, namely $37, 31, 23$, and $19$.

For example if $p = 37$, then $y \leq \dfrac{25}{37-16}$ or $y \leq \dfrac{25}{21}$, and so $y = 1$. However, when $p = 37$ and $y = 1$, we get $x = \dfrac{16(1)}{37(1)-25} = \dfrac{16}{12}$ which is not an integer, and thus $p \neq 37$.

We summarize similar work for $p = 31, 23, 19$ in the table below noting that when $y = 1$ and $p = 23$ or $p = 19$ we get $py < 25$ (earlier we showed $py > 25$), and thus we need not consider these two cases.

| $p$ | $y \leq \frac{25}{p-16}$ | Possible integer values of $y$ | Corresponding values of $x = \frac{16y}{py-25}$ |
|---|---|---|---|
| 31 | $y \leq \frac{25}{31-16} = \frac{25}{15}$ | $y = 1$ | $x = \frac{16}{6}$ |
| 23 | $y \leq \frac{25}{23-16} = \frac{25}{7}$ | $y = 2, 3$ | $x = \frac{32}{21}, \frac{48}{44}$ |
| 19 | $y \leq \frac{25}{19-16} = \frac{25}{3}$ | $y = 2, 3, 4, 5, 6, 7, 8$ | $x = \frac{32}{13}, \frac{48}{32}, \frac{64}{51}, \frac{80}{70}, \frac{96}{89}, \frac{112}{108}, \frac{128}{127}$ |

Since there are no integer values of $x$, then $p \neq 19, 23, 31, 37$.

The final remaining value to check is $p = 11$.

As noted earlier, $py > 25$ and so when $p = 11$, we get $y > \frac{25}{11}$ or $y \geq 3$ (since $y$ is an integer).

Trying $y = 3$, we get $x = \dfrac{16(3)}{11(3)-25} = \dfrac{48}{8} = 6$ and so when $p = 11$, $(x, y) = (6, 3)$ is a solution to the equation.

Summarizing, the values of $p$ for which there is at least one ordered pair of positive integers $(x, y)$ that is a solution to the equation are $5, 7, 11, 13, 17, 29$, and $41$.

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# *2020 Galois Contest*

**Wednesday, April 15, 2020**
**(in North America and South America)**

**Thursday, April 16, 2020**
**(outside of North America and South America)**

*Solutions*

1. (a) The number of letters in each row after the first is twice the number of letters in the previous row.
Since Row 4 has 8 letters, then Row 5 has $2 \times 8 = 16$ letters, and Row 6 has $2 \times 16 = 32$ letters.

Alternatively, we can continue the pattern to Row 6 as shown.

Row 1 $A$
Row 2 $BB$
Row 3 $AAAA$
Row 4 $BBBBBBBB$
Row 5 $AAAAAAAAAAAAAAAA$
Row 6 $BBBBBBBBBBBBBBBBBBBBBBBBBBBBBBBB$

(b) If the pattern consists of 6 rows, the total number of letters is $1+2+4+8+16+32 = 63$.

(c) *Solution 1*
If the total number of letters in the pattern is 63, then there are 6 rows in the pattern (as we saw in part (b)).
Counting, we get that there are $1 + 4 + 16 = 21$ $A$'s, and $2 + 8 + 32 = 42$ $B$'s.

*Solution 2*
Notice that in Row 2 there are twice as many $B$'s as there are $A$'s in Row 1, and in Row 4 there are twice as many $B$'s as there are $A$'s in Row 3.
Further, the rows alternate between $A$'s and $B$'s and the number of letters in each row is twice the number of letters in the previous row, and so this pattern continues.
Thus, if there are an even number of rows in the pattern, then the total number of $B$'s in the pattern is twice the total number of $A$'s, and so in this case $\frac{1}{3}$ of the letters in the pattern are $A$'s and $\frac{2}{3}$ of the letters are $B$'s.
If the total number of letters in the pattern is 63, then there are 6 rows in the pattern (as we saw in part (b)), and so the number of $A$'s in the pattern is $\frac{1}{3} \times 63 = 21$ and the number of $B$'s is $\frac{2}{3} \times 63 = 2 \times 21 = 42$.

(d) *Solution 1*
We begin by determining the number of rows in the pattern given that the total number of letters is 4095.
We may do this by counting the number of $A$'s and $B$'s in each row and keeping a running total of the number of letters in the pattern after each complete row.

| Row Number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of $A$'s | 1 | 0 | 4 | 0 | 16 | 0 | 64 | 0 | 256 | 0 | 1024 | 0 |
| Number of $B$'s | 0 | 2 | 0 | 8 | 0 | 32 | 0 | 128 | 0 | 512 | 0 | 2048 |
| Number of Letters | 1 | 3 | 7 | 15 | 31 | 63 | 127 | 255 | 511 | 1023 | 2047 | 4095 |

If the pattern has 12 complete rows, there are a total of 4095 letters, of which $1 + 4 + 16 + 64 + 256 + 1024 = 1365$ are $A$'s and $2 + 8 + 32 + 128 + 512 + 2048 = 2730$ are $B$'s.
Thus, if there are 4095 letters in the pattern, the difference between the number of $A$'s and the number of $B$'s is $2730 - 1365 = 1365$.

*Solution 2*
We begin by determining the number of rows in the pattern given that the total number of letters is 4095.
Since $4095 = 1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + 2048$ and the sum

on the right side of this equation has 12 terms, then a pattern with 4095 letters contains exactly 12 complete rows.
Since 12 is an even number of rows, we may use the result from Solution 2 in part (c) to determine that the pattern has $\frac{1}{3} \times 4095 = 1365$ $A$'s and $\frac{2}{3} \times 4095 = 2 \times 1365 = 2730$ $B$'s.
Thus, if there are 4095 letters in the pattern, the difference between the number of $A$'s and the number of $B$'s is $2730 - 1365 = 1365$.
(Alternatively, we may have concluded that if $\frac{2}{3}$ of the letters are $B$'s and $\frac{1}{3}$ are $A$'s, then the difference between the number of $A$'s and $B$'s is $\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ of the total number of letters, or $\frac{1}{3} \times 4095 = 1365$.)

2. (a) The surface area of a rectangular prism is given by the formula $A = 2\ell w + 2\ell h + 2wh$.
Thus, the rectangular prism with length 2 cm, width 5 cm, and height 9 cm has surface area $2(2)(5) + 2(2)(9) + 2(5)(9) = 20 + 36 + 90 = 146$ cm$^2$.

(b) The volume of a rectangular prism is given by the formula $V = \ell wh$.
If the rectangular prism has a square base, then $\ell = w$ and so $V = \ell^2 h$.
Substituting $V = 160$ cm$^3$ and $h = 10$ cm, we get $160 = \ell^2(10)$ or $\ell^2 = 16$, and so $\ell = 4$ cm (since $\ell > 0$).
Therefore, the side length of the square base of a rectangular prism with height 10 cm and volume 160 cm$^3$ is 4 cm.

(c) If a rectangular prism has a square base, then $\ell = w$.
Since the area of the base is 36 cm$^2$, then $36 = \ell \cdot w = \ell^2$, and so $\ell = w = \sqrt{36} = 6$ cm (since $\ell > 0$).
If the surface area of this prism is 240 cm$^2$, then substituting, we get $240 = 2(6)(6) + 2(6)h + 2(6)h$ or $240 = 72 + 24h$, and so $h = \dfrac{240 - 72}{24} = 7$ cm.
Thus, the volume of the prism is $\ell wh = (6)(6)(7) = 252$ cm$^3$.

(d) Substituting into the formula for volume, we get $x = k(2k)(3k)$ or $x = 6k^3$.
Substituting into the formula for surface area, we get $x = 2(k)(2k) + 2(k)(3k) + 2(2k)(3k)$ or $x = 4k^2 + 6k^2 + 12k^2 = 22k^2$.
Equating the two expressions that are each equal to $x$ and solving, we get

$$\begin{aligned} 6k^3 &= 22k^2 \\ 6k^3 - 22k^2 &= 0 \\ 2k^2(3k - 11) &= 0 \end{aligned}$$

Since $k > 0$, then $3k - 11 = 0$ and so $k = \dfrac{11}{3}$.

3. (a) The original product is $2 \times 8 = 16$.
If we add 1 to each of 2 and 8, we get 3 and 9 and a new product of $3 \times 9 = 27$.
The ones digit of this new product is 1 more than the ones digit of the original product $(7 - 6 = 1)$.
The tens digit of this new product is 1 more than the tens digit of the original product $(2 - 1 = 1)$.
Since the positive integer $d = 1$ satisfies the required conditions, $(2, 8)$ is a RadPair.

(b) The original product is $3 \times 6 = 18$.
If we add 1 to each of 3 and 6, we get 4 and 7 and a new product of $4 \times 7 = 28$.
The ones digit of this new product is not 1 more than the ones digit of the original product $(8 - 8 \neq 1)$, and thus $d = 1$ does not satisfy the required conditions for $(3, 6)$ to

be a RadPair.
If we add 2 to each of 3 and 6, we get 5 and 8 and a new product of $5 \times 8 = 40$.
The ones digit of this new product is not 2 more than the ones digit of the original product ($0 - 8 \neq 2$), and thus $d = 2$ does not satisfy the required conditions for $(3, 6)$ to be a RadPair.
Adding 3 to each of 3 and 6, we get 6 and 9 and a new product of $6 \times 9 = 54$.
The ones digit of this new product is not 3 more than the ones digit of the original product ($4 - 8 \neq 3$), and thus $d = 3$ does not satisfy the required conditions for $(3, 6)$ to be a RadPair.
Adding 4 to each of 3 and 6, we get 7 and 10 and a new product of $7 \times 10 = 70$.
The ones digit of this new product is not 4 more than the ones digit of the original product ($0 - 8 \neq 4$), and thus $d = 4$ does not satisfy the required conditions for $(3, 6)$ to be a RadPair.
Adding 5 to each of 3 and 6, we get 8 and 11 and a new product of $8 \times 11 = 88$.
The ones digit of this new product is not 5 more than the ones digit of the original product ($8 - 8 \neq 5$), and thus $d = 5$ does not satisfy the required conditions for $(3, 6)$ to be a RadPair.
Adding 6 to each of 3 and 6, we get 9 and 12 and a new product of $9 \times 12 = 108$.
Since this new product is not a two-digit integer, then $d = 6$ does not satisfy the required conditions for $(3, 6)$ to be a RadPair.
All integer values of $d > 6$ will give products that are greater than 108, and thus $(3, 6)$ is not a RadPair.

(c) *Solution 1*
In part (b) we showed that $(3, 6)$ was not a RadPair.
If $x = 1$, then the product $1 \times 6 = 6$ is not a two-digit integer and so $(1, 6)$ is also not a RadPair.
For each of the remaining values, $x = 2, 4, 5, 6$, we try different values for $d$ (as was done in part (b)), to determine if $(x, 6)$ is a RadPair.
We summarize this work in the table that follows.

| $(x, 6)$ | Original Product $6x$ | $d$ | New Product | RadPair? |
|---|---|---|---|---|
| $(2, 6)$ | 12 | 3 | $(2 + 3) \times (6 + 3) = 45$ | Yes, since $4 - 1 = 5 - 2 = 3$ |
| $(4, 6)$ | 24 | 1 | $(4 + 1) \times (6 + 1) = 35$ | Yes, since $3 - 2 = 5 - 4 = 1$ |
| $(5, 6)$ | 30 | | | No $d$ satisfies the conditions. |
| $(6, 6)$ | 36 | | | No $d$ satisfies the conditions. |

Therefore, the positive integers $x \leq 6$ for which $(x, 6)$ is a RadPair are $x = 2$ and $x = 4$.

*Solution 2*
If $(x, 6)$ is a RadPair, then the original product $6x$ is a two-digit integer and thus can be written as $10m + n$ for some integers $1 \leq m \leq 9$ and $0 \leq n \leq 9$.
That is, $6x = 10m + n$ and so $m$ is the tens digit of the original product $6x$ and $n$ is the ones digit.
Further, if $(x, 6)$ is a RadPair, then there exists a positive integer $d$ such that

- the product $(x + d)(6 + d)$ is a two-digit integer, and
- the ones digit of the product $(x + d)(6 + d)$ equals $d$ plus the ones digit, $n$, of the product $6x$, and
- the tens digit of the product $(x + d)(6 + d)$ equals $d$ plus the tens digit, $m$, of the

product $6x$.

The second bullet gives that the ones digit of the product $(x+d)(6+d)$ equals $n+d$.
The third bullet gives that the tens digit of the product $(x+d)(6+d)$ equals $m+d$.
Since the product $(x+d)(6+d)$ has tens digit $m+d$ and ones digit $n+d$, we have $(x+d)(6+d) = 10(m+d)+(n+d)$.
Expanding each side of this equation and using the earlier fact that $6x = 10m+n$, we get

$$\begin{aligned}
(x+d)(6+d) &= 10(m+d)+(n+d) \\
6x+xd+6d+d^2 &= 10m+10d+n+d \\
6x+xd+6d+d^2 &= 10m+n+11d \\
6x+xd+6d+d^2 &= 6x+11d \quad (\text{since } 6x = 10m+n) \\
xd+6d+d^2 &= 11d \\
xd+d^2 &= 5d
\end{aligned}$$

Since $d>0$, we may divide this final equation by $d$ to give $x+d=5$.
If $(x,6)$ is a RadPair, then $x+d=5$.
That is, the condition $x+d=5$ is necessary for $(x,6)$ to be a RadPair, but it does not guarantee that $(x,6)$ is a RadPair.
We saw an example of this in part (b) when we showed that $(3,6)$ was not a RadPair.
If $x=3$ and $x+d=5$, then $d=2$.
However $3\times 6 = 18$ and $(3+2)\times(6+2) = 40$ and since $4-1 \neq 2$, then the condition $x+d=5$ ($x=3$ and $d=2$) did not produce a RadPair.
As we saw in Solution 1, $(1,6)$ is not a RadPair since $1\times 6 = 6$ is not a two-digit integer, and thus we are left to check $x = 2,4,5,6$.
If $x=2$, then $d=5-2=3$ and so $(2,6)$ may be a RadPair with $d=3$.
If $x=4$, then $d=5-4=1$ and so $(4,6)$ may be a RadPair with $d=1$.
If $x=5$, then $d=5-5=0$, but $d>0$ and so $(5,6)$ is not a RadPair.
If $x=6$, then $d=5-6=-1$, but $d>0$ and so $(6,6)$ is not a RadPair.
We can verify that each of $(2,6)$ and $(4,6)$ is indeed a RadPair (see Solution 1).
The positive integers $x\leq 6$ for which $(x,6)$ is a RadPair are $x=2$ and $x=4$.

(d) As in part (c) Solution 2, we will proceed algebraically to determine a condition that is required for $(a,b)$ to be a RadPair, but that does not guarantee that $(a,b)$ is a RadPair. To then obtain an accurate count of the number of RadPairs, we will need to show that a specific value for $d$ satisfies each of the given requirements.

If $(a,b)$ with $a\leq b$ is a RadPair, then the original product $ab$ is a two-digit integer and thus can be written as $10h+k$ for some integers $1\leq h\leq 9$ and $0\leq k\leq 9$.
That is, $ab = 10h+k$ and so $h$ is the tens digit of the original product $ab$ and $k$ is the ones digit.
Further, if $(a,b)$ is a RadPair, then there exists a positive integer $d$ such that

- the product $(a+d)(b+d)$ is a two-digit integer, and
- the ones digit of the product $(a+d)(b+d)$ equals $d$ plus the ones digit, $k$, of the product $ab$, and
- the tens digit of the product $(a+d)(b+d)$ equals $d$ plus the tens digit, $h$, of the product $ab$.

The second bullet gives that the ones digit of the product $(a+d)(b+d)$ equals $k+d$.
The third bullet gives that the tens digit of the product $(a+d)(b+d)$ equals $h+d$.

Therefore, the product $(a+d)(b+d)$ has tens digit $h+d$ and ones digit $k+d$, and thus $(a+d)(b+d) = 10(h+d) + (k+d)$.
Expanding each side of this equation and using the earlier fact that $ab = 10h + k$, we get

$$\begin{aligned} (a+d)(b+d) &= 10(h+d) + (k+d) \\ ab + ad + bd + d^2 &= 10h + 10d + k + d \\ ab + ad + bd + d^2 &= 10h + k + 11d \\ ab + ad + bd + d^2 &= ab + 11d \\ ad + bd + d^2 &= 11d \end{aligned}$$

Since $d > 0$, we may divide this final equation by $d$ to give $a + b + d = 11$.
If $(a, b)$ is a RadPair, then $a + b + d = 11$.
Again, this condition $a + b + d = 11$ is necessary for $(a, b)$ to be a RadPair, but it does not guarantee that $(a, b)$ is a RadPair.
When $a = 1$, there is no value for the digit $b$ such that $ab$ is a two-digit integer.
Thus, there are no RadPairs $(a, b)$ with $a = 1$.
When $a \geq 6$, then $a + b > 11$ (since $a \leq b$).
If $a + b > 11$ and $a + b + d = 11$, then $d < 0$ which is not possible.
Thus, there are no RadPairs $(a, b)$ with $a \geq 6$.
In the table below, we use the condition $a + b + d = 11$ and the definition of a RadPair to determine all RadPairs $(a, b)$ with $a \leq b$.
We may exclude checking some pairs by recalling that:

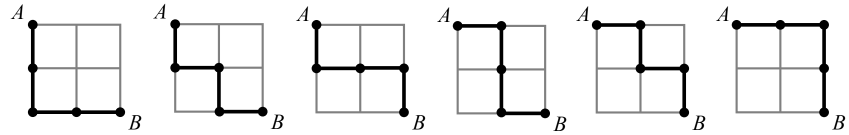
- $ab \geq 10$ and so we may omit pairs such as $(1, 8)$ and $(2, 4)$
- $a + b < 11$ and so we may omit pairs such as $(2, 9)$ and $(3, 9)$
- we have previously shown that $(3, 6)$ is not a RadPair

| $(a, b)$ | $d$ | $ab$ | $(a+d)(b+d)$ | Check |
|---|---|---|---|---|
| $(2,5)$ | 4 | 10 | $(2+4)\times(5+4) = 54$ | $5-1 = 4-0 = 4$ |
| $(2,6)$ | 3 | 12 | $(2+3)\times(6+3) = 45$ | $4-1 = 5-2 = 3$ |
| $(2,7)$ | 2 | 14 | $(2+2)\times(7+2) = 36$ | $3-1 = 6-4 = 2$ |
| $(2,8)$ | 1 | 16 | $(2+1)\times(8+1) = 27$ | $2-1 = 7-6 = 1$ |
| $(3,4)$ | 4 | 12 | $(3+4)\times(4+4) = 56$ | $5-1 = 6-2 = 4$ |
| $(3,5)$ | 3 | 15 | $(3+3)\times(5+3) = 48$ | $4-1 = 8-5 = 3$ |
| $(3,7)$ | 1 | 21 | $(3+1)\times(7+1) = 32$ | $3-2 = 2-1 = 1$ |
| $(4,4)$ | 3 | 16 | $(4+3)\times(4+3) = 49$ | $4-1 = 9-6 = 3$ |
| $(4,5)$ | 2 | 20 | $(4+2)\times(5+2) = 42$ | $4-2 = 2-0 = 2$ |
| $(4,6)$ | 1 | 24 | $(4+1)\times(6+1) = 35$ | $3-2 = 5-4 = 1$ |
| $(5,5)$ | 1 | 25 | $(5+1)\times(5+1) = 36$ | $3-2 = 6-5 = 1$ |

These are the only possibilities satisfying the given requirements, and so there are exactly 11 RadPairs.

4. Throughout the solution to this question, we will represent a path as a sequence of moves, down ($D$), up ($U$), right ($R$), left ($L$), between adjacent vertices, begining at $A$, ending at $B$.

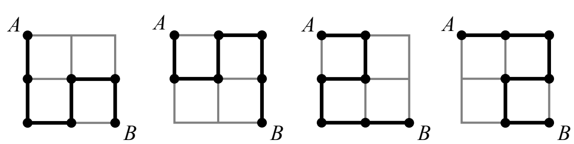
   (a) Every path in a $2 \times 2$ grid contains a minimum of 2 moves down and 2 moves right (since $B$ is 2 edges down and 2 edges right from $A$).
   Thus, every path can be represented by a sequence containing at least 2 $D$'s and 2 $R$'s, and has a length of at least 4 moves.
   We begin by determining the number of paths in a $2 \times 2$ grid whose length is exactly 4.



There are 6 such paths, as shown, and these paths can be represented by the sequences of moves $DDRR$, $DRDR$, $DRRD$, $RDDR$, $RDRD$, $RRDD$, respectively.
We note that these are the only possible arrangements of exactly 2 $D$'s and 2 $R$'s.

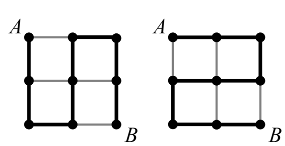
Can a path in a $2 \times 2$ grid have length 5?
If in addition to the required 2 moves down and 2 moves right, a path contained a move up, then this move would need to be "undone" by a move in the opposite direction, down.
That is, the net result of every path must be 2 $D$'s and 2 $R$'s, and so each $U$ must be paired with an additional $D$. Similarly, each $L$ must be paired with an additional $R$.
To summarize, every path in a $2 \times 2$ grid must contain 2 $D$'s, 2 $R$'s, and any additional moves must occur in pairs of opposite moves, $U/D$ or $L/R$.
Thus, every path in a $2 \times 2$ grid contains 4 moves (2 $D$'s, 2 $R$'s) and possibly additional pairs of moves, and thus has an even length.
A path of length 5 is not possible.

Next, we determine the number of paths whose length is 6.
Each of these paths contains 2 $D$'s, 2 $R$'s, and either 1 $U/D$ pair or 1 $L/R$ pair.
How many arrangements are there of 3 $R$'s, 1 $L$ and 2 $D$'s which give a path from $A$ to $B$?
To begin, consider the number of arrangements of the 3 $R$'s and the 1 $L$ (the horizontal moves).
The $L$ cannot appear before the first $R$ and it cannot appear after the last $R$ since each of these sequences ($LRRR$ and $RRRL$) leaves the $2 \times 2$ grid.
Thus, there are 2 possible arrangements of these 4 letters, namely $RLRR$ and $RRLR$.
Next, we count the number of possible placements of the 2 $D$'s among each of the above sequences.
A path must pass through each vertex at most once, and so the $L$ cannot be followed by an $R$ and vice versa ($RL$ or $LR$ would mean "backtracking").
For this reason, 1 $D$ must be placed on each side of the $L$.
Thus, there are 2 possible arrangements of 3 $R$'s, 1 $L$ and 2 $D$'s which give a path from $A$ to $B$, namely $RDLDRR$ and $RRDLDR$.
A similar argument can be made for the arrangement of 3 $D$'s, 1 $U$ and 2 $R$'s, and so there are 4 paths of length 6 from $A$ to $B$.



These 4 paths, respectively represented by the sequences of moves $DDRURD$, $DRURDD$, $RDLDRR$, $RRDLDR$, are shown.

Finally, we determine the number of paths whose length is 8.



There are 2 such paths, as shown, and these paths can be represented by the sequences of

moves $DDRUURDD$, $RRDLLDRR$, respectively.
Can you justify why these are the only paths of length 8?

Is a path of length 10 or greater possible?
The first edge of a path touches two vertices, and each additional edge touches one new vertex.
Thus, a path of length 8 passes through $2+7=9$ vertices (as seen in the two grids above), and a path of length 10 (or greater) would pass through at least $2+9=11$ vertices.
However, a path must pass through each vertex at most once, and since a $2\times 2$ grid contains $3\times 3=9$ vertices, it is not possible to have a path of length 10 or greater.

In a $2\times 2$ grid, the number of paths of any length from $A$ to $B$ is $6+4+2=12$.

(b) Following the reasoning in part (a), a path from $A$ to $B$ in a $10\times 10$ grid contains a minimum of 10 moves down and 10 moves right (since $B$ is 10 edges down and 10 edges right from $A$).
Further, any additional moves must occur in opposite pairs ($U$ and $D$ or $L$ and $R$), and so every path in a $10\times 10$ grid contains 20 moves (10 $D$'s, 10 $R$'s) and possibly additional pairs of moves, and thus has an even length.

(c) *Solution 1*
In a $4\times 4$ grid, a path from $A$ to $B$ has at least 4 moves down (4 $D$'s) and at least 4 moves right (4 $R$'s).
Thus, each path of length 10 has these 8 moves and exactly one additional pair of opposite moves, either left paired with right or up paired with down.
That is, each path has exactly 4 vertical moves ($U/D$) and 6 horizontal moves ($L/R$), or it has exactly 4 horizontal moves and 6 vertical moves.
By symmetry, each of these two cases will give an equal number of paths from $A$ to $B$, and so we count the number of paths for one of the cases and double that number to determine the total.

Assume that a path from $A$ to $B$ contains 4 vertical moves and 6 horizontal moves.
Each of the 4 vertical moves must be down.
The 6 horizontal moves must be 5 right and 1 left (4 moves right and 1 left/right opposite pair).
Thus, each path may be given by a string of 10 letters (4 $D$'s, 5 $R$'s and 1 $L$), and we may determine the number of paths by counting the number of arrangements of these 10 letters that correspond to a path from $A$ to $B$.
To begin, consider the number of arrangements of the 5 $R$'s and the 1 $L$ (the horizontal moves).
Since $L$ cannot be the first move and it cannot be the last move, there are 4 possible arrangements of these 6 letters, namely $RLRRRR$, $RRLRRR$, $RRRLRR$, and $RRRRLR$.
Next, we count the number of possible placements of the 4 $D$'s within each of the above sequences.
A path must pass through each vertex at most once, and so the $L$ cannot be followed by an $R$ and vice versa ($RL$ or $LR$ would mean "backtracking").
For this reason, 1 $D$ must be placed on each side of the $L$ which gives the 4 sequences $RDLDRRRR$, $RRDLDRRR$, $RRRDLDRR$, and $RRRRDLDR$.
In each of these sequences, there are 2 $D$'s remaining to be placed.
The 2 $D$'s can be placed in each of the above 4 sequences in one of two ways: adjacent to one another or not adjacent to one another.
How many ways can the 2 $D$'s be placed if they are adjacent?

The 2 $D$'s can be placed before the first $R$ or after the last $R$ (so 2 locations), between any adjacent pair of $R$'s (there are 3 of these in each sequence), between the $R$ and $D$ (1 location), or between the $D$ and $R$ (1 location).
Note that placing the 2 $D$'s between the $D$ and $L$ (or between the $L$ and $D$) gives the same sequence as when the 2 $D$'s are placed between the $R$ and $D$ (or between the $D$ and $R$).
Thus, there are $2+3+1+1=7$ ways that 2 adjacent $D$'s can be placed into each of the 4 sequences $RDLDRRRR$, $RRDLDRRR$, $RRRDLDRR$, and $RRRRDLDR$.
How many ways can the 2 $D$'s be placed if they are not adjacent?
One of the $D$'s must be placed into any one of the 7 locations previously given (for the same reason), and the second $D$ can be placed into any of the remaining 6 locations (so that it is not adjacent to the first $D$).
These 2 $D$'s are identical and so switching these $D$'s within the arrangement gives the same sequence, and thus we must divide by 2.
Therefore, there are $\frac{7\times 6}{2}=21$ ways that 2 non-adjacent $D$'s can be placed into each of the 4 sequences.
In total, there are $7+21=28$ ways to place the 2 $D$'s into each of 4 sequences, and so there are $28\times 4=112$ arrangements of 4 $D$'s, 5 $R$'s and 1 $L$.
In a $4\times 4$ grid, the number of paths from $A$ to $B$ having 4 vertical moves and 6 horizontal moves is thus 112.
A similar argument can be made for the arrangement of 4 $R$'s, 5 $D$'s and 1 $U$, and so there are an additional 112 paths having 4 horizontal moves and 6 vertical moves.
Thus, the total number of paths of length 10 from $A$ to $B$ in a $4\times 4$ grid is $112\times 2=224$.

*Solution 2*
In a $4\times 4$ grid, a path from $A$ to $B$ has at least 4 moves down (4 $D$'s) and at least 4 moves right (4 $R$'s).
Thus, each path of length 10 has these 8 moves and exactly one additional pair of opposite moves, either left paired with right or up paired with down.
That is, each path has exactly 4 vertical moves ($U/D$) and 6 horizontal moves ($L/R$), or it has exactly 4 horizontal moves and 6 vertical moves.
By symmetry, each of these two cases will give an equal number of paths from $A$ to $B$, and so we count the number of paths for one of the cases and double that number to determine the total.

Assume that a path from $A$ to $B$ contains 4 vertical moves and 6 horizontal moves.
Each of the 4 vertical moves must be down.
The 6 horizontal moves must be 5 right and 1 left (4 moves right and 1 left/right opposite pair).
Thus, each path may be given by a string of 10 letters (4 $D$'s, 5 $R$'s and 1 $L$), and we may determine the number of paths by counting the number of arrangements of these 10 letters that correspond to a path from $A$ to $B$.
A path must pass through each vertex at most once, and so the $L$ cannot be followed by an $R$ and vice versa ($RL$ or $LR$ would mean "backtracking").
Further, $L$ cannot be the first move and it cannot be the last move.
Thus the $L$ cannot have an $R$ on either side of it, and it cannot be the first or last letter, and so each arrangement must contain the string $DLD$.
How many arrangements of 5 $R$'s, 2 $D$'s, and 1 $DLD$ are there?

We treat $DLD$ as a single letter, and so there are 8 letters for which there would be 8! possible arrangements if the 8 letters were all distinct.
However, the 5 $R$'s are identical, and thus each of the 5! arrangements of the $R$'s within the 8 letters gives the same sequence of moves and so we must divide 8! by 5!.
Similarly, the 2 $D$'s are identical and so switching these $D$'s within the arrangement gives the same sequence, and thus we must also divide by 2.
The total number of arrangements of 5 $R$'s, 2 $D$'s, and 1 $DLD$ is $\frac{8!}{(5!)(2)} = \frac{8 \cdot 7 \cdot 6}{2} = 168$.
Are each of these 168 paths possible?
A sequence of edges that revisit a vertex is not permitted, however forcing the path to contain the moves $DLD$ ensures that this cannot happen.
A sequence of edges that leaves the grid is also not permitted.
Since each sequence contains exactly 4 $D$'s and no $U$'s, it is not possible that the path leaves the top or the bottom of the grid.
However, each sequence of moves contains 5 $R$'s, and if these each appear before $DLD$ (before a move left), then our path will leave the right side of the $4 \times 4$ grid.
Similarly, if the $DLD$ appears in the sequence before an $R$, then the path leaves the left side of the grid.
Since each of these is possible within the arrangements counted above, some of the 168 arrangements are not possible.
We can count the number of such arrangements which leave the grid and subtract this number from the total number 168.
There are exactly 2 ways that a path leaves the grid: all 5 $R$'s appear in the sequence before $DLD$, or all 5 $R$'s appear after $DLD$ (the $L$ occurs before the first $R$).
By symmetry, each of these two cases gives an equal number of paths, and thus we count one case and double the result.
How many of the 168 sequences have 5 $R$'s before $DLD$?
We can count these paths by considering the following three cases.

<u>Case (i): the 5 $R$'s and the 2 $D$'s occur before $DLD$</u>

The number of such paths is equal to the number of arrangements of 5 $R$'s and 2 $D$'s, which is $\frac{7!}{(5!)(2)} = \frac{7 \cdot 6}{2} = 21$.

<u>Case (ii): the 5 $R$'s and 1 $D$ occur before $DLD$ (1 $D$ occurs after the $DLD$)</u>

The number of such paths is equal to the number of arrangements of 5 $R$'s and 1 $D$ (since the location of the other $D$ is forced), which is $\frac{6!}{5!} = 6$.

<u>Case (iii): the 5 $R$'s occur before $DLD$ and the 2 $D$'s occur after</u>

In this case, there is only 1 only such path.

Thus, there are a total of $21 + 6 + 1 = 28$ paths with 5 $R$'s before $DLD$ and 28 more with 5 $R$'s following $DLD$, for a total of 56 paths which leave the grid.
In a $4 \times 4$ grid, the number of paths from $A$ to $B$ having 4 vertical moves and 6 horizontal moves is thus $168 - 56 = 112$.
Since there are an additional 112 paths having 4 horizontal moves and 6 vertical moves, the total number of paths of length 10 from $A$ to $B$ in a $4 \times 4$ grid is $112 \times 2 = 224$.

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# *2019 Galois Contest*

**Wednesday, April 10, 2019**
**(in North America and South America)**

**Thursday, April 11, 2019**
**(outside of North America and South America)**

*Solutions*

1. (a) The total of the listed prices is $\$7.50 + \$5.00 + \$3.00 = \$15.50$.
Sales tax of 10% on \$15.50 is equal to $\$15.50 \times 0.10 = \$1.55$.
After sales tax is included, the total of Becky's bill is $\$15.50 + \$1.55 = \$17.05$.
Alternately, we could add 10% sales tax directly to Becky's bill by multiplying $\$15.50 \times 1.10 = \$17.05$.

(b) After 10% sales tax is included, the cost of one \$6.00 burrito is $\$6.00 \times 1.10 = \$6.60$.
At a cost of \$6.60 for each burrito, the total cost of 7 burritos is $\$6.60 \times 7 = \$46.20$, and the total cost of 8 burritos is $\$6.60 \times 8 = \$52.80$.
Since Jackson has \$50.00, the greatest number of burritos that he can buy is 7.

(c) On Monday, Chase spent $\$5.00 + \$4.50 = \$9.50$ plus 10% tax for two hotdogs.
Including tax, Chase spent $\$9.50 \times 1.10 = \$10.45$ on Monday.

On Tuesday, Chase spent $\$5.00 + \$5.00 = \$10.00$ plus 5% tax for two hotdogs.
Including tax, Chase spent $\$10.00 \times 1.05 = \$10.50$ on Tuesday.

Thus, Chase spent less money on Monday.

2. (a) The $y$-intercept of the line with equation $y = -2x + 12$ is 12 and so $OA = 12$.
The $x$-intercept of this line is determined by letting $y = 0$ and solving for $x$. We get $0 = -2x + 12$ or $2x = 12$ and so $x = 6$.
The $x$-intercept is 6, and so $OB = 6$.
The area of $\triangle AOB$ is $\frac{1}{2}(OB)(OA) = \frac{1}{2}(6)(12) = 36$.

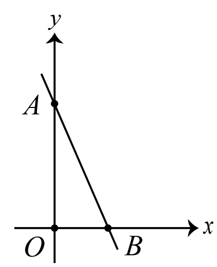

Figure 1

(b) *Solution 1*
We begin by determining the equation of the line passing through $O$ and $C$.
This line is perpendicular to the line with equation $y = -2x + 12$, and so its slope is the negative reciprocal of $-2$, which is $\frac{1}{2}$.
This line passes through the origin and so it has $y$-intercept 0 and equation $y = \frac{1}{2}x$.

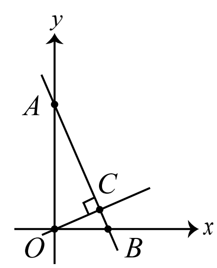

Figure 2

Point $C$ is the point of intersection of the lines $y = \frac{1}{2}x$ and $y = -2x + 12$.

Substituting the equation of the first line into the second, we get $\frac{1}{2}x = -2x + 12$ or $\frac{5}{2}x = 12$ and so $x = \frac{24}{5}$.
When $x = \frac{24}{5}$, the equation $y = \frac{1}{2}x$ gives $y = \frac{1}{2}\left(\frac{24}{5}\right) = \frac{12}{5}$, and so the coodinates of $C$ are $\left(\frac{24}{5}, \frac{12}{5}\right)$.

*Solution 2*

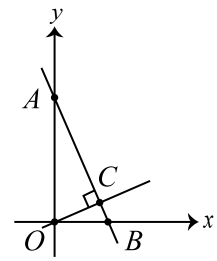

Figure 2

As in Solution 1, we begin by recognizing that the line passing through $O$ and $C$ has slope $\frac{1}{2}$.
Point $C$ lies on the line with equation $y = -2x + 12$ and so if the $x$-coordinate of $C$ is $a$, then the $y$-coordinate is $-2a + 12$.
The slope of the line through $O(0, 0)$ and $C(a, -2a + 12)$ is $\frac{-2a + 12}{a}$ and must equal $\frac{1}{2}$.
Solving, we get $\frac{-2a + 12}{a} = \frac{1}{2}$ or $2(-2a + 12) = a$ or $24 = 5a$, and so $a = \frac{24}{5}$.
When $a = \frac{24}{5}$, we get $-2a+12 = -2\left(\frac{24}{5}\right)+12 = -\frac{48}{5}+12 = \frac{12}{5}$, and so the coordinates of $C$ are $\left(\frac{24}{5}, \frac{12}{5}\right)$.

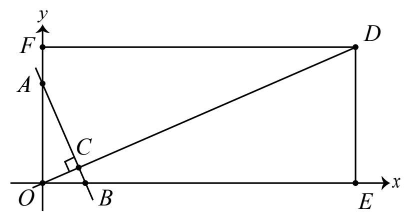

Figure 3

(c) From part (b) Solution 1, the equation of the line passing through $O$ and $C$ is $y = \frac{1}{2}x$. Point $D$ lies on this line and so if the $x$-coordinate of $D$ is $n$, then the $y$-coordinate of $D$ is $\frac{1}{2}n$, so $D$ has coordinates $\left(n, \frac{1}{2}n\right)$.
Point $E$ lies vertically below $D$ and thus has the same $x$-coordinate as $D$.
That is, the coordinates of $E$ are $(n, 0)$ and so $OE = n$.
Similarly, $F$ is positioned horizontally from $D$ and thus has the same $y$-coordinate as $D$.
That is, the coordinates of $F$ are $\left(0, \frac{1}{2}n\right)$ and so $OF = \frac{1}{2}n$.
The area of $DEOF$ is 1352, and so $(OE)(OF) = 1352$ or $n\left(\frac{1}{2}n\right) = 1352$ or $n^2 = 2704$, and so $n = \sqrt{2704} = 52$ (since $n > 0$), and $\frac{1}{2}n = 26$.
If the area of $DEOF$ is 1352, the coordinates of $D$ are $(52, 26)$.

3. (a) This question is asking for the greatest number of factors of 2 in 9!.
Expressing 9! as a product of prime factors, we get

$$9! = 9(8)(7)(6)(5)(4)(3)(2)(1) = (3^2)(2^3)(7)(2 \cdot 3)(5)(2^2)(3)(2)(1).$$

Simplifying, $9! = 7(5)(3^4)(2^7)$ and so 7 is the largest positive integer $m$ for which $2^m$ is a divisor of 9!.

(b) To be divisible by $7^2$, $n!$ must have at least two factors of 7.
Multiples of 7 are the only integers which have a factor of 7.
The smallest positive multiples of 7 are 7 and 14, and each of these contributes exactly one 7 to the prime factorization of 14!, and so if $n \leq 13$, then $n!$ cannot possibly have two factors of 7.
Thus, the smallest value of $n$ for which $n!$ is divisible by $7^2$ is 14.

(c) A positive integer equal to $n!$ is divisible by $7^7$ but not by $7^8$ if it has exactly seven 7s in its prime factorization (in addition to having other prime factors).
Multiples of 7 are the only integers which have a factor of 7.
The first six positive multiples of 7 $(7, 14, 21, 28, 35, 42)$, each have exactly one factor of 7, and thus each contributes exactly one 7 to the prime factorization of 42!.

That is, 42! is divisible by $7^6$ but not by $7^7$ and so $n > 42$.
The next value of $n > 42$ which is a multiple of 7 (and thus contributes 7s to its prime factorization) is 49.
However, 49 contributes two additional 7s to the prime factorization of 49! (since $49 = 7^2$), and so 49! is divisible by $7^{6+2} = 7^8$.
For each positive integer $n \geq 49$, $n!$ is divisible by $7^8$ and possibly some higher power of 7.
For each positive integer $n < 49$, the greatest power of 7 that divides $n!$ is at most $7^6$.
Thus, there is no positive integer $n$ for which $n!$ is divisible by $7^7$ but is not divisible by $7^8$.

(d) Multiples of 13 are the only integers which have a factor of 13.
Since $n!$ has two factors of 13, then $n$ must be at least 26 (one factor from 13 and a second from 26).
Since 29 is a prime number and does not appear in the prime factorization of $n!$, then $n \leq 28$.
We note that for each of the possible values $n = 26, 27$ and 28, $n!$ has two factors of 11, two factors of 13, and one factor of each of 17,19 and 23, as required.
In the table below, we determine the number of factors of $2, 3, 5$, and 7 in 26!.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Numbers containing factors of 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 |
| Number of factors of 2 in each | 1 | 2 | 1 | 3 | 1 | 2 | 1 | 4 | 1 | 2 | 1 | 3 | 1 |
| Numbers containing factors of 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | | | | | |
| Number of factors of 3 in each | 1 | 1 | 2 | 1 | 1 | 2 | 1 | 1 | | | | | |
| Numbers containing factors of 5 | 5 | 10 | 15 | 20 | 25 | | | | | | | | |
| Number of factors of 5 in each | 1 | 1 | 1 | 1 | 2 | | | | | | | | |
| Numbers containing factors of 7 | 7 | 14 | 21 | | | | | | | | | | |
| Number of factors of 7 in each | 1 | 1 | 1 | | | | | | | | | | |

Recall that $n! = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$, and $a + b + c + d = 45$.
Since $n!$ has $a$ factors of 2, then counting in the table above, $a = 23$ when $n = 26$.
Similarly, 26! has $b = 10$, $c = 6$, and $d = 3$, and so $a + b + c + d = 23 + 10 + 6 + 3 = 42$, and so $n \neq 26$.
Next, we determine the values of $a, b, c, d$ for 27!.
Since 27! has all of the prime factors that 26! has, in addition to the prime factors of $27 = 3^3$, then $a + b + c + d = 23 + (10 + 3) + 6 + 3 = 45$, as required.
Finally, we determine the values of $a, b, c, d$ for 28!.
Since 28! has all of the prime factors that 27! has, in addition to the prime factors of $28 = 2^2 \cdot 7$, then $a + b + c + d = (23 + 2) + 13 + 6 + (3 + 1) = 48$, and so $n \neq 28$.
Therefore, $n = 27$ is the only positive integer for which $n! = 2^a \cdot 3^b \cdot 5^c \cdot 7^d \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23$, and $a + b + c + d = 45$.

4. (a) In order for a positive integer to be divisible by 10, its ones digit must be 0.
For a positive integer to be digit-balanced, we need each digit $d$ to appear at most $d$ times.
In the case that $d = 0$, this means a digit balanced number has at most 0 zeros.
This means the number of zeros must be 0.
Since a multiple of 10 has at least one digit which is 0 (the ones digit), it cannot be digit balanced.

(b) Every four-digit positive integer has digits: $x, x, x, x$ or $x, x, x, y$ or $x, x, y, y$ or $x, x, y, z$ or $x, y, z, w$ for some distinct digits $w, x, y, z$.
In this part, we exclude the possibility that a digit could equal 0.

A positive integer with distinct digits $w, x, y, z$ is always digit-balanced since no digit appears more than once.

A positive integer with digits $x, x, x, x$ is not digit-balanced if $x = 1$ or $x = 2$ or $x = 3$. There are 3 such integers.

A positive integer with digits $x, x, x, y$ is not digit-balanced if $x = 1$ or $x = 2$, with $y$ being any non-zero digit not equal to $x$.
There are 2 choices for $x$, 8 choices for $y$ (any non-zero digit not equal to $x$), and 4 possible locations for the digit $y$ (thousands, hundreds, tens, or ones digit), after which the digits $x$ are placed without further choice.
Thus, in this case, there are $2 \times 8 \times 4 = 64$ integers that are not digit-balanced.

A positive integer with digits $x, x, y, y$ is not digit-balanced if either $x = 1$ or $y = 1$. Assume that $x = 1$ and $y \neq 1$.
There are 8 choices for $y$ and 6 choices of positions for $x$. (If the integer has digits $abcd$, then $x$ could be positions $a, b$ or $a, c$ or $a, d$ or $b, c$ or $b, d$ or $c, d$.)
Thus, in this case, there are $8 \times 6 = 48$ integers that are not digit-balanced.

A positive integer with digits $x, x, y, z$ is not digit-balanced if $x = 1$. Here, we assume that $y \neq 1$ and $z \neq 1$ and $y \neq z$.
There 6 choices of positions for the two $x$'s. (If the integer has digits $abcd$, then $x$ could be positions $a, b$ or $a, c$ or $a, d$ or $b, c$ or $b, d$ or $c, d$.)
There are then 8 choices for the left-most of the other digits (any non-zero digit other than 1) and 7 choices for the remaining digit (any non-zero digit other than 1 or $y$).
Thus, in this case, there are $6 \times 8 \times 7 = 336$ integers that are not digit-balanced.

In total, there are $336 + 48 + 64 + 3 = 451$ four-digit integers with all non-zero digits that are not digit-balanced.

(c) We assume $n$ and $m$ are digit-balanced $k$-digit numbers so that $m + n = 10^k$ where $k$ is as large as possible. First, we will deduce that $k \leq 21$. After that, we will explain how to produce digit-balanced integers with $k$ digits whose sum is $10^k$ for any $k$ with $1 \leq k \leq 21$, which will show that the answer to the problem is 21.

We set $n = n_k n_{k-1} n_{k-2} \cdots n_2 n_1$ and $m = m_k m_{k-1} m_{k-2} \cdots m_2 m_1$ so that $n_k, n_{k-1}, \ldots, n_1$ and $m_k, m_{k-1}, \ldots, m_1$ are the digits of $n$ and $m$, respectively. We argued in part (a) that a digit-balanced number cannot have any digits which are 0, so the digits of $m$ and $n$ are all between 1 and 9 inclusive.

Using the condition that $m + n = 10^k$, we note that $m_1 + n_1$ must end in 0 and hence, be a multiple of 10. Since $1 \leq m_1 \leq 9$ and $1 \leq n_1 \leq 9$, we have that $2 \leq m_1 + n_1 \leq 18$. The only multiple of 10 in this range is 10 itself, so it must be that $m_1 + n_1 = 10$. This means there is a carry in the addition:

$$\begin{array}{rcccccc} & & & & & {\scriptstyle 1} & \\ & n_k & n_{k-1} & \cdots & n_4 & n_3 & n_2 & n_1 \\ + & m_k & m_{k-1} & \cdots & m_4 & m_3 & m_2 & m_1 \\ \hline & & & & & & & 0 \end{array}$$

Looking at the tens digit, we need $n_2 + m_2 + 1$ to be a multiple of 10. Like the situation above, $1 \le n_2 \le 9$ and $1 \le m_2 \le 9$, so $2 \le n_2+m_2 \le 18$, which means $3 \le n_2+m_2+1 \le 19$. The only multiple of 10 in this range is 10 itself, which means $n_2 + m_2 + 1 = 10$ or $n_2 + m_2 = 9$.

Continuing the addition, we have

$$\begin{array}{rccccccc} & & & & & {\scriptstyle 1} & {\scriptstyle 1} & \\ & n_k & n_{k-1} & \cdots & n_4 & n_3 & n_2 & n_1 \\ + & m_k & m_{k-1} & \cdots & m_4 & m_3 & m_2 & m_1 \\ \hline & & & & & & 0 & 0 \end{array}$$

Following the same reasoning, we have that $n_3 + m_3 = 9$, $n_4 + m_4 = 9$, and so on.

For now, we will not worry about the units digit of $n$. Instead, we will focus on the digits $n_k$ through $n_2$. We will refer to these as the "first $k - 1$ digits" of $n$. Similarly, we will refer to $m_k$ through $m_2$ as the first $k - 1$ digits of $m$.

Since each of $n$ and $m$ is digit-balanced, no more than one of the first $k - 1$ digits of $n$ or $m$ can be 1.

Similarly, at most two of the first $k - 1$ digits of $m$ and $n$ are 2, at most three of the first $k - 1$ digits are 3, and at most four of the first $k - 1$ digits are 4.

Suppose $n_r = 5$ for some $r > 1$. Then we must have $m_r = 4$ since $n_r + m_r = 9$. Since at most four of the first $k - 1$ digits of $m$ are 4, this means at most four of the first $k - 1$ digits of $n$ are 5.

If $n_r = 6$ for some $r > 1$, then $m_r = 3$ since $n_r + m_r = 9$. Since $m$ is digit-balanced, this means at most three of the first $k - 1$ digits of $n$ are 6.

Similarly, at most two of the first $k - 1$ digits of $n$ are 7 and at most one of the first $k - 1$ digits of $n$ are 8.

If $n_r = 9$ for some $r > 1$, we have $m_r = 0$, but $m$ is digit-balanced, so it does not have any digits which are 0. Therefore, none of the first $k - 1$ digits of $n$ can be 9.

The following table summarizes the maximum number of occurrences of each digit in the first $k - 1$ digits of $n$:

| digit | maximum number of occurrences in $n$ |
|---|---|
| 1 | 1 |
| 2 | 2 |
| 3 | 3 |
| 4 | 4 |
| 5 | 4 |
| 6 | 3 |
| 7 | 2 |
| 8 | 1 |
| 9 | 0 |

Since every digit of $n$ must be between 1 and 9, this means $k-1$ is no larger than

$$1+2+3+4+4+3+2+1+0=20$$

This means $k-1 \leq 20$, so $k \leq 21$. This establishes that if $n$ and $m$ are both digit-balanced, each have $k$ digits, and $m+n=10^k$, then $k \leq 21$. We will now explain how to construct digit-balanced numbers with $k$ digits for each $k$ satisfying $1 \leq k \leq 21$.

To construct digit-balanced numbers $n$ and $m$ which each have 21 digits and whose sum is $m+n=10^{21}$, we must use the maximum number of each digit in the first 20 digits.

If the units digits are $n_1=m_1=5$, this satisfies $n_1+m_1=10$, and each of $m$ and $n$ only has four digits which are 5 in the first 20 digits, so having the units digits equal to 5 will result in $m$ and $n$ being digit-balanced. Can you see why the units digits of $m$ and $n$ must equal 5 for $k=21$?

Taking $n_1=m_1=5$ will give us what we want. For example, we could take:

$$n = 877666555544443332215 \quad \text{and} \quad m = 122333444455556667785$$

Indeed, each of these numbers has 21 digits, is digit-balanced, and their sum is $10^{21}$.

To produce pairs $m$ and $n$ of digit-balanced integers with $k$ digits whose sum is $10^k$ with $k$ smaller than 21, we can simply remove digits from the left of $n$ and $m$ one at a time. For example,

$$n = 77666555544443332215 \quad \text{and} \quad m = 22333444455556667785$$

is a pair of digit-balanced numbers which each have twenty digits and whose sum is $10^{20}$.

The pair

$$n = 44443332215 \quad \text{and} \quad m = 55556667785$$

is a pair of digit-balanced numbers which each have eleven digits and whose sum is $10^{11}$.

The pair $n=5$ and $m=5$ is a pair of digit-balanced numbers which each have one digit and whose sum is $10=10^1$.

There exist digit-balanced positive integers $m$ and $n$, where $m+n=10^k$ and $m$ and $n$ each have $k$ digits for all integer values of $k$ from 1 to 21 inclusive.

Thus, the number of possible values of $k$ is 21.

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# *2018 Galois Contest*

**Thursday, April 12, 2018**
**(in North America and South America)**

**Friday, April 13, 2018**
**(outside of North America and South America)**

*Solutions*

1. (a) Simplifying, we get $\dfrac{12x^2}{3x} = 4x$ for $x \neq 0$.

(b) Since $\dfrac{12x^2}{3x} = 4x$, the value of the expression $\dfrac{12x^2}{3x}$ is equal to the value of the simplified expression $4x$ for all values of $x \neq 0$.

So when $x = 5$, the value of the expression $\dfrac{12x^2}{3x}$ is equal to $4(5) = 20$.

(c) Simplifying, we get $\dfrac{8mn}{3m^2} = \dfrac{8n}{3m}$ for $m \neq 0$.

The value of the expression $\dfrac{8mn}{3m^2}$ is equal to the value of the simplified expression $\dfrac{8n}{3m}$ for all values of $m \neq 0$.

Substituting $n = 2m$ into $\dfrac{8n}{3m}$ and simplifying, we get $\dfrac{8(2m)}{3m} = \dfrac{16m}{3m} = \dfrac{16}{3}$.

When $n = 2m$ and $m \neq 0$, the value of the expression $\dfrac{8mn}{3m^2}$ is $\dfrac{16}{3}$.

(d) Simplifying, we get $\dfrac{8p^2q}{5pq^2} = \dfrac{8p}{5q}$ for $p \neq 0$, $q \neq 0$.

When $q = 6$, we get $\dfrac{8p}{5q} = \dfrac{8p}{5(6)} = \dfrac{8p}{30} = \dfrac{4p}{15}$.

That is, when $q = 6$ (and $p \neq 0$) the expression $\dfrac{8p^2q}{5pq^2}$ is equal to $\dfrac{4p}{15}$, and so the solution to $3 \leq \dfrac{8p^2q}{5pq^2} \leq 4$ is equivalent to the solution to $3 \leq \dfrac{4p}{15} \leq 4$.

Solving $3 \leq \dfrac{4p}{15} \leq 4$, we get $45 \leq 4p \leq 60$ or $\dfrac{45}{4} \leq p \leq \dfrac{60}{4}$, and so $11.25 \leq p \leq 15$.

Since $p$ is a positive integer, then $p = 12, 13, 14, 15$.

Note: In each of the solutions to (b), (c) and (d), we chose to simplify the expression before substituting. Changing the order to substitution followed by simplification would also allow us to solve these problems.

2. (a) In $\triangle ABC$, $\angle ABC = 90^\circ$.

Using the Pythagorean Theorem, we get $AC^2 = AB^2 + BC^2$ or $AC^2 = 8^2 + 15^2$, and so $AC = \sqrt{64 + 225} = \sqrt{289} = 17$ (since $AC > 0$).

(b) In Figure 2, $EF$ is a diameter and so its length is twice the radius or 26.

From the second fact, we know that $\angle EDF = 90^\circ$.

Using the Pythagorean Theorem, we get $DF^2 = EF^2 - DE^2$ or $DF^2 = 26^2 - 24^2$, and so $DF = \sqrt{676 - 576} = \sqrt{100} = 10$ (since $DF > 0$).

(c) Since $SQ$ is a diameter, then $\angle SPQ = \angle SRQ = 90^\circ$.

In $\triangle SPQ$, $SP = PQ$ which means that $\triangle SPQ$ is isosceles and so

$\angle PQS = \angle PSQ = \dfrac{180^\circ - 90^\circ}{2} = 45^\circ$.

Since $\angle RQP = 80^\circ$, then $\angle RQO = \angle RQP - \angle PQS = 80^\circ - 45^\circ = 35^\circ$.

In $\triangle ROQ$, $OR = OQ$ (both are radii) and so $\angle QRO = \angle RQO = 35^\circ$ and $\angle ROQ = 180^\circ - 2 \times 35^\circ = 110^\circ$.

In $\triangle SRQ$, we get $\angle RSQ = 180^\circ - \angle SRQ - \angle RQS = 180^\circ - 90^\circ - 35^\circ = 55^\circ$.

3. (a) A cylinder having radius $r$ and height $h$ has volume $\pi r^2 h$.
Cylinder A has radius 12 and height 25, and so its volume is $\pi(12)^2(25) = 3600\pi$.
Before Cylinder B is lowered into Cylinder A, the height of water in Cylinder A is 19, and so initially the volume of water in Cylinder A is $\pi(12)^2(19) = 2736\pi$.
The height of Cylinder B, 30, is greater than the height of Cylinder A, and so it is not possible for water to pour out of Cylinder A and into Cylinder B.
When Cylinder B is lowered to the bottom of Cylinder A, the portion of Cylinder B lying inside Cylinder A has radius 9 and height 25 (the height of Cylinder A).
Thus, the volume that Cylinder B occupies within Cylinder A is $\pi(9)^2(25) = 2025\pi$.
Since water cannot pour into Cylinder B, the space available for water within Cylinder A (and outside Cylinder B) is the difference between the volume of Cylinder A and the volume of Cylinder B lying inside Cylinder A, or $3600\pi - 2025\pi = 1575\pi$.
The volume of water in Cylinder A was initially $2736\pi$ and once Cylinder B is lowered to the bottom of Cylinder A, the space available for water in Cylinder A becomes $1575\pi$.
Therefore, the volume of water that spills out of Cylinder A and onto the ground is $2736\pi - 1575\pi = 1161\pi$.

(b) As Cylinder B is lowered into Cylinder A, water spills out of Cylinder A and onto the ground when:

(i) the volume of water in Cylinder A exceeds the volume inside Cylinder A and outside Cylinder B, and
(ii) the top of Cylinder B lies above the top of Cylinder A.
(See Figure 1 given in the question.)

As Cylinder B is lowered into Cylinder A, water spills out of Cylinder A and into Cylinder B when:

(i) the top of Cylinder B lies below the top of Cylinder A, and
(ii) the volume of water in Cylinder A (and outside Cylinder B) exceeds the volume inside Cylinder A that lies below the top of Cylinder B and outside Cylinder B, and
(iii) Cylinder B is not full of water.
(See Figure 2 given in the question.)

In Figure 3 shown, the top of Cylinder B has been lowered to the same level as the top of Cylinder A.
At this point, the volume of space inside Cylinder A and outside Cylinder B is $\pi(12)^2(25) - \pi(9)^2(20) = 3600\pi - 1620\pi = 1980\pi$.
The initial volume of water in Cylinder A was $2736\pi$, and so at this point the volume of water that has spilled out of Cylinder A and onto the ground is $2736\pi - 1980\pi = 756\pi$.

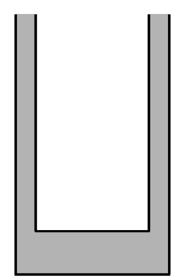

Figure 3

(Since the top of Cylinder B is not below the top of Cylinder A, no water has spilled out of Cylinder A and into Cylinder B at this point.)
As Cylinder B is lowered below this level, water will spill out of Cylinder A and into Cylinder B. How much water will spill into Cylinder B?

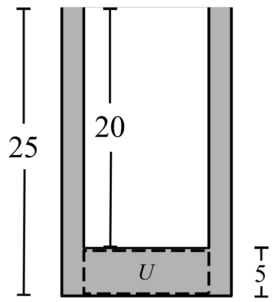

Figure 4

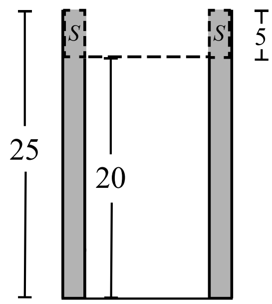

Figure 5

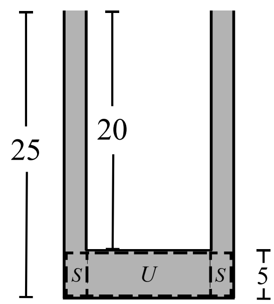

Figure 6

In Figure 4, the volume of water labelled $U$ (lying directly underneath Cylinder B) will be displaced by Cylinder B when it is lowered to the bottom of Cylinder A.
This volume of water will spill into Cylinder B (since the top of Cylinder B will be below the top of Cylinder A).
The shape of the water labelled $U$ is cylindrical, has radius equal to that of Cylinder B, 9, and has height $25 - 20 = 5$.
So the volume of the water labelled $U$ is $\pi(9)^2(5) = 405\pi$.
In addition, the water labelled $S$ in Figure 5 will also spill into Cylinder B when it is lowered to the bottom of Cylinder A.
The shape of the water labelled $S$ is a cylindrical ring, inside Cylinder A and outside Cylinder B, having height $25 - 20 = 5$, and so has volume $\pi(12)^2(5) - \pi(9)^2(5) = 315\pi$.
The volume of water that spills from Cylinder A into Cylinder B is $405\pi + 315\pi = 720\pi$.
The depth, $d$, of water in Cylinder B when it is on the bottom of Cylinder A is given by $\pi(9)^2(d) = 720\pi$ and so $d = \dfrac{720\pi}{81\pi} = \dfrac{80}{9}$.
Note: We could have determined the volume of water that spills into Cylinder B by noticing that the volume labelled $S$ (in Figure 5), is equal to the volume of water surrounding the water labelled $U$ (see Figure 6).
Since both volumes have height 5, their combined volume is equal to that of a cylinder with radius 12 and height 5, and so $V = \pi(12)^2(5) = 720\pi$, as we previously determined.

(c) *Solution 1*

We begin by finding the range of values of $h$ for which some water will spill out of Cylinder A when Cylinder B is lowered to the bottom of Cylinder A.
Consider lowering Cylinder B into Cylinder A until the water level reaches the top of Cylinder A, as shown in Figure 7 (we know this is possible for some values of $h$ since it occured in part (a)).
Let $y$ be the distance between the bottoms of the two cylinders, and so the distance between the top of Cylinder A and the bottom of Cylinder B is $25 - y$.

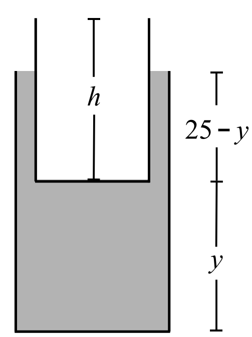

Figure 7

The volume of water, $V_w$, is equal to the volume of Cylinder A that lies below the bottom of Cylinder B, or $\pi(12)^2(y)$, added to the volume inside Cylinder A and outside Cylinder B between the top of Cylinder A and the bottom of Cylinder B, or $\pi(12)^2(25-y) - \pi(9)^2(25-y) = \pi(12^2 - 9^2)(25-y)$.
That is, $V_w = \pi(12)^2(y) + \pi(12^2 - 9^2)(25-y) = 144\pi y + 63\pi(25-y) = 81\pi y + 1575\pi$.
From part (a), the initial volume of water is $2736\pi$, and so we get $81\pi y + 1575\pi = 2736\pi$ or $81\pi y = 1161\pi$, and so $y = \frac{43}{3}$.
So if $h > 25 - y = 25 - \frac{43}{3} = \frac{32}{3}$, then water will spill out of Cylinder A onto the ground.
What if $h \leq \frac{32}{3}$?

When $h \le \frac{32}{3}$, Cylinder B may be lowered so that its top is level with the top of Cylinder A without any water spilling out of Cylinder A onto the ground.
In this case when $h \le \frac{32}{3}$, then $y \ge 25 - \frac{32}{3} = \frac{43}{3}$, and so $y > h$.
That is, when Cylinder B is lowered so that its top is level with the top of Cylinder A, the volume of water that lies directly below Cylinder B is greater that the volume of Cylinder B and so Cylinder B will be completely full of water when it is lowered to the bottom of Cylinder A.
In this question, we require that Cylinder B not be full and so $h > \frac{32}{3}$ and water will spill out of Cylinder A onto the ground before the top of Cylinder B is level with the top of Cylinder A.

Next, we will further restrict the range of values of $h$ so that when Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full.
Consider lowering Cylinder B to the point where the tops of the two cylinders are level with one another (so then $h \le 25$).
Some water has spilled out of Cylinder A.
When Cylinder B is lowered beyond this point (so then we require $h < 25$), water will spill from Cylinder A into Cylinder B (and not onto the ground).

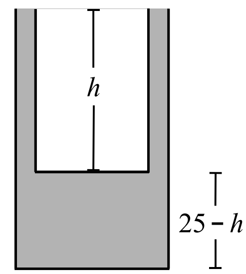

Figure 8

From the solution in part (b), recall that when Cylinder B is lowered to the bottom of Cylinder A, the volume of water that will spill from Cylinder A into Cylinder B is equal to the volume of water inside Cylinder A that lies below the bottom of Cylinder B (as in Figure 8).
This cylinder has radius 12 and height $25 - h$, and so has volume $\pi(12)^2(25 - h)$.
Assume that when this volume of water has spilled into Cylinder B, it fills Cylinder B to a depth of $d$.
Once Cylinder B is lowered to the bottom of Cylinder A, the volume of water in Cylinder B, $\pi(9)^2(d)$, must equal $\pi(12)^2(25 - h)$.
Solving for $d$, we get $81\pi d = 144\pi(25 - h)$ or $d = \dfrac{3600 - 144h}{81}$, and so $d = \dfrac{400 - 16h}{9}$.
The depth of water in Cylinder B must be less than the height of Cylinder B (Cylinder B cannot be full), so then $d < h$ or $\dfrac{400 - 16h}{9} < h$ or $400 - 16h < 9h$ or $400 < 25h$, and so $16 < h$.
As noted earlier, no water can spill into Cylinder B unless its height is less than that of Cylinder A, and so $h < 25$.
When Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full when $16 < h < 25$.

*Solution 2*
Let the volume of Cylinder A be $V_A$, the volume of Cylinder B be $V_B$, and the initial volume of water be $V_W$.
As we determined in Solution 1, $V_A = 3600\pi$, $V_B = 81\pi h$, and $V_W = 2736\pi$.
If $V_W + V_B > V_A$, then water spills out of the large cylinder onto the ground.
This gives $2736\pi + 81\pi h > 3600\pi$ or $81\pi h > 864\pi$, and so $h > \frac{32}{3}$.
If $\frac{32}{3} < h < 25$, water will spill onto the ground and then into B. (If $h \le \frac{32}{3}$, B will actually be full of water when lowered into A, since no water spills out of A and the height of B is less than the initial height of water.)

Assume that $\frac{32}{3} < h < 25$.
Then the volume of water that spills out of Cylinder A onto the ground is

$$V_{\text{water on ground}} = V_W + V_B - V_A = 81\pi h - 864\pi.$$

When the tops of the two cylinders are at the same level (Figure 9), no water has spilled into Cylinder B, and so the volume of water in Cylinder A is the initial volume of water less the volume of water that has spilled out onto the ground.

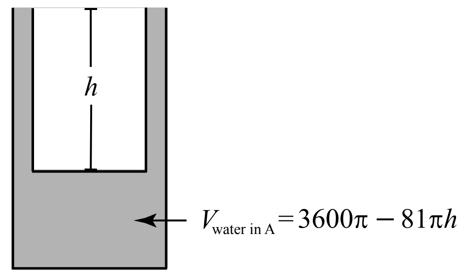

Figure 9

That is,

$$V_{\text{water in A}} = 2736\pi - V_{\text{water on ground}} = 2736\pi - (81\pi h - 864\pi) = 3600\pi - 81\pi h.$$

From this point on, all water stays in Cylinder B or in Cylinder A.
When Cylinder B is on the bottom of Cylinder A (Figure 10), the volume of water outside of Cylinder B (but inside Cylinder A), is the volume of Cylinder A that lies below the top of Cylinder B less the volume of Cylinder B.

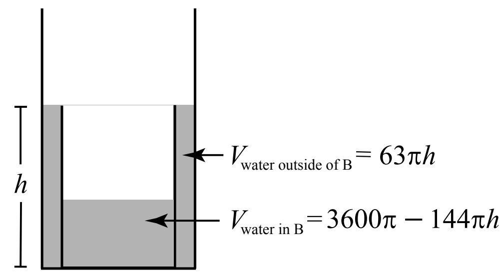

Figure 10

That is,

$$V_{\text{water outside of B}} = \pi(12^2)h - \pi(9^2)h = 63\pi h.$$

Further, the volume of water in Cylinder A, $3600\pi - 81\pi h$, must be equal to the volume of water outside of Cylinder B plus the volume of water inside of Cylinder B.
That is,

$$\begin{aligned} V_{\text{water in A}} &= V_{\text{water outside of B}} + V_{\text{water in B}} \\ V_{\text{water in B}} &= V_{\text{water in A}} - V_{\text{water outside of B}} \\ &= 3600\pi - 81\pi h - 63\pi h \\ &= 3600\pi - 144\pi h. \end{aligned}$$

The volume of water in Cylinder B must be less than the volume of Cylinder B, and so $3600\pi - 144\pi h < 81\pi h$ or $3600\pi < 225\pi h$, and thus $16 < h$.
When Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full when $16 < h < 25$.

4. (a) As a sum of one of more consecutive positive integers, 45 can be written as

45, 22+23, 14+15+16, 7+8+9+10+11, 5+6+7+8+9+10, and 1+2+3+4+5+6+7+8+9,

and there are no other such lists.
The value of $C(45)$ is 6.

(b) The sum of the positive integers from 1 to $n$ is given by the formula $\frac{1}{2}n(n+1)$.
The sum of the positive integers from 4 to $n$ ($n \geq 4$) is equal to the sum of the positive

integers from 1 to $n$ less the sum of the positive integers from 1 to 3, or $1+2+3=6$.
Therefore,

$$\begin{aligned} m &= 4+5+6+\cdots+n \\ &= \tfrac{1}{2}n(n+1)-6 \\ &= \tfrac{1}{2}(n(n+1)-12) \\ &= \tfrac{1}{2}(n^2+n-12) \\ &= \tfrac{1}{2}(n-3)(n+4). \end{aligned}$$

Since $m=\frac{1}{2}(n+a)(n+b)$ with $a<b$, then $a=-3$ and $b=4$.

(c) If $m=(a+1)+(a+2)+\cdots+n$, for integers $a\geq 0$ and $n\geq a+1$, then $m$ is equal to the sum of the integers from 1 to $n$ less the sum of the integers from 1 to $a$.
That is, $m=\frac{1}{2}n(n+1)-\frac{1}{2}a(a+1)$.
Simplifying, we get

$$\begin{aligned} m &= \tfrac{1}{2}n(n+1)-\tfrac{1}{2}a(a+1) \\ &= \tfrac{1}{2}(n^2+n-a^2-a) \\ &= \tfrac{1}{2}(n^2-a^2+n-a) \\ &= \tfrac{1}{2}((n-a)(n+a)+n-a) \\ &= \tfrac{1}{2}(n-a)(n+a+1) \end{aligned}$$

Each pair of integers $(a,n)$ ($a\geq 0$ and $n\geq a+1$) for which $m=\frac{1}{2}(n-a)(n+a+1)$ gives a unique sum of one or more consecutive positive integers from $a+1$ to $n$ whose sum is $m$.
In this question, we are asked to determine the number of such pairs $(a,n)$ given that $m=2\times 3^4\times 5^6$.
Since $m=\frac{1}{2}(n-a)(n+a+1)$, then $2m=(n-a)(n+a+1)$.
That is, $2m$ can be expressed as the product of two positive integers $n+a+1$ and $n-a$.
The difference between these two integers is $(n+a+1)-(n-a)=2a+1$, which is an odd integer for all integers $a\geq 0$.
Since the difference between $n+a+1$ and $n-a$ is odd, then one of these integers must be even and the other odd (we say that they have different *parity*).
Thus, the problem of evaluating $C(m)$ appears to be equivalent to counting the number of factor pairs of $2m$ ($n+a+1$ and $n-a$) that have different parity.
At this point, we have shown that each pair of integers $(a,n)$ ($a\geq 0$ and $n\geq a+1$) for which $m=\frac{1}{2}(n-a)(n+a+1)$ gives a factor pair with different parity.
We must now show that the converse is also true; that is, each factor pair with different parity gives a unique pair $(a,n)$.
Suppose that $2m=d\cdot e$ for some positive odd integer $d$ and positive even integer $e$.
We show that each pair $d$ and $e$ will give a pair of integers $a$ and $n$.
If $d>e$, suppose that $d=n+a+1$ and $e=n-a$ (since $n+a+1>n-a$).
Adding the equations $n+a+1=d$ and $n-a=e$, we get $2n+1=d+e$ or $n=\frac{1}{2}(d+e-1)$.
Subtracting the two equations $n+a+1=d$ and $n-a=e$, we get $2a+1=d-e$ or $a=\frac{1}{2}(d-e-1)$.
Since $d$ and $e$ have different parity, then each of $d+e$ and $d-e$ is odd, and so each of $d+e-1$ and $d-e-1$ is even.
Therefore, each of $n=\frac{1}{2}(d+e-1)$ and $a=\frac{1}{2}(d-e-1)$ is an integer and $n>a$.
(If we assume that $d<e$, we can make a similar argument to show there exist corresponding integers $a$ and $n$ with $n>a$.)

That is, each factor pair $(d, e)$ having different parity gives a unique pair $(a, n)$ with $n > a$. This confirms that evaluating $C(m)$ is equivalent to counting the number of factor pairs of $2m$ that have different parity.

Before evaluating $C(2 \times 3^4 \times 5^6)$, we apply this to part (a) to confirm that $C(45) = 6$, and to demonstrate that for each odd factor of $2 \times 45$, there exists a corresponding unique list of consecutive positive integers whose sum is 45.

Since $m = 45 = 3^2 \times 5$, then $2m = 2 \times 3^2 \times 5$, and so the odd factors of $2 \times 3^2 \times 5$ must be of the form $3^i \times 5^j$ for integers $0 \leq i \leq 2$ and $0 \leq j \leq 1$ (that is, odd numbers have only odd divisors).
Since there are 3 choices for $i$ (0,1,2), and 2 choices for $j$ (0,1), there are $3 \times 2 = 6$ odd factors of $2 \times 3^2 \times 5$ (these are $1, 3, 5, 9, 15$, and 45).
Next, we demonstrate that each of these 6 odd factors gives a unique pair $(a, n)$ for which:

(i) $(n - a)(n + a + 1) = 2 \times 45$, and
(ii) $n + a - 1$ and $n - a$ have different parity, and
(iii) $(a + 1) + (a + 2) + \cdots + n = 45$.

The odd factors $1, 3, 5, 9, 15, 45$ give the factor pairs $(1, 90), (3, 30), (5, 18), (9, 10), (15, 6)$, and $(45, 2)$ (we notice that the two numbers in each pair do indeed have different parity).
Next we note that since $a \geq 0$, then $n + a + 1 > n - a$ and so for example, using the factor pair $(5, 18)$, we get $n - a = 5$ and $n + a + 1 = 18$.
Adding the two equations to solve this system of equations, we get $2n + 1 = 23$ and so $n = 11$ and $a = 6$.
This pair $(6, 11)$ gives the sum $7 + 8 + 9 + 10 + 11 = 45$.
We summarize the results using the other factor pairs in the table below.

| Factor Pair | $a - n$ | $a + n + 1$ | $n$ | $a$ | $(a + 1) + (a + 2) + \cdots + n$ |
|---|---|---|---|---|---|
| $(1, 90)$ | 1 | 90 | 45 | 44 | 45 |
| $(3, 30)$ | 3 | 30 | 16 | 13 | $14 + 15 + 16 = 45$ |
| $(5, 18)$ | 5 | 18 | 11 | 6 | $7 + 8 + 9 + 10 + 11 = 45$ |
| $(9, 10)$ | 9 | 10 | 9 | 0 | $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$ |
| $(15, 6)$ | 6 | 15 | 10 | 4 | $5 + 6 + 7 + 8 + 9 + 10 = 45$ |
| $(45, 2)$ | 2 | 45 | 23 | 21 | $22 + 23 = 45$ |

Comparing this table to our answer in part (a), we see that indeed each odd factor of $2 \times 45$ gives a unique list of consecutive positive integers whose sum is 45.

Finally, we turn our focus to evaluating $C(2 \times 3^4 \times 5^6)$, that is, counting the number of odd factors of $2^2 \times 3^4 \times 5^6$.
The odd factors of $2^2 \times 3^4 \times 5^6$ are of the form $3^i \times 5^j$ for integers $0 \leq i \leq 4$ and $0 \leq j \leq 6$.
Since there are 5 choices for $i$ and 7 choices for $j$, there are $5 \times 7 = 35$ odd factors of $2^2 \times 3^4 \times 5^6$, and so $C(2 \times 3^4 \times 5^6) = 35$.

(d) We would like to determine the smallest positive integer $k$ for which $C(k) = 215 = 5 \times 43$ (both 5 and 43 are prime numbers).
If $k = 2^a$ for some non-negative integer $a$, then $C(k) = 1$, and so $k$ must have some odd prime factors $p_1, p_2, \ldots, p_n$. Can you see why?
That is, $k = 2^a \cdot p_1^{q_1} \cdot p_2^{q_2} \cdots p_n^{q_n}$ for distinct prime numbers $p_i, 1 \leq i \leq n$ and positive integers $q_j, 1 \leq j \leq n$.

From part (c), we know that $C(k) = (q_1 + 1)(q_2 + 1) \cdots (q_n + 1)$.
Since $C(k) = 5 \times 43 = (q_1 + 1)(q_2 + 1) \cdots (q_n + 1)$, then we let $n = 2$ and $q_1 + 1 = 5$ or $q_1 = 4$, and $q_2 + 1 = 43$ or $q_2 = 42$.
To minimize $k$, we let $a = 0$, and choose the smallest distinct odd primes $p_1 = 5$ and $p_2 = 3$ ($p_1 = 3$ and $p_2 = 5$ gives $k = 3^4 \times 5^{42}$, which is a much larger value for $k$).
The smallest positive integer $k$ for which $C(k) = 215$ is $k = 5^4 \times 3^{42}$.

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# *2017 Galois Contest*

**Wednesday, April 12, 2017**
**(in North America and South America)**

**Thursday, April 13, 2017**
**(outside of North America and South America)**

*Solutions*

1. (a) In Box E, 6 of the 30 cups were purple.
The percentage of purple cups in Box E was $\frac{6}{30} \times 100\% = \frac{2}{10} \times 100\% = 20\%$.

(b) On Monday, 30% of Daniel's 90 cups or $30\% \times 90 = \frac{30}{100} \times 90 = 27$ cups were purple.
Daniel had 9 purple cups in Box D and 6 purple cups in Box E.
Therefore, the number of purple cups in Box F was $27 - 9 - 6 = 12$.

(c) Daniel had 27 purple cups and 90 cups in total.
On Tuesday, Avril added 9 more purple cups to Daniel's cups, bringing the number of purple cups to $27 + 9 = 36$, and the total number of cups to $90 + 9 = 99$.
Barry brought some yellow cups and included them with the 99 cups.
Let the number of yellow cups that Barry brought be $y$.
The total number of cups was then $99 + y$, while the number of purple cups was still 36 (since Barry brought yellow cups only).
Since the percentage of cups that were purple was again 30% or $\frac{30}{100}$, then $\frac{30}{100}$ of $99 + y$ must equal 36.
Solving, we get $\frac{30}{100} \times (99 + y) = 36$ or $30(99 + y) = 3600$ or $99 + y = 120$, and so $y = 21$.
Therefore, Barry brought 21 cups.

2. (a) Abdi arrived at 5:02 a.m., and so Abdi paid \$5.02.
Caleigh arrived at 5:10 a.m., and so Caleigh paid \$5.10.
In total, Abdi and Caleigh paid $\$5.02 + \$5.10 = \$10.12$.

(b) If both Daniel and Emily had arrived at the same time, then they each would have paid the same amount, or $\$12.34 \div 2 = \$6.17$.
In this case, they would have both arrived at 6:17 a.m.
If Daniel arrived 5 minutes earlier, at 6:12 a.m., and Emily arrived 5 minutes later, at 6:22 a.m., then they would have arrived 10 minutes apart and in total they would have still paid \$12.34.
(We may check that these arrival times are 10 minutes apart, and that Daniel and Emily's total price is $\$6.12 + \$6.22 = \$12.34$, as required.)

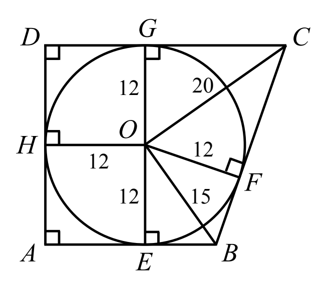
(c) To minimize the amount that Karla could have paid, we maximize the amount that Isaac and Jacob pay.
Isaac and Jacob arrived together and Karla arrived after.
Since Karla arrived at a later time than Isaac and Jacob, then Karla paid more than Isaac and Jacob.
If Isaac and Jacob both arrived together at 6:18 a.m., then they would each have paid \$6.18, and Karla would have paid $\$18.55 - \$6.18 - \$6.18 = \$6.19$.
This is the minimum amount that Karla could have paid. Why?
If Isaac and Jacob arrived at 6:19 a.m. or later, then Karla would have arrived at a time earlier than 6:19 a.m. (since $\$18.55 - \$6.19 - \$6.19 = \$6.17$).
Since Karla arrived after Isaac and Jacob, this is not possible.
If Isaac and Jacob arrived earlier than 6:18 a.m., then they would have each paid less than \$6.18, and so Karla would have paid more than \$6.19.
Therefore, the minimum amount that Karla could have paid is \$6.19.

(d) If Larry arrived earlier than 5:39 a.m., then he would have paid less than \$5.39 and so Mio would have paid more than $\$11.98 - \$5.39 = \$6.59$.
Since Mio arrived during the time of the special pricing, it is not possible for Mio to have

paid more than \$6.59, and so Larry must have arrived at 5:39 a.m. or later.
If Larry arrived between 5:39 a.m. and 5:59 a.m. (inclusive), then Larry would have paid the amount between \$5.39 and \$5.59 corresponding to his arrival time.
Therefore, Mio would have paid an amount between \$11.98 − \$5.59 = \$6.39 and \$11.98 − \$5.39 = \$6.59 (inclusive).
Each of the amounts between \$6.39 and \$6.59 corresponds to an arrival time for Mio between 6:39 a.m. and 6:59 a.m., each of which is a possible time that Mio could have arrived during the special pricing period.
That is, each arrival time for Larry from 5:39 a.m. to 5:59 a.m. corresponds to an arrival time for Mio from 6:39 a.m. to 6:59 a.m.
Each of these times is during the period of the special pricing and each corresponding pair of times gives a total price of \$11.98.
To see this, consider that if Larry arrived $x$ minutes after 5:39 a.m. (where $x$ is an integer and $0 \leq x \leq 20$), then Mio arrived $x$ minutes before 6:59 a.m., and in total they paid $\$5.39 + x¢ + \$6.59 - x¢ = \$11.98$.
Since Larry's arrival time and Mio's arrival time may be switched to give the same total, \$11.98, then Larry could also have arrived between 6:39 a.m. and 6:59 a.m.
The only times left to consider are those from 6:00 a.m. to 6:38 a.m.
If Larry arrived at one of these times, his price would have been between \$6.00 and \$6.38, and so Mio's price would have been between \$11.98 − \$6.38 = \$5.60 and \$11.98 − \$6.00 = \$5.98.
Since there are no arrival times which correspond to Mio having to pay an amount between \$5.60 and \$5.98, then it is not possible that Larry arrived at any time from 6:00 a.m. to 6:38 a.m.
Therefore, the ranges of times during which Larry could have arrived are 5:39 a.m. to 5:59 a.m or 6:39 a.m. to 6:59 a.m.

3. (a) Since $\angle OPQ = 90°$, then $\triangle OPQ$ is a right-angled triangle.
By the Pythagorean Theorem, $OQ^2 = OP^2 + PQ^2 = 18^2 + 24^2 = 900$, and so $OQ = \sqrt{900} = 30$ (since $OQ > 0$).
Line segment $OS$ is a radius of the circle and thus has length 18.
Therefore, $SQ = OQ - OS = 30 - 18 = 12$.

(b) Sides $AB, BC, CD$, and $DA$ are tangent to the circle at points $E, F, G$, and $H$, respectively.
Therefore, radii $OE$, $OF$, $OG$, and $OH$ are perpendicular to their corresponding sides, as shown.
In quadrilateral $DHOG$, $\angle OGD = \angle GDH = \angle DHO = 90°$ and so $\angle GOH = 90°$.
Since $OH = OG = 12$ (they are radii of the circle), then $DHOG$ is a square with side length 12.

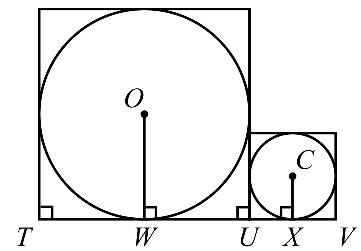



Similarly, $HAEO$ is also a square with side length 12.
Since $\angle OGC = 90°$, then $\triangle OGC$ is a right-angled triangle.
By the Pythagorean Theorem, $GC^2 = OC^2 - OG^2 = 20^2 - 12^2 = 256$, and so $GC = \sqrt{256} = 16$ (since $GC > 0$).
It can be similarly shown that $FC = 16$.
Since $\angle OEB = 90°$, then $\triangle OEB$ is a right-angled triangle.
By the Pythagorean Theorem, $EB^2 = OB^2 - OE^2 = 15^2 - 12^2 = 81$, and so $EB = \sqrt{81} = 9$ (since $EB > 0$).
It can be similarly shown that $FB = 9$.

Therefore, the perimeter of $ABCD$ is $GD + DH + HA + AE + EB + BF + FC + CG$ or $4 \times 12 + 2 \times 9 + 2 \times 16 = 98$.

(c) In Figure 1:

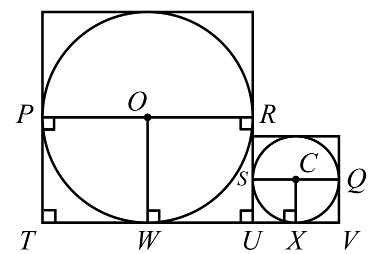
Figure 1




Since the circles are inscribed in their respective squares, then $TU$ is a tangent to the larger circle and $UV$ is a tangent to the smaller circle.
Let $TU$ touch the larger circle at $W$, and let $UV$ touch the smaller circle at $X$.
The radius $OW$ is perpendicular to $TU$, and the radius $CX$ is perpendicular to $UV$.

In Figure 2:

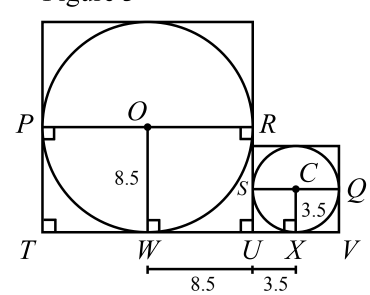
Figure 2




The diameter of the larger circle is equal to the side length of the larger square.
To see this, label the points $P$ and $R$ where the vertical sides of the larger square touch the larger circle.
Join $P$ to $O$ and join $R$ to $O$.
The radius $OP$ is perpendicular to $PT$ and the radius $OR$ is perpendicular to $RU$.
In quadrilateral $PTWO$, $\angle OPT = \angle PTW = \angle TWO = 90^\circ$, and so $\angle POW = 90^\circ$.
Similarly, in quadrilateral $RUWO$, $\angle ROW = 90^\circ$.
Therefore, $\angle POW + \angle ROW = 180^\circ$ and so $PR$ passes through $O$ and is thus a diameter of the larger circle.
In quadrilateral $PTUR$, all 4 interior angles measure $90^\circ$, and so $PTUR$ is a rectangle.
It can similarly be shown that if $S$ and $Q$ are the points where the vertical sides of the smaller square touch the smaller circle, then $SQ$ is a diameter of the smaller circle and $SUVQ$ is a rectangle.

In Figure 3:

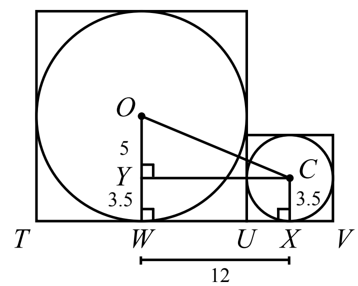
Figure 3




The area of the larger square is 289, and so each side of the larger square has length $\sqrt{289} = 17$.
The diameter of the larger circle is equal to the side length of the larger square, or $PR = TU = 17$.
Since $O$ is the midpoint of $PR$, and $OW$ is perpendicular to $TU$, then $W$ is the midpoint of $TU$.
Therefore, $WU = OR = OW = 17 \div 2 = 8.5$.
The area of the smaller square is 49, and so each side of the smaller square has length $\sqrt{49} = 7$.
Similarly, $X$ is the midpoint of $UV$ and so $UX = SC = CX = 7 \div 2 = 3.5$.

In Figure 4:
Finally, we construct the line segment from $C$, parallel to $XW$, and meeting $OW$ at $Y$.
In quadrilateral $YWXC$, $CY$ is parallel to $XW$, $YW$ is perpendicular to $XW$, and $CX$ is perpendicular to $XW$, and so $YWXC$ is a rectangle.
Thus, $CX = YW = 3.5$, and
$CY = XW = XU + WU = 3.5 + 8.5 = 12$.
Since $\angle OYC = 90^\circ$, then $\triangle OYC$ is a right-angled triangle with $CY = 12$, and $OY = OW - YW = 8.5 - 3.5 = 5$.

Figure 4




By the Pythagorean Theorem, $OC^2 = CY^2 + OY^2 = 12^2 + 5^2 = 144 + 25 = 169$, and so $OC = \sqrt{169} = 13$ (since $OC > 0$).

4. (a) The total area of the $m = 14$ by $n = 10$ Koeller-rectangle is $m \times n = 14 \times 10 = 140$.
The dimensions of the shaded area inside a Koeller-rectangle are $(m-2)$ by $(n-2)$ since the 1 by 1 squares along the sides are unshaded, so each dimension is reduced by 2.
Therefore, the shaded area of a 14 by 10 Koeller-rectangle is $(14-2)\times(10-2) = 12\times 8 = 96$.
The unshaded area is the difference between the total area and the shaded area, or $mn-(m-2)(n-2) = mn-(mn-2m-2n+4) = 2m+2n-4$ or $2\times 14+2\times 10-4 = 44$.
Finally, $r$ is the ratio of the shaded area to the unshaded area, or $\frac{96}{44} = \frac{24}{11}$ (or $24 : 11$).

(b) As we saw in part (a), the shaded area of an $m$ by $n$ Koeller-rectangle is $(m-2)(n-2)$, and the unshaded area is $2m + 2n - 4$.
Therefore, $r = \frac{(m-2)(n-2)}{2m+2n-4}$. When $n = 4$, $r = \frac{2(m-2)}{2m+4} = \frac{2(m-2)}{2(m+2)} = \frac{m-2}{m+2}$.
We rewrite $\frac{m-2}{m+2}$ as $\frac{m+2-4}{m+2} = \frac{m+2}{m+2} - \frac{4}{m+2} = 1 - \frac{4}{m+2}$.
We must determine all possible integer values of $u$ for which $r = 1 - \frac{4}{m+2} = \frac{u}{77}$, for some integer $m \geq 3$.
Simplifying this equation, we get

$$\begin{aligned}
1 - \frac{4}{m+2} &= \frac{u}{77} \\
1 - \frac{u}{77} &= \frac{4}{m+2} \\
\frac{77-u}{77} &= \frac{4}{m+2} \\
(m+2)(77-u) &= 4 \times 77
\end{aligned}$$

Both $u$ and $m$ are integers, and so $(m+2)(77-u)$ is the product of two integers.
If $a$ and $b$ are positive integers so that $ab = 4 \times 77 = 2^2 \times 7 \times 11$, then there are 6 possible factor pairs $(a, b)$ with $a < b$.
These are: $(1, 308), (2, 154), (4, 77), (7, 44), (11, 28)$, and $(14, 22)$.
Since $m \geq 3$, then $m + 2 \geq 5$ and so $m + 2$ cannot equal $1, 2$ and $4$.
However, $m+2$ can equal any of the remaining 9 divisors: $7, 11, 14, 22, 28, 44, 77, 154, 308$.
In the table below, we determine the possible values for $u$ given that $(m + 2)(77 - u) = 2^2 \times 7 \times 11$, and $m + 2 \geq 5$.

| $m+2$ | 7 | 11 | 14 | 22 | 28 | 44 | 77 | 154 | 308 |
|---|---|---|---|---|---|---|---|---|---|
| $77-u$ | 44 | 28 | 22 | 14 | 11 | 7 | 4 | 2 | 1 |
| $u$ | 33 | 49 | 55 | 63 | 66 | 70 | 73 | 75 | 76 |

The integer values of $u$ for which there exists a Koeller-rectangle with $n=4$ and $r=\dfrac{u}{77}$, are $u=33,49,55,63,66,70,73,75,76$.

(For example, the 5 by 4 Koeller-rectangle has $r=\dfrac{m-2}{m+2}=\dfrac{3}{7}=\dfrac{33}{77}$, and so $u=33$.)

(c) As in part (b), $r=\dfrac{(m-2)(n-2)}{2m+2n-4}$, and when $n=10$, $r=\dfrac{8(m-2)}{2m+16}=\dfrac{4(m-2)}{m+8}$.

Rearranging this equation, we get

$$\begin{aligned}
r &= \frac{4(m-2)}{m+8}\\
\frac{r}{4} &= \frac{m-2}{m+8}\\
\frac{r}{4} &= \frac{m+8-10}{m+8}\\
\frac{r}{4} &= \frac{m+8}{m+8}-\frac{10}{m+8}\\
\frac{r}{4} &= 1-\frac{10}{m+8}\\
\frac{10}{m+8} &= 1-\frac{r}{4}\\
\frac{10}{m+8} &= 1-\frac{u}{4p^2} \quad (\text{since } r=\frac{u}{p^2})\\
\frac{10}{m+8} &= \frac{4p^2-u}{4p^2}\\
40p^2 &= (m+8)(4p^2-u)
\end{aligned}$$

Since $p,u$ and $m$ are integers, then $(m+8)(4p^2-u)$ is the product of two integers.

We must determine all prime numbers $p$ for which there are exactly 17 positive integer values of $u$ for Koeller-rectangles satisfying this equation $40p^2=(m+8)(4p^2-u)$.

For $p=2,3,5,7$, and then $p\geq 11$, we proceed with the following strategy:

- determine the value of $40p^2$
- count the number of divisors of $40p^2$
- eliminate possible values of $m+8$, thus eliminating possible values of $4p^2-u$
- count the number of values of $u$ by counting the number of values of $4p^2-u$

If $p=2$, then $40p^2=40\times 2^2=2^5\times 5$, and so $2^5\times 5=(m+8)(16-u)$.

Each divisor of $2^5\times 5$ is of the form $2^i\times 5^j$, for integers $0\leq i\leq 5$ and $0\leq j\leq 1$.

That is, there are 6 choices for $i$ (each of the integers from 0 to 5) and 2 choices for $j$ (0 or 1), and so there are $6\times 2=12$ different divisors of $2^5\times 5$.

Since $2^5\times 5=(m+8)(16-u)$, then there are at most 12 different integer values of $16-u$ (the 12 divisors of $2^5\times 5$), and so there are at most 12 different integer values of $u$.

Therefore, when $p=2$, there cannot be exactly 17 positive integer values of $u$.

If $p=5$, then $40p^2=40\times 5^2=2^3\times 5^3$, and so $2^3\times 5^3=(m+8)(100-u)$.

Each divisor of $2^3\times 5^3$ is of the form $2^i\times 5^j$, for integers $0\leq i\leq 3$ and $0\leq j\leq 3$.

That is, there are 4 choices for $i$ and 4 choices for $j$, and so there are $4\times 4=16$ different divisors of $2^3\times 5^3$.

Since $2^3\times 5^3=(m+8)(100-u)$, then there are at most 16 different integer values of

$100 - u$, and so there are at most 16 different integer values of $u$.
Therefore, when $p = 5$, there cannot be exactly 17 positive integer values of $u$.

If $p = 3$, then $40p^2 = 40 \times 3^2 = 2^3 \times 3^2 \times 5$, and so $2^3 \times 3^2 \times 5 = (m+8)(36-u)$.
Each divisor of $2^3 \times 3^2 \times 5$ is of the form $2^i \times 3^j \times 5^k$, for integers $0 \le i \le 3$, $0 \le j \le 2$, and $0 \le k \le 1$.
That is, there are $4 \times 3 \times 2 = 24$ different divisors of $2^3 \times 3^2 \times 5$.
Since $m \ge 3$, then $m+8 \ge 11$, and so the divisors of $2^3 \times 3^2 \times 5$ which $m+8$ cannot equal are: $1, 2, 3, 4, 5, 6, 8, 9$, and 10.
Since there are 9 divisors which $m+8$ cannot equal, then there are 9 divisors that $36-u$ cannot equal. (These divisors can be determined by dividing $2^3 \times 3^2 \times 5$ by each of the 9 divisors $1, 2, 3, 4, 5, 6, 8, 9$, and 10.)
So then there are $24 - 9 = 15$ different integer values of $36 - u$, and so there are exactly 15 different integer values of $u$ when $p = 3$.
Therefore, there are not 17 positive integer values of $u$ when $p = 3$.

If $p = 7$, then $40p^2 = 40 \times 7^2 = 2^3 \times 5 \times 7^2$, and so $2^3 \times 5 \times 7^2 = (m+8)(196-u)$.
Each divisor of $2^3 \times 5 \times 7^2$ is of the form $2^i \times 5^j \times 7^k$, for integers $0 \le i \le 3$, $0 \le j \le 1$, and $0 \le k \le 2$.
That is, there are $4 \times 2 \times 3 = 24$ different divisors of $2^3 \times 5 \times 7^2$.
Since $m+8 \ge 11$, then the divisors of $2^3 \times 5 \times 7^2$ that $m+8$ cannot equal are: $1, 2, 4, 5, 7, 8$, and 10.
Since there are 7 divisors which $m+8$ cannot equal, then there are 7 divisors that $196-u$ cannot equal. (These divisors can be determined by dividing $2^3 \times 5 \times 7^2$ by each of the 7 divisors $1, 2, 4, 5, 7, 8$, and 10. We also note that each of the remaining divisors that $196-u$ can equal, is less than 196, giving a positive integer value for $u$.)
So then there are $24 - 7 = 17$ different integer values of $196 - u$, and so there are exactly 17 different integer values of $u$ when $p = 7$.

For all remaining primes $p \ge 11$, we get $40p^2 = 2^3 \times 5 \times p^2$, and so $2^3 \times 5 \times p^2 = (m+8)(4p^2 - u)$.
Since $p \ne 2$ and $p \ne 5$, each divisor of $2^3 \times 5 \times p^2$ is of the form $2^i \times 5^j \times p^k$, for integers $0 \le i \le 3$, $0 \le j \le 1$, and $0 \le k \le 2$.
That is, there are $4 \times 2 \times 3 = 24$ different divisors of $2^3 \times 5 \times p^2$.
Since $m+8 \ge 11$ and $p \ge 11$, then the divisors of $2^3 \times 5 \times p^2$ that $m+8$ cannot equal are: $1, 2, 4, 5, 8$, and 10.
Since there are 6 divisors which $m+8$ cannot equal, then there are 6 divisors which $4p^2 - u$ cannot equal. (These divisors can be determined by dividing $2^3 \times 5 \times p^2$ by each of the 6 divisors $1, 2, 4, 5, 8$, and 10. We also note that each of the remaining divisors that $4p^2 - u$ can equal, is less than $4p^2$, giving a positive integer value for $u$.)
So then there are $24 - 6 = 18$ different integer values of $4p^2 - u$, and so there are exactly 18 different integer values of $u$ for all prime numbers $p \ge 11$.

Therefore, $p = 7$ is the only prime number for which there are exactly 17 positive integer values of $u$ for Koeller-rectangles with $n = 10$ and $r = \dfrac{u}{p^2}$.

The CENTRE for EDUCATION
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# *2016 Galois Contest*

**Wednesday, April 13, 2016**
**(in North America and South America)**

**Thursday, April 14, 2016**
**(outside of North America and South America)**

*Solutions*

1. (a) The first bucket contains 7 red discs.
Each bucket after the first contains 3 more red discs than the previous bucket.
Thus, the second bucket contains $7+3 = 10$ red discs, the third bucket contains $10+3 = 13$ red discs, and the fourth bucket contains $13 + 3 = 16$ red discs.

(b) *Solution 1*
Let $b$ represent the number of buckets after the first.
Since the first bucket contains 17 green discs and each bucket after the first contains 1 more green disc than the previous bucket, then there are $17 + b$ green discs inside the bucket $b$ after the first.
Since the first bucket contains 7 red discs and each bucket after the first contains 3 more red discs than the previous bucket, then there are $7+3b$ red discs inside the bucket $b$ after the first.
The number of green discs in a bucket is equal to the number of red discs inside the same bucket when $17 + b = 7 + 3b$ or when $2b = 10$, and so when $b = 5$.
Thus, there are an equal number of red discs and green discs in the bucket 5 after the first bucket, which is the $6^{th}$ bucket.

*Solution 2*
Using the fact that the first bucket contains 17 green discs and 7 red discs, and each bucket after the first contains 1 more green disc and 3 more red discs than the previous bucket, then we may summarize the number of green and red discs inside each bucket.

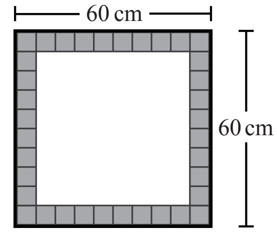
| Bucket Number | Number of green discs | Number of red discs |
|---|---|---|
| 1 | 17 | 7 |
| 2 | 18 | 10 |
| 3 | 19 | 13 |
| 4 | 20 | 16 |
| 5 | 21 | 19 |
| 6 | 22 | 22 |

Therefore, the $6^{th}$ bucket contains an equal number of red discs and green discs.
(Note that since the number of red discs is increasing by 3 each bucket and the number of green discs is increasing by 1 each bucket, then this is the only bucket in which the number of red discs and green discs will be equal.)

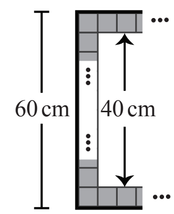
*Solution 3*
In the first bucket, there are 17 green discs and 7 red discs. Each bucket after the first contains 1 more green disc and 3 more red discs than the previous bucket.
Since there are 2 more red discs than green being put in, then the difference between the numbers of green and red discs will decrease by 2 for each bucket after the first.
Since the original difference is $17 - 7 = 10$, then it takes $10 \div 2 = 5$ more buckets to arrive at a bucket where the numbers of green and red discs will be equal.
Therefore, the $6^{th}$ bucket contains an equal number of red discs and green discs.

(c) As in part (b), there are $17 + b$ green discs and $7 + 3b$ red discs inside the bucket $b$ after the first.
The number of red discs in a bucket is equal to twice the number of green discs inside the same bucket when $7 + 3b = 2(17 + b)$ or when $7 + 3b = 34 + 2b$, and so when $b = 27$.
In the $27^{th}$ bucket after the first (the $28^{th}$ bucket), there are $17 + 27 = 44$ green discs and $7 + 3(27) = 88$ red discs. (We note that 88 is indeed twice 44.)
The total number of discs in this bucket is $44 + 88 = 132$.

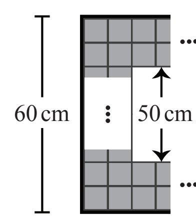
2. (a) A plate with 36 shaded squares has 10 shaded squares along each side of the plate, as shown.
We can see this from the diagram, or by considering a plate with $s$ squares along each side.
In this case, we can count $s$ squares on the top edge and $s$ squares on the bottom edge, plus $s - 2$ new squares on each of the left edge and the right edge. (The 2 corner squares on each of these edges are already counted.)
This means that there are $2s + 2(s - 2) = 4s - 4$ squares along the edges.
Here, we want $4s - 4 = 36$ or $4s = 40$, and so $s = 10$.
There are 10 squares along each side of the plate and the side length of the square plate is 60 cm, thus the side length of each of the shaded squares is $\frac{60}{10} = 6$ cm.




(b) Since the plate is a square, and there are an equal number of identical shaded squares along each edge of the plate, then the unshaded area in the centre of the plate is also a square.
The area of this unshaded square in the centre of the plate is $1600\,\text{cm}^2$, and so each of its sides has length $\sqrt{1600} = 40$ cm, as shown.
Consider the row of squares along the left edge of the plate.
Since the side length of the square plate is 60 cm and the side length of the inner square is 40 cm, then the sum of the side lengths of the two shaded corner squares is $60 - 40 = 20$ cm.
Therefore, each shaded corner square (and thus each shaded square) has side length $\frac{20}{2} = 10$ cm.



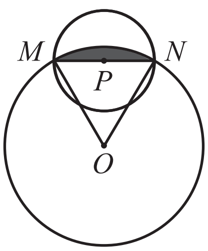

(c) Using the same argument as in part (b), the area of the unshaded square in the centre of the plate is $2500\,\text{cm}^2$, and so each of its sides has length $\sqrt{2500} = 50$ cm, as shown.
The side length of the square plate is 60 cm and so the sum of the side lengths of 4 shaded squares (2 stacked vertically in the top two rows and 2 stacked vertically in the bottom two rows) is $60 - 50 = 10$ cm.
Therefore, each of these shaded squares (and thus each shaded square on the plate) has side length $\frac{10}{4} = \frac{5}{2}$ cm.
The side length of the square plate is 60 cm and each shaded square has length $\frac{5}{2}$ cm, and so along an outside edge of the plate there are $60 \div \frac{5}{2} = 60 \times \frac{2}{5} = 12 \times 2 = 24$ shaded squares.
There are 2 rows that each contain 24 shaded squares along each of the top and bottom of the square, and 2 additional rows that each contain $24 - 4 = 20$ shaded squares along the left and right sides of the square.
That is, the total number of shaded squares on the plate is $4 \times 24 + 4 \times 20 = 96 + 80 = 176$.




3. (a) Triangle $ABC$ is equilateral with side length 6, and so $AB = BC = CA = 6$.
Since $D$ is the midpoint of $BC$, then $BD = DC = 3$.
In $\triangle ADC$, $\angle ADC = 90^\circ$ and so by the Pythagorean Theorem $AD^2 = AC^2 - DC^2$.
Therefore, $h^2 = 6^2 - 3^2 = 36 - 9 = 27$ and so $h = \sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$, since $h > 0$.

(b) The shaded region lies inside the circle and outside the hexagon and thus its area is determined by subtracting the area of the hexagon from the area of the circle.
First we find the area of the hexagon.
Each vertex of hexagon $EFGHIJ$ lies on the circle.
Since the circle has centre $O$ and radius 6, then $OE = OF = OG = OH = OI = OJ = 6$.
Each side length of the hexagon is also 6, and so the hexagon is formed by six congruent equilateral triangles with side length 6. (For example, $\triangle OGH$ is one these 6 triangles.)
Each of these triangles is congruent to $\triangle ABC$ from part (a) and thus has height $h = 3\sqrt{3}$.
The area of each of the six congruent triangles is $\frac{1}{2}(6)(3\sqrt{3}) = 9\sqrt{3}$.
Therefore, the area of hexagon $EFGHIJ$ is $6 \times 9\sqrt{3} = 54\sqrt{3}$.
The area of the circle with centre $O$ and radius 6 is $\pi(6)^2 = 36\pi$.
Finally, the area of the shaded region is $36\pi - 54\sqrt{3}$.

(c) Let the area of the shaded region that we are required to find be $A$.
Let the area of the shaded region in the diagram to the right be $S$.
We may determine $A$ by subtracting $S$ from the area of the semi-circle with centre $P$.




First we determine $S$.
Consider the circle with centre $O$. The shaded region having area $S$ lies inside sector $MON$ of this circle, but outside $\triangle MON$.
That is, $S$ is determined by subtracting the area of $\triangle MON$ from the area of sector $MON$.
In $\triangle MON$, $MN = ON = OM = r$ (since $ON$ and $OM$ are radii), and so the triangle is equilateral. Join $O$ to $P$.
Since $ON = OM$ and $P$ is the midpoint of $MN$, then $OP$ is the altitude (height) of $\triangle MON$ with base $MN$.
In $\triangle OPN$, $\angle OPN = 90^\circ$ and so by the Pythagorean Theorem $OP^2 = ON^2 - PN^2$.
Since $PN = \frac{1}{2}(MN) = \frac{1}{2}r$, $OP^2 = r^2 - (\frac{1}{2}r)^2 = r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$, and so $OP = \sqrt{\frac{3}{4}r^2} = \frac{\sqrt{3}}{2}r$.
Therefore, the area of $\triangle MON$ is $\frac{1}{2}(MN)(OP) = \frac{1}{2}(r)\left(\frac{\sqrt{3}}{2}r\right) = \frac{\sqrt{3}}{4}r^2$.
Next, we determine the area of sector $MON$.
Since $\triangle MON$ is equilateral, then $\angle MON = 60^\circ$.
Thus, the area of sector $MON$ is $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of the area of the circle with centre $O$ and radius $r$, or $\frac{1}{6}\pi r^2$.
Therefore, $S = \frac{1}{6}\pi r^2 - \frac{\sqrt{3}}{4}r^2$.
Finally, one-half of the area of the circle with centre $P$ and radius $PN = \frac{1}{2}r$ is $\frac{1}{2}\pi\left(\frac{1}{2}r\right)^2 = \frac{1}{8}\pi r^2$, and so $A = \frac{1}{8}\pi r^2 - S = \frac{1}{8}\pi r^2 - \left(\frac{1}{6}\pi r^2 - \frac{\sqrt{3}}{4}r^2\right) = \left(\frac{1}{8}\pi - \frac{1}{6}\pi + \frac{\sqrt{3}}{4}\right)r^2$.
Simplifying further, the exact area of the shaded region is $\frac{6\sqrt{3}-\pi}{24}r^2$.

4. (a) The prime factorization of 126 is $126 = 2^1 3^2 7^1$.
Given an input of $126 = 2^1 3^2 7^1$, the output from the Barbeau Process is

$$126\left(\frac{1}{2} + \frac{2}{3} + \frac{1}{7}\right) = 126\left(\frac{21+28+6}{42}\right) = 3(55) = 165\ .$$

(b) Given an input of $p^2 q$, the output from the Barbeau Process is $p^2 q\left(\frac{2}{p} + \frac{1}{q}\right) = 2pq + p^2$.
We are told that this output is equal to 135.
Since $135 = 3^3 5$, then $2pq + p^2 = 3^3 5$ or $p(2q + p) = 3^3 5$.

Since $p$ is a prime number that is a divisor of $3^3 5$, then $p = 3$ or $p = 5$.
If $p = 3$, then $3(2q + 3) = 3^3 5$ or $2q + 3 = 45$ and so $q = 21$.
However $q$ is a prime number and 21 is not a prime number, so $p \neq 3$.
If $p = 5$, then $5(2q + 5) = 3^3 5$ or $2q + 5 = 27$ and so $q = 11$.
Therefore, the only pair $(p, q)$ of different prime numbers that satisfies the given conditions is $(5, 11)$.

(c) *Solution 1*
Given an input of $2^a 3^b 5^c$, the output from the Barbeau Process is

$$2^a 3^b 5^c \left(\frac{a}{2} + \frac{b}{3} + \frac{c}{5}\right) = 2^a 3^b 5^c \left(\frac{15a + 10b + 6c}{30}\right) .$$

We are told that this output is equal to $4 \times 2^a 3^b 5^c$.
Comparing these gives $\dfrac{15a + 10b + 6c}{30} = 4$ or $15a + 10b + 6c = 120$.
Since $a, b, c$ are positive integers, and $15 \times 8 = 120$, $10 \times 12 = 120$, and $6 \times 20 = 120$, then $1 \le a \le 7, 1 \le b \le 11$, and $1 \le c \le 19$.
Further, $10b, 6c$ and 120 are divisible by 2, and so $15a = 120 - 10b - 6c$ is divisible by 2.
Therefore, $a$ is divisible by 2 and since $1 \le a \le 7$, then $a$ equals $2, 4$ or 6.
Similarly, $15a, 6c$, and 120 are divisible by 3, and so $10b$ is divisible by 3.
Therefore, $b$ is divisible by 3 and since $1 \le b \le 11$, then $b$ equals $3, 6$ or 9.
Finally, $15a, 10b$, and 120 are divisible by 5, and so $6c$ is divisible by 5.
Therefore, $c$ is divisible by 5 and since $1 \le c \le 19$, then $c$ equals $5, 10$ or 15.
Since $a \ge 2, b \ge 3$, and $c \ge 5$, then $15a \ge 15 \times 2 = 30, 10b \ge 10 \times 3 = 30$, and $6c \ge 6 \times 5 = 30$.
That is, each of the terms $15a, 10b, 6c$ is at least 30, and so each of the terms is at most $120 - 2(30) = 60$ (for example, $15a = 120 - 10b - 6c \le 120 - 30 - 30 = 60$).
Therefore, $15a \le 60$ and so $a$ is equal to 2 or 4, $10b \le 60$ and so $b$ is equal to 3 or 6, and $6c \le 60$ and so $c$ is equal to 5 or 10.
If $a = 2$ and $b = 3$, then $6c = 120 - 15(2) - 10(3) = 60$, and so $c = 10$.
If $a = 2$ and $b = 6$, then $6c = 120 - 15(2) - 10(6) = 30$, and so $c = 5$.
If $a = 4$ and $b = 3$, then $6c = 120 - 15(4) - 10(3) = 30$, and so $c = 5$.
If $a = 4$ and $b = 6$, then $6c = 120 - 15(4) - 10(6) = 0$, which is not possible.
Therefore, the triples $(a, b, c)$ of positive integers which satisfy the given conditions are $(2, 3, 10), (2, 6, 5)$, and $(4, 3, 5)$.

*Solution 2*
Given an input of $2^a 3^b 5^c$, the output from the Barbeau Process is

$$2^a 3^b 5^c \left(\frac{a}{2} + \frac{b}{3} + \frac{c}{5}\right) = 2^a 3^b 5^c \left(\frac{15a + 10b + 6c}{30}\right) .$$

We are told that this output is equal to $4 \times 2^a 3^b 5^c$.
Comparing these gives $\dfrac{15a + 10b + 6c}{30} = 4$ or $15a + 10b + 6c = 120$.
Since $10b$, $6c$ and 120 are divisible by 2 and $15a = 120 - 10b - 6c$, then $15a$ is divisible by 2 which means that $a$ is divisible by 2. Thus, we set $a = 2A$ for some positive integer $A$.
Since $15a$, $6c$ and 120 are divisible by 3 and $10b = 120 - 15a - 6c$, then $10b$ is divisible by 3 which means that $b$ is divisible by 3. Thus, we set $b = 3B$ for some positive integer $B$.
Since $15a$, $10b$ and 120 are divisible by 5 and $6c = 120 - 15a - 10b$, then $6c$ is divisible by 5

which means that $c$ is divisible by 5. Thus, we set $c = 5C$ for some positive integer $C$.
Therefore, the equation $15a + 10b + 6c = 120$ becomes $30A + 30B + 30C = 120$ or $A + B + C = 4$.
Since $A$, $B$ and $C$ are positive integers that add to 4, the possible values for $(A, B, C)$ are $(2, 1, 1)$, $(1, 2, 1)$, and $(1, 1, 2)$, because each is at least 1 which only leaves 1 "left over" to make up the total of 4.
Since $(a, b, c) = (2A, 3B, 5C)$, then the possible triples $(a, b, c)$ are $(4, 3, 5)$, $(2, 6, 5)$, and $(2, 3, 10)$.

(d) We proceed in a number of steps.

Step 1: Each exponent in the prime factorization must be a multiple of its prime
We show that, since the output is an integer multiple of the input, each exponent in the prime factorization must be an integer multiple of its corresponding prime.
This generalizes what we found in Solution 2 to part (c).
In other words, suppose that $n = p_1^{a_1} p_2^{a_2} p_3^{a_3} \dots p_k^{a_k}$ where $p_1, p_2, ..., p_k$ are different prime numbers and $a_1, a_2, ..., a_k$ are integers that are each at least 0.
By definition, the output of the Barbeau Process is $n\left(\dfrac{a_1}{p_1} + \dfrac{a_2}{p_2} + \dfrac{a_3}{p_3} + \cdots + \dfrac{a_k}{p_k}\right)$.
Note that we are allowing each of $a_1, a_2, \dots, a_k$ to be 0, which does not affect the value of the output.
Here, we are told that the output of the Barbeau process is $3n$.
Therefore, $n\left(\dfrac{a_1}{p_1} + \dfrac{a_2}{p_2} + \dfrac{a_3}{p_3} + \cdots + \dfrac{a_k}{p_k}\right) = 3n$ or $\dfrac{a_1}{p_1} + \dfrac{a_2}{p_2} + \dfrac{a_3}{p_3} + \cdots + \dfrac{a_k}{p_k} = 3$.
Rearranging, we obtain $\dfrac{a_1}{p_1} = 3 - \dfrac{a_2}{p_2} - \dfrac{a_3}{p_3} - \cdots - \dfrac{a_k}{p_k}$.
Multiplying both sides by $p_2 p_3 \cdots p_k$, we obtain

$$\frac{a_1 p_2 p_3 \cdots p_k}{p_1} = 3p_2 p_3 \cdots p_k - a_2 p_3 \cdots p_k - a_3 p_2 p_4 \cdots p_k - \cdots - a_k p_2 p_3 \cdots p_{k-1}$$

Since every term on the right side is an integer, then $\dfrac{a_1 p_2 p_3 \cdots p_k}{p_1}$ must be an integer.
Since the prime numbers $p_1, p_2, \dots, p_k$ are all distinct, then $a_1$ must be a multiple of $p_1$ to make this fraction actually equal an integer.
A similar argument will show that $a_2$ is a multiple of $p_2$, $a_3$ is a multiple of $p_3$, and so on.

Step 2: $n$ does not have any prime factor larger than 7
Suppose that $n$ does have a prime factor that is larger than 7.
In other words, suppose that $n$ has a prime factor $p_i$ that is at least 11.
In this case, the factor $p_i^{a_i}$ in the prime factorization of $n$ is at least $11^{11}$ since $p_i \geq 11$ and $a_i$ is a multiple of $p_i$ and so must also be at least 11.
In this case, $n \geq 11^{11}$.
But $n$ is restricted to be less than $10^{10}$, so this is not possible.
Therefore, $n$ does not have any prime factor larger than 7.

Step 3: Simplifying the algebra
Combining Steps 1 and 2, $n$ must be of the form $n = 2^a 3^b 5^c 7^d$ for some non-negative integers $a$, $b$, $c$, and $d$ that are multiples of 2, 3, 5, and 7, respectively.
As in Solution 2 to part (c), we set $a = 2A$, $b = 3B$, $c = 5C$, and $d = 7D$ for some non-negative integers $A, B, C, D$.
Therefore, the equation $\dfrac{a_1}{p_1} + \dfrac{a_2}{p_2} + \dfrac{a_3}{p_3} + \cdots + \dfrac{a_k}{p_k} = 3$ becomes $\dfrac{a}{2} + \dfrac{b}{3} + \dfrac{c}{5} + \dfrac{d}{7} = 3$, which

becomes $A+B+C+D=3$.
Therefore, we want to find all of the non-negative integer solutions to $A+B+C+D=3$ which give $n=2^{2A}3^{3B}5^{5C}7^{7D}<10^{10}$.

Step 4: Restrictions on $A,B,C,D$
Since $A+B+C+D=3$ and each of $A,B,C,D$ is non-negative, then each of $A,B,C,D$ is at most 3.
If $D=2$ or $D=3$, then $n$ is divisible by $7^{14}$ or $7^{21}$, each of which is larger than $10^{10}$.
Therefore, $D\leq 1$.
If $C=3$, then $n$ is divisible by $5^{15}$, which is larger than $10^{10}$.
Therefore, $C\leq 2$.

Step 5: Determining the values of $n$
Since $A+B+C+D=3$ and each of $A,B,C,D$ is non-negative, then $A,B,C,D$ could be (i) $3,0,0,0$ in some order, or (ii) $2,1,0,0$ in some order, or (iii) $1,1,1,0$, in some order. (If one value is 3, the rest must be 0. If one value is 2, then there must be one 1 and two 0s. If no value is 2, then there must be three 1s and one 0.)
Using the facts that $C\leq 2$ and $D\leq 1$, we enumerate the possibilities:

| Category | $A$ | $B$ | $C$ | $D$ | $n$ | Less than $10^{10}$? |
|---|---|---|---|---|---|---|
| (i) | 3 | 0 | 0 | 0 | $2^6$ | Yes |
| (i) | 0 | 3 | 0 | 0 | $3^9$ | Yes |
| (ii) | 2 | 1 | 0 | 0 | $2^4 3^3$ | Yes |
| (ii) | 2 | 0 | 1 | 0 | $2^4 5^5$ | Yes |
| (ii) | 2 | 0 | 0 | 1 | $2^4 7^7$ | Yes |
| (ii) | 1 | 2 | 0 | 0 | $2^2 3^6$ | Yes |
| (ii) | 0 | 2 | 1 | 0 | $3^6 5^5$ | Yes |
| (ii) | 0 | 2 | 0 | 1 | $3^6 7^7$ | Yes |
| (ii) | 1 | 0 | 2 | 0 | $2^2 5^{10}$ | Yes |
| (ii) | 0 | 1 | 2 | 0 | $3^3 5^{10}$ | Yes |
| (ii) | 0 | 0 | 2 | 1 | $5^{10} 7^7$ | No |
| (iii) | 1 | 1 | 1 | 0 | $2^2 3^3 5^5$ | Yes |
| (iii) | 1 | 1 | 0 | 1 | $2^2 3^3 7^7$ | Yes |
| (iii) | 1 | 0 | 1 | 1 | $2^2 5^5 7^7$ | No |
| (iii) | 0 | 1 | 1 | 1 | $3^3 5^5 7^7$ | No |

In each case, we can check whether the possible value of $n$ is smaller than $10^{10}$ using a calculator. (Which of these calculations can you "reason" through without using a calculator?)

In summary, the possible values of $n$ are

$$2^6, 3^9, 2^4 3^3, 2^4 5^5, 2^4 7^7, 2^2 3^6, 3^6 5^5, 3^6 7^7, 2^2 5^{10}, 3^3 5^{10}, 2^2 3^3 5^5, 2^2 3^3 7^7$$

The CENTRE for EDUCATION
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# *2015 Galois Contest*

**Thursday, April 16, 2015**
**(in North America and South America)**

**Friday, April 17, 2015**
**(outside of North America and South America)**

*Solutions*

1. (a) The $x$-intercept of a line occurs at the point whose $y$-coordinate is 0.
Substituting $y = 0$ into the equation of line 1, we get $0 = 2x + 6$ or $2x = -6$ and so $x = -3$.
Line 1 has $x$-intercept $-3$. (Point $P$ has coordinates $(-3, 0)$.)

(b) *Solution 1*
The equation of line 2 with slope $-3$ and $y$-intercept $b$ is $y = -3x + b$.
Line 2 passes through $Q(3, 12)$ and so $x = 3$ and $y = 12$ satisfies the equation of line 2.
Substituting, we get $12 = -3(3) + b$ or $12 = -9 + b$ and so $b = 21$.
The equation of line 2 is $y = -3x + 21$.

*Solution 2*
A line with slope $m$ and passing through the point $(x_1, y_1)$ has equation $y-y_1 = m(x-x_1)$.
Since line 2 has slope $m = -3$ and passes through $Q(3, 12)$, the slope-point equation of line 2 is $y - 12 = -3(x - 3)$.

(c) We find the coordinates of point $R$ by substituting $y = 0$ into the equation of line 2.
This gives $0 = -3x + 21$ or $3x = 21$ and so $x = 7$.
That is, line 2 has $x$-intercept 7 (point $R$ has coordinates $(7, 0)$).
If we let the base of $\triangle PQR$ be side $PR$, then the height of the triangle is the vertical distance from $Q$ to $PR$.
Since $Q$ has $y$-coordinate 12, then this height is 12.
The $x$-coordinate of $P$ is $-3$ and the $x$-coordinate of $R$ is 7. (Both $y$-coordinates are 0.)
Therefore, $PR$ has length $7 - (-3) = 10$.
Finally, the area of $\triangle PQR$ is $\frac{1}{2} \times 10 \times 12 = 60$.

2. (a) The total number of students at School A is the sum of the number of students who received a ride and the number of students who did not, or $330 + 420 = 750$.
Since 330 of 750 students received a ride, the percentage of students at School A who received a ride is $\frac{330}{750} \times 100\% = 0.44 \times 100\% = 44\%$.

(b) *Solution 1*
At school B, 30% of 240 students or $\frac{30}{100} \times 240 = \frac{7200}{100} = 72$ students received a ride.
If 50% or $\frac{1}{2}$ of the students in School B were to receive a ride, then $\frac{1}{2} \times 240 = 120$ students would get a ride.
Therefore, $120-72 = 48$ more students needed to receive a ride so that 50% of the students in School B got a ride.

*Solution 2*
As a percent, the difference between 50% of students receiving a ride and the 30% of students who did receive a ride is 20%.
Therefore, 20% of 240 students or $\frac{20}{100} \times 240 = \frac{4800}{100} = 48$ additional students would need to receive a ride so that 50% of students in School B got a ride.

(c) *Solution 1*
At school C, 45% of 200 students or $\frac{45}{100} \times 200 = \frac{9000}{100} = 90$ students received a ride.
At school D, $x$% of 300 students or $\frac{x}{100} \times 300 = \frac{300x}{100} = 3x$ students received a ride.
The total number of students at School C and School D is $200 + 300 = 500$.
The total number of students receiving a ride at School C and School D is $90 + 3x$.
Since 57.6% of the combined group of students from the two schools received a ride, then $\frac{90+3x}{500} = \frac{57.6}{100}$.
Multiplying both sides of this equation by 500, we get $90+3x = 57.6 \times 5$ or $90+3x = 288$ or $3x = 198$ and so $x = 66$.

*Solution 2*

At school C, 45% of 200 students or $\frac{45}{100} \times 200 = \frac{9000}{100} = 90$ students received a ride.

The total number of students at School C and School D is $200 + 300 = 500$.

Since 57.6% of the combined group of students from the two schools received a ride, then $\frac{57.6}{100} \times 500 = \frac{28800}{100} = 288$ students received a ride.

Out of the 288 students who received a ride, 90 students were from School C and so the remaining $288 - 90 = 198$ students were from School D.

Since there are 300 students at School D, the percentage of students receiving a ride is $\frac{198}{300} \times 100\% = 0.66 \times 100\% = 66\%$.

Therefore, the value of $x$ is 66.

(d) At school E, $n\%$ of 200 students or $\frac{n}{100} \times 200 = \frac{200n}{100} = 2n$ students received a ride.

At school F, $2n\%$ of 250 students or $\frac{2n}{100} \times 250 = \frac{500n}{100} = 5n$ students received a ride.

The total number of students at School E and School F is $200 + 250 = 450$.

The total number of students receiving a ride at School E and School F is $2n + 5n = 7n$.

Between 55% and 60% of the 450 students from the two schools received a ride.

Since 55% of 450 is 247.5 and 60% of 450 is 270, then $7n > 247.5$ and $7n < 270$.

Solving $7n > 247.5$ we get $n > 35.35$, after rounding, and $7n < 270$ gives $n < 38.57$, after rounding.

Since $n$ is a positive integer and $n > 35.35$ and $n < 38.57$, then the possible values of $n$ are $36, 37$ and $38$.

3. (a) Since 5 is an odd integer, then $n$ must be an odd integer for the sum $n + 5$ to be an even integer.

(If $n$ was an even integer, then $n + 5$ would be the sum of an even integer and an odd integer, which is an odd integer.)

(b) We first note that the product of an even integer and any other integers, even or odd, is always an even integer.

Let $N = cd(c + d)$.

If $c$ or $d$ is an even integer (or both $c$ and $d$ are even integers), then $N$ is the product of an even integer and some other integers and thus is even.

The only remaining possibility is that both $c$ and $d$ are odd integers.

If $c$ and $d$ are odd integers, then the sum $c + d$ is an even integer.

In this case, $N$ is again the product of an even integer and some other integers and so it is an even integer.

Therefore, for any integers $c$ and $d$, $cd(c + d)$ is always an even integer.

(c) Since $e$ and $f$ are positive integers so that $ef = 300$, then we may begin by determining the factor pairs of positive integers whose product is 300.

Written as ordered pairs $(x, y)$ with $x < y$, these are:

$$(1, 300), (2, 150), (3, 100), (4, 75), (5, 60), (6, 50), (10, 30), (12, 25), (15, 20).$$

It is also required that the sum $e + f$ be odd and so exactly one of $e$ or $f$ must be odd.

Therefore, the factor pairs whose sum is odd are:

$$(1, 300), (3, 100), (4, 75), (5, 60), (12, 25), (15, 20).$$

There are 6 ordered pairs $(e, f)$ satisfying the given conditions.

(d) Since both $m$ and $n$ are positive integers, then $2n > 1$ and so $2n + m > m + 1$.
Let $a = m + 1$ and $b = 2n + m$ or $a = 2n + m$ and $b = m + 1$ so that $ab = 9000$.
We must first determine all factor pairs $(a, b)$ of positive integers whose product is 9000.
We begin by considering the parity (whether each is even or odd) of the factors $a$ and $b$.

Since 2 is even, then $2n$ is even for all positive integers $n$.
If $m$ is even then $2n + m$ is even since the sum of two even integers is even.
However if $m$ is even, then $m + 1$ is odd since the sum of an even integer and an odd integer is odd.
That is, if $m$ is even, then $a$ is odd and $b$ is even or $a$ is even and $b$ is odd.
We say that the factors $a$ and $b$ have *different parity* since one is even and one is odd.

If $m$ is odd then $2n + m$ is odd. If $m$ is odd then $m + 1$ is even.
That is, if $m$ is odd, then $a$ is even and $b$ is odd or $a$ is odd and $b$ is even and so the factors $a$ and $b$ have different parity for all possible values of $m$.

Now we are searching for all factor pairs $(a, b)$ of positive integers whose product is 9000 with $a$ and $b$ having different parity.
Written as a product of its prime factors, $9000 = 2^3 \times 3^2 \times 5^3$ and so $ab = 2^3 \times 3^2 \times 5^3$.
Since exactly one of $a$ or $b$ is odd, then one of them does not have a factor of 2 and so the other must have all factors of 2.
That is, either $a = 2^3 r = 8r$ and $b = s$, or $a = r$ and $b = 8s$ for positive integers $r$ and $s$.
In both cases, $ab = 8rs = 9000$ and so $rs = \frac{9000}{8} = 1125 = 3^2 5^3$.
We now determine all factor pairs $(r, s)$ of positive integers whose product is 1125.
These are $(r, s) = (1, 1125), (3, 375), (5, 225), (9, 125), (15, 75), (25, 45)$.
Therefore $(a, b) = (8r, s) = (8, 1125), (24, 375), (40, 225), (72, 125), (120, 75), (200, 45)$, or $(a, b) = (r, 8s) = (1, 9000), (3, 3000), (5, 1800), (9, 1000), (15, 600), (25, 360)$.
Since $2n + m > m + 1 > 1$, then the pair $(1, 9000)$ is not possible.
This leaves 11 factor pairs $(a, b)$ such that $ab = 9000$ with $a$ and $b$ having different parity.
Each of these 11 factor pairs $(a, b)$ gives an ordered pair $(m, n)$.
To see this, let $m + 1$ equal the smaller of $a$ and $b$, and let $2n + m$ equal the larger (since $2n + m > m + 1$).
For example when $(a, b) = (8, 1125)$, then $m+1 = 8$ or $m = 7$ and so $2n+m = 2n+7 = 1125$ or $2n = 1118$ or $n = 559$.
That is, the factor pair $(a, b) = (8, 1125)$ corresponds to the ordered pair $(m, n) = (7, 559)$ so that $(m + 1)(2n + m) = 9000$.
Each of the 11 pairs $(a, b)$ gives an ordered pair $(m, n)$ such that $(m + 1)(2n + m) = 9000$.
We determine the corresponding ordered pair $(m, n)$ for each $(a, b)$ in the table below (although this work is not necessary since we were only asked for the number of ordered pairs).

| $(a, b)$ | $m + 1$ | $2n + m$ | $(m, n)$ |
|---|---|---|---|
| $(8, 1125)$ | 8 | 1125 | $(7, 559)$ |
| $(24, 375)$ | 24 | 375 | $(23, 176)$ |
| $(40, 225)$ | 40 | 225 | $(39, 93)$ |
| $(72, 125)$ | 72 | 125 | $(71, 27)$ |
| $(120, 75)$ | 75 | 120 | $(74, 23)$ |
| $(200, 45)$ | 45 | 200 | $(44, 78)$ |

| $(a, b)$ | $m + 1$ | $2n + m$ | $(m, n)$ |
|---|---|---|---|
| $(3, 3000)$ | 3 | 3000 | $(2, 1499)$ |
| $(5, 1800)$ | 5 | 1800 | $(4, 898)$ |
| $(9, 1000)$ | 9 | 1000 | $(8, 496)$ |
| $(15, 600)$ | 15 | 600 | $(14, 293)$ |
| $(25, 360)$ | 25 | 360 | $(24, 168)$ |

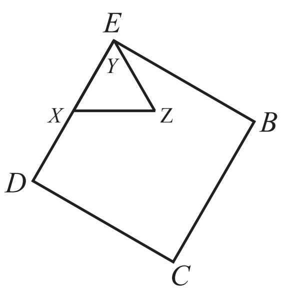
There are 11 ordered pairs $(m, n)$ of positive integers satisfying $(m + 1)(2n + m) = 9000$.

4. (a) Since $EXD$ is a straight line, then $\angle YXE + \angle YXZ = 180^\circ$.
Since $\triangle XYZ$ is an equilateral triangle, then $\angle YXZ = 60^\circ$.
Thus, $\angle YXE = 180^\circ - 60^\circ = 120^\circ$.

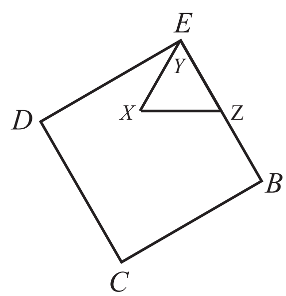
(b) We make a table to track the position of the square as it rotates about the triangle:

| After Move # | Coinciding Vertices | Second Vertex of $\triangle XYZ$ on Side of Square | Centre of Rotation of Next Move | Angle of Rotation of Next Move |
|---|---|---|---|---|
| Initial | $D$ and $Z$ | $X$ on $DE$ | $X$ | $120^\circ$ |
| 1 | $E$ and $Y$ | $X$ on $DE$ | $Y$ | $30^\circ$ |
| 2 | $E$ and $Y$ | $Z$ on $EB$ | $Z$ | $120^\circ$ |
| 3 | $B$ and $X$ | $Z$ on $EB$ | $X$ | $30^\circ$ |
| 4 | $B$ and $X$ | $Y$ on $BC$ | $Y$ | $120^\circ$ |
| 5 | $C$ and $Z$ | $Y$ on $BC$ | $Z$ | $30^\circ$ |
| 6 | $C$ and $Z$ | $X$ on $CD$ | $X$ | $120^\circ$ |
| 7 | $D$ and $Y$ | $X$ on $CD$ | $Y$ | $30^\circ$ |
| 8 | $D$ and $Y$ | $Z$ on $DE$ | $Z$ | $120^\circ$ |

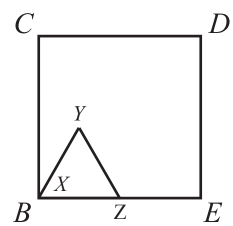
The information in this table comes from the diagrams here:



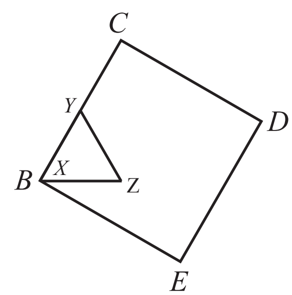
End of First Move



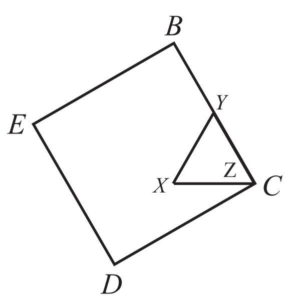
End of Second Move



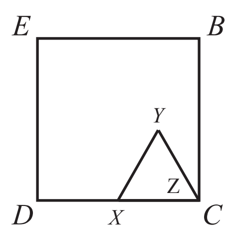
End of Third Move



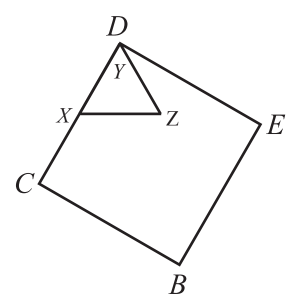
End of Fourth Move



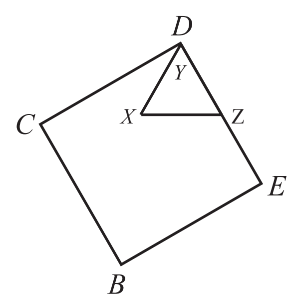
End of Fifth Move



End of Sixth Move



End of Seventh Move



End of Eighth Move

Each angle of rotation is either $180^\circ - 60^\circ = 120^\circ$ or $90^\circ - 60^\circ = 30^\circ$.
Therefore, $D$ next coincides with a vertex of the triangle after 7 moves.
(We continued the table through the 8th move to be more useful in part (c).)

(c) After 0 moves, the square has $D$ at $Z$ and $X$ on $DE$ and the next rotation is a rotation of $120^\circ$ about $X$.
After 8 moves, the square has $D$ at $Y$ and $Z$ on $DE$ and the next rotation is a rotation of $120^\circ$ about $Z$.
Therefore, through 8 moves, the net change of position of the square was moving clockwise around 2 sides of the triangle. After these 8 moves, the square is in a similar position, relative to the triangle, to its initial position: $D$ is at a vertex of the triangle and $DE$ lies along a side of the triangle.
Starting from after the 8th move, the square starts in this similar position, and so 8 more moves will take the square again to a similar position. The net change will again be that the square has moved clockwise around 2 sides of the triangle. (We note from part (b) that after 8 moves is the first time (after the original position) that vertex $D$ coincides with a vertex of the triangle at the same relative position.)
Thus, after 16 moves, the square will have $D$ at $X$ and $Y$ on $DE$ and the next rotation is a rotation of $120^\circ$ about $Y$. (This is the second time that vertex $D$ coincides with a vertex of the triangle at the same relative position.)
Using similar reasoning, after 24 moves, the square will have $D$ at $Z$ and $X$ on $DE$ and the next rotation is a rotation of $120^\circ$ about $X$, which is the original position.
Therefore, we need to determine the total distance travelled by $E$ through these 24 moves.
This total distance equals 3 times the distance travelled by $E$ through the first 8 moves.
This is because the relative position of the square with respect to the triangle ($D$ at a vertex of the triangle, $DE$ along a side of the triangle) is the same after 8 moves as it was after 0 moves, so the relative sequence of rotations undergone by $E$ from after move 8 to after move 16, and from after move 16 to after move 24, will be the same as they were for the first 8 moves.

Since $EBCD$ has side length 2, then $ED = EB = 2$.
Also, the distance from $E$ to the midpoint of $ED$ is 1, as is the distance from $E$ to the midpoint of $EB$.
Since $EBCD$ has side length 2, then $EC = 2\sqrt{2}$.
Also, the distance from $E$ to the midpoint of each of $DC$ and $BC$ is $\sqrt{2^2+1^2} = \sqrt{5}$, by the Pythagorean Theorem.
We make a chart of the rotation undergone by $E$ through each of the first 8 moves:

| Move # | Centre of Rotation | Distance of Centre from $E$ | Angle of Rotation of Square |
|---|---|---|---|
| 1 | $X$ | 1 | $120^\circ$ |
| 2 | $Y$ | 0 | $30^\circ$ |
| 3 | $Z$ | 1 | $120^\circ$ |
| 4 | $X$ | 2 | $30^\circ$ |
| 5 | $Y$ | $\sqrt{5}$ | $120^\circ$ |
| 6 | $Z$ | $2\sqrt{2}$ | $30^\circ$ |
| 7 | $X$ | $\sqrt{5}$ | $120^\circ$ |
| 8 | $Y$ | 2 | $30^\circ$ |

During each rotation, each point on the square (except for the point of the square in contact with the centre of rotation) is rotated about the centre of rotation through the angle of rotation of the square.
Thus, during each rotation, the distance travelled by $E$ is the fraction of the circumference of the whole circle given by the angle of rotation as compared to $360^\circ$.

Therefore, the distance travelled by $E$ through the first 8 moves is

$$\frac{120^\circ}{360^\circ}2\pi(1)+\frac{30^\circ}{360^\circ}2\pi(0)+\frac{120^\circ}{360^\circ}2\pi(1)+\frac{30^\circ}{360^\circ}2\pi(2)+$$

$$\frac{120^\circ}{360^\circ}2\pi(\sqrt{5})+\frac{30^\circ}{360^\circ}2\pi(2\sqrt{2})+\frac{120^\circ}{360^\circ}2\pi(\sqrt{5})+\frac{30^\circ}{360^\circ}2\pi(2)$$

which simplifies to

$$\tfrac{2}{3}\pi+0+\tfrac{2}{3}\pi+\tfrac{1}{3}\pi+\tfrac{2}{3}\sqrt{5}\pi+\tfrac{1}{3}\sqrt{2}\pi+\tfrac{2}{3}\sqrt{5}\pi+\tfrac{1}{3}\pi$$

or

$$2\pi+\tfrac{4}{3}\sqrt{5}\pi+\tfrac{1}{3}\sqrt{2}\pi$$

Therefore, when the square returns to its initial position after 24 moves, $E$ has travelled 3 times this far, or a total distance of $6\pi+4\sqrt{5}\pi+\sqrt{2}\pi$.

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# *2014 Galois Contest*

**Wednesday, April 16, 2014**
**(in North America and South America)**

**Thursday, April 17, 2014**
**(outside of North America and South America)**

*Solutions*

1. (a) The three angles shown in the pie chart are $(2x)^\circ$, $(3x)^\circ$ and $90^\circ$.
Since these three angles form a complete circle, then $(2x)^\circ+(3x)^\circ+90^\circ = 360^\circ$, or $5x = 270$ and so $x = 54$.

(b) The ratio of the number of bronze medals to the number of silver medals to the number of gold medals is equal to the ratio of the sector angles, $(2x)^\circ$ to $(3x)^\circ$ to $90^\circ$, respectively.
Since $x = 54$, then the required ratio is $2(54) : 3(54) : 90$ or $108 : 162 : 90$.
Dividing each term by 18 the ratio becomes $6 : 9 : 5$, which is written in lowest terms.

(c) Since the ratio of the number of bronze to silver to gold medals is $6 : 9 : 5$, let the number of bronze, silver and gold medals in the trophy case be $6x$, $9x$ and $5x$ respectively.
Since the total number of medals in the trophy case is 80, then $6x + 9x + 5x = 80$ or $20x = 80$ and so $x = 4$.
Thus, there are $6 \times 4 = 24$ bronze medals, $9 \times 4 = 36$ silver medals, and $5 \times 4 = 20$ gold medals in the trophy case.

(d) The trophy case begins with 24, 36 and 20 bronze, silver and gold medals, respectively.
Recall that the number of medals is in the ratio $6 : 9 : 5$.
For the ratio of the final number of medals to remain unchanged, we claim that the number of medals added by the teacher must also be in the ratio $6 : 9 : 5$.
(We will prove this claim is true at the end of the solution.)
Since $6 : 9 : 5$ is in lowest terms, the smallest number of medals that the teacher could have added is 6 bronze, 9 silver and 5 gold.
Therefore, the smallest number of medals that could now be in the trophy case is $80 + 6 + 9 + 5$ or 100 medals.
We note that the number of bronze, silver and gold medals is now 30, 45 and 25, which is still in the ratio $6 : 9 : 5$.

Proof of Claim: Let the number of bronze, silver and gold medals added be $b, s$ and $g$ respectively. When these are added to the existing medals, the number of bronze, silver and gold medals becomes $(24 + b)$, $(36 + s)$ and $(20 + g)$. The claim is that for the new ratio, $(24 + b) : (36 + s) : (20 + g)$, to remain unchanged (that is, to equal $24 : 36 : 20$), then $b : s : g$ must equal $6 : 9 : 5$. If $(24 + b) : (36 + s) : (20 + g) = 24 : 36 : 20$, then $\dfrac{36+s}{24+b} = \dfrac{36}{24}$ and $\dfrac{20+g}{36+s} = \dfrac{20}{36}$. From the first equation, $24(36) + 24s = 36(24) + 36b$ and so $24s = 36b$ or $\dfrac{s}{b} = \dfrac{36}{24} = \dfrac{9}{6}$. Similarly, from the second equation we can show that $\dfrac{g}{s} = \dfrac{5}{9}$. Thus, $b : s : g = 6 : 9 : 5$ as claimed.

2. (a) *Solution 1*
Each of the 200 passengers who checked exactly one bag is charged \$20 to do so.
Each of the 45 passengers who checked exactly two bags is charged \$20 for the first bag plus \$7 for the second bag, or \$27 in total for the two bags.
Thus, the total charge for all checked bags is $(200 \times \$20) + (45 \times \$27)$ or \$5215.

*Solution 2*
All 245 passengers checked at least one bag.
They were each charged \$20 to check this first bag.
The 45 passengers who checked a second bag were each charged an additional \$7 to do so.
Thus, the total charge for all checked bags is $(245 \times \$20) + (45 \times \$7)$ or \$5215.

(b) *Solution 1*

Since each of the 245 passengers checked at least one bag, then the total baggage fees collected for the first bag is $245 \times \$20 = \$4900$.
A total of $\$5173 - \$4900 = \$273$ in baggage fees remains to be collected.
Since all passengers checked exactly one or exactly two bags, then the remaining \$273 in baggage fees is collected from the passengers who checked a second bag.
The cost to check a second bag is \$7.
Thus, the number of passengers who checked exactly two bags is $\frac{273}{7} = 39$.

*Solution 2*

Let the number of passengers who checked exactly one bag be $n$.
Since there were 245 passengers on board, and each checked exactly one bag or exactly two bags, then the remaining $(245 - n)$ passengers checked exactly two bags.
Each of the $n$ passengers who checked exactly one bag is charged \$20 to do so.
Each of the $(245-n)$ passengers who checked exactly two bags is charged \$20 for the first bag plus \$7 for the second bag, or \$27 in total for the two bags.
Since the total charge for all checked bags is \$5173, then $(n \times 20) + ((245-n) \times 27) = 5173$.
Solving, $20n + 6615 - 27n = 5173$ or $1442 = 7n$, and so $n = 206$.
That is, $245 - n = 245 - 206 = 39$ passengers checked exactly two bags.

*Solution 3*

All 245 passengers checked at least one bag.
They were each charged \$20 to check this first bag.
Let the number of passengers who checked exactly two bags be $m$.
The $m$ passengers who checked a second bag were each charged an additional \$7 to do so.
Thus, the total charge for all checked bags is $(245 \times \$20) + (m \times \$7)$, so $4900 + 7m = 5173$ or $7m = 273$ and $m = 39$. Therefore, 39 passengers checked exactly two bags.

(c) Assume that each of the 245 passengers checked at most two bags.
The charge to check exactly two bags is \$27, so in this case the total baggage fees collected could not have exceeded $245 \times \$27 = \$6615$.
Since \$6825 (which is greater than \$6615) was collected in baggage fees on this third flight, then at least one passenger must have checked at least three bags.
(It is possible to have baggage fees total \$6825 if 215 passengers check exactly 2 bags, and 30 passengers check exactly 3 bags.
Here, the total baggage fees collected would be $(215 \times \$27) + (30 \times \$34) = \$6825$.)

(d) Assume that each passenger (of which there are at most 245), checked at most two bags.
Let the number of passengers who checked exactly one bag be $a$ and the number of passengers who checked exactly two bags be $b$.
While it may be the case that there are passengers who checked no bags, they don't contribute to the \$142 collected and so we may ignore them.
Each of the $a$ passengers who checked exactly one bag is charged \$20, while each of the $b$ passengers who checked exactly two bags is charged \$27.
Since the total fees collected was \$142, then $20a + 27b = 142$.
Solving for $a$ we get, $a = \frac{142 - 27b}{20}$ and since both $a$ and $b$ must be non-negative integers, we systematically try values for $b$ in the table below to see if any gives a non-negative integer value for $a$. Since $27b$ is at most 142, but $27(5) = 135$ and $27(6) = 162$, then $b$ is at most 5 ($27b$ is larger than 162 when $b$ is larger than 6).

| Value of $b$ | Calculation of $a$ |
|---|---|
| 0 | $a = \frac{142-27(0)}{20} = 7.1$ |
| 1 | $a = \frac{142-27(1)}{20} = 5.75$ |
| 2 | $a = \frac{142-27(2)}{20} = 4.4$ |
| 3 | $a = \frac{142-27(3)}{20} = 3.05$ |
| 4 | $a = \frac{142-27(4)}{20} = 1.7$ |
| 5 | $a = \frac{142-27(5)}{20} = 0.35$ |

Each of the values of $a$ calculated above is not a non-negative integer.
Thus, there are no non-negative integers $a$ and $b$ that make $20a + 27b = 142$.
Therefore, there is no combination of passengers who check at most two bags such that the baggage fees collected total \$142.
Therefore, there must be at least one passenger who checked at least 3 bags.
(It is possible to have baggage fees total \$142 if 4 passengers check exactly 2 bags, and 1 passenger checks exactly 3 bags. Here, the total baggage fees collected would be $(4 \times \$27) + (1 \times \$34) = \$142$.)

3. (a) *Solution 1*
The cards numbered 1 and 7 are in Emily's set and since $1 + 7 = 8$, then we have found one pair that she can select.
To maintain a sum of 8, we must decrease 7 by 1 when increasing 1 by 1.
That is, the cards numbered 2 and 6 have a sum of 8 and both are in Emily's set so we have found a second pair that she can select.
Repeating the process again, we get the third pair of cards numbered 3 and 5.
The 3 pairs that Emily can select from her set, each having a sum of 8, are $(1, 7), (2, 6)$ and $(3, 5)$.
(Note that an attempt to repeat the process one more time gives $(4, 4)$, however there is only 1 card numbered 4 in Emily's set.)

*Solution 2*
If we let the smaller card be numbered $a$ and the larger card be numbered $b$, then $a+b = 8$ or $b = 8 - a$.
Since $a < b$, then $a < 8 - a$ or $2a < 8$ and so $a < 4$.
Since $a \geq 1$, then the only possible values for $a$ are $1, 2, 3$.
Thus, the three pairs having a sum of 8 are $(1, 7)$, $(2, 6)$ and $(3, 5)$.

(b) *Solution 1*
As in part (a), we first attempt to use the card numbered 1 (the smallest numbered card in the set) to form a pair whose sum is 13.
However, the largest number that we can select to pair 1 with is 10, and this gives a sum of $1 + 10 = 11$ which is less than the required sum of 13.

In a similar manner, begin by first selecting the largest numbered card in the set, 10.
When paired with the card numbered 10, the card numbered 3 gives a sum of 13.
Thus, $(3, 10)$ is one pair that Silas may select.
As in part (a), if we increase the lower numbered card by 1 and decrease the higher numbered card by 1, then we maintain a constant sum, 13.
This gives the pairs $(4, 9), (5, 8), (6, 7)$.
We can not continue this process once we reach $(6, 7)$ since the lower numbered card would then become the higher (we would have the pair $(7, 6)$) and we need pairs $(a, b)$ where $a < b$.
Thus, there are exactly 4 pairs, $(3, 10), (4, 9), (5, 8), (6, 7)$, that Silas may select.

*Solution 2*
If we let the smaller card be numbered $a$ and the larger card be numbered $b$, then $a+b = 13$ or $b = 13 - a$.
Since $a < b$, then $a < 13 - a$ or $2a < 13$ and so $a < 6.5$.
Also, since $b \leq 10$, then $13 - a \leq 10$ or $3 \leq a$.
Since $3 \leq a < 6.5$, then the only possible values for $a$ are $3, 4, 5, 6$.
Thus, there are exactly 4 pairs, $(3, 10), (4, 9), (5, 8), (6, 7)$, that Silas may select.

(c) If $k \leq 50$, then the maximum sum of any pair is $49 + 50 = 99$.
Therefore to achieve a sum of 100, it must be the case that $k > 50$.
If $k = 51$, then the pair $(49, 51)$ has sum 100.
However, this is the only pair having sum 100.
If $k = 52$, then the pairs $(49, 51)$ and $(48, 52)$ both have sum 100, but these are the only 2 pairs that sum to 100.
Each time we increase $k$ by 1 starting from 51, we obtain one additional pair whose sum is 100, because there is an additional value of $b$ (the larger numbered card in the pair) that can be used.
If $k = 51 + 9 = 60$, then we have the following ten pairs whose sum is 100:
$(49, 51)$, $(48, 52)$, $(47, 53)$, $(46, 54)$, $(45, 55)$, $(44, 56)$, $(43, 57)$, $(42, 58)$, $(41, 59)$, $(40, 60)$.
If we increase $k$ again to $k = 61$, then an additional pair, $(39, 61)$, increases the number of pairs whose sum is 100 to 11.
Thus, Daniel must have a set of $k = 60$ cards numbered consecutively from 1 to 60.

(d) We show that the possible values of $S$ are $S = 67, 68, 84, 85$.

Suppose that $S$ is odd; that is, $S = 2k + 1$ for some integer $k \geq 0$.
The pairs of positive integers $(a, b)$ with $a < b$ and $a + b = S$ are

$$(1, 2k), (2, 2k - 1), (3, 2k - 2), \ldots, (k - 1, k + 2), (k, k + 1)$$

(Since $a < b$, then $a$ is less than half of $S$ (or $k + \frac{1}{2}$) so the possible values of $a$ are 1 to $k$.)
These pairs satisfy all of the requirements, except possibly the fact that $a \leq 75$ and $b \leq 75$.
Since $a < b$, then we only need to consider whether or not $b \leq 75$.
If $2k \leq 75$, then each of these pairs is an allowable pair, and there are $k$ such pairs.
For there to be 33 such pairs, we have $k = 33$, which gives $S = 2(33) + 1 = 67$.
If $2k > 75$, then not all of these pairs are allowable pairs, as some have $b$ values which are too large.
Counting from the left, the first pair with an allowable $b$ value has $b = 75$, which gives $a = S - 75 = (2k + 1) - 75 = 2k - 74$.
This means that the allowable pairs are

$$(2k - 74, 75), (2k - 73, 74), \ldots, (k - 1, k + 2), (k, k + 1)$$

There are $k - (2k - 74) + 1 = 75 - k$ such pairs.
For there to be 33 such pairs, we have $k = 42$, which gives $S = 2(42) + 1 = 85$.
To summarize the case where $S = 2k + 1$ is odd, there are $k$ allowable pairs when $2k \leq 75$ and $75 - k$ allowable pairs when $2k > 75$, giving possible values of $S$ of 67 and 85.

Suppose that $S$ is even; that is, $S = 2k$ for some integer $k \geq 1$.
The pairs of positive integers $(a, b)$ with $a < b$ and $a + b = S$ are

$$(1, 2k - 1), (2, 2k - 2), (3, 2k - 3), \ldots, (k - 2, k + 2), (k - 1, k + 1)$$

(Since $a < b$, then $a$ is less than half of $S$ (or $k$) so the possible values of $a$ are 1 to $k - 1$.)
If $2k - 1 \leq 75$, then each of these pairs is an allowable pair, and there are $k - 1$ such pairs.
For there to be 33 such pairs, we have $k = 34$, which gives $S = 2(34) = 68$.
If $2k - 1 > 75$, then not all of these pairs are allowable pairs, as some have $b$ values which are too large.
Counting from the left, the first pair with an allowable $b$ value has $b = 75$, which gives $a = S - 75 = 2k - 75$.
This means that the allowable pairs are

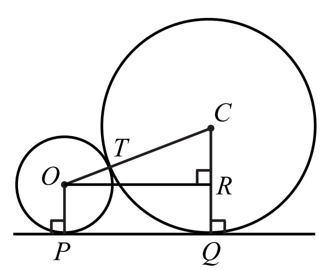
$$(2k - 75, 75), (2k - 74, 74), \ldots, (k - 2, k + 2), (k - 1, k + 1)$$

There are $(k - 1) - (2k - 75) + 1 = 75 - k$ such pairs.
For there to be 33 such pairs, we have $k = 42$, which gives $S = 2(42) = 84$.
To summarize the case where $S = 2k$ is even, there are $k - 1$ allowable pairs when $2k - 1 \leq 75$ and $75 - k$ allowable pairs when $2k - 1 > 75$, giving possible values of $S$ of 68 and 84.

Overall, the possible values of $S$ are 67, 68, 84, and 85.

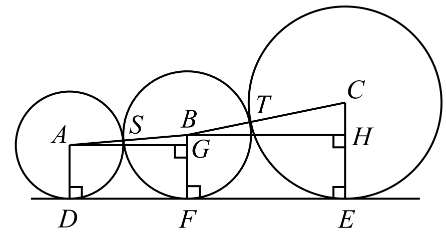
When $S = 67$, the 33 pairs are: $(1, 66), (2, 65), (3, 64), \ldots, (31, 36), (32, 35), (33, 34)$.
When $S = 68$, the 33 pairs are: $(1, 67), (2, 66), (3, 65), \ldots, (31, 37), (32, 36), (33, 35)$.
When $S = 84$, the 33 pairs are: $(9, 75), (10, 74), (11, 73), \ldots, (39, 45), (40, 44), (41, 43)$.
When $S = 85$, the 33 pairs are: $(10, 75), (11, 74), (12, 73), \ldots, (40, 45), (41, 44), (42, 43)$.

4. (a) As suggested, we begin by constructing the segment from $O$, parallel to $PQ$, meeting $CQ$ at $R$.
Both $OP$ and $CQ$ are perpendicular to $PQ$ and since $OR$ is parallel to $PQ$, then $OR$ is also perpendicular to $OP$ and $CQ$.
That is, $ORQP$ is a rectangle (it has 4 right angles).
The radius of the small circle is 2 and so $OP = OT = 2$ (since both are radii).
The radius of the large circle is 5 and so $CQ = CT = 5$ (since both are radii).




Since $O, T, C$ are collinear with $OT = 2$ and $CT = 5$, then $OC = OT + CT = 2 + 5 = 7$.
In rectangle $ORQP$, $RQ = OP = 2$.
Therefore, $CR = CQ - RQ = 5 - 2 = 3$.
In right-angled $\triangle OCR$, we have $OC^2 = CR^2 + OR^2$ by the Pythagorean Theorem.
Thus, $OR^2 = OC^2 - CR^2 = 7^2 - 3^2 = 40$, and so $OR = \sqrt{40} = 2\sqrt{10}$ (since $OR > 0$).
Finally, $PQ = OR = 2\sqrt{10}$ (since $ORQP$ is a rectangle).

(b) *Solution 1*




Let the centres of the circles be $A, B, C$, as shown.
Let $F$ be the point of tangency between the third circle and the horizontal line.
As in part (a), we construct line segments $AG$ and $BH$ parallel to $DE$ and line segments $AD, BF, CE$ perpendicular to $DE$.
Label $S$ and $T$, the points of tangency, so then $A, S, B$ are collinear as are $B, T, C$ collinear.
Let the radius of the third circle be $r$ so that $BF = BS = BT = r$.
The radius of the small circle is 4, so $AS = AD = 4$.
The radius of the large circle is 9, so $CT = CE = 9$.
Let $DF = y$. Then $FE = DE - DF = 24 - y$.
As in part (a), $GF = AD = 4$ and $DF = AG = y$ (since $AGFD$ is a rectangle).
Similarly, $HE = BF = r$ and $BH = FE = 24 - y$ (since $BHEF$ is a rectangle).
In right-angled $\triangle ABG$, $AB = AS + BS = 4 + r$ and $BG = BF - GF = r - 4$.
By the Pythagorean Theorem, $AB^2 = AG^2 + BG^2$ or $(4 + r)^2 = y^2 + (r - 4)^2$.
(Note that in the diagram we have assumed that $r > 4$, however if $r < 4$, then $G$ would be placed on $AD$ such that $AG = AD - GD = 4 - r$. In this case, we get $AB^2 = AG^2 + BG^2$ or $(4+r)^2 = (4-r)^2 + y^2$. Since $(4-r)^2 = (r-4)^2$, the equation given by the Pythagorean Theorem is not dependent on which of these two circles has a larger radius.)
In right-angled $\triangle BCH$, $BC = BT + CT = r + 9$ and $CH = CE - HE = 9 - r$.
By the Pythagorean Theorem, $BC^2 = BH^2 + CH^2$ or $(r + 9)^2 = (24 - y)^2 + (9 - r)^2$.
(Note that in the diagram we have assumed that $r < 9$, however if $r > 9$, then $H$ would be placed on $BF$ such that $BH = BF - HF = r - 9$. In this case, we get $BC^2 = BH^2 + CH^2$ or $(r + 9)^2 = (9 - r)^2 + (24 - y)^2$. Since $(9 - r)^2 = (r - 9)^2$, the equation given by the Pythagorean Theorem is not dependent on which of these two circles has a larger radius.)

Next, we solve the system of equations

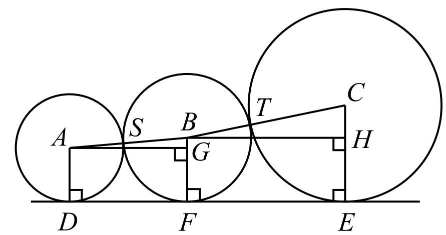
$$\begin{aligned} (4+r)^2 &= y^2 + (r-4)^2 \qquad (1) \\ (r+9)^2 &= (24-y)^2 + (9-r)^2 \qquad (2) \end{aligned}$$

Equation (1) becomes $y^2 = (4 + r)^2 - (r - 4)^2$.
Expanding and simplifying we get $y^2 = 16 + 8r + r^2 - r^2 + 8r - 16$ or $y^2 = 16r$.
Equation (2) becomes $(24 - y)^2 = (r + 9)^2 - (9 - r)^2$.
Instead of expanding, we can factor the right side as a difference of squares, so that $(24 - y)^2 = (r + 9 + 9 - r)(r + 9 - 9 + r) = (18)(2r) = 36r$.
Thus the system of equations simplifies to

$$\begin{aligned} y^2 &= 16r \qquad (3) \\ (24-y)^2 &= 36r \qquad (4) \end{aligned}$$

Since $y^2 = 16r = \frac{4}{9}(36r) = \frac{4}{9}(24 - y)^2$, then $y = \pm\frac{2}{3}(24 - y)$.
Solving these two equations, $y = \frac{2}{3}(24 - y)$ and $y = -\frac{2}{3}(24 - y)$, gives $y = \frac{48}{5}$ or $y = -48$.
Since $y > 0$, then $y = \frac{48}{5}$.
Finally, we substitute $y = \frac{48}{5}$ into (3) to get $16r = \left(\frac{48}{5}\right)^2$, so then $r = \frac{48^2}{5^2} \times \frac{1}{16} = \frac{144}{25}$.
The radius of the third circle is $\frac{144}{25}$.

*Solution 2*




Let the centres of the circles be $A, B, C$, as shown.
Let $F$ be the point of tangency between the third circle and the horizontal line.
As in part (a), we construct line segments $AG$ and $BH$ parallel to $DE$ and line segments $AD, BF, CE$ perpendicular to $DE$.
Label $S$ and $T$, the points of tangency, so then $A, S, B$ are collinear as are $B, T, C$ collinear.
We begin by considering the more general case in which we let the radius of the circle with centre $A$ be $r_1$, the radius of the circle with centre $B$ be $r_2$, and the radius of the circle with centre $C$ be $r_3$. (As was discussed in Solution 1, we may assume that $r_1 < r_2 < r_3$.)
We then have $AD = AS = r_1$, $BS = BF = BT = r_2$, and $CT = CE = r_3$.
As in part (a), $GF = AD = r_1$ and $DF = AG$ (since $AGFD$ is a rectangle).
Similarly, $HE = BF = r_2$ and $BH = FE$ (since $BHEF$ is a rectangle).
In right-angled $\triangle ABG$, $AB = r_1 + r_2$ and $BG = BF - GF = r_2 - r_1$.
By the Pythagorean Theorem, $AG^2 = AB^2 - BG^2 = (r_1 + r_2)^2 - (r_2 - r_1)^2$.
Expanding and simplifying, we get

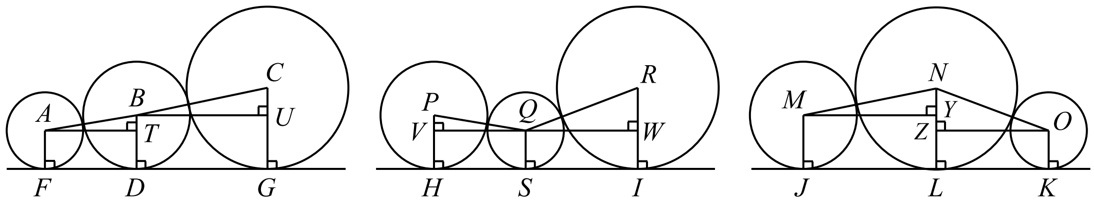
$$\begin{aligned} AG^2 &= r_1^2 + 2r_1r_2 + r_2^2 - r_2^2 + 2r_1r_2 - r_1^2 \\ &= 4r_1r_2 \\ \therefore AG &= 2\sqrt{r_1r_2} \quad \text{since } AG > 0 \end{aligned}$$

Similarly, in right-angled $\triangle BCH$, $BC = r_2 + r_3$ and $CH = CE - HE = r_3 - r_2$.
By the Pythagorean Theorem, $BH^2 = BC^2 - CH^2 = (r_2 + r_3)^2 - (r_3 - r_2)^2$.
Factoring the right side as a difference of squares, we get

$$\begin{aligned} BH^2 &= (r_2 + r_3 + r_3 - r_2)(r_2 + r_3 - r_3 + r_2) \\ &= (2r_3)(2r_2) \\ &= 4r_2r_3 \\ \therefore BH &= 2\sqrt{r_2r_3} \quad \text{since } BH > 0 \end{aligned}$$

Thus $DE = DF + FE = AG + BH = 2\sqrt{r_1r_2} + 2\sqrt{r_2r_3}$.
Given that $DE = 24$, $r_1 = 4$ and $r_3 = 9$, we substitute to get $24 = 2\sqrt{4r_2} + 2\sqrt{9r_2}$.
Simplifying, we get $12 = \sqrt{4r_2} + \sqrt{9r_2}$ or $12 = 2\sqrt{r_2} + 3\sqrt{r_2}$ or $12 = 5\sqrt{r_2}$ and so $\sqrt{r_2} = \frac{12}{5}$.
Finally, we square both sides to get $r_2 = \frac{12^2}{5^2} = \frac{144}{25}$.
Therefore, the radius of the third circle is $\frac{144}{25}$.

(c) We begin by constructing line segments as in part (b), and label the diagram as shown.




As we did in part (b) Solution 2, we can show that

$$FG = FD + DG = AT + BU = 2\sqrt{r_1 r_2} + 2\sqrt{r_2 r_3}$$
$$HI = HS + SI = VQ + QW = 2\sqrt{r_1 r_2} + 2\sqrt{r_1 r_3}$$
$$JK = JL + LK = MY + ZO = 2\sqrt{r_2 r_3} + 2\sqrt{r_1 r_3}$$

(Note that we could show each of these results algebraically as we did in part (b), or we could notice that the circles with centres $P$ and $Q$ have the same radii as those with centres $B$ and $A$ respectively, and so $PQ = AB$ or more importantly, $VQ = AT$.)

Since $r_2 < r_3$, then $r_1 r_2 < r_1 r_3$ and so $2\sqrt{r_1 r_2} < 2\sqrt{r_1 r_3}$.
Since $r_1 < r_2$, then $r_1 r_3 < r_2 r_3$ and so $2\sqrt{r_1 r_3} < 2\sqrt{r_2 r_3}$.
Let $x = \sqrt{r_1 r_2}, y = \sqrt{r_1 r_3}$ and $z = \sqrt{r_2 r_3}$.
Then $x < y < z$.
Since $y < z$, then $x + y < x + z$ so $HI < FG$.
Since $x < y$, then $x + z < y + z$ so $FG < JK$.
Since the lengths of $FG, HI, JK$ are $18, 20, 22$ in some order and $HI < FG < JK$, then $HI = 18, FG = 20$ and $JK = 22$.
Thus, our 3 equations become

$$\begin{aligned} 2x + 2y &= 18 & & & x + y &= 9 \qquad (1)\\ 2x + 2z &= 20 & &\text{or} & x + z &= 10 \qquad (2)\\ 2z + 2y &= 22 & & & z + y &= 11 \qquad (3) \end{aligned}$$

Adding equations (1), (2), (3) we get $2(x + y + z) = 30$, and so $x + y + z = 15$ (4).
Subtracting equation (1) from equation (4) gives $z = (x + y + z) - (x + y) = 15 - 9 = 6$.
Similarly, subtracting each of the equations (2) and (3) from equation (4) in turn, we get $y = 5$ and $x = 4$.
Since $z = 6$, then $\sqrt{r_2 r_3} = 6$ or $r_2 r_3 = 6^2$.
Since $y = 5$, then $\sqrt{r_1 r_3} = 5$ or $r_1 r_3 = 5^2$.
Since $x = 4$, then $\sqrt{r_1 r_2} = 4$ or $r_1 r_2 = 4^2$.
Multiplying these 3 equations together gives $r_1^2 r_2^2 r_3^2 = 4^2 \cdot 5^2 \cdot 6^2$ or $(r_1 r_2 r_3)^2 = (4 \cdot 5 \cdot 6)^2$ and so $r_1 r_2 r_3 = 4 \cdot 5 \cdot 6 = 120$ (since $r_1, r_2, r_3 > 0$).
Finally, dividing this equation by $r_2 r_3 = 6^2$ gives $r_1 = \frac{r_1 r_2 r_3}{r_2 r_3} = \frac{120}{6^2} = \frac{10}{3}$.
Similarly, we get $r_2 = \frac{r_1 r_2 r_3}{r_1 r_3} = \frac{120}{5^2} = \frac{24}{5}$ and $r_3 = \frac{r_1 r_2 r_3}{r_1 r_2} = \frac{120}{4^2} = \frac{15}{2}$.

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# *2013 Galois Contest*

**Thursday, April 18, 2013**
**(in North America and South America)**

**Friday, April 19, 2013**
**(outside of North America and South America)**

*Solutions*

1. (a) The slope of the line passing through the points $(2,0)$ and $(0,4)$ is $\frac{4-0}{0-2} = \frac{4}{-2} = -2$.
Since the line passes through the point $(0,4)$, the $y$-intercept of this line is 4.
Therefore, an equation of the line is $y = -2x + 4$.

(b) Rearranging the equation from part (a), $y = -2x + 4$ becomes $2x + y = 4$.
Dividing both sides of the equation by 4 we get $\frac{2x+y}{4} = \frac{4}{4}$ or $\frac{2x}{4} + \frac{y}{4} = 1$ and so the required form of the equation is $\frac{x}{2} + \frac{y}{4} = 1$.

(c) To determine the $x$-intercept, we set $y = 0$ and solve for $x$.
Thus, $\frac{x}{3} + \frac{y}{10} = 1$ becomes $\frac{x}{3} + \frac{0}{10} = 1$ or $\frac{x}{3} = 1$, and so $x = 3$.
The $x$-intercept is 3.
To determine the $y$-intercept, we let $x = 0$ and solve for $y$.
Thus, $\frac{x}{3} + \frac{y}{10} = 1$, becomes $\frac{0}{3} + \frac{y}{10} = 1$ or $\frac{y}{10} = 1$, and so $y = 10$.
The $y$-intercept is 10.
(Note that the intercepts are the denominators of the two fractions.)

(d) *Solution 1*
The slope of the line passing through the points $(8,0)$ and $(2,3)$ is $\frac{3-0}{2-8} = \frac{3}{-6} = -\frac{1}{2}$.
Thus, an equation of the line is $y = -\frac{1}{2}x + b$.
To find the $y$-intercept $b$, we substitute $(8,0)$ into the equation and solve for $b$.
The equation becomes, $0 = -\frac{1}{2}(8) + b$, or $0 = -4 + b$ and so $b = 4$.
Therefore an equation of the line is $y = -\frac{1}{2}x + 4$.
Rearranging this equation, $y = -\frac{1}{2}x + 4$ becomes $\frac{1}{2}x + y = 4$.
Multiplying both sides of the equation by 2, we get $x + 2y = 8$.
Dividing both sides of the equation by 8 we get, $\frac{x+2y}{8} = \frac{8}{8}$ or $\frac{x}{8} + \frac{2y}{8} = 1$ and so the required form of the equation is $\frac{x}{8} + \frac{y}{4} = 1$.

*Solution 2*
We recognize from the previous parts of the question that a line with equation written in the form $\frac{x}{e} + \frac{y}{f} = 1$, has $x$-intercept $e$ and $y$-intercept $f$.
Since the line passes through $(8,0)$, then its $x$-intercept is 8 and so $e = 8$.
Substituting the point $(2,3)$ into the equation $\frac{x}{8} + \frac{y}{f} = 1$ gives $\frac{2}{8} + \frac{3}{f} = 1$ or $\frac{3}{f} = 1 - \frac{1}{4}$ or $\frac{3}{f} = \frac{3}{4}$, and so $f = 4$.
Therefore, the equation of the line is $\frac{x}{8} + \frac{y}{4} = 1$.

2. (a) *Solution 1*

A 100 cm tall red candle takes 600 minutes to burn completely.

Therefore, the red candle burns at a rate of $\frac{100 \text{ cm}}{600 \text{ min}} = \frac{1}{6}$ cm/min.

After 180 minutes, the height of the red candle will have decreased by $\frac{1}{6}$ cm/min $\times$ 180 min $= 30$ cm.

*Solution 2*

A 100 cm tall red candle takes 600 minutes to burn completely.

The fraction of the red candle that burns in 180 minutes is $\frac{180}{600} = \frac{3}{10}$.

Since the candle was initially 100 cm tall, then $\frac{3}{10} \times 100$ cm $= 30$ cm will have burned after 180 minutes.

Therefore, the height of the red candle will have decreased by 30 cm, 180 minutes after being lit.

(b) To reach a height of 80 cm, the green candle will have decreased by $100 - 80 = 20$ cm.

Since 20 cm is $\frac{20}{100} = \frac{1}{5}$ of the candle's original height, then it will take $\frac{1}{5}$ of the total time to decrease in height to this point.

Since it takes the green candle 480 minutes to burn completely, it will take $\frac{1}{5} \times 480 = 96$ minutes after being lit to decrease to a height of 80 cm.

(c) Since the red candle takes 600 minutes to burn completely, 60 minutes is $\frac{60}{600}$ or $\frac{1}{10}$ of its total burning time.

Therefore after 60 minutes, the red candle will have decreased by $\frac{1}{10} \times 100 = 10$ cm in height. That is, the red candle will be 90 cm tall after burning for 60 minutes.

Since the green candle takes 480 minutes to burn completely, 60 minutes is $\frac{60}{480}$ or $\frac{1}{8}$ of its total burning time.

Therefore after 60 minutes, the green candle will have decreased by $\frac{1}{8} \times 100 = 12.5$ cm in height. That is, the green candle will be 87.5 cm tall after burning for 60 minutes.

The red candle will be $90 - 87.5 = 2.5$ cm taller than the green candle 60 minutes after they are lit.

(d) *Solution 1*

From part (c), the green candle will decrease in height by 2.5 cm more than the red candle every 60 minutes (since their heights decrease at constant rates).

A difference of 2.5 cm in height every 60 minutes written as a fraction is $\frac{2.5}{60}$ cm/min., which is equivalent to $\frac{5}{120} = \frac{1}{24}$ cm/min.

That is, the green candle will decrease in height by 1 cm more than the red candle every 24 minutes after being lit.

Therefore, the red candle will be 7 cm taller than the green candle $7 \times 24 = 168$ minutes after they are lit.

*Solution 2*

The red candle burns at a rate of 100 cm every 600 minutes or $\frac{1}{6}$ cm/min.

The green candle burns at a rate of 100 cm every 480 minutes or $\frac{5}{24}$ cm/min.
In $t$ minutes after being lit, $\frac{1}{6}t$ cm of the red candle will have burned.
In $t$ minutes after being lit, $\frac{5}{24}t$ cm of the green candle will have burned.
Since both candles began with the same 100 cm height, then the red candle is 7 cm taller than the green candle when $(100 - \frac{1}{6}t) - (100 - \frac{5}{24}t) = 7$.
Simplifying this equation, we get $\frac{5}{24}t - \frac{1}{6}t = 7$ and by multiplying both sides by 24, $5t - 4t = 7 \times 24$, and so $t = 168$.
Therefore, the red candle is 7cm taller than the green candle 168 minutes after being lit.

3. (a) *Solution 1*
The last number in the $7^{th}$ row is $7 \times 8 = 56$.
Since the $7^{th}$ row has 7 numbers in it, we list the 7 even integers decreasing from 56, which are $56, 54, 52, 50, 48, 46, 44$.
Written in the order they will appear in the table, the numbers in the $7^{th}$ row are, $44, 46, 48, 50, 52, 54, 56$.

*Solution 2*
The last number in the $6^{th}$ row is $6 \times 7 = 42$.
Therefore the next even integer, 44, will appear as the first number in the $7^{th}$ row of the table.
Since the $7^{th}$ row has 7 numbers in it, we list the 7 even integers increasing from 44.
Thus the numbers in the $7^{th}$ row are, $44, 46, 48, 50, 52, 54, 56$.

(b) The last number in the $100^{th}$ row is $100 \times 101 = 10\,100$.
The last number in the $99^{th}$ row is $99 \times 100 = 9900$.
Therefore the next even integer, 9902, will appear as the first number in the $100^{th}$ row of the table.
The first and last numbers in the $100^{th}$ row of the table are 9902 and 10 100, respectively.

(c) The last number in row $r$ is equal to $r(r + 1)$, so $L = r(r + 1)$.
The first number in row $(r + 2)$ is 2 more than the last number in row $(r + 1)$.
The last number in row $(r + 1)$ is $(r + 1)(r + 2)$, so $F = (r + 1)(r + 2) + 2$.
We require $F + L$ to be at least 2013, so $F + L = (r + 1)(r + 2) + 2 + r(r + 1) \geq 2013$.
To determine the smallest value for $r$ such that $F+L = (r+1)(r+2)+2+r(r+1) \geq 2013$, we solve the following inequality:

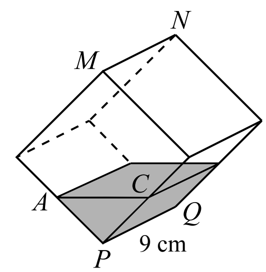
$$\begin{aligned}
(r+1)(r+2) + 2 + r(r+1) &\geq 2013 \\
r^2 + 3r + 2 + 2 + r^2 + r &\geq 2013 \\
2r^2 + 4r + 4 &\geq 2013 \\
r^2 + 2r + 2 &\geq 1006.5 \\
r^2 + 2r + 1 &\geq 1006.5 - 1 \\
(r+1)^2 &\geq 1005.5 \\
\therefore\ r + 1 \geq +\sqrt{1005.5} &\quad \text{or} \quad r + 1 \leq -\sqrt{1005.5}
\end{aligned}$$

Since $r$ is positive, $r + 1 \geq \sqrt{1005.5}$ and so $r \geq +\sqrt{1005.5} - 1 \approx 30.7096$.
Thus, the smallest possible value of the integer $r$ such that $F + L$ is at least 2013 is 31.
*Check*: Since $L$ is the last number in row $r = 31$, then $L = 31 \times 32 = 992$.
Since $F$ is the first number in row $r + 2 = 33$, then $F$ is 2 more than the last number in row 32, or $F = (32 \times 33) + 2 = 1058$.
Therefore, $F + L = 1058 + 992 = 2050 \geq 2013$ as required.

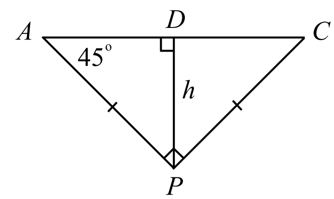
We must also check if 31 is the smallest value of $r$ such that $F + L \geq 2013$.
Since the numbers are arranged in the rows in a strictly increasing way, we need only check that when $r = 30$, $F + L < 2013$.
When $r = 30$, $F + L = ((31 \times 32) + 2) + (30 \times 31) = 994 + 930 = 1924 < 2013$.

4. (a) The shape formed by the water is a rectangular prism.
The area of the base of this rectangular prism is the same as the area of the base of the cube, which is $9 \times 9 = 81$ cm$^2$.
Since the height of the water is 1 cm, then the volume of water in the cube is $81 \times 1 = 81$ cm$^3$.

(b) When the cube is rotated 45° about edge $PQ$, edge $MN$ will lie directly above $PQ$.
In this position, the water now has the shape of a triangular prism, as shown.
Moreover, the triangular face is a right-angled isosceles triangle. It is right-angled since the adjacent faces of the cube are perpendicular to one another, and it is isosceles as a result of the symmetry created by rotating the cube such that $MN$ lies directly above $PQ$. (Put another way, by symmetry, the water level is such that $AP = CP$.)

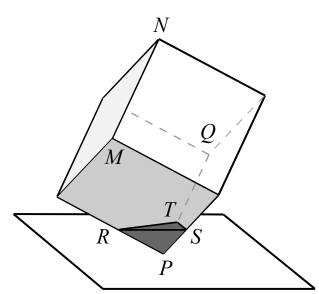



As shown in the second diagram, the depth of the water, $h$, is equal to the length of altitude $PD$ in triangle $APC$.
In $\triangle APD$, $\angle DAP = 45°$ (since $\triangle APC$ is a right-angled isosceles triangle) and so $\angle APD = 180° - 90° - 45° = 45°$.
That is, $\triangle APD$ is also a right-angled isosceles triangle with $AD = DP = h$.

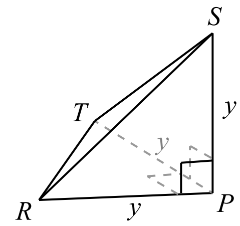



The volume of the water is given by the area of this right-angled isosceles triangle $APC$ multiplied by the length of the prism, $PQ$, which is 9 cm.
Since $AD = DP = h$ and $\triangle ADP$ is congruent to $\triangle CDP$, then $AC = 2AD = 2h$.
Thus, the area of $\triangle APC$ is $\frac{1}{2}AC \times DP = \frac{1}{2}(2h) \times h = h^2$.
Therefore, the volume of water is equal to $(h^2 \times 9)$ cm$^3$.
In part (a), we found the volume of water to be equal to 81 cm$^3$.
Since no water has been lost, the volume is the same in this new orientation of the cube.
That is, $h^2 \times 9 = 81$ so $h^2 = 9$ and $h = 3$ (since $h > 0$).
Therefore, the depth of water in the cube is 3 cm.

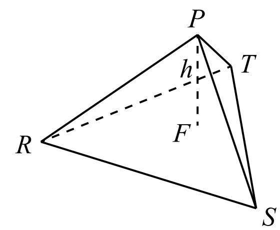
(c) In this new position, the water now has the shape of a tetrahedron. By symmetry the water is the same distance "up" the edges and so $PR = PS = PT$.
Three of the faces of this tetrahedron, $\triangle PRS, \triangle PST, \triangle PTR$, are right triangles since adjacent edges of a cube are perpendicular ($\angle RPS = \angle SPT = \angle TPR = 90°$).
That is, the tetrahedron has three congruent, right isosceles triangular faces, $\triangle PRS, \triangle PST$, and $\triangle PTR$.




Since these three triangles are congruent, then the sides $RS, ST$ and $TR$ are all equal in length and thus the fourth face, $\triangle RST$, is equilateral.

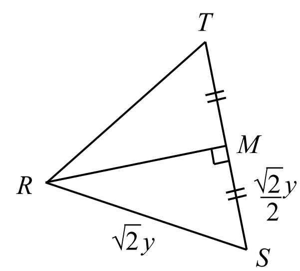



In the second diagram, the tetrahedron has been repositioned to sit on $\triangle PRT$. In this position, we call the base of the tetrahedron $\triangle PRT$ and thus the height of the tetrahedron is $PS$ since $PS$ is perpendicular to $\triangle PRT$ ($PS$ is perpendicular to both $PR$ and $PT$).
Since the tetrahedron is a triangular-based pyramid, its volume is $\frac{1}{3}|\triangle PRT| \times PS$ (where $|\triangle PRT|$ denotes the area of $\triangle PRT$).
Suppose that $PR = PS = PT = y$, as shown. (Recall that these are all equal in length by symmetry.)
In $\triangle PRT$, $PR$ and $PT$ are perpendicular and so $|\triangle PRT| = \frac{1}{2}PR \times PT = \frac{1}{2}y^2$.
Therefore, the volume of the tetrahedron is $\frac{1}{3}|\triangle PRT| \times PS = \frac{1}{3} \times \frac{1}{2}y^2 \times y = \frac{1}{6}y^3$.
In part (a) we found the volume of water to be equal to 81 cm$^3$, and since no water has been lost, the volume is the same in this new orientation of the cube.
That is, $\frac{1}{6}y^3 = 81$ or $y^3 = 486$ and so $y = \sqrt[3]{486}$.




In the third diagram shown, the tetrahedron has been repositioned again so that it is easier to visualize its vertical height, $PF = h$, the length that we are asked to find.
Since opposite corner $N$ is directly above corner $P$, the line segment $PN$ is perpendicular to the ground.
We call the point of intersection of $PN$ and the top surface of the water, point $F$.

Using the Pythagorean Theorem in $\triangle PRS$, we get $RS^2 = PR^2 + PS^2 = y^2 + y^2 = 2y^2$.
Since $RS > 0$, then $RS = \sqrt{2}y$ and so $RS = ST = TR = \sqrt{2}y$.
(We could have used the fact that $\triangle PRS$ is a special $45^\circ - 45^\circ - 90^\circ$ triangle and so $PR : PS : RS = 1 : 1 : \sqrt{2}$.)
Again, since the tetrahedron is a triangular-based pyramid its volume is $\frac{1}{3}|\triangle RST| \times h$.
We first determine the area of $\triangle RST$ by constructing altitude $RM$, as shown in the fourth diagram.
Point $M$ is the midpoint of $TS$ and thus $MS = \frac{\sqrt{2}}{2}y$.




By the Pythagorean Theorem, $RS^2 = RM^2 + MS^2$ or $RM^2 = (\sqrt{2}y)^2 - \left(\frac{\sqrt{2}}{2}y\right)^2$.
So $RM^2 = 2y^2 - \frac{1}{2}y^2 = \frac{3}{2}y^2$, and therefore $RM = \sqrt{\frac{3}{2}}y$ (since $RM > 0$).
(We could have used the fact that $\triangle RMS$ is a special $30^\circ - 60^\circ - 90^\circ$ triangle and so $MS : SR : RM = 1 : 2 : \sqrt{3}$.)
The area of $\triangle RST$ is then $\frac{1}{2} \times TS \times RM = \frac{1}{2} \times \sqrt{2}y \times \sqrt{\frac{3}{2}}y = \frac{\sqrt{3}}{2}y^2$, and so the volume of the tetrahedron (the volume of the water) is $\frac{1}{3}\left(\frac{\sqrt{3}}{2}y^2\right)h$ or $\frac{\sqrt{3}}{6}y^2h$.
Finally we substitute $y = \sqrt[3]{486}$ into $\frac{\sqrt{3}}{6}y^2h$, our expression for the volume of the tetrahedron, so that $\frac{\sqrt{3}}{6}y^2h$ becomes $\frac{\sqrt{3}}{6}(\sqrt[3]{486})^2h$.
Again, in part (a) we found the volume of water to be equal to 81 cm$^3$, and since no water has been lost, the volume is the same in this new orientation of the cube.
That is, $\frac{\sqrt{3}}{6}(\sqrt[3]{486})^2h = 81$ or $h = \dfrac{6 \times 81}{\sqrt{3}(\sqrt[3]{486})^2} = \dfrac{486}{\sqrt{3} \times 486^{\frac{2}{3}}} = \dfrac{486^{\frac{1}{3}}}{\sqrt{3}} \approx 4.539$.
To the nearest hundredth of a centimetre, the depth of water in the cube is 4.54 cm.

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# *2012 Galois Contest*

**Thursday, April 12, 2012**
**(in North America and South America)**

**Friday, April 13, 2012**
**(outside of North America and South America)**

*Solutions*

1. (a) In $\triangle JPA$, $\angle JPA = 90^\circ$.
Using the Pythagorean Theorem, $AJ^2 = 15^2 + 20^2$ or $AJ^2 = 225 + 400 = 625$ and so $AJ = \sqrt{625} = 25$, since $AJ > 0$.
The distance from $A$ to $J$ is 25.

(b) In $\triangle BAQ$, $\angle BAQ = 90^\circ$.
Using the Pythagorean Theorem, $39^2 = BA^2 + 15^2$ or $BA^2 = 1521 - 225 = 1296$ and so $BA = \sqrt{1296} = 36$, since $BA > 0$.
The distance from $B$ to $A$ is 36.

(c) In $\triangle BJA$, $\angle BJA = 90^\circ$.
Using the Pythagorean Theorem, $BA^2 = BJ^2 + AJ^2$ or $1296 = BJ^2 + 625$ or $BJ^2 = 1296 - 625 = 671$ and so $BJ = \sqrt{671} \approx 25.904$, since $BJ > 0$.
Thus, the distance from Budan's ball to the jack ball is approximately 25.904.
In part (a), we determined the distance between Adam's ball and the jack ball to be 25.
Therefore, Adam's ball is closer to the jack ball.

2. (a) The average of any two numbers is found by adding the two numbers and dividing by two.
Thus, the three averages when the numbers are taken in pairs are

$$\frac{25+5}{2} = \frac{30}{2} = 15, \quad \frac{5+29}{2} = \frac{34}{2} = 17, \text{ and } \frac{25+29}{2} = \frac{54}{2} = 27.$$

(b) The average of 2 and 6 is $\frac{2+6}{2} = 4$.
Since 6 is greater than 2, then the average of 6 and $n$ is greater than the average of 2 and $n$.
Therefore, the average of 6 and $n$ is 13 and the average of 2 and $n$ is 11.
Since the average of 6 and $n$ is 13, then $\frac{6+n}{2} = 13$ or $6 + n = 26$ and so $n = 20$.
We can check that $n = 20$ is correct by recognizing that the average of 2 and 20 is indeed 11.

(c) When each of the three numbers is added to the average of the other two, the resulting three expressions are

$$2 + \frac{a+b}{2}, \quad a + \frac{2+b}{2}, \quad b + \frac{2+a}{2}.$$

To determine which of these expressions is equal to which of the results, 14, 17, 21, we must order the three expressions from smallest to largest.
Since $2 < a < b$, then $2 + (2 + a + b) < a + (2 + a + b) < b + (2 + a + b)$
or $4 + a + b < 2a + 2 + b < 2b + 2 + a$.
Dividing by 2, $\frac{4+a+b}{2} < \frac{2a+2+b}{2} < \frac{2b+2+a}{2}$ or $\frac{4}{2} + \frac{a+b}{2} < \frac{2a}{2} + \frac{2+b}{2} < \frac{2b}{2} + \frac{2+a}{2}$
and so $2 + \frac{a+b}{2} < a + \frac{2+b}{2} < b + \frac{2+a}{2}$.
Since $2 + \frac{a+b}{2}$ is the smallest of the three expressions, then it must equal the smallest of the three results, 14.
Since $b + \frac{2+a}{2}$ is the largest of the three expressions, then it must equal the largest of the three results, 21.

We now solve the following system of two equations and two unknowns.

$$2 + \frac{a+b}{2} = 14 \qquad (1)$$

$$b + \frac{2+a}{2} = 21 \qquad (2)$$

Multiplying each equation by 2,

$$4 + a + b = 28 \qquad (3)$$

$$2b + 2 + a = 42 \qquad (4)$$

Thus,

$$a + b = 24 \qquad (5)$$

$$a + 2b = 40 \qquad (6)$$

Subtracting equation (5) from equation (6), we get $b = 16$.
Substituting $b = 16$ into equation (5), $a + 16 = 24$, and so $a = 8$.
(We may check that our solution is correct by substituting $a = 8$ and $b = 16$ into the third expression $a + \frac{2+b}{2}$ to get the third result, 17.)

3. (a) The slope of the line is $m = -3$; thus its equation is $y = -3x + b$ with $y$-intercept $b$.
Since the line passes through the point $(2, 6)$, then $x = 2$ and $y = 6$ satisfy the equation of the line.
Substituting $x = 2$ and $y = 6$ into the equation of the line, then $6 = -3(2) + b$ and so $b = 12$.
The equation of the line is $y = -3x + 12$ and the line has $y$-intercept 12.
To find the $x$-intercept, we let $y = 0$ and solve for $x$.
Thus, $0 = -3x + 12$ or $3x = 12$, and so the line has $x$-intercept 4.

(b) The slope of the line is $m$; thus its equation is $y = mx + b$ with $y$-intercept $b$.
Since the line passes through the point $(2, 6)$, then $x = 2$ and $y = 6$ satisfy the equation of the line.
Substituting $x = 2$ and $y = 6$ into the equation of the line, then $6 = 2m + b$ and so $b = 6 - 2m$.
The equation of the line is $y = mx + (6 - 2m)$ and the line has $y$-intercept $6 - 2m$.
To find the $x$-intercept, we let $y = 0$ and solve for $x$.
Thus, $0 = mx + (6 - 2m)$ or $mx = 2m - 6$ or $x = \frac{2m-6}{m}$, and so the line has $x$-intercept $2 - \frac{6}{m}$.
(We require $m \neq 0$, otherwise the line is horizontal and the $x$-intercept does not exist.)

(c) The line through the point $(2, 6)$ with slope $m$ has $x$-intercept $2 - \frac{6}{m}$ and $y$-intercept $6 - 2m$, as determined in part (b).
(We require $m \neq 0$, otherwise the line is horizontal and the $x$-intercept, $P$, does not exist.)
Since $P$ is the $x$-intercept of this line, $OP$ has length $2 - \frac{6}{m}$.
Since $Q$ is the $y$-intercept of this line, $OQ$ has length $6 - 2m$.
Therefore, the area of $\triangle POQ$ is given by $\frac{1}{2}(OP)(OQ) = \frac{1}{2}\left(2 - \frac{6}{m}\right)(6 - 2m)$.

Since the area of $\triangle POQ$ is 25, then $\frac{1}{2}\left(2-\frac{6}{m}\right)(6-2m)=25$.
Solving for $m$,

$$\begin{aligned}
\frac{1}{2}\left(2-\frac{6}{m}\right)(6-2m) &= 25\\
\left(2-\frac{6}{m}\right)(6-2m) &= 50\\
(2m-6)(6-2m) &= 50m\\
12m-4m^2-36+12m &= 50m\\
4m^2+26m+36 &= 0\\
2m^2+13m+18 &= 0\\
(2m+9)(m+2) &= 0
\end{aligned}$$

Therefore, two possible values are $m=-\frac{9}{2}$ and $m=-2$.
Since $P$ and $Q$ lie on the positive $x$-axis and the positive $y$-axis respectively, we must check that these two values for $m$ give $2-\frac{6}{m}>0$ and $6-2m>0$.
When $m=-\frac{9}{2}$, $2-\frac{6}{m}=2+\frac{6}{\frac{9}{2}}$ which is greater than 0.
When $m=-\frac{9}{2}$, $6-2m=6+2(\frac{9}{2})$ which is also greater than 0.
When $m=-2$, $2-\frac{6}{m}=2+\frac{6}{2}$ which is greater than 0.
When $m=-2$, $6-2m=6+2(2)$ which is also greater than 0.
Therefore, the two values of $m$ for which $P$ and $Q$ lie on the positive $x$-axis and the positive $y$-axis, respectively, and for which $\triangle POQ$ has area 25, are $m=-\frac{9}{2}$ and $m=-2$.
*Note*: If we remove the restriction that $P$ and $Q$ both be located on their respective *positive* axes, then there are two more values of $m$ for which $\triangle POQ$ has area 25. Can you determine these?

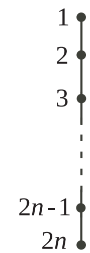
4. (a) Let point $P$ be the location where the students should meet such that the total distance travelled by all five is as small as possible.
Label the intersections where Abe, Bo, Carla, Denise, and Ernie initially begin, $A$, $B$, $C$, $D$, and $E$, respectively.

If $P$ is located west (left) of $A$, then every student must walk farther to reach $P$ than they would if $P$ was moved to $A$.
Similarly, if $P$ is located east (right) of $E$, then every student must walk farther to reach $P$ than they would if $P$ was moved to $E$.
That is, $P$ must lie on the east-west street somewhere from $A$ to $E$, inclusive.
To minimize the total distance travelled by all five students, we need to minimize $AP+BP+CP+DP+EP$.
For any point $P$ located between $A$ and $E$ inclusive, $AP+EP=14$, since $A$ and $E$ combined walk the entire length of the road to meet at $P$.
Thus, we need to minimize $BP+CP+DP$.

If $P$ is located west (left) of $B$, then the three students Bo, Carla and Denise must walk farther to reach $P$ than they would if $P$ was moved to $B$ (and the combined distance for $A$ and $E$ is still 14).
Similarly, if $P$ is located east (right) of $D$, then these three students must walk farther to reach $P$ than they would if $P$ was moved to $D$.
That is, $P$ must lie on the east-west street somewhere from $B$ to $D$, inclusive.
For any such point $P$ located between $B$ and $D$ inclusive, $BP + DP = 6$.
Thus, we need to minimize $CP$.
This is done by locating $P$ at $C$ such that $CP = 0$.

The minimum total distance travelled by all five students is $14 + 6$ or 20, and this only occurs when the students meet at $C$, the intersection at which Carla begins.

(b) Let the number of students be $2n$ (since there are an even number of them).
Let point $P$ be the location where the students should meet such that the total distance travelled by all is as small as possible.
Label the students and their starting intersections in order from 1 to $2n$ beginning with the student who is farthest north, as shown.




As in part (a), if $P$ is located north of Student 1, then every student must walk farther to reach $P$ than they would if $P$ was moved to Intersection 1.
Similarly, if $P$ is located south of Student $2n$, then every student must walk farther to reach $P$ than they would if $P$ was moved to Intersection $2n$.
Thus, we may conclude that $P$ must lie on the north-south street somewhere from Intersection 1 to Intersection $2n$, inclusive.
For any such location of $P$, the combined distance travelled by Student 1 and Student $2n$ is constant (it's the distance between Intersection 1 and Intersection $2n$).
Thus, to minimize the total distance travelled by all students, we must minimize the distance travelled by Students $2, 3, 4, \ldots, (2n - 1)$.

Again, if $P$ is located north of Student 2, then each of these $(2n - 2)$ students must walk farther to reach $P$ than they would if $P$ was moved to Intersection 2.
Similarly, if $P$ is located south of Student $(2n - 1)$, then each of these students must walk farther to reach $P$ than they would if $P$ was moved to Intersection $(2n - 1)$.
Thus, we may conclude that $P$ must lie on the north-south street somewhere from Intersection 2 to Intersection $(2n - 1)$, inclusive.
For any such location of $P$, the combined distance travelled by Student 2 and Student $(2n - 1)$ is constant (it's the distance between Intersection 2 and Intersection $(2n - 1)$).
Thus, to minimize the total distance travelled by all students, we must minimize the distance travelled by Students $3, 4, 5, \ldots, (2n - 2)$.

We may continue the argument in this manner, explaining why $P$ must next be located between Intersection 3 and Intersection $(2n - 2)$, then Intersection 4 and Intersection $(2n - 3)$, then Intersection 5 and Intersection $(2n - 4)$, and so on.
Ultimately we conclude that $P$ must be located between the middle two intersections, Intersection $n$ and Intersection $(n + 1)$, inclusive (since there are $(n - 1)$ intersections north of Intersection $n$ and $(n - 1)$ intersections south of Intersection $(n + 1)$).
For any such location of $P$, the combined distance travelled by Student $n$ and Student $(n + 1)$ is constant (it's the distance between Intersection $n$ and Intersection $(n + 1)$).
Thus, to minimize the total distance travelled by all students, they should meet anywhere between Intersection $n$ and Intersection $(n + 1)$, inclusive.

(c) Since each student must walk along the streets, their total distance travelled is the sum of their distance travelled east or west and their distance travelled north or south.
Further, if for example a student is required to travel 5 km east and 4 km north, this may be achieved by travelling along many different paths.
However, the one thing that all these routes of minimum distance have in common is that the total distance travelled east is 5 km and the total distance travelled north is 4 km.
That is, the distance travelled east-west is independent of the distance travelled north-south, and as such, we may minimize them independently in order to arrive at the minimum total distance travelled.
Strictly speaking, this works only if we can minimize both distances at the same point, which we can here.
This reduces the problem into two distinct parts.
In Part 1, we will find the east-west location (an $x$-coordinate) that minimizes the total distance that the students must travel horizontally in the plane.
In Part 2, we will find the north-south location (a $y$-coordinate) that minimizes the total distance that the students must travel vertically in the plane.
We can then combine these to determine one location that minimizes both, so minimizes the total distance.

*Part 1*
Since we are attempting to locate the intersection that minimizes total east-west travel for all 100 students, we need only consider their starting east-west locations, that is, their $x$-coordinates.
The $x$-coordinates for the first 50 students are, $2, 4, 8, 16, 32, 64, \ldots, 2^{49}, 2^{50}$.
The $x$-coordinates for the students numbered 51 to 100 are, $1, 2, 3, 4, 5, 6, \ldots, 49, 50$.
Finding the $x$ location that will make the total horizontal distance travelled as small as possible, is equivalent to the problem that we solved in part (b).
We must order the 100 $x$-coordinates from lowest to highest and then determine the two $x$ values that are in the middle of the ordered list (as we did in part (b)).
Since there are 100 $x$ values, we are looking for the 50th and 51st numbers in the ordered list.
There are 5 numbers less than 45 in the list $2, 4, 8, 16, 32, 64, \ldots, 2^{49}, 2^{50}$.
These 5 numbers, $2, 4, 8, 16, 32$ along with the 44 numbers $1, 2, 3, 4, 5, 6, \ldots, 44$ from the second list, will be the first 49 numbers in the ordered list.
Thus, the 50th and 51st numbers are 45 and 46.
These $x$-coordinates are the horizontal positions that will minimize the total distance travelled by all students in an east-west direction.

*Part 2*
Since we are attempting to locate the intersection that minimizes total north-south travel for all 100 students, we need only consider their starting north-south locations, that is, their $y$-coordinates.
The $y$-coordinates for the first 50 students are, $1, 2, 3, 4, 5, 6, \ldots, 49, 50$.
The $y$-coordinates for the students numbered 51 to 100 are, $2, 4, 6, 8, 10, 12, \ldots, 98, 100$.
Finding the $y$ location that will make the total vertical distance travelled as small as possible, is also equivalent to the problem that we solved in part (b).
We must order the 100 $y$-coordinates from lowest to highest and then determine the two $y$ values that are in the middle of the ordered list.
Since there are 100 $y$ values, we are looking for the 50th and 51st numbers in the ordered

list.
There are 49 numbers in the ordered list that are less than or equal to 33 (33 of these from the first list and 16 from the second).
Thus, the 50th and 51st numbers in the ordered list are both 34 (since 34 appears in both the first list and the second).
Since the 50th and 51st numbers are both 34, there is just one $y$-coordinate that will minimize the total distance travelled by all students in a north-south direction.

The total horizontal travel is minimized when $x = 45$ or $x = 46$, and the total vertical travel is minimized when $y = 34$.
Thus, the intersections at which the students should meet in order to make the total distance travelled by all students as small as possible are $(45, 34)$ or $(46, 34)$.

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

# 2011 Galois Contest

**Wednesday, April 13, 2011**

*Solutions*

1. (a) Using Jackson's rule, the second term of Fabien's sequence is $\frac{1}{1-2} = \frac{1}{-1} = -1$.

   (b) Since the second term is $-1$, the third term is $\frac{1}{1-(-1)} = \frac{1}{1+1} = \frac{1}{2}$.

   Since the third term is $\frac{1}{2}$, the fourth term is $\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$.

   Since the fourth term is 2, the fifth term is $\frac{1}{1-2} = \frac{1}{-1} = -1$.

   (c) Since the fourth term, 2, is equal to the first term and each term depends only on the previous term, then the sequence of terms repeats every 3 terms.
   That is, the sequence of numbers produced is $2, -1, \frac{1}{2}, 2, -1, \frac{1}{2}, 2, \ldots$.
   Since the terms of the sequence $2, -1, \frac{1}{2}$ repeat every three terms, then we must determine how many groups of three terms there are in the first 2011 terms.
   Since $2011 = 670 \times 3 + 1$, the sequence $2, -1, \frac{1}{2}$ repeats 670 times (giving the first 2010 terms), with the $2011^{th}$ term being 2.
   That is, there are 671 terms equal to 2 in Fabien's sequence.

   (d) The repeating cycle identified in part (c) has a sum of $2 + (-1) + \frac{1}{2} = \frac{3}{2}$.
   This complete cycle repeats 670 times.
   Thus, the sum of the first 2010 terms in the sequence is $670 \times \frac{3}{2} = 1005$.
   Since the $2011^{th}$ term is 2, the sum of all terms in Fabien's sequence is $1005 + 2$ or 1007.

2. (a) We organize the possibilities that may appear on the coins in the table below.

| '5 coin' | '7 coin' | '10 coin' | Score |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 5 |
| 0 | 7 | 0 | 7 |
| 0 | 0 | 10 | 10 |
| 5 | 7 | 0 | 12 |
| 5 | 0 | 10 | 15 |
| 0 | 7 | 10 | 17 |
| 5 | 7 | 10 | 22 |

   The other possible scores are $0, 5, 7, 10, 12, 15,$ and 22.

   (b) *Solution 1*
   Since the three given scores are different from one another, a different coin must be showing a 0 on each of the three tosses.
   That is, after the three tosses each coin has had its zero side appear once, and its non-zero side appear twice.
   This means that the total of the scores from all three tosses, $60 + 110 + 130 = 300$, represents twice the sum of the number on the non-zero sides of the three coins.
   If twice the sum of the non-zero numbers on the three coins equals 300, then the sum of the non-zero numbers on the three coins is $300 \div 2$ or 150.
   Since the maximum possible score occurs when the non-zero number appears on each of the three coins, then the maximum possible score is 150.

   *Solution 2*
   Since the three given scores are different from one another, a different coin must be showing a 0 on each of the three tosses.

Let the non-zero number appearing on each of the three coins be $a$, $b$ and $c$.
Since exactly one of the three coins shows a zero on each of the three tosses, we may assume without loss of generality that $a + b = 60$, $a + c = 110$, and $b + c = 130$.
Adding the left sides of these three equations gives $a + b + a + c + b + c$ or $2a + 2b + 2c$.
Adding the right sides of the three equations gives $60 + 110 + 130$ or $300$.
Since $2a + 2b + 2c = 300$, then $2(a + b + c) = 300$ and so $a + b + c = 150$.
This sum, 150, represents the score when the non-zero number appears on each of the three coins.
Since the maximum possible score occurs when the non-zero number appears on each of the three coins, then the maximum possible score is 150.

(c) We organize the possibilities that may appear on the third coin in the table below, accounting for all of the possible combinations of values from the first two coins:

| Appearing on the '25 coin' | Appearing on the '50 coin' | Appearing on the $3^{rd}$ coin |
|---|---|---|
| 0 | 0 | $170 - 0 = 170$ |
| 25 | 0 | $170 - 25 = 145$ |
| 0 | 50 | $170 - 50 = 120$ |
| 25 | 50 | $170 - 75 = 95$ |

The possible non-zero numbers that may appear on the third coin are $170, 145, 120$, and $95$.

3. (a) Since $\angle ABP = 90^\circ$, $\triangle ABP$ is a right-angled triangle.
By the Pythagorean Theorem, $BP^2 = AP^2 - AB^2$ or $BP^2 = 20^2 - 16^2$ or $BP^2 = 144$ and so $BP = 12$, since $BP > 0$.
Since $\angle QTP = 90^\circ$, $\triangle QTP$ is a right-angled triangle with $PT = 12$.
Since $PT = BP = 12$, then by the Pythagorean Theorem, $QT^2 = QP^2 - PT^2$ or $QT^2 = 15^2 - 12^2$ or $QT^2 = 81$ and so $QT = 9$, since $QT > 0$.

(b) In triangles $PQT$ and $DQS$, $\angle PTQ = \angle DSQ = 90^\circ$.
Also, $\angle PQT$ and $\angle DQS$ are vertically opposite angles and are therefore equal.
Since $\angle PTQ = \angle DSQ$, $\angle PQT = \angle DQS$, and the sum of the 3 angles in any triangle is $180^\circ$, then the third pair of corresponding angles, $\angle QPT$ and $\angle QDS$, are also equal.
Since the corresponding angles in these two triangles are equal, then $\triangle PQT$ and $\triangle DQS$ are similar triangles.

(c) Since $ABCD$ is a rectangle and $TS$ is perpendicular to $BC$, then $ABTS$ is also a rectangle.
Thus, $TS = BA = 16$ and $QS = TS - QT = 16 - 9 = 7$.
As shown in part (b), $\triangle PQT$ and $\triangle DQS$ are similar triangles.
Therefore, the ratios of corresponding side lengths in these two triangles are equal.
That is, $\dfrac{SD}{TP} = \dfrac{QS}{QT}$ or $\dfrac{SD}{12} = \dfrac{7}{9}$ or $SD = 12 \times \dfrac{7}{9} = \dfrac{28}{3}$.

(d) *Solution 1*
In $\triangle QAS$ and $\triangle RAD$, $\angle QAS$ and $\angle RAD$ are common (the same) angles and thus are equal.
Since $ABCD$ is a rectangle, $\angle RDA = 90^\circ = \angle QSA$.
Since $\angle QAS = \angle RAD$, $\angle RDA = \angle QSA$, and the sum of the 3 angles in any triangle is $180^\circ$, then the third pair of corresponding angles, $\angle SQA$ and $\angle DRA$, are also equal.
Since the corresponding angles in these two triangles are equal, then $\triangle QAS$ and $\triangle RAD$ are similar triangles.

Therefore, the ratios of corresponding side lengths in these two triangles are equal.

That is, $\dfrac{RD}{QS} = \dfrac{DA}{SA}$ or $RD = QS \times \dfrac{DA}{SA}$.

However, $DA = AS + SD = 24 + \dfrac{28}{3} = \dfrac{100}{3}$, and so $RD = 7 \times \dfrac{\left(\frac{100}{3}\right)}{24} = 7 \times \dfrac{100}{72}$ or $RD = \dfrac{175}{18}$.

Since $\triangle QAS$ and $\triangle RAD$ are similar triangles, then $\dfrac{RA}{QA} = \dfrac{RD}{QS}$.

Thus, $RA = QA \times \dfrac{RD}{QS} = 25 \times \dfrac{\left(\frac{175}{18}\right)}{7}$ or $RA = 25 \times \dfrac{25}{18}$, and so $RA = \dfrac{625}{18}$.

Since $QR = RA - QA$, then $QR = \dfrac{625}{18} - 25$ or $QR = \dfrac{625 - 450}{18}$, and so $QR = \dfrac{175}{18}$.

Therefore, $QR = RD$.

*Solution 2*

In triangles $PQA$ and $TQP$, the ratios of corresponding side lengths are equal.

That is, $\dfrac{PA}{TP} = \dfrac{PQ}{TQ} = \dfrac{QA}{QP}$ or $\dfrac{20}{12} = \dfrac{15}{9} = \dfrac{25}{15} = \dfrac{5}{3}$.

Therefore, $\triangle PQA$ and $\triangle TQP$ are similar triangles and thus their corresponding angles are equal.

That is, $\angle PQA = \angle TQP = \alpha$.

Since $\angle RQD$ and $\angle PQA$ are vertically opposite angles, then $\angle RQD = \angle PQA = \alpha$.

Since $CD$ and $TS$ are parallel, then by the Parallel Lines Theorem $\angle RDQ = \angle TQP = \alpha$.

Therefore, $\angle RDQ = \angle RQD$ and so $\triangle RQD$ is an isosceles triangle with $QR = RD$.

4. (a) Since $T(4) = 10$ and $T(10) = 55$, then $T(a) = T(10) - T(4) = 45$.

That is, $\dfrac{a(a+1)}{2} = 45$ or $a^2 + a = 90$, and so $a^2 + a - 90 = 0$.

Since $a > 0$ and $(a-9)(a+10) = 0$, then $a = 9$.

(b) The left side of the equation, $T(b+1) - T(b)$, gives $\dfrac{(b+1)(b+2)}{2} - \dfrac{b(b+1)}{2}$, which simplifies to $\dfrac{b^2 + 3b + 2 - b^2 - b}{2}$ or $\dfrac{2b+2}{2}$ or $b + 1$.

That is, $b + 1$ is equal to $T(x)$, a triangular number.

Since $b > 2011$, we are looking for the the smallest triangular number greater than 2012.

After some trial and error, we observe that $T(62) = 1953$ and $T(63) = 2016$, and so $b + 1 = 2016$ or $b = 2015$ is the smallest value that works.

(c) Since $T(28) = 406$, the second equation gives $c + d + e = 406$ or $e = 406 - (c + d)$.

Next, we simplify the first equation.

$$\begin{aligned} T(c) + T(d) &= T(e) \\ \frac{c(c+1)}{2} + \frac{d(d+1)}{2} &= \frac{e(e+1)}{2} \\ c(c+1) + d(d+1) &= e(e+1) \end{aligned}$$

We now substitute $e = 406 - (c+d)$ into this equation above and simplify.

$$\begin{aligned} c(c+1)+d(d+1) &= e(e+1) \\ c(c+1)+d(d+1) &= (406-(c+d))(407-(c+d)) \\ c^2+c+d^2+d &= 406 \times 407 - 406(c+d) - 407(c+d) + (c+d)^2 \\ c^2+c+d^2+d &= 406 \times 407 - 813(c+d) + (c+d)^2 \\ c^2+c+d^2+d &= 406 \times 407 - 813(c+d) + c^2 + 2cd + d^2 \\ c+d &= 406 \times 407 - 813(c+d) + 2cd \\ 2cd &= c+d+813(c+d) - 406 \times 407 \\ 2cd &= 814(c+d) - 406 \times 407 \\ cd &= 407(c+d) - 203 \times 407 \\ cd &= 407(c+d-203), \end{aligned}$$

as required.

(d) *Solution 1*

Using the result from part (c), we are looking to find all triples $(c, d, e)$ of positive integers, where $c \leq d \leq e$, such that $cd = 407(c+d-203)$.

Since the right side of this equation is divisible by 407, then the left side must also be divisible by 407.

Observe that $407 = 37 \times 11$.

Since $cd$ is divisible by 407 and 407 is divisible by 37, then $cd$ is divisible by 37.

Since 37 is a prime number, then one of $c$ or $d$ must be divisible by 37.

Since $c+d+e = 406$ then $d+e \leq 406$.

Since $d \leq e$, then $d+d \leq 406$ or $d \leq 203$.

Therefore, $c \leq d \leq 203$.

Thus, one of $c$ or $d$ is a multiple of 37 that is less than 203.

The largest multiple of 37 less than 203 is $5 \times 37 = 185$.

Next, we try the values $d = 37, 74, 111, 148, 185$ in the equation $cd = 407(c+d-203)$ to see if we get an integer value for $c$.

The system of equations that we are solving is symmetric in $c$ and $d$.

That is, exchanging $c$ and $d$ in the two equations yields the same two equations and thus the same solutions, but with $c$ and $d$ switched.

Therefore, if we happened to get a value of $c$ larger than the value of $d$ that we were trying, then we could just switch them.

In trying the possible values $d = 37, 74, 111, 148, 185$, we only obtain an integer value for $c$ when $d = 185$.

The only triple $(c, d, e)$, where $c \leq d \leq e$, such that $cd = 407(c+d-203)$ is $(33, 185, 188)$.

*Solution 2*

Using the result from part (c), we are looking to find all triples $(c, d, e)$ of positive integers, where $c \leq d \leq e$, such that $cd = 407(c+d-203)$.

Since the right side of this equation is divisible by 407, then the left side must also be divisible by 407.

Observe that $407 = 37 \times 11$.

Since $cd$ is divisible by 407 and 407 is divisible by 37, then $cd$ is divisible by 37.

Since 37 is a prime number, then one of $c$ or $d$ must be divisible by 37.

Suppose that $d$ is divisible by 37, or that $d = 37n$ for some positive integer $n$.

(We will consider the possibility that it is $c$ that is divisible by 37 later in the solution.)
Since $c + d + e = 406$ and $c, d, e$ are positive integers, then $1 \le d \le 404$ or $1 \le n \le 10$.
With $d = 37n$ our equation $cd = 407(c + d - 203)$ becomes $37cn = 407(c + 37n - 203)$.
Dividing through by 37, we get $cn = 11(c+37n-203)$ or $cn - 11c = 11 \times 37n - 11 \times 203$.
Isolating $c$ in this equation we have $c(n - 11) = 407n - 2233$ or $c = \dfrac{407n - 2233}{n - 11}$.
Since the numerator $407n-2233$ can be written as $407n-4477+2244$ or $407(n-11)+2244$, then we have $c = \dfrac{407(n - 11) + 2244}{n - 11}$ or $c = \dfrac{407(n - 11)}{n - 11} + \dfrac{2244}{n - 11}$ or $c = 407 + \dfrac{2244}{n - 11}$.
Since $c$ is a positive integer, then $n - 11$ must divide 2244.
Since $1 \le n \le 10$, then $-10 \le n - 11 \le -1$.
Thus, the only possibilities for $n - 11$ are $-1, -2, -3, -4$, and $-6$.
However, of these 5 possibilities only $n - 11 = -6$ gives a positive value for $c$.
Since $n - 11 = -6$, then $n = 5$, $d = 37 \times 5 = 185$, $c = 33$ and $e = 406 - (c + d) = 188$.
A triple $(c, d, e)$, where $c \le d \le e$, such that $cd = 407(c + d - 203)$ is $(33, 185, 188)$.

Earlier in this solution we made the assumption that $d$ was divisible by 37.
Suppose that it is $c$ that is divisible by 37 or that $c = 37n$ for some positive integer $n$.
Since $c + d + e = 406$ and $c, d, e$ are positive integers, then $1 \le c \le 404$ or $1 \le n \le 10$.
With $c = 37n$ our equation $cd = 407(c + d - 203)$ becomes $37dn = 407(37n + d - 203)$.
Dividing through by 37, we get $dn = 11(37n+d-203)$ or $dn - 11d = 11 \times 37n - 11 \times 203$.
Isolating $d$ in this equation we have $d(n - 11) = 407n - 2233$ or $d = \dfrac{407n - 2233}{n - 11}$.
Since the numerator $407n-2233$ can be written as $407n-4477+2244$ or $407(n-11)+2244$, then we have $d = \dfrac{407(n - 11) + 2244}{n - 11}$ or $d = \dfrac{407(n - 11)}{n - 11} + \dfrac{2244}{n - 11}$ or $d = 407 + \dfrac{2244}{n - 11}$.
Since $d$ is a positive integer, then $n - 11$ must divide 2244.
Since $1 \le n \le 10$, then $-10 \le n - 11 \le -1$.
Thus, the only possibilities for $n - 11$ are $-1, -2, -3, -4$, and $-6$.
However, of these 5 possibilities only $n - 11 = -6$ gives a positive value for $d$.
Since $n - 11 = -6$, then $n = 5$, $c = 37 \times 5 = 185$, $d = 33$ and $e = 406 - (c + d) = 188$.
Since there is a restriction that $c \le d \le e$, then this solution is not possible.

The only triple $(c, d, e)$, where $c \le d \le e$, such that $cd = 407(c+d-203)$ is $(33, 185, 188)$.

Canadian
Mathematics
Competition
*An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario*

# 2010 Galois Contest

## Friday, April 9, 2010

*Solutions*

1. (a) Emily's new showerhead uses 13 L of water per minute.
At this rate, it will take Emily $\frac{260}{13} = 20$ minutes of showering to use 260 L of water.

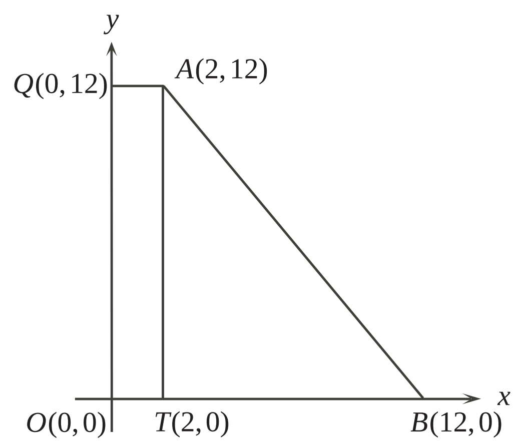
(b) By using the new showerhead, Emily is using $18 - 13 = 5$ L of water per minute less than when she used the old showerhead.
Thus for a 10 minute shower using the new showerhead, Emily uses $10 \times 5 = 50$ L less water.

(c) From part (b), we know that Emily saves 5 L of water per minute by using the new showerhead.
For a 15 minute shower, Emily saves $15 \times 5 = 75$ L of water.
Since Emily is charged 8 cents per 100 L of water, she will save $\frac{8}{100} \times 75 = 6$ cents in water costs for a 15 minute shower.

(d) *Solution 1*
Emily is charged 8 cents per 100 L of water and is saving 5 L of water per minute.
Thus, she is saving $\frac{8}{100} \times 5 = \frac{8}{20} = \frac{2}{5}$ of a cent per minute by using the new showerhead.
To save \$30 or 3000 cents, it will take Emily $3000 \div \frac{2}{5} = 3000 \times \frac{5}{2} = 7500$ minutes of showering.

*Solution 2*
From part (c), Emily saves 6 cents in 15 minutes of showering. Since $3000 \div 6 = 500$, it takes $15 \times 500 = 7500$ minutes to save \$30.

2. (a) *Solution 1*
If point $T$ is placed at $(2, 0)$, then $T$ is on $OB$ and $AT$ is perpendicular to $OB$.
Since $QO$ is perpendicular to $OB$, then $QO$ is parallel to $AT$.
Both $QA$ and $OT$ are horizontal, so then $QA$ is parallel to $OT$.
Therefore, $QATO$ is a rectangle.
The area of rectangle $QATO$ is $QA \times QO$ or $(2 - 0) \times (12 - 0) = 24$.
Since $AT$ is perpendicular to $TB$, we can treat $AT$ as the height of $\triangle ATB$ and $TB$ as the base.
The area of $\triangle ATB$ is $\frac{1}{2} \times TB \times AT$ or $\frac{1}{2} \times (12 - 2) \times (12 - 0) = \frac{1}{2} \times 10 \times 12 = 60$.
The area of $QABO$ is the sum of the areas of rectangle $QATO$ and $\triangle ATB$, or $24 + 60 = 84$.




*Solution 2*
Both $QA$ and $OB$ are horizontal, so then $QA$ is parallel to $OB$.
Thus, $QABO$ is a trapezoid.
Since $QO$ is perpendicular to $OB$, we can treat $QO$ as the height of the trapezoid.
Then, $QABO$ has area $\frac{1}{2} \times QO \times (QA + OB) = \frac{1}{2} \times 12 \times (2 + 12) = \frac{1}{2} \times 12 \times 14 = 84$.

(b) Since $CO$ is perpendicular to $OB$, we can treat $CO$ as the height of $\triangle COB$ and $OB$ as the base. The area of $\triangle COB$ is $\frac{1}{2} \times OB \times CO$ or $\frac{1}{2} \times (12 - 0) \times (p - 0) = \frac{1}{2} \times 12 \times p = 6p$.

(c) Since $QA$ is perpendicular to $QC$, we can treat $QC$ as the height of $\triangle QCA$ and $QA$ as the base. The area of $\triangle QCA$ is $\frac{1}{2} \times QA \times QC$ or $\frac{1}{2} \times (2 - 0) \times (12 - p) = \frac{1}{2} \times 2 \times (12 - p) = 12 - p$.

(d) The area of $\triangle ABC$ can be found by subtracting the area of $\triangle COB$ and the area of $\triangle QCA$ from the area of quadrilateral $QABO$.
From parts (a), (b) and (c), the area of $\triangle ABC$ is thus $84 - 6p - (12 - p) = 72 - 5p$.
Since the area of $\triangle ABC$ is 27, then $72 - 5p = 27$ or $5p = 45$, so $p = 9$.

3. (a) We solve the system of equations by the method of elimination.
Adding the first equation to the second, we get $2x = 52$, and so $x = 26$.
Substituting $x = 26$ into the first equation, we get $26 + y = 42$, and so $y = 16$.
The solution to the system of equations is $(x, y) = (26, 16)$.

(b) *Solution 1*
We proceed as in part (a) by solving the system of equations by the method of elimination.
Adding the first equation to the second, we get $2x = p + q$, and so $x = \frac{p+q}{2}$.
We are given that $p$ is an even integer and that $q$ is an odd integer.
The sum of an even integer and an odd integer is always an odd integer.
Thus, $\frac{p+q}{2}$ is an odd integer divided by two, which is never an integer.
Therefore, the given system of equations has no positive integer solutions.

*Solution 2*
We proceed as in part (a) by solving the system of equations by the method of elimination.
Adding the first equation to the second, we get $2x = p+q$. Since the sum of an even integer and an odd integer is always an odd integer, the right side of the equation $2x = p+q$ is always odd. However, the left side of this equation is always even for any integer $x$.
Therefore, the given system of equations has no positive integer solutions.

(c) The left side of the equation, $x^2 - y^2$, is a difference of squares.
Factoring $x^2 - y^2$, then the equation $x^2 - y^2 = 420$ becomes $(x + y)(x - y) = 420$.
Since $x$ and $y$ are positive integers, then $x + y$ is a positive integer.
Since $(x + y)(x - y) = 420$ and $x + y$ is a positive integer, then $x - y$ is a positive integer.
Since $x$ and $y$ are positive integers, then $x + y > x - y$.
Thus, we are searching for pairs of positive integers whose product is equal to 420.
We list all of the possibilities below where $x + y > x - y$:

| $x+y$ | $x-y$ | $(x+y)(x-y)$ |
|---|---|---|
| 420 | 1 | 420 |
| 210 | 2 | 420 |
| 140 | 3 | 420 |
| 105 | 4 | 420 |
| 84 | 5 | 420 |
| 70 | 6 | 420 |
| 60 | 7 | 420 |
| 42 | 10 | 420 |
| 35 | 12 | 420 |
| 30 | 14 | 420 |
| 28 | 15 | 420 |
| 21 | 20 | 420 |

Each of the pairs of factors listed above determines a system of equations.
For example the first pair, 420 and 1, gives the system of equations:

$$\begin{aligned} x + y &= 420 \\ x - y &= 1 \end{aligned}$$

From part (b), we know that for positive integer solutions $(x, y)$ of this system of equations to exist, one of the factors cannot be odd if the other is even.
Thus, we may eliminate the pairs of factors that have different parity (one factor is odd and the other factor is even) from our table above.
The following possibilities remain:

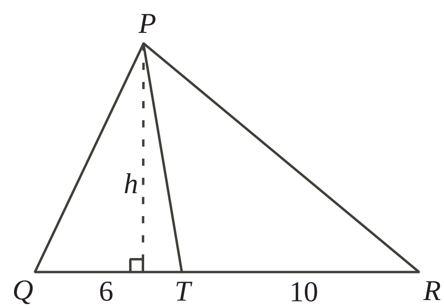
| $x+y$ | $x-y$ | $(x+y)(x-y)$ |
|---|---|---|
| 210 | 2 | 420 |
| 70 | 6 | 420 |
| 42 | 10 | 420 |
| 30 | 14 | 420 |

We also know from part (b), that to determine $x$ for each of the 4 systems of equations generated by the table above, we add the two factors and then divide the sum by 2.
The value of $y$ is then determined by substituting $x$ back into either equation.
We complete the solutions in the table below:

| $x+y$ | $x-y$ | $(x+y)(x-y)$ | $2x$ | $x$ | $y$ |
|---|---|---|---|---|---|
| 210 | 2 | 420 | 212 | 106 | 104 |
| 70 | 6 | 420 | 76 | 38 | 32 |
| 42 | 10 | 420 | 52 | 26 | 16 |
| 30 | 14 | 420 | 44 | 22 | 8 |

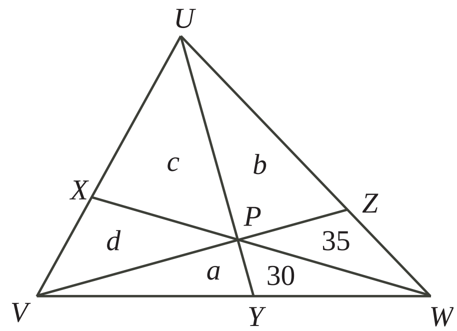
Therefore, the pairs of positive integers $(x, y)$ that satisfy $x^2 - y^2 = 420$ are $(106, 104), (38, 32), (26, 16)$, and $(22, 8)$.

4. (a) Construct the altitude of $\triangle PQT$ from $P$ to $QT$.
Let the length of the altitude be $h$.
Note that this altitude of $\triangle PQT$ is also the altitude of $\triangle PTR$.
The ratio of the area of $\triangle PQT$ to the area of $\triangle PTR$ is $\dfrac{\frac{1}{2} \times QT \times h}{\frac{1}{2} \times TR \times h} = \dfrac{QT}{TR} = \dfrac{6}{10} = \dfrac{3}{5}$.




(b) From part (a), we can generalize the fact that if two triangles have their bases along the same straight line and they share a common vertex that is not on this line, then the ratio of their areas is equal to the ratio of the lengths of their bases.
This generalization will be used throughout the solutions to parts (b) and (c).
We will also adopt the notation $|\triangle XYZ|$ to represent the area of $\triangle XYZ$.
Using the fact above, $\dfrac{|\triangle AEF|}{|\triangle DEF|} = \dfrac{AF}{FD} = \dfrac{3}{1}$. Thus, $|\triangle AEF| = 3 \times |\triangle DEF| = 3(17) = 51$.
Then, $|\triangle AED| = |\triangle AEF| + |\triangle DEF| = 51 + 17 = 68$.
Also, $\dfrac{|\triangle ECD|}{|\triangle AED|} = \dfrac{EC}{AE} = \dfrac{2}{1}$. Thus, $|\triangle ECD| = 2 \times |\triangle AED| = 2(68) = 136$.
Then, $|\triangle DCA| = |\triangle ECD| + |\triangle AED| = 136 + 68 = 204$.
Since $D$ is the midpoint of $BC$, $\dfrac{BD}{DC} = \dfrac{1}{1}$, and $\dfrac{|\triangle BDA|}{|\triangle DCA|} = \dfrac{BD}{DC} = \dfrac{1}{1}$.
Then, $|\triangle BDA| = |\triangle DCA| = 204$ and $|\triangle ABC| = |\triangle BDA| + |\triangle DCA| = 204 + 204 = 408$.

(c) Let the area of $\triangle PYV$, $\triangle PZU$, $\triangle UXP$, and $\triangle XVP$, be $a$, $b$, $c$, and $d$, respectively.




Since $\dfrac{|\triangle PYV|}{|\triangle PYW|} = \dfrac{VY}{YW} = \dfrac{4}{3}$,
then $|\triangle PYV| = \frac{4}{3} \times |\triangle PYW| = \frac{4}{3}(30) = 40$.
Thus, $a = 40$.
Also, $\dfrac{|\triangle VZW|}{|\triangle VZU|} = \dfrac{ZW}{ZU} = \dfrac{|\triangle PZW|}{|\triangle PZU|}$ or $|\triangle VZW| \times |\triangle PZU| = |\triangle PZW| \times |\triangle VZU|$.

Thus, $\dfrac{|\triangle VZU|}{|\triangle PZU|} = \dfrac{|\triangle VZW|}{|\triangle PZW|} = \dfrac{35+30+40}{35} = \dfrac{105}{35} = \dfrac{3}{1}$.
Therefore, $\dfrac{|\triangle VZU|}{|\triangle PZU|} = \dfrac{3}{1}$, or $\dfrac{b+c+d}{b} = \dfrac{3}{1}$ or $b+c+d = 3b$ and $c+d = 2b$.
Next, $\dfrac{|\triangle UVY|}{|\triangle UYW|} = \dfrac{VY}{YW} = \dfrac{4}{3}$, so $\dfrac{40+c+d}{30+35+b} = \dfrac{4}{3}$.
Since $c+d = 2b$, we have $3(40+2b) = 4(65+b)$, so $120+6b = 260+4b$,
then $2b = 140$ and $b = 70$.
Next, $\dfrac{|\triangle UXW|}{|\triangle XVW|} = \dfrac{UX}{XV} = \dfrac{|\triangle UXP|}{|\triangle XVP|}$, or $\dfrac{35+b+c}{30+a+d} = \dfrac{c}{d}$.
Since $b = 70$ and $a = 40$, $\dfrac{105+c}{70+d} = \dfrac{c}{d}$, or $d(105+c) = c(70+d)$.
Thus, $105d + cd = 70c + cd$ or $105d = 70c$, and $\dfrac{d}{c} = \dfrac{70}{105} = \dfrac{2}{3}$ or $d = \dfrac{2}{3}c$.
Since $c+d = 2b = 2(70) = 140$, we have $c+d = c + \frac{2}{3}c = \frac{5}{3}c = 140$, or $c = \frac{3}{5}(140) = 84$.
Therefore, the area of $\triangle UXP$ is 84.

**Canadian Mathematics Competition**

***An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario***

# *2009 Galois Contest*

## Wednesday, April 8, 2009

*Solutions*

1. (a) Since the total number of students in the class is $8+7+3+2=20$, then the fraction of students having blonde hair is $\frac{8}{20}$.
Thus, $\frac{8}{20}\times 100\% = 40\%$ of the students in the class have blonde hair.

   (b) Since there are 3 students with red hair and 2 students with black hair, then the number of students that have red or black hair is 5.
Thus, the fraction of students with red or black hair is $\frac{5}{20}$.
The percentage of students in the class with red or black hair is $\frac{5}{20}\times 100\% = 25\%$.

   (c) First note that if some number of students in the class with blonde hair dye their hair black, the total number of students in the class remains unchanged at 20.
For 20% of the class to have black hair, 20% of 20 or 4 students need to have black hair.
Since there are presently 2 students with black hair, 2 students with blonde hair would need to dye their hair black.

   (d) If $x$ students with red hair join the class, the number of students with red hair will be $3+x$, and the total number of students in the class will be $20+x$.
Thus, the new fraction of students having red hair will be $\frac{3+x}{20+x}$.
Since 32% is equivalent to $\frac{32}{100}=\frac{8}{25}$, we need to find the value of $x$ satisfying $\frac{3+x}{20+x}=\frac{8}{25}$.
Solving, we get $25(3+x)=8(20+x)$ or $75+25x=160+8x$ or $17x=85$ or $x=5$.
Thus, 5 students with red hair would have to join the class so the percentage of students in the class with red hair is equal to 32%.

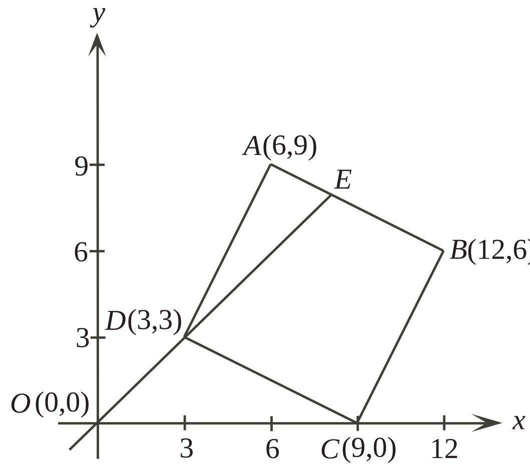
2. (a) *Solution 1*
Since $ABCD$ is a square, the path travelling from $A$ to $B$ is the same as the path travelling from $D$ to $C$. To travel from $A$ to $B$, we go 6 units right and 3 units down.
Thus, $C$ has coordinates $(3+6, 3-3)=(9,0)$, so $t=9$.

   (b) *Solution 2*
The slope of $CD$ is $\frac{3-0}{3-t}$ and the slope of $AB$ is $\frac{6-9}{12-6}=\frac{-3}{6}=-\frac{1}{2}$.
Since $ABCD$ is a square, $CD$ is parallel to $AB$.
Since parallel line segments have equal slopes, we get

$$\begin{aligned}\frac{3}{3-t} &= -\frac{1}{2}\\ (3)(2) &= (-1)(3-t)\\ 6 &= -3+t\\ t &= 9\end{aligned}$$

Therefore, the $x$-coordinate of vertex $C$ is 9.

(c)




First, we find the equation of the line through $O$ and $D$.
The slope of the line through $O(0,0)$ and $D(3,3)$ is $\dfrac{3-0}{3-0} = 1$.
Since this line has $y$-intercept 0, the equation of the line is $y = x$.
Next, we find the equation of the line through $A$ and $B$.
As in part (a), the slope of the line through $A$ and $B$ is $-\frac{1}{2}$.
Therefore, the line has equation $y = -\frac{1}{2}x + b$ for some $b$.
Since $B(12, 6)$ lies on this line, then $6 = -\frac{1}{2}(12) + b$ so $6 = -6 + b$ or $b = 12$.
Thus, the equation of the line is $y = -\frac{1}{2}x + 12$.
These lines, $y = x$ and $y = -\frac{1}{2}x + 12$, intersect when $x = -\frac{1}{2}x + 12$ or $\frac{3}{2}x = 12$ or $x = 8$.
Therefore, point $E$ has coordinates (8,8).

(d) The required lengths are as follows,

$$ED = \sqrt{(8-3)^2 + (8-3)^2} = \sqrt{50} = 5\sqrt{2}$$

and

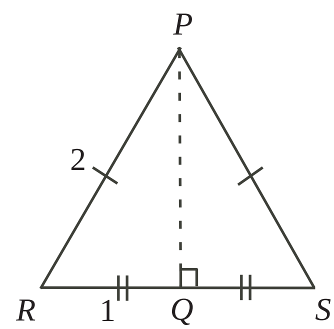
$$EB = \sqrt{(8-12)^2 + (8-6)^2} = \sqrt{20} = 2\sqrt{5}$$

and

$$CD = CB = \sqrt{(12-9)^2 + (6-0)^2} = \sqrt{45} = 3\sqrt{5}.$$

Thus, the perimeter of quadrilateral $EBCD$ is $5\sqrt{2} + 2\sqrt{5} + 2 \times 3\sqrt{5} = 5\sqrt{2} + 8\sqrt{5}$.

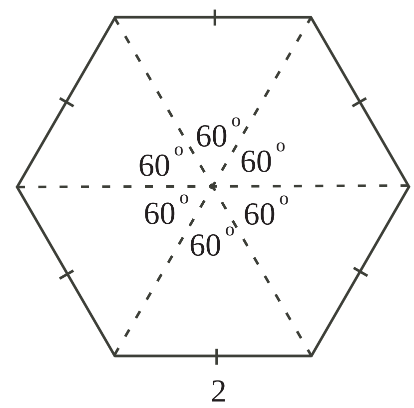
3. (a) Equilateral triangle $PRS$ has side lengths equal to 2.
Since $PR = PS$, the perpendicular from $P$ meets $RS$ at its midpoint $Q$ as shown. Thus, $RQ = QS = 1$ and $\triangle PRQ$ is a right triangle.
Using the Pythagorean Theorem,



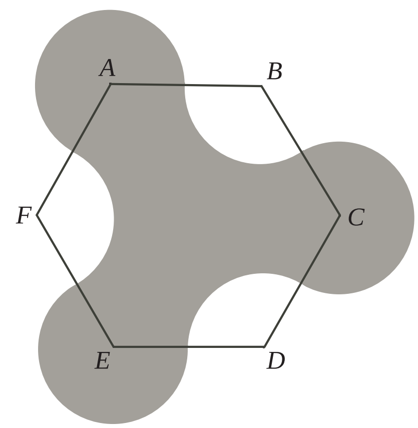

$$\begin{aligned} PR^2 &= RQ^2 + QP^2 \\ 2^2 &= 1^2 + QP^2 \\ 4 &= 1 + QP^2 \\ 3 &= QP^2 \\ QP &= \sqrt{3} \quad (\text{since } QP > 0) \end{aligned}$$

The area of an equilateral triangle with side length 2 is, $\frac{1}{2}(RS)(QP) = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$.

(b) Using 6 equilateral triangles with side length 2, we can create the regular hexagon as shown.
Let us justify that these 6 equilateral triangles will meet in a common point at the hexagon centre without any overlap or gaps between the triangles.
The angle at each vertex of an equilateral triangle is $60^\circ$. When 6 of these vertices meet at a common point, the sum of the angles is $6 \times 60^\circ = 360^\circ$, a complete rotation as required. Also, note that the sides of the hexagon formed each have length 2 and each interior angle of the hexagon measures $60^\circ + 60^\circ = 120^\circ$. Thus, we can be assured that exactly 6 equilateral triangles with side length 2 can produce a regular hexagon with side length 2.
Then, the area of the regular hexagon is 6 times the area of the equilateral triangle from part (a) or
$A = 6 \times \sqrt{3} = 6\sqrt{3}$.




(c) The construction of the hexagon in part (b) gives each interior angle measuring $120^\circ$.
Since the interior angle at each of $B$, $D$ and $F$ is $120^\circ$, then the unshaded sector inside the hexagon at each of these points is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a full circle.
Thus, the area of each of these sectors is $\frac{1}{3} \times \pi(1)^2 = \frac{1}{3}\pi$.
Therefore, the total area of the 3 unshaded sectors inside the hexagon, is $3 \times \frac{1}{3}\pi = \pi$.
Since each interior angle of the hexagon measures $120^\circ$, the measure of the reflex angle at each of $A$, $C$ and $E$ is $360^\circ - 120^\circ = 240^\circ$.
Thus, the shaded sector outside the hexagon at each of these points is $\frac{240^\circ}{360^\circ} = \frac{2}{3}$ of a full circle.
Thus, the area of each of these sectors is $\frac{2}{3} \times \pi(1)^2 = \frac{2}{3}\pi$.
Therefore, the total area of the 3 shaded sectors outside the hexagon, is $3 \times \frac{2}{3}\pi = 2\pi$.
The shaded area consists of the entire hexagon, minus the three unshaded sectors at $B$, $D$ and $F$, plus the three shaded sectors at $A$, $C$ and $E$, or $6\sqrt{3} - \pi + 2\pi = 6\sqrt{3} + \pi$.




4. (a) The largest positive integer $N$ that can be written in this form is obtained by maximizing the values of the integers $a$, $b$, $c$, $d$, and $e$. Thus, $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $e = 5$, which gives $N = 1(1!) + 2(2!) + 3(3!) + 4(4!) + 5(5!) = 1 + 2(2) + 3(6) + 4(24) + 5(120) = 719$.

(b) For any two positive integers $n$ and $m$, it is always possible to write a division statement of the form,

$$n = qm + r,$$

where the *quotient* $q$ and *remainder* $r$ are non-negative integers and $0 \leq r < m$.
The following table shows some examples of this.

| $n$ | $m$ | $q$ | $r$ | $n = qm + r$ |
|---|---|---|---|---|
| 20 | 6 | 3 | 2 | $20 = 3(6) + 2$ |
| 12 | 13 | 0 | 12 | $12 = 0(13) + 12$ |
| 9 | 7 | 1 | 2 | $9 = 1(7) + 2$ |
| 36 | 9 | 4 | 0 | $36 = 4(9) + 0$ |

Notice that in each of the 4 examples, the inequality $0 \le r < m$ has been satisfied.
We can always satisfy this inequality by beginning with $n$ and then subtracting multiples of $m$ from it until we get a number in the range 0 to $m - 1$. We let $r$ be this number, or $r = n - qm$, so that $n = qm + r$.
Further, this process is repeatable. For example, beginning with $n = 653$ and $m = 5! = 120$, we get $653 = 5(120) + 53$. We can now repeat the process using remainder $r = 53$ as our next $n$, and $4! = 24$ as our next $m$. This process is shown in the table below with each new remainder becoming our next $n$ and $m$ taking the successive values of 5!, 4!, 3!, 2!, and 1!.

| $n$ | $m$ | $q$ | $r$ | $n = qm + r$ |
|---|---|---|---|---|
| 653 | 120 | 5 | 53 | $653 = 5(120) + 53$ |
| 53 | 24 | 2 | 5 | $53 = 2(24) + 5$ |
| 5 | 6 | 0 | 5 | $5 = 0(6) + 5$ |
| 5 | 4 | 1 | 1 | $5 = 2(2) + 1$ |
| 1 | 1 | 1 | 0 | $1 = 1(1) + 0$ |

From the 5th column of the table above,

$$\begin{aligned} 653 &= 5(120) + 53 \\ &= 5(120) + 2(24) + 5 \\ &= 5(120) + 2(24) + 0(6) + 5 \\ &= 5(120) + 2(24) + 0(6) + 2(2) + 1 \\ &= 5(120) + 2(24) + 0(6) + 2(2) + 1(1) + 0 \\ &= 5(5!) + 2(4!) + 0(3!) + 2(2!) + 1(1!) \end{aligned}$$

Thus, $n = 653$ is written in the required form with $a = 1$, $b = 2$, $c = 0$, $d = 2$, and $e = 5$.

(c) Following the process used in (b) above, we obtain the more general result shown here.

| $n$ | $m$ | $q$ | $r$ | $n = qm + r$ | restriction on $r$ |
|---|---|---|---|---|---|
| $n$ | 120 | $e$ | $r_1$ | $n = e(120) + r_1$ | $0 \le r_1 < 120$ |
| $r_1$ | 24 | $d$ | $r_2$ | $r_1 = d(24) + r_2$ | $0 \le r_2 < 24$ |
| $r_2$ | 6 | $c$ | $r_3$ | $r_2 = c(6) + r_3$ | $0 \le r_3 < 6$ |
| $r_3$ | 2 | $b$ | $r_4$ | $r_3 = b(2) + r_4$ | $0 \le r_4 < 2$ |
| $r_4$ | 1 | $a$ | $r_5$ | $r_4 = a(1) + r_5$ | $0 \le r_5 < 1$ |

From the 5th column of this table,

$$\begin{aligned} n &= e(120) + r_1 \\ &= e(120) + d(24) + r_2 \\ &= e(120) + d(24) + c(6) + r_3 \\ &= e(120) + d(24) + c(6) + b(2) + r_4 \\ &= e(120) + d(24) + c(6) + b(2) + a(1) + r_5 \\ &= e(5!) + d(4!) + c(3!) + b(2!) + a(1!) \text{ (since } r_5 = 0) \end{aligned}$$

We must justify that the integers $a$, $b$, $c$, $d$, and $e$ satisfy their required inequality.
From part (b), each of these quotients is a non-negative integer. Therefore, it remains to show that $a \leq 1$, $b \leq 2$, $c \leq 3$, $d \leq 4$, and $e \leq 5$.
From part (a), $N = 719$, therefore $0 \leq n < 720$.
From the table above, we have $n = e(120)+r_1$. Therefore $e(120)+r_1 < 720$ or $e(120) < 720$ (since $r_1 \geq 0$), and so $e < 6$. Thus, $e \leq 5$, as required.
Also from the table above, $r_1 < 120$, so $d(24) + r_2 < 120$ or $d(24) < 120$ (since $r_2 \geq 0$), and therefore $d < 5$. Thus, $d \leq 4$, as required.
Also, $r_2 < 24$, so $c(6) + r_3 < 24$ or $c(6) < 24$ (since $r_3 \geq 0$), and therefore $c < 4$.
Thus, $c \leq 3$, as required.
Continuing, $r_3 < 6$, so $b(2) + r_4 < 6$ or $b(2) < 6$ (since $r_4 \geq 0$), and therefore $b < 3$.
Thus, $b \leq 2$, as required.
Finally, $r_4 < 2$, so $a(1) + r_5 < 2$ or $a(1) < 2$ (since $r_5 = 0$), and therefore $a < 2$.
Thus, $a \leq 1$, as required.
Therefore, all integers $n$, with $0 \leq n \leq N$, can be written in the required form.

(d) Since $c = 0$, we are required to find the sum of all integers $n$ of the form $n = a + 2b + 24d + 120e$, with the stated restrictions on the integers $a$, $b$, $d$, and $e$.
Since $n = a + 2b + 24d + 120e = (a + 2b) + 24(d + 5e)$, let $n_1 = a + 2b$ and $n_2 = d + 5e$ so that $n = n_1 + 24n_2$. First, consider all possible values of $n_1$.
Since $0 \leq a \leq 1$ and $0 \leq b \leq 2$ and $n_1 = a + 2b$, we have that $n_1$ can equal any of the numbers in the set $\{0, 1, 2, 3, 4, 5\}$. Each of these comes from exactly one pair $(a, b)$.
Next, find all possible values for $n_2 = d + 5e$. Since $0 \leq d \leq 4$ and $0 \leq e \leq 5$, we have that $d + 5e$ can equal any of the numbers in the set $\{0, 1, 2, 3, 4, 5, 6, 7, \ldots, 29\}$.
Each of these comes from exactly one pair $(d, e)$.
Therefore, $24n_2$ can equal any of the numbers in the set $\{24 \times 0, 24 \times 1, 24 \times 2, \ldots, 24 \times 29\} = \{0, 24, 48, \ldots, 696\}$, the multiples of 24 from 0 to 696.
Adding each of these possible values of $24n_2$ in turn to each of the 6 possible values of $n_1$, we get the set of all possible $n = n_1 + 24n_2$:

$$\{0, 1, 2, 3, 4, 5, 24, 25, 26, 27, 28, 29, 48, 49, 50, 51, 52, 53, \ldots, 696, 697, 698, 699, 700, 701\}$$

Because each of the 6 possible values of $n_1$ comes from exactly one pair $(a, b)$ and each of the 30 possible values of $n_2$ comes from exactly one pair $(d, e)$, then each of these integers above occurs exactly once as $a$, $b$, $d$, and $e$ move through their possible values.
It remains to find the sum of these possible values for $n$:

$$\begin{aligned}
& 0 + 1 + 2 + 3 + 4 + 5 + 24 + 25 + 26 + 27 + 28 + 29 + 48 + 49 + \cdots + 699 + 700 + 701 \\
= \ & 0 + 1 + 2 + 3 + 4 + 5 + (24 + 0) + (24 + 1) + (24 + 2) + (24 + 3) + (24 + 4) + (24 + 5) \\
& +(48 + 0) + (48 + 1) + \cdots + (696 + 3) + (696 + 4) + (696 + 5) \\
= \ & (0 + 1 + 2 + 3 + 4 + 5) + 24 \times 6 + (0 + 1 + 2 + 3 + 4 + 5) + 48 \times 6 \\
& +(0 + 1 + 2 + 3 + 4 + 5) + \cdots + 696 \times 6 + (0 + 1 + 2 + 3 + 4 + 5) \\
= \ & 30(0 + 1 + 2 + 3 + 4 + 5) + 24 \times 6 + 48 \times 6 + \cdots + 696 \times 6 \\
= \ & 30(15) + 24(6)[1 + 2 + 3 + \cdots + 29] \\
= \ & 30(15) + 24(6)\left[\frac{29 \times 30}{2}\right] \\
= \ & 63090
\end{aligned}$$

Canadian
Mathematics
Competition
*An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario*

# *2008 Galois Contest*

## Wednesday, April 16, 2008

*Solutions*

1. (a) Since $(36, 25, x)$ is an O'Hara triple, then $\sqrt{36} + \sqrt{25} = x$, or $x = 6 + 5 = 11$.

   (b) Since $(a, 9, 5)$ is an O'Hara triple, then $\sqrt{a} + \sqrt{9} = 5$, or $\sqrt{a} + 3 = 5$, so $\sqrt{a} = 2$ or $a = 4$.

   (c) We want to find integers $a$ and $b$ with $\sqrt{a} + \sqrt{b} = 6$.
   We can find five such pairs by trying

   - $\sqrt{a} = 5$ with $\sqrt{b} = 1$ so $a = 25$ and $b = 1$,
   - $\sqrt{a} = 4$ with $\sqrt{b} = 2$ so $a = 16$ and $b = 4$,
   - $\sqrt{a} = 3$ with $\sqrt{b} = 3$ so $a = 9$ and $b = 9$,
   - $\sqrt{a} = 2$ with $\sqrt{b} = 4$ so $a = 4$ and $b = 16$,
   - $\sqrt{a} = 1$ with $\sqrt{b} = 5$ so $a = 1$ and $b = 25$.

   Therefore, five O'Hara triples with $x = 6$ are $(25, 1, 6)$, $(16, 4, 6)$, $(9, 9, 6)$, $(4, 16, 6)$, $(1, 25, 6)$.
   (Note that we are not asked to prove that these are the only triples, only to find five of them.)

2. (a) The line has slope $\dfrac{9-5}{6-0} = \dfrac{4}{6} = \dfrac{2}{3}$.
   Since the line passes through $P(0, 5)$, then its $y$-intercept is 5.
   Thus, the equation of the line is $y = \frac{2}{3}x + 5$.

   (b) A line that is perpendicular to the line from (a) must have slope equal to the negative reciprocal of $\frac{2}{3}$. That is, its slope equals $-\dfrac{1}{\frac{2}{3}} = -\dfrac{3}{2}$.
   Thus, this line has equation $y = -\frac{3}{2}x + b$ for some real number $b$.
   Since the line passes through $Q(6, 9)$, then the point $(6, 9)$ satisfies the equation of the line, so $9 = -\frac{3}{2}(6) + b$ or $9 = -9 + b$ or $b = 18$.
   Therefore, the equation of the line is $y = -\frac{3}{2}x + 18$.

   (c) Since every point on the $x$-axis has $y$-coordinate equal to 0, we find the $x$-coordinate of $R$ by using the equation of the line $y = -\frac{3}{2}x + 18$ and setting $y = 0$ to get $0 = -\frac{3}{2}x + 18$ or $\frac{3}{2}x = 18$ or $x = \frac{2}{3}(18) = 12$.
   Thus, the coordinates of $R$ are $(12, 0)$.

   (d) *Solution 1*
   Triangle $PQR$ is right-angled at $Q$, so its area is $\frac{1}{2}(PQ)(QR)$.
   Since the coordinates of $P$ are $(0, 5)$, of $Q$ are $(6, 9)$, and of $R$ are $(12, 0)$, then

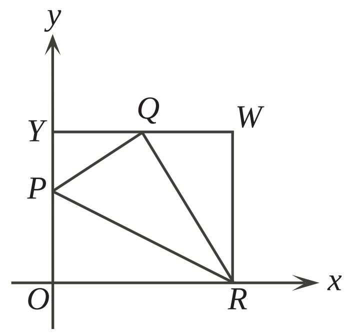
   $$PQ = \sqrt{(6-0)^2 + (9-5)^2} = \sqrt{6^2 + 4^2} = \sqrt{52} = 2\sqrt{13}$$

   and

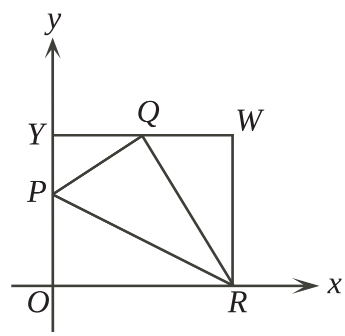
   $$QR = \sqrt{(6-12)^2 + (9-0)^2} = \sqrt{6^2 + 9^2} = \sqrt{117} = 3\sqrt{13}$$

   Therefore, the area is $\frac{1}{2}(2\sqrt{13})(3\sqrt{13}) = 3(13) = 39$.

*Solution 2*
We "complete the rectangle" by drawing a horizontal line through $Q$ meeting the $y$-axis at $Y(0,9)$ and a vertical line through $R$ meeting the previous horizontal line at $W(12,9)$.

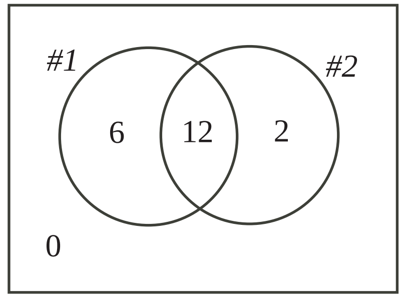



The rectangle $ORWY$ has width 12 and height 9 and so has area $12(9) = 108$.
The area of $\triangle PQR$ is the area of this rectangle minus the areas of $\triangle POR$, $\triangle PYQ$ and $\triangle QWR$.
Each of the these last three triangles is right-angled as it has two sides parallel to the axes.




From the diagram, the area of $\triangle POR$ is $\frac{1}{2}(5)(12) = 30$.
Also, the area of $\triangle PYQ$ is $\frac{1}{2}(4)(6) = 12$.
Lastly, the area of $\triangle QWR$ is $\frac{1}{2}(6)(9) = 27$.
From this information, the area of $\triangle PQR$ is $108 - 30 - 12 - 27 = 39$.

3. (a) The largest possible number who could have answered both questions correctly is 14. (This would happen if all who answered question 2 correctly also answered question 1 correctly.)
There cannot be more than 14 people who answered both questions correctly, because only 14 people answered question 2 correctly.
To find the smallest possible number who answered both questions correctly, we find the largest possible number who answered at least one question incorrectly. To do this, we try to ensure that different students answered each question incorrectly.
Here, 2 students answered question 1 incorrectly and 6 students answered question 2 incorrectly, so at most 8 students answered one question incorrectly. (There would be fewer than 8 students if there was overlap between those who answered each question incorrectly.)
Since at most 8 students answered one question incorrectly, then at least $20 - 8 = 12$ students answered both questions correctly.
We can actually achieve this number by arranging the students as seen in the Venn diagram, where the circles show the number of students who answered each question correctly.



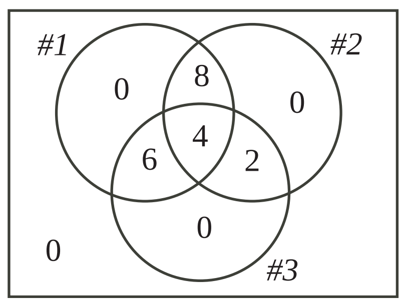

(b) The largest possible number who could have answered all three questions correctly is 12. (This would happen if all who answered question 3 correctly also answered questions 1 and 2 correctly.) There cannot be more than 12 people who answered all three questions correctly, because only 12 people answered question 3 correctly.

To find the smallest possible number who answered all three questions correctly, we find the largest possible number who answered at least one question incorrectly. To do this, we try to ensure that different students answered each question incorrectly.

Here, 2 students answered question 1 incorrectly, 6 students answered question 2 incorrectly, and 8 students answered question 3 incorrectly, so at most 16 students answered one question incorrectly.

Since at most 16 students answered one question incorrectly, then at least $20 - 16 = 4$ students answered all three questions correctly.

We can actually achieve this number by arranging the students as seen in the Venn diagram, where the circles show the number of students who answered each question correctly.




(c) We model our approach from (b).

To find the smallest possible number who answered all three questions correctly, we find the largest possible number who answered at least one question incorrectly. To do this, we try to ensure that different students answered each question incorrectly.

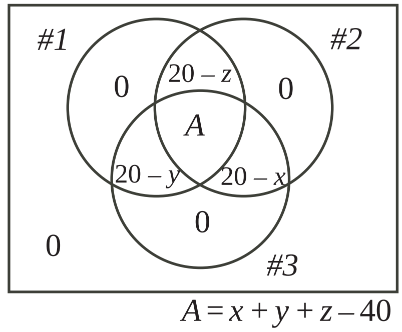
Here, $20-x$ students answered question 1 incorrectly, $20-y$ students answered question 2 incorrectly, and $20-z$ students answered question 3 incorrectly, so at most $60-x-y-z$ students answered one question incorrectly.

(Since $40 \leq x+y+z \leq 60$, then $0 \leq 60-x-y-z \leq 20$, so it makes sense to talk about $60-x-y-z$ students in this context.)

Since at most $60-x-y-z$ students answered one question incorrectly, then at least $20-(60-x-y-z) = x+y+z-40$ students answered all three questions correctly.

(Note that $0 \leq x+y+z-40 \leq 20$ since $40 \leq x+y+z \leq 60$, so $x+y+z-40$ is an admissible number of students.)

We can actually achieve this number by arranging the students as seen in the Venn diagram, where the circles show number of the students who answered each question correctly.




4. (a) The list starts as 1, 2, 3, 4, 5, 6.
If Carolyn removes 2, then Paul removes the remaining positive divisor of 2 (that is, 1) to leave the list 3, 4, 5, 6.
Carolyn must remove a number from this list that has at least one positive divisor other than itself remaining.
The only such number is 6, so Carolyn removes 6 and so Paul removes the remaining positive divisor of 6 (that is, 3), to leave the list 4, 5.
Carolyn cannot remove either of the remaining numbers as neither has a positive divisor other than itself remaining.
Thus, Paul removes 4 and 5.
In summary, Carolyn removes 2 and 6 for a sum of $2 + 6 = 8$ and Paul removes 1, 3, 4, and 5 for a sum of $1 + 3 + 4 + 5 = 13$.

(b) Since Carolyn removes a single number on each turn in such a way that Paul must be able to remove a number from the list, then Carolyn can remove at most half of the numbers of the list. In this case, Carolyn can remove at most five numbers.
The maximum possible five numbers that Carolyn could remove are the largest five numbers from the list (that is, 6, 7, 8, 9, 10), whose sum is 40.
This is the maximum possible without referring to all of the rules of the game. So is it possible for her to remove these five numbers?
In order to do so, she must remove them in an order which forces Paul to remove only one number on each of his turns.
If Carolyn removes 7 first, Paul removes only 1.
If Carolyn removes 9 next, Paul removes only 3.
If Carolyn removes 6 next, Paul removes only 2.
If Carolyn removes 8 next, Paul removes only 4.
If Carolyn removes 10 next, Paul removes only 5.
(Carolyn could have switched her last two turns.)
Therefore, Carolyn can indeed remove the five largest numbers, so her maximum possible final sum is 40.

(c) As in (b), Carolyn can remove at most half of the numbers from the list, so can remove at most 7 numbers.
Can she possibly remove 7 numbers?
If Carolyn removes 7 numbers, then Paul must also remove 7 numbers since he removes at least one number for each one that Carolyn removes and there are only 14 numbers.
Since Paul always removes numbers that are divisors of the number just removed by Carolyn, then Paul can never remove a number larger than $\frac{1}{2}n$. (For him to do so, Carolyn would have to have removed a number larger than $2 \times \frac{1}{2}n = n$. This is impossible.)
Therefore, if Carolyn actually removes 7 numbers, then she must remove the 7 numbers larger than $\frac{1}{2}n$ (that is, 8, 9, 10, 11, 12, 13, 14).

Whichever number Carolyn removes first, Paul will remove 1 on his first turn, as it is a positive divisor of every positive integer. (Paul might remove other numbers too.)

At this stage, at least one of 11 and 13 is left in the list (depending on whether Carolyn removed one of these on her first turn).

For the sake of argument, assume that 11 is still left in the list. (The argument is the same if 13 is left.)

Carolyn cannot now remove 11 from the list. This is because 11 is a prime number and its only positive divisors are 1 and 11, so 11 does not have a positive divisor other than itself left in the list, so by the last rule, Carolyn cannot remove 11.

Thus, Carolyn cannot remove all of the numbers from 8 to 14, so cannot remove 7 numbers from the list.

**Canadian Mathematics Competition**
***An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario***

# *2007 Galois Contest*

## Wednesday, April 18, 2007

*Solutions*

1. (a) Let $A$ be the price of an Apple in cents, $B$ the price of a Banana in cents, and $C$ the price of a Cherry in cents.
From the given information, $A + C = 62$ and $B + C = 66$.
Since each combination includes a Cherry and one other piece of fruit, then the price of a Banana must be $66 - 62 = 4$ cents higher than that of an Apple.
(We could have also subtracted these equations to obtain $B - A = 4$.)
Therefore, the difference in prices of an Apple and a Banana is 4 cents, with the Banana having a higher price.

(b) *Solution 1*
Let $M$ be the price of a Mango in cents, $N$ the price of a Nectarine in cents, and $P$ the price of a Pear in cents.
From the given information, $M + N = 60$, $P + N = 60$ and $M + P = 68$.
Looking at the first two equations, $M$ and $P$ must be equal (since $M = 60 - N$ and $P = 60 - N$).
Looking then at the third equation, using $M = P$, we get $2P = 68$, so $P = 34$, or the price of a Pear is 34 cents.

*Solution 2*
Let $M$ be the price of a Mango in cents, $N$ the price of a Nectarine in cents, and $P$ the price of a Pear in cents.
From the given information, $M + N = 60$, $P + N = 60$ and $M + P = 68$.
Adding the second and third equations, we obtain $2P + M + N = 60 + 68 = 128$.
Since $M + N = 60$ (from the first equation), then $2P + 60 = 128$ or $2P = 68$ or $P = 34$.
Therefore, the price of a Pear is 34 cents.

*Solution 3*
Let $M$ be the price of a Mango in cents, $N$ the price of a Nectarine in cents, and $P$ the price of a Pear in cents.
From the given information, $M + N = 60$, $P + N = 60$ and $M + P = 68$.
Adding the three equations together, we obtain $2M + 2N + 2P = 188$.
Dividing by 2, we obtain $M + N + P = 94$.
Since $M + N = 60$ and $M + N + P = 94$, then $P = 94 - 60 = 34$.
Therefore, the price of a Pear is 34 cents.

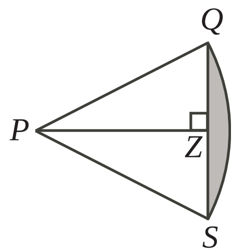
(c) Let $T$ be the price of a Tangerine in cents, $L$ the price of a Lemon in cents, and $G$ the price of a Grapefruit in cents.
From the given information, $T + L = 60$, $T - G = 6$ and $G + T + L = 94$.
Using the first equation to substitute into the third equation, $G + 60 = 94$, so $G = 34$.
Since $G = 34$ and $T - G = 6$, then $T = 34 + 6 = 40$.
Since $T = 40$ and $T + L = 60$, then $L = 20$, so the price of a Lemon is 20 cents.
(There are lots of other ways to combine these equations to get $L = 20$.)

2. (a) In the diagram, the radius of the sector is 12 so $OA = OB = 12$.
Since the angle of the sector is $60^\circ$, then the sector is $\dfrac{60^\circ}{360^\circ} = \dfrac{1}{6}$ of the total circle.
Therefore, arc $AB$ is $\frac{1}{6}$ of the total circumference of a circle of radius 12, so has length $\frac{1}{6}(2\pi(12)) = 4\pi$.
Therefore, the perimeter of the sector is $12 + 12 + 4\pi = 24 + 4\pi$.

(b) Each of sector $ABD$ and $BDC$ is one-sixth of a full circle of radius 12, so has area one-sixth of the area of a circle of radius 12.
Therefore, each sector has area $\frac{1}{6}(\pi(12^2)) = \frac{1}{6}(144\pi) = 24\pi$.
Thus, the area of figure $ABCD$ is $2(24\pi) = 48\pi$.

(c) Since $OY$ is a radius of the circle with centre $O$, then $OY = 12$.
To find the length of $XY$, we must find the length of $OX$.
Since $OA = OB$, then $\triangle OAB$ is isosceles.
Since $\angle AOB = 60^\circ$, then $\angle OAB = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
Therefore, $\angle AOX = 180^\circ - 60^\circ - 90^\circ = 30^\circ$, so $\triangle OAX$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Since $OA = 12$, then $AX = \frac{1}{2}OA = 6$ and $OX = \sqrt{3}AX = 6\sqrt{3}$.
Thus, $XY = OY - OX = 12 - 6\sqrt{3} \approx 1.61$.

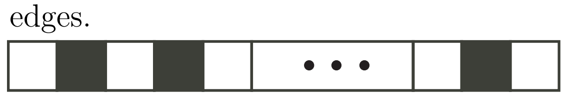
(d) By symmetry, the areas of the two parts of the shaded region are equal.
Consider the right part of the shaded region and the left triangle.




The shaded area is equal to the area of sector $PQS$ minus the area of triangle $PQS$.
The area of sector $PQS$ is $24\pi$ as in part (b).
Using our work from part (c), $\triangle PQS$ is equilateral, so $QS = 12$.
Draw a perpendicular from $P$ to $Z$ on $QS$.
From (c) again, $PZ = 6\sqrt{3}$.
Therefore, we can consider $\triangle PQS$ as having a base $QS$ of length 12 and height $PZ$ of length $6\sqrt{3}$.
Therefore, the area of $\triangle PQS$ is $\frac{1}{2}(12)(6\sqrt{3}) = 36\sqrt{3}$.
Therefore, the area of the right part of the shaded region is $24\pi - 36\sqrt{3}$, so the area of the entire shaded region is $2(24\pi - 36\sqrt{3}) = 48\pi - 72\sqrt{3} \approx 26.1$.

3. (a) The 1 by 1 by 1 cubes that have at least two painted faces are those along the edges of the 5 by 5 by 5 including those at the corner.
The large cube has 12 edges and has 3 of the small cubes along each edge which are not at the corners.
The large cube has 8 corners.
Therefore, there are $3 \times 12 + 8 = 44$ small cubes that have at least two painted faces.

(b) i. The small cubes that have exactly two white faces are those along the edges of the larger cubes that are not at the corners.
Each of the 12 edges of the large cube has $2k + 1$ smaller cubes along it.
The "white cubes" (that is, white on two faces) that we want are those that are not at the end of one of the edges.
Consider one of these edges.



If we remove the white cube at one end of the edge of $2k + 1$ cubes, half of the remaining cubes (that is, $k$ of the remaining cubes) are white, and all but one of them

is not at the end of the edge.
Therefore, there are $k-1$ cubes not at the end of this edge, so there are $k-1$ cubes on this edge that are white on exactly two faces.
Since there are 12 edges, there are exactly $12(k-1) = 12k - 12$ small cubes in the large cube that have exactly two white faces.

ii. The cubes that have at least two white faces are those from part (i) that have exactly two white faces and those with exactly three white faces.
The cubes with exactly three white faces are the 8 cubes at the corners of the large cube.
Therefore, there are $12k - 12 + 8 = 12k - 4$ cubes with at least two white faces.
Could $12k - 4$ be equal to 2006?
If so, $12k - 4 = 2006$ or $12k = 2010$ or $k = 167.5$, which is not an integer.
Therefore, there is no value of $k$ for which the number of cubes having at least two white faces is 2006.

4. (a) Suppose that we use $x$ yellow and $y$ green rods.
We thus want $5x + 3y = 62$ and each of $x$ and $y$ is a non-negative integer or $3y = 62 - 5x$.
The easiest approach is to try the possible values of $x$ and check if $y$ is an integer. We make a table:

| $x$ | $62-5x$ | $62-5x$ divisible by 3? | $y$ |
|---|---|---|---|
| 0 | 62 | No | |
| 1 | 57 | Yes | 19 |
| 2 | 52 | No | |
| 3 | 47 | No | |
| 4 | 42 | Yes | 14 |
| 5 | 37 | No | |
| 6 | 32 | No | |
| 7 | 27 | Yes | 9 |
| 8 | 22 | No | |
| 9 | 17 | No | |
| 10 | 12 | Yes | 4 |
| 11 | 7 | No | |
| 12 | 2 | No | |

Therefore, there are 4 different sets (1 yellow and 19 green, 4 yellow and 14 green, 7 yellow and 9 green, 10 yellow and 4 green) that can be used.

(b) Green rods are 3 cm in length, yellow rods are 5 cm in length, black rods are 7 cm in length, and red rods are 9 cm in length.
If we use $a$ green rods and $b$ red rods, then the total length of the pole is $3a + 9b$ cm, which is always divisible by 3.
Therefore, we cannot make a pole of length 62 cm with green and red rods, since 62 is not divisible by 3.
(We can check that each other pair of colours will allow us to make a pole of length 62 cm: 19 green and 1 yellow, 16 green and 2 black, 11 yellow and 1 black, 7 yellow and 3 red, and 5 black and 3 red rods each make a pole of length 62 cm.)

(c) Since we use at least 81 of each colour of rods, then let us suppose that we use $81 + a$ green, $81 + b$ pink, $81 + c$ violet, and $81 + d$ red rods, where each of $a$, $b$, $c$, and $d$ is a non-negative integer.

For the total length to be 2007 cm, we must have

$$\begin{aligned} 3(81+a)+4(81+b)+8(81+c)+9(81+d) &= 2007 \\ 3a+4b+8c+9d+81(3+4+8+9) &= 2007 \\ 3a+4b+8c+9d &= 2007-1944 \\ 3a+4b+8c+9d &= 63 \end{aligned}$$

We want to find the total number of non-negative integer solutions to this equation.
To do this, we group the terms on the left-hand side as $3(a+3d)+4(b+2c)$ and let $x=a+3d$ and $y=b+2c$ and look at the equation $3x+4y=63$, where $x$ and $y$ are non-negative integers.
The possible solutions to this are $(x,y)=(1,15),(5,12),(9,9),(13,6),(17,3),(21,0)$.
If $x=a+3d=1$, then $(a,d)=(1,0)$, so there is 1 combination.
If $x=a+3d=5$, then $(a,d)=(5,0),(2,1)$, so there are 2 combinations.
If $x=a+3d=9$, then $(a,d)=(9,0),(6,1),(3,2),(0,3)$, so there are 4 combinations.
If $x=a+3d=13$, then $(a,d)=(13,0),(10,1),(7,2),(4,3),(1,4)$, so there are 5 combinations.
If $x=a+3d=17$, then $(a,d)=(17,0),(14,1),(11,2),(8,3),(5,4),(2,5)$, so there are 6 combinations.
If $x=a+3d=21$, then $(a,d)=(21,0),(18,1),(15,2),(12,3),(9,4),(6,5),(3,6),(0,7)$, so there are 8 combinations.
Similarly, if $y=b+2c$ runs through the possible values of 15, 12, 9, 6, 3, and 0, there are 8, 7, 5, 4, 2, and 1 combinations, respectively, for $b$ and $c$.

Now we must combine the combinations.
If $x=1$ and $y=15$, there is 1 combination for $a$ and $d$, and 8 combinations for $b$ and $c$, so there are $1\times 8=8$ combinations for $a$, $b$, $c$, and $d$.
If $x=5$ and $y=12$, there are 2 combinations for $a$ and $d$, and 7 combinations for $b$ and $c$, so there are $2\times 7=14$ combinations for $a$, $b$, $c$, and $d$ (since each combination for $b$ and $c$ works with each combination for $a$ and $d$).
With the remaining pairs for $x$ and $y$, there are $4\times 5=20$, $5\times 4=20$, $6\times 2=12$, $8\times 1=8$ combinations.
In total, there are thus $8+14+20+20+12+8=82$ combinations for $a$, $b$, $c$, and $d$, so there are 82 such sets of rods.

**Canadian Mathematics Competition**

***An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario***

# *2006 Galois Contest*

## Thursday, April 20, 2006

*Solutions*

1. (a) The largest possible difference comes when one of Amelie and Bob chooses the slips with the three largest numbers and the other choose the slips with the three smallest numbers. Thus, one of them chooses 1, 2 and 3 (for a total of 6) and the other chooses 4, 5 and 6 (for a total of 15).
The difference in the totals is 9.

(b) The total of the numbers on the six slips is $1 + 2 + 3 + 4 + 5 + 6 = 21$.
For Amelie's total to be one more than Bob's total, her total must be 11 and Bob's must be 10 (since the sum of their totals is 21).
The possible groups of three slips giving totals of 11 are

$$1, 4, 6 \quad 2, 3, 6 \quad 2, 4, 5$$

These are the possible groups that Amelie can choose.

(c) When Amelie and Bob each choose three of the slips, the sum of their totals is the sum of all of the numbers on the slips, or 21.
If they each had the same total, the sum of their totals would be even, so could not be 21.
Therefore, they cannot have the same total.

(d) Since Amelie and Bob must choose half of the slips, the total number of slips must be even.
Therefore, the smallest value that $n$ could take is 8.
If $n = 8$, can they obtain the same total?
If $n = 8$, the sum of the numbers on the eight slips is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$, which is also even.
Here, Amelie and Bob could obtain the same total if Amelie chooses 1, 2, 7, 8 (and so Bob chooses 3, 4, 5, 6).
Therefore, the smallest value of $n$ that works is $n = 8$.

2. (a) *Solution 1*
Since $DE = EF = 4$ and $\angle DEF = 90°$, then by the Pythagorean Theorem,
$DF^2 = DE^2 + EF^2 = 4^2 + 4^2 = 32$, so $DF = \sqrt{32} = 4\sqrt{2}$.

*Solution 2*
Since $\triangle DEF$ is right-angled and isosceles, its angles are $45°$, $45°$ and $90°$.
Therefore, $DF = \sqrt{2}(DE) = 4\sqrt{2}$.

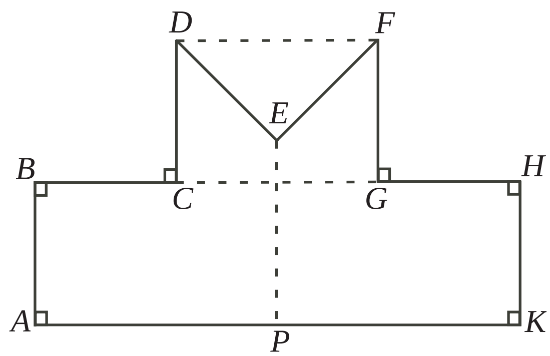
(b) *Solution 1*
Since $\triangle DEF$ is isosceles with $DE = EF$ and $EM$ is perpendicular to $DF$, then
$DM = MF = \frac{1}{2}DF = 2\sqrt{2}$.
Since $\triangle DME$ is right-angled, then by the Pythagorean Theorem,
$EM^2 = DE^2 - DM^2 = 4^2 - (2\sqrt{2})^2 = 16 - 8 = 8$, so $EM = \sqrt{8} = 2\sqrt{2}$.

*Solution 2*
Since $\triangle DEF$ is isosceles with $DE = EF$ and $EM$ is perpendicular to $DF$, then
$DM = MF = \frac{1}{2}DF = 2\sqrt{2}$.

Since $\triangle DEF$ is isosceles and right-angled, then $\angle EDF = 45^\circ$, so $\triangle DME$ is also isosceles and right-angled.
Therefore, $EM = DM = 2\sqrt{2}$.

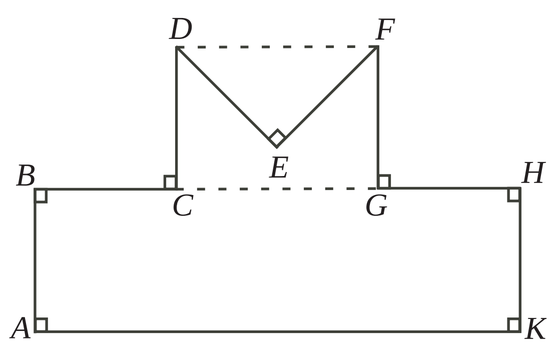
*Solution 3*
Since $DE$ and $EF$ are perpendicular, the area of $\triangle DEF$ is $\frac{1}{2}(DE)(EF) = \frac{1}{2}(4)(4) = 8$.
Since $DF$ and $ME$ are perpendicular, the area of $\triangle DEF$ is also $\frac{1}{2}(DF)(ME)$, so
$\frac{1}{2}(4\sqrt{2})(ME) = 8$ or $ME = \dfrac{8}{2\sqrt{2}} = 2\sqrt{2}$.

(c) Join $DF$ and $CG$.




Each of $CGFD$ and $AKHB$ is a rectangle, as each has 4 right angles.
The height of rectangle $CGFD$ is 4 since $DC = FG = 4$, and the height of rectangle $AKHB$ is also 4 since $AB = HK = 4$.
The length of $EP$ is thus the distance of $E$ to $CG$ plus the height of the bottom rectangle.
The distance of $E$ to $CG$ is the difference between the height of rectangle $CGFD$ and the length of $EM$, or $4 - 2\sqrt{2}$.
Thus, the length of $EP$ is $4 + (4 - 2\sqrt{2}) = 8 - 2\sqrt{2}$.

(d) The area of the figure is equal to the area of rectangle $AKHB$ plus the area of rectangle $CGFD$ minus the area of triangle $DEF$.




Since the length of $DF$ is $4\sqrt{2}$ and the length of $CD$ is 4, the area of rectangle $CGFD$ is $4(4\sqrt{2}) = 16\sqrt{2}$.
Now $BH = BC + CG + GH = 4 + DF + 4 = 8 + 4\sqrt{2}$.
Since $AB = 4$, the area of rectangle $AKHB$ is $4(8 + 4\sqrt{2}) = 32 + 16\sqrt{2}$.
From Solution 3 to part (b), the area of $\triangle DEF$ is 8.
Thus, the area of figure $ABCDEFGHK$ is $(16\sqrt{2}) + (32 + 16\sqrt{2}) - 8 = 24 + 32\sqrt{2}$.

3. (a) Since $A$ has coordinates $(0, 16)$ and $B$ has coordinates $(8, 0)$, the slope of the line through $A$ and $B$ is $\dfrac{16-0}{0-8} = -2$.
Since the line passes through the $y$-axis at $A(0, 16)$, then its $y$-intercept is 16, so the line has equation $y = -2x + 16$.

(b) Suppose that $P$ has coordinates $(c, d)$.
Since $P$ lies on the line $y = -2x+16$, then $d = -2c+16$, so $P$ has coordinates $(c, -2c+16)$.
For $PDOC$ to be a square, $PD = PC$.
But $PD$ is the distance of $P$ from the $y$-axis, so $PD = c$ and $PC$ is the distance of $P$ from the $x$-axis, so $PC = -2c + 16$.
Therefore, $c = -2c + 16$ or $3c = 16$ or $c = \frac{16}{3}$.
Thus, $P$ has coordinates $\left(\frac{16}{3}, \frac{16}{3}\right)$.

(c) *Solution 1*
As in (b), we may suppose that $P$ has coordinates $(c, -2c + 16)$.
The area of rectangle $PDOC$ is $PD \times PC$, or $c(-2c + 16)$.
Since the area is 30, then

$$\begin{aligned} 30 &= c(-2c+16) \\ 30 &= -2c^2 + 16c \\ 2c^2 - 16c + 30 &= 0 \\ c^2 - 8c + 15 &= 0 \\ (c-3)(c-5) &= 0 \end{aligned}$$

so $c = 3$ or $c = 5$.
Therefore, the two possible points $P$ are $(3, 10)$ and $(5, 6)$.
(We can check that each gives a rectangle of area 30.)

*Solution 2*
For the area of rectangle $PDOC$ to be 30, the coordinates of $P$ are $\left(c, \dfrac{30}{c}\right)$.
For $P$ to lie on the line $y = -2x + 16$,

$$\begin{aligned} \frac{30}{c} &= -2c + 16 \\ 30 &= -2c^2 + 16c \\ c^2 - 8c + 15 &= 0 \\ (c-3)(c-5) &= 0 \end{aligned}$$

so $c = 3$ or $c = 5$.
Therefore, the two possible points $P$ are $(3, 10)$ and $(5, 6)$.

4. (a) Suppose that we start with the 2 digit integer $\underline{a}\,\underline{b} = 10a + b$ and reverse the order of its digits to obtain $\underline{b}\,\underline{a} = 10b + a$.
The difference is $(10b + a) - (10a + b) = 9b - 9a = 9(b - a)$.
For this difference to equal 27, we must have $9(b - a) = 27$ or $b - a = 3$ or $b = a + 3$. That is, the second digit of the original number is 3 larger than the first digit.

Thus, the possible beginning numbers are 14, 25, 36, 47, 58, and 69.

(b) *Solution 1*
Start with $\underline{a}\,\underline{b}\,\underline{c} = 100a + 10b + c$ and reverse the order of the digits to obtain $\underline{c}\,\underline{b}\,\underline{a} = 100c + 10b + a$.
We may assume, without loss of generality, that the first number is larger than the second number (otherwise, we simply reverse the roles of the numbers).
Then their difference is

$$\underline{r}\,\underline{s}\,\underline{t} = \underline{a}\,\underline{b}\,\underline{c} - \underline{c}\,\underline{b}\,\underline{a} = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 99(a - c)$$

Since $a$ and $c$ are distinct digits, then the possible values of $a - c$ are 1 through 9, so the possible values for the integer $\underline{r}\,\underline{s}\,\underline{t}$ are 99 times the numbers 1 through 9. We show the possible values, their reverses and the sums in the table:

| $\underline{r}\,\underline{s}\,\underline{t}$ | 099 | 198 | 297 | 396 | 495 | 594 | 693 | 792 | 891 |
|---|---|---|---|---|---|---|---|---|---|
| $\underline{t}\,\underline{s}\,\underline{r}$ | 990 | 891 | 792 | 693 | 594 | 495 | 396 | 297 | 198 |
| Sum | 1089 | 1089 | 1089 | 1089 | 1089 | 1089 | 1089 | 1089 | 1089 |

Therefore, the required sum is always 1089.

*Solution 2*
Start with $\underline{a}\,\underline{b}\,\underline{c} = 100a + 10b + c$ and reverse the order of the digits to obtain $\underline{c}\,\underline{b}\,\underline{a} = 100c + 10b + a$.
We may assume, without loss of generality, that the first number is larger than the second number (otherwise, we simply reverse the roles of the numbers).
Then their difference is

$$\underline{r}\,\underline{s}\,\underline{t} = \underline{a}\,\underline{b}\,\underline{c} - \underline{c}\,\underline{b}\,\underline{a} = (100a + 10b + c) - (100c + 10b + a) = 99a - 99c = 99(a - c)$$

Since $a$ and $c$ are distinct digits, then the possible values of $a - c$ are the integers 1 through 9.
In any of these cases, $\underline{r}\,\underline{s}\,\underline{t} = 99(a - c) = 100(a - c) - (a - c)$, that is, is $a - c$ less than $100(a - c)$ and so has hundreds digit $a - c - 1$, tens digit 9 and units digit $10 - (a - c)$.
In other words, $\underline{r}\,\underline{s}\,\underline{t} = \underline{a - c - 1}\ \ \underline{9}\ \ \underline{10 - (a - c)}$.
When the order of the digits is reversed, we obtain $\underline{t}\,\underline{s}\,\underline{r} = \underline{10 - (a - c)}\ \ \underline{9}\ \ \underline{a - c - 1}$.
Adding these numbers,

$$\begin{array}{cccc} & \underline{a - c - 1} & \underline{9} & \underline{10 - (a - c)} \\ + & \underline{10 - (a - c)} & \underline{9} & \underline{a - c - 1} \\ \hline \underline{1} & \underline{0} & \underline{8} & \underline{9} \end{array}$$

(Note that a 1 has been "carried" from the tens column to the hundreds column, and also from the hundreds column to the thousands column.)

(c) Since $N = \underline{a}\,\underline{b}\,\underline{c}\,\underline{d} = 1000a + 100b + 10c + d$, then $M = \underline{d}\,\underline{c}\,\underline{b}\,\underline{a} = 1000d + 100c + 10b + a$.

Then

$$\begin{aligned} P &= M - N \\ &= (1000d + 100c + 10b + a) - (1000a + 100b + 10c + d) \\ &= 1000(d-a) + 100(c-b) + 10(b-c) + (a-d) \\ &= 999d + 90c - 90b - 999a \\ &= 999(d-a) + 90(c-b) \end{aligned}$$

Since $a \leq b \leq c \leq d$, then $d - a \geq 0$ and $c - b \geq 0$.
(This tells us that while the third line of the equations above looks like it represents $P$ in terms of its digits, two of the digits are possibly negative.)
We notice also that $d - a \geq c - b$.

<u>Case 1: $a = b = c = d$</u>
In this case, $P = 0$ so $Q = 0$, so $P + Q = 0$.

<u>Case 2: $d - a = 1$</u>
In this case, $c - b$ can only be 0 or 1, so the two possible values of $P$ are $P = 999 = 0999$ and $P = 999 + 90 = 1089$.
Reversing these, we obtain $Q = 9990$ and $Q = 9801$, giving $P + Q = 999 + 9990 = 10\,989$ and $P + Q = 1089 + 9801 = 10\,890$.

<u>Case 3: $d - a > 1$, $c - b = 0$</u>
In this case, $P = 999(d-a) = 1000(d-a)-(d-a)$, so has digits $\underline{d-a-1}$ $\underline{9}$ $\underline{9}$ $\underline{10-(d-a)}$.
Thus, $Q$ has digits $\underline{10-(d-a)}$ $\underline{9}$ $\underline{9}$ $\underline{d-a-1}$, and so adding $P$ and $Q$, we obtain

$$\begin{array}{cccccc} & & \underline{d-a-1} & \underline{9} & \underline{9} & \underline{10-(d-a)} \\ + & & \underline{10-(d-a)} & \underline{9} & \underline{9} & \underline{d-a-1} \\ \hline & \underline{1} & \underline{0} & \underline{9} & \underline{8} & \underline{9} \end{array}$$

Alternatively, we could write

$$\begin{aligned} P &= 999(d-a) \\ &= 1000(d-a) - (d-a) \\ &= 1000(d-a-1) + 1000 - (d-a) \\ &= 1000(d-a-1) + 100(9) + 100 - (d-a) \\ &= 1000(d-a-1) + 100(9) + 10(9) + (10 - (d-a)) \end{aligned}$$

where we are writing out the "borrowing" process explicitly.
Here, the digits are $d - a - 1$, 9, 9 and $10 - (d - a)$ and each digit is at least 0.
Thus, $Q = 1000(10 - (d - a)) + 100(9) + 10(9) + (d - a - 1)$ and so

$$P + Q = 1000(9) + 100(18) + 10(18) + 9 = 10\,989$$

Case 4: $d-a>1$, $c-b>0$

In this case,

$$\begin{aligned}
P &= 999(d-a)+90(c-b) \\
&= 1000(d-a)-(d-a)+100(c-b)-10(c-b) \\
&= \underline{d-a-1}\ \underline{9}\ \underline{9}\ \underline{10-(d-a)}+\underline{c-b-1}\ \underline{10-(c-b)}\ \underline{0} \\
&= \underline{d-a-1}\ \underline{9+c-b-1}\ \underline{9+10-(c-b)}\ \underline{10-(d-a)+0} \\
&= \underline{d-a-1}\ \underline{9+c-b-1+1}\ \underline{9-(c-b)}\ \underline{10-(d-a)} \\
&= \underline{d-a}\ \underline{c-b-1}\ \underline{9-(c-b)}\ \underline{10-(d-a)}
\end{aligned}$$

where some "carrying" has been done in the last two lines.
Therefore, $Q=\underline{10-(d-a)}\ \underline{9-(c-b)}\ \underline{c-b-1}\ \underline{d-a}$.
Thus, adding $P$ and $Q$ we obtain

$$\begin{array}{cccccc}
 & & \underline{d-a} & \underline{c-b-1} & \underline{9-(c-b)} & \underline{10-(d-a)} \\
+ & & \underline{10-(d-a)} & \underline{9-(c-b)} & \underline{c-b-1} & \underline{d-a} \\
\hline
 & \underline{1} & \underline{0} & \underline{8} & \underline{9} & \underline{0}
\end{array}$$

Alternatively, we could write

$$\begin{aligned}
P &= 999(d-a)+90(c-b) \\
&= 1000(d-a)+100(c-b)-10(c-b)-(d-a) \\
&= 1000(d-a)+100(c-b-1)+100-10(c-b)-(d-a) \\
&= 1000(d-a)+100(c-b-1)+10(10-(c-b))-(d-a) \\
&= 1000(d-a)+100(c-b-1)+10(9-(c-b))+(10-(d-a))
\end{aligned}$$

where we are writing out the "borrowing" process explicitly.
Here, the digits are $d-a$, $c-b-1$, $9-(c-b)$ and $10-(d-a)$ and each digit is at least 0.
Thus, $Q=1000(10-(d-a))+100(9-(c-b))+10(c-b-1)+(d-a)$ and so

$$P+Q=1000(10)+100(8)+10(8)+10=10\,890$$

Therefore, the possible values for $P+Q$ are 0, 10 890 and 10 989.

**Canadian Mathematics Competition**

**An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario**

# 2005 Galois Contest

**Wednesday, April 20, 2005**

*Solutions*

1. (a) *Solution 1*

We list the first 11 terms, obtaining each by adding 5 to the previous term:

$$17, 22, 27, 32, 37, 42, 47, 52, 57, 62, 67$$

Therefore, the 11th term is 67.

*Solution 2*

Since the constant that we add to each term to obtain the next is 5, then the 11th term is $17 + 5(10) = 67$, since we must add the common difference 10 times to the 1st term to obtain the 11th term.

(b) *Solution 1*

Every term in the first sequence ends in a 7 or a 2. This is because the difference between consecutive terms is always 5, and whenever we add 5 to a number ending in 7, we obtain a number ending in 2, and vice versa.

Every term in the second sequence ends in a 3 or an 8. This is because the difference between consecutive terms is always 15, and whenever we add 15 to a number ending in 3, we obtain a number ending in 8, and vice versa.

No number can occur in both sequences, since no number can end in two different digits.

*Solution 2*

Every number in the first sequence is of the form $17+5n$ for some non-negative integer $n$.

Every number in the second sequence is of the form $13+15m$ for some non-negative integer $m$.

For a number to occur in both sequences, we would need $17 + 5n = 13 + 15m$ or $4 = 15m - 5n = 5(3m - n)$.

But the right side is a multiple of 5, and the left side is not, so this is impossible.

Therefore, there is no number which occurs in both sequences.

(c) *Solution 1*

We see first that 22 occurs in both sequences.

Since the difference between consecutive terms is 5 in the first sequence and 6 in the second sequence, then a number $5 \times 6 = 30$ greater than a number which occurs in both sequences will again occur in both sequences. (This is because adding 30 to a term in the first sequence moves us six terms along the sequence and adding 30 to a term in the second sequence moves us five terms along.)

Therefore, terms which occur in both sequences are 22, 52, 82, 112, 142, 172, 202, 232, 262, 292, 322, 352, 382, and 412.

Thus, 412 is a number between 400 and 420 which occurs in both sequences.

*Solution 2*

Since the difference between consecutive terms in the first sequence is 5, then the first sequence contains every positive integer which is at least 17 and ends in a 2 or a 7.

Therefore, the numbers in the first sequence which are between 400 and 420 are 402, 407, 412 and 417.

Which of these four possibilities occurs in the second sequence?

Since the difference between consecutive terms in the second sequence is 6, then every number in the second sequence is of the form $16 + 6n$, where $n$ is a non-negative integer.

Does $16 + 6n = 402$ have a solution where $n$ is an integer? This would mean $6n = 386$, but 386 is not divisible by 6, so there is no solution.

Does $16 + 6n = 407$ have a solution where $n$ is an integer? No, since the left side is even and the right side is odd.
Does $16 + 6n = 412$ have a solution where $n$ is an integer? Yes, since if $16 + 6n = 412$ then $6n = 396$ or $n = 66$.
Therefore, 412 occurs in both sequences.
(We could also check that 417 cannot occur in the second sequence, since it is odd.)

2. (a) The tiles which remain to be placed are one 5 and two 6s. There are only 3 different ways in which these tiles can be placed. Here are the three ways along with the totals of each column and row:

| | | | |
|---|---|---|---|
| 1 | 3 | 6 | 5 |
| 3 | 1 | 2 | 2 |
| 5 | 4 | 4 | 6 |

(Column totals: 9, 8, 12, 13; Row totals: 15, 8, 19)

| | | | |
|---|---|---|---|
| 1 | 3 | 5 | 5 |
| 3 | 1 | 2 | 2 |
| 6 | 4 | 4 | 6 |

(Column totals: 10, 8, 11, 13; Row totals: 14, 8, 20)

| | | | |
|---|---|---|---|
| 1 | 3 | 6 | 5 |
| 3 | 1 | 2 | 2 |
| 6 | 4 | 4 | 5 |

(Column totals: 10, 8, 12, 12; Row totals: 15, 8, 19)

Of these the ones which give Omar more points than Emilia is the first

| | | | |
|---|---|---|---|
| 1 | 3 | 6 | 5 |
| 3 | 1 | 2 | 2 |
| 5 | 4 | 4 | 6 |

where Omar gets 4 points and Emilia gets 3.

(b) The total number of points available is 7, since there are 4 columns and 3 rows and each column and each row gives 1 point to one of the players.
So Emilia's score and Omar's score must add up to 7.
Since 7 is odd, Emilia's score and Omar's score cannot be the same.

(c) *Solution 1*
The tiles that remain to be placed are 2, 2, 4 and 5, ie. three even numbers and one odd number.
In the grid, the fourth column is already complete and has an even sum (16), so Emilia already has one point.
Consider rows 1 and 3 and columns 1 and 3.
Each has only one space left open, and each has an even sum so far.
In any of these cases, if an even number is placed in the empty space, the row or column will be complete and the sum will be even, giving Emilia a point. If an odd number is placed in the empty space, the row or column will be complete and the sum will be odd, giving Omar a point.
But since there is only one odd-numbered tile left to play, then only one of these four rows and columns can end up with an odd score, and three are guaranteed to be even.
So no matter where Omar places the 5, Emilia will get 3 more points, for a total of at least 4 points.

Since there are only 7 points available, then Emilia is guaranteed to have more points than Omar, because Omar will get at most 3 points.

*Solution 2*

The tiles that remain to be placed are 2, 2, 4 and 5, ie. three even numbers and one odd number.
Suppose Omar places the 5 in the leftmost empty box. Then the numbers which go into the other three empty spaces are the 2, 2 and 4, so are all even.
The sum of the top row will then be $1 + 3 + 6$ plus either 2 or 4, so will be even.
The sum of the bottom row will be $3 + 1 + 6$ plus either 2 or 4, so will be even.
The sum of the fourth column is already 16, so is even.
The sum of the third column will be $3 + 1$ plus either 2 or 4, so will be even.
Therefore, in this case Emilia will get at least 4 of the 7 possible points, so will have more points than Omar.

If Omar places the 5 in the rightmost empty box, the argument is exactly the same, except we look at the first column instead of the third. In this case, again Emilia gets at least 4 points.

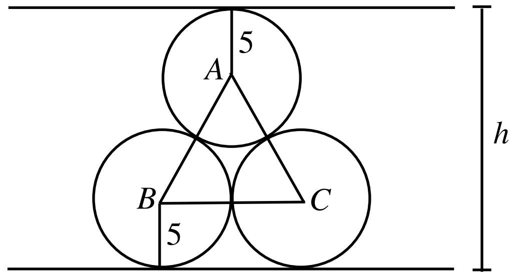
Suppose Omar places the 5 in the uppermost empty box. Then the numbers which go into the other three empty spaces are the 2, 2, and 4, so are all even.
The sum of the bottom row will be $3 + 1 + 6$ plus either 2 or 4, so will be even.
The sum of the fourth column is already 16, so is even.
The sum of each of the first and third columns is $3 + 1$ plus either 2 or 4, so will be even.
Therefore, in this case Emilia will get at least 4 of the 7 possible points, so will have more points than Omar.

If Omar places the 5 in the lowermost empty box, the argument is exactly the same as this previous case, except we look at the top row instead of the bottom row. In this case, again Emilia gets at least 4 points.

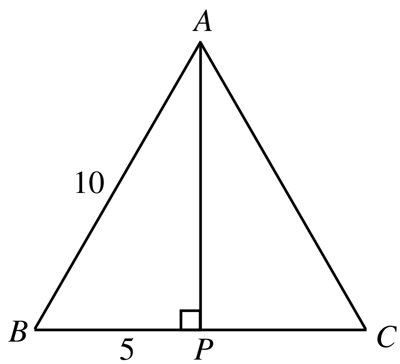
Therefore, having looked at all of the possible cases, Emilia will always get at least 4 points, so will always have more points than Omar, no matter where Omar places the 5.

3. (a) In Crate A, there are 10 pipes in each row. Since there are 200 pipes to pack, there will be 20 rows in total.
In Crate B, there are 10 pipes in every other row and 9 pipes in the in-between rows. Thus, there are 19 pipes in every pair of neighbouring rows. So with 20 rows, there will be 190 pipes, and the topmost row will have 9 pipes in it, since even-numbered rows contain 9 pipes. Thus, if we add one more row, we will add 10 more pipes, for 200 in total. So in Crate B, there will be 21 rows in total.

(b) Join the centres $A$, $B$ and $C$ of the three circles.
The lines $AB$, $BC$ and $CA$ will pass through the points where the circles touch, so will each have length 10 cm (that is, twice the radius of one of the circles).
We can break the height of the pile into three pieces: the distance from the bottom of the pile to the line $BC$, the height of the equilateral triangle $ABC$, and the distance $A$ to the top of the pile.



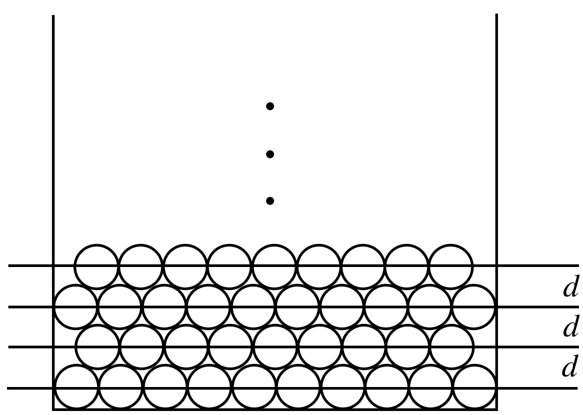

The first and last of these distances are each equal to the radius of one of the circles, that is, 5 cm.
So we must determine the height of $\triangle ABC$, which is an equilateral triangle with side length 10 cm.
There are many ways to do this.
Drop a perpendicular from $A$ to $P$ on $BC$.
Since $AB = AC$, then $P$ is the midpoint of $BC$, so $BP = 5$ cm.




Then $\triangle ABP$ is a 30°-60°-90° triangle, so $AP = \sqrt{3}BP = 5\sqrt{3}$ cm.
Thus, the height of the pile is $5 + 5\sqrt{3} + 5 = (10 + 5\sqrt{3})$ cm.

(c) In Crate A, we have 20 rows of 10 pipes packed directly on top of each other. So the height of the packing is 20 times the diameter of a single pipe, or 200 cm.

In Crate B, draw a horizontal line through the centres of the 9 or 10 pipes in each row. By symmetry, the distance between each consecutive pair of these 21 lines will be the same, say equal to $d$. There will be 20 such distances.

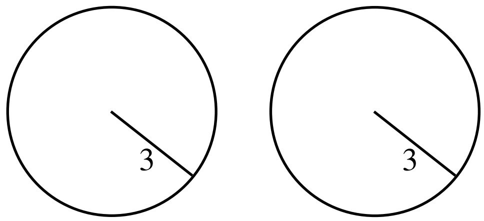



The distance of the bottom line from the bottom of the crate is equal to the radius of a pipe, and the distance of the top line from the top of the top row is also equal to the radius of a pipe.
Thus, the total height of the packing in Crate B is equal to $(10 + 20d)$ cm.

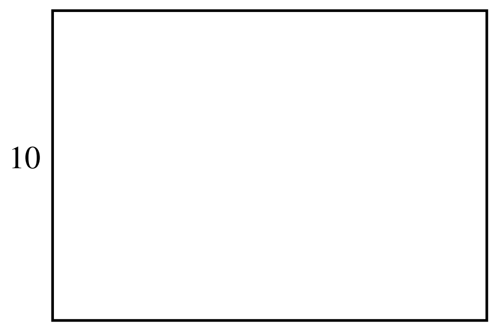
What is $d$ equal to?
If we extract three pairwise touching pipes from two consecutive rows, we get a pile exactly as in (b), so $d$ is equal to the height of the equilateral triangle whose height we calculated in (b), ie. $d = 5\sqrt{3}$ cm.
Therefore, the total height of this packing is $(10 + 100\sqrt{3})$ cm, which is approximately 183.2 cm.

Therefore, the difference in the total heights of the two packings is $200 - (10 + 100\sqrt{3}) = 190 - 100\sqrt{3}$ cm, or about 16.8 cm, with the packing in Crate A being the higher one.

4. (a) To calculate the total surface area of the cylinder, we cut off the two ends to obtain two circles of radius 3.




The two ends combined have an area of $2\pi r^2 = 2\pi(3^2) = 18\pi$.
Next, we must calculate the lateral surface area.
To do this, we make a vertical cut through this area, and unroll the lateral surface.
When we do this, we obtain a rectangle with height 10. The width of the rectangle (ie. the length of the top edge) is equal to the circumference of one of the ends, since the top edge of this rectangle lay exactly along the circumference of the top end.




The circumference of one of the ends is $2\pi r = 2\pi(3) = 6\pi$, so the width of the rectangle is $6\pi$.
Therefore, the area of this rectangle is $10 \times 6\pi = 60\pi$.
So the total surface area of the cylinder is $18\pi + 60\pi = 78\pi$.
The volume of the cylinder is equal to $\pi r^2 h = \pi(3^2)(10) = 90\pi$.

(b) Since the volume of the sphere is $\frac{4}{3}\pi r^3$ and the volume of the cylinder is $\pi r^2 H$ and these are equal, then $\frac{4}{3}\pi r^3 = \pi r^2 H$ or $H = \frac{4}{3}r$ (cancelling the common factor of $\pi r^2$).
Looking at our work from (a) more generally, we can see that the total surface area of a cylinder of radius $r$ and height $H$ is $2\pi r^2 + 2\pi r H$ (area of the two ends plus lateral area).
We are told that the surface area of a cone with radius $r$, height $h$ and slant height $s$ is $\pi r^2 + \pi r s$.
Since the total surface areas of the cone and cylinder are equal, then

$$\begin{aligned}
2\pi r^2 + 2\pi r H &= \pi r^2 + \pi r s \\
2r + 2H &= r + s \qquad \text{(cancelling the common factor of } \pi r\text{)} \\
r + 2H &= s
\end{aligned}$$

Since we want to determine some information about $h$ and $H$, we should try to eliminate the $s$.
Look at the cone. Join the tip of the cone to the centre of the circular base. This line will be perpendicular to the base, so will form a right-angled triangle with legs $r$ and $h$ and hypotenuse $s$.
Thus, by the Pythagorean Theorem, $s = \sqrt{r^2 + h^2}$.
Therefore, $r + 2H = \sqrt{r^2 + h^2}$.
We can now use the fact that $H = \frac{4}{3}r$ or $r = \frac{3}{4}H$ to write this equation in terms of $H$ and $h$ only:

$$\begin{aligned}
r + 2H &= \sqrt{r^2 + h^2} \\
\frac{3}{4}H + 2H &= \sqrt{\left(\frac{3}{4}H\right)^2 + h^2} \\
\frac{11}{4}H &= \sqrt{\frac{9}{16}H^2 + h^2} \\
\frac{121}{16}H^2 &= \frac{9}{16}H^2 + h^2 \qquad \text{(squaring both sides)} \\
\frac{112}{16}H^2 &= h^2 \\
7H^2 &= h^2
\end{aligned}$$

Since $h^2 = 7H^2$, then $h = \sqrt{7}H$ and so we cannot have both $h$ and $H$ integers, since $\sqrt{7}$ is irrational.

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2004 Solutions
# Galois Contest (Grade 10)

1. (a) Since at least one of each type of prize is given out, then these four prizes account for $\$5 + \$25 + \$125 + \$625 = \$780$.
Since there are five prizes given out which total \$905, then the fifth prize must have a value of $\$905 - \$780 = \$125$.
Thus, the Fryer Foundation gives out one \$5 prize, one \$25 prize, two \$125 prizes, and one \$625 prize.

(b) As in (a), giving out one of each type of prize accounts for \$780.
The fifth prize could be a \$5 prize for a total of $\$780 + \$5 = \$785$.
The fifth prize could be a \$25 prize for a total of $\$780 + \$25 = \$805$.
The fifth prize could be a \$625 prize for a total of $\$780 + \$625 = \$1405$.
(We already added an extra \$125 prize in (a).)

(c) *Solution 1*
Since at least one of each type of prize is given out, this accounts for \$780. So we must figure out how to distribute the remaining $\$880 - \$780 = \$100$ using at most 5 of each type of prize. We cannot use any \$125 or \$625 prizes, since these are each greater than the remaining amount.
We could use four additional \$25 prizes to make up the \$100.
Could we use fewer than four \$25 prizes? If we use three additional \$25 prizes, this accounts for \$75, which leaves \$25 remaining in \$5 prizes, which can be done by using five additional \$5 prizes.
Could we use fewer than three \$25 prizes? If so, then we would need to make at least \$50 with \$5 prizes, for which we need at least ten such prizes. But we can use at most six \$5 prizes in total, so this is impossible.
Therefore, the two ways of giving out \$880 in prizes under the given conditions are:
i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes
We can check by addition that each of these totals \$880.

*Solution 2*
We know that the possible total values using at least one of each type of prize and exactly five prizes are \$785, \$805, \$905 and \$1405.
We try starting with \$785 and \$805 to get to \$880. (Since \$905 and \$1405 are already larger than \$880, we do not need to try these.)

Starting with \$785, we need to give out an additional \$95 to get to \$880. Using three \$25 prizes accounts for \$75, leaving \$20 to be split among four \$5 prizes. (Using fewer than three \$25 prizes will mean we need more than six \$5 prizes in total.) So in this way, we need one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (since there were already two included in the \$785).

Starting with \$805, we need to give out an additional \$75 to get to \$880. Using three \$25 prizes will accomplish this, for a total of one \$625 prize, one \$125 prize, five \$25 prizes, and one \$5 prize. We could also use two \$25 prizes and five \$5 prizes to make up the \$75, for a total of one \$625 prize, one \$125 prize, four \$25 prizes, and six \$5 prizes (which is the same as we obtained above starting with \$785). If we use fewer than two additional \$25 prizes, we would need too many \$5 prizes.

Therefore, the two ways of giving out \$880 in prizes under the given conditions are:

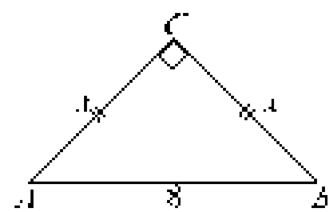
i) one \$625 prize, one \$125 prize, five \$25 prizes, one \$5 prize
ii) one \$625 prize, one \$125 prize, four \$25 prizes, six \$5 prizes

We can check by addition that each of these totals \$880.

2. (a) *Solution 1*

Let $AC = x$. Then $BC = x$ since triangle $ABC$ is isosceles.
Since triangle $ABC$ is right-angled, then, by the Pythagorean Theorem,

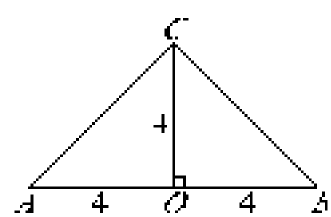
$$
\begin{aligned}
x^2 + x^2 &= 8^2 \\
2x^2 &= 64 \\
x^2 &= 32 \\
x &= \sqrt{32} = 4\sqrt{2}
\end{aligned}
$$



(In fact, we won't actually need to know $x$ – we will only need to know $x^2$.)
Therefore, the area of the triangle is $\frac{1}{2}(AC)(BC) = \frac{1}{2}x^2 = 16$.

*Solution 2*

Let $O$ be the midpoint $AB$. Thus, $O$ is the centre of the semi-circle, and the semi-circle has radius 4.
Join $O$ to $C$. Since triangle $ABC$ is isosceles, $OC$ is perpendicular to $AB$. Since $C$ is on the circle, then $OC$ is a radius, and so $OC = 4$.
Therefore, the area of the triangle is
$\frac{1}{2}(AB)(OC) = \frac{1}{2}(8)(4) = 16$.



(b) The total area of the two shaded regions is equal to the difference between the area of the semi-circle and the area of the triangle.
We know that the area of the triangle is 16, and the semi-circle has radius 4, so its area is $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4)^2 = 8\pi$.
Therefore, the total area of the two shaded regions is $8\pi - 16$.

(c) In (b), we saw that the area of the semi-circle drawn on $AB$ is $8\pi$.
Since $AC = CB$, then the areas of the semi-circles drawn on $AC$ and $CB$ are equal.
Since $AC = 4\sqrt{2}$, then the diameter of the semi-circle on $AC$ is $4\sqrt{2}$, and so the radius of the semi-circle is $2\sqrt{2}$. Therefore, the area of the semi-circle on $AC$ is $\frac{1}{2}\pi\left(2\sqrt{2}\right)^2 = \frac{1}{2}\pi(8) = 4\pi$.
So the sum of the areas of the semi-circles on $AC$ and $CB$ is $4\pi + 4\pi = 8\pi$, which is the area of the semi-circle on $AB$, as required.

3. (a) If Bob places a 3, then the total of the two numbers so far is 8, so Avril should place a 7 to bring the total up to 15.
Since Bob can place a 3 in any the eight empty circles, Avril should place a 7 in the circle directly opposite the one in which Bob places the 3. This allows Avril to win on her next turn.

(b) As in (a), Bob can place any of the numbers 1, 2, 3, 4, 6, 7, 8, 9 in any of the eight empty circles. On her next turn, Avril should place a disc in the circle directly opposite the one in which Bob put his number. What number should Avril use? Avril should place the number that brings the total up to 15, as shown below:

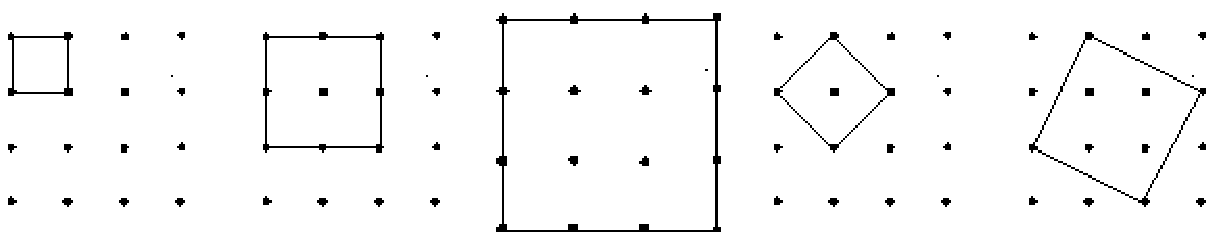
| Bob's First Turn | Total so far | Avril's Second Turn |
|---|---|---|
| 1 | 6 | 9 |
| 2 | 7 | 8 |
| 3 | 8 | 7 |
| 4 | 9 | 6 |
| 6 | 11 | 4 |
| 7 | 12 | 3 |
| 8 | 13 | 2 |
| 9 | 14 | 1 |

Since each of these possibilities is available to Avril on her second turn (since 5 is not in the list and none is equal to Bob's number), then she can always win on her second turn.

(c) Bob can place any of the numbers 4, 5, 6, 7, 8, 9 in any of the six empty circles. Which of these numbers can we pair up so that the sum of the pair is 14 (so that placing these numbers in the two circles at opposite ends of a line gives 15)? We can pair 5 and 9, and

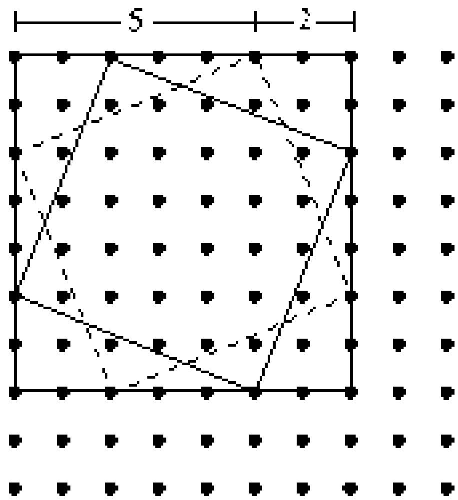
6 and 8 in this way. However, the 4 and 7 do not pair up with any other number to give a sum of 14.
So if Bob on his next turn places 5, 6, 8, or 9, Avril should place the second number from the pair (that is, 9, 8, 6, or 5, respectively) opposite the number Bob places and she will win.
If Bob places the 4 or the 7, Avril should then place the 7 or the 4, respectively, in the opposite circle. She will not win on this turn, but this forces Bob to then place one of the remaining 4 paired numbers on his next turn, so Avril can win for sure on her following turn.

4. (a) There are three rows of three 1 by 1 squares, or 9 in total.
There are two rows of two 2 by 2 squares, or 4 in total.
There is one 3 by 3 square.



There are squares which have side length $\sqrt{2}$, formed by cutting off the four corners of a 2 by 2 square, as in the third example in the 3 by 3 grid. Each 2 by 2 square can be cut to give one $\sqrt{2}$ by $\sqrt{2}$ square, so there are 4 of these.
We can also cut the 3 by 3 square. We can do by cutting off right-angled triangles which have legs 1 and 2 (and hypotenuse $\sqrt{5}$) to give a square of side-length $\sqrt{5}$. We can do this cutting in two different ways – either by starting the long side of the triangle horizontal, or with the long side of the triangle vertical. So there are two squares which are $\sqrt{5}$ by $\sqrt{5}$.
This gives $9+4+1+4+2=20$ squares in total.

(b) We start by noting that there are nine squares that measure 7 by 7. These squares have sides that are parallel to the sides of the 10 by 10 grid. For each of these squares, there are two squares that have a side length of $\sqrt{29}$. We have drawn one such example to illustrate this.




(c) First we count the number of squares with sides parallel to the sides of the 10 by 10 grid.
There are nine rows of nine 1 by 1 square, or $9^2$ squares in total.
There are eight rows of eight 2 by 2 squares, or $8^2$ squares in total.
This pattern continues, until we find that there are two rows of two 8 by 8 squares, or $2^2$ in total, and there is only $1 = 1^2$ 9 by 9 square.
As in (b), each of these squares starting with the 2 by 2 can be cut to give squares whose sides are not parallel to the side of the grid.
Each 2 by 2 square can have its sides divided 1 and 1 to give a square of side length $\sqrt{2}$. So the 2 by 2 squares account for $2(8^2)$ squares in total (the original 2 by 2 squares, plus the squares of side length $\sqrt{2}$).
Each 3 by 3 square can have its sides divided 1 and 2 (to give a square of side length $\sqrt{5}$) or 2 and 1 (to give a square of side length $\sqrt{5}$). So the 3 by 3 squares account for $3(7^2)$ squares in total.
Each 4 by 4 square can have its sides divided 1 and 3, 2 and 2, or 3 and 1 to give squares of side length $\sqrt{10}$, $\sqrt{8}$ or $\sqrt{10}$, respectively. So the 4 by 4 squares account for $4(6^2)$ squares in total.
We continue this pattern up to the 9 by 9 squares. Each 9 by 9 square can have its sides divided 1 and 8, 2 and 7, 3 and 6, 4 and 5, 5 and 4, 6 and 3, 7 and 2, or 8 and 1, to give squares of side length $\sqrt{65}$, $\sqrt{53}$, $\sqrt{45}$, $\sqrt{41}$, $\sqrt{41}$, $\sqrt{45}$, $\sqrt{53}$ or $\sqrt{65}$. So the 9 by 9 squares account for $9(1^2)$ squares in total, since there is only one 9 by 9 square, and it gives rise to 9 different squares.
Therefore, the total number of squares that can be formed is

$$1(9^2)+2(8^2)+3(7^2)+4(6^2)+5(5^2)+6(4^2)+7(3^2)+8(2^2)+9(1^2).$$

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An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2003 Solutions
# Galois Contest (Grade 10)

1. (a) *Solution 1*

Since we are looking for 5 consecutive perfect squares which add to 1815, then the middle one of these squares should be close to $\frac{1}{5}(1815) = 363$.

What perfect square is closest to 363? Using a calculator, $\sqrt{363} \approx 19.05$, so $19^2 = 361$ is closest to 363.

So we try

$$17^2 + 18^2 + 19^2 + 20^2 + 21^2 = 289 + 324 + 361 + 400 + 441 = 1815$$

as we wanted.

So the largest of these integers is 21.

*Solution 2*

Let $n$ be the smallest of these 5 consecutive positive integers.

Then

$$\begin{aligned} n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 + (n+4)^2 &= 1815 \\ n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 + n^2 + 6n + 9 + n^2 + 8n + 16 &= 1815 \\ 5n^2 + 20n + 30 &= 1815 \\ n^2 + 4n + 6 &= 363 \\ n^2 + 4n - 357 &= 0 \\ (n+21)(n-17) &= 0 \end{aligned}$$

Since $n$ is positive, then $n = 17$, and so the largest of the integers is $n + 4 = 21$.

*Solution 3*

Let $m$ be the middle of the 5 consecutive positive integers. (This will make the algebra easier.)

Then the 5 consecutive integers are $m-2$, $m-1$, $m$, $m+1$, and $m+2$, and so we have

$$\begin{aligned} (m-2)^2 + (m-1)^2 + m^2 + (m+1)^2 + (m+2)^2 &= 1815 \\ m^2 - 4m + 4 + m^2 - 2m + 1 + m^2 + m^2 + 2m + 1 + m^2 + 4m + 4 &= 1815 \\ 5m^2 + 10 &= 1815 \\ 5m^2 &= 1805 \\ m^2 &= 361 \\ m &= 19 \end{aligned}$$

since $m$ is positive. Therefore, the largest of the integers is $m + 2 = 21$.

(b) *Solution 1*

Let $n$ be the smallest of the 5 consecutive integers.

Then the sum of their squares is

$$\begin{aligned} &n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2 \\ &= n^2+n^2+2n+1+n^2+4n+4+n^2+6n+9+n^2+8n+16 \\ &= 5n^2+20n+30 \\ &= 5\left(n^2+4n+6\right) \end{aligned}$$

Since we have been able to factor out a 5 from the expression, then the sum of the squares of any 5 consecutive integers is divisible by 5.
(Note that $n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2=5\left(n^2+4n+6\right)$ is divisible by 5 because the expression left in the parentheses after the 5 is factored out has to be an integer since $n$ is an integer.)

*Solution 2*
Let $m$ be the middle of the 5 consecutive integers. (This will make the algebra easier.) Then the 5 consecutive integers are $m-2$, $m-1$, $m$, $m+1$, and $m+2$, and so the sum of their squares is

$$\begin{aligned} &(m-2)^2+(m-1)^2+m^2+(m+1)^2+(m+2)^2 \\ &= m^2-4m+4+m^2-2m+1+m^2+m^2+2m+1+m^2+4m+4 \\ &= 5m^2+10 \\ &= 5\left(m^2+2\right) \end{aligned}$$

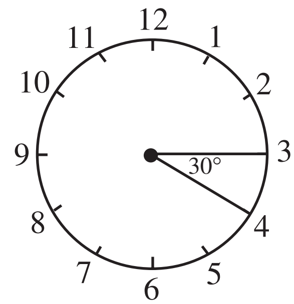
which is divisible by 5, since we are able to factor out a 5.
(As before, note that $m^2+2$ is always an integer because $m$ itself is an integer.)

*Extension*

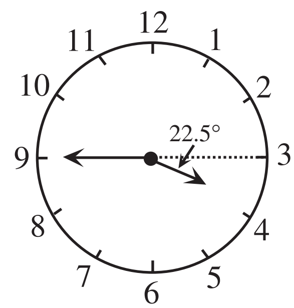
In (a), we saw that $17^2+18^2+19^2+20^2+21^2=1815$.
To express 1815 as the sum of 5 consecutive integers, we first figure out what the "middle" (or average) of the 5 integers is. The average is $\frac{1}{5}(1815)=363$, so
$361+362+363+364+365=1815$.
We want to determine the next integer larger than 1815 with this same property.
What is the next larger positive integer that is the sum of the squares of 5 consecutive integers?
In (b), we saw that if $m$ is the middle of 5 consecutive integers, then the sum of their squares is $(m-2)^2+(m-1)^2+m^2+(m+1)^2+(m+2)^2=5m^2+10$, and if $m=19$, then we get 1815 as the sum. The next larger value that is the sum of 5 squares will be when $m=20$, giving a sum of $5(20)^2+10=2010$.
Is 2010 the sum of 5 consecutive integers?
Here, the average number should be $\frac{1}{5}(2010)=402$, and in fact
$400+401+402+403+404=2010$.

Therefore, 2010 is the next integer larger than 1815 that is the sum of both 5 consecutive integers, and the sum of the squares of 5 consecutive integers.
(From what we have done here, it appears as if *any* multiple of 5 is the sum of 5 consecutive integers. Can you prove this?)

2. (a) Between 3:00 p.m. and 3:45 p.m., three-quarters of an hour has passed so the hour hand has moved three-quarters of the way from the "3" to the "4".
If we join each of the "3" and the "4" to the centre of the clock, the central angle of the sector formed is $\frac{1}{12}$ of the total way around the circle, or $30^\circ$.
Therefore, the angle through which the hour hand passes is $\frac{3}{4}\left(30^\circ\right) = 22.5^\circ$.




(b) *Solution 1*
If the hour hand was pointing exactly at the "3" at 3:45 p.m., the angle would be $180^\circ$, because the minute hand points exactly at the "9".
However, the hour hand has moved $22.5^\circ$ beyond the "3" (while the minute hand is pointing exactly at the "9"), so the angle between the hands is $180^\circ - 22.5^\circ = 157.5^\circ$.




*Solution 2*
For every hour that passes, the minute hand passes through $360^\circ$ and the hour hand passes through $30^\circ$. Therefore, in any one hour period, the minute hand passes through $330^\circ$ more than the hour hand. This implies that for every minute that passes the minute hand gains $5.5^\circ$ on the hour hand.
Between 3:00 p.m. and 3:45 p.m., the minute hand would gain $5.5^\circ \times 45 = 247.5^\circ$ on the hour hand.
Since the minute hand was $90^\circ$ behind the hour hand at 3:00 p.m., then at 3:45 p.m., the minute hand is $247.5^\circ - 90^\circ = 157.5^\circ$ ahead of the hour hand. This is the required angle.

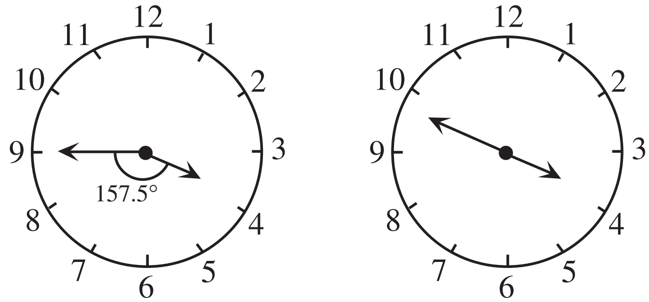
(c) *Solution 1*
At 3:45 p.m., the angle between the hands is $157.5^\circ$ and is *increasing*, because the minute hand moves faster than the hour hand, and the minute hand is "ahead" of the hour hand.
In one minute, the minute hand moves $\frac{1}{60}$ of the way around the clock, or $6^\circ$.
In one minute, the hour hand moves $\frac{1}{60}$ between two hour markings, or $\frac{1}{60}$ of $\frac{1}{12}$ of the total way around the clock, or $0.5^\circ$.

Therefore, the angle between the hands is increasing at a rate of $6^\circ - 0.5^\circ = 5.5^\circ$ per minute, since the hands are moving in the same direction.

From 3:45 p.m., the angle needs to increase by $180^\circ - 157.5^\circ = 22.5^\circ$ to reach $180^\circ$.

Therefore, it will take $\dfrac{22.5^\circ}{5.5^\circ} \approx 4.09$ minutes (or 4 minutes and 5 seconds) after 3:45 p.m. for the angle to become $180^\circ$.

Thus, the angle is $180^\circ$ at approximately 3:49 p.m.

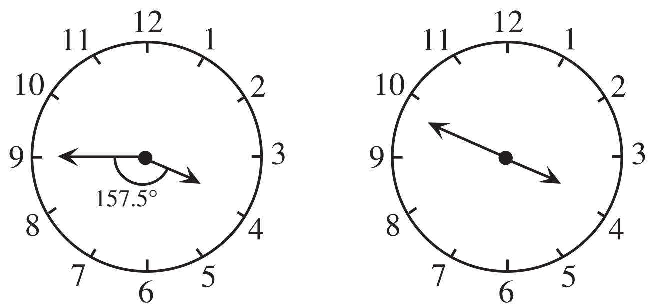



*Solution 2*

From Solution 2 in (b), we know that the minute hand gains $5.5^\circ$ every minute on the hour hand. At 3:00 p.m., the minute hand is $90^\circ$ behind the hour hand and we want it to be $180^\circ$ ahead of the hour hand.

In total, then, the minute hand must "make up" $270^\circ$ which will take $\dfrac{270^\circ}{5.5^\circ} = 49\frac{1}{11}$ minutes.

Thus, the angle is $180^\circ$ at approximately 3:49 p.m.

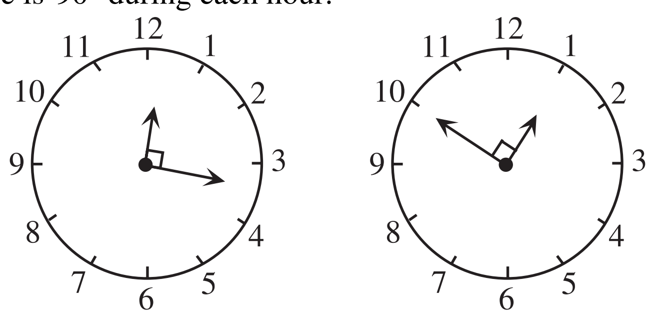



*Extension*

In any 12 hour period, every time between 12:00 and 11:59 will occur exactly once, so we can say that the 12 hour period begins at 12:00 and ends at 12:00.

We look for positions where the angle is $90^\circ$ during each hour.

Between 12:00 and 1:00, the hour hand will be between the 12 and 1. Thus, there will be one desired time when the minute hand is between the 3 and 4, and one where it is between the 9 and 10 (ie. one desired time between 12:15 and 12:20 and another between 12:45 and 12:50).



Between 1:00 and 2:00, the hour hand will be between the 1 and 2. Thus, there will be one desired time when the minute hand is between the 4 and 5, and one where it is between the 10 and 11.
We can continue with this reasoning for each of the 12 hours, and obtain 2 desired times for each hour slot. However, if one of the desired times occurs exactly on an hour, we will have counted it twice! (For instance, between 2:00 and 3:00 we will have counted one desired time when the minute hand is between the 11 and 12, while between 3:00 and 4:00 we will have counted one desired time when the minute hand is between the 12 and 1. These are in fact the same time – 3:00!)

At which hours is the angle between the hands equal to $90^\circ$? At 3:00 and 9:00. Therefore, we have double-counted two times, so the actually number of desired times is $2(12)-2=22$.

3. (a) We complete the chart, including at each step a description of the step.

This is one possible way for the game to be played in 8 moves.

| | | | | | |
|---|---|---|---|---|---|
| *Start* | *Q* | *Q* | | *D* | *D* |
| *Shift* | *Q* | | *Q* | *D* | *D* |
| *Jump* | *Q* | *D* | *Q* | | *D* |
| *Shift* | *Q* | *D* | *Q* | *D* | |
| *Jump* | *Q* | *D* | | *D* | *Q* |
| *Jump* | | *D* | *Q* | *D* | *Q* |
| *Shift* | *D* | | *Q* | *D* | *Q* |
| *Jump* | *D* | *D* | *Q* | | *Q* |
| *Shift* | *D* | *D* | | *Q* | *Q* |

There is in fact only one other possibility (which simply reverses whether a dime or quarter is being moved at each step).

(We can notice that, at each step, there is only one possible move that can be made to avoid having to backtrack.)

| | | | | | |
|---|---|---|---|---|---|
| *Start* | *Q* | *Q* | | *D* | *D* |
| *Shift* | *Q* | *Q* | *D* | | *D* |
| *Jump* | *Q* | | *D* | *Q* | *D* |
| *Shift* | | *Q* | *D* | *Q* | *D* |
| *Jump* | *D* | *Q* | | *Q* | *D* |
| *Jump* | *D* | *Q* | *D* | *Q* | |
| *Shift* | *D* | *Q* | *D* | | *Q* |
| *Jump* | *D* | | *D* | *Q* | *Q* |
| *Shift* | *D* | *D* | | *Q* | *Q* |

(b) In a game starting with 3 quarters and 3 dimes, the game board will have 7 squares. Since there are only two possible types of moves that can be made ("Jumps" and "Shifts"), then it is impossible for the 3 quarters to switch their order (ie. we are not allowed to jump one quarter over a second quarter).
Thus, the quarter that starts in the third square ends in the seventh square, the quarter that starts in the second square ends in the sixth square, and the quarter that starts in the first square ends in the fifth square.
Each quarter then moves 4 spaces, making a total of 12 squares moved. Similarly, each dime moves 4 spaces, making a total of 12 squares moved. In total, the coins move 24 squares. If this was done using only shifts, this would require 24 moves.

However, the game cannot be played in this fashion, because it is necessary to make jumps. Since we want the dimes and quarters to change positions, each dime needs to jump over (or be jumped over by) each of the 3 quarters. In other words, there need to 9 jumps made. Since each jump results in a move of 2 spaces, this "saves" 9 shifts.
Therefore, the number of required moves is at least $24-9=15$.
We have assumed here that no "backtracking" is done, so the game cannot be played in fewer than 15 moves.
(Can you construct the diagram to show how the game can be played in 15 moves?)

*Extension*

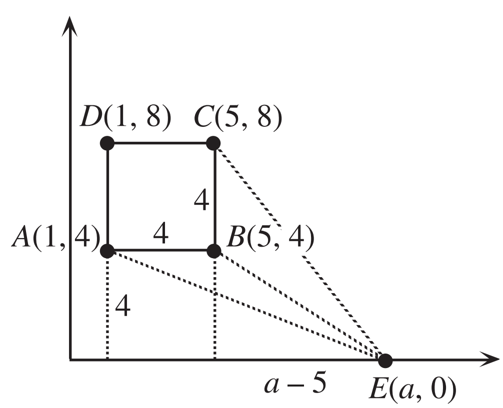
We use the same strategy as in (b).
In a game starting with $n$ quarters and $n$ dimes, the game board will have $2n+1$ slots.

Since there are only two possible types of moves that can be made ("Jumps" and "Shifts"), then it is impossible for the $n$ quarters to switch their order (ie. we are not allowed to jump on quarter over another).
Thus, the quarter that starts in the first square ends in the $(n+2)$th square, the quarter that starts in the second square ends in the $(n+3)$th square, and so on, with the quarter that starts in the $n$th square ending in the $(2n+1)$th square.
Each of the $n$ quarters has moved a total of $n+1$ squares, so the quarters have therefore moved a total of $n(n+1)$ squares.
Similarly, the dimes have moved a total $n(n+1)$ squares.
So the coins have moved a total of $2n(n+1)=2n^2+2n$ squares. If this was done using only shifts, this would require $2n^2+2n$ moves.
Using exactly the same reasoning as in (c), the required number of moves is
$2n^2+2n-n^2=n^2+2n=n(n+2)$.
(A question that should be asked is "Can the game be played in exactly $n(n+2)$ moves?" In order to answer this question, we would need to come up with a general strategy that would allow us to play the game in this number of moves, no matter what value $n$ takes.)

4. (a) Since $ABCD$ is a square and $AD$ has side length 4, then each of the sides of $ABCD$ has length 4. We can also conclude that $B$ has coordinates $(5,4)$ and $C$ has coordinates $(5,8)$. (Since $AD$ is parallel to the $y$-axis, then $AB$ is parallel to the $x$-axis.) The area of square $ABCD$ is thus $4^2=16$.

Since the $\Delta CBE$ and $\Delta ABE$ lie entirely outside square $ABCD$ then $E$ must lie below the line $AB$ (which it does), and to the right of the line $CB$, so $a>5$.

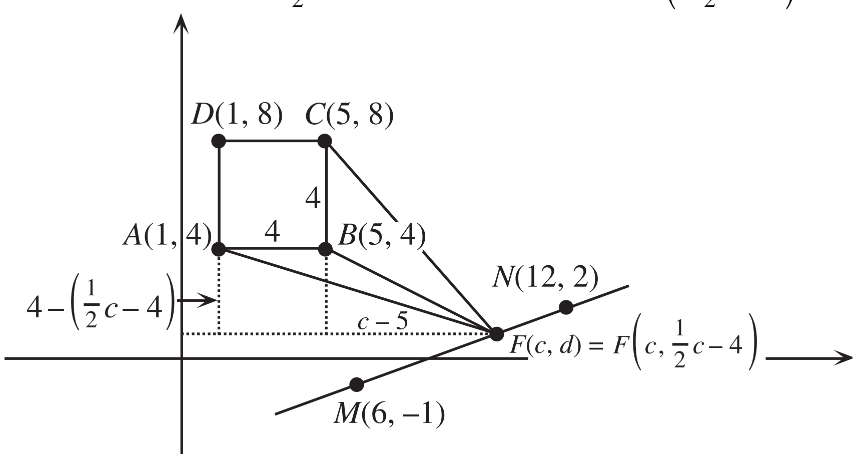
First, we consider $\Delta ABE$. We can think of $AB$, which has length 4, as being the base of $\Delta ABE$. The length of the height of $\Delta ABE$ is the distance from $E$ to the line through $A$ and $B$. This distance is 4, since $AB$ is parallel to the $x$-axis. Thus, the area of $\Delta ABE$ is $\frac{1}{2}bh=\frac{1}{2}(4)(4)=8$. (Note that the area of $\Delta ABE$ is *always* 8, provided that $a>5$.)




Next, we look at $\Delta CBE$. We can think of $CB$, which has length 4, as being the base of $\Delta CBE$. The length of the height of $\Delta CBE$ is the distance from $E$ to the line through $C$ and $B$. This distance is $a-5$, since $BC$ is parallel to the $y$-axis. Therefore, the area of $\Delta CBE$ is $\frac{1}{2}bh=\frac{1}{2}(4)(a-5)=2a-10$.

Since we would like the sum of the areas of these two triangles to be equal to the area of the square, then $8+(2a-10)=16$ or $a=9$.
Therefore, $a=9$.
[Note that this can also be done by subtraction of areas.]

(b) Let the point $F$ have coordinates $(c,d)$.
Since triangle $CBF$ lies entirely outside the square, $F$ lies to the right of the square, ie. $c>5$. Since triangle $ABF$ lies entirely outside the square, $F$ lies below the square, ie. $d<4$.
Now $F$ lies on the line passing through both $M$ and $N$. What is the equation of this line?
The slope is $\frac{2-(-1)}{12-6}=\frac{1}{2}$, so the equation of the line is $y-2=\frac{1}{2}(x-12)$ or $y=\frac{1}{2}x-4$.
Since $F$ lies on this line, then $d=\frac{1}{2}c-4$, so $F$ has coordinates $\left(c,\frac{1}{2}c-4\right)$.



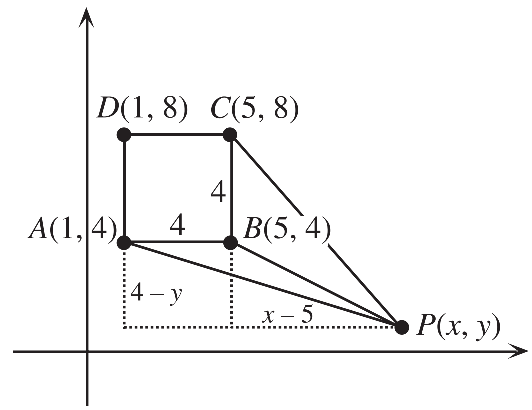

We consider next $\Delta ABF$. This triangle has base $AB$ of length 4. The height of $\Delta ABF$ is the vertical distance from $F$ to the line through $A$ and $B$, which is $4-\left(\frac{1}{2}c-4\right)=8-\frac{1}{2}c$.
Therefore, the area of $\Delta ABF$ is $\frac{1}{2}bh=\frac{1}{2}(4)\left(8-\frac{1}{2}c\right)=16-c$.

$\Delta CBF$ has base $CB$ of length 4. The height of $\Delta CBF$ is the horizontal distance from $F$ to the line through $C$ and $B$, which is $c-5$. Therefore, the area of $\Delta CBF$ is $\frac{1}{2}bh=\frac{1}{2}(4)(c-5)=2c-10$.

Thus, we are looking for the value of $c$ for which $16-c+2c-10=16$, and so $c=10$.
Therefore, $F$ is the point $(10,1)$.

*Extension*

Since triangle *CBP* lies entirely outside the square, *P* lies to the right of the square, ie. $x > 5$. Since triangle *ABP* lies entirely outside the square, *P* lies below the square, ie. $y < 4$.




We consider next $\Delta ABP$. This triangle has base *AB* of length 4. The height of $\Delta ABP$ is the vertical distance from *P* to the line through *A* and *B*, which is $4-y$. Therefore, the area of $\Delta ABP$ is $\frac{1}{2}bh = \frac{1}{2}(4)(4-y) = 8-2y$.

$\Delta CBP$ has base *CB* of length 4. The height of $\Delta CBP$ is the horizontal distance from *P* to the line through *C* and *B*, which is $x-5$. Therefore, the area of $\Delta CBP$ is $\frac{1}{2}bh = \frac{1}{2}(4)(x-5) = 2x-10$.

Since the sum of the areas of $\Delta ABP$ and $\Delta CBP$ is equal to the area of the square, then

$$\begin{aligned}(8-2y)+(2x-10) &= 16\\ 2x-18 &= 2y\\ y &= x-9\end{aligned}$$

Therefore, the points *P* which satisfy the condition that the sum of the areas of $\Delta ABP$ and $\Delta CBP$ is equal to the area of the rectangle are all of the points on the line $y = x-9$ with $x > 5$ and $y < 4$.

[Note that we could include the two endpoints $(5,-4)$ and $(13,4)$ if we allowed the area of one of the two triangles to be equal to 0.]