The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2024 Fermat Contest*

**(Grade 11)**

**Wednesday, February 28, 2024**
**(in North America and South America)**

**Thursday, February 29, 2024**
**(outside of North America and South America)**

*Solutions*

1. Calculating, $3\left(\frac{5}{3}-\frac{1}{3}\right)=3\cdot\frac{5}{3}-3\cdot\frac{1}{3}=5-1=4$.
   Alternatively, $3\left(\frac{5}{3}-\frac{1}{3}\right)=3\cdot\frac{4}{3}=4$.

   ANSWER: (D)

2. Simplifying, $4x^2-3x^2=x^2$. When $x=2$, this expression equals 4.
   Alternatively, when $x=2$, we have $4x^2-3x^2=4\cdot 2^2-3\cdot 2^2=16-12=4$.

   ANSWER: (C)

3. The volume of a $1\times 1\times 1$ cube is 1.
   The volume of a $2\times 2\times 2$ cube is 8.
   Thus, 8 of the smaller cubes are needed to make the larger cube.

   ANSWER: (E)

4. For there to be equal numbers of each colour of candy, there must be at most 3 red candies and at most 3 yellow candies, since there are 3 blue candies to start.
   Thus, Shuxin ate at least 7 red candies and at least 4 yellow candies.
   This means that Shuxin ate at least $7+4=11$ candies.
   We note that if Shuxin eats 7 red candies, 4 yellow candies, and 0 blue candies, there will indeed be equal numbers of each colour.

   ANSWER: (A)

5. Square $PQRS$ is made up of 16 equal-sized small squares.
   Of these, 2 are fully shaded and 8 are half-shaded.
   This shading is equivalent to fully shading $2+8\cdot\frac{1}{2}=2+4=6$ of the 16 small squares.
   Thus, square $PQRS$ is $\frac{6}{16}=\frac{3}{8}$ shaded.

   ANSWER: (E)

6. Using a calculator, $\sqrt{15}\approx 3.87$ and $\sqrt{50}\approx 7.07$.
   The integers between these real numbers are 4, 5, 6, 7, of which there are 4.
   Alternatively, we could note that integers between $\sqrt{15}$ and $\sqrt{50}$ correspond to values of $\sqrt{n}$ where $n$ is a perfect square and $n$ is between 15 and 50. The perfect squares between 15 and 50 are 16, 25, 36, 49, of which there are 4.

   ANSWER: (B)

7. *Solution 1*
   When a line is reflected in the $y$-axis, its $y$-intercept does not change (since it is on the line of reflection) and its slope is multiplied by $-1$.
   Therefore, the new line has slope $-3$ and $y$-intercept 6, which means that its equation is $y=-3x+6$.
   The $x$-intercept of this new line is found by setting $y=0$ and solving for $x$ which gives $0=-3x+6$ or $3x=6$ or $x=2$.

   *Solution 2*
   The $x$-intercept of the original line is found by setting $y=0$ in the equation of the line and solving for $x$, which gives $0=3x+6$ or $3x=-6$ or $x=-2$.
   When the line is reflected in the $y$-axis, the $x$-intercept of the new line is the reflection of the original line in the $y$-axis, and thus is $x=2$.

   ANSWER: (A)

8. Using exponent laws, $1000^{20} = (10^3)^{20} = 10^{60}$ and so $n = 60$.

Answer: (B)

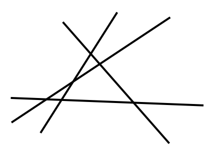
9. Since $O$ is the centre of the circle, then $OA = OB = OC$.
This means that $\triangle AOB$ and $\triangle COB$ are both isosceles with $\angle ABO = \angle BAO = \angle BAC = 25^\circ$.
Thus, $\angle AOB = 180^\circ - \angle ABO - \angle BAO = 130^\circ$.
Since $\angle AOC$ is a straight angle, then $\angle BOC = 180^\circ - \angle AOB = 180^\circ - 130^\circ = 50^\circ$.

Answer: (D)

10. After David is seated, there are 4 seats in which Pedro can be seated, of which 2 are next to David.
Thus, the probability that Pedro is next to David is $\frac{2}{4}$ or $\frac{1}{2}$.

Answer: (C)

11. Each of the 4 lines can intersect each of the other 3 lines at most once.
This might appear to create $4 \times 3 = 12$ points of intersection, but each point of intersection is counted twice – one for each of the 2 lines.
Thus, the maximum number of intersection points is $\dfrac{4 \times 3}{2} = 6$.
The diagram below demonstrates that 6 intersection points are indeed possible:



Answer: (D)

12. When a list of 5 numbers $a$, $b$, $c$, $d$, $e$ has the property that $a + b + c = c + d + e$, it is also true that $a + b = d + e$.
With the given list of 5 numbers, it is likely easier to find two pairs with no overlap and with equal sum than to find two triples with one overlap and equal sum.
After some trial and error, we can see that $6 + 21 = 10 + 17$, and so the list 6, 21, 5, 10, 17 has the given property, which means that 5 is in the middle.
(We note that these two pairs are the only such pairs, after allowing for switching the numbers in each pair and/or switching the pairs.)

Answer: (A)

13. Expanding, $(x + m)(x + n) = x^2 + nx + mx + mn = x^2 + (m + n)x + mn$.
The constant term of this quadratic expression is $mn$, and so $mn = -12$.
Since $m$ and $n$ are integers, they are each divisors of $-12$ and thus of 12.
Of the given possibilities, only 5 is not a divisor of 12, and so $m$ cannot equal 5.
We can check that each of the other four choices is a possible value of $m$.

Answer: (E)

14. We note first that $\triangle ACB$ has a right angle and a $60^\circ$ angle and so it is a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Since $AB = \sqrt{3}$, then using the known ratios of side lengths, we can see that $BC = 1$ and $AC = 2$.
Next, we note that $\triangle ACE$ has two $45^\circ$ angles and so is an isosceles right-angled triangle.
This means that $CE = AC = 2$ and $\angle ACE = 90^\circ$.

Also, $AE = \sqrt{2}AC = 2\sqrt{2}$.
Further, since $\angle BCD$ is a straight angle, then

$$\angle ECD = 180^\circ - \angle ACB - \angle ACE = 180^\circ - 60^\circ - 90^\circ = 30^\circ$$

Since $\triangle CED$ has a $30^\circ$ angle and a right-angle, it is also a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Using the known ratios of sides, since $CE = 2$, we have $DE = 1$ and $CD = \sqrt{3}$.
Therefore, the perimeter of $ABDE$ is

$$AB + BC + CD + DE + AE = \sqrt{3} + 1 + \sqrt{3} + 1 + 2\sqrt{2} = 2 + 2\sqrt{2} + 2\sqrt{3}$$

Answer: (E)

15. We first note that $197 = 8 \cdot 24 + 5$.
This tells us that the time that is 197 hours from now is 8 days and 5 hours.
Since Anila's grandmother's activities are the same every day, then in 197 hours and 5 minutes she will be doing the same thing as she is doing in 5 hours and 5 minutes, at which point she is doing yoga.

Answer: (C)

16. Of the row and column products, only 135 and 160 are divisible by 5. This means that 5 must go in the square in the 2nd row, 3rd column.
Of the row and column products, only 21 and 56 are divisible by 7. This means that 7 must go in the square in the 1st row, 1st column.
Of the row and column products, only 108 and 135 are divisible by 9. This means that 9 must go in the square in the 2nd row, 2nd column.
So far, this gives the following grid:

| | | | |
|---|---|---|---|
| 7 | | | 56 |
| | 9 | 5 | 135 |
| | | | 48 |
| 21 | 108 | 160 | |

In the 2nd row, the product is 135 which means that the missing entry is $\frac{135}{5 \cdot 9} = 3$.
In the 1st column, the product is 21 which means that the missing entry is $\frac{21}{7 \cdot 3} = 1$.
The 3rd row, whose product is 48, thus includes 1 and two more integers between 1 and 9. The only divisor pair of 48 with both divisors less than 10 is $48 = 6 \cdot 8$.
Since 8 is not a divisor of 108, then $N$ must be 6.

We can complete the square as follows:

| | | | |
|---|---|---|---|
| 7 | 2 | 4 | 56 |
| 3 | 9 | 5 | 135 |
| 1 | 6 | 8 | 48 |
| 21 | 108 | 160 | |

Answer: (C)

17. Since $b + d > a + d$, then $b > a$. This means that $a$ does not have the greatest value.
Since $c + e > b + e$, then $c > b$. This means that $b$ does not have the greatest value.
Since $b + d = c$ and each of $b$, $c$, $d$ is positive, then $d < c$, which means that $d$ does not have the greatest value.
Consider the last equation $a + c = b + e$ along with the fact that $a < b < c$.
From this, we see that $e = c + (a - b)$.
Since $a < b$, then $a - b$ is negative and so $e < c$.
This means that $c$ has the greatest value.

Answer: (C)

18. Since $3x + 2y = 6$, then $(3x + 2y)^2 = 6^2$ or $9x^2 + 12xy + 4y^2 = 36$.
Since $9x^2 + 4y^2 = 468$, then

$$12xy = (9x^2 + 12xy + 4y^2) - (9x^2 + 4y^2) = 36 - 468 = -432$$

and so $xy = \frac{-432}{12} = -36$.
(With some additional work, we can find that the solutions to the system of equations are $(x, y) = (-4, 9)$ and $(x, y) = (6, -6)$.)

Answer: (A)

19. Suppose that when the three dice are rolled, the numbers rolled are $x$, $y$ and $z$.
Since there are 6 possibilities for each of $x$, $y$ and $z$, there are $6 \cdot 6 \cdot 6 = 216$ possible outcomes.
Also, the sum, $S$, of the three rolls is at least $3 \cdot 1 = 3$ and at most $3 \cdot 6 = 18$.
The outcome "$S > 5$" is the complement of the outcome "$S \leq 5$".
Thus, the probability that $S > 5$ is 1 minus the probability that $S \leq 5$.
It is easier to compute the probability that $S \leq 5$ directly by listing the rolls that give this.
If $S = 3$, then $x + y + z = 3$ and so $(x, y, z) = (1, 1, 1)$.
If $S = 4$, then $x + y + z = 4$ and so $x$, $y$ and $z$ must be 1, 1 and 2 in some order. Thus, $(x, y, z) = (2, 1, 1)$ or $(1, 2, 1)$ or $(1, 1, 2)$.
If $S = 5$, then $x + y + z = 5$ and so $x$, $y$ and $z$ must be 1, 1 and 3, or 1, 2 and 2 in some order. There are 3 arrangements in each case and so 6 triples in total.
Therefore, there are $1 + 3 + 6 = 10$ triples with $S \leq 5$, and so the probability that $S > 5$ is equal to $1 - \frac{10}{216} \approx 0.954$.
Of the given choices, this is closest to 0.95.

Answer: (B)

20. Suppose that the radius of cylinder is $r$ and the height of the cylinder is $h$.
This means that the volume of the cylinder is $\pi r^2 h$; the volume of half of the cylinder is $\frac{1}{2}\pi r^2 h$.
Also, the radius of the cone is $\frac{1}{2}r$ and the height of the cone is $h$.
This means that the volume of the cone is $\frac{1}{3}\pi\left(\frac{1}{2}r\right)^2 h$ or $\frac{1}{12}\pi r^2 h$.
When the cone is divided into two pieces by a horizontal plane at half of its height, the top portion of the cone is a cone with the same proportions, but with dimensions $\frac{1}{2}$ of those of the larger cone.
This means that the volume of the top portion is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ of that of the cone, which equals $\frac{1}{8}\cdot\frac{1}{12}\pi r^2 h$ or $\frac{1}{96}\pi r^2 h$.
To see this in another way, we note that this top portion of the cone has height $\frac{1}{2}h$ and should have radius $\frac{1}{2}\cdot\frac{1}{2}r$ (because the radius decreases proportionally to the height). This means that the volume of this portion is $\frac{1}{3}\pi\left(\frac{1}{4}r\right)^2\cdot\frac{1}{2}h$ which is again $\frac{1}{96}\pi r^2 h$.
Using this information, the bottom portion of the cone has volume $\frac{7}{8}\cdot\frac{1}{12}\pi r^2 h = \frac{7}{96}\pi r^2 h$.
Now, when the cone is in the cylinder and the cylinder is filled with water to half of its height, the volume of the bottom half of the cylinder is filled with the bottom portion of the cone and with the water.
Therefore, the volume of water is the difference between half of the volume of the cylinder and the volume of the bottom portion of the cone, or $\frac{1}{2}\pi r^2 h - \frac{7}{96}\pi r^2 h = \frac{48}{96}\pi r^2 h - \frac{7}{96}\pi r^2 h = \frac{41}{96}\pi r^2 h$.
When the cone is removed, the water then occupies a cylinder with radius $r$ and volume $\frac{41}{96}\pi r^2 h$.
If the depth of the water in this configuration is $d$, then $\pi r^2 d = \frac{41}{96}\pi r^2 h$ and so $d = \frac{41}{96}h$, which means that the depth of the water is $\frac{41}{96}$ of the height of the cylinder.

ANSWER: (B)

21. Since the second column includes the number 1, then step (ii) was never used on the second column, otherwise each entry would be at least 2.
To generate the 1, 3 and 2 in the second column, we thus need to have used step (i) 1 time on row 1, 3 times on row 2, and 2 times on row 3.
This gives:

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 3 | 3 | 3 |
| 2 | 2 | 2 |

We cannot use step (i) any more times, otherwise the entries in column 2 will increase. Thus, $a = 1 + 3 + 2 = 6$.
To obtain the final grid from this current grid using only step (ii), we must increase each entry in column 1 by 6 (which means using step (ii) 3 times) and increase each entry in column 3 by 4 (which means using step (ii) 2 times). Thus, $b = 3 + 2 = 5$.
Therefore, $a + b = 11$.

ANSWER: 11

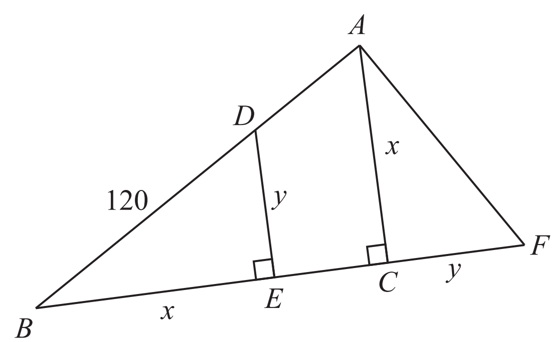
22. We note that

$$ac + bd - ad - bc = ac - ad - bc + bd = a(c - d) - b(c - d) = (a - b)(c - d)$$

Since each of $a$, $b$, $c$, $d$ is taken from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then $a - b \leq 9$ since the greatest possible difference between two numbers in the set is 9.
Similarly, $c - d \leq 9$.
Now, if $a - b = 9$, we must have $a = 10$ and $b = 1$.
In this case, $c$ and $d$ come from the set $\{2, 3, 4, 5, 6, 7, 8, 9\}$ and so $c - d \leq 7$.
Therefore, if $a - b = 9$, we have $(a - b)(c - d) \leq 9 \cdot 7 = 63$.
If $a - b = 8$, then either $a = 9$ and $b = 1$, or $a = 10$ and $b = 2$.
In both cases, we cannot have $c - d = 9$ but we could have $c - d = 8$ by taking the other of these two pairs with a difference of 8.
Thus, if $a - b = 8$, we have $(a - b)(c - d) \leq 8 \cdot 8 = 64$.
Finally, if $a - b \leq 7$, the original restriction $c - d \leq 9$ tells us that $(a - b)(c - d) \leq 7 \cdot 9 = 63$.
In summary, the greatest possible value for $ac + bd - ad - bc$ is 64 which occurs, for example, when $a = 9$, $b = 1$, $c = 10$, and $d = 2$.

ANSWER: 64

23. *Solution 1*
Suppose that $BE = AC = x$ and $DE = y$.
Extend $BC$ to point $F$ so that $BC = DE = y$.

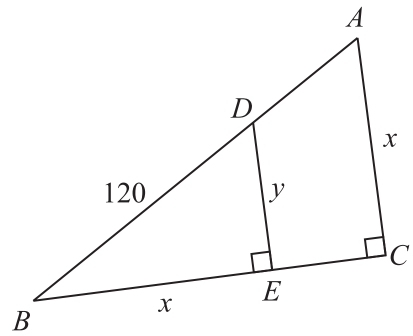



Since $BC + DE = 288$, then $BF = BC + CF = BC + DE = 288$.
Also, $\triangle BED$ is congruent to $\triangle ACF$ by side-angle-side.
Therefore,

$$\angle BAF = \angle BAC + \angle ACF = \angle BDE + \angle DBE = 90^\circ$$

since $DE$ and $AC$ are parallel.
Next, $\triangle BED$ is similar to $\triangle BAF$ since both are right-angled and they share an angle at $B$.
Therefore, $\dfrac{DE}{BD} = \dfrac{FA}{BF}$ and so $\dfrac{DE}{120} = \dfrac{120}{288}$, which gives $DE = \dfrac{120 \cdot 120}{288} = 50$, as required.

*Solution 2*
Suppose that $BE = AC = x$ and $DE = y$.
Since $DE + BC = 288$, then $BC = 288 - y$.




We note that $\triangle BED$ is similar to $\triangle BCA$ because each is right-angled and their angles at $B$ are common.
Therefore, $\dfrac{BE}{DE} = \dfrac{BC}{AC}$ and so $\dfrac{x}{y} = \dfrac{288 - y}{x}$.
Manipulating, we obtain $x^2 = y(288 - y)$ and so $x^2 = 288y - y^2$ or $x^2 + y^2 = 288y$.
Also, using the Pythagorean Theorem in $\triangle BED$ gives $x^2 + y^2 = 120^2$.
Since $x^2 + y^2 = 288y$ and $x^2 + y^2 = 120^2$, then $288y = 120^2$ which gives $2 \cdot 12 \cdot 12 \cdot y = 120 \cdot 120$ and so $2y = 10 \cdot 10$ or $y = 50$.
Therefore, $DE = 50$.

Answer: 50

24. Throughout this solution, we use the fact that if $N$ is a positive integer with $N > 1$ and $N$ has prime factorization $p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ for some distinct prime numbers $p_1, p_2, \ldots, p_m$ and positive integers $a_1, a_2, \ldots, a_m$, then the number of positive divisors of $N$ including 1 and $N$ is equal to $(1 + a_1)(1 + a_2) \cdots (1 + a_m)$.
We are told that $N$ is a positive multiple of 2024.
Now, $2024 = 8 \cdot 253 = 2^3 \cdot 11 \cdot 23$.
This means that $N$ has at least 3 prime factors (namely 2, 11 and 23) and that at least one of these prime factors has an exponent of at least 3.
Let $D$ be the number of positive divisors that $N$ has. We are told that $100 < D < 110$.
Since $D = (1 + a_1)(1 + a_2) \cdots (1 + a_m)$ and $N$ has at least 3 prime factors, then $D$ is a positive integer that can be written as the product of at least 3 positive integers each greater than 2.
$D$ cannot equal 101, 103, 107, or 109, since each of these is prime (and so cannot be written as the product of 3 integers each at least 2).
$D$ also cannot equal 106 because $106 = 2 \cdot 53$ (both 2 and 53 are prime), which means that 106 cannot be written as the product of three integers each greater than 1.
The possible values of $D$ that remain are 102, 104, 105, 108.
We note that $102 = 2 \cdot 3 \cdot 17$ and $104 = 2^3 \cdot 13$ and $105 = 3 \cdot 5 \cdot 7$ and $108 = 2^2 \cdot 3^3$.

Case 1: $D = 102$
Since the prime factors of $N$ include at least 2, 11 and 23, then the prime factorization of $N$ includes factors of $2^a$, $11^b$ and $23^c$ for some positive integers $a$, $b$ and $c$ with $a \geq 3$.
If a fourth prime power $p^e$ was also a factor of $N$, then $D$ would be divisible by $(1 + a)(1 + b)(1 + c)(1 + e)$. ($D$ could have more factors if $N$ had more prime factors.)
Since $D = 102 = 2 \cdot 3 \cdot 17$ has only 3 prime factors, it cannot be written as the product of 4 integers each greater than 1.

Thus, $N$ cannot have a fourth prime factor.
This means that $N = 2^a 11^b 23^c$, which gives $D = (1+a)(1+b)(1+c) = 2 \cdot 3 \cdot 17$.
This means that $1+a$, $1+b$ and $1+c$ are equal to 2, 3 and 17, in some order, and so $a$, $b$ and $c$ are equal to 1, 2 and 16, in some order.
For $D$ to be as small as possible, the largest exponent goes with the smallest prime, the next largest exponent with the next smallest prime, and so on. (Can you see why this makes $N$ as small as possible?)
Therefore, the smallest possible value of $N$ in this case is $N = 2^{16} 11^2 23^1 = 182\,386\,688$.

Case 2: $D = 105$
Using a similar argument, we can determine that $N = 2^a 11^b 23^c$ with $1+a$, $1+b$ and $1+c$ equal to 3, 5 and 7 in some order, meaning that $a$, $b$ and $c$ equal 2, 4, 6 in some order.
Therefore, the minimum value of $N$ is this case is $N = 2^6 11^4 23^2 = 495\,685\,696$.

Case 3: $D = 104$
Since $D = 2^3 \cdot 13$ has 4 prime factors, then $N$ cannot have more than 4 prime factors. (If $N$ had 5 or more prime factors, then the product equal to $D$ would include at least 5 integers, each at least 2.)
Therefore, $N = 2^a 11^b 23^c$ and $D = (1+a)(1+b)(1+c)$, or $N = 2^a 11^b 23^c p^e$ for some prime $p \neq 2, 11, 23$ and $D = (1+a)(1+b)(1+c)(1+e)$.
This means that $(1+a)(1+b)(1+c) = 2^3 \cdot 13$ or $(1+a)(1+b)(1+c)(1+e) = 2^3 \cdot 13$.
In the case that $N$ has three prime factors, we note that $104 = 26 \cdot 2 \cdot 2 = 13 \cdot 4 \cdot 2$ are the only two ways of writing 104 as the product of 3 integers each of which is at least 2.
These give corresponding minimum values of $N$ of $N = 2^{25} \cdot 11 \cdot 23 = 8\,489\,271\,296$ and $N = 2^{12} \cdot 11^3 \cdot 23 = 125\,390\,848$.
In the case that $N$ has four prime factors, then $(1+a)(1+b)(1+c)(1+e) = 2 \cdot 2 \cdot 2 \cdot 13$ means that $a$, $b$, $c$ and $e$ are 1, 1, 1, 12 in some order.
This in turn means that the corresponding smallest possible value of $N$ is

$$N = 2^{12} \cdot 3 \cdot 11 \cdot 23 = 3\,108\,864$$

We note here that the prime power $p^e$ has become $3^1$ in order to minimize both $p$ (since $p > 2$) and its exponent.

Case 4: $D = 108$
Since $D = 2^2 \cdot 3^3$ has 5 prime factors, then $N$ cannot have more than 5 prime factors.
If $N$ has 5 prime factors, then we need to use the factorization $D = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$.
This is not possible, however, because the power $2^a$ must have $a \geq 3$ which would mean that one of the five factors of $D$ would have to be at least 4.
If $N$ has 4 prime factors, then $D$ must be partitioned as $9 \cdot 3 \cdot 2 \cdot 2$ or $6 \cdot 3 \cdot 3 \cdot 2$ or $4 \cdot 3 \cdot 3 \cdot 3$. (Since two of the prime factors have to be combined, either two 2s, two 3s, or a 2 and a 3 are combined.)
These give minimum values of $N = 2^8 \cdot 3^2 \cdot 11 \cdot 23 = 582\,912$ and $N = 2^5 \cdot 3^2 \cdot 11^2 \cdot 23 = 801\,504$ and $N = 2^3 \cdot 3^2 \cdot 11^2 \cdot 23^2 = 4\,408\,648$.
If $N$ has 3 prime factors, then we must use one of the factorizations $D = 27 \cdot 2 \cdot 2$ or $D = 18 \cdot 3 \cdot 2$ or $D = 12 \cdot 3 \cdot 3$ or $D = 9 \cdot 4 \cdot 3$ or $D = 6 \cdot 6 \cdot 3$.

These gives corresponding minimum values

$$\begin{aligned} N &= 2^{26} \cdot 11 \cdot 23 = 16\,978\,542\,592 \\ N &= 2^{17} \cdot 11^2 \cdot 23 = 364\,773\,376 \\ N &= 2^{11} \cdot 11^2 \cdot 23^2 = 131\,090\,432 \\ N &= 2^{8} \cdot 11^3 \cdot 23^2 = 180\,249\,344 \\ N &= 2^{5} \cdot 11^5 \cdot 23^2 = 2\,726\,271\,328 \end{aligned}$$

Combining Cases 1 through 4, the minimum possible value of $N$ is 582 912.
The sum of the digits of 582 912 is $5 + 8 + 2 + 9 + 1 + 2 = 27$.

ANSWER: 27

25. Suppose that, for some integer $n \geq 2$, we have $a_n = x$ and $a_{n-1} = y$.
The equation $a_n + a_{n-1} = \frac{5}{2}\sqrt{a_n \cdot a_{n-1}}$ can be re-written as $x + y = \frac{5}{2}\sqrt{xy}$.
Since $x > 0$ and $y > 0$, squaring both sides of the equation gives an equivalent equation which is $(x + y)^2 = \frac{25}{4}xy$.
Manipulating algebraically, we obtain the following equivalent equations:

$$\begin{aligned} (x+y)^2 &= \tfrac{25}{4}xy \\ 4(x^2 + 2xy + y^2) &= 25xy \\ 4x^2 - 17xy + 4y^2 &= 0 \\ (4x - y)(x - 4y) &= 0 \end{aligned}$$

Therefore, the given relationship is equivalent to $4x = y$ or $x = 4y$.
Returning to the sequence notation, we now know that it is the case that $4a_n = a_{n-1}$ (that is, $a_n = \frac{1}{4}a_{n-1}$) or $a_n = 4a_{n-1}$.
Putting this another way, each term in the sequence can be obtained from the previous term either by multiplying by 4 or by dividing by 4.
We are told that $a_1 = 4$ and $a_{11} = 1024$. We note $\dfrac{a_{11}}{a_1} = \dfrac{1024}{4} = 256 = 4^4$.
We can think of moving along the sequence from $a_1$ to $a_{11}$ by making 10 "steps", each of which involves either multiplying by 4 or dividing by 4.

If there are $m$ steps in which we multiply by 4 and $10 - m$ steps in which we divide by 4, then $\dfrac{4^m}{4^{10-m}} = 4^4$ which gives $4^{2m-10} = 4^4$ or $2m - 10 = 4$ and so $m = 7$.
In other words, the sequence involves 7 steps of multiplying by 4 and 3 steps of dividing by 4.
These steps completely define the sequence.
The number of possible sequences, $S$, equals the number of ways of arranging these 10 steps, which equals $\dbinom{10}{3}$.
(If combinatorial notation is unfamiliar, we could systematically count the number of arrangements instead.)
Therefore, $S = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 5 \cdot 3 \cdot 8 = 120$. The rightmost two digits of $S$ are 20.

ANSWER: 20

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
***cemc.uwaterloo.ca***

# *2023 Fermat Contest*

**(Grade 11)**

**Wednesday, February 22, 2023**
**(in North America and South America)**

**Thursday, February 23, 2023**
**(outside of North America and South America)**

*Solutions*

1. Evaluating, $0.3 + 0.03 = 0.33$.

   Answer: (D)

2. Since $3 + x = 5$, then $x = 2$.
   Since $-3 + y = 5$, then $y = 8$.
   Thus, $x + y = 10$.
   Alternatively, we could have added the original two equations to obtain $(3+x)+(-3+y) = 5+5$ which simplifies to $x + y = 10$.

   Answer: (E)

3. When $x = 2$, we obtain $2x^2 + 3x^2 = 5x^2 = 5 \cdot 2^2 = 5 \cdot 4 = 20$.

   Answer: (E)

4. There are 60 minutes in an hour and 24 hours in a day.
   Thus, there are $60 \cdot 24 = 1440$ minutes in a day.
   Since there are 7 days in a week, the number of minutes in a week is $7 \cdot 1440 = 10\,080$.
   Of the given choices, this is closest to (C) 10 000.

   Answer: (C)

5. Using the given rule, the output of the machine is $2 \times 0 + 2 \times 3 = 0 + 6 = 6$.

   Answer: (D)

6. Since there are 3 doors and 2 colour choices for each door, there are $2^3 = 8$ ways of painting the three doors.
   Using "B" to represent black and "G" to represent gold, these ways are BBB, BBG, BGB, BGG, GBB, GBG, GGB, and GGG.

   Answer: (A)

7. Since juice boxes come in packs of 3, Danny needs to buy at least 6 packs for the 17 players. (If Danny bought 5 packs, he would have 15 juice boxes which is not enough; with 6 packs, he would have 18 juice boxes.)
   Since apples come in bags of 5, Danny needs to buy at least 4 bags. (We note that $3 \cdot 5 = 15$ is too small, and $4 \cdot 5 = 20$, which is enough.)
   Therefore, the minimum amount that Danny can spend is $6 \cdot \$2.00 + 4 \cdot \$4.00 = \$28.00$.

   Answer: (B)

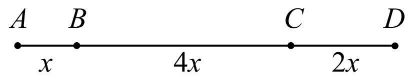
8. Riding at 15 km/h, Bri finishes the 30 km in $\dfrac{30\text{ km}}{15\text{ km/h}} = 2$ h.
   Riding at 20 km/h, Ari finishes the 30 km in $\dfrac{30\text{ km}}{20\text{ km/h}} = 1.5$ h.
   Therefore, Bri finishes 0.5 h after Ari, which is 30 minutes.

   Answer: (C)

9. In total, the three tanks contain $3600\text{ L} + 1600\text{ L} + 3800\text{ L} = 9000\text{ L}$.
   If the water is divided equally between the three tanks, each will contain $\frac{1}{3} \cdot 9000\text{ L} = 3000\text{ L}$.
   Therefore, $3600\text{ L} - 3000\text{ L} = 600\text{ L}$ needs to be moved from Tank A to Tank B.
   (We note that 800 L would also need to be moved from Tank C to Tank B, and at this point, the three tanks will contain 3000 L.)

   Answer: (B)

10. Suppose that $AB = x$ for some $x > 0$.
Since $AB : AC = 1 : 5$, then $AC = 5x$.
This means that $BC = AC - AB = 5x - x = 4x$.
Since $BC : CD = 2 : 1$ and $BC = 4x$, then $CD = 2x$.




Therefore, $AB : CD = x : 2x = 1 : 2$.

Answer: (B)

11. Suppose that Mathilde had $m$ coins at the start of last month and Salah and $s$ coins at the start of last month.
From the given information, 100 is 25% more than $m$, so $100 = 1.25m$ which means that $m = \dfrac{100}{1.25} = 80$.
From the given information, 100 is 20% less than $s$, so $100 = 0.80s$ which means that $s = \dfrac{100}{0.80} = 125$.
Therefore, at the beginning of last month, they had a total of $m + s = 80 + 125 = 205$ coins.

Answer: (E)

12. A rectangle with length 8 cm and width $\pi$ cm has area $8\pi$ cm$^2$.
Suppose that the radius of the semi-circle is $r$ cm.
The area of a circle with radius $r$ cm is $\pi r^2$ cm$^2$ and so the area of the semi-circle is $\frac{1}{2}\pi r^2$ cm$^2$.
Since the rectangle and the semi-circle have the same area, then $\frac{1}{2}\pi r^2 = 8\pi$ and so $\pi r^2 = 16\pi$ or $r^2 = 16$.
Since $r > 0$, then $r = 4$ and so the radius of the semi-circle is 4 cm.

Answer: (B)

13. The equation $a(x + 2) + b(x + 2) = 60$ has a common factor of $x + 2$ on the left side.
Thus, we can re-write the equation as $(a + b)(x + 2) = 60$.
When $a + b = 12$, we obtain $12 \cdot (x + 2) = 60$ and so $x + 2 = 5$ which gives $x = 3$.

Answer: (A)

14. The line with a slope of 2 and $y$-intercept 6 has equation $y = 2x + 6$.
To find its $x$-intercept, we set $y = 0$ to obtain $0 = 2x + 6$ or $2x = -6$, which gives $x = -3$.
The line with a slope of $-4$ and $y$-intercept 6 has equation $y = -4x + 6$.
To find its $x$-intercept, we set $y = 0$ to obtain $0 = -4x + 6$ or $4x = 6$, which gives $x = \frac{6}{4} = \frac{3}{2}$.
The distance between the points on the $x$-axis with coordinates $(-3, 0)$ and $(\frac{3}{2}, 0)$ is $3 + \frac{3}{2}$ which equals $\frac{6}{2} + \frac{3}{2}$ or $\frac{9}{2}$.

Answer: (E)

15. The 1st term is 16.
Since 16 is even, the 2nd term is $\frac{1}{2} \cdot 16 + 1 = 9$.
Since 9 is odd, the 3rd term is $\frac{1}{2}(9 + 1) = 5$.
Since 5 is odd, the 4th term is $\frac{1}{2}(5 + 1) = 3$.
Since 3 is odd, the 5th term is $\frac{1}{2}(3 + 1) = 2$.
Since 2 is even, the 6th term is $\frac{1}{2} \cdot 2 + 1 = 2$.
This previous step shows us that when one term is 2, the next term will also be 2.
Thus, the remaining terms in this sequence are all 2.
In particular, the 101st term is 2.

Answer: (B)

16. The given arrangement has 14 zeroes and 11 ones showing.
Loron can pick any row or column in which to flip the 5 cards over. Furthermore, the row or column that Loron chooses can contain between 0 and 5 of the cards with different numbers on their two sides.
Of the 5 rows and 5 columns, 3 have 4 zeroes and 1 one, 2 have 3 zeroes and 2 ones, and 5 have 2 zeroes and 3 ones.
This means that the number of zeroes cannot decrease by more than 4 when the cards in a row or column are flipped, since the only way that the zeroes could decrease by 5 is if all five cards in the row or column had 0 on the top face and 1 on the bottom face.
Therefore, there cannot be as few as $14 - 5 = 9$ zeroes after Loron flips the cards, which means that the ratio cannot be $9 : 16$, or (C). This means that the answer to the given problem is (C).

For completeness, we will show that the other ratios are indeed achievable.
If Loron chooses the first column and if this column includes 3 cards with ones on both sides, and 2 cards with zeroes on one side (facing up) and ones on the reverse side, then flipping the cards in this column yields $14 - 2 = 12$ zeroes and $11 + 2 = 13$ ones.
Thus, the ratio $12 : 13$ (choice (A)) is possible.
If Loron chooses the fifth column and if this column includes 1 card with a one on both sides and 4 cards with zeroes on one side (facing up) and ones on the reverse side, then flipping the cards in this column yields $14 - 4 = 10$ zeroes and $11 + 4 = 15$ ones.
Thus, the ratio $10 : 15 = 2 : 3$ (choice (B)) is possible.
If Loron chooses the first column and if the top 4 cards in this column have the same numbers on both sides and the bottom card has a one on the top side and a zero on the reverse side, then flipping the cards in this column yields $14 + 1 = 15$ zeroes and $11 - 1 = 10$ ones.
Thus, the ratio $15 : 10 = 3 : 2$ (choice (D)) is possible.
If Loron chooses the first column and if the first, fourth and fifth cards in this column have the same numbers on both sides and the second and third cards each has a one on the top side and a zero on the reverse side, then flipping the cards in this column yields $14 + 2 = 16$ zeroes and $11 - 2 = 9$ ones.
Thus, the ratio $16 : 9$ (choice (E)) is possible.
Therefore, the only ratio of the five that are given that is not possible is $9 : 16$, or (C).

Answer: (C)

17. We start by finding the prime factors of 1184:

$$1184 = 2 \cdot 592 = 2^2 \cdot 296 = 2^3 \cdot 148 = 2^4 \cdot 74 = 2^5 \cdot 37$$

The positive divisors of 1184 cannot contain prime factors other than 2 and 37, and cannot contain more than 5 factors of 2 or 1 factor of 37.
Thus, the positive divisors are

$$1, 2, 4, 8, 16, 32, 37, 74, 148, 296, 592, 1184$$

(The first five of these divisors have 0 factors of 37 and 0 through 5 factors of 2, while the last five have 1 factor 37 and 0 through 5 factors of 2.)
The sum, $S$, of these divisors is

$$\begin{aligned} S &= 1 + 2 + 4 + 8 + 16 + 32 + 37 + 74 + 148 + 296 + 592 + 1184 \\ &= (1 + 2 + 4 + 8 + 16 + 32) + 37 \cdot (1 + 2 + 4 + 8 + 16 + 32) \\ &= (1 + 2 + 4 + 8 + 16 + 32) \cdot (1 + 37) \\ &= 63 \cdot 38 \\ &= 2394 \end{aligned}$$

Answer: (A)

18. Each group of four jumps takes the grasshopper 1 cm to the east and 3 cm to the west, which is a net movement of 2 cm to the west, and 2 cm to the north and 4 cm to the south, which is a net movement of 2 cm to the south.
In other words, we can consider each group of four jumps, starting with the first, as resulting in a net movement of 2 cm to the west and 2 cm to the south.
We note that $158 = 2 \times 79$.
Thus, after 79 groups of four jumps, the grasshopper is $79 \times 2 = 158$ cm to the west and 158 cm to the south of its original position. (We need at least 79 groups of these because the grasshopper cannot be 158 cm to the south of its original position before the end of 79 such groups.)
The grasshopper has made $4 \times 79 = 316$ jumps so far.
After the 317th jump (1 cm to the east), the grasshopper is 157 cm west and 158 cm south of its original position.
After the 318th jump (2 cm to the north), the grasshopper is 157 cm west and 156 cm south of its original position.
After the 319th jump (3 cm to the west), the grasshopper is 160 cm west and 156 cm south of its original position.
After the 320th jump (4 cm to the south), the grasshopper is 160 cm west and 160 cm south of its original position.
After the 321st jump (1 cm to the east), the grasshopper is 159 cm west and 160 cm south of its original position.
After the 322nd jump (2 cm to the north), the grasshopper is 159 cm west and 158 cm south of its original position.
After the 323rd jump (3 cm to the west), the grasshopper is 162 cm west and 158 cm south of its original position, which is the desired position.
As the grasshopper continues jumping, each of its positions will always be at least 160 cm south of its original position, so this is the only time that it is at this position.
Therefore, $n = 323$. The sum of the squares of the digits of $n$ is $3^2 + 2^2 + 3^2 = 9 + 4 + 9 = 22$.

Answer: (A)

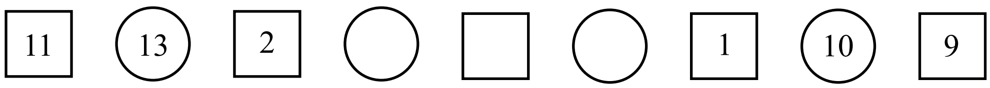
19. If $x$ and $y$ satisfy $2x^2 + 8y = 26$, then $x^2 + 4y = 13$ and so $4y = 13 - x^2$.
Since $x$ and $y$ are integers, then $4y$ is even and so $13 - x^2$ is even, which means that $x$ is odd.
Since $x$ is odd, we can write $x = 2q + 1$ for some integer $q$.
Thus, $4y = 13 - x^2 = 13 - (2q + 1)^2 = 13 - (4q^2 + 4q + 1) = 12 - 4q^2 - 4q$.
Since $4y = 12 - 4q^2 - 4q$, then $y = 3 - q^2 - q$.
Thus, $x - y = (2q + 1) - (3 - q^2 - q) = q^2 + 3q - 2$.
When $q = 4$, we obtain $x - y = q^2 + 3q - 2 = 4^2 + 3 \cdot 4 - 2 = 26$.
We note also that, when $q = 4$, $x = 2q + 1 = 9$ and $y = 3 - q^2 - q = -17$ which satisfy $x^2 + 4y = 13$.
We can also check that there is no integer $q$ for which $q^2 + 3q - 2$ is equal to any of $-8$, $-16$, 22, or 30. (For example, if $q^2 + 3q - 2 = -16$, then $q^2 + 3q + 14 = 0$, and this quadratic equation has no integer solutions.)

Answer: (B)

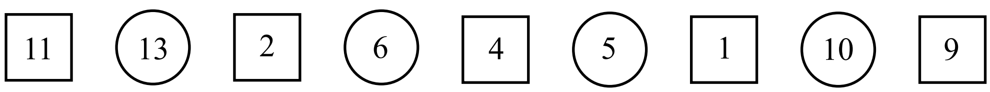
20. If $n!$ ends with exactly $m$ zeroes, then $n!$ is divisibe by $10^m$ but not divisible by $10^{m+1}$. (If $n!$ were divisible by $10^{m+1}$, it would end with at least $m + 1$ zeroes.)
In this case, we can write $n! = 10^m \cdot q$ where $q$ is not divisible by 10. This in turn means that either $q$ is not divisible by 2 or not divisible by 5 or both.
Since $2 < 5$, when $n \geq 2$, the product $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n - 1) \cdot n$ includes more multiples of 2 than of 5 among the $n$ integers in its product, so $n!$ includes more factors of 2 than of 5.
This in turn means that, if $n!$ ends in exactly $m$ zeroes, then $n! = 10^m \cdot q$ where $q$ is not divisible by 5, and so the number of zeroes at the end of $n!$ is exactly equal to the number of prime factors of 5 in the prime factorization of $n!$.
We note also that as $n$ increases, the number of zeroes at the end of $n!$ never decreases since the number of factors of 5 either stays the same or increases as $n$ increases.
For $n = 1$ to $n = 4$, the product $n!$ includes 0 multiples of 5, so $n!$ ends in 0 zeroes.
For $n = 5$ to $n = 9$, the product $n!$ includes 1 multiple of 5 (namely 5), so $n!$ ends in 1 zero.
For $n = 10$ to $n = 14$, the product $n!$ includes 2 multiples of 5 (namely 5 and 10), so $n!$ ends in 2 zeroes.
For $n = 15$ to $n = 19$, the product $n!$ includes 3 multiples of 5 (namely 5, 10 and 15), so $n!$ ends in 3 zeroes.
For $n = 20$ to $n = 24$, the product $n!$ includes 4 multiples of 5 (namely 5, 10, 15, and 20), so $n!$ ends in 4 zeroes.
For $n = 25$ to $n = 29$, the product $n!$ includes 5 multiples of 5 (namely 5, 10, 15, 20, and 25) and includes 6 factors of 5 (since 25 contributes 2 factors of 5), so $n!$ ends in 6 zeroes.
For $n = 30$ to $n = 34$, $n!$ ends in 7 zeroes. For $n = 35$ to $n = 39$, $n!$ ends in 8 zeroes.
For $n = 40$ to $n = 44$, $n!$ ends in 9 zeroes. For $n = 45$ to $n = 49$, $n!$ ends in 10 zeroes.
For $n = 50$ to $n = 54$, $n!$ ends in 12 zeroes, since the product $n!$ includes 10 multiples of 5, two of which include 2 factors of 5.
For $n = 55$ to $n = 74$, $n!$ will end in 13, 14, 15, 16 zeroes as $n$ increases.
For $n = 75$ to $n = 79$, $n!$ ends in 18 zeroes.
For $n = 80$ to $n = 99$, $n!$ ends of 19, 20, 21, 22 zeroes as $n$ increases.
For $n = 100$ to $n = 104$, $n!$ ends in 24 zeroes.
For $n = 105$ to $n = 124$, $n!$ ends in 25, 26, 27, 28 zeroes.
For $n = 125$, $n!$ ends in 31 zeroes since 125 includes 3 factors of 5, so 125! ends in 3 more than zeroes than 124!.
Of the integers $m$ with $1 \leq m \leq 30$, there is no value of $n$ for which $n!$ ends in $m$ zeroes when $m = 5, 11, 17, 23, 29, 30$, which means that $30 - 6 = 24$ of the values of $m$ are possible.

Answer: (D)

21. From the given information, if $a$ and $b$ are in two consecutive squares, then $a+b$ goes in the circle between them.
Since all of the numbers that we can use are positive, then $a+b$ is larger than both $a$ and $b$.
This means that the largest integer in the list, which is 13, cannot be either $x$ or $y$ (and in fact cannot be placed in any square). This is because the number in the circle next to it must be smaller than 13 (because 13 is the largest number in the list) and so cannot be the sum of 13 and another positive number from the list.
Thus, for $x+y$ to be as large as possible, we would have $x$ and $y$ equal to 10 and 11 in some order. But here we have the same problem: there is only one larger number from the list (namely 13) that can go in the circles next to 10 and 11, and so we could not fill in the circle next to both 10 and 11.
Therefore, the next largest possible value for $x+y$ is when $x=9$ and $y=11$. (We could also swap $x$ and $y$.)
Here, we could have $13 = 11+2$ and $10 = 9+1$, giving the following partial list:




The remaining integers (4, 5 and 6) can be put in the shapes in the following way that satisfies the requirements.




This tells us that the largest possible value of $x+y$ is 20.

ANSWER: 20

22. *Solution 1*
Starting with the given relationship between $x$ and $y$ and manipulating algebraically, we obtain successively

$$\begin{aligned}
\frac{1}{x+y} &= \frac{1}{x} - \frac{1}{y} \\
xy &= (x+y)y - (x+y)x \qquad \text{(multiplying by } xy(x+y)) \\
xy &= xy + y^2 - x^2 - xy \\
x^2 + xy - y^2 &= 0 \\
\frac{x^2}{y^2} + \frac{x}{y} - 1 &= 0 \qquad \text{(dividing by } y^2 \text{ which is non-zero)} \\
t^2 + t - 1 &= 0
\end{aligned}$$

where $t = \dfrac{x}{y}$.
Since $x > 0$ and $y > 0$, then $t > 0$. Using the quadratic formula

$$t = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

Since $t > 0$, then $\dfrac{x}{y} = t = \dfrac{\sqrt{5}-1}{2}$.
Therefore,

$$\begin{aligned}
\left(\frac{x}{y} + \frac{y}{x}\right)^2 &= \left(\frac{\sqrt{5}-1}{2} + \frac{2}{\sqrt{5}-1}\right)^2 \\
&= \left(\frac{\sqrt{5}-1}{2} + \frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)}\right)^2 \\
&= \left(\frac{\sqrt{5}-1}{2} + \frac{2(\sqrt{5}+1)}{4}\right)^2 \\
&= \left(\frac{\sqrt{5}-1}{2} + \frac{\sqrt{5}+1}{2}\right)^2 \\
&= (\sqrt{5})^2 \\
&= 5
\end{aligned}$$

*Solution 2*
Since $x, y > 0$, the following equations are equivalent:

$$\begin{aligned}
\frac{1}{x+y} &= \frac{1}{x} - \frac{1}{y} \\
1 &= \frac{x+y}{x} - \frac{x+y}{y} \\
1 &= \frac{x}{x} + \frac{y}{x} - \frac{x}{y} - \frac{y}{y} \\
1 &= 1 + \frac{y}{x} - \frac{x}{y} - 1 \\
1 &= \frac{y}{x} - \frac{x}{y} \\
-1 &= \frac{x}{y} - \frac{y}{x}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(\frac{x}{y} + \frac{y}{x}\right)^2 &= \frac{x^2}{y^2} + 2 \cdot \frac{x}{y} \cdot \frac{y}{x} + \frac{y^2}{x^2} \\
&= \frac{x^2}{y^2} + 2 + \frac{y^2}{x^2} \\
&= \frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} + 4 \\
&= \frac{x^2}{y^2} - 2 \cdot \frac{x}{y} \cdot \frac{y}{x} + \frac{y^2}{x^2} + 4 \\
&= \left(\frac{x}{y} - \frac{y}{x}\right)^2 + 4 \\
&= (-1)^2 + 4 \\
&= 5
\end{aligned}$$

ANSWER: 05

23. We write an integer $n$ with $100 \le n \le 999$ as $n = 100a + 10b + c$ for some digits $a$, $b$ and $c$.
That is, $n$ has hundreds digit $a$, tens digit $b$, and ones digit $c$.
For each such integer $n$, we have $s(n) = a + b + c$.
We want to count the number of such integers $n$ with $7 \le a + b + c \le 11$.
When $100 \le n \le 999$, we know that $1 \le a \le 9$ and $0 \le b \le 9$ and $0 \le c \le 9$.

First, we count the number of $n$ with $a + b + c = 7$.
If $a = 1$, then $b + c = 6$ and there are 7 possible pairs of values for $b$ and $c$. These pairs are $(b, c) = (0, 6), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)$.
If $a = 2$, then $b + c = 5$ and there are 6 possible pairs of values for $b$ and $c$.
Similarly, when $a = 3, 4, 5, 6, 7$, there are 5, 4, 3, 2, 1 pairs of values, respectively, for $b$ and $c$.
In other words, the number of integers $n$ with $a+b+c = 7$ is equal to $7+6+5+4+3+2+1 = 28$.

Using a similar process, we can determine that the number of such integers $n$ with $s(n) = 8$ is $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ and the number of such integers $n$ with $s(n) = 9$ is $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$.

We have to be more careful counting the number of integers $n$ with $s(n) = 10$ and $s(n) = 11$, because none of the digits can be greater than 9.
Consider the integers $n$ with $a + b + c = 10$.
If $a = 1$, then $b + c = 9$ and there are 10 possible pairs of values for $b$ and $c$. These pairs are $(b, c) = (0, 9), (1, 8), \ldots, (8, 1), (9, 0)$.
If $a = 2$, then $b + c = 8$ and there are 9 possible pairs of values for $b$ and $c$.
As $a$ increases from 1 to 9, we find that there are $10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 54$ such integers $n$.
(Note that when $a = 9$, we have $b + c = 1$ and there are 2 pairs of values for $b$ and $c$.)
Finally, we consider the integers $n$ with $a + b + c = 11$.
If $a = 1$, then $b + c = 10$ and there are 9 possible pairs of values for $b$ and $c$. These pairs are $(b, c) = (1, 9), (2, 8), \ldots, (8, 2), (9, 1)$.
If $a = 2$, then $b + c = 9$ and there are 10 possible pairs of values for $b$ and $c$.
If $a = 3$, then $b + c = 8$ and there are 9 possible pairs of values for $b$ and $c$.
Continuing in this way, we find that there are $9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 = 61$ such integers $n$.

Having considered all cases, we see that the number of such integers $n$ is

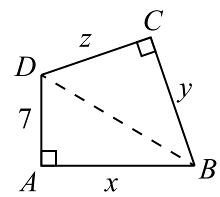
$$S = 28 + 36 + 45 + 54 + 61 = 224$$

The rightmost two digits of $S$ are 24.

ANSWER: 24

24. *Solution 1*

Suppose that $AB = x$, $BC = y$, $CD = z$, and $DA = 7$. (It does not matter to which side length we assign the fixed length of 7.)
We are told that $x$, $y$ and $z$ are integers.
Since the perimeter of $ABCD$ is 224, we have $x + y + z + 7 = 224$ or $x + y + z = 217$.
Join $B$ to $D$.




The area of $ABCD$ is equal to the sum of the areas of $\triangle DAB$ and $\triangle BCD$.
Since these triangles are right-angled, then $2205 = \frac{1}{2} \cdot DA \cdot AB + \frac{1}{2} \cdot BC \cdot CD$.
Multiplying by 2, we obtain $4410 = 7x + yz$.
Finally, we also note that, using the Pythagorean Theorem twice, we obtain

$$DA^2 + AB^2 = DB^2 = BC^2 + CD^2$$

and so $49 + x^2 = y^2 + z^2$.
We need to determine the value of $S = x^2 + y^2 + z^2 + 7^2$.
Since $x + y + z = 217$, then $x = 217 - y - z$.
Substituting into $4410 = 7x + yz$ and proceeding algebraically, we obtain successively

$$\begin{aligned}
4410 &= 7x + yz \\
4410 &= 7(217 - y - z) + yz \\
4410 &= 1519 - 7y - 7z + yz \\
2891 &= yz - 7y - 7z \\
2891 &= y(z - 7) - 7z \\
2891 &= y(z - 7) - 7z + 49 - 49 \\
2940 &= y(z - 7) - 7(z - 7) \\
2940 &= (y - 7)(z - 7)
\end{aligned}$$

Therefore, $y - 7$ and $z - 7$ form a positive divisor pair of 2940. (Since their product is postiive, they are either both positive or both negative. Since $y$ and $z$ are positive, if both of $y - 7$ and $z - 7$ are negative, we would have $0 < y < 7$ and $0 < z < 7$ which could not be large enough to allow for a feasible value of $x$.)
We note that $y + z = 217 - x$ and so $y + z < 217$ which means that $(y - 7) + (z - 7) < 203$.
Since

$$2940 = 20 \cdot 147 = 2^2 \cdot 5 \cdot 3 \cdot 7^2$$

then the divisors of 2940 are the positive integers of the form $2^r \cdot 3^s \cdot 5^t \cdot 7^u$ where $0 \le r \le 2$ and $0 \le s \le 1$ and $0 \le t \le 1$ and $0 \le u \le 2$.
Thus, these divisors are

$$1, 2, 3, 4, 5, 6, 7, 10, 12, 14, 15, 20, 21, 28, 30, 35, 42, 49,$$

$$60, 70, 84, 98, 105, 140, 147, 196, 210, 245, 294, 420, 490, 588, 735, 980, 1470, 2940$$

We can remove divisor pairs from this list whose sum is greater than 203. This gets us to the shorter list

$$20, 21, 28, 30, 35, 42, 49, 60, 70, 84, 98, 105, 140, 147$$

This means that there are 7 divisor pairs remaining to consider. We can assume that $y < z$. Using the fact that $x + y + z = 217$, we can solve for $x$ in each case. These values of $x$, $y$ and $z$ will satisfy the perimeter and area conditions, but we need to check the Pythaogrean condition. We make a table:

| $y-7$ | $z-7$ | $y$ | $z$ | $x = 217 - y - z$ | $y^2 + z^2 - x^2$ |
|---|---|---|---|---|---|
| 20 | 147 | 27 | 154 | 36 | 23 149 |
| 21 | 140 | 28 | 147 | 42 | 20 629 |
| 28 | 105 | 35 | 112 | 70 | 8869 |
| 30 | 98 | 37 | 105 | 75 | 6769 |
| 35 | 84 | 42 | 91 | 84 | 2989 |
| 42 | 70 | 49 | 77 | 91 | 49 |
| 49 | 60 | 56 | 67 | 94 | −1211 |

Since we need $y^2 + z^2 - x^2 = 49$, then we must have $y = 49$ and $z = 77$ and $x = 91$.
This means that $S = x^2 + y^2 + z^2 + 7^2 = 91^2 + 49^2 + 77^2 + 7^2 = 16\,660$.
The rightmost two digits of $S$ are 60.

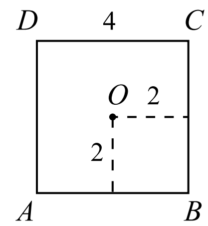
*Solution 2*
As in Solution 1, we have $x + y + z = 217$ and $4410 = 7x + yz$ and $x^2 + 49 = y^2 + z^2$.
Re-arranging and squaring the first equation and using the second and third equations, we obtain

$$\begin{aligned}
y + z &= 217 - x \\
y^2 + z^2 + 2yz &= x^2 - 434x + 217^2 \\
(x^2 + 49) + 2(4410 - 7x) &= x^2 - 434x + 217^2 \\
49 + 8820 - 14x &= -434x + 217^2 \\
420x &= 217^2 - 8820 - 49 \\
420x &= 38\,220 \\
x &= 91
\end{aligned}$$

Thus, $y + z = 217 - 91 = 126$ and $yz = 4410 - 7 \cdot 91 = 3773$.
This gives, $y(126 - y) = 3773$ and so $y^2 - 126y + 3773 = 0$ or $(y - 49)(y - 77) = 0$.
Therefore, $y = 49$ (which means $z = 77$) or $y = 77$ (which means $z = 49$).
We note that $y^2 + z^2 = 49^2 + 77^2 = 8330 = 91^2 + 7^2 = x^2 + 7^2$ which verifies the remaining equation.
This means that $S = x^2 + y^2 + z^2 + 7^2 = 91^2 + 49^2 + 77^2 + 7^2 = 16\,660$.
The rightmost two digits of $S$ are 60.

ANSWER: 60

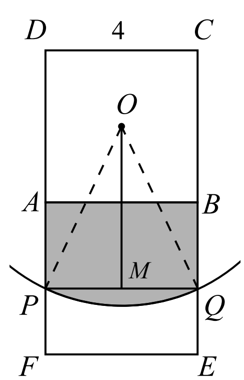
25. Throughout this solution, we will not explicitly include units, but will assume that all lengths are in metres and all areas are in square metres.
The top face of the cube is a square, which we label $ABCD$, and we call its centre $O$.
Since the cube has edge length 4, then the side length of square $ABCD$ is 4.
This means that $O$ is a perpendicular distance of 2 from each of the sides of square $ABCD$, and thus is a distance of $\sqrt{2^2+2^2}=\sqrt{8}$ from each of the vertices of $ABCD$.




These vertices are the farthest points on $ABCD$ from $O$.
Since $\sqrt{8}\approx 2.8$, then the loose end of the rope of length 5 can reach every point on $ABCD$, which has area 16.

Next, the rope cannot reach to the bottom face of the cube because the shortest distance along the surface of the cube from $O$ to the bottom face is 6 and the rope has length 5. We will confirm this in another way shortly.
Also, since the rope is anchored to the centre of the top face and all of the faces are square, the rope can reach the same area on each of the four side faces.
Suppose that the area of one of the side faces that can be reached is $a$. Since the rope can reach the entire area of the top face, then the total area that can be reached is $16+4a$.
We thus need to determine the value of $a$.

Suppose that one of the side faces is square $ABEF$, which has side length 4. Consider the figure created by square $ABCD$ and square $ABEF$ together. We can think of this as an "unfolding" of part of the cube.




When the rope is stretched tight, its loose end traces across square $ABEF$ an arc of a circle centred at $O$ and with radius 5.
Notice that the farthest that the rope can reach down square $ABEF$ is a distance of 3, since its anchor is a distance of 2 from $AB$. This confirms that the rope cannot reach the bottom face of the cube since it would have to cross $FE$ to do so.
Suppose that this arc cuts $AF$ at $P$ and cuts $BE$ at $Q$.

We want to determine the area of square $ABEF$ above arc $PQ$ (the shaded area); the area of this region is $a$.

We will calculate the value of $a$ by determining the area of rectangle $ABQP$ and adding the area of the region between the circular arc and line segment $PQ$.
We will calculate this latter area by determining the area of sector $OPQ$ and subtracting the area of $\triangle OPQ$.
We note that $PQ = 4$. Let $M$ be the midpoint of $PQ$; thus $PM = MQ = 2$.
Since $\triangle OPQ$ is isosceles with $OP = OQ = 5$, then $OM$ is perpendicular to $PQ$.
By the Pythagorean Theorem, $OM = \sqrt{OP^2 - PM^2} = \sqrt{5^2 - 2^2} = \sqrt{21}$.
Thus, the area of $\triangle OPQ$ is $\frac{1}{2} \cdot PQ \cdot OM = \frac{1}{2} \cdot 4 \cdot \sqrt{21} = 2\sqrt{21}$.
Furthermore, since $O$ is a distance of 2 from $AB$ and $OM = \sqrt{21}$, then the height of rectangle $ABQP$ is $\sqrt{21} - 2$.
Thus, the area of rectangle $ABQP$ is $4 \cdot (\sqrt{21} - 2) = 4\sqrt{21} - 8$.
To find the area of sector $OPQ$, we note that the area of a circle with radius 5 is $\pi \cdot 5^2$, and so the area of the sector is $\dfrac{\angle POQ}{360^\circ} \cdot 25\pi$.
Now, $\angle POQ = 2\angle POM = 2\sin^{-1}(2/5)$, since $\triangle POM$ is right-angled at $M$ which means that $\sin(\angle POM) = \dfrac{PM}{OP}$.
Thus, the area of the sector is $\dfrac{2\sin^{-1}(2/5)}{360^\circ} \cdot 25\pi$.
Putting this all together, we obtain

$$\begin{aligned}
100A &= 100(16 + 4a) \\
&= 1600 + 400a \\
&= 1600 + 400\left((4\sqrt{21} - 8) + \frac{2\sin^{-1}(2/5)}{360^\circ} \cdot 25\pi - 2\sqrt{21}\right) \\
&= 1600 + 400\left(2\sqrt{21} - 8 + \frac{2\sin^{-1}(2/5)}{360^\circ} \cdot 25\pi\right) \\
&= 800\sqrt{21} - 1600 + \frac{800\sin^{-1}(2/5) \cdot 25\pi}{360^\circ} \\
&\approx 6181.229
\end{aligned}$$

(Note that we have not switched to decimal approximations until the very last step in order to avoid any possible rounding error.)
Therefore, the integer closest to $100A$ is 6181, whose rightmost two digits are 81.

Answer: 81

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2022 Fermat Contest*

**(Grade 11)**

**Wednesday, February 23, 2022**
**(in North America and South America)**

**Thursday, February 24, 2022**
**(outside of North America and South America)**

*Solutions*

1. Evaluating, $6 + (3 \times 6) - 12 = 6 + 18 - 12 = 12$.

   Answer: (C)

2. *Solution 1*
   Since the average of two numbers is 7, their sum is $2 \times 7 = 14$.
   Since one of the numbers is 5, the other is $14 - 5 = 9$.

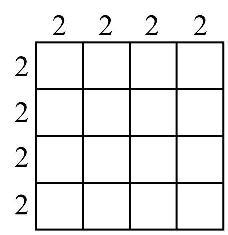
   *Solution 2*
   The average of two numbers is 7 and one of the numbers is 5.
   Since 5 is 2 less than 7, the other number must be 2 more than 7, and so is 9.

   Answer: (E)

3. From the day on which she walks 500 m to the day on which she walks 4500 m, Gauravi increases her distance by (4500 m) − (500 m) = 4000 m.
   Since Gauravi increases her distance by 500 m each day, then it takes $\dfrac{4000\text{ m}}{500\text{ m}} = 8$ days to increase from 500 m to 4500 m.
   Starting from Monday and counting forward by 8 days (which is 1 week and 1 day) gets to Tuesday, and so Gauravi walks exactly 4500 m on a Tuesday.

   Answer: (C)

4. By arranging 4 rows of 4 squares of side length 2, a square of side length 8 can be formed.

   Thus, $4 \cdot 4 = 16$ squares can be arranged in this way. Since these smaller squares completely cover the larger square, it is impossible to use more $2 \times 2$ squares, so 16 is the largest possible number.

   Answer: (C)

5. Since the list includes 15 integers, then an integer has a probability of $\frac{1}{3}$ of being selected if it occurs $\frac{1}{3} \cdot 15 = 5$ times in the list.
   The integer 5 occurs 5 times in the list and no other integer occurs 5 times, so $n = 5$.

   Answer: (E)

6. The given triangle can be considered to have base $PQ$ (which is vertical along the line $x = 2$) and perpendicular height which runs from $R$ horizontally to $PQ$.
   The line segment joining $P(2, 6)$ to $Q(2, 2)$ has length 4.
   Point $R(8, 5)$ is 6 units to the right of the line with equation $x = 2$.
   Thus, $\triangle PQR$ has area $\frac{1}{2} \cdot 4 \cdot 6 = 12$.

   Answer: (D)

7. Evaluating, $(1 + 2 + 3)(1 + \frac{1}{2} + \frac{1}{3}) = 6(1 + \frac{1}{2} + \frac{1}{3}) = 6 + \frac{6}{2} + \frac{6}{3} = 6 + 3 + 2 = 11$.

   Answer: (B)

8. Since $10x + y = 75$ and $10y + x = 57$, then

$$(10x + y) + (10y + x) = 75 + 57$$

and so

$$11x + 11y = 132$$

Dividing by 11, we get $x + y = 12$.
(We could have noticed initially that $(x, y) = (7, 5)$ is a pair that satisfies the two equations, thence concluding that $x + y = 12$.)

Answer: (A)

9. Since Pearl digs 4 holes in 7 days and $\frac{21}{7} = 3$, then in 21 days, Pearl digs $3 \cdot 4 = 12$ holes.
Since Miguel digs 2 holes in 3 days and $\frac{21}{3} = 7$, then in 21 days, Miguel digs $7 \cdot 2 = 14$ holes.
In total, they dig $12 + 14 = 26$ holes in 21 days.

Answer: (D)

10. Manipulating the left side, $2^{11} \times 6^5 = 2^{11} \times (2 \times 3)^5 = 2^{11} \times 2^5 \times 3^5 = 2^{16} \times 3^5$.
Since $4^x \times 3^y = 2^{16} \times 3^5$ and $x$ and $y$ are positive integers, then $y = 5$ (because $4^x$ has no factors of 3).
This also means that $4^x = 2^{16}$.
Since $4^x = (2^2)^x = 2^{2x}$, then $4^x = 2^{16}$ gives $2^{2x} = 2^{16}$ and so $2x = 16$ or $x = 8$.
Therefore, $x + y = 8 + 5 = 13$.

Answer: (D)

11. We use $A$, $B$, $C$, $D$, and $E$ to represent Andy, Bev, Cao, Dhruv, and Elcim, respectively.
We use the notation $D > B$ to represent the fact "Dhruv is older than Bev".
The five sentences give $D > B$ and $B > E$ and $A > E$ and $B > A$ and $C > B$. These show us that Dhruv and Cao are older than Bev, and Elcim and Andy are younger than Bev.
This means that two people are older than Bev and two people are younger than Bev, which means that Bev must be the third oldest.

Answer: (B)

12. Since $d$ is an odd integer, then $d + d$ is even and $d \times d$ is odd.
Since $e$ is an even integer, then $e + e$ is even, which means that $(e + e) \times d$ is even.
Also, $e + d$ is odd, which means that $d \times (e + d)$ is odd.
Thus, 2 of the 4 expressions are equal to an odd integer.

Answer: (C)

13. Suppose that each of the small rectangles has shorter sides of length $x$.
Then the height of the rectangle in Figure A is $2x$, which means that the longer side of each small rectangle has length $2x$.
Therefore, the rectangle in Figure A has height $2x$ and width $2x + x = 3x$, which means that its perimeter is $2(3x + 2x) = 10x$. Also, the rectangle in Figure B has height $2x$ and width $x + 2x + x = 4x$, so its perimeter is $2(4x + 2x) = 12x$.

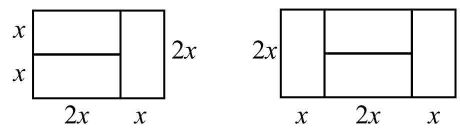



Figure A Figure B

Therefore, the ratio of the perimeter of Figure A to the perimeter of Figure B is $10x : 12x$ which is equal to $5 : 6$, since $x \neq 0$.

Answer: (E)

14. Zebadiah must remove at least 3 shirts.
If he removes 3 shirts, he might remove 2 red shirts and 1 blue shirt.
If he removes 4 shirts, he might remove 2 red shirts and 2 blue shirts.
Therefore, if he removes fewer than 5 shirts, it is not guaranteed that he removes either 3 of the same colour or 3 of different colours.
Suppose that he removes 5 shirts. If 3 are of the same colour, the requirements are satisfied.
If no 3 of the 5 shirts are of the same colour, then at most 2 are of each colour (for example, 2 red, 2 blue and 1 green). This means that he must remove shirts of 3 colours, since if he only removed shirts of 2 colours, he would remove at most $2 + 2 = 4$ shirts.
In other words, if he removes 5 shirts, it is guaranteed that there are either 3 of the same colour or shirts of all 3 colours.
Thus, the minimum number is 5.

Answer: (D)

15. If $a$ is odd, the output is $a + 3$, which is even because it is the sum of two odd integers.
If $a$ is even, the output is $a + 5$, which is odd, because it is the sum of an even integer and an odd integer.
Starting with $a = 15$ and using the machine 2 times, we obtain $15 \to 15+3 = 18 \to 18+5 = 23$.
Starting with 23 and using the machine 2 times, we obtain $23 \to 23 + 3 = 26 \to 26 + 5 = 31$.
Starting with an odd integer and using the machine 2 times, the net result is adding 8 to the input, because the odd input generates a first output that is 3 larger (and so even) and a second output that is 5 larger than the first output.
This generates a net result that is $3 + 5$ larger than the input.
Therefore, using the machine 46 more times (that is, repeating the 2 steps a total of 23 more times), we add 8 a total of 23 more times to obtain the output $31 + 23 \cdot 8 = 215$.
To this point, the machine has been used 50 times.
Using the machine for the 51st time, $215 \to 215 + 3 = 218$ and so the final output is 218.

Answer: (B)

16. Since the remainder when 111 is divided by $n$ is 6, then $111 - 6 = 105$ is a multiple of $n$ and $n > 6$ (since, by definition, the remainder must be less than the divisor).
Since $105 = 3 \cdot 5 \cdot 7$, the positive divisors of 105 are $1, 3, 5, 7, 15, 21, 35, 105$.
Therefore, the possible values of $n$ are 7, 15, 21, 35, 105, of which there are 5.

Answer: (A)

17. Suppose that the original can has radius $r$ cm and height $h$ cm.
Since the surface area of the original can is 300 cm$^2$, then $2\pi r^2 + 2\pi rh = 300$.
When the radius of the original can is doubled, its new radius is $2r$ cm, and so an expression for its surface area, in cm$^2$, is $2\pi(2r)^2+2\pi(2r)h$ which equals $8\pi r^2+4\pi rh$, and so $8\pi r^2+4\pi rh = 900$.
When the height of the original can is doubled, its new height is $2h$ cm, and so an expression for its surface area, in cm$^2$, is $2\pi r^2 + 2\pi r(2h)$ which equals $2\pi r^2 + 4\pi rh$.
Multiplying $2\pi r^2 + 2\pi rh = 300$ by 3, we obtain $6\pi r^2 + 6\pi rh = 900$.
Since $8\pi r^2 + 4\pi rh = 900$, we obtain

$$\begin{aligned} 6\pi r^2 + 6\pi rh &= 8\pi r^2 + 4\pi rh \\ 2\pi rh &= 2\pi r^2 \\ \pi rh &= \pi r^2 \end{aligned}$$

Since $2\pi r^2+2\pi rh = 300$ and $\pi rh = \pi r^2$, then $2\pi r^2+2\pi r^2 = 300$ and so $4\pi r^2 = 300$ or $\pi r^2 = 75$.
Since $\pi rh = \pi r^2 = 75$, then $2\pi r^2 + 4\pi rh = 6 \cdot 75 = 450$, and so the surface area of the cylinder with its height doubled is 450 cm$^2$.

Answer: (A)

18. Let $A$ be Aria's starting point, $B$ be Bianca's starting point, and $M$ be their meeting point.
It takes Aria 42 minutes to walk from $A$ to $M$ and 28 minutes from $M$ to $B$.
(Note that 9:00 a.m. is 18 minutes after 8:42 a.m., and 9:10 a.m. is 10 minutes after 9:00 a.m..)
Since Aria walks at a constant speed, then the ratio of the distance $AM$ to the distance $MB$ is equal to the ratio of times, or 42 : 28, which is equivalent to 3 : 2.
Since it takes 42 minutes for Bianca to walk from $B$ to $M$, the ratio of distances $AM$ to $MB$ is 3 : 2, and Bianca walks at a constant speed, then it takes Bianca $\frac{3}{2} \times 42 = 63$ minutes to walk from $M$ to $A$.
Therefore, Bianca arrives at Aria's starting point at 9:45 a.m.
(Note that 9:00 a.m. is 18 minutes after 8:42 a.m., and 9:45 a.m. is 45 minutes after 9:00 a.m..)

Answer: (D)

19. Since $\triangle PQR$ is right-angled at $R$, then by the Pythagorean Theorem,

$$PQ^2 = PR^2 + QR^2 = 12^2 + 16^2 = 144 + 256 = 400$$

Since $PQ > 0$, then $PQ = 20$.
Since $M$ is the midpoint of $PQ$, then $MQ = \frac{1}{2}PQ = 10$.
Now $\triangle NMQ$ is similar to $\triangle PRQ$, since each is right-angled and they share a common angle at $Q$.
Therefore, $\dfrac{NQ}{PQ} = \dfrac{MQ}{RQ}$ and so $\dfrac{NQ}{20} = \dfrac{10}{16}$ which gives $NQ = \dfrac{20 \cdot 10}{16} = \dfrac{25}{2}$.
Thus, $RN = RQ - NQ = 16 - \dfrac{25}{2} = \dfrac{32}{2} - \dfrac{25}{2} = \dfrac{7}{2}$.
Since $\triangle PNR$ is right-angled at $R$, its area equals $\dfrac{1}{2} \cdot PR \cdot RN = \dfrac{1}{2} \cdot 12 \cdot \dfrac{7}{2} = 21$.

Answer: (A)

20. We note that

$$t_1 = \frac{1}{1} - \frac{1}{3} = \frac{2}{3} \approx 0.67$$

$$t_1 + t_2 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{2}{3} + \frac{1}{4} = \frac{11}{12} \approx 0.92$$

$$\begin{aligned} t_1 + t_2 + t_3 &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \\ &= \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = 1.05 \end{aligned}$$

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} \\ &= \frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \approx 1.13 \end{aligned}$$

This means that the sum of the first $k$ terms is less than 1.499 for $k = 1, 2, 3, 4$.
When $k > 4$, we can extend the pattern that we saw for $k = 3$ and $k = 4$ to note that

$$\begin{aligned} t_1 + t_2 + t_3 + \ldots + t_{k-1} + t_k &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots \\ &\qquad + \left(\frac{1}{k-1} - \frac{1}{k+1}\right) + \left(\frac{1}{k} - \frac{1}{k+2}\right) \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots - \frac{1}{k+1} - \frac{1}{k+2} \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots + \frac{1}{k-1} - \frac{1}{k-1} + \frac{1}{k} - \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+2} \\ &= \frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2} \\ &= 1.500 - \frac{1}{k+1} - \frac{1}{k+2} \end{aligned}$$

This means that the sum of the first $k$ terms is less than 1.499 exactly when $\frac{1}{k+1} + \frac{1}{k+2}$ is greater than 0.001.
As $k$ increases from 4, each of $\frac{1}{k+1}$ and $\frac{1}{k+2}$ decreases, which means that their sum decreases as well.
When $k = 1998$, $\frac{1}{k+1} + \frac{1}{k+2} = \frac{1}{1999} + \frac{1}{2000} > \frac{1}{2000} + \frac{1}{2000} = \frac{1}{1000} = 0.001$.
When $k = 1999$, $\frac{1}{k+1} + \frac{1}{k+2} = \frac{1}{2000} + \frac{1}{2001} < \frac{1}{2000} + \frac{1}{2000} = \frac{1}{1000} = 0.001$.
This means that $\frac{1}{k+1} + \frac{1}{k+2}$ is greater than 0.001 exactly when $k \leq 1998$ and is less than 0.001 when $k \geq 1999$.

In other words, the sum of the first $k$ terms is less than 1.499 for $k = 1, 2, 3, 4$ as well as for $5 \le k \le 1998$, which is the same as saying that this is true for $1 \le k \le 1998$.
Therefore, $k = 1998$ is the largest positive integer for which the sum of the first $k$ terms is less than 1.499.

ANSWER: (E)

21. The total mass of the six steel bars in the bags is at least $1+2+3+4+5+6=21$ kg and at most $10+11+12+13+14+15=75$ kg. This is because the masses of the 15 given bars are 1 kg, 2 kg, 3 kg, ..., 14 kg, and 15 kg.
Since the six bars are divided between three bags with the same total mass in each bag, then the total mass in each bag is at least $21 \div 3 = 7$ kg and at most $75 \div 3 = 25$ kg.
There are $25 - 7 + 1 = 19$ masses that are an integer number of kilograms in this range (7 kg, 8 kg, 9 kg, ..., 23 kg, 24 kg, 25 kg).
Each of 19 these masses is indeed possible. To see this, we note that

$$1+6=2+5=3+4=7 \qquad 1+7=2+6=3+5=8 \qquad 1+8=2+7=3+6=9$$

which shows that 7, 8 and 9 are possible values of $M$.
Continuing to increase the larger values to 15, 14, 13, we eventually obtain

$$1+15=2+14=3+13=16$$

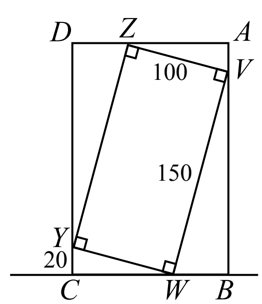
and also that each value of $M$ between 10 and 16, inclusive, will be a possible value of $M$.
Now, we increase the smaller values, starting from the last three pairs:

$$2{+}15=3{+}14=4{+}13=17 \qquad 3{+}15=4{+}14=5{+}13=18 \qquad \cdots \qquad 10{+}15=11{+}14=12{+}13=25$$

which shows that 17, 18, 19, 20, 21, 22, 23, 24, and 25 are also possible values of $M$.
This shows that every integer value of $M$ with $7 \le M \le 25$ is possible.
In summary, there are 19 possible values of $M$.

ANSWER: 19

22. We enclose the given rectangle in a larger rectangle with horizontal and vertical sides so that the vertices of the smaller rectangle lie on the sides of the larger rectangle.
We will also remove the units from the problem and deal with dimensionless quantities.




Since $VWYZ$ is a rectangle, then $YW = ZV = 100$ and $ZY = VW = 150$.
Since $\triangle YCW$ is right-angled at $C$, by the Pythagorean Theorem,

$$CW^2 = YW^2 - YC^2 = 100^2 - 20^2 = 10\,000 - 400 = 9600$$

Since $CW > 0$, then $CW = \sqrt{9600} = \sqrt{1600 \cdot 6} = \sqrt{1600} \cdot \sqrt{6} = 40\sqrt{6}$.
The height of $Z$ above the horizontal line is equal to the length of $DC$, which equals $DY + YC$ which equals $DY + 20$.
Now $\triangle ZDY$ is right-angled at $D$ and $\triangle YCW$ is right-angled at $C$.
Also, $\angle DYZ + \angle ZYW + \angle WYC = 180^\circ$, which means that $\angle DYZ + \angle WYC = 90^\circ$, since $\angle ZYW = 90^\circ$.
Since $\angle CWY + \angle WYC = 90^\circ$ as well (using the sum of the angles in $\triangle YCW$), we obtain $\angle DYZ = \angle CWY$, which tells us that $\triangle ZDY$ is similar to $\triangle YCW$.
Therefore, $\dfrac{DY}{ZY} = \dfrac{CW}{YW}$ and so $DY = \dfrac{ZY \cdot CW}{YW} = \dfrac{150 \cdot 40\sqrt{6}}{100} = 60\sqrt{6}$.
Finally, $DC = DY + 20 = 60\sqrt{6} + 20 \approx 166.97$. Rounded to the nearest integer, $DC$ is 167.
Since the length of $DC$ to the nearest integer is $100 + x$ and this must equal 167, then $x = 67$.

ANSWER: 67

23. Suppose that $k$ is a fixed, but unknown, positive integer.
Suppose also that the lines with equations $9x + 4y = 600$ and $kx - 4y = 24$ intersect at the point with positive integer coordinates $(x, y)$.
Since $9x + 4y = 600$ and $kx - 4y = 24$, adding these equations, we get $9x + kx = 624$ and so $(9 + k)x = 624$.
Since $x$ and $y$ are to be positive integers and $k > 0$, then $9 + k$ and $x$ are a positive divisor pair of 624 with $9 + k > 9$.
Now $624 = 6 \cdot 104 = 6 \cdot 8 \cdot 13 = 2^4 3^1 13^1$, and so the positive divisors of 624 are

$$1, 2, 3, 4, 6, 8, 12, 13, 16, 24, 26, 39, 48, 52, 78, 104, 156, 208, 312, 624$$

We also want the value of $y$ to be a positive integer.
Since the point $(x, y)$ lies on the line with equation $9x + 4y = 600$, then $4y = 600 - 9x$ which gives $y = 150 - \frac{9}{4}x$, which is an integer exactly when $x$ is a multiple of 4.
Therefore, we want $x$ to be a positive divisor of 624 which is a multiple of 4.
Thus, the possible values of $x$ are $4, 8, 12, 16, 24, 48, 52, 104, 156, 208, 312, 624$.
The corresponding values of $9 + k$ are $156, 78, 52, 39, 26, 13, 12, 6, 4, 3, 2, 1$.
Since $9 + k > 9$, we eliminate $6, 4, 3, 2, 1$ from this list.
Thus, the possible values of $9 + k$ are $156, 78, 52, 39, 26, 13, 12$.
The corresponding values of $k$ are $147, 69, 43, 30, 17, 4, 3$.
These correspond to the following values of $x$: $4, 8, 12, 16, 24, 48, 52$.
Using $y = 150 - \frac{9}{4}x$, these give the following values of $y$: $141, 132, 123, 114, 96, 42, 33$. These are indeed all positive.
This means that there are 7 values of $k$ with the required properties.

ANSWER: 07

24. Since $f(p) = 17$, then $ap^2 + bp + c = 17$.
Since $f(q) = 17$, then $aq^2 + bq + c = 17$.
Subtracting these two equations, we obtain $a(p^2 - q^2) + b(p - q) = 0$.
Since $p^2 - q^2 = (p - q)(p + q)$, this becomes $a(p - q)(p + q) + b(p - q) = 0$.
Since $p < q$, then $p - q \neq 0$, so we divide by $p - q$ to get $a(p + q) + b = 0$.
Since $f(p + q) = 47$, then $a(p + q)^2 + b(p + q) + c = 47$ and so $(p + q)(a(p + q) + b) + c = 47$.
Since $a(p + q) + b = 0$, then $(p + q)(0) + c = 47$ which tells us that $c = 47$.
Since $ap^2 + bp + c = 17$, then $ap^2 + bp = -30$ and so $p(ap + b) = -30$.
Similarly, $q(aq + b) = -30$.
Since $p$ and $q$ are prime numbers and $a$ and $b$ are integers, then $p$ and $q$ must be prime divisors of $-30$. We note that $30 = 2 \cdot 3 \cdot 5$ and also that $p$ and $q$ must be distinct.
Since $p < q$, then $p = 2$ and $q = 3$, or $p = 2$ and $q = 5$, or $p = 3$ and $q = 5$.

Alternatively, we could note that since $f(p) = f(q) = 17$, then $f(p) - 17 = f(q) - 17 = 0$.
Therefore, $f(x) - 17$ is a quadratic polynomial with roots $p$ and $q$, which means that we can write $f(x) - 17 = a(x - p)(x - q)$, since the quadratic polynomial has leading coefficient $a$.
Since $f(p + q) = 47$, then $f(p + q) - 17 = a(p + q - p)(p + q - q)$ which gives $47 - 17 = aqp$ or $apq = 30$.
As above, $p = 2$ and $q = 3$, or $p = 2$ and $q = 5$, or $p = 3$ and $q = 5$.

If $p = 2$ and $q = 3$, the equations $p(ap + b) = -30$ becomes $2(2a + b) = -30$ (or $2a + b = -15$) and the equation $q(aq + b) = -30$ becomes $3(3a + b) = -30$ (or $3a + b = -10$).
Subtracting $2a + b = -15$ from $3a + b = -10$, we obtain $a = 5$ (note that $a > 0$) which gives $b = -15 - 2 \cdot 5 = -25$.
Therefore, $f(x) = 5x^2 - 25x + 47$.
Since $pq = 6$, then $f(pq) = 5(6^2) - 25(6) + 47 = 77$.

If $p = 2$ and $q = 5$, we get $2a + b = -15$ and $5a + b = -6$.
Subtracting the first of these from the second, we obtain $3a = 9$ which gives $a = 3$ (note that $a > 0$) and then $b = -15 - 2 \cdot 3 = -21$.
Therefore, $f(x) = 3x^2 - 21x + 47$.
Since $pq = 10$, then $f(pq) = 3(10^2) - 21(10) + 47 = 137$.

If $p = 3$ and $q = 5$, we get $3a + b = -10$ and $5a + b = -6$.
Subtracting the first of these from the second, we obtain $2a = 4$ which gives $a = 2$ (note that $a > 0$) and then $b = -10 - 3 \cdot 2 = -16$.
Therefore, $f(x) = 2x^2 - 16x + 47$.
Since $pq = 15$, then $f(pq) = 2(15^2) - 16(15) + 47 = 257$.

The sum of these values of $f(pq)$ is $77 + 137 + 257 = 471$.
The rightmost two digits of this integer are 71.

ANSWER: 71

25. Consider the grid as laid out in the problem:

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $d$ | 5 | $e$ |
| $f$ | $g$ | $h$ |

We know that the sums of the integers along each row, along each column, and along the two main diagonals are all divisible by 5.
We start by removing all but the integers 5, $a$, $c$, $f$, and $h$ from the grid.

| | | |
|---|---|---|
| $a$ | | $c$ |
| | 5 | |
| $f$ | | $h$ |

There are 9 choices for each of $a$ and $c$.
Since the sum of the entries on each diagonal is a multiple of 5, then $a + 5 + h$ is a multiple of 5, which is equivalent to saying that $a + h$ is a multiple of 5.
Note that each of $a$ and $h$ is between 1 and 9.
If $a = 1$, then $h = 4$ or $h = 9$. If $a = 6$, then $h = 4$ or $h = 9$.
If $a = 2$, then $h = 3$ or $h = 8$. If $a = 7$, then $h = 3$ or $h = 8$.
If $a = 3$, then $h = 2$ or $h = 7$. If $a = 8$, then $h = 2$ or $h = 7$.
If $a = 4$, then $h = 1$ or $h = 6$. If $a = 9$, then $h = 1$ or $h = 6$.
If $a = 5$, then $h = 5$.
We write $h = \overline{a}$ to show that $h$ depends on $a$. (Note that $h$ does not depend on $c$.) We will remember later that there might be 1 or 2 possible values for $h$, depending on the value of $a$.

Similarly, $c+5+f$ is a multiple of 5, or equivalently that $c+f$ is a multiple of 5 and this same combinations of possible values for $c$ and $f$ exist as for $a$ and $h$.
We write $f = \overline{c}$. This gives us

| | | |
|---|---|---|
| $a$ | | $c$ |
| | 5 | |
| $\overline{c}$ | | $\overline{a}$ |

Since $a + b + c$ is a multiple of 5, then $b + (a + c)$ is a multiple of 5. We write $b = \overline{a + c}$, since $b$ depends on $a + c$.
This gives us

| | | |
|---|---|---|
| $a$ | $\overline{a+c}$ | $c$ |
| | 5 | |
| $\overline{c}$ | | $\overline{a}$ |

Since $a$ and $c$ are each between 1 and 9, then $a + c$ is between 2 and 18. Recall that $b = \overline{a + c}$ is also between 1 and 9.
If $a + c$ is one of 2, 7, 12, and 17, the possible values for $b = \overline{a + c}$ are 3 and 8.
If $a + c$ is one of 3, 8, 13, and 18, the possible values for $b = \overline{a + c}$ are 2 and 7.
If $a + c$ is one of 4, 9, and 14, the possible values for $b = \overline{a + c}$ are 1 and 6.
If $a + c$ is one of 5, 10, and 15, then $b = \overline{a + c} = 5$.
If $a + c$ is one of 6, 11, and 16, the possible values for $b = \overline{a + c}$ are 4 and 9.

We can now start to consider a number of cases. Because we have seen above that the number of possibilities for some of the entries depend on whether or not $a$ and $c$ are 5, we look at (i) $a = c = 5$, (ii) $a = 5$ and $c \neq 5$, (iii) $c = 5$ and $a \neq 5$, and (iv) $a \neq 5$ and $c \neq 5$.

Case 1: $a = c = 5$
From above, there is only one choice for each of $\overline{a}$ and $\overline{c}$: each must equal 5.
Also, $a + c = 10$ and so $\overline{a + c}$ must also equal 5, giving the grid:

| | | |
|---|---|---|
| 5 | 5 | 5 |
| | 5 | |
| 5 | | 5 |

Similarly, each of the remaining cells can only be filled with 5, so there is only 1 way of completing the grid in this case.

Case 2: $a = 5$ and $c \neq 5$
Since $a = 5$, then $\overline{a} = 5$.
Also, since $a = 5$, then $a + c = 5 + c$ which means that $\overline{a + c}$ is the same as $\overline{c}$, giving the grid:

| | | |
|---|---|---|
| 5 | $\overline{c}$ | $c$ |
| | 5 | |
| $\overline{c}$ | | 5 |

Here, there are 8 choices for $c$ (everything but 5) and 2 choices for each occurrence of $\overline{c}$ (since $c$ is not 5).
Furthermore, the possibilities for the 3 empty cells are determined by either the value of $c$ or by the value of $\overline{c}$, neither of which can be a multiple of 5.
Thus, there are 2 possibilities for each of these 3 empty cells.
Combining this information, in this case, there are thus $1^2 \cdot 8 \cdot 2^2 \cdot 2^3 = 2^8$ grids.

Case 3: $c = 5$ and $a \neq 5$
If $c = 5$ and $a \neq 5$, there are also $2^8$ grids.

Next, we consider the situation when $a \neq 5$ and $c \neq 5$.
We know here that there are two possible values for each of $\overline{a}$ and $\overline{c}$. However, the number of possible values for $\overline{a + c}$ depends on whether $a + c$ is a multiple of 5. Additionally, the number of possibilities for the 3 unlabelled cells also depend on the values of combinations of $a$, $c$, $\overline{a}$, and $\overline{c}$. This leads to three more cases in which $a \neq 5$ and $c \neq 5$.

Case 4: $a \neq 5$ and $c \neq 5$ and $a + c$ is a multiple of 5
There are 8 choices for $a$ (everything but 5).
There are then 2 choices for $c$ (either of the choices that makes $a + c$ a multiple of 5).
There are 2 choices for each of $\overline{a}$ and $\overline{c}$, since neither $a$ nor $c$ is 5.
Also, $\overline{a + c} = 5$ since $a + c$ is a multiple of 5.
This gives the grid:

| | | |
|---|---|---|
| $a$ | 5 | $c$ |
| | 5 | |
| $\overline{c}$ | | $\overline{a}$ |

The empty cell in the bottom row must be filled with a 5 to make the sum of the middle column a multiple of 5.

We now examine the first and third columns and see that neither $a + \overline{c}$ nor $c + \overline{a}$ can be a multiple of 5.
One way to justify this is to note that, since $a + c$ is a multiple of 5, the remainders when $a$ and $c$ are divided by 5 must add to 5.
This means that $a$ and $\overline{c}$ have the same non-zero remainder when divided by 5, which in turn means that their sum is not divisible by 5.
Therefore, the remaining 2 empty cells each have 2 possible entries to make their column sums multiples of 5.
There are 8 choices for $a$, 2 choices for $c$, 2 cells which must be filled with 5, and 2 choices for each of the remaining 4 cells.
In this case, there are thus $8 \cdot 2 \cdot 1^2 \cdot 2^4 = 2^8$ grids.

Finally, we look at the grids where $a \neq 5$ and $c \neq 5$ and $a + c$ is not a multiple of 5, separating the situations where $a - c$ is a multiple of 5 and $a - c$ is not a multiple of 5.

Case 5: $a \neq 5$ and $c \neq 5$ and $a + c$ is a not multiple of 5 and $a - c$ is a multiple of 5
There are 8 choices for $a$.
There are then 2 choices for $c$: either $c$ with the same remainder as $a$ when divided by 5.
There are 2 choices for each of $\overline{a}$ and $\overline{c}$ and $\overline{a+c}$ since none of $a$, $c$ and $a + c$ is a multiple of 5.

| | | |
|---|---|---|
| $a$ | $\overline{a+c}$ | $c$ |
| | 5 | |
| $\overline{c}$ | | $\overline{a}$ |

Since $a - c$ is a multiple of 5, then $a + \overline{c}$ and $c + \overline{a}$ are both multiples of 5.
To see this, note that $\overline{c} = 5 - c$ or $\overline{c} = 10 - c$ or $\overline{c} = 15 - c$, and so $a + \overline{c}$ is equal to one of $5 + a - c$ or $10 + a - c$ or $15 + a - c$ which are all multiples of 5 since $a - c$ is.
This means that each of the empty side cells must be filled with 5.
Finally, there are 2 choices for the bottom entry (since $\overline{a+c}$ is not a multiple of 5).
In this case, there are $8 \cdot 2 \cdot 2^3 \cdot 1^2 \cdot 2 = 2^8$ grids.

Case 6: $a \neq 5$ and $c \neq 5$ and $a + c$ is a not multiple of 5 and $a - c$ is not a multiple of 5
There are 8 choices for $a$.
There are then 4 choices for $c$ (not 5, not either choice that makes $a + c$ a multiple of 5, not either choice that makes $a - c$ a multiple of 5).
There 2 choices for each of $\overline{a}$, $\overline{c}$, and $\overline{a+c}$.
There are also 2 choices for each of the 3 remaining entries in the grid since the two entries in each of the first column, third column and third row do not add to a multiple of 5.
In this case, there are $8 \cdot 4 \cdot 2^3 \cdot 2^3 = 2^{11}$ grids.

Combining all of the cases, the number of possible ways to complete the grid is

$$N = 1 + 2^8 + 2^8 + 2^8 + 2^8 + 2^{11} = 1 + 4 \cdot 2^8 + 2^{11} = 3073$$

The rightmost two digits of $N$ are 73.

Answer: 73

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2021 Fermat Contest*

(Grade 11)

**Tuesday, February 23, 2021**
**(in North America and South America)**

**Wednesday, February 24, 2021**
**(outside of North America and South America)**

*Solutions*

1. A rectangle with width 2 cm and length 3 cm has area $2 \text{ cm} \times 3 \text{ cm} = 6 \text{ cm}^2$.

   Answer: (E)

2. Calculating, $2 + 3 \times 5 + 2 = 2 + 15 + 2 = 19$.

   Answer: (A)

3. Expressed as a fraction, 25% is equivalent to $\frac{1}{4}$.
   Since $\frac{1}{4}$ of 60 is 15, then 25% of 60 is 15.

   Answer: (B)

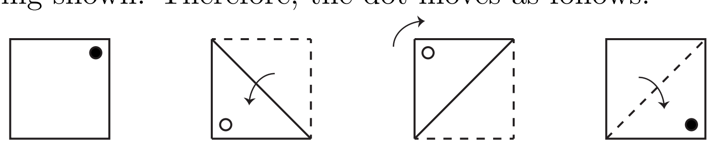
4. When $x \neq 0$, we obtain $\dfrac{4x}{x+2x} = \dfrac{4x}{3x} = \dfrac{4}{3}$.
   Thus, when $x = 2021$, we have $\dfrac{4x}{x+2x} = \dfrac{4}{3}$.
   Alternatively, we could substitute $x = 2021$ to obtain $\dfrac{4x}{x+2x} = \dfrac{8084}{2021+4042} = \dfrac{8084}{6063} = \dfrac{4}{3}$.

   Answer: (B)

5. We note that $6 = 2 \times 3$ and $27 = 3 \times 9$ and $39 = 3 \times 13$ and $77 = 7 \times 11$, which means that each of 6, 27, 39, and 77 can be written as the product of two integers, each greater than 1. Thus, 53 must be the integer that cannot be written in this way. We can check that 53 is indeed a prime number.

   Answer: (C)

6. We draw an unshaded dot to represent the location of the dot when it is on the other side of the sheet of paper being shown. Therefore, the dot moves as follows:



   It is worth noting that folding and unfolding the paper have no net effect on the figure. Thus, the resulting figure can be determined by rotating the original figure by 90° clockwise.

   Answer: (E)

7. When $x = -2$, we get $x^2 = 4$. Here, $x < x^2$.
   When $x = -\frac{1}{2}$, we get $x^2 = \frac{1}{4}$. Here, $x < x^2$.
   When $x = 0$, we get $x^2 = 0$. Here, $x = x^2$.
   When $x = \frac{1}{2}$, we get $x^2 = \frac{1}{4}$. Here, $x > x^2$.
   When $x = 2$, we get $x^2 = 4$. Here, $x < x^2$.
   This means that $x = \frac{1}{2}$ is the only choice where $x > x^2$.

   Answer: (D)

8. Suppose that the original integer has tens digit $a$ and ones (units) digit $b$.
   This integer is equal to $10a + b$.
   When the digits are reversed, the tens digit of the new integer is $b$ and the ones digit is $a$.
   This new integer is equal to $10b + a$.
   Since the new two-digit integer minus the original integer is 54, then $(10b + a) - (10a + b) = 54$ and so $9b - 9a = 54$ which gives $b - a = 6$.
   Thus, the positive difference between the two digits of the original integer is 6. An example of a pair of such integers is 71 and 17.

   Answer: (C)

9. The line with equation $y = 2x - 6$ has slope 2. When this line is translated, the slope does not change.
The line with equation $y = 2x - 6$ has $y$-intercept $-6$. When this line is translated upwards by 4 units, its $y$-intercept is translated upwards by 4 units and so becomes $-2$.
This means that the new line has equation $y = 2x - 2$.
To find its $x$-intercept, we set $y = 0$ to obtain $0 = 2x - 2$ and so $2x = 2$ or $x = 1$.
Thus, the $x$-intercept is 1.

ANSWER: (D)

10. Using exponent laws, $3^{x+2} = 3^x \cdot 3^2 = 3^x \cdot 9$.
Since $3^x = 5$, then $3^{x+2} = 3^x \cdot 9 = 5 \cdot 9 = 45$.

ANSWER: (E)

11. Since the second number being added is greater than 300 and the sum has hundreds digit $R$, then $R$ cannot be 0.
From the ones column, we see that the ones digit of $R + R$ is 0. Since $R \neq 0$, then $R = 5$.
This makes the sum

$$\begin{array}{cccc} & P & \overset{1}{7} & 5 \\ + & 3 & 9 & 5 \\ \hline & 5 & Q & 0 \end{array}$$

Since $1 + 7 + 9 = 17$, we get $Q = 7$ and then $1 + P + 3 = 5$ and so $P = 1$, giving the final sum

$$\begin{array}{cccc} & \overset{1}{1} & \overset{1}{7} & 5 \\ + & 3 & 9 & 5 \\ \hline & 5 & 7 & 0 \end{array}$$

Therefore, $P + Q + R = 1 + 7 + 5 = 13$.

ANSWER: (A)

12. A perfect square is divisible by 9 exactly when its square root is divisible by 3.
In other words, $n^2$ is divisible by 9 exactly when $n$ is divisible by 3.
In the list $1, 2, 3, \ldots, 19, 20$, there are 6 multiples of 3.
Therefore, in the list $1^2, 2^2, 3^2, \ldots, 19^2, 20^2$, there are 6 multiples of 9.

ANSWER: (E)

13. In an isosceles right-angled triangle, the ratio of the length of the hypotenuse to the length of each of the shorter sides is $\sqrt{2} : 1$.
Consider $\triangle WZX$ which is isosceles and right-angled at $Z$.
Here, $WX : WZ = \sqrt{2} : 1$. Since $WX = 6\sqrt{2}$, then $WZ = \dfrac{6\sqrt{2}}{\sqrt{2}} = 6$.
Since $\triangle WZX$ is isosceles, then $XZ = WZ = 6$.
Consider $\triangle XYZ$ which is isosceles and right-angled at $Y$.
Here, $YZ = \dfrac{XZ}{\sqrt{2}} = \dfrac{6}{\sqrt{2}} = \dfrac{6}{\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{6\sqrt{2}}{2} = 3\sqrt{2}$.
Since $\triangle XYZ$ is isosceles, then $XY = YZ = 3\sqrt{2}$.
Therefore, the perimeter of $WXYZ$ is

$$WX + XY + YZ + WZ = 6\sqrt{2} + 3\sqrt{2} + 3\sqrt{2} + 6 = 12\sqrt{2} + 6 \approx 22.97$$

Of the given choices, this is closest to 23.

ANSWER: (C)

14. Suppose that Natascha runs at $r$ km/h.
Since she cycles 3 times as fast as she runs, she cycles at $3r$ km/h.
In 1 hour of running, Natascha runs $(1\text{ h}) \cdot (r\text{ km/h}) = r$ km.
In 4 hours of cycling, Natascha cycles $(4\text{ h}) \cdot (3r\text{ km/h}) = 12r$ km.
Thus, the ratio of the distance that she cycles to the distance that she runs is equivalent to the ratio $12r$ km : $r$ km which is equivalent to $12 : 1$.

Answer: (A)

15. *Solution 1*
Since $a$ is a positive integer, $45a$ is a positive integer.
Since $b$ is a positive integer, $45a$ is less than 2021.
The largest multiple of 45 less than 2021 is $45 \times 44 = 1980$. (Note that $45 \cdot 45 = 2025$ which is greater than 2021.)
If $a = 44$, then $b = 2021 - 45 \cdot 44 = 41$.
Here, $a + b = 44 + 41 = 85$.
If $a$ is decreased by 1, the value of $45a + b$ is decreased by 45 and so $b$ must be increased by 45 to maintain the same value of $45a + b$, which increases the value of $a + b$ by $-1 + 45 = 44$.
Therefore, if $a < 44$, the value of $a + b$ is always greater than 85.
If $a > 44$, then $45a > 2021$ which makes $b$ negative, which is not possible.
Therefore, the minimum possible value of $a + b$ is 85.

*Solution 2*
We re-write $45a + b = 2021$ as $44a + (a + b) = 2021$.
Since $a$ and $b$ are positive integers, $44a$ and $a + b$ are positive integers.
In particular, this tells us that $44a$, which is a multiple of 44, is less than 2021.
Since the sum of $44a$ and $a + b$ is constant, to minimize $a + b$, we can try to maximize $44a$.
Since $44 \cdot 45 = 1980$ and $44 \cdot 46 = 2024$, the largest multiple of 44 less than 2021 is 1980.
This means that $a + b \geq 2021 - 1980 = 41$.
However, $a+b$ cannot equal 41 since we would need $44a = 1980$ and so $a = 45$ (making $b = -4$) to make this possible.
The next multiple of 44 less than 1980 is $44 \cdot 44 = 1936$.
If $a = 44$, then $a + b = 2021 - 44a = 85$.
If $a = 44$ and $a + b = 85$, then $b = 41$ which is possible.
Since $a + b = 41$ is not possible and 85 is the next smallest possible value for $a + b$, then the minimum possible value for $a + b$ is 85.

Answer: (C)

16. The first few values of $n!$ are

$$
\begin{aligned}
1! &= 1 \\
2! &= 2(1) = 2 \\
3! &= 3(2)(1) = 6 \\
4! &= 4(3)(2)(1) = 24 \\
5! &= 5(4)(3)(2)(1) = 120
\end{aligned}
$$

We note that

$$\begin{aligned} 2! - 1! &= 1 \\ 4! - 1! &= 23 \\ 3! - 1! &= 5 \\ 5! - 1! &= 119 \end{aligned}$$

This means that if $a$ and $b$ are positive integers with $b > a$, then 1, 3, 5, 9 are all possible ones (units) digits of $b! - a!$.
This means that the only possible answer is choice (D), or 7.

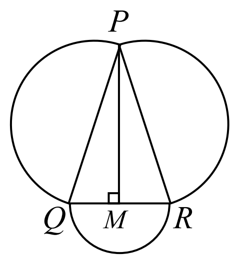
To be complete, we explain why 7 cannot be the ones (units) digit of $b! - a!$.
For $b! - a!$ to be odd, one of $b!$ and $a!$ is even and one of them is odd.
The only odd factorial is $1!$, since every other factorial has a factor of 2.
Since $b > a$, then if one of $a$ and $b$ is 1, we must have $a = 1$.
For the ones (units) digit of $b! - 1$ to be 7, the ones (units) digit of $b!$ must be 8.
This is impossible as the first few factorials are shown above and every greater factorial has a ones (units) digit of 0, because it is a multiple of both 2 and 5.

Answer: (D)

17. Since the average of the two smallest integers in $S$ is 5, their sum is $2 \cdot 5 = 10$.
Since the average of the two largest integers in $S$ is 22, their sum is $2 \cdot 22 = 44$.
Suppose that the other five integers in the set $S$ are $p < q < r < t < u$. (Note that the integers in $S$ are all distinct.)
The average of the nine integers in $S$ is thus equal to $\dfrac{10 + 44 + p + q + r + t + u}{9}$ which equals $6 + \dfrac{p + q + r + t + u}{9}$.
We would like this average to be as large as possible.
To make this average as large as possible, we want $\dfrac{p + q + r + t + u}{9}$ to be as large as possible, which means that we want $p + q + r + t + u$ to be as large as possible.
What is the maximum possible value of $u$?
Let $x$ and $y$ be the two largest integers in $S$, with $x < y$. Since $x$ and $y$ are the two largest integers, then $u < x < y$.
Since $x + y = 44$ and $x < y$ and $x$ and $y$ are integers, then $x \leq 21$.
For $u$ to be as large as possible (which will allow us to make $p$, $q$, $r$, $t$ as large as possible), we set $x = 21$.
In this case, we can have $u = 20$.
To make $p$, $q$, $r$, $t$ as large as possible, we can take $p = 16$, $q = 17$, $r = 18$, $t = 19$.
Here, $p + q + r + t + u = 90$.
If $x < 21$, then $p + q + r + t + u$ will be smaller and so not give the maximum possible value.
This means that the maximum possible average of the integers in $S$ is $6 + \dfrac{90}{9} = 16$.

Answer: (B)

18. Suppose that $PQ = PR = 2x$ and $QR = 2y$.
The semi-circles with diameters $PQ$ and $PR$ thus have radii $x$ and the radius of the semi-circle with diameter $QR$ is $y$.
The area of each semi-circle with radius $x$ is $\frac{1}{2}\pi x^2$ and the area of the semi-circle with radius $y$ is $\frac{1}{2}\pi y^2$.
Since the sum of the areas of the three semi-circles equals 5 times the area of the semi-circle with diameter $QR$, then
$$\tfrac{1}{2}\pi x^2 + \tfrac{1}{2}\pi x^2 + \tfrac{1}{2}\pi y^2 = 5 \cdot \tfrac{1}{2}\pi y^2$$
which gives $x^2 + x^2 + y^2 = 5y^2$ and so $2x^2 = 4y^2$ which gives $x^2 = 2y^2$ and so $x = \sqrt{2}y$.
Suppose that $M$ is the midpoint of $QR$ and that $P$ is joined to $M$.



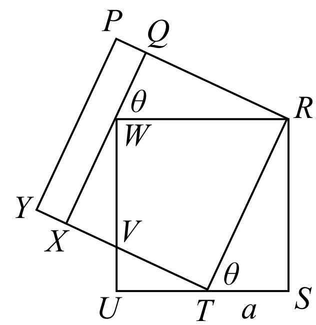

Since $\triangle PQR$ is isosceles with $PQ = PR$, then $PM$ is perpendicular to $QR$.
In other words, $\triangle PMQ$ is right-angled at $M$.
Therefore, $\cos(\angle PQR) = \cos(\angle PQM) = \dfrac{QM}{PQ} = \dfrac{\frac{1}{2}QR}{PQ} = \dfrac{y}{2x} = \dfrac{y}{2\sqrt{2}y} = \dfrac{1}{2\sqrt{2}} = \dfrac{1}{\sqrt{4 \cdot 2}} = \dfrac{1}{\sqrt{8}}$.

Answer: (B)

19. Since $x + y = 7$, then $x + y + z = 7 + z$.
Thus, the equation $(x + y + z)^2 = 4$ becomes $(7 + z)^2 = 4$.
Since the square of $7 + z$ equals 4, then $7 + z = 2$ or $7 + z = -2$.
If $7 + z = 2$, then $z = -5$.
In this case, since $xz = -180$, we get $x = \dfrac{-180}{-5} = 36$ which gives $y = 7 - x = -29$.
If $7 + z = -2$, then $z = -9$.
In this case, since $xz = -180$, we get $x = \dfrac{-180}{-9} = 20$ which gives $y = 7 - x = -13$.
We can check by direct substitution that $(x, y, z) = (36, -29, -5)$ and $(x, y, z) = (20, -13, -9)$ are both solutions to the original system of equations.
Since $S$ is the sum of the possible values of $y$, we get $S = (-29)+(-13) = -42$ and so $-S = 42$.

Answer: (E)

20. Let $ST = a$ and let $\angle STR = \theta$.
Since $\triangle RST$ is right-angled at $S$, then $\angle TRS = 90^\circ - \theta$.
Since $PRTY$ and $WRSU$ are squares, then $\angle PRT = \angle WRS = 90^\circ$.
Thus, $\angle QRW + \angle WRT = \angle WRT + \angle TRS$ and so $\angle QRW = \angle TRS = 90^\circ - \theta$.
Since $PQXY$ is a rectangle, then $\angle PQX = 90^\circ$, which means that $\triangle WQR$ is right-angled at $Q$.
This means that $\angle QWR = 90^\circ - \angle QRW = 90^\circ - (90^\circ - \theta) = \theta$.




Consider $\triangle RST$.
Since $ST = a$ and $\angle STR = \theta$, then $\cos\theta = \dfrac{a}{RT}$ and so $RT = \dfrac{a}{\cos\theta}$.
Also, $\tan\theta = \dfrac{RS}{a}$ and so $RS = a\tan\theta$.
Since $PRTY$ is a square, then $PY = PR = RT = \dfrac{a}{\cos\theta}$.
Since $WRSU$ is a square, then $RW = RS = a\tan\theta$.
Next, consider $\triangle QRW$.
Since $RW = a\tan\theta$ and $\angle QWR = \theta$, then $\sin\theta = \dfrac{QR}{a\tan\theta}$ and so $QR = a\tan\theta\sin\theta$.
This means that

$$\begin{aligned}
PQ &= PR - QR \\
&= \frac{a}{\cos\theta} - a\tan\theta\sin\theta \\
&= \frac{a}{\cos\theta} - a\cdot\frac{\sin\theta}{\cos\theta}\cdot\sin\theta \\
&= \frac{a}{\cos\theta} - \frac{a\sin^2\theta}{\cos\theta} \\
&= \frac{a(1-\sin^2\theta)}{\cos\theta} \\
&= \frac{a\cos^2\theta}{\cos\theta} \qquad (\text{since } \sin^2\theta + \cos^2\theta = 1) \\
&= a\cos\theta \qquad (\text{since } \cos\theta \neq 0)
\end{aligned}$$

Since the area of rectangle $PQXY$ is 30, then $PQ \cdot PY = 30$ and so $a\cos\theta \cdot \dfrac{a}{\cos\theta} = 30$ which gives $a^2 = 30$.
Since $a > 0$, we get $a = \sqrt{30} \approx 5.48$. Of the given choices, $ST = a$ is closest to 5.5.

Answer: (C)

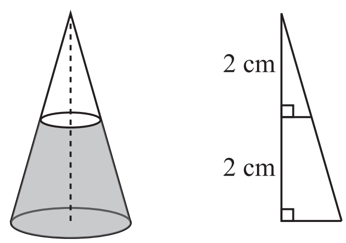
21. Since $f(2) = 5$ and $f(mn) = f(m) + f(n)$, then $f(4) = f(2 \cdot 2) = f(2) + f(2) = 10$.
Since $f(3) = 7$, then $f(12) = f(4 \cdot 3) = f(4) + f(3) = 10 + 7 = 17$.

While this answers the question, is there actually a function that satisfies the requirements? The answer is yes.
One function that satisfies the requirements of the problem is the function $f$ defined by $f(1) = 0$ and $f(2^p 3^q r) = 5p + 7q$ for all non-negative integers $p$ and $q$ and all positive integers $r$ that are not divisible by 2 or by 3. Can you see why this function satisfies the requirements?

Answer: (A)

22. The total surface area of the cone includes the circular base and the lateral surface.
For the given unpainted cone, the base has area $\pi r^2 = \pi(3\text{ cm})^2 = 9\pi\text{ cm}^2$ and the lateral surface has area $\pi rs = \pi(3\text{ cm})(5\text{ cm}) = 15\pi\text{ cm}^2$.
Thus, the total surface area of the unpainted cone is $9\pi\text{ cm}^2 + 15\pi\text{ cm}^2 = 24\pi\text{ cm}^2$.
Since the height and base are perpendicular, the lengths $s$, $h$ and $r$ form a right-angled triangle with hypotenuse $s$.
For this cone, by the Pythagorean Theorem, $h^2 = s^2 - r^2 = (5\text{ cm})^2 - (3\text{ cm})^2 = 16\text{ cm}^2$ and so $h = 4$ cm.

When the unpainted cone is placed in the container of paint so that the paint rises to a depth of 2 cm, the base of the cone (area $9\pi\text{ cm}^2$) is covered in paint.
Also, the bottom portion of the lateral surface is covered in paint.




The unpainted portion of the cone is itself a cone with height 2.
When we take a vertical cross-section of the cone through its top vertex and a diameter of the base, the triangle formed above the paint is similar to the original triangle and has half its dimensions. The triangles are similar because both are right-angled and they share an equal angle at the top vertex. The ratio is $2 : 1$ since their heights are 4 cm and 2 cm.
Therefore, the unpainted cone has radius 1.5 cm (half of the original radius of 3 cm) and slant height 2.5 cm (half of the original slant height of 5 cm).
Thus, the unpainted lateral surface area is the lateral surface area of a cone with radius 1.5 cm and slant height 2.5 cm, and so has area $\pi(1.5\text{ cm})(2.5\text{ cm}) = 3.75\pi\text{ cm}^2$.
This means that the painted lateral surface area is $15\pi\text{ cm}^2 - 3.75\pi\text{ cm}^2 = 11.25\pi\text{ cm}^2$. (This is in fact three-quarters of the total surface area. Can you explain why this is true in a different way?)
Thus, the fraction of the total surface area of the cone that is painted is

$$\frac{9\pi\text{ cm}^2 + 11.25\pi\text{ cm}^2}{24\pi\text{ cm}^2} = \frac{20.25}{24} = \frac{81}{96} = \frac{27}{32}$$

Since $\frac{27}{32}$ is in lowest terms, then $p + q = 27 + 32 = 59$.

Answer: (A)

23. Since each Figure is formed by placing two copies of the previous Figure side-by-side along the base and then adding other pieces above, the number of dots in the base of each Figure is two times as many as in the previous Figure.
Since each Figure is an equilateral triangle, then the number of dots in the Figure equals the sum of the positive integers from 1 to the number of dots in the base, inclusive. In other words, if the base of a Figure consists of $b$ dots, then the Figure includes $1 + 2 + 3 + \cdots + (b - 1) + b$ dots. This sum is equal to $\frac{1}{2}b(b + 1)$. (If this formula for the sum is unfamiliar, can you argue why it is true?)
Since each Figure is formed by using three copies of the previous Figure and any new dots added are shaded dots, the number of unshaded dots in each Figure is exactly three times the number of unshaded dots in the previous Figure.
Since each dot is either shaded or unshaded, the number of shaded dots equals the total number of dots minus the number of unshaded dots.
Using these statements, we construct a table:

| Figure | Dots in base | Dots in Figure | Unshaded dots | Shaded dots |
|---|---|---|---|---|
| 1 | 2 | 3 | 3 | 0 |
| 2 | 4 | 10 | 9 | 1 |
| 3 | 8 | 36 | 27 | 9 |
| 4 | 16 | 136 | 81 | 55 |
| 5 | 32 | 528 | 243 | 285 |
| 6 | 64 | 2080 | 729 | 1351 |
| 7 | 128 | 8256 | 2187 | 6069 |
| 8 | 256 | 32 896 | 6561 | 26 335 |
| 9 | 512 | 131 328 | 19 683 | 111 645 |

Therefore, the smallest value of $n$ for which Figure $n$ includes at least 100 000 dots is $n = 9$.

We note that since the number of dots in the base of Figure 1 is 2 and the number of dots in the base of each subsequent Figure is double the number of dots in the previous Figure, then the number of dots in the base of Figure $n$ is equal to $2^n$.
Since the number of unshaded dots in Figure 1 is 3 and the number of unshaded dots in each subsequent Figure is three times the number of unshaded dots in the previous Figure, then the number of unshaded dots in Figure $n$ is $3^n$.
Therefore, a formula for the number of unshaded dots in Figure $n$ is $\frac{1}{2}2^n(2^n + 1) - 3^n$ which can be re-written as $2^{2n-1} + 2^{n-1} - 3^n$, which agrees with the numbers in the table above.

ANSWER: (B)

24. The curves with equations $y = ax^2 + 2bx - a$ and $y = x^2$ intersect exactly when the equation

$$ax^2 + 2bx - a = x^2$$

has at least one real solution.
This equation has at least one real solution when the quadratic equation

$$(a - 1)x^2 + 2bx - a = 0$$

has at least one real solution.
(Note that when $a = 1$, this equation is actually linear as long as $b \neq 0$ and so will have at least one real solution.)

This quadratic equation has at least one real solution when its discriminant is non-negative; that is, when

$$(2b)^2 - 4(a-1)(-a) \geq 0$$

Manipulating algebraically, we obtain the equivalent inequalities:

$$\begin{aligned} 4a^2 - 4a + 4b^2 &\geq 0 \\ a^2 - a + b^2 &\geq 0 \\ a^2 - a + \tfrac{1}{4} + b^2 &\geq \tfrac{1}{4} \\ a^2 - 2a(\tfrac{1}{2}) + (\tfrac{1}{2})^2 + b^2 &\geq \tfrac{1}{4} \\ (a - \tfrac{1}{2})^2 + b^2 &\geq (\tfrac{1}{2})^2 \end{aligned}$$

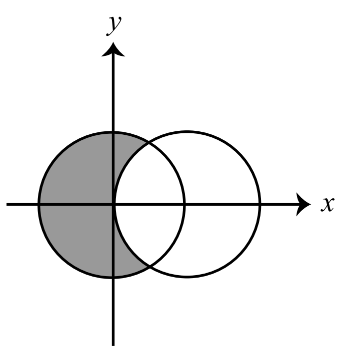
Therefore, we want to determine the probability that a point $(a,b)$ satisfies $(a-\frac{1}{2})^2 + b^2 \geq (\frac{1}{2})^2$ given that it satisfies $a^2 + b^2 \leq (\frac{1}{2})^2$.
The equation $(x-\frac{1}{2})^2 + y^2 = (\frac{1}{2})^2$ represents a circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$.

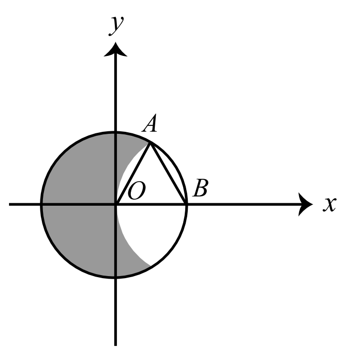
The inequality $(x-\frac{1}{2})^2 + y^2 \geq (\frac{1}{2})^2$ represents the region outside of this circle.

The equation $x^2 + y^2 = (\frac{1}{2})^2$ represents a circle with centre $(0,0)$ and radius $\frac{1}{2}$.

The inequality $x^2 + y^2 \leq (\frac{1}{2})^2$ represents the region inside this circle.

Re-phrasing the problem in an equivalent geometric way, we want to determine the probability that a point $(a,b)$ is *outside* the circle with centre $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ given that it is *inside* the circle with centre $(0,0)$ and radius $\frac{1}{2}$.
Note that each of these two circles passes through the centre of the other circle.




Putting this another way, we want to determine the fraction of the area of the circle centred at the origin that is outside the circle centred at $(\frac{1}{2}, 0)$. This region is shaded in the first diagram above.
Let $r = \frac{1}{2}$.
The area of each circle is $\pi r^2$.
To determine the shaded area, we subtract the unshaded area from the area of the circle. By symmetry, the unshaded area below the $x$-axis will equal the unshaded area above the $x$-axis.
In the second diagram above, the origin is labelled $O$, the point where the circle centred at $O$ intersects the $x$-axis is labelled $B$, and the point of intersect in the first quadrant of the two cicles is labelled $A$. The unshaded region above the $x$-axis consists of $\triangle AOB$ plus two curvilinear regions.
Since $OA$ and $OB$ are radii of the circle centred at $O$, then $OA = OB = r$.
Since $AB$ is a radius of the circle centred at $B$, then $AB = r$.
This means that $\triangle OAB$ is equilateral, and so it has three $60^\circ$ angles.

When $\triangle OAB$ is combined with the curvilinear region above and to the right of $\triangle OAB$, we thus obtain a sector of the circle with centre $O$ and central angle $60^\circ$.
Since $60^\circ$ is $\frac{1}{6}$ of $360^\circ$, this sector is $\frac{1}{6}$ of the entire circle and so has area $\frac{1}{6}\pi r^2$.
Similarly, when $\triangle OAB$ is combined with the curvilinear region above and to the left of $\triangle OAB$, we obtain a sector of the circle with centre $B$ and central angle $60^\circ$; this sector has area $\frac{1}{6}\pi r^2$.
If $K$ is the area of equilateral $\triangle OAB$, the area of each curvilinear region is $\frac{1}{6}\pi r^2 - K$ and so the unshaded area above the $x$-axis is equal to $2(\frac{1}{6}\pi r^2 - K) + K$ which simplifies to $\frac{1}{3}\pi r^2 - K$.
Thus, the total unshaded area inside the circle of area $\pi r^2$ is $\frac{2}{3}\pi r^2 - 2K$, which means that the shaded area inside the circle is $\pi r^2 - (\frac{2}{3}\pi r^2 - 2K)$ which is equal to $\frac{1}{3}\pi r^2 + 2K$.
Finally, we need to use the area of an equilateral triangle with side length $r$. This area is equal to $\frac{\sqrt{3}}{4}r^2$. (To see this, we could use an altitude to divide the equilateral triangle into two $30^\circ$-$60^\circ$-$90^\circ$ triangles. Using the ratio of side lengths in these special triangles, this altitude has length $\frac{\sqrt{3}}{2}r$ and so the area of the equilateral triangle is $\frac{1}{2}r(\frac{\sqrt{3}}{2})$ or $\frac{\sqrt{3}}{4}r^2$.)
Therefore, the fraction of the circle that is shaded is

$$\frac{\frac{1}{3}\pi r^2 + \frac{\sqrt{3}}{2}r^2}{\pi r^2} = \frac{\frac{1}{3}\pi + \frac{\sqrt{3}}{2}}{\pi} = \frac{2\pi + 3\sqrt{3}}{6\pi} \approx 0.609$$

This means that the probability, $p$, that a point $(a, b)$ satisfies the given conditions is approximately 0.609 and so $100p \approx 60.9$.
Of the given choices, this is closest to 61, or (E).

Answer: (E)

25. Suppose that $a$, $b$ and $c$ are positive integers with $abc = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2$.
We determine the number of triples $(a, b, c)$ with this property. (We are temporarily ignoring the size ordering condition in the original question.)
Since the product $abc$ has two factors of 2, then $a$, $b$ and $c$ have a total of two factors of 2.
There are 6 ways in which this can happen: both factors in $a$, both factors in $b$, both in $c$, one each in $a$ and $b$, one each in $a$ and $c$, and one each in $b$ and $c$.
Similarly, there are 6 ways of distributing each of the other squares of prime factors.
Since $abc$ includes exactly 8 squares of prime factors and each can be distributed in 6 ways, there are $6^8$ ways of building triples $(a, b, c)$ using the prime factors, and so there are $6^8$ triples $(a, b, c)$ with the required product.

Next, we include the condition no pair of $a$, $b$ and $c$ should be equal. (We note that $a$, $b$ and $c$ cannot all be equal, since their product is not a perfect cube.)
We count the number of triples with one pair equal, and subtract this number from $6^8$.
We do this by counting the number of these triples with $a = b$. By symmetry, the number of triples with $a = c$ and with $b = c$ will be equal to this total.
In order to have $a = b$ and $a \neq c$ and $b \neq c$, for each of the squared prime factors $p^2$ of $abc$, either $p^2$ is distributed as $p$ and $p$ in each of $a$ and $b$, or $p^2$ is distributed to $c$.
Thus, for each of the 8 squared prime factors $p^2$, there are 2 ways to distribute, and so $2^8$ triples $(a, b, c)$ with $a = b$ and $a \neq c$ and $b \neq c$.
Similarly, there will be $2^8$ triples with $a = c$ and $2^8$ triples with $b = c$.
This means that there are $6^8 - 3 \cdot 2^8$ triples $(a, b, c)$ with the required product and with no two of $a$, $b$, $c$ equal.

The original problem asked us to the find the number of triples $(x, y, z)$ with the given product and with $x < y < z$.
To convert triples $(a, b, c)$ with no size ordering to triples $(x, y, z)$ with $x < y < z$, we divide

by 6. (Each triple $(x, y, z)$ corresponds to 6 triples $(a, b, c)$ of distinct positive integers with no size ordering.)
Therefore, the total number of triples $(x, y, z)$ with the required properties is

$$N = \frac{1}{6}(6^8 - 3 \cdot 2^8) = 6^7 - 2^7 = 279\,808$$

When $N$ is divided by 100, the remainder is 8.

Answer: (C)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2020 Fermat Contest*

**(Grade 11)**

**Tuesday, February 25, 2020**
**(in North America and South America)**

**Wednesday, February 26, 2020**
**(outside of North America and South America)**

*Solutions*

1. Since $OPQR$ is a rectangle with two sides on the axes, then its sides are horizontal and vertical. Since $PQ$ is horizontal, the $y$-coordinate of $Q$ is the same as the $y$-coordinate of $P$, which is 3. Since $QR$ is vertical, the $x$-coordinate of $Q$ is the same as the $x$-coordinate of $R$, which is 5. Therefore, the coordinates of $Q$ are $(5,3)$.

   Answer: (B)

2. Calculating,

$$3 \times 2020 + 2 \times 2020 - 4 \times 2020 = 2020 \times (3 + 2 - 4) = 2020 \times 1 = 2020$$

   Alternatively,

$$3 \times 2020 + 2 \times 2020 - 4 \times 2020 = 6060 + 4040 - 8080 = 10\,100 - 8080 = 2020$$

   Answer: (E)

3. Expanding and simplifying, $(x+1)^2 - x^2 = (x^2 + 2x + 1) - x^2 = 2x + 1$.

   Answer: (A)

4. Ewan's sequence starts with 3 and each following number is 11 larger than the previous number. Since every number in the sequence is some number of 11s more than 3, this means that each number in the sequence is 3 more than a multiple of 11. Furthermore, every such positive integer is in Ewan's sequence.
   Since $110 = 11 \times 10$ is a multiple of 11, then $113 = 110 + 3$ is 3 more than a multiple of 11, and so is in Ewan's sequence.
   Alternatively, we could write Ewan's sequence out until we get into the correct range:

$$3, 14, 25, 36, 47, 58, 69, 80, 91, 102, 113, 124, \ldots$$

   Answer: (A)

5. Calculating, $\sqrt{\dfrac{\sqrt{81} + \sqrt{81}}{2}} = \sqrt{\dfrac{9+9}{2}} = \sqrt{9} = 3$.

   Answer: (A)

6. Since 12 and 21 are multiples of 3 ($12 = 4 \times 3$ and $21 = 7 \times 3$), the answer is not (A) or (D).
   16 is a perfect square ($16 = 4 \times 4$) so the answer is not (C).
   The sum of the digits of 26 is 8, which is not a prime number, so the answer is not (E).
   Since 14 is not a multiple of a three, 14 is not a perfect square, and the sum of the digits of 14 is $1 + 4 = 5$ which is prime, then the answer is (B) 14.

   Answer: (B)

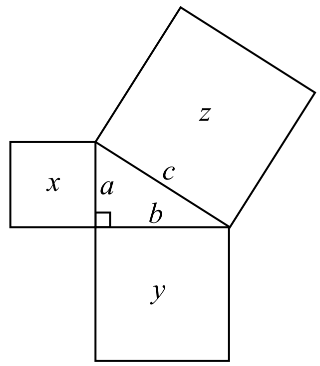
7. Since $WXY$ is a straight angle, then $p^\circ + q^\circ + r^\circ + s^\circ + t^\circ = 180^\circ$ and so $p + q + r + s + t = 180$.
   To calculate the average of $p$, $q$, $r$, $s$, and $t$, we add the five numbers and divide by 5.
   Therefore, the average of $p$, $q$, $r$, $s$, and $t$ is $\dfrac{p+q+r+s+t}{5} = \dfrac{180}{5} = 36$.

   Answer: (B)

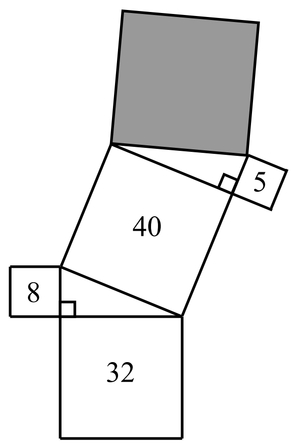
8. Since $8 = 2 \times 2 \times 2 = 2^3$, then $8^{20} = (2^3)^{20} = 2^{3 \times 20} = 2^{60}$.
   Thus, if $2^n = 8^{20}$, then $n = 60$.

   Answer: (B)

9. The Pythagorean Theorem tells us that if a right-angled triangle has sides of length $a$, $b$ and $c$, with $c$ the hypotenuse, then $a^2 + b^2 = c^2$.
Since the area of a square of side length $a$ is $a^2$, the Pythagorean Theorem can be re-phrased to say that the sum of the areas of the squares that can be drawn on the two shorter sides equals the area of the square that can be drawn on the hypotenuse. (In the figure below, this says that $x + y = z$ where $x$, $y$ and $z$ are the areas of the squares, as shown.)




In the given diagram, this means that the area of the unmarked, unshaded square is $8+32 = 40$.




This means that the area of the shaded square is $40 + 5 = 45$.

Answer: (B)

10. We are given that $s$ and $t$ are positive integers and that $s(s - t) = 29$.
Since $s$ and $t$ are positive, then $s - t$ is less than $s$.
Since $s$ is positive and 29 is positive and $s(s - t) = 29$, then $s - t$ must also be positive.
Since 29 is a prime number, the only way that it can be written as a product of two positive integers is $29 = 29 \cdot 1$.
Since $s(s - t) = 29$ and $s > s - t$, then we must have $s = 29$ and $s - t = 1$.
Since $s = 29$ and $s - t = 1$, we obtain $t = 28$.
Therefore, $s + t = 29 + 28 = 57$.

Answer: (C)

11. Each of the first and second columns has 4 X's in it, which means that at least 2 X's need to be moved. We will now show that this can be actually done by moving 2 X's.
Each of the first and second rows has 4 X's in it, so we move the two X's on the main diagonals, since this will remove X's from the first and second columns and the first and second rows simultaneously.
The fifth column starts with one X in it, so we move the two X's to the fifth column into the rows that only contain 2 X's. Doing this, we obtain:

| | | | | |
|---|---|---|---|---|
| O | X | X | X | |
| X | O | X | | X |
| X | X | | | X* |
| X | X | | X | |
| | | X | X | X* |

(The cells from which X's have been removed are marked with O's; the cells to which X's are moved are marked with X*'s.)
Therefore, the smallest number of X's that must be moved is 2.

Answer: (B)

12. Since Harriet ran 720 m at 3 m/s, then this segment took her $\dfrac{720 \text{ m}}{3 \text{ m/s}} = 240$ s.
In total, Harriet ran 1000 m in 380 s, so the remaining part of the course was a distance of 1000 m $-$ 720 m $=$ 280 m which she ran in 380 s $-$ 240 s $=$ 140 s.
Since she ran this section at a constant speed of $v$ m/s, then $\dfrac{280 \text{ m}}{140 \text{ s}} = v$ m/s which means that $v = 2$.

Answer: (A)

13. Since the sum of any two adjacent numbers is constant, then $2 + x = x + y$.
This means that $y = 2$ and makes the list $2, x, 2, 5$.
This means that the sum of any two adjacent numbers is $2 + 5 = 7$, and so $x = 5$.
Therefore, $x - y = 5 - 2 = 3$.

Answer: (C)

14. If $\frac{2}{7}$ of the roses are to be yellow, then the remaining $\frac{5}{7}$ of the roses are to be red.
Since there are 30 red roses and these are to be $\frac{5}{7}$ of the roses, then $\frac{1}{7}$ of the total number of roses would be $30 \div 5 = 6$, which means that there would be $6 \times 7 = 42$ roses in total.
If there are 42 roses of which 30 are red and the rest are yellow, then there are $42 - 30 = 12$ yellow roses.
Since there are 19 yellow roses to begin, then $19 - 12 = 7$ yellow roses are removed.

Answer: (E)

15. When $N = 3x + 4y + 5z$ with each of $x$, $y$ and $z$ equal to either 1 or $-1$, there are 8 possible combinations of values for $x$, $y$ and $z$:

| $x$ | $y$ | $z$ | $N$ |
|---|---|---|---|
| 1 | 1 | 1 | 12 |
| 1 | 1 | $-1$ | 2 |
| 1 | $-1$ | 1 | 4 |
| 1 | $-1$ | $-1$ | $-6$ |
| $-1$ | 1 | 1 | 6 |
| $-1$ | 1 | $-1$ | $-4$ |
| $-1$ | $-1$ | 1 | $-2$ |
| $-1$ | $-1$ | $-1$ | $-12$ |

From this information, $N$ cannot equal 0, $N$ is never odd, $N$ can equal 4, and $N$ is always even. Therefore, exactly one of the four given statements is true.

ANSWER: (B)

16. We note that $\dfrac{x+y}{x} = \dfrac{x}{x} + \dfrac{y}{x} = 1 + \dfrac{y}{x}$.
The greatest possible value of $\dfrac{x+y}{x} = 1 + \dfrac{y}{x}$ thus occurs when $\dfrac{y}{x}$ is as great as possible.
Since $x$ is always negative and $y$ is always positive, then $\dfrac{y}{x}$ is negative.
Therefore, for $\dfrac{y}{x}$ to be as great as possible, it is as least negative as possible (i.e. closest to 0 as possible).
Since $x$ is negative and $y$ is positive, this happens when $x$ is as negative as possible and $y$ is as small as possible – that is, when $x = -4$ and $y = 2$.
Therefore, the greatest possible value of $\dfrac{x+y}{x}$ is $1 + \dfrac{2}{-4} = \dfrac{1}{2}$.

ANSWER: (E)

17. Since $\triangle PQR$ is right-angled at $Q$, its area equals $\frac{1}{2} \cdot PQ \cdot QR$.
Since its area is 30 and $PQ = 5$, then $\frac{1}{2} \cdot 5 \cdot QR = 30$ and so $QR = 30 \cdot \frac{2}{5} = 12$.
By the Pythagorean Theorem, we know that

$$PR^2 = PQ^2 + QR^2 = 5^2 + 12^2 = 25 + 144 = 169$$

Since $PR > 0$, then $PR = \sqrt{169} = 13$.
If we now consider $\triangle PQR$ as having base $PR$ and perpendicular height $QS$, we see that its area equals $\frac{1}{2} \cdot PR \cdot QS$.
Since its area is 30 and $PR = 13$, then $\frac{1}{2} \cdot 13 \cdot QS = 30$ which gives $QS = 30 \cdot \frac{2}{13} = \frac{60}{13}$.

ANSWER: (A)

18. Suppose that the four teams in the league are called W, X, Y, and Z.
Then there is a total of 6 games played:

W against X, W against Y, W against Z, X against Y, X against Z, Y against Z

In each game that is played, either one team is awarded 3 points for a win and the other is awarded 0 points for a loss (for a total of 3 points between the two teams), or each team is awarded 1 point for a tie (for a total of 2 points between the two teams).
Since 6 games are played, then the theoretical maximum number of points that could be awarded is $6 \cdot 3 = 18$ and the theoretical minimum number of points that can be awarded is $6 \cdot 2 = 12$.
In particular, this means that it is not possible for the total number of points to be 11.
We can show that each of the possibilities from 12 to 18 points, inclusive, is actually possible.
Therefore, $S$ cannot equal 11.

Answer: (C)

19. When $(3 + 2x + x^2)(1 + mx + m^2x^2)$ is expanded, the terms that include an $x^2$ will come from multiplying a constant with a term that includes $x^2$ or multiplying two terms that includes $x$.
In other words, the term that includes $x^2$ will be

$$3 \cdot m^2x^2 + 2x \cdot mx + x^2 \cdot 1 = 3m^2x^2 + 2mx^2 + x^2 = (3m^2 + 2m + 1)x^2$$

From the condition that the coefficient of this term equals 1, we see that $3m^2 + 2m + 1 = 1$ which gives $3m^2 + 2m = 0$ or $m(3m + 2) = 0$, which means that $m = 0$ or $m = -\frac{2}{3}$.
The sum of these possible values of $m$ is $-\frac{2}{3}$.

Answer: (B)

20. When a dot is removed from a face with an even number of dots, that face then has an odd number of dots.
When a dot is removed from a face with an odd number of dots, that face then has an even number of dots.
Initially, there are 3 faces with an even number of dots and 3 faces with an odd number of dots.
If a dot is removed from a face with an even number of dots, there are then 4 faces with an odd number of dots and 2 faces with an even number of dots. This means that the probability of rolling an odd number after a dot is removed is $\frac{4}{6}$ in this case.
If a dot is removed from a face with an odd number of dots, there are then 2 faces with an odd number of dots and 4 faces with an even number of dots. This means that the probability of rolling an odd number after a dot is removed is $\frac{2}{6}$ in this case.
Since there are $2 + 3 + 4 + 5 + 6 + 7 = 27$ dots on the faces, then the probability that a dot is removed from the face with 2 dots is $\frac{2}{27}$, from the face with 3 dots is $\frac{3}{27}$, and so on.
Thus, the probability that a dot is removed from the face with 2 dots and then an odd number is rolled is the product of the probabilities, which is $\frac{2}{27} \cdot \frac{2}{3}$, since there are now 4 odd faces and 2 even faces.
Similarly, the probability that a dot is removed from the face with 3 dots and then an odd number is rolled is $\frac{3}{27} \cdot \frac{1}{3}$.
Continuing in this way, the probability of rolling an odd number after a dot is removed is $\frac{2}{27} \cdot \frac{2}{3} + \frac{3}{27} \cdot \frac{1}{3} + \frac{4}{27} \cdot \frac{2}{3} + \frac{5}{27} \cdot \frac{1}{3} + \frac{6}{27} \cdot \frac{2}{3} + \frac{7}{27} \cdot \frac{1}{3}$.
This equals $\frac{2}{3} \cdot (\frac{2}{27} + \frac{4}{27} + \frac{6}{27}) + \frac{1}{3}(\frac{3}{27} + \frac{5}{27} + \frac{7}{27}) = \frac{2}{3} \cdot \frac{12}{27} + \frac{1}{3} \cdot \frac{15}{27} = \frac{8}{27} + \frac{5}{27} = \frac{13}{27}$.

Answer: (C)

21. If the product of three numbers $x$, 36 and $y$ is 2592, then $x \cdot 36 \cdot y = 2592$ and so $xy = \dfrac{2592}{36} = 72$.
If $x$ and $y$ are positive integers with $xy = 72$, then we have the following possibilities:

| $x$ | $y$ | $x+y$ |
|---|---|---|
| 72 | 1 | 73 |
| 36 | 2 | 38 |
| 24 | 3 | 27 |
| 18 | 4 | 22 |
| 12 | 6 | 18 |
| 9 | 8 | 17 |

We have assumed that $x > y$ since we have not assigned an order to $x$, 36 and $y$.
In the given problem, we want to put four pairs of numbers in the outer circles so that the 9 numbers are different and the sum of the 9 numbers is as large as possible.
Putting this another way, we want to choose 4 of the 6 pairs in the table above (knowing that we cannot choose the pair 36 and 2 since 36 is already in the middle circle) to make the sum as large as possible.
Since we know the sums of the pairs, we choose the pairs with the four largest sums.
This means that the sum of the 9 numbers will be $(72+1)+(24+3)+(18+4)+(12+6)+36$ which equals $73+27+22+18+36$ or 176.

Answer: (B)

22. Since $x^2+3xy+y^2 = 909$ and $3x^2+xy+3y^2 = 1287$, then

$$\begin{aligned}(x^2+3xy+y^2)+(3x^2+xy+3y^2) &= 909+1287\\ 4x^2+4xy+4y^2 &= 2196\\ x^2+xy+y^2 &= 549\end{aligned}$$

Since $x^2+3xy+y^2 = 909$ and $x^2+xy+y^2 = 549$, then

$$\begin{aligned}(x^2+3xy+y^2)-(x^2+xy+y^2) &= 909-549\\ 2xy &= 360\\ xy &= 180\end{aligned}$$

Since $x^2+3xy+y^2 = 909$ and $xy = 180$, then

$$\begin{aligned}(x^2+3xy+y^2)-xy &= 909-180\\ x^2+2xy+y^2 &= 729\\ (x+y)^2 &= 27^2\end{aligned}$$

Therefore, $x+y = 27$ or $x+y = -27$. This also shows that $x+y$ cannot equal any of 39, 29, 92, and 41.
(We can in fact solve the system of equations $x+y = 27$ and $xy = 180$ for $x$ and $y$ to show that there do exist real numbers $x$ and $y$ that are solutions to the original system of equations.)
Therefore, a possible value for $x+y$ is (A) 27.

Answer: (A)

23. *Solution 1*

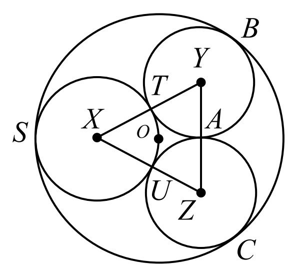
Since $f(x) = ax + b$ for all real numbers $x$, then $f(t) = at + b$ for some real number $t$.
When $t = bx + a$, we obtain $f(bx + a) = a(bx + a) + b = abx + (a^2 + b)$.
We also know that $f(bx + a) = x$ for all real numbers $x$.
This means that $abx + (a^2 + b) = x$ for all real numbers $x$ and so $(ab - 1)x + (a^2 + b) = 0$ for all real numbers $x$.
For this to be true, it must be the case that $ab = 1$ and $a^2 + b = 0$.
From the second equation $b = -a^2$ which gives $a(-a^2) = 1$ and so $a^3 = -1$, which means that $a = -1$.
Since $b = -a^2$, then $b = -1$ as well, which gives $a + b = -2$.

*Solution 2*

Since $f(x) = ax + b$ for all $x$, then when $x = a$, we obtain $f(a) = a^2 + b$.
Since $f(bx + a) = x$ for all $x$, then when $x = 0$, we obtain $f(a) = 0$.
Comparing values for $f(a)$, we obtain $a^2 + b = 0$ or $b = -a^2$.
This gives $f(x) = ax - a^2$ for all real numbers $x$ and $f(-a^2x + a) = x$ for all real numbers $x$.
Since $f(-a^2x + a) = x$ for all $x$, then when $x = -1$, we obtain $f(a^2 + a) = -1$.
Since $f(x) = ax - a^2$ for all $x$, then when $x = a^2 + a$, we obtain $f(a^2 + a) = a(a^2 + a) - a^2$.
Comparing values for $f(a^2 + a)$, we obtain $a(a^2 + a) - a^2 = -1$ or $a^3 = -1$.
Since $a$ is a real number, then $a = -1$.
Since $b = -a^2$, then $b = -1$, which gives $a + b = -2$.
Checking, we see that if $f(x) = -x - 1$, then $f(-x - 1) = -(-x - 1) - 1 = x$, as required.

ANSWER: (E)

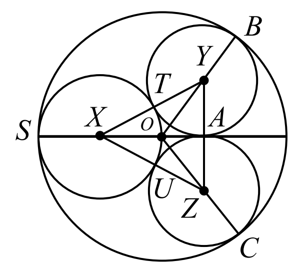
24. Suppose the centre of the largest circle is $O$.
Suppose that the circle with centre $X$ touches the largest circle at $S$ and the two circles with centres $Y$ and $Z$ at $T$ and $U$, respectively.
Suppose that the circles with centres $Y$ and $Z$ touch each other at $A$, and the largest circle at $B$ and $C$, respectively.
Join $X$ to $Y$, $X$ to $Z$, and $Y$ to $Z$.




(Note that the diagram has been re-drawn here so that the circle with centre $X$ actually appears to pass through the centre of the largest circle.)
Since the circles are tangent at points $T$ and $U$, line segments $XY$ and $XZ$ pass through $T$ and $U$, respectively.
Further, $XY = XT + TY = 1 + r$, since the circles with centres $X$ and $Y$ have radii 1 and $r$, respectively.
Similarly, $XZ = 1 + r$.
Also, $YA = ZA = YB = ZC = r$, since these are radii of the two circles.
When one circle is inside another circle, and the two circles touch at a point, then the radii of the two circles that pass through this point lie on top of each other. This is because the

circles have a common tangent at the point where they touch and this common tangent will be perpendicular to each of the radii.
Since the circle with centre $X$ touches the largest circle at $S$, then $X$ lies on $OS$.
In the largest circle, consider the diameter that passes through $X$.
Since the circle with centre $X$ passes through $O$, then the radius of the largest circle is twice that of the circle with centre $X$, or 2.
It is also the case that $XO = 1$.
Next, we join $O$ to $B$. Since the circles with centres $O$ and $Y$ touch at $B$, then $OB$ passes through $Y$. This means that $OY = OB - BY = 2 - r$. Similarly, $OZ = 2 - r$.
Further, by symmetry in the largest circle, the diameter through $X$ also passes through $A$, the point at which the two smallest circles touch:

> To see this more formally, draw the common tangent through $A$ to the circles with centres $Y$ and $Z$.
> This line is perpendicular to $YZ$, since it is tangent to both circles.
> Since $\triangle OYZ$ is isosceles with $OY = OZ$, the altitude through the midpoint $A$ of its base passes through $O$.
> Similarly, $\triangle XYZ$ is isosceles with $XY = XZ$ and so its altitude through $A$ passes through $X$.
> Since the line perpendicular to $YZ$ at $A$ passes through both $O$ and $X$, it is the diameter that passes through $X$.




Now, we consider $\triangle XYA$ and $\triangle OYA$, each of which is right-angled at $A$.
By the Pythagorean Theorem,

$$OA = \sqrt{OY^2 - YA^2} = \sqrt{(2-r)^2 - r^2} = \sqrt{4 - 4r + r^2 - r^2} = \sqrt{4 - 4r}$$

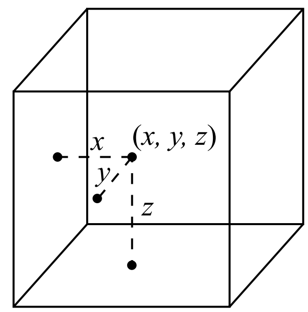
Again, using the Pythagorean Theorem,

$$\begin{aligned}
XA^2 + YA^2 &= XY^2 \\
(XO + OA)^2 + r^2 &= (1+r)^2 \\
(1 + \sqrt{4-4r})^2 &= 1 + 2r + r^2 - r^2 \\
1 + 2\sqrt{4-4r} + (4 - 4r) &= 1 + 2r \\
2\sqrt{4-4r} &= 6r - 4 \\
\sqrt{4-4r} &= 3r - 2 \\
4 - 4r &= (3r-2)^2 \qquad \text{(squaring both sides)} \\
4 - 4r &= 9r^2 - 12r + 4 \\
8r &= 9r^2
\end{aligned}$$

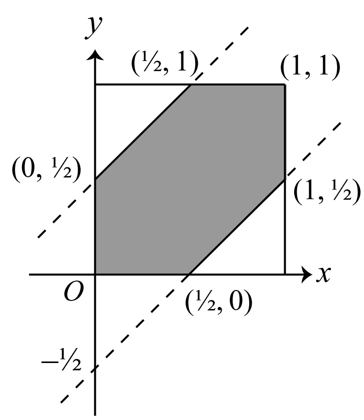
Since $r \neq 0$, then $9r = 8$ and so $r = \frac{8}{9} \approx 0.889$.
Of the given choices, this is closest to (E) 0.89.

ANSWER: (E)

25. Consider a $1 \times 1 \times 1$ cube.
We associate a triple $(x, y, z)$ of real numbers with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ and $0 \leq z \leq 1$ with a point inside this cube by letting $x$ be the perpendicular distance of a point from the left face, $y$ the perpendicular distance of a point from the front face, and $z$ the perpendicular distance from the bottom face.
We call this point $(x, y, z)$. Choosing $x$, $y$ and $z$ randomly and independently between 0 and 1 is equivalent to randomly and uniformly choosing a point $(x, y, z)$ on or inside the cube.



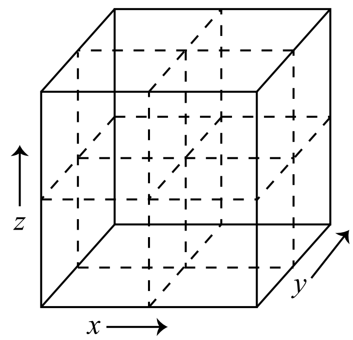

The conditions that $-\frac{1}{2} < x - y < \frac{1}{2}$ and $-\frac{1}{2} < x - z < \frac{1}{2}$ restrict the values of $x$, $y$ and $z$ that can be chosen, which translates into restricting the points inside the cube that satisfy these conditions. Hence, these restrictions determine a region inside this cube.
The probability that a point randomly chosen inside this cube satisfies the given conditions will be equal to the volume of the region defined by the conditions divided by the volume of the entire cube.
Since the volume of the cube is 1, then the probability will equal the volume of the region defined by those conditions.
Consider now the region in the $xy$-plane defined by $-\frac{1}{2} < x - y < \frac{1}{2}$.
Re-arranging these inequalities, we obtain $x - \frac{1}{2} < y < x + \frac{1}{2}$, which means that a point $(x, y)$ that satisfies these conditions lies above the line with equation $y = x - \frac{1}{2}$ and below the line with equation $y = x + \frac{1}{2}$.
Restricting to $0 \leq x \leq 1$ and $0 \leq y \leq 1$, we obtain the region shown:



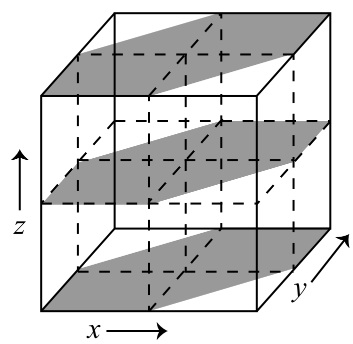

Since a point $(x, y, z)$ in the region satisifes $-\frac{1}{2} < x - y < \frac{1}{2}$, these conditions allow us to "slice" the cube from above keeping the portion that looks like the region above. The points that remain are exactly those that satisfy this condition.
Similarly, the conditions $-\frac{1}{2} < x - z < \frac{1}{2}$ give $x - \frac{1}{2} < z < x + \frac{1}{2}$, which has the same shape in the $xz$-plane.

Therefore, we can slice the cube from front to back to look like this shape. Now, we need to determine the volume of the remaining region.

To determine the volume of the region, we split the $1 \times 1 \times 1$ cube into eight cubes each measuring $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$.




When this cube is sliced by the restrictions corresponding to $x - \frac{1}{2} < y < x + \frac{1}{2}$, the back left and front right cubes on the top and bottom layers are sliced in half.



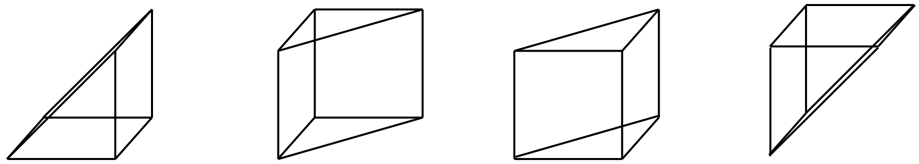

When this cube is sliced by the restrictions corresponding to $x - \frac{1}{2} < z < x + \frac{1}{2}$, the top left and bottom right cubes in the front and back are sliced in half.
The eight little cubes are sliced as follows:

| Little cube | Sliced by $x - \frac{1}{2} < y < x + \frac{1}{2}$ | Sliced by $x - \frac{1}{2} < z < x + \frac{1}{2}$ |
|---|---|---|
| Bottom front left | No | No |
| Bottom front right | Yes | Yes |
| Bottom back left | Yes | No |
| Bottom back right | No | Yes |
| Top front left | No | Yes |
| Top front right | Yes | No |
| Top back left | Yes | Yes |
| Top back right | No | No |

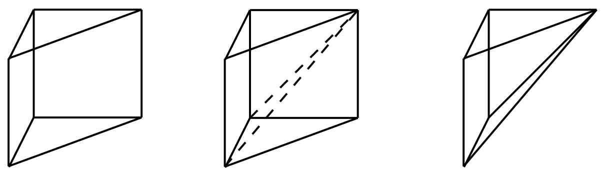
This means that we can consider the little cubes as follows:

- Bottom front left and top back right: these cubes are not sliced in either direction and so contribute $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ to the volume of the solid.

- Bottom back left, bottom back right, top front left, top front right: these cubes are sliced in half in one direction and are not sliced in the other direction, and so contribute $\frac{1}{2}$ of their volume (or $\frac{1}{16}$ each) to the solid.



- Top back left and bottom front right: Each of these cubes is sliced in half in two directions. The first slice cuts the cube into a triangular prism, whose volume is half of the volume of the little cube, or $\frac{1}{16}$. The second slice creates a square-based pyramid out of this prism. The pyramid has base with edge length $\frac{1}{2}$ and height $\frac{1}{2}$, and so has volume $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{24}$.



Therefore, the volume of the solid is $2 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + 2 \cdot \frac{1}{24} = \frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{7}{12}$.
Finally, this means that the required probability is $\frac{7}{12}$.

Answer: (B)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2019 Fermat Contest*

(Grade 11)

**Tuesday, February 26, 2019**
**(in North America and South America)**

**Wednesday, February 27, 2019**
**(outside of North America and South America)**

*Solutions*

1. Since the largest multiple of 5 less than 14 is 10 and $14 - 10 = 4$, then the remainder when 14 is divided by 5 is 4.

   Answer: (E)

2. Simplifying, we see that $20(x + y) - 19(y + x) = 20x + 20y - 19y - 19x = x + y$ for all values of $x$ and $y$.

   Answer: (B)

3. Evaluating, $8 - \dfrac{6}{4-2} = 8 - \dfrac{6}{2} = 8 - 3 = 5$.

   Answer: (A)

4. The segment of the number line between 3 and 33 has length $33 - 3 = 30$.
   Since this segment is divided into six equal parts, then each part has length $30 \div 6 = 5$.
   The segment $PS$ is made up of 3 of these equal parts, and so has length $3 \times 5 = 15$.
   The segment $TV$ is made up of 2 of these equal parts, and so has length $2 \times 5 = 10$.
   Thus, the sum of the lengths of $PS$ and $TV$ is $15 + 10$ or 25.

   Answer: (A)

5. Since 1 hour equals 60 minutes, then 20 minutes equals $\frac{1}{3}$ of an hour.
   Since Mike rides at 30 km/h, then in $\frac{1}{3}$ of an hour, he travels $\frac{1}{3} \times 30$ km $= 10$ km.

   Answer: (E)

6. Suppose that $SU = UW = WR = b$ and $PS = h$.
   Since the width of rectangle $PQRS$ is $3b$ and its height is $h$, then its area is $3bh$.
   Since $SU = b$ and the distance between parallel lines $PQ$ and $SR$ is $h$, then the area of $\triangle STU$ is $\frac{1}{2}bh$. Similarly, the area of each of $\triangle UVW$ and $\triangle WXR$ is $\frac{1}{2}bh$.
   Therefore, the fraction of the rectangle that is shaded is $\dfrac{3 \times \frac{1}{2}bh}{3bh}$ which equals $\dfrac{1}{2}$.

   Answer: (C)

7. Since Cans is north of Ernie, then Ernie cannot be the town that is the most north.
   Since Dundee is south of Cans, then Dundee cannot be the town that is the most north.
   Since Arva is south of Blythe, then Arva cannot be the town that is the most north.
   Since Arva is north of Cans, then Cans cannot be the town that is the most north.
   The only remaining possibility is that Blythe is the town that is the most north.
   The following arrangement is the unique one that satisfies the given conditions:

   Blythe
   Arva
   Cans
   Dundee
   Ernie

   Answer: (B)

8. We note that $8 \times 48 \times 81 = 2^3 \times (2^4 \times 3) \times 3^4 = 2^7 \times 3^5 = 2^2 \times 2^5 \times 3^5 = 2^2 \times (2 \times 3)^5 = 2^2 \times 6^5$.
   After $6^5$ is divided out from $8 \times 48 \times 81$, the quotient has no factors of 3 and so no further factors of 6 can be divided out.
   Therefore, the largest integer $k$ for which $6^k$ is a divisor of $8 \times 48 \times 81$ is $k = 5$.

   Answer: (C)

9. The average of $\frac{1}{8}$ and $\frac{1}{6}$ is $\dfrac{\frac{1}{8}+\frac{1}{6}}{2} = \dfrac{\frac{3}{24}+\frac{4}{24}}{2} = \frac{1}{2} \times \frac{7}{24} = \frac{7}{48}$.

Answer: (E)

10. We find the smallest such integer greater than 30 000 and the largest such integer less than 30 000 and then determine which is closest to 30 000.
Let $M$ be the smallest integer greater than 30 000 that is formed using the digits 2, 3, 5, 7, and 8, each exactly once.
Since $M$ is greater than 30 000, its ten thousands digit is at least 3.
To make $M$ as small as possible (but greater than 30 000), we set its ten thousands digit to 3.
To make $M$ as small as possible, its thousands digit should be as small as possible, and thus equals 2.
Continuing in this way, its hundreds, tens and ones digits are 578. Thus, $M = 32\,578$.
Let $m$ be the largest integer less than 30 000 that is formed using the digits 2, 3, 5, 7, and 8, each exactly once.
Since $m$ is less than 30 000, its ten thousands digit is less than 3 and must thus be 2.
To make $m$ as large as possible (but less than 30 000), its thousands digit should be as large as possible, and thus equals 8.
Continuing in this way, its hundreds, tens and ones digits are 7, 5 and 3, respectively. Thus, $m = 28\,753$.
Since $M - 30\,000 = 2578$ and $30\,000 - m = 1247$, then $m$ is closer to 30 000.
Thus, $N = m = 28\,753$. The tens digit of $N$ is 5.

Answer: (B)

11. The line with equation $y = x - 3$ has slope 1.
To find the $x$-intercept of the line with equation $y = x - 3$, we set $y = 0$ and solve for $x$ to obtain $x - 3 = 0$ or $x = 3$. Thus, line $\ell$ also has $x$-intercept 3.
Further, since the two lines are perpendicular, the slopes of the two lines have a product of $-1$, which means that the slope of $\ell$ is $-1$.
Line $\ell$ has slope $-1$ and passes through $(3, 0)$.
This means that $\ell$ has equation $y - 0 = -1(x - 3)$ or $y = -x + 3$.
Therefore, the $y$-intercept of line $\ell$ is 3.

Answer: (C)

12. Alberto answered 70% of 30 questions correctly in the first part.
Thus, Alberto answered $\frac{70}{100} \times 30 = 21$ questions correctly in the first part.
Alberto answered 40% of 50 questions correctly in the second part.
Thus, Alberto answered $\frac{40}{100} \times 50 = 20$ questions correctly in the second part.
Overall, Alberto answered $21 + 20 = 41$ of $30 + 50 = 80$ questions correctly.
This represents a percentage of $\frac{41}{80} \times 100\% = 51.25\%$.
Of the given choices, this is closest to 51%.

Answer: (D)

13. The number of minutes between 7:00 a.m. and the moment when Tanis looked at her watch was $8x$, and the number of minutes between the moment when Tanis looked at her watch and 8:00 a.m. was $7x$.
The total number of minutes between 7:00 a.m. and 8:00 a.m. is 60.
Therefore, $8x + 7x = 60$ and so $15x = 60$ or $x = 4$.
The time at that moment was $8x = 32$ minutes after 7:00 a.m., and so was 7:32 a.m. (We note that 7:32 a.m. is $28 = 7x$ minutes before 8:00 a.m.)

Answer: (C)

14. Each letter A, B, C, D, E appears exactly once in each column and each row.
The entry in the first column, second row cannot be A or E or B (the entries already present in that column) and cannot be C or A (the entries already present in that row).
Therefore, the entry in the first column, second row must be D.
This means that the entry in the first column, fourth row must be C.
The entry in the fifth column, second row cannot be D or C or A or E and so must be B.
This means that the entry in the second column, second row must be E.
Using similar arguments, the entries in the first row, third and fourth columns must be D and B, respectively.
This means that the entry in the second column, first row must be C.
Using similar arguments, the entries in the fifth row, second column must be A.
Also, the entry in the third row, second column must be D.
This means that the letter that goes in the square marked with $*$ must be B.
We can complete the grid as follows:

| | | | | |
|---|---|---|---|---|
| A | C | D | B | E |
| D | E | C | A | B |
| E | D | B | C | A |
| C | B | A | E | D |
| B | A | E | D | C |

Answer: (B)

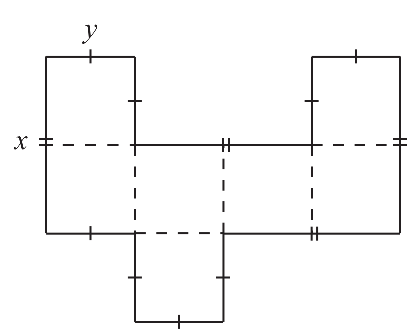
15. Since 4 balls are chosen from 6 red balls and 3 green balls, then the 4 balls could include:

- 4 red balls, or
- 3 red balls and 1 green ball, or
- 2 red balls and 2 green balls, or
- 1 red ball and 3 green balls.

There is only 1 different-looking way to arrange 4 red balls.
There are 4 different-looking ways to arrange 3 red balls and 1 green ball: the green ball can be in the 1st, 2nd, 3rd, or 4th position.
There are 6 different-looking ways to arrange 2 red balls and 2 green balls: the red balls can be in the 1st/2nd, 1st/3rd, 1st/4th, 2nd/3rd, 2nd/4th, or 3rd/4th positions.
There are 4 different-looking ways to arrange 1 red ball and 3 green balls: the red ball can be in the 1st, 2nd, 3rd, or 4th position.
In total, there are $1 + 4 + 6 + 4 = 15$ different-looking arrangements.

Answer: (A)

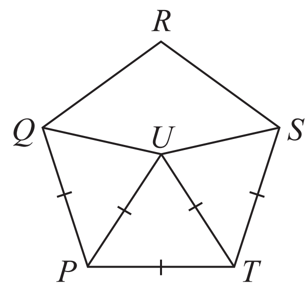
16. Since $x = 2y$, then by drawing dotted lines parallel to the line segments in the given figure, some of which start at midpoints of the current sides, we can divide the figure into 7 squares, each of which is $y$ by $y$.



Since the area of the given figure is 252, then $7y^2 = 252$ or $y^2 = 36$.
Since $y > 0$, then $y = 6$.
The perimeter of the figure consists of 16 segments of length $y$.
Therefore, the perimeter is $16 \times 6 = 96$.

Answer: (A)

17. Join $QU$ and $SU$.
Since $\triangle PUT$ is equilateral, then $PU = UT = TP$.
Since pentagon $PQRST$ is regular, then $QP = PT = TS$.
Thus, $PU = QP$ and $UT = TS$, which means that $\triangle QPU$ and $\triangle STU$ are isosceles.




Each interior angle in a regular pentagon measures $108^\circ$.
Since $\angle UPT = 60^\circ$, then $\angle QPU = \angle QPT - \angle UPT = 108^\circ - 60^\circ = 48^\circ$.
Since $\triangle QPU$ is isosceles with $QP = PU$, then $\angle PQU = \angle PUQ$.
Thus, $\angle PUQ = \frac{1}{2}(180^\circ - \angle QPU) = \frac{1}{2}(180^\circ - 48^\circ) = 66^\circ$.
By symmetry, $\angle TUS = 66^\circ$.
Finally, $\angle QUS = 360^\circ - \angle PUQ - \angle PUT - \angle TUS = 360^\circ - 66^\circ - 60^\circ - 66^\circ = 168^\circ$.

Answer: (B)

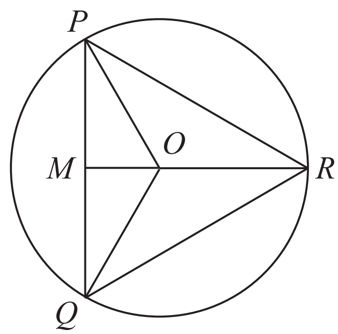
18. Let $n$ be a 7-digit positive integer made up of the digits 0 and 1 only, and that is divisible by 6.
The leftmost digit of $n$ cannot be 0, so must be 1.
Since $n$ is divisible by 6, then $n$ is even, which means that the rightmost digit of $n$ cannot be 1, and so must be 0.
Therefore, $n$ has the form $1\,pqr\,st0$ for some digits $p, q, r, s, t$ each equal to 0 or 1.
$n$ is divisible by 6 exactly when it is divisible by 2 and by 3.
Since the ones digit of $n$ is 0, then it is divisible by 2.
$n$ is divisible by 3 exactly when the sum of its digits is divisible by 3.
The sum of the digits of $n$ is $1 + p + q + r + s + t$.
Since each of $p, q, r, s, t$ is 0 or 1, then $1 \leq 1 + p + q + r + s + t \leq 6$.
Thus, $n$ is divisible by 3 exactly when $1 + p + q + r + s + t$ is equal to 3 or to 6.
That is, $n$ is divisible by 3 exactly when either 2 of $p, q, r, s, t$ are 1s or all 5 of $p, q, r, s, t$ are 1s.
There are 10 ways for 2 of these to be 1s.
These correspond to the pairs $pq, pr, ps, pt, qr, qs, qt, rs, rt, st$.
There is 1 way for all 5 of $p, q, r, s, t$ to be 1s.
Thus, there are $1 + 10 = 11$ such 7-digit integers.

Answer: (B)

19. We use the functional equation $f(2x+1) = 3f(x)$ repeatedly.
Setting $x = 1$, we get $f(3) = 3f(1) = 3 \times 6 = 18$.
Setting $x = 3$, we get $f(7) = 3f(3) = 3 \times 18 = 54$.
Setting $x = 7$, we get $f(15) = 3f(7) = 3 \times 54 = 162$.
Setting $x = 15$, we get $f(31) = 3f(15) = 3 \times 162 = 486$.
Setting $x = 31$, we get $f(63) = 3f(31) = 3 \times 486 = 1458$.

Answer: (D)

20. Suppose that a circle with centre $O$ has radius 2 and that equilateral $\triangle PQR$ has its vertices on the circle.
Join $OP$, $OQ$ and $OR$.
Join $O$ to $M$, the midpoint of $PQ$.




Since the radius of the circle is 2, then $OP = OQ = OR = 2$.
By symmetry, $\angle POQ = \angle QOR = \angle ROP$.
Since these three angles add to $360^\circ$, then $\angle POQ = \angle QOR = \angle ROP = 120^\circ$.
Since $\triangle POQ$ is isosceles with $OP = OQ$ and $M$ is the midpoint of $PQ$, then $OM$ is an altitude and an angle bisector.
Therefore, $\angle POM = \frac{1}{2}\angle POQ = 60^\circ$ which means that $\triangle POM$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Since $OP = 2$ and is opposite the $90^\circ$ angle, then $OM = 1$ and $PM = \sqrt{3}$.
Since $PM = \sqrt{3}$, then $PQ = 2PM = 2\sqrt{3}$.
Therefore, the area of $\triangle POQ$ is $\frac{1}{2} \cdot PQ \cdot OM = \frac{1}{2} \cdot 2\sqrt{3} \cdot 1 = \sqrt{3}$.
Since $\triangle POQ$, $\triangle QOR$ and $\triangle ROP$ are congruent, then they each have the same area.
This means that the area of $\triangle PQR$ is three times the area of $\triangle POQ$, or $3\sqrt{3}$.

Answer: (A)

21. *Solution 1*

We start with the ones digits.
Since $4 \times 4 = 16$, then $T = 6$ and we carry 1 to the tens column.
Looking at the tens column, since $4 \times 6 + 1 = 25$, then $S = 5$ and we carry 2 to the hundreds column.
Looking at the hundreds column, since $4 \times 5 + 2 = 22$, then $R = 2$ and we carry 2 to the thousands column.
Looking at the thousands column, since $4 \times 2 + 2 = 10$, then $Q = 0$ and we carry 1 to the ten thousands column.
Looking at the ten thousands column, since $4 \times 0 + 1 = 1$, then $P = 1$ and we carry 0 to the hundred thousands column.
Looking at the hundred thousands column, $4 \times 1 + 0 = 4$, as expected.
This gives the following completed multiplication:

$$\begin{array}{r} 1\,0\,2\,5\,6\,4 \\ \times \qquad\qquad 4 \\ \hline 4\,1\,0\,2\,5\,6 \end{array}$$

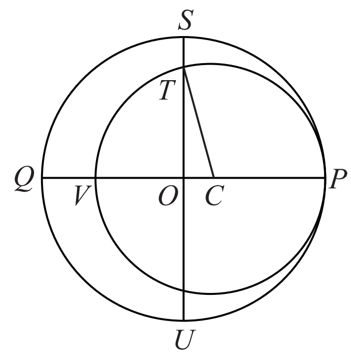
Finally, $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

*Solution 2*

Let $x$ be the five-digit integer with digits $PQRST$.
This means that $PQRST0 = 10x$ and so $PQRST4 = 10x + 4$.
Also, $4PQRST = 400\,000 + PQRST = 400\,0000 + x$.
From the given multiplication, $4(10x + 4) = 400\,000 + x$ which gives $40x + 16 = 400\,000 + x$ or $39x = 399\,984$.
Thus, $x = \dfrac{399\,984}{39} = 10\,256$.
Since $PQRST = 10\,256$, then $P + Q + R + S + T = 1 + 0 + 2 + 5 + 6 = 14$.

Answer: (A)

22. Let $D$ be the length of the diameter of the larger circle and let $d$ be the length of the diameter of the smaller circle.
Since $QP$ and $VP$ are diameters of the larger and smaller circles, then $QV = QP - VP = D - d$.
Since $QV = 9$, then $D - d = 9$.
Let $C$ be the centre of the smaller circle and join $C$ to $T$. Since $D > d$, $C$ is to the right of $O$ along $QP$.




Since $CT$ is a radius of the smaller circle, then $CT = \frac{1}{2}d$.
Also, $OC = OP - CP$. Since $OP$ and $CP$ are radii of the two circles, then $OC = \frac{1}{2}D - \frac{1}{2}d$.
Since $SO$ is a radius of the larger circle and $ST = 5$, then $TO = SO - ST = \frac{1}{2}D - 5$.
Since $QP$ and $SU$ are perpendicular, then $\triangle TOC$ is right-angled at $O$.
By the Pythagorean Theorem,

$$\begin{aligned}
TO^2 + OC^2 &= CT^2 \\
\left(\tfrac{1}{2}D - 5\right)^2 + \left(\tfrac{1}{2}D - \tfrac{1}{2}d\right)^2 &= \left(\tfrac{1}{2}d\right)^2 \\
4\left(\tfrac{1}{2}D - 5\right)^2 + 4\left(\tfrac{1}{2}D - \tfrac{1}{2}d\right)^2 &= 4\left(\tfrac{1}{2}d\right)^2 \\
(D-10)^2 + (D-d)^2 &= d^2 \\
(D-10)^2 + 9^2 &= d^2 \\
81 &= d^2 - (D-10)^2 \\
81 &= (d - (D-10))(d + (D-10)) \\
81 &= (d - D + 10)(d + (D-10)) \\
81 &= (10 - (D-d))(d + D - 10) \\
81 &= (10-9)(d + D - 10) \\
81 &= d + D - 10 \\
91 &= d + D
\end{aligned}$$

and so the sum of the diameters is 91.

Answer: (B)

23. We consider first the integers that can be expressed as the sum of exactly 4 consecutive positive integers.
The smallest such integer is $1+2+3+4 = 10$. The next smallest such integer is $2+3+4+5 = 14$.
We note that when we move from $k+(k+1)+(k+2)+(k+3)$ to $(k+1)+(k+2)+(k+3)+(k+4)$, we add 4 to the total (this equals the difference between $k+4$ and $k$ since the other three terms do not change).
Therefore, the positive integers that can be expressed as the sum of exactly 4 consecutive positive integers are those integers in the arithemetic sequence with first term 10 and common difference 4.
Since $n \leq 100$, these integers are

$$10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, 66, 70, 74, 78, 82, 86, 90, 94, 98$$

There are 23 such integers.
Next, we consider the positive integers $n \leq 100$ that can be expressed as the sum of exactly 5 consecutive positive integers.
The smallest such integer is $1 + 2 + 3 + 4 + 5 = 15$ and the next is $2 + 3 + 4 + 5 + 6 = 20$.
Using an argument similar to that from above, these integers form an arithemetic sequence with first term 15 and common difference 5.
Since $n \leq 100$, these integers are $15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95, 100$.
When we exclude the integers already listed above (30, 50, 70, 90), we obtain

$$15, 20, 25, 35, 40, 45, 55, 60, 65, 75, 80, 85, 95, 100$$

There are 14 such integers.
Next, we consider the positive integers $n \leq 100$ that can be expressed as the sum of exactly 6 consecutive positive integers.
These integers form an arithmetic sequence with first term 21 and common difference 6.
Since $n \leq 100$, these integers are $21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81, 87, 93, 99$.
When we exclude the integers already listed above (45, 75), we obtain

$$21, 27, 33, 39, 51, 57, 63, 69, 81, 87, 93, 99$$

There are 12 such integers.
Since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 105$ and this is the smallest integer that can be expressed as the sum of 14 consecutive positive integers, then no $n \leq 100$ is the sum of 14 or more consecutive positive integers. (Any sum of 15 or more consecutive positive integers will be larger than 105.)
Therefore, if an integer $n \leq 100$ can be expressed as the sum of $s \geq 4$ consecutive integers, then $s \leq 13$.
We make a table to enumerate the $n \leq 100$ that come from values of $s$ with $7 \leq s \leq 13$ that we have not yet counted:

| $s$ | Smallest $n$ | Possible $n \leq 100$ | New $n$ |
|---|---|---|---|
| 7 | 28 | $28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98$ | $28, 49, 56, 77, 84, 91$ |
| 8 | 36 | $36, 44, 52, 60, 68, 76, 84, 92, 100$ | $36, 44, 52, 68, 76, 92$ |
| 9 | 45 | $45, 54, 63, 72, 81, 90, 99$ | 72 |
| 10 | 55 | $55, 65, 75, 85, 95$ | None |
| 11 | 66 | $66, 77, 88, 99$ | 88 |
| 12 | 78 | $78, 90$ | None |
| 13 | 91 | 91 | None |

In total, there are $23 + 14 + 12 + 6 + 6 + 1 + 1 = 63$ such $n$.
What do you notice about the $n$ that cannot expressed in this way?

Answer: (B)

24. A quadratic equation has two distinct real solutions exactly when its discriminant is positive. For the quadratic equation $x^2 - (r+7)x + r + 87 = 0$, the discriminant is

$$\Delta = (r+7)^2 - 4(1)(r+87) = r^2 + 14r + 49 - 4r - 348 = r^2 + 10r - 299$$

Since $\Delta = r^2 + 10r - 299 = (r+23)(r-13)$ which has roots $r = -23$ and $r = 13$, then $\Delta > 0$ exactly when $r > 13$ or $r < -23$. (To see this, we could picture the parabola with equation $y = x^2 + 10x - 299 = (x+23)(x-13)$ and see where it lies above the $x$-axis.)

We also want both of the solutions of the original quadratic equation to be negative.
If $r > 13$, then the equation $x^2 - (r+7)x + r + 87 = 0$ is of the form $x^2 - bx + c = 0$ with each of $b$ and $c$ positive.
In this case, if $x < 0$, then $x^2 > 0$ and $-bx > 0$ and $c > 0$ and so $x^2 - bx + c > 0$.
This means that, if $r > 13$, there cannot be negative solutions.
Thus, it must be the case that $r < -23$. This does not guarantee negative solutions, but is a necessary condition.
So we consider $x^2 - (r+7)x + r + 87 = 0$ along with the condition $r < -23$.
This quadratic is of the form $x^2 - bx + c = 0$ with $b < 0$. We do not yet know whether $c$ is positive, negative or zero.
We know that this equation has two distinct real solutions.
Suppose that the quadratic equation $x^2 - bx + c = 0$ has real solutions $s$ and $t$.
This means that the factors of $x^2 - bx + c$ are $x - s$ and $x - t$.
In other words, $(x-s)(x-t) = x^2 - bx + c$.
Now,

$$(x-s)(x-t) = x^2 - tx - sx + st = x^2 - (s+t)x + st$$

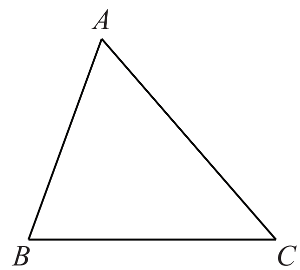
Since $(x-s)(x-t) = x^2 - bx + c$, then $x^2 - (s+t)x + st = x^2 - bx + c$ for all values of $x$, which means that $b = (s+t)$ and $c = st$.
Since $b < 0$, then it cannot be the case that $s$ and $t$ are both positive, since $b = s + t$.
If $c = 0$, then it must be the case that $s = 0$ or $t = 0$.
If $c < 0$, then it must be the case that one of $s$ and $t$ is positive and the other is negative.
If $c = st$ is positive, then $s$ and $t$ are both positive or both negative, but since $b < 0$, then $s$ and $t$ cannot both be positive, hence are both negative.
Knowing that the equation $x^2 - bx + c = 0$ has two distinct real roots and that $b < 0$, the condition that the two roots are negative is equivalent to the condition that $c > 0$.
Here, $c = r + 87$ and so $c > 0$ exactly when $r > -87$.
Finally, this means that the equation $x^2 - (r+7)x + r + 87 = 0$ has two distinct real roots which are both negative exactly when $-87 < r < -23$.
This means that $p = -87$ and $q = -23$ and so $p^2 + q^2 = 8098$.

ANSWER: (E)

25. In this solution, we will use two geometric results:

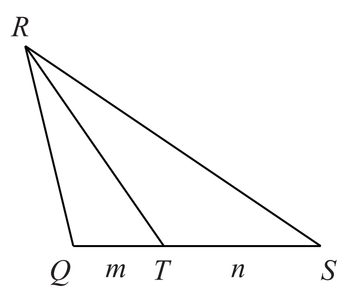
(i) The *Triangle Inequality*
This result says that, in $\triangle ABC$, each of the following inequalities is true:

$$AB + BC > AC \qquad AC + BC > AB \qquad AB + AC > BC$$




This result comes from the fact that the shortest distance between two points is the length of the straight line segment joining those two points.
For example, the shortest distance between the points $A$ and $C$ is the length of the line segment $AC$. Thus, the path from $A$ to $C$ through a point $B$ not on $AC$, which has length $AB + BC$, is longer. This explanation tells us that $AB + BC > AC$.

(ii) The *Angle Bisector Theorem*
In the given triangle, we are told that $\angle QRT = \angle SRT$. This tells us that $RT$ is an angle bisector of $\angle QRS$. The Angle Bisector Theorem says that, since $RT$ is the angle bisector of $\angle QRS$, then $\dfrac{QT}{TS} = \dfrac{RQ}{RS}$.




The Angle Bisector Theorem can be proven using the sine law:

In $\triangle RQT$, we have $\dfrac{RQ}{\sin(\angle RTQ)} = \dfrac{QT}{\sin(\angle QRT)}$.

In $\triangle RST$, we have $\dfrac{RS}{\sin(\angle RTS)} = \dfrac{TS}{\sin(\angle SRT)}$.

Dividing the first equation by the second, we obtain

$$\frac{RQ\sin(\angle RTS)}{RS\sin(\angle RTQ)} = \frac{QT\sin(\angle SRT)}{TS\sin(\angle QRT)}$$

Since $\angle QRT = \angle SRT$, then $\sin(\angle QRT) = \sin(\angle SRT)$.
Since $\angle RTQ = 180^\circ - \angle RTS$, then $\sin(\angle RTQ) = \sin(\angle RTS)$.
Combining these three equalities, we obtain $\dfrac{RQ}{RS} = \dfrac{QT}{TS}$, as required.

We now begin our solution to the problem.

By the Angle Bisector Theorem, $\dfrac{RQ}{RS} = \dfrac{QT}{TS} = \dfrac{m}{n}$.

Therefore, we can set $RQ = km$ and $RS = kn$ for some real number $k > 0$.
By the Triangle Inequality, $RQ + RS > QS$.
This is equivalent to the inequality $km + kn > m + n$ or $k(m + n) > m + n$.
Since $m + n > 0$, this is equivalent to $k > 1$.
Using the Triangle Inequality a second time, we know that $RQ + QS > RS$.
This is equivalent to $km + m + n > kn$, which gives $k(n - m) < n + m$.
Since $n > m$, then $n - m > 0$ and so we obtain $k < \dfrac{n+m}{n-m}$.
(Since we already know that $RS > RQ$, a third application of the Triangle Inequality will not give any further information. Can you see why?)

The perimeter, $p$, of $\triangle QRS$ is $RQ + RS + QS = km + kn + m + n = (k + 1)(m + n)$.
Since $k > 1$, then $p > 2(m + n)$.
Since $2(m + n)$ is an integer, then the smallest possible integer value of $p$ is $2m + 2n + 1$.
Since $k < \dfrac{n+m}{n-m}$, then $p < \left(\dfrac{n+m}{n-m} + 1\right)(n + m)$.
Since $n+m$ is a multiple of $n-m$, then $\left(\dfrac{n+m}{n-m} + 1\right)(n+m)$ is an integer, and so the largest possible integer value of $p$ is $\left(\dfrac{n+m}{n-m} + 1\right)(n + m) - 1$.
Every possible value of $p$ between $2m + 2n + 1$ and $\left(\dfrac{n+m}{n-m} + 1\right)(n + m) - 1$, inclusive, can actually be achieved. We can see this by starting with point $R$ almost at point $T$ and then continously pulling $R$ away from $QS$ while keeping the ratio $\dfrac{RQ}{RS}$ fixed until the triangle is almost flat with $RS$ along $RQ$ and $QS$.
We know that the smallest possible integer value of $p$ is $2m + 2n + 1$ and the largest possible integer value of $p$ is $\left(\dfrac{n+m}{n-m} + 1\right)(n + m) - 1$.
The number of integers in this range is

$$\left(\left(\frac{n+m}{n-m} + 1\right)(n + m) - 1\right) - (2m + 2n + 1) + 1$$

From the given information, the number of possible integer values of $p$ is $m^2 + 2m - 1$.
Therefore, we obtain the following equivalent equations:

$$\begin{aligned}
\left(\left(\frac{n+m}{n-m} + 1\right)(n + m) - 1\right) - (2m + 2n + 1) + 1 &= m^2 + 2m - 1 \\
\left(\left(\frac{n+m}{n-m} + 1\right)(n + m)\right) - (2m + 2n) &= m^2 + 2m \\
\left(\left(\frac{n+m}{n-m} + \frac{n-m}{n-m}\right)(n + m)\right) - (2m + 2n) &= m^2 + 2m \\
\left(\frac{2n}{n-m}\right)(n + m) - 2m - 2n &= m^2 + 2m \\
\frac{2n^2 + 2nm}{n-m} - 2m - 2n &= m^2 + 2m
\end{aligned}$$

$$\begin{aligned}
\frac{2n^2+2nm}{n-m} - \frac{2(n+m)(n-m)}{n-m} &= m^2+2m \\
\frac{2n^2+2nm}{n-m} - \frac{2n^2-2m^2}{n-m} &= m^2+2m \\
\frac{2m^2+2nm}{n-m} &= m^2+2m \\
\frac{2m+2n}{n-m} &= m+2 \qquad (\text{since } m \neq 0) \\
2m+2n &= (m+2)(n-m) \\
2m+2n &= nm+2n-m^2-2m \\
0 &= nm-m^2-4m \\
0 &= m(n-m-4)
\end{aligned}$$

Since $m>0$, then $n-m-4=0$ and so $n-m=4$.

For an example of such a triangle, suppose that $m=2$ and $n=6$.

Here, $\dfrac{n+m}{n-m}=2$ and so the minimum possible perimeter is $2n+2m+1=17$ and the maximum possible perimeter is $\left(\dfrac{n+m}{n-m}+1\right)(n+m)-1=23$.

The number of integers between 17 and 23, inclusive, is 7, which equals $m^2+2m-1$ or $2^2+2(2)-1$, as expected.

Answer: (A)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2018 Fermat Contest*

(Grade 11)

**Tuesday, February 27, 2018**
**(in North America and South America)**

**Wednesday, February 28, 2018**
**(outside of North America and South America)**

*Solutions*

1. Evaluating,

$$\begin{aligned} 2016 - 2017 + 2018 - 2019 + 2020 &= 2016 + (2018 - 2017) + (2020 - 2019) \\ &= 2016 + 1 + 1 \\ &= 2018 \end{aligned}$$

Answer: (D)

2. Since the maximum temperature was $14°\text{C}$ and the minimum temperature was $-11°\text{C}$, then the range of temperatures was $14°\text{C} - (-11°\text{C}) = 25°\text{C}$.

Answer: (B)

3. The expression $(3x + 2y) - (3x - 2y)$ is equal to $3x + 2y - 3x + 2y$ which equals $4y$.
When $x = -2$ and $y = -1$, this equals $4(-1)$ or $-4$.

Answer: (A)

4. The fraction $\frac{5}{7}$ is between 0 and 1.
The fraction $\frac{28}{3}$ is equivalent to $9\frac{1}{3}$ and so is between 9 and 10.
Therefore, the integers between these two fractions are $1, 2, 3, 4, 5, 6, 7, 8, 9$, of which there are 9.

Answer: (B)

5. If $\heartsuit = 1$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 1 \times 1 \times 1 = 1$, which is not possible since $\nabla$ and $\heartsuit$ must be different positive integers.
If $\heartsuit = 2$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 2 \times 2 \times 2 = 8$, which is possible.
If $\heartsuit = 3$, then $\nabla = \heartsuit \times \heartsuit \times \heartsuit = 3 \times 3 \times 3 = 27$, which is not possible since $\nabla$ is less than 20.
If $\heartsuit$ is greater than 3, then $\nabla$ will be greater than 27 and so $\heartsuit$ cannot be greater than 3.
Thus, $\heartsuit = 2$ and so $\nabla = 8$.
This means that $\nabla \times \nabla = 8 \times 8 = 64$.

Answer: (D)

6. Since $\angle QRT = 158°$, then $\angle QRP = 180° - \angle QRT = 180° - 158° = 22°$.
Since $\angle PRS = \angle QRS$ and $\angle QRP = \angle PRS + \angle QRS$, then $\angle QRS = \frac{1}{2}\angle QRP = \frac{1}{2}(22°) = 11°$.
Since $\triangle QSR$ is right-angled at $Q$, then $\angle QSR = 180° - 90° - \angle QRS = 90° - 11° = 79°$.

Answer: (E)

7. Since Bev has driven 312 km and still has 858 km left to drive, the distance from Waterloo to Marathon is $312 \text{ km} + 858 \text{ km} = 1170 \text{ km}$.
The halfway point of the drive is $\frac{1}{2}(1170 \text{ km}) = 585 \text{ km}$ from Waterloo.
To reach this point, she still needs to drive $585 \text{ km} - 312 \text{ km} = 273 \text{ km}$.

Answer: (B)

8. A line segment joining two points is parallel to the $x$-axis exactly when the $y$-coordinates of the two points are equal.
Here, this means that $2k + 1 = 4k - 5$ and so $6 = 2k$ or $k = 3$.
(We can check that when $k = 3$, the coordinates of the points are $(3, 7)$ and $(8, 7)$.)

Answer: (A)

9. Since the area of rectangle $PQRS$ is 180 and $SR = 15$, then $PS = \dfrac{180}{15} = 12$.
Since $PS = 12$ and $US = 4$, then $PU = PS - US = 12 - 4 = 8$.
Since $\triangle PUT$ is right-angled at $U$, then by the Pythagorean Theorem,

$$TU = \sqrt{PT^2 - PU^2} = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$$

since $TU > 0$.
In $\triangle PTS$, we can consider base $PS$ and height $TU$.
Therefore, its area is $\frac{1}{2}(PS)(TU) = \frac{1}{2}(12)(6) = 36$.

Answer: (B)

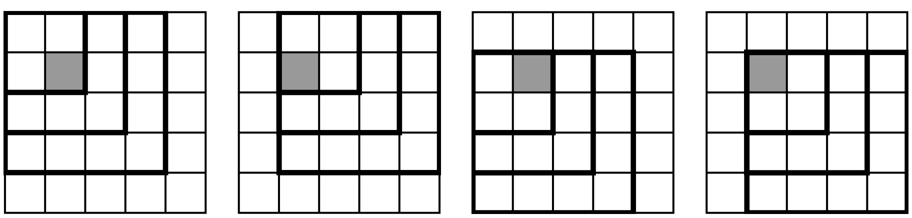
10. For any real number $x$ not equal to 0, $x^2 > 0$.
Since $-1 < x < 0$, then $x^2 < (-1)^2 = 1$, and so $0 < x^2 < 1$.
Of the given points, only $C$ is between 0 and 1.

Answer: (C)

11. Since $\frac{5}{6}$ of the balls are white and the remainder of the balls are red, then $\frac{1}{6}$ of the balls are red.
Since the 8 red balls represent $\frac{1}{6}$ of the total number of balls and $\frac{5}{6} = 5 \cdot \frac{1}{6}$, then the number of white balls is $5 \cdot 8 = 40$.

Answer: (C)

12. There is 1 square that is $1 \times 1$ that contains the shaded square (namely, the square itself).
There are 4 squares of each of the sizes $2 \times 2$, $3 \times 3$ and $4 \times 4$ that contain the shaded square.



Finally, there is 1 square that is $5 \times 5$ that contains the shaded square (namely, the $5 \times 5$ grid itself).
In total, there are thus $1 + 4 + 4 + 4 + 1 = 14$ squares that contain the shaded $1 \times 1$ square.

Answer: (E)

13. We would like to find the first time after 4:56 where the digits are consecutive digits in increasing order.
It would make sense to try 5:67, but this is not a valid time.
Similarly, the time cannot start with 6, 7, 8 or 9.
No time starting with 10 or 11 starts with consecutive increasing digits.
Starting with 12, we obtain the time 12:34. This is the first such time.
We need to determine the length of time between 4:56 and 12:34.
From 4:56 to 11:56 is 7 hours, or $7 \times 60 = 420$ minutes.
From 11:56 to 12:00 is 4 minutes.
From 12:00 to 12:34 is 34 minutes.
Therefore, from 4:56 to 12:34 is $420 + 4 + 34 = 458$ minutes.

Answer: (A)

14. The line with equation $y = x$ has slope 1 and passes through $(0, 0)$.
When this line is translated, its slope does not change.
When this line is translated 3 units to the right and 2 units down, every point on the line is translated 3 units to the right and 2 units down. Thus, the point $(0, 0)$ moves to $(3, -2)$.
Therefore, the new line has slope 1 and passes through $(3, -2)$.
Thus, its equation is $y - (-2) = 1(x - 3)$ or $y + 2 = x - 3$ or $y = x - 5$.
The $y$-intercept of this line is $-5$.

Answer: (C)

15. Each entry in the grid must be a divisor of the product of the numbers in its row and the product of the numbers in its column.

| | | | |
|---|---|---|---|
| | | | 56 |
| | | | 135 |
| | $N$ | | 48 |
| 21 | 108 | 160 | |

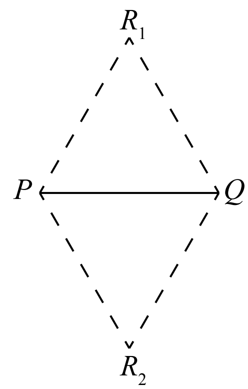
Only two of the products are multiples of 5, namely 160 and 135.
This means that the 5 must go in the second row and third column.
From this, we can see that the product of the other two numbers in the second row is $\frac{135}{5} = 27$.
Since all of the entries are between 1 and 9, then the remaining two numbers in this row must be 3 and 9.
Since 9 is not a divisor of 21, then 9 must be in the middle column.
This means that the product of the remaining numbers in the middle column is $\frac{108}{9} = 12$.
This means that the remaining digits in the middle column are 3 and 4, or 2 and 6. (These are the only factor pairs of 12 from the list of possible entries.)
Since 3 already occurs in the second row, then the entries in the second column must be 2 and 6.
Since 6 is not a divisor of 56, then 6 cannot go in the first row.
This means that 6 goes in the third row and so $N = 6$.
We can complete the grid as follows:

| | | | |
|---|---|---|---|
| 7 | 2 | 4 | 56 |
| 3 | 9 | 5 | 135 |
| 1 | 6 | 8 | 48 |
| 21 | 108 | 160 | |

Answer: (D)

16. *Solution 1*

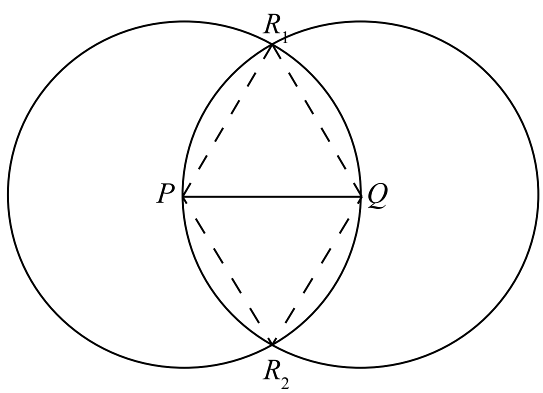
If point $R$ is placed so that $PQ = QR = PR$, then the resulting $\triangle PQR$ is equilateral.
Since points $P$ and $Q$ are fixed, then there are two possible equilateral triangles with $PQ$ as a side – one on each side of $PQ$.




One way to see this is to recognize that there are two possible lines through $P$ that make an angle of $60^\circ$ with $PQ$.

*Solution 2*

Consider the line segment $PQ$. Draw a circle with centre $P$ that passes through $Q$ and a circle with centre $Q$ that passes through $P$.




Suppose that the point $R$ satisfies $PQ = QR = PR$.
Since $PQ = QR$, then $P$ and $R$ are the same distance from $Q$, so $R$ lies on the circle with centre $Q$ that passes through $P$.
Since $PQ = PR$, then $R$ lies on the circle with centre $P$ that passes through $Q$.
In other words, point $R$ is on both circles in the diagram.
Since these two circles intersect in exactly two points, then there are two possible locations for $R$.

Answer: (C)

17. The side length of the square is 2 and $M$ and $N$ are midpoints of sides.
Thus, $SM = MR = QN = NR = 1$.
Using the Pythagorean Theorem in $\triangle PSM$, we get $PM = \sqrt{PS^2 + SM^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$ since $PM > 0$.
Similarly, $PN = \sqrt{5}$.
Using the Pythagorean Theorem in $\triangle MNR$, we get $MN = \sqrt{MR^2 + NR^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$ since $MN > 0$.
Using the cosine law in $\triangle PMN$, we get

$$\begin{aligned} MN^2 &= PM^2 + PN^2 - 2(PM)(PN)\cos(\angle MPN) \\ 2 &= 5 + 5 - 2(\sqrt{5})(\sqrt{5})\cos(\angle MPN) \\ 2 &= 10 - 10\cos(\angle MPN) \\ 10\cos(\angle MPN) &= 8 \\ \cos(\angle MPN) &= \tfrac{8}{10} = \tfrac{4}{5} \end{aligned}$$

Answer: (A)

18. Suppose that $\sqrt{7 + \sqrt{48}} = m + \sqrt{n}$.
Squaring both sides, we obtain $7 + \sqrt{48} = (m + \sqrt{n})^2$.
Since $(m + \sqrt{n})^2 = m^2 + 2m\sqrt{n} + n$, then $7 + \sqrt{48} = (m^2 + n) + 2m\sqrt{n}$.
Let's make the assumption that $m^2 + n = 7$ and $2m\sqrt{n} = \sqrt{48}$.
Squaring both sides of the second equation, we obtain $4m^2n = 48$ or $m^2n = 12$.
So we have $m^2 + n = 7$ and $m^2n = 12$.
By inspection, we might see that $m = 2$ and $n = 3$ is a solution.
If we didn't see this by inspection, we could note that $n = 7 - m^2$ and so $m^2(7 - m^2) = 12$ or $m^4 - 7m^2 + 12 = 0$.
Factoring, we get $(m^2 - 3)(m^2 - 4) = 0$.
Since $m$ is an integer, then $m^2 \neq 3$.
Thus, $m^2 = 4$ which gives $m = \pm 2$. Since $m$ is a positive integer, then $m = 2$.
When $m = 2$, we get $n = 7 - m^2 = 3$.
Therefore, $m = 2$ and $n = 3$, which gives $m^2 + n^2 = 13$.
We note that $m + \sqrt{n} = 2 + \sqrt{3}$ and that $(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} = 7 + \sqrt{48}$, as required. This means that, while the assumption we made at the beginning was not fully general, it did give us an answer to the problem.

Answer: (E)

19. *Solution 1*
Over the first 3 minutes of the race, Peter ran 48 m farther than Radford. Here is why:

> We note that at a time of 0 minutes, Radford was at the 30 m mark.
> If Radford ran $d$ m over these 3 minutes, then he will be at the $(d+30)$ m mark after 3 minutes.
> Since Peter is 18 m ahead of Radford after 3 minutes, then Peter is at the $(d+30+18)$ m mark.
> This means that, in 3 minutes, Peter ran $(d + 48)$ m which is 48 m farther than Radford's $d$ m.

Since each runs at a constant speed, then Peter runs $\dfrac{48\text{ m}}{3\text{ min}} = 16$ m/min faster than Radford.
Since Peter finishes the race after 7 minutes, then Peter runs for another 4 minutes.

Over these 4 minutes, he runs $(4\text{ min}) \cdot (16\text{ m/min}) = 64$ m farther than Radford.
After 3 minutes, Peter was 18 m ahead of Radford.
Therefore, after 7 minutes, Peter is $18\text{ m} + 64\text{ m} = 82$ m farther ahead than Radford, and so Radford is 82 m from the finish line.

*Solution 2*
As in Solution 1, suppose that Radford ran $d$ m over the first 3 minutes and so Peter runs $(d + 48)$ m over these first 3 minutes.
Since Peter's speed is constant, he runs $\frac{4}{3}(d + 48)$ m over the next 4 minutes.
Since Radford's speed is constant, he runs $\frac{4}{3}d$ over these next 4 minutes.
This means that Peter runs a total of $(d + 48)\text{ m} + \frac{4}{3}(d + 48)\text{ m} = \frac{7}{3}(d + 48)$ m.
Also, Radford is $(30 + d + \frac{4}{3}d)$ m from the start after 7 minutes, since he had a 30 m head start.
Thus, Radford's distance from the finish line, in metres, is

$$\tfrac{7}{3}(d + 48) - (30 + d + \tfrac{4}{3}d) = \tfrac{7}{3}d + 112 - 30 - d - \tfrac{4}{3}d = 82$$

Answer: (D)

20. We count the positive integers $x$ for which the product

$$(x - 2)(x - 4)(x - 6) \cdots (x - 2016)(x - 2018) \qquad (*)$$

equals 0 and is less than 0 separately.
The product $(*)$ equals 0 exactly when one of the factors equals 0.
This occurs exactly when $x$ equals one of $2, 4, 6, \ldots, 2016, 2018$.
These are the even integers from 2 to 2018, inclusive, and there are $\frac{2018}{2} = 1009$ such integers.
The product $(*)$ is less than 0 exactly when none of its factors is 0 and an odd number of its factors are negative.
We note further that for every integer $x$ we have

$$x - 2 > x - 4 > x - 6 > \cdots > x - 2016 > x - 2018$$

When $x = 1$, we have $x - 2 = -1$ and so all 1009 factors are negative, making $(*)$ negative.
When $x = 3$, we have $x - 2 = 1$, $x - 4 = -1$ and all of the other factors are negative, giving 1008 negative factors and so a positive product.
When $x = 5$, we have $x - 2 = 3$, $x - 4 = 1$ and $x - 6 = -1$ and all of the other factors are negative, giving 1007 negative factors and so a negative product.
This pattern continues giving a negative value for $(*)$ for $x = 1, 5, 9, 13, \ldots, 2013, 2017$.
There are $1 + \frac{2017-1}{4} = 505$ such values (starting at 1, these occur every 4 integers).
When $x \geq 2019$, each factor is positive and so $(*)$ is positive.
Therefore, there are $1009 + 505 = 1514$ positive integers $x$ for which the product $(*)$ is less than or equal to 0.

We should further justify the pattern that we found above.
Suppose that $x = 4n + 1$ for $n = 0, 1, 2, \ldots, 504$. (These are the integers $1, 5, 9, 13, \ldots, 2017$.)
Then $(*)$ becomes

$$(4n - 1)(4n - 3)(4n - 5) \cdots (4n - 2015)(4n - 2017)$$

The $2k$th factor is $(n - (4k - 1))$ and so when $n = 4k$, this factor is positive and the next factor is negative.
In other words, when $n = 2k$, the first $2k$ of these factors are positive and the remaining factors

are negative.
In other words, when $n = 2k$, there is an even number of positive factors.
Since the total number of factors is 1009, which is odd, then the number of negative factors is odd and so the product is negative.
In a similar way, we can show that if $x = 4n + 3$ for $n = 0, 1, 2, \ldots, 503$ (these are the integers $3, 7, 11, \ldots, 2011, 2015$), then the product is positive.
This confirms that this pattern continues.

Answer: (C)

21. Substituting $n = 1$ into the equation $a_{n+1} = a_n + a_{n+2} - 1$ gives $a_2 = a_1 + a_3 - 1$.
Since $a_1 = x$ and $a_3 = y$, then $a_2 = x + y - 1$.
Rearranging the given equation, we obtain $a_{n+2} = a_{n+1} - a_n + 1$ for each $n \geq 1$.
Thus,

$$\begin{aligned}
a_4 &= a_3 - a_2 + 1 = y - (x + y - 1) + 1 = 2 - x \\
a_5 &= a_4 - a_3 + 1 = (2 - x) - y + 1 = 3 - x - y \\
a_6 &= a_5 - a_4 + 1 = (3 - x - y) - (2 - x) + 1 = 2 - y \\
a_7 &= a_6 - a_5 + 1 = (2 - y) - (3 - x - y) + 1 = x \\
a_8 &= a_7 - a_6 + 1 = x - (2 - y) + 1 = x + y - 1
\end{aligned}$$

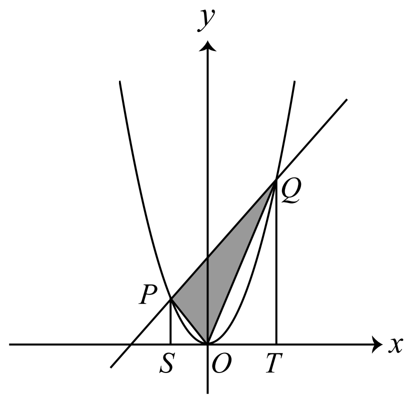
Since $a_7 = a_1$ and $a_8 = a_2$ and each term in the sequence depends only on the previous two terms, then the sequence repeats each 6 terms.
(For example, $a_9 = a_8 - a_7 + 1 = a_2 - a_1 + 1 = a_3$ and so on.)
Now

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = x + (x + y - 1) + y + (2 - x) + (3 - x - y) + (2 - y) = 6$$

which means that the sum of each successive group of 6 terms is also equal to 6.
We note that $2016 = 6 \cdot 336$ and so the 2016th term is the end of a group of 6 terms, which means that the sum of the first 2016 terms in the sequence is $6 \cdot 336 = 2016$.
Finally, $a_{2017} = a_1 = x$ and $a_{2018} = a_2 = x + y - 1$.
Thus, the sum of the first 2018 terms is $2016 + x + (x + y - 1) = 2x + y + 2015$.

Answer: (E)

22. First, we find the coordinates of the points $P$ and $Q$ in terms of $k$ by finding the points of intersection of the graphs with equations $y = x^2$ and $y = 3kx + 4k^2$.
Equating values of $y$, we obtain $x^2 = 3kx + 4k^2$ or $x^2 - 3kx - 4k^2 = 0$.
We rewrite the left side as $x^2 - 4kx + kx + (-4k)(k) = 0$ which allows us to factor and obtain $(x - 4k)(x + k) = 0$ and so $x = 4k$ or $x = -k$.
Since $k > 0$, $P$ is in the second quadrant and $Q$ is in the first quadrant, then $P$ has $x$-coordinate $-k$ (which is negative).
Since $P$ lies on $y = x^2$, then its $y$-coordinate is $(-k)^2 = k^2$ and so the coordinates of $P$ are $(-k, k^2)$.
Since $Q$ lies on $y = x^2$ and has $x$-coordinate $4k$, then its $y$-coordinate is $(4k)^2 = 16k^2$ and so the coordinates of $Q$ are $(4k, 16k^2)$.
Our next step is to determine the area of $\triangle OPQ$ in terms of $k$.
Since the area of $\triangle OPQ$ is numerically equal to 80, this will give us an equation for $k$ which will allow us to find the slope of the line.
To find the area of $\triangle OPQ$ in terms of $k$, we drop perpendiculars from $P$ and $Q$ to $S$ and $T$, respectively, on the $x$-axis.




The area of $\triangle OPQ$ is equal to the area of trapezoid $PSTQ$ minus the areas of $\triangle PSO$ and $\triangle QTO$.
Trapezoid $PSTQ$ has parallel bases $SP$ and $TQ$ and perpendicular height $ST$.
Since the coordinates of $P$ are $(-k, k^2)$, then $SP = k^2$.
Since the coordinates of $Q$ are $(4k, 16k^2)$, then $TQ = 16k^2$.
Also, $ST = 4k - (-k) = 5k$.
Thus, the area of trapezoid $PSTQ$ is $\frac{1}{2}(SP + TQ)(ST) = \frac{1}{2}(k^2 + 16k^2)(5k) = \frac{85}{2}k^3$.
$\triangle PSO$ is right-angled at $S$ and so has area $\frac{1}{2}(SP)(SO) = \frac{1}{2}(k^2)(0 - (-k)) = \frac{1}{2}k^3$.
$\triangle QTO$ is right-angled at $T$ and so has area $\frac{1}{2}(TQ)(TO) = \frac{1}{2}(16k^2)(4k - 0) = 32k^3$.
Combining these, the area of $\triangle POQ$ equals $\frac{85}{2}k^3 - \frac{1}{2}k^3 - 32k^3 = 10k^3$.
Since this area equals 80, then $10k^3 = 80$ or $k^3 = 8$ and so $k = 2$.
This means that the slope of the line is $3k$ which equals 6.

Answer: (D)

23. We are told that $(x - a)(x - 6) + 3 = (x + b)(x + c)$ for all real numbers $x$.
In particular, this equation holds when $x = 6$.
Substituting $x = 6$ gives $(6 - a)(6 - 6) + 3 = (6 + b)(6 + c)$ or $3 = (6 + b)(6 + c)$.
Since $b$ and $c$ are integers, then $6 + b$ and $6 + c$ are integers, which means that $6 + b$ is a divisor of 3.
Therefore, the possible values of $6 + b$ are $3, 1, -1, -3$.
These yield values for $b$ of $-3, -5, -7, -9$.
We need to confirm that each of these values for $b$ gives integer values for $a$ and $c$.
If $b = -3$, then $6 + b = 3$. The equation $3 = (6 + b)(6 + c)$ tells us that $6 + c = 1$ and so $c = -5$.
When $b = -3$ and $c = -5$, the original equation becomes $(x - a)(x - 6) + 3 = (x - 3)(x - 5)$.
Expanding the right side gives $(x-a)(x-6)+3 = x^2-8x+15$ and so $(x-a)(x-6) = x^2-8x+12$.
The quadratic $x^2 - 8x + 12$ factors as $(x-2)(x-6)$ and so $a = 2$ and this equation is an identity that is true for all real numbers $x$.
Similarly, if $b = -5$, then $c = -3$ and $a = 2$. (This is because $b$ and $c$ are interchangeable in the original equation.)
Also, if $b = -7$, then $c = -9$ and we can check that $a = 10$.
Similarly, if $b = -9$, then $c = -7$ and $a = 10$.
Therefore, the possible values of $b$ are $b = -3, -5, -7, -9$.
The sum of these values is $(-3) + (-5) + (-7) + (-9) = -24$.

Answer: (B)

24. We use the notation "$a/b/c$" to mean $a$ pucks in one bucket, $b$ pucks in a second bucket, and $c$ pucks in the third bucket, ignoring the order of the buckets.

Yellow buckets

1/0/0: With 1 puck to distribute, the distribution will always be 1/0/0.

Blue buckets

Since there are 2 pucks to distribute amongst the three buckets, then there is a total of $3^2 = 9$ ways of doing this. (There are 3 possibilities for each of 2 pucks.)

2/0/0: There are 3 ways in which the 2 pucks end up in the same bucket (1 way for each of the 3 buckets). The probability of this is $\frac{3}{9}$.

1/1/0: Thus, there are $9 - 3 = 6$ ways in which the 2 pucks are distributed with 1 puck in each of two buckets and 0 pucks in the third bucket. The probability of this is $\frac{6}{9}$.

Red buckets

With 3 pucks to distribute amongst 3 buckets, there is a total of $3^3 = 27$ ways.

3/0/0: There are 3 ways in which the 3 pucks end up in the same bucket (1 way for each of the 3 buckets). The probability of this is $\frac{3}{27}$.

1/1/1: There are $3 \cdot 2 \cdot 1 = 6$ ways in which the 3 pucks end up with one in each bucket (3 choices of bucket for the first puck, 2 for the second, and 1 for the third). The probability of this is $\frac{6}{27}$.

2/1/0: Thus, there are $27 - 3 - 6 = 18$ ways in which the 2 pucks are distributed with 2 pucks in 1 bucket, 1 puck in 1 bucket, and 0 pucks in 1 bucket. The probability of this is $\frac{18}{27}$.

Green buckets

With 4 pucks to distribute amongst 3 buckets, there is a total of $3^4 = 81$ ways.

4/0/0: There are 3 ways in which the 4 pucks end up in the same bucket (1 way for each of the 3 buckets). The probability of this is $\frac{3}{81}$.

3/1/0: There are $4 \times 3 \times 2 = 24$ ways in which the pucks end up with 3 in one bucket and 1 in another (4 ways to choose a puck to be on its own, 3 ways to choose the bucket for this puck, and 2 ways to choose the bucket for the 3 pucks). The probability of this is $\frac{24}{81}$.

2/1/1: There are 6 ways of choosing two of the four pucks. (If they are labelled W, X, Y, Z, then we can choose WX, WY, WZ, XY, XZ, or YZ.) There are $6 \times 3 \times 2 = 36$ ways in which the pucks can be distributed with 2 pucks in one bucket and 1 puck in each of the remaining buckets (6 ways to choose the 2 pucks that go together, 3 ways to choose the bucket, and 2 ways in which the remaining 2 pucks can be assigned to the remaining 2 buckets). The probability of this is $\frac{36}{81}$.

2/2/0: Thus, there are $81 - 3 - 24 - 36 = 18$ ways in which the 4 pucks are distributed with 2 pucks in each of 2 buckets. The probability of this is $\frac{18}{81}$.

For a green bucket to contain more pucks than each of the other 11 buckets, the following possible distributions exist with probabilities as shown:

| Green | Red | Blue | Yellow | Probability |
|---|---|---|---|---|
| 4/0/0 ($p = \frac{3}{81}$) | Any ($p = 1$) | Any ($p = 1$) | Any ($p = 1$) | $\frac{3}{81}$ |
| 3/1/0 ($p = \frac{24}{81}$) | Any but 3/0/0 ($p = 1 - \frac{3}{27}$) | Any ($p = 1$) | Any ($p = 1$) | $\frac{24}{81} \cdot \frac{24}{27}$ |
| 2/1/1 ($p = \frac{36}{81}$) | 1/1/1 ($p = \frac{6}{27}$) | 1/1/0 ($p = \frac{6}{9}$) | Any ($p = 1$) | $\frac{36}{81} \cdot \frac{6}{27} \cdot \frac{6}{9}$ |

A 2/2/0 distribution of pucks among green buckets cannot satisfy the desired conditions because there would be not be a single green bucket with more pucks in it than any other bucket, as there would be two green buckets containing the same number of pucks.

Therefore, the overall probability is $\frac{3}{81} + \frac{24}{81} \cdot \frac{24}{27} + \frac{36}{81} \cdot \frac{6}{27} \cdot \frac{6}{9} = \frac{1}{27} + \frac{8}{27} \cdot \frac{8}{9} + \frac{4}{9} \cdot \frac{2}{9} \cdot \frac{2}{3} = \frac{9}{243} + \frac{64}{243} + \frac{16}{243} = \frac{89}{243}$.

Answer: (B)

25. Suppose that $D$ is a digit and $k$ is a positive integer. Then

$$D_{(k)} = \underbrace{DD\cdots DD}_{k \text{ times}} = D \cdot \underbrace{11\cdots 11}_{k \text{ times}} = D \cdot \tfrac{1}{9} \cdot \underbrace{99\cdots 99}_{k \text{ times}} = D \cdot \tfrac{1}{9} \cdot (1\underbrace{00\cdots 00}_{k \text{ times}} - 1) = D \cdot \tfrac{1}{9} \cdot (10^k - 1)$$

Therefore, the following equations are equivalent:

$$\begin{aligned}
P_{(2k)} - Q_{(k)} &= \left(R_{(k)}\right)^2 \\
P \cdot \tfrac{1}{9} \cdot (10^{2k} - 1) - Q \cdot \tfrac{1}{9} \cdot (10^k - 1) &= \left(R \cdot \tfrac{1}{9} \cdot (10^k - 1)\right)^2 \\
P \cdot \tfrac{1}{9} \cdot (10^{2k} - 1) - Q \cdot \tfrac{1}{9} \cdot (10^k - 1) &= R^2 \cdot \tfrac{1}{81} \cdot (10^k - 1)^2 \\
9P \cdot (10^{2k} - 1) - 9Q \cdot (10^k - 1) &= R^2 \cdot (10^k - 1)^2 \\
9P \cdot (10^k - 1)(10^k + 1) - 9Q \cdot (10^k - 1) &= R^2 \cdot (10^k - 1)^2 \\
9P \cdot (10^k + 1) - 9Q &= R^2 \cdot (10^k - 1) \qquad \text{(since } 10^k - 1 \neq 0\text{)} \\
9P \cdot 10^k + 9K - 9Q &= R^2 \cdot 10^k - R^2 \\
9P - 9Q + R^2 &= 10^k(R^2 - 9P)
\end{aligned}$$

We consider three cases: $3 \leq k \leq 2018$, $k = 1$, and $k = 2$.

<u>Case 1: $3 \leq k \leq 2018$</u>
Suppose that $R^2 - 9P \neq 0$.
Since $k \geq 3$, then $10^k(R^2 - 9P) > 1000$ if $R^2 - 9P > 0$ and $10^k(R^2 - 9P) < -1000$ if $R^2 - 9P < 0$.
Since $P, Q, R$ are digits, then $9P - 9Q + R^2$ is at most $9(9) - 9(0) + 9^2 = 162$ and $9P - 9Q + R^2$ is at least $9(0) - 9(9) + 0^2 = -81$.
This means that if $R^2 - 9P \neq 0$, we cannot have $9P - 9Q + R^2 = 10^k(R^2 - 9P)$ since the possible values do not overlap.
So if $3 \leq k \leq 2018$, we must have $R^2 - 9P = 0$ and so $9P - 9Q + R^2 = 0$.
If $R^2 = 9P$, then $R^2$ is a multiple of 3 and so $R$ is a multiple of 3.
Since $R$ is a positive digit, then $R = 3$ or $R = 6$ or $R = 9$.
If $R = 3$, then $9P = R^2 = 9$ and so $P = 1$.
Since $9P - 9Q + R^2 = 0$, then $9Q = 9(1) + 9 = 18$ and so $Q = 2$.
If $R = 6$, then $9P = R^2 = 36$ and so $P = 4$.
Since $9P - 9Q + R^2 = 0$, then $9Q = 9(4) + 36 = 72$ and so $Q = 8$.
If $R = 9$, then $9P = R^2 = 81$ and so $P = 9$.
Since $9P - 9Q + R^2 = 0$, then $9Q = 9(9) + 81 = 162$ and so $Q$ cannot be a digit.
Therefore, in the case where $3 \leq k \leq 2018$, we obtain the quadruples $(P, Q, R, k) = (1, 2, 3, k)$ and $(P, Q, R, k) = (4, 8, 9, k)$.
Since there are $2018 - 3 + 1 = 2016$ possible values of $k$, then we have $2 \cdot 2016 = 4032$ quadruples so far.

<u>Case 2: $k = 1$</u>
Here, the equation $9P - 9Q + R^2 = 10^k(R^2 - 9P)$ becomes $9P - 9Q + R^2 = 10R^2 - 90P$ or $99P = 9R^2 + 9Q$ or $11P = R^2 + Q$.
For each possible value of $P$ from 1 to 9, we determine the possible values of $Q$ and $R$ by looking for perfect squares that are at most 9 less than $11P$.
$P = 1$: Here, $11P = 11$ which is close to squares 4 and 9. We obtain $(R, Q) = (2, 7), (3, 2)$.
$P = 2$: Here, $11P = 22$ which is close to the square 16. We obtain $(R, Q) = (4, 6)$.
$P = 3$: Here, $11P = 33$ which is close to the square 25. We obtain $(R, Q) = (5, 8)$.
$P = 4$: Here, $11P = 44$ which is close to the square 36. We obtain $(R, Q) = (6, 8)$.
$P = 5$: Here, $11P = 55$ which is close to the square 49. We obtain $(R, Q) = (7, 6)$.

$P = 6$: Here, $11P = 66$ which is close to the square 64. We obtain $(R, Q) = (8, 2)$.
$P = 7$: There are no perfect squares between 68 and 76, inclusive.
$P = 8$: Here, $11P = 88$ which is close to the square 81. We obtain $(R, Q) = (9, 7)$.
$P = 9$: There are no perfect squares between 90 and 98, inclusive.
Since $k = 1$ in each of these cases, we obtain an additional 8 quadruples.

Case 3: $k = 2$
Here, the equation $9P - 9Q + R^2 = 10^k(R^2 - 9P)$ becomes $9P - 9Q + R^2 = 100R^2 - 900P$ or $909P = 99R^2 + 9Q$ or $101P = 11R^2 + Q$.
As $P$ ranges from 1 to 9, the possible values of $101P$ are $101, 202, 303, 404, 505, 606, 707, 808, 909$.
As $R$ ranges from 1 to 9, the possible values of $11R^2$ are $11, 44, 99, 176, 275, 396, 539, 704, 891$.
The pairs of integers in the first and second lists that differ by at most 9 are

(i) 101 and 99 (which give $(P, Q, R) = (1, 2, 3)$),

(ii) 404 and 396 (which give $(P, Q, R) = (4, 8, 6)$), and

(iii) 707 and 704 (which give $(P, Q, R) = (7, 3, 8)$).

Since $k = 2$ in each of these cases, we obtain an additional 3 quadruples.

In total, there are thus $N = 4032 + 8 + 3 = 4043$ quadruples.
The sum of the digits of $N$ is $4 + 0 + 4 + 3 = 11$.

Answer: (C)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2017 Fermat Contest*

**(Grade 11)**

**Tuesday, February 28, 2017**
**(in North America and South America)**

**Wednesday, March 1, 2017**
**(outside of North America and South America)**

*Solutions*

1. Evaluating, $6 \times 2017 - 2017 \times 4 = 2017(6 - 4) = 2017(2) = 4034$.

   ANSWER: (D)

2. In the diagram, there are 7 rows in which there are shaded squares, and there are 7 shaded squares in each row.
   Thus, there are $7 \cdot 7 = 49$ shaded squares.

   ANSWER: (E)

3. The sum of 2, 3 and 6 is $2 + 3 + 6 = 11$. Their product is $2 \cdot 3 \cdot 6 = 36$.

   ANSWER: (C)

4. Since 300 litres drains in 25 hours, then the rate at which water is leaving the tank equals $\dfrac{300 \text{ L}}{25 \text{ h}}$ or 12 L/h.

   ANSWER: (A)

5. The graph of $y = -2x^2 + 4$ is a parabola.
   Since the coefficient of $x^2$ is negative, the parabola opens downwards.
   Since the constant term is positive, the $y$-intercept of the parabola (that is, the value of $y$ when $x = 0$) is positive.
   Of the given graphs, only (D) has these properties. (Since the coefficient of $x$ is 0 in the given equation, then the graph should be symmetric about the $y$-axis, as the graph in (D) is.)

   ANSWER: (D)

6. Since the average of 5 and 9 is $\dfrac{5+9}{2} = 7$, then the averages of 5 and $x$ and of $x$ and 9 must be 10 and 12.
   In other words, $\dfrac{5+x}{2}$ and $\dfrac{x+9}{2}$ are equal to 10 and 12 in some order.
   Adding these, we obtain $\dfrac{5+x}{2} + \dfrac{x+9}{2} = 10 + 12$ or $\dfrac{14+2x}{2} = 22$ and so $7 + x = 22$ or $x = 15$.
   (We could have also noted that $\dfrac{5+x}{2} < \dfrac{x+9}{2}$ since $5 < 9$, and so $\dfrac{x+9}{2} = 12$.)

   ANSWER: (B)

7. Since $x = 1$ is a solution of the equation $x^2 + ax + 1 = 0$, then $1^2 + a(1) + 1 = 0$ or $2 + a = 0$ and so $a = -2$.

   ANSWER: (E)

8. Since $\dfrac{1}{2n} + \dfrac{1}{4n} = \dfrac{2}{4n} + \dfrac{1}{4n} = \dfrac{3}{4n}$, then the given equation becomes $\dfrac{3}{4n} = \dfrac{3}{12}$ or $4n = 12$.
   Thus, $n = 3$.

   ANSWER: (E)

9. We need to determine the time 100 hours before 5 p.m. Friday.
Since there are 24 hours in 1 day and since $100 = 4(24) + 4$, then 100 hours is equal to 4 days plus 4 hours.
Starting at 5 p.m. Friday, we move 4 hours back in time to 1 p.m. Friday and then an additional 4 days back in time to 1 p.m. Monday.
Thus, Kamile turned her computer on at 1 p.m. Monday.

Answer: (D)

10. Suppose that the integers $a < b < c < n$ have $a + b + c + n = 100$.
Since $a < b < c < n$, then $a + b + c + n < n + n + n + n = 4n$. Thus, $100 < 4n$ and so $n > 25$.
Since $n$ is an integer, then $n$ is at least 26.
Could $n$ be 26? In this case, we would have $a + b + c = 100 - 26 = 74$.
If $n = 26$, then $a + b + c$ is at most $23 + 24 + 25 = 72$, which means that we cannot have $a + b + c = 74$.
Therefore, $n$ cannot be 26.
Could $n$ be 27? In this case, we would have $a + b + c = 100 - 27 = 73$.
Here, we could have $a + b + c = 23 + 24 + 26 = 73$, and so $n = 27$ is possible, which means that the smallest possible value of $n$ is 27. (There are other values of $a, b, c$ that work with $n = 27$ as well.)

Answer: (D)

11. Each student brought exactly one of an apple, a banana, and an orange.
Since 20% of the students brought an apple and 35% brought a banana, then the percentage of students who brought an orange is $100\% - 20\% - 35\% = 45\%$.
Therefore, the 9 students who brought an orange represent 45% of the class.
This means that 1 student represents $45\% \div 9 = 5\%$ of the class.
Thus, the class has $100\% \div 5\% = 20$ students in it.

Answer: (D)

12. The question is equivalent to asking how many three-digit positive integers beginning with 2 are larger than 217.
These integers are 218 through 299 inclusive.
There are $299 - 217 = 82$ such integers.

Answer: (B)

13. The line through $R(2, 4)$ and $Q(4, 0)$ has slope $\frac{4-0}{2-4} = -2$.
Since it passes through $(4, 0)$, this line has equation $y - 0 = -2(x - 4)$ or $y = -2x + 8$.
The line with equation $y = -2x + 8$ has $y$-intercept 8, and so the coordinates of $P$ are $(0, 8)$.
Now, $\triangle OPQ$ is right-angled at $O$ and so its area is $\frac{1}{2}(OQ)(OP) = \frac{1}{2}(4)(8) = 16$.

Answer: (E)

14. The expression

$$\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\left(1+\tfrac{1}{5}\right)\left(1+\tfrac{1}{6}\right)\left(1+\tfrac{1}{7}\right)\left(1+\tfrac{1}{8}\right)\left(1+\tfrac{1}{9}\right)$$

is equal to

$$\left(\tfrac{3}{2}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{5}\right)\left(\tfrac{7}{6}\right)\left(\tfrac{8}{7}\right)\left(\tfrac{9}{8}\right)\left(\tfrac{10}{9}\right)$$

which equals

$$\frac{3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}$$

Removing common factors from the numerator and denominator, we obtain $\frac{10}{2}$ or 5.

Answer: (A)

15. Since $\angle XMZ = 30^\circ$, then $\angle XMY = 180^\circ - \angle XMZ = 180^\circ - 30^\circ = 150^\circ$.
Since the angles in $\triangle XMY$ add to $180^\circ$, then

$$\angle YXM = 180^\circ - \angle XYZ - \angle XMY = 180^\circ - 15^\circ - 150^\circ = 15^\circ$$

(Alternatively, since $\angle XMZ$ is an exterior angle of $\triangle XMY$, then $\angle XMZ = \angle YXM + \angle XYM$ which also gives $\angle YXM = 15^\circ$.)
Since $\angle XYM = \angle YXM$, then $\triangle XMY$ is isosceles with $MX = MY$.
But $M$ is the midpoint of $YZ$, and so $MY = MZ$.
Since $MX = MY$ and $MY = MZ$, then $MX = MZ$.
This means that $\triangle XMZ$ is isosceles with $\angle XZM = \angle ZXM$.
Therefore, $\angle XZY = \angle XZM = \frac{1}{2}(180^\circ - \angle XMZ) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.

Answer: (A)

16. Since $x + 2y = 30$, then

$$\begin{aligned}
\frac{x}{5} + \frac{2y}{3} + \frac{2y}{5} + \frac{x}{3} &= \frac{x}{5} + \frac{2y}{5} + \frac{x}{3} + \frac{2y}{3} \\
&= \frac{1}{5}x + \frac{1}{5}(2y) + \frac{1}{3}x + \frac{1}{3}(2y) \\
&= \frac{1}{5}(x + 2y) + \frac{1}{3}(x + 2y) \\
&= \frac{1}{5}(30) + \frac{1}{3}(30) \\
&= 6 + 10 \\
&= 16
\end{aligned}$$

Answer: (B)

17. Suppose that the base of the prism is $b$ cm by $w$ cm and the height of the prism is $h$ cm.
Since Aaron has 144 cubes with edge length 1 cm, then the volume of the prism is 144 cm$^3$, and so $bwh = 144$.
Since the perimeter of the base is 20 cm, then $2b + 2w = 20$ or $b + w = 10$.
Since $b$ and $w$ are positive integers, then we can make a chart of the possible combinations of $b$ and $w$ and the resulting values of $h = \dfrac{144}{bw}$, noting that since $b$ and $w$ are symmetric, then we can assume that $b \leq w$:

| $b$ | $w$ | $h$ |
|---|---|---|
| 1 | 9 | 16 |
| 2 | 8 | 9 |
| 3 | 7 | $\frac{48}{7}$ |
| 4 | 6 | 6 |
| 5 | 5 | $\frac{144}{25}$ |

Since $h$ must itself be a positive integer, then the possible values of $h$ are 16, 9 and 6.
The sum of the possible heights is 16 cm + 9 cm + 6 cm = 31 cm.

Answer: (A)

18. For any positive real number $x$, $\lfloor x \rfloor$ equals the largest integer less than or equal to $x$ and so $\lfloor x \rfloor \leq x$.
In particular, $\lfloor x \rfloor \cdot x \leq x \cdot x = x^2$.
Thus, if $\lfloor x \rfloor \cdot x = 36$, then $36 \leq x^2$.
Since $x > 0$, then $x \geq 6$.
In fact, if $x = 6$, then $\lfloor x \rfloor = \lfloor 6 \rfloor = 6$ and so $\lfloor x \rfloor \cdot x = x^2 = 36$. Therefore, $x = 6$. (Note that if $x > 6$, then $\lfloor x \rfloor \cdot x > 6 \cdot 6 = 36$.)
Also, since $\lfloor y \rfloor \cdot y = 71$, then $y^2 \geq 71$.
Since $y > 0$, then $y \geq \sqrt{71} \approx 8.43$.
Since $y \geq \sqrt{71} \approx 8.43$, then $\lfloor y \rfloor \geq 8$.
Suppose that $\lfloor y \rfloor = 8$.
In this case, $y = \dfrac{71}{\lfloor y \rfloor} = \dfrac{71}{8} = 8.875$. Note that if $y = \dfrac{71}{8}$, then $\lfloor y \rfloor = 8$, so $y = \dfrac{71}{8}$ is a solution.
(In fact, it is the only solution with $y > 0$. Can you see why?)
Therefore, $x + y = 6 + \dfrac{71}{8} = \dfrac{119}{8}$.

Answer: (B)

19. If $a > 0$, the distance from the vertical line with equation $x = a$ to the $y$-axis is $a$.
If $a < 0$, the distance from the vertical line with equation $x = a$ to the $y$-axis is $-a$.
In each case, there are exactly two points on the vertical line with equation $x = a$ that are also a distance of $a$ or $-a$ (as appropriate) from the $x$-axis: $(a, a)$ and $(a, -a)$. These points lie on the horizontal lines with equations $y = a$ and $y = -a$, respectively.
(If $a = 0$, the line $x = a$ coincides with the $y$-axis and the unique point on this line that is equidistant from the coordinate axes is the origin $(0, 0)$ which does not lie on the line with equation $3x + 8y = 24$.)
If the point $(a, a)$ lies on the line $3x + 8y = 24$, then $3a + 8a = 24$ or $a = \frac{24}{11}$.
If the point $(a, -a)$ lies on the line $3x + 8y = 24$, then $3a - 8a = 24$ or $a = -\frac{24}{5}$.
The sum of these values of $a$ is $\frac{24}{11} + \left(-\frac{24}{5}\right) = \frac{120-264}{55} = -\frac{144}{55}$.

Answer: (B)

20. Since $m$ and $n$ are positive integers with $n > 1$ and $m^n = 2^{25} \times 3^{40}$, then 2 and 3 are prime factors of $m$ (since they are prime factors of $m^n$) and must be the only prime factors of $m$ (since if there were other prime factors of $m$, then there would be other prime factors of $m^n$).
Therefore, $m = 2^a \times 3^b$ for some positive integers $a$ and $b$ and so $m^n = (2^a \times 3^b)^n = 2^{an} \times 3^{bn}$.
Since $m^n = 2^{25} \times 3^{40}$, then we must have $an = 25$ and $bn = 40$.
Since $a, b, n$ are positive integers, then $n$ is a common divisor of 25 and 40.
Since $n > 1$, then $n = 5$, which means that $a = 5$ and $b = 8$.
In this case, $m = 2^5 \times 3^8 = 32 \times 6561 = 209\,952$, which gives $m + n = 209\,952 + 5 = 209\,957$.

Answer: (C)

21. Since $WXYZ$ is a four-digit positive integer, then $WXYZ \leq 9999$. (In fact $WXYZ$ cannot be this large since all of its digits must be different.)
Since $WXYZ \leq 9999$, then $TWUYV \leq 2(9999) = 19998$.
Since $T \neq 0$, then $T = 1$.
Next, we note that the "carry" from any column to the next cannot be larger than 1. (Since $Z \leq 9$, then $Z + Z \leq 18$ and so the carry from the ones column to the tens column is 0 or 1. Similarly, since $Y + Y \leq 18$, then the largest sum of the digits plus carry in the tens column is 19 and so the maximum carry to the hundreds column is 1. This reasoning continues in the columns to the left.)
Thus, we make a chart of possible digits $d$ and the resulting units digit in the sum from $d + d$ with and without a carry of 1:

| $d$ | Units digit of $d + d$ with no carry | Units digit of $d + d$ with carry of 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 2 | 3 |
| 2 | 4 | 5 |
| 3 | 6 | 7 |
| 4 | 8 | 9 |
| 5 | 0 | 1 |
| 6 | 2 | 3 |
| 7 | 4 | 5 |
| 8 | 6 | 7 |
| 9 | 8 | 9 |

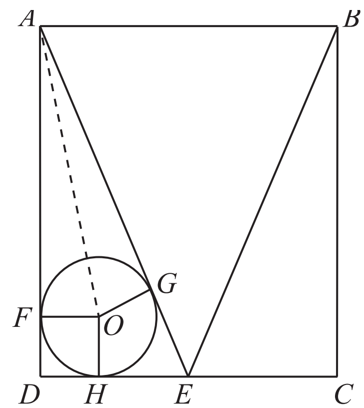
We use this table to first determine the digits $W$ and $Y$.
Since the digits in the thousands column are all the same, then the digit $W$ must be 9, since it must be at least 5 to produce a carry to the ten thousands column. We note further that this means that $X \geq 5$ to produce a carry into this column.
Also, the digit $Y$ must equal 0 (since the digits $T, U, V, W, X, Y, Z$ are different).
This means that there is no carry from the ones column to the tens column.
We summarize what we know so far:

$$\begin{array}{cccccc} & & 9 & X & 0 & Z \\ + & & 9 & X & 0 & Z \\ \hline & 1 & 9 & U & 0 & V \end{array}$$

and $X \geq 5$ and $Z \leq 4$.
Since $T = 1$ and $W = 9$, then $Z$ can be 2, 3 or 4, and $X$ can be 5, 6, 7, or 8.
Note that if $X = 5$, then we have $U = 0 = Y$, which is not possible, so $X \neq 5$.
If $Z = 2$, then $V = 4$. In this case, we cannot have $X = 6$ (which would give $U = 2 = Z$) or $X = 7$ (which would give $U = 4 = V$) and so $X = 8$, which gives $U = 6$.
If $Z = 3$, then $V = 6$. In this case, $X$ cannot equal 6 or 8 and so $X = 7$ (which gives $U = 4$).
If $Z = 4$, then $V = 8$. In this case, $X$ cannot equal 7 or 8 and so $X = 6$ (which gives $U = 2$).
In summary, there are 3 possible values for $U$, namely, 2, 4 and 6.
We can check that the sums $9802 + 9802 = 19604$ and $9703 + 9703 = 19406$ and $9604 + 9604 = 19208$ all satisfy the original problem.

Answer: (C)

22. We slice the cylinder, cone and sphere using a vertical plane that passes through the centres of the top and bottom faces of the cylinder and through the centre of the sphere.
The resulting cross-sections of the cylinder, cone and sphere are a rectangle, triangle and circle, respectively.
Since the sphere is touching the cylinder and the cone, then by slicing the cylinder in this way, the resulting circle is tangent to two sides of the rectangle (at $F$ and $H$) and a side of the triangle (at $G$).
Join $O$ to $F$, $G$ and $H$. Since radii are perpendicular to tangents at the resulting points of tangency, then $OF$ is perpendicular to $AD$, $OG$ is perpendicular to $AE$, and $OH$ is perpendicular to $DE$.




Let the radius of the sphere (now a circle) be $r$. Then $OF = OG = OH = r$.
Since the radius of the cylinder is 12, then $DE = 12$.
Since the height of the cylinder is 30, then $AD = 30$.
Since $FOHD$ has right angles at $F$, $D$ and $H$, then it must have four right angles, and so is a rectangle.
Since $OF = OH = r$, then $FOHD$ is in fact a square with $DH = DF = r$.
Since $DE = 12$ and $DH = r$, then $EH = 12 - r$.
Since $AD = 30$ and $DF = r$, then $AF = 30 - r$.
Since $AG$ and $AF$ are tangents to the circle from the same point, then $AG = AF = 30 - r$.
(To see this, note that $\triangle AFO$ and $\triangle AGO$ are both right-angled, have a common side $AO$ and equal sides $FO$ and $GO$, which means that they are congruent.)
Similarly, $EG = EH = 12 - r$.
Finally, $AE = AG + GE$.
By the Pythagorean Theorem in $\triangle ADE$, $AE = \sqrt{12^2 + 30^2} = \sqrt{1044}$.
Thus, $\sqrt{1044} = (30 - r) + (12 - r)$ and so $2r = 42 - \sqrt{1044}$ or $r = 21 - \frac{1}{2}\sqrt{1044} \approx 4.8445$.
Of the given choices, this is closest to 4.84.

ANSWER: (A)

23. Since $a$ is a positive integer and $a + \dfrac{b}{c}$ is a positive integer, then $\dfrac{b}{c}$ is a positive integer. In other words, $b$ is a multiple of $c$.
Similarly, since $\dfrac{a}{c} + b$ is a positive integer and $b$ is a positive integer, then $a$ is a multiple of $c$.
Thus, we can write $a = Ac$ and $b = Bc$ for some positive integers $A$ and $B$.
Therefore, $a + \dfrac{b}{c} = 101$ becomes $Ac + B = 101$ and $\dfrac{a}{c} + b = 68$ becomes $A + Bc = 68$.
Adding these new equations gives $Ac + B + A + Bc = 101 + 68$ or $A(c + 1) + B(c + 1) = 169$ and so $(A + B)(c + 1) = 169$.
Since $(A + B)(c + 1) = 169$, then $c + 1$ is a divisor of 169.
Since $169 = 13^2$, then the positive divisors of 169 are 1, 13, 169.
Since $A, B, c$ are positive integers, then $A + B \geq 2$ and $c + 1 \geq 2$.
Since neither $A + B$ nor $c + 1$ can equal 1, then $A + B = c + 1 = 13$.
Finally, $\dfrac{a + b}{c} = \dfrac{Ac + Bc}{c} = A + B = 13$ and so $k = 13$.

ANSWER: (A)

24. We label the 8 teams as F, G, H, J, K, L, M, N.
We first determine the total number of games played.
Since each pair of teams plays exactly one game, then each team plays 7 games (one against each of the other 7 teams). Since there are 8 teams, then it seems as if there are $8 \cdot 7 = 56$ games, except that each game has been counted twice in this total (since, for example, we have counted G playing K and K playing G). Thus, there are in fact $\frac{8 \cdot 7}{2} = 28$ games played.
Since there are 28 games played and there are 2 equally likely outcomes for each game, then there are $2^{28}$ possible combinations of outcomes. (We can consider that the games are numbered from 1 to 28 and that F plays G in game 1, F plays H in game 2, and so on. A possible combination of outcomes for the tournament can be thought of as a "word" with 28 letters, the first letter being F or G (depending on the winner of the first game), the second letter being F or H (depending on the winner of the second game), and so on. There are two choices for each letter in the word, and so $2^{28}$ possible words.)
To determine the probability that every team loses at least one game and every team wins at least one game, we determine the probability that there is a team that loses 0 games or a team that wins 0 games and subtract this probability from 1.
Since we know the total number of possible combinations of outcomes, we determine the probability by counting the number of combinations of outcomes in which there is a team that loses 0 games (that is, wins all of its games) or a team that wins 0 games (that is, loses all of its games), or both.
To determine the number of combinations of outcomes in which there is a team that wins all of its games, we note that there are 8 ways to choose this team. Once a team is chosen (we call this team X), the results of the 7 games played by X are determined (X wins all of these) and the outcomes of the remaining $28 - 7 = 21$ games are undetermined.
Since there are two possible outcomes for each of these 21 undetermined games, then there are $8 \cdot 2^{21}$ combinations of outcomes in which there is a team that wins all of its games. (Note that there cannot be two teams that win all of their games, since these two teams have to play a game.) Similarly, there are $8 \cdot 2^{21}$ combinations of outcomes in which there is a team that loses all of its games. (Can you see why?)
Before arriving at our conclusion, we note that there might be combinations of outcomes that are included in both of these counts. That is, there might be combinations of outcomes in which there is a team that wins all of its games *and* in which there is a team that loses all of its games.
Since this total has been included in both sets of $8 \cdot 2^{21}$ combinations of outcomes, we need to determine this total and subtract it once to leave these combinations included exactly once in our total.
To determine the number of combinations of outcomes in this case, we choose a team (X) to win all of its games and a team (Y) to lose all of its games.
Once X is chosen, the outcomes of its 7 games are all determined (X wins).
Once Y is chosen, the outcomes of its 6 additional games are all determined (Y loses these 6 games plus the game with X that has already been determined).
The outcomes of the remaining $28 - 7 - 6 = 15$ games are undetermined.
Therefore, the number of combinations of outcomes is $8 \cdot 7 \cdot 2^{15}$ since there are 8 ways of choosing X, and then 7 ways of choosing Y (any team but X), and then $2^{15}$ combinations of outcomes for the undetermined games.
Thus, there are $8 \cdot 2^{21} + 8 \cdot 2^{21} - 8 \cdot 7 \cdot 2^{15}$ combinations of outcomes in which either one team loses 0 games or one team wins 0 games (or both).

Therefore, the probability that one team loses 0 games or one team wins 0 games is

$$\frac{8\cdot 2^{21}+8\cdot 2^{21}-8\cdot 7\cdot 2^{15}}{2^{28}}=\frac{2^{15}(8\cdot 2^{6}+8\cdot 2^{6}-8\cdot 7)}{2^{28}}=\frac{2^{3}\cdot 2^{6}+2^{3}\cdot 2^{6}-2^{3}\cdot 7}{2^{13}}=\frac{2^{6}+2^{6}-7}{2^{10}}$$

This means that the probability that every team loses at least one game and wins at least one game is $1-\frac{64+64-7}{1024}=1-\frac{121}{1024}=\frac{903}{1024}$.

Answer: (D)

25. Suppose that $r=\sqrt{\frac{\sqrt{53}}{2}+\frac{3}{2}}$.
Thus, $r^2=\frac{\sqrt{53}}{2}+\frac{3}{2}$ and so $2r^2=\sqrt{53}+3$ or $2r^2-3=\sqrt{53}$.
Squaring both sides again, we obtain $(2r^2-3)^2=53$ or $4r^4-12r^2+9=53$ which gives $4r^4-12r^2-44=0$ or $r^4-3r^2-11=0$ or $r^4=3r^2+11$.

Suppose next that

$$r^{100}=2r^{98}+14r^{96}+11r^{94}-r^{50}+ar^{46}+br^{44}+cr^{40} \qquad (*)$$

for some positive integers $a,b,c$.
Since $r\neq 0$, we can divide by $r^{40}$ to obtain

$$r^{60}=2r^{58}+14r^{56}+11r^{54}-r^{10}+ar^{6}+br^{4}+c$$

Now using the relationship $r^4=3r^2+11$, we can see that

$$\begin{aligned} r^{60}-2r^{58}-14r^{56}-11r^{54} &= r^{54}(r^6-2r^4-14r^2-11)\\ &= r^{54}(r^2(3r^2+11)-2r^4-14r^2-11)\\ &= r^{54}(3r^4+11r^2-2r^4-14r^2-11)\\ &= r^{54}(r^4-3r^2-11)\\ &= r^{54}(0)\\ &= 0 \end{aligned}$$

Therefore, the equation $(*)$ is equivalent to the much simpler equation

$$r^{10}=ar^6+br^4+c$$

Next, we express $r^{10}$ and $r^6$ as combinations of $r^2$ and constant terms. (To do this, we will need to express $r^8$ in this way too.)

$$r^6=r^2r^4=r^2(3r^2+11)=3r^4+11r^2=3(3r^2+11)+11r^2=20r^2+33$$
$$r^8=r^2r^6=r^2(20r^2+33)=20r^4+33r^2=20(3r^2+11)+33r^2=93r^2+220$$
$$r^{10}=r^2r^8=r^2(93r^2+220)=93r^4+220r^2=93(3r^2+11)+220r^2=499r^2+1023$$

Therefore, the equation

$$r^{10}=ar^6+br^4+c$$

is equivalent to

$$499r^2+1023=a(20r^2+33)+b(3r^2+11)+c$$

Rearranging, we obtain

$$0=r^2(20a+3b-499)+(33a+11b+c-1023)$$

Therefore, if $20a + 3b = 499$ and $33a + 11b + c = 1023$, then the equation is satisfied. (It also turns out that if the equation is satisfied, then it must be the case that $20a + 3b = 499$ and $33a + 11b + c = 1023$. This is because $r^2$ is an irrational number.)
So the original problem is equivalent to finding positive integers $a, b, c$ with $20a+3b = 499$ and $33a + 11b + c = 1023$.
We proceed by finding pairs $(a, b)$ of positive integers that satisfy $20a + 3b = 499$ and then checking to see if the value of $c = 1023 - 33a - 11b$ is positive. Since we need to find one triple $(a, b, c)$ of positive integers, we do not have to worry greatly about justifying that we have all solutions at any step.
Since $20a$ has a ones digit of 0 and $20a + 3b = 499$, then the ones digit of $3b$ must be 9, which means that the ones digit of $b$ must be 3.
If $b = 3$, we obtain $20a = 499 - 3b = 490$ and so $a$ is not an integer.
If $b = 13$, we obtain $20a = 499 - 3b = 460$ and so $a = 23$.
Note that, from $(a, b) = (23, 13)$, we can obtain additional solutions by noticing that $20(3) = 3(20)$ and so if we decrease $a$ by 3 and increase $b$ by 20, the sum $20a + 3b$ does not change.
However, it turns out that if $(a, b) = (23, 13)$, then $c = 1023 - 33(23) - 11(13) = 121$.
Since we are only looking for a unique triple $(a, b, c)$, then $(a, b, c) = (23, 13, 121)$.
Finally, $a^2 + b^2 + c^2 = 23^2 + 13^2 + 121^2 = 15\,339$.

Answer: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2016 Fermat Contest*

(Grade 11)

**Wednesday, February 24, 2016**
**(in North America and South America)**

**Thursday, February 25, 2016**
**(outside of North America and South America)**

*Solutions*

1. Since $x = 3$ and $y = 2x$, then $y = 2 \cdot 3 = 6$.
   Since $y = 6$ and $z = 3y$, then $z = 3 \cdot 6 = 18$.

   Answer: (D)

2. A square-based pyramid has 8 edges: 4 edges that form the square base and 1 edge that joins each of the four vertices of the square base to the top vertex.

   Answer: (C)

3. Evaluating, $\dfrac{20 + 16 \times 20}{20 \times 16} = \dfrac{20 + 320}{320} = \dfrac{340}{320} = \dfrac{17}{16}$.
   Alternatively, we could notice that each of the numerator and denominator is a multiple of 20, and so $\dfrac{20 + 16 \times 20}{20 \times 16} = \dfrac{20(1 + 16)}{20 \times 16} = \dfrac{1 + 16}{16} = \dfrac{17}{16}$.

   Answer: (E)

4. The 7th oblong number is the number of dots in retangular grid of dots with 7 columns and 8 rows.
   Thus, the 7th oblong number is $7 \times 8 = 56$.

   Answer: (C)

5. *Solution 1*
   Let the coordinates of $R$ be $(a, b)$.
   Since $Q$ is the midpoint of $P$ and $R$, then the difference between the $x$-coordinates of $Q$ and $P$ equals the difference between the $x$-coordinates of $R$ and $Q$.
   In other words, $a - 4 = 4 - 1$ and so $a = 7$.
   Similarly, $b - 7 = 7 - 3$ and so $b = 11$.
   Thus, the coordinates of $R$ are $(7, 11)$.

   *Solution 2*
   Let the coordinates of $R$ be $(a, b)$.
   The midpoint of $P(1, 3)$ and $R(a, b)$ has coordinates $\left(\frac{1}{2}(1 + a), \frac{1}{2}(3 + b)\right)$.
   Since $Q(4, 7)$ is the midpoint of $PR$, then $4 = \frac{1}{2}(1 + a)$ (which gives $7 = 1 + a$ or $a = 7$) and $14 = \frac{1}{2}(3 + b)$ (which gives $14 = 3 + b$ or $b = 11$).
   Therefore, the coordinates of $R$ are $(7, 11)$.

   Answer: (B)

6. Each week, Carrie sends 5 messages to her brother on each of 2 days, for a total of 10 messages.
   Each week, Carrie sends 2 messages to her brother on each of the remaining 5 days, for a total of 10 messages.
   Therefore, Carrie sends $10 + 10 = 20$ messages per week.
   In four weeks, Carrie sends $4 \cdot 20 = 80$ messages.

   Answer: (D)

7. Evaluating, $(-2)^3 - (-3)^2 = -8 - 9 = -17$.

   Answer: (A)

8. Since $\sqrt{25 - \sqrt{n}} = 3$, then $25 - \sqrt{n} = 9$.
   Thus, $\sqrt{n} = 16$ and so $n = 16^2 = 256$.

   Answer: (E)

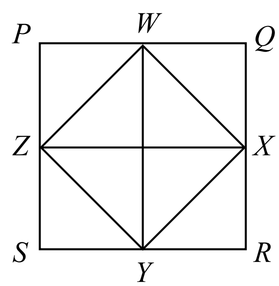
9. *Solution 1*
Since $x\%$ of 60 is 12, then $\dfrac{x}{100} \cdot 60 = 12$ or $x = \dfrac{12 \cdot 100}{60} = 20$.
Therefore, 15% of $x$ is 15% of 20, or $0.15 \cdot 20 = 3$.

*Solution 2*
Since $x\%$ of 60 is 12, then $\dfrac{x}{100} \cdot 60 = 12$ or $\dfrac{60x}{100} = 12$.
In terms of $x$, 15% of $x$ equals $\dfrac{15}{100}x$ or $\dfrac{15x}{100}$.
Since $\dfrac{60x}{100} = 12$, then $\dfrac{15x}{100} = \dfrac{1}{4}\left(\dfrac{60x}{100}\right) = \dfrac{1}{4} \cdot 12 = 3$.

Answer: (D)

10. *Solution 1*
Since square $PQRS$ has side length 2, then $PQ = QR = RS = SP = 2$.
Since $W$, $X$, $Y$, $Z$ are the midpoints of the sides of $PQRS$, then $PW = PZ = 1$.
Since $\angle ZPW = 90^\circ$, then $WZ = \sqrt{PW^2 + PZ^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$.
Therefore, square $WXYZ$ has side length $\sqrt{2}$.
The area of square $WXYZ$ is $(\sqrt{2})^2 = 2$ and the area of square $PQRS$ is $2^2 = 4$.
The ratio of these areas is $2 : 4$ or $1 : 2$.

*Solution 2*
Join $W$ to $Y$ and $X$ to $Z$.




Since $PQRS$ is a square and $W$, $X$, $Y$, and $Z$ are the midpoints of its sides, then $WY$ and $ZX$ divide the square into four identical squares.
Each of these four squares is divided into two triangles of equal area by its diagonal. (These diagonals are $WZ$, $WX$, $XY$, $YZ$.)
Square $WXYZ$ is made up of 4 of these triangles of equal area.
Square $PQRS$ is made up of 8 of these triangles of equal area.
Therefore, the ratio of these areas is $4 : 8$ or $1 : 2$.

Answer: (A)

11. By the Pythagorean Theorem in $\triangle PRS$,

$$PS = \sqrt{RS^2 - PR^2} = \sqrt{13^2 - 12^2} = \sqrt{169 - 144} = \sqrt{25} = 5$$

since $PS > 0$.
Thus, $PQ = PS + SQ = 5 + 11 = 16$.
By the Pythagorean Theorem in $\triangle PRQ$,

$$RQ = \sqrt{PR^2 + PQ^2} = \sqrt{12^2 + 16^2} = \sqrt{144 + 256} = \sqrt{400} = 20$$

since $RQ > 0$.
Therefore, the perimeter of $\triangle QRS$ is $RS + SQ + RQ = 13 + 11 + 20 = 44$.

Answer: (B)

12. Since $128 = 2^7$, its positive divisors are

$$2^0 = 1 \qquad 2^1 = 2 \qquad 2^2 = 4 \qquad 2^3 = 8 \qquad 2^4 = 16 \qquad 2^5 = 32 \qquad 2^6 = 64 \qquad 2^7 = 128$$

Of these, the integers $1, 4, 16, 64$ are perfect squares, which means that 128 has three positive divisors that are perfect squares larger than 1.

Answer: (D)

13. Since $4x, 2x - 3, 4x - 3$ form an arithmetic sequence, then the differences between consecutive terms are equal, or $(2x - 3) - 4x = (4x - 3) - (2x - 3)$.
Thus, $-2x - 3 = 2x$ or $4x = -3$ and so $x = -\frac{3}{4}$.

Answer: (E)

14. Since the average of the four numbers $4, a, b, 22$ is 13, then $\dfrac{4 + a + b + 22}{4} = 13$ and so $4 + a + b + 22 = 52$ or $a + b = 26$.
Since $a > 4$ and $a$ is an integer, then $a \geq 5$.
Since $a + b = 26$ and $a < b$, then $a$ is less than half of 26, or $a < 13$.
Since $a$ is an integer, then $a \leq 12$.
Therefore, we have $5 \leq a \leq 12$.
There are 8 choices for $a$ in this range: 5, 6, 7, 8, 9, 10, 11, 12. (Note that $12 - 5 + 1 = 8$.)
These give the pairs $(a, b) = (5, 21), (6, 20), (7, 19), (8, 18), (9, 17), (10, 16), (11, 15), (12, 14)$.
Thus, there are 8 possible pairs.

Answer: (B)

15. When Hicham runs 10 km at an average speed of 12 km/h, he takes $\frac{10}{12} = \frac{5}{6}$ hours to run this distance.
Since Hicham runs for a total of 1.5 hours, then he runs the last 6 km in $\frac{3}{2} - \frac{5}{6} = \frac{9}{6} - \frac{5}{6} = \frac{4}{6} = \frac{2}{3}$ hours.
Since he runs 6 km in $\frac{2}{3}$ hours, his average speed for this segment is $\frac{6}{2/3} = 9$ km/h.

Answer: (B)

16. *Solution 1*
Since $x = 18$ is a solution to the equation $x^2 + 12x + c = 0$, then $x = 18$ satisfies this equation.
Thus, $18^2 + 12(18) + c = 0$ and so $324 + 216 + c = 0$ or $c = -540$.
Therefore, the original equation becomes $x^2 + 12x - 540 = 0$ or $(x - 18)(x + 30) = 0$.
Therefore, the other solution is $x = -30$.

*Solution 2*
We use the fact that the sum of the roots of an equation of the form $x^2 + bx + c = 0$ is $-b$.
If the roots of the equation $x^2 + 12x + c = 0$ are 18 and $r$, then $18 + r = -12$ or $r = -30$.
Therefore, the other solution is $x = -30$.

Answer: (C)

17. The number of points on the circle equals the number of spaces between the points around the circle.
Moving from the point labelled 7 to the point labelled 35 requires moving $35 - 7 = 28$ points and so 28 spaces around the circle.
Since the points labelled 7 and 35 are diametrically opposite, then moving along the circle from 7 to 35 results in travelling halfway around the circle.
Since 28 spaces makes half of the circle, then $2 \cdot 28 = 56$ spaces make the whole circle.
Thus, there are 56 points on the circle, and so $n = 56$.

Answer: (C)

18. The first equation $\dfrac{x-y}{x+y} = 9$ gives $x - y = 9x + 9y$ and so $-8x = 10y$ or $-4x = 5y$.
The second equation $\dfrac{xy}{x+y} = -60$ gives $xy = -60x - 60y$.
Multiplying this equation by 5 gives $5xy = -300x - 300y$ or $x(5y) = -300x - 60(5y)$.
Since $5y = -4x$, then $x(-4x) = -300x - 60(-4x)$ or $-4x^2 = -60x$.
Rearranging, we obtain $4x^2 - 60x = 0$ or $4x(x - 15) = 0$.
Therefore, $x = 0$ or $x = 15$.
Since $y = -\frac{4}{5}x$, then $y = 0$ or $y = -12$.
From the first equation, it cannot be the case that $x = y = 0$.
We can check that the pair $(x, y) = (15, -12)$ satisfies both equations.
Therefore, $(x + y) + (x - y) + xy = 3 + 27 + (-180) = -150$.

Answer: (B)

19. *Solution 1*
Suppose that, when the $n$ students are put in groups of 2, there are $g$ complete groups and 1 incomplete group.
Since the students are being put in groups of 2, an incomplete group must have exactly 1 student in it.
Therefore, $n = 2g + 1$.
Since the number of complete groups of 2 is 5 more than the number of complete groups of 3, then there were $g - 5$ complete groups of 3.
Since there was still an incomplete group, this incomplete group must have had exactly 1 or 2 students in it.
Therefore, $n = 3(g - 5) + 1$ or $n = 3(g - 5) + 2$.
If $n = 2g+1$ and $n = 3(g-5)+1$, then $2g+1 = 3(g-5)+1$ or $2g+1 = 3g-14$ and so $g = 15$.
In this case, $n = 2g + 1 = 31$ and there were 15 complete groups of 2 and 10 complete groups of 3.
If $n = 2g+1$ and $n = 3(g-5)+2$, then $2g+1 = 3(g-5)+2$ or $2g+1 = 3g-13$ and so $g = 14$.
In this case, $n = 2g + 1 = 29$ and there were 14 complete groups of 2 and 9 complete groups of 3.
If $n = 31$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
If $n = 29$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
Since the difference between the number of complete groups of 3 and the number of complete groups of 4 is given to be 3, then it must be the case that $n = 31$.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

*Solution 2*

Since the $n$ students cannot be divided exactly into groups of 2, 3 or 4, then $n$ is not a multiple of 2, 3 or 4.

The first few integers larger than 1 that are not divisible by 2, 3 or 4 are 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35.

In each case, we determine the number of complete groups of each size:

| $n$ | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 25 | 29 | 31 | 35 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| # of complete groups of 2 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 | 15 | 17 |
| # of complete groups of 3 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| # of complete groups of 4 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 7 | 8 |

Since the number of complete groups of 2 is 5 more than the number of complete groups of 3 which is 3 more than the number of complete groups of 4, then of these possibilities, $n = 31$ works.

In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

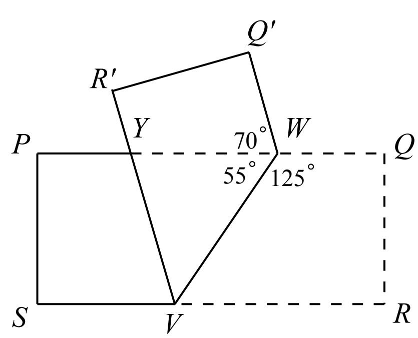
(Since the problem is a multiple choice problem and we have found a value of $n$ that satisfies the given conditions and for which an answer is present, then this answer must be correct. Solution 1 shows why $n = 31$ is the only value of $n$ that satisfies the given conditions.)

Answer: (B)

20. Since points $Y$, $W$ and $Q$ form a straight line segment, then $\angle YWV = 180^\circ - \angle VWQ$ and so $\angle YWV = 180^\circ - 125^\circ = 55^\circ$.

Since $Q'$ is the final position of $Q$ after folding, then $\angle Q'WV$ is the final position of $\angle QWV$ after folding, and so $\angle Q'WV = \angle QWV$.

Thus, $\angle Q'WV = \angle QWV = 125^\circ$ and so $\angle Q'WY = \angle Q'WV - \angle YWV = 125^\circ - 55^\circ = 70^\circ$.

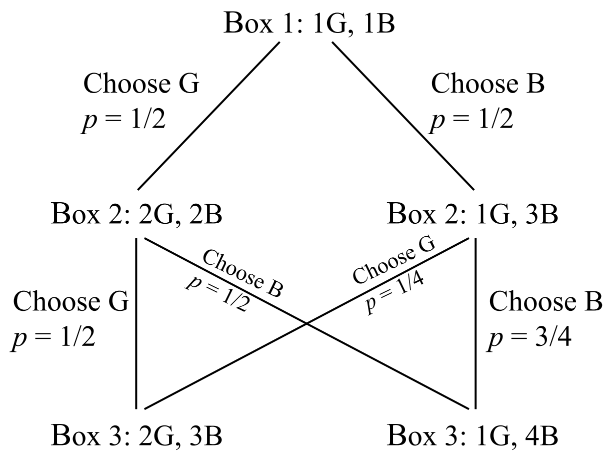



Since $Q'W$ and $R'Y$ are parallel sides of the piece of paper, then $\angle R'YW + \angle Q'WY = 180^\circ$, and so $\angle R'YW = 180^\circ - \angle Q'WY = 180^\circ - 70^\circ = 110^\circ$.

Finally, $\angle PYV$ is opposite $\angle R'YW$ so $\angle PYV = \angle R'YW = 110^\circ$.

Answer: (A)

21. When a marble is chosen from Box 1, the probability is $\frac{1}{2}$ that it will be gold and the probability is $\frac{1}{2}$ that it will be black.
Thus, after this choice is made, there is a probability of $\frac{1}{2}$ that Box 2 contains 2 gold marbles and 2 black marbles, and there is a probability of $\frac{1}{2}$ that Box 2 contains 1 gold marble and 3 black marbles.
In the first case (which occurs with probability $\frac{1}{2}$), the probability that a gold marble is chosen from Box 2 is $\frac{2}{4} = \frac{1}{2}$ and the probability that a black marble is chosen from Box 2 is $\frac{2}{4} = \frac{1}{2}$.
In the second case (which occurs with probability $\frac{1}{2}$), the probability that a gold marble is chosen from Box 2 is $\frac{1}{4}$ and the probability that a black marble is chosen from Box 2 is $\frac{3}{4}$.




Therefore, the probability that a gold marble is chosen from Box 2 is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$ and the probability that a black marble is chosen from Box 2 is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{3}{4} = \frac{5}{8}$.
Thus, after this choice is made, there is a probability of $\frac{3}{8}$ that Box 3 contains 2 gold marbles and 3 black marbles, and a probability of $\frac{5}{8}$ that Box 3 contains 1 gold marble and 4 black marbles.
Finally, the probability that a gold marble is chosen from Box 3 equals the probability that Box 3 contains 2 gold marbles and 3 black marbles times the probability of choosing a gold marble in this situation (that is, $\frac{3}{8} \cdot \frac{2}{5}$) plus the probability that Box 3 contains 1 gold marble and 4 black marbles times the probability of choosing a gold marble in this situation.
In other words, this probability is $\frac{3}{8} \cdot \frac{2}{5} + \frac{5}{8} \cdot \frac{1}{5} = \frac{11}{40}$.

ANSWER: (A)

22. *Solution 1*

We expand the given expression to obtain

$$(x^2+6x+9)+2(y^2-4y+4)+4(x^2-14x+49)+(y^2+8y+16)$$

We expand further to obtain

$$x^2+6x+9+2y^2-8y+8+4x^2-56x+196+y^2+8y+16$$

We simplify to obtain

$$5x^2-50x+3y^2+229$$

We remove a common factor of 5 from the first two terms

$$5(x^2-10x)+3y^2+229$$

and then complete the square to obtain

$$5(x^2-10x+5^2-5^2)+3y^2+229$$

This gives

$$5(x-5)^2-125+3y^2+229$$

or

$$5(x-5)^2+3y^2+104$$

Since $(x-5)^2 \geq 0$ for all real numbers $x$ and $3y^2 \geq 0$ for all real numbers $y$, then the minimum value of $5(x-5)^2+3y^2+104$ (and hence of the original expression) is $5(0)+3(0)+104$ or 104. We note that this value is actually achieved when $x=5$ (which gives $(x-5)^2=0$) and $y=0$ (which gives $3y^2=0$).

*Solution 2*

We expand the given expression to obtain

$$(x^2+6x+9)+2(y^2-4y+4)+4(x^2-14x+49)+(y^2+8y+16)$$

We expand further to obtain

$$x^2+6x+9+2y^2-8y+8+4x^2-56x+196+y^2+8y+16$$

The terms involving $x$ are

$$x^2+6x+9+4x^2-56x+196=5x^2-50x+205=5x^2-50x+125+80=5(x-5)^2+80$$

The terms involving $y$ are

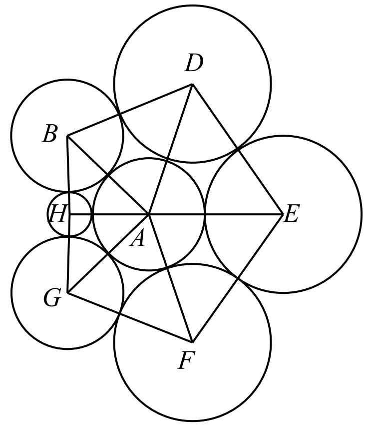
$$2y^2-8y+8+y^2+8y+16=3y^2+24$$

Since $(x-5)^2 \geq 0$ for all real numbers $x$, then the minimum value of $5(x-5)^2+80$ is 80.

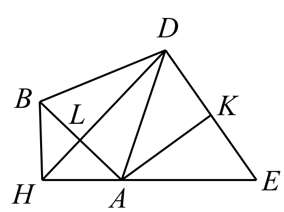
Since $y^2 \geq 0$ for all real numbers $y$, then the minimum value of $3y^2+24$ is 24.
Since the minimum value of $(x-3)^2+4(x-7)^2$ is 80 and the minimum value of $2(y-2)^2+(y+4)^2$ is 24, then the minimum value of $(x-3)^2+2(y-2)^2+4(x-7)^2+(y+4)^2$ is $80+24=104$.

Answer: (E)

23. We label the centres of the coins $A, B, D, E, F, G, H$, as shown, and we join $AB$, $AD$, $AE$, $AF$, $AG$, $AH$, $BD$, $DE$, $EF$, $FG$, $GH$, and $HB$.




The radius of the circles with centres $D$, $E$ and $F$ is 3 cm and the radius of the circles with centres $A$, $B$ and $G$ is 2 cm. (From this point until the very last step of the problem, we will not include the units, which are always centimetres.)
Let the radius of the circle with centre $H$ be $r$. We want to determine $r$.
When we join the centres of two circles that are just touching, the resulting line segment passes through the point at which the circles touch and the length of this line segment is the sum of the radii of the two circles.
Thus, $AB = 2+2 = 4$. Similarly, $AG = 4$, $BD = AD = AE = AF = GF = 5$, $DE = EF = 6$, and $HA = HB = HG = r + 2$.
Now $\triangle ADE$ and $\triangle AFE$ each have side lengths 5, 5 and 6, which means that these triangles are congruent.
Similarly, $\triangle ADB$ and $\triangle AFG$ are congruent, and $\triangle ABH$ and $\triangle AGH$ are congruent.
Since corresponding angles in congruent triangles are equal, then $\angle HAB + \angle BAD + \angle DAE$ is equal to $\angle HAG + \angle GAF + \angle FAE$.
But these six angles surround point $A$, so the sum of these six angles is $360°$.
Thus, $\angle HAB + \angle BAD + \angle DAE = \angle HAG + \angle GAF + \angle FAE = 180°$.
Now, we remove the circles from the diagram and focus on the top half of the diagram.
We join $A$ to $K$, the midpoint of $DE$, and $D$ and $H$ to $L$, the midpoint of $AB$.




Consider $\triangle ADE$.
Since $DE = 6$ and $K$ is the midpoint of $DE$, then $DK = KE = 3$.
Also, since $AD = AE = 5$, then $\triangle ADE$ is isosceles and so $AK$ is perpendicular to $DE$ and $AK$ bisects $\angle DAE$.
Similarly, $\triangle AHB$ is isosceles with $HA = HB$ and $\triangle ABD$ is isosceles with $DA = DB$.
Since $L$ is the midpoint of $AB$ and $AB = 4$, then $AL = LB = 2$ and $DL$ and $HL$ are perpendicular to $AB$ at $L$.
We know that $\angle HAB + \angle BAD + \angle DAE = \angle HAL + \angle LAD + 2\angle EAK = 180°$.

But $\sin(\angle EAK) = \dfrac{EK}{AE} = \dfrac{3}{5}$, so $\angle EAK \approx 36.87^\circ$.
Also, $\cos(\angle LAD) = \dfrac{AL}{AD} = \dfrac{2}{5}$, so $\angle LAD \approx 66.42^\circ$.
Thus, $\angle HAL = 180^\circ - \angle LAD - 2\angle EAK \approx 180^\circ - 66.42^\circ - 2(36.87^\circ) \approx 39.84^\circ$.
Finally, we know that $\cos(\angle HAL) = \dfrac{AL}{HA} = \dfrac{2}{r+2}$.
Since $\cos(39.84^\circ) \approx 0.7679$, then $\dfrac{2}{r+2} \approx 0.7679$.
From this we obtain $r \approx \dfrac{2}{0.7679} - 2$.
$Since \dfrac{2}{0.7679} - 2 \approx 0.60451$, then of the given choices, this radius is closest to 0.605 cm.
(If more decimal places are carried through from earlier calculations, we obtain $r \approx 0.60466$, which is still closest to 0.605 cm.)

Answer: (D)

24. We begin by understanding and describing $\lfloor \sqrt{k} \rfloor$ in a different way.
Each positive integer $k$ can be placed between two consecutive perfect squares. More precisely, for each positive integer $k$, there exists a unique positive integer $n$ with $n^2 \leq k < (n+1)^2$.
Since $n^2 \leq k < (n+1)^2$, then $n^2 \leq k < n^2 + 2n + 1$ and $n \leq \sqrt{k} < n + 1$.
Since $n$ and $n+1$ are consecutive integers, then $\lfloor \sqrt{k} \rfloor = n$.
In other words, $n$ is the largest positive integer less than or equal to $\sqrt{k}$.
So we want to calculate the sum of all integers $k$ with $1 \leq k \leq 999\,999$ for which $k$ is a multiple of its corresponding $n$.
Now $n^2 = n \cdot n$ and $n^2 + 2n + 1 = n(n+2) + 1$.
This means that the possible values of $k$ with $n^2 \leq k < n^2 + 2n + 1$ are the multiples of $n$ in this range, or $k = n \cdot n$, $k = n(n+1)$, and $k = n(n+2)$.
Since $k \leq 999\,999$, then $k < 1000^2 = 1\,000\,000$, and so $n \leq 999$.
So the given problem is equivalent to determining the sum of $n^2$, $n(n+1)$ and $n(n+2)$ for each $n$ from 1 to 999.
Since $n^2 + n(n+1) + n(n+2) = 3n^2 + 3n$, then we want to determine the sum of $3n^2 + 3n$ for $n = 1$ to $n = 999$ inclusive.
Thus,

$$\begin{aligned} S &= (3(1^2) + 3(1)) + (3(2^2) + 3(2)) + \cdots + (3(998^2) + 3(998)) + (3(999^2) + 3(999)) \\ &= 3(1^2 + 2^2 + \cdots + 998^2 + 999^2) + 3(1 + 2 + \cdots + 998 + 999) \end{aligned}$$

Since $1+2+\cdots+(m-1)+m = \dfrac{m(m+1)}{2}$ and $1^2+2^2+\cdots+(m-1)^2+m^2 = \dfrac{m(m+1)(2m+1)}{6}$ for every positive integer $m$, then

$$\begin{aligned} S &= 3 \cdot \frac{999(1000)(1999)}{6} + 3 \cdot \frac{999(1000)}{2} \\ &= \frac{3(999)(1000)}{6}(1999 + 3) \\ &= \frac{999(1000)}{2}(2002) \\ &= 999(1000)(1001) \end{aligned}$$

and so $S = 999\,999\,000$.

Answer: (C)

25. We begin by partitioning the set $A$ into disjoint subsets of the form

$$P_b = \{b, 3b, 9b, \ldots, 3^k b\}$$

where $b$ is a positive integer with $1 \leq b \leq 2045$ and $k$ is the largest non-negative integer with $3^k b \leq 2045$.
The first two of these sets are $P_1 = \{1, 3, 9, 27, 81, 243, 729\}$ and $P_2 = \{2, 6, 18, 54, 162, 486, 1458\}$.
These sets have $b = 1$ and $b = 2$, and thus $k = 6$.
We will see that each element of $A$ is an element of exactly one of these sets.
Since $3^7 = 2187$, then $3^7 b \geq 2187 > 2045$ for every positive integer $b$, so $k = 6$ is the largest possible value of $k$ that we can have.
For each $k$ with $0 \leq k \leq 6$, we determine the range of values of $b$ that give that $k$:

- $k = 6$: $1 \leq b \leq 2$ since $2 \cdot 3^6 = 1458 < 2045$ and $3 \cdot 3^6 = 2187 > 2045$
- $k = 5$: $3 \leq b \leq 8$ since $8 \cdot 3^5 = 1944 < 2045$ and $9 \cdot 3^5 = 2187 > 2045$
- $k = 4$: $9 \leq b \leq 25$ since $25 \cdot 3^4 = 2025 < 2045$ and $26 \cdot 3^4 = 2106 > 2045$
- $k = 3$: $26 \leq b \leq 75$ since $75 \cdot 3^3 = 2025 < 2045$ and $76 \cdot 3^3 = 2052 > 2045$
- $k = 2$: $76 \leq b \leq 227$ since $227 \cdot 3^2 = 2043 < 2045$ and $228 \cdot 3^2 = 2052 > 2045$
- $k = 1$: $228 \leq b \leq 681$ since $681 \cdot 3^1 = 2043 < 2045$ and $682 \cdot 3^1 = 2046 > 2045$
- $k = 0$: $682 \leq b \leq 2045$ since $2045 \cdot 3^0 = 2045$ and $2046 \cdot 3^0 = 2046 > 2045$

Since we want to create disjoint sets $P_b$ whose union is the set $A = \{1, 2, 3, \ldots, 2044, 2045\}$, then we exclude all $b$'s that are multiples of 3. (If $b$ were a multiple of 3, it would appear as an element of a set $P_r$ with $r \leq b$.)
For each of the ranges above, we count the number of multiples of 3 that we need to exclude:

| $k$ | Range of values of $b$ | # of multiples of 3 | # of remaining $b$ | # of elements in each $P_b$ |
|---|---|---|---|---|
| 6 | $1 \leq b \leq 2$ | 0 | $2 - 0 = 2$ | 7 |
| 5 | $3 \leq b \leq 8$ | 2 | $6 - 2 = 4$ | 6 |
| 4 | $9 \leq b \leq 25$ | 6 | $17 - 6 = 11$ | 5 |
| 3 | $26 \leq b \leq 75$ | 17 | $50 - 17 = 33$ | 4 |
| 2 | $76 \leq b \leq 227$ | 50 | $152 - 50 = 102$ | 3 |
| 1 | $228 \leq b \leq 681$ | 152 | $454 - 152 = 302$ | 2 |
| 0 | $682 \leq b \leq 2045$ | 454 | $1364 - 454 = 910$ | 1 |

Note for example that the range $26 \leq b \leq 75$ contains $75 - 25 = 50$ integers and includes the multiples of 3 from $9 \cdot 3$ to $25 \cdot 3$, inclusive, so includes $25 - 8 = 17$ multiples of 3, and the range $228 \leq b \leq 681$ contains $681 - 227 = 454$ integers and includes the multiples of 3 from $76 \cdot 3$ to $227 \cdot 3$, inclusive, so includes $227 - 75 = 152$ multiples of 3.
An integer $b$ in $A$ that is not a multiple of 3 generates a set $P_b$ and cannot appear in a different set $P_r$. Any multiple of 3, say $m$, will appear in exactly one $P_b$ where the value of $b$ (which is itself not a multiple of 3) is obtained by dividing out all of factors of 3 from $m$.
We note that $2 \cdot 7 + 4 \cdot 6 + 11 \cdot 5 + 33 \cdot 4 + 102 \cdot 3 + 302 \cdot 2 + 910 \cdot 1 = 2045$ so the union of these sets $P_b$ includes enough elements to be the entire set $A$. No element of $A$ appears in more than one $P_b$, so the union of these sets $P_b$ is the entire set $A$.

Why are these sets useful in the given problem?
A subset of $A$ is triple-free if it does not contain two elements, one of which is three times the other.

This means that a subset of $A$ is triple-free exactly when it does not contain two consecutive elements of any $P_b$.
A triple-free subset $T$ of $A$ that contains as many elements as possible will contain as many elements as possible from each of the $P_b$ defined above.
Since no two consecutive elements of any $P_b$ can appear in $T$, then a $P_b$ with 7 elements can contribute at most 4 elements to $T$ (every other element starting with the first), a $P_b$ with 6 elements can contribute at most 3 elements to $T$ (it cannot contain 4 elements without having two consecutive elements), and so on.
For each $k$ from $k = 7$ to $k = 0$, each $P_b$ above can contribute at most the following number of elements to $T$:

| $k$ | # of elements in each $P_b$ | Number of elements that can be chosen |
|---|---|---|
| 6 | 7 | 4 |
| 5 | 6 | 3 |
| 4 | 5 | 3 |
| 3 | 4 | 2 |
| 2 | 3 | 2 |
| 1 | 2 | 1 |
| 0 | 1 | 1 |

This means that $T$ can contain at most $2\cdot4+4\cdot3+11\cdot3+33\cdot2+102\cdot2+302\cdot1+910\cdot1 = 1535$ elements. This agrees with the information in the problem statement.
Where is there choice in creating $T$?
If a $P_b$ contains 1 element, then this element must be chosen for $T$.
If a $P_b$ contains 3 elements, then there is 1 way to choose 2 elements for $T$, since in this case the 1st and 3rd elements (in increasing order) must be chosen.
If a $P_b$ contains 5 elements, then there is 1 way to choose 3 elements for $T$, since in this case the 1st, 3rd and 5th elements (in increasing order) must be chosen.
If a $P_b$ contains 7 elements, then there is 1 way to choose 4 elements for $T$, since in this case the 1st, 3rd, 5th and 7th (in increasing order) elements must be chosen.
If a $P_b$ contains 2 elements, then there are 2 ways to choose 1 element for $T$.
If a $P_b$ contains 4 elements, then there are 3 ways to choose 2 elements for $T$. (If we want to choose 2 elements from $\{A, B, C, D\}$ without choosing consecutive elements, we can choose $A$ and $C$, or $A$ and $D$, or $B$ and $D$.)
If a $P_b$ contains 6 elements, then there are 4 ways to choose 3 elements for $T$. (If we want to choose 3 elements from $\{A, B, C, D, E, F\}$ without choosing consecutive elements, we can choose $A$ and $C$ and $E$, or $A$ and $C$ and $F$, or $A$ and $D$ and $F$, or $B$ and $D$ and $F$.)
This means that the number of ways of choosing a triple-free subset $T$ of $A$ with as many elements as possible is $1^2 \cdot 4^4 \cdot 1^{11} \cdot 3^{33} \cdot 1^{102} \cdot 2^{302} \cdot 1^{910}$.
This is because there are 2 sets $P_b$ with $k = 6$ and 1 way of choosing 4 elements from each of them (giving $1^2$ choices in total), 4 set $P_b$ with $k = 5$ and 4 ways of choosing 3 elements from each of them (giving $4^4$ choices in total), and so on.
The seven factors in this product give, from $k = 6$ to $k = 0$, the number of ways of choosing the maximum number of elements from a set $P_b$ corresponding to that value of $k$ raised to the power of the number of such sets $P_b$.
Therefore, the number of ways equals $4^4 \cdot 3^{33} \cdot 2^{302} = 2^{310} \cdot 3^{33}$.
From the statement of the problem, we thus have $N = 2^2 + 3^2 + 310^2 + 33^2 = 97\,202$.
The final three digits of this integer are 202.

Answer: (A)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2015 Fermat Contest*

(Grade 11)

**Tuesday, February 24, 2015**
**(in North America and South America)**

**Wednesday, February 25, 2015**
**(outside of North America and South America)**

*Solutions*

1. The average of the five given numbers is $\dfrac{8+9+10+11+12}{5} = \dfrac{50}{5} = 10$.
   Alternatively, since there are an odd number of consecutive integers, the average is the middle number, which is 10.

   Answer: (E)

2. Evaluating, $\dfrac{2 \times 3 + 4}{2+3} = \dfrac{6+4}{5} = \dfrac{10}{5} = 2$.

   Answer: (A)

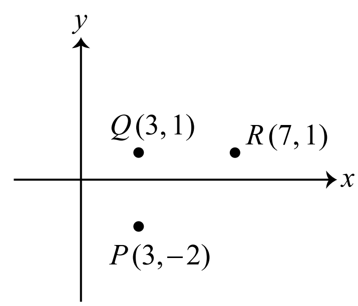
3. Suppose that the distance between consecutive points on the path is $d$.
   Then, walking from $P$ to $U$, Emily walks a total distance of $5d$.
   Walking back from $U$ to $P$, she also walks a distance of $5d$, for a total distance walked of $10d$.
   Since 70% of 10 is 7, then she has completed 70% of her walk after walking a distance of $7d$ (that is, after having walked 7 segments).
   She has walked a distance of $7d$ after walking a distance of $2d$ back from $U$, which gets her to $S$.

   Answer: (D)

4. Evaluating, $(x-3)^2 = (-3-3)^2 = (-6)^2 = 36$.

   Answer: (B)

5. Based on the diagram shown, the order of the vertices of the rectangle must be $PQRS$. (In any other configuration, either $\angle QRP$ or $\angle QPR$ would be an angle in the rectangle, which is not possible.)
   Since $P$ and $Q$ have the same $x$-coordinate, then side $PQ$ of the rectangle is vertical.
   This means that side $SR$ must also be vertical, and so the $x$-coordinate of $S$ is the same as the $x$-coordinate of $R$, which is 7.

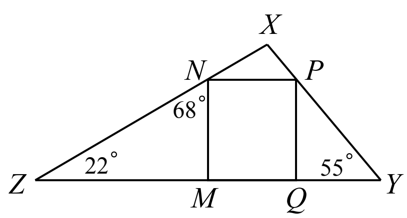
   
   

   Since $Q$ and $R$ have the same $y$-coordinate, then side $QR$ of the rectangle is horizontal.
   This means that side $PS$ must also be horizontal, and so the $y$-coordinate of $S$ is the same as the $y$-coordinate of $P$, which is $-2$.
   Thus, the coordinates of $S$ are $(7, -2)$.

   Answer: (B)

6. Since $MNPQ$ is a rectangle, then $\angle NMQ$ is a right angle and thus so is $\angle NMZ$.
   The sum of the angles in $\triangle NMZ$ is $180^\circ$, so

   $$\angle NZM = 180^\circ - \angle ZNM - \angle ZMN = 180^\circ - 68^\circ - 90^\circ = 22^\circ$$

   Since the sum of the angles in $\triangle ZXY$ is also $180^\circ$, then

   $$\angle YXZ = 180^\circ - \angle XZY - \angle XYZ = 180^\circ - 22^\circ - 55^\circ = 103^\circ$$

   Answer: (E)

7. Violet starts with one-half of the money that she needed to buy the necklace.
After her sister gives her money, she has three-quarters of the amount that she needs.
This means that her sister gave her $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ of the total amount that she needs.
Since she now has three-quarters of the amount that she needs, then she still needs one-quarter of the total cost.
In other words, her father will give her the same amount that her sister gave her, or \$30.

ANSWER: (D)

8. Since $15^2 = 225$ and $15 = 3 \cdot 5$, then $225 = 15^2 = (3 \cdot 5)^2 = 3^2 \cdot 5^2$.
Therefore, $x = 2$ and $y = 2$, so $x + y = 4$.

ANSWER: (B)

9. *Solution 1*
The two teams include a total of $25 + 19 = 44$ players.
There are exactly 36 students who are at least one team.
Thus, there are $44 - 36 = 8$ students who are counted twice.
Therefore, there are 8 students who play both baseball and hockey.

*Solution 2*
Suppose that there are $x$ students who play both baseball and hockey.
Since there are 25 students who play baseball, then $25-x$ of these play baseball and not hockey.
Since there are 19 students who play hockey, then $19-x$ of these play hockey and not baseball.
Since 36 students play either baseball or hockey or both, then

$$(25 - x) + (19 - x) + x = 36$$

(The left side is the sum of the numbers of those who play baseball and not hockey, those who play hockey and not baseball, and those who play both.)
Therefore, $44 - x = 36$ and so $x = 44 - 36 = 8$.
Thus, 8 students play both baseball and hockey.

ANSWER: (B)

10. Since Bruce drove 200 km at a speed of 50 km/h, this took him $\frac{200}{50} = 4$ hours.
Anca drove the same 200 km at a speed of 60 km/h with a stop somewhere along the way.
Since Anca drove 200 km at a speed of 60 km/h, the time that the driving portion of her trip took was $\frac{200}{60} = 3\frac{1}{3}$ hours.
The length of Anca's stop is the difference in driving times, or $4 - 3\frac{1}{3} = \frac{2}{3}$ hours.
Since $\frac{2}{3}$ hours equals 40 minutes, then Anca stops for 40 minutes.

ANSWER: (A)

11. For each of the three digits of such a positive integer, there are three choices of digit (7, 8 or 9).
Therefore, there are $3 \cdot 3 \cdot 3 = 27$ possible integers that use no digits other than 7, 8 or 9.
(We note that there are 9 such integers beginning with each of 7, 8 and 9.
The 9 such integers beginning with 7 are 777, 778, 779, 787, 788, 789, 797, 798, 799.)

ANSWER: (E)

12. Since $\cos 60^\circ = \frac{1}{2}$ and $\cos 45^\circ = \frac{1}{\sqrt{2}}$, then the given equation $\cos 60^\circ = \cos 45^\circ \cos\theta$ becomes $\frac{1}{2} = \frac{1}{\sqrt{2}}\cos\theta$.
Therefore, $\cos\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.
Since $0^\circ \leq \theta \leq 90^\circ$, then $\theta = 45^\circ$.

ANSWER: (D)

13. We make a table of the total amount of money that each of Steve and Wayne have at the end of each year. After the year 2000, each entry in Steve's column is found by doubling the previous entry and each entry in Wayne's column is found by dividing the previous entry by 2. We stop when the entry in Steve's column is larger than that in Wayne's column:

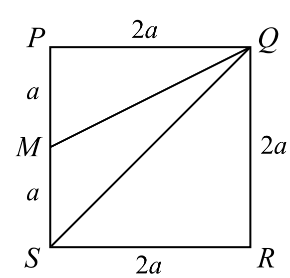
| Year | Steve | Wayne |
|---|---|---|
| 2000 | \$100 | \$10 000 |
| 2001 | \$200 | \$5000 |
| 2002 | \$400 | \$2500 |
| 2003 | \$800 | \$1250 |
| 2004 | \$1600 | \$625 |

Therefore, 2004 is the first time at which Steve has more money than Wayne at the end of the year.

ANSWER: (C)

14. *Solution 1*
Since $PQRS$ is a square, then its diagonal $SQ$ cuts it into two equal areas.
Therefore, the ratio of the area of $\triangle PQS$ to the area of square $PQRS$ is $1 : 2$.
$\triangle PQS$ can be viewed as having base $PS$ and height $PQ$.
$\triangle MQS$ can be viewed as having base $MS$ and height $PQ$. (This is because $PQ$ is perpendicular to the line containing $MS$.)
Since $MS = \frac{1}{2}PS$, then the area of $\triangle MQS$ is one-half of the area of $\triangle PQS$.
Since the ratio of the area of $\triangle PQS$ to the area of square $PQRS$ is $1 : 2$, then the ratio of the area of $\triangle QMS$ to the area of square $PQRS$ is $1 : 4$.

*Solution 2*
Suppose that the side length of square $PQRS$ is $2a$.
Then the area of square $PQRS$ is $(2a)^2 = 4a^2$.
Since $M$ is the midpoint of side $PS$, then $PM = MS = a$.
Then $\triangle QMS$ can be seen as having base $MS$ and height $PQ$. (This is because $PQ$ is perpendicular to the line containing $MS$.)
Since $MS = a$ and $PQ = 2a$, then the area of $\triangle QMS$ is $\frac{1}{2}(MS)(PQ) = \frac{1}{2}a(2a) = a^2$.
Therefore, the ratio of the area of $\triangle QMS$ to the area of square $PQRS$ is $a^2 : 4a^2$ which equals $1 : 4$.




ANSWER: (B)

15. *Solution 1*

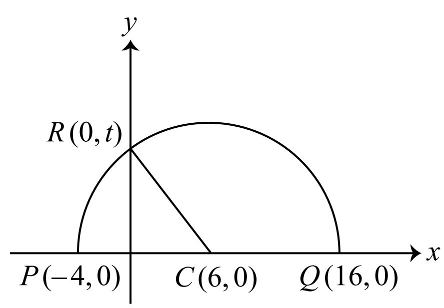
Zoltan answered 45 questions.
If all of his answers had been correct, his score would have been $45(4) = 180$ points.
Since his score was 135 points, then he lost $180 - 135 = 45$ points.
For each incorrect answer, he lost 5 points compared to a correct answer, since a correct answer adds 4 points and an incorrect answer subtracts 1 point.
Therefore, Zoltan had $45 \div 5 = 9$ incorrect answers.
(We can check that 36 correct, 9 incorrect, and 5 unanswered gives a score of $4(36) - 1(9) + 0(5)$ or 135 points.)

*Solution 2*

Suppose that Zoltan answered $x$ questions incorrectly.
Since he answered 45 questions in total, then he answered $45 - x$ questions correctly.
Since the test included 50 questions and he answered 45, then he did not answer 5 questions.
Using the marking scheme, his score was $4(45 - x) - 1(x) + 0(5)$.
We are told that his score was 135 points.
Hence, $4(45 - x) - 1(x) + 0(5) = 135$ and so $180 - 4x - x = 135$.
Thus, $5x = 45$ or $x = 9$.
Therefore, Zoltan answered 9 questions incorrectly.
(We can check that 36 correct, 9 incorrect, and 5 unanswered gives a score of $4(36) - 1(9) + 0(5)$ or 135 points.)

ANSWER: (A)

16. Since $P(-4, 0)$ and $Q(16, 0)$ are endpoints of a diameter of the semi-circle, then the length of the diameter is $16 - (-4) = 20$.
Since a diameter of the semi-circle has length 20, then the radius of the semi-circle is $\frac{1}{2}(20) = 10$.
Also, the centre $C$ is the midpoint of diameter $PQ$ and so has coordinates $\left(\frac{1}{2}(-4 + 16), \frac{1}{2}(0 + 0)\right)$ or $(6, 0)$.




Now the distance between $C(6, 0)$ and $R(0, t)$ is 10, since $CR$ is a radius.
Therefore,

$$\begin{aligned} \sqrt{(6-0)^2 + (0-t)^2} &= 10 \\ 36 + t^2 &= 100 \\ t^2 &= 64 \end{aligned}$$

(Alternatively, we could have noted that if $O$ is the origin, then $\triangle ROC$ is right-angled with $RO = t$, $RC = 10$ and $OC = 6$ and then used the Pythagorean Theorem to obtain $t^2 + 6^2 = 10^2$, which gives $t^2 = 64$.)
Since $t > 0$, then $t = 8$.

ANSWER: (C)

17. Since $\dfrac{a+b}{a-b}=3$, then $a+b=3(a-b)$ or $a+b=3a-3b$.
Thus, $4b=2a$ and so $2b=a$ or $2=\dfrac{a}{b}$.
(Note that $b\neq 0$, since otherwise the original equation would become $\dfrac{a}{a}=3$, which is not true.)

ANSWER: (D)

18. The equation $x^2+2kx+7k-10=0$ has two equal real roots precisely when the discriminant of this quadratic equation equals 0.
The discriminant, $\Delta$, equals

$$\Delta=(2k)^2-4(1)(7k-10)=4k^2-28k+40$$

For the discriminant to equal 0, we have $4k^2-28k+40=0$ or $k^2-7k+10=0$ or $(k-2)(k-5)=0$.
Thus, $k=2$ or $k=5$.
We check that each of these values gives an equation with the desired property.
When $k=2$, the equation is $x^2+4x+4=0$ which is equivalent to $(x+2)^2=0$ and so only has one solution for $x$.
When $k=5$, the equation is $x^2+10x+25=0$ which is equivalent to $(x+5)^2=0$ and so only has one solution for $x$.
The sum of these values of $k$ is $2+5=7$.

ANSWER: (E)

19. Suppose the slope of the three parallel lines is $m$.
The equation of a line with slope $m$ and $y$-intercept 2 is $y=mx+2$.
To find the $x$-intercept in terms of $m$, we set $y=0$ and solve for $x$.
Doing this, we obtain $mx+2=0$ or $x=-\dfrac{2}{m}$.
Similarly, the line with slope $m$ and $y$-intercept 3 has $x$-intercept $-\dfrac{3}{m}$.
Also, the line with slope $m$ and $y$-intercept 4 has $x$-intercept $-\dfrac{4}{m}$.
Since the sum of the $x$-intercepts of these lines is 36, then $\left(-\dfrac{2}{m}\right)+\left(-\dfrac{3}{m}\right)+\left(-\dfrac{4}{m}\right)=36$.
Multiplying both sides by $m$, we obtain $-2-3-4=36m$ and so $36m=-9$ or $m=-\dfrac{1}{4}$.

ANSWER: (E)

20. First, we factor $a^{2014}+a^{2015}$ as $a^{2014}(1+a)$.
If $a=5$ or $a=10$, then the factor $a^{2014}$ is a multiple of 5, so the original expression is divisible by 5.
If $a=4$ or $a=9$, then the factor $(1+a)$ is a multiple of 5, so the original expression is divisible by 5.
If $a=1,2,3,6,7,8$, then neither $a^{2014}$ nor $(1+a)$ is a multiple of 5.
Since neither factor is a multiple of 5, which is a prime number, then the product $a^{2014}(1+a)$ is not divisible by 5.
Therefore, there are four integers $a$ in the range $1\leq a\leq 10$ for which $a^{2014}+a^{2015}$ is divisible by 5.

ANSWER: (C)

21. If Amina wins, she can win on her first turn, on her second turn, or on her third turn.
If she wins on her first turn, then she went first and tossed tails.
This occurs with probability $\frac{1}{2}$.
If she wins on her second turn, then she tossed heads, then Bert tossed heads, then Amina tossed tails. This gives the sequence HHT. The probability of this sequence of tosses occurring is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. (Note that there is only one possible sequence of Ts and Hs for which Amina wins on her second turn, and the probability of a specific toss on any turn is $\frac{1}{2}$.)
Similarly, if Amina wins on her third turn, then the sequence of tosses that must have occurred is HHHHT, which has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.
Therefore, the probability that Amina wins is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{16+4+1}{32} = \frac{21}{32}$.

Answer: (A)

22. Since $a$, $b$ and $c$ form an arithmetic sequence in this order, then $a = b - d$ and $c = b + d$ for some real number $d$.
We note that $d \neq 0$, since otherwise we would have $a = b = c$ and then $abc = 17\,955$ would tell us that $b^3 = 17\,955$ or $b = \sqrt[3]{17\,955}$, which is not an integer.
Writing the terms of the geometric sequence in terms of $b$ and $d$, we have

$$3a+b = 3(b-d)+b = 4b-3d \qquad 3b+c = 3b+(b+d) = 4b+d \qquad 3c+a = 3(b+d)+(b-d) = 4b+2d$$

Since $3a + b$, $3b + c$ and $3c + a$ form a geometric sequence in this order, then

$$\begin{aligned} \frac{3b+c}{3a+b} &= \frac{3c+a}{3b+c} \\ (3b+c)^2 &= (3a+b)(3c+a) \\ (4b+d)^2 &= (4b-3d)(4b+2d) \\ 16b^2 + 8bd + d^2 &= 16b^2 - 4bd - 6d^2 \\ 12bd &= -7d^2 \\ 12b &= -7d \qquad \text{(since } d \neq 0\text{)} \\ d &= -\tfrac{12}{7}b \end{aligned}$$

Therefore, $a = b - d = b - (-\frac{12}{7}b) = \frac{19}{7}b$ and $c = b + d = b + (-\frac{12}{7}b) = -\frac{5}{7}b$.
Since $abc = 17\,955$, then $(\frac{19}{7}b)(b)(-\frac{5}{7}b) = 17\,955$ or $-\frac{95}{49}b^3 = 17\,955$ or $b^3 = -9261$ and so $b = -21$.
Thus, $a = \frac{19}{7}b = \frac{19}{7}(-21) = -57$ and $c = -\frac{5}{7}b = -\frac{5}{7}(-21) = 15$.
We can check that $a = -57$, $b = -21$ and $c = 15$ have a product of $17\,955$, that $-57, -21, 15$ is indeed an arithmetic sequence (with common difference 36), and that $3a + b = -192$, $3b + c = -48$, and $3c + a = -12$ form a geometric sequence (with common ratio $\frac{1}{4}$).
Therefore, $a + b + c = (-57) + (-21) + 15 = -63$.

Answer: (A)

23. Starting from the given equation, we obtain the equivalent equations

$$\begin{aligned} 5x^2 - 4xy + 2x + y^2 &= 624 \\ 5x^2 - 4xy + 2x + y^2 + 1 &= 625 \\ 4x^2 - 4xy + y^2 + x^2 + 2x + 1 &= 625 \\ (2x - y)^2 + (x+1)^2 &= 625 \end{aligned}$$

Note that $625 = 25^2$.
Since $x$ and $y$ are both integers, then the left side of the given equation is the sum of two perfect squares. Since any perfect square is non-negative, then each of these perfect squares is at most $625 = 25^2$.
The perfect squares from $0^2$ to $25^2$ are:

$$0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144,$$

$$169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625$$

The pairs of perfect squares from this list that have a sum of 625 are

$$625 = 625 + 0 = 576 + 49 = 400 + 225$$

(We can verify this relatively quickly by checking the difference between 625 and each of the perfect squares in the list from 324 to 625 to see if this difference is itself a perfect square. We do not need to check anything less than 324, since 625 is odd and so one of the two squares adding to 625 must be larger than the other and hence larger than half of 625.)
Therefore, $(2x - y)^2$ and $(x + 1)^2$ equal $25^2$ and $0^2$ in some order, or $24^2$ and $7^2$ in some order, or $20^2$ and $15^2$ in some order.
Thus, $2x - y$ and $x + 1$ equal $\pm 25$ and 0 in some order, or $\pm 24$ and $\pm 7$ in some order, or $\pm 20$ and $\pm 15$ in some order.
Since $x \geq 0$, then $x+1 \geq 1$, so we need to consider the possibilities that $x+1 = 25, 24, 7, 20, 15$:

- If $x + 1 = 25$, then $x = 24$. If $2x - y = 0$ and $x = 24$, then $y = 48$.
- If $x + 1 = 24$, then $x = 23$. If $2x - y = 7$ and $x = 23$, then $y = 39$; if $2x - y = -7$ and $x = 23$, then $y = 53$.
- If $x + 1 = 7$, then $x = 6$. If $2x - y = 24$ and $x = 6$, then $y = -12$; if $2x - y = -24$ and $x = 6$, then $y = 36$.
- If $x + 1 = 20$, then $x = 19$. If $2x - y = 15$ and $x = 19$, then $y = 23$; if $2x - y = -15$ and $x = 19$, then $y = 53$.
- If $x + 1 = 15$, then $x = 14$. If $2x - y = 20$ and $x = 14$, then $y = 8$; if $2x - y = -20$ and $x = 14$, then $y = 48$.

From this list, the pairs of non-negative integers $(x, y)$ that satisfy the condition $0 \leq x \leq y$ are $(x, y) = (24, 48), (23, 39), (23, 53), (6, 36), (19, 23), (19, 53), (14, 48)$.
There are 7 such pairs. (We can check by direct substitution that each pair satisfies the original equation.)

ANSWER: (E)

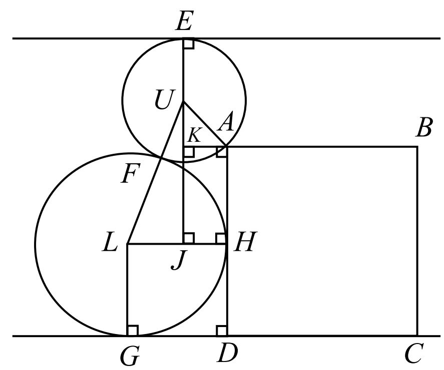
24. Let $r$ be the radius of the lower circle.

We label the square as $ABCD$, the centre of the upper circle as $U$, and the centre of the lower circle as $L$. We call $E$ the point at which the upper circle touches the top line, $G$ the point at which the lower circle touches the bottom line, $F$ the point at which the two circles touch, and $H$ the point at which the lower circle touches the square.

We join $EU$, $UF$, $FL$, $LG$, $LH$, and $UA$.

We need to use two facts about circles:

- When the centre of a circle is joined to a point of tangency, the resulting line segment is perpendicular to the tangent line. Thus, $UE$ is perpendicular to the top line, $LG$ is perpendicular to the bottom line, and $LH$ is perpendicular to $AD$.
- When two circles are tangent, the line segment joining their centres passes through the point of tangency between the circles. Thus, $UFL$ is a straight line segment.




Extend $EU$ downwards to meet $LH$ at $J$. Since $EU$ and $AD$ are perpendicular to the parallel lines and $LH$ is perpendicular to $AD$, then $EJ$ is perpendicular to $LH$.

Extend $BA$ to meet $EJ$ at $K$. Using a similar argument, $AK$ is perpendicular to $EJ$.

We focus on $\triangle UJL$, which is right-angled at $J$.

Since the radius of the upper circle is 65, then we know that $EU = UA = UF = 65$.

Since the radius of the lower circle is $r$, then we know that $FL = LH = LG = r$.

Thus, $UL = UF + FL = 65 + r = r + 65$.

Since the top and bottom lines are parallel, $EJ$ and $LG$ are perpendicular to these lines, and $LJ$ is parallel to these lines, then $EU + UJ + LG$ equals the distance between the lines.

Thus, $65 + UJ + r = 400$ and so $UJ = 335 - r$.

Next, we note that $LJ = LH - JH = r - JH = r - AK$, since $AKJH$ is a rectangle.

Now $\triangle UKA$ is right-angled and has $UA = 65$.

Also, $UK = EK - EU = EK - 65$.

But $EK$ equals the difference between the distance between the lines and the side length of the square, or $400 - 279 = 121$.

Thus, $UK = 121 - 65 = 56$.

Using the Pythagorean Theorem in $\triangle UKA$, we obtain $AK^2 = UA^2 - UK^2 = 65^2 - 56^2$ and so $AK^2 = 1089 = 33^2$.

Since $AK > 0$, then $AK = 33$.

Thus, $LJ = r - 33$.

Finally, using the Pythagorean Theorem in $\triangle UJL$, we obtain

$$\begin{aligned}
UJ^2 + LJ^2 &= UL^2 \\
(335 - r)^2 + (r - 33)^2 &= (r + 65)^2 \\
r^2 - 670r + 335^2 + r^2 - 66r + 33^2 &= r^2 + 130r + 65^2 \\
r^2 - 866r + 109089 &= 0
\end{aligned}$$

By the quadratic formula, $r = \dfrac{866 \pm \sqrt{866^2 - 4(1)(109089)}}{2} = \dfrac{866 \pm 560}{2} = 153$ or $713$.

Since $r$ must be less than the distance between the two lines, which is 400, then $r = 153$. Of the given answers, this is closest to 153.

ANSWER: (C)

25. This solution is written in as mathematically complete a way as possible. Of course, while doing a multiple choice contest, those who attempt and/or complete this problem would be unlikely to work through all of these details.

*Step 1: Using information about decimal equivalents to rewrite fractions*
Consider a real number $x$ whose decimal equivalent is of the form $0.g_1g_2\dots g_p\overline{r_1r_2\dots r_q}$ for some integers $p \geq 0$ and $q > 0$ and digits $g_1, g_2, \dots, g_p, r_1, r_2, \dots, r_q$.
(Note that if $p = 0$, then $0.g_1g_2\dots g_p\overline{r_1r_2\dots r_q} = 0.\overline{r_1r_2\dots r_q}$.)
Then $x = \dfrac{c}{10^p(10^q - 1)}$ for some positive integer $c$.
We demonstrate this in a specific example and leave the full algebraic derivation to the end of this solution. If $x = 0.12\overline{745}$, then

$$\begin{aligned}
x &= 0.12\overline{745} \\
100x &= 12.\overline{745} \\
10^2x - 12 &= 0.\overline{745} \\
1000(10^2x - 12) &= 745.\overline{745} \\
10^3(10^2x - 12) - 745 &= 0.\overline{745} \\
10^3(10^2x - 12) - 745 &= 10^2x - 12 \\
10^3(10^2x - 12) - (10^2x - 12) &= 745 \\
(10^3 - 1)(10^2x - 12) &= 745 \\
10^2x - 12 &= \frac{745}{10^3 - 1} \\
10^2x &= 12 + \frac{745}{10^3 - 1} \\
x &= \frac{12}{10^2} + \frac{745}{10^2(10^3 - 1)} \\
x &= \frac{(10^3 - 1)12 + 745}{10^2(10^3 - 1)}
\end{aligned}$$

Suppose that a fraction $\dfrac{m}{n}$ has $m$ and $n$ positive integers with $m < n$. Then $0 < \dfrac{m}{n} < 1$.
Suppose now that a fraction $\dfrac{m}{n}$ of positive integers with $0 < \dfrac{m}{n} < 1$ has the property that there is a sequence of consecutive digits of length 6 in its decimal equivalent that repeats consecutively and indefinitely. That is, suppose that $\dfrac{m}{n} = 0.g_1g_2\dots g_p\overline{r_1r_2\dots r_6}$ for some integer $p \geq 0$ and digits $g_1, g_2, \dots, g_p, r_1, r_2, r_3, r_4, r_5, r_6$.
From above, $\dfrac{m}{n} = \dfrac{c}{10^p \cdot 999999}$ for some positive integer $c$. (Note that $10^6 - 1 = 999999$.)

*Step 2: Using further conditions given to analyze $n$*
Continuing from above, $cn = 10^p \cdot 999999m$.
Since we are also told that $\dfrac{m}{n}$ is in lowest terms, then $m$ and $n$ have no common divisors larger than 1, and so $n$ must be a divisor of $10^p \cdot 999999$.
Note that $10^p \cdot 999999 = 2^p \cdot 5^p \cdot 999 \cdot 1001 = 2^p \cdot 5^p \cdot (3^3 \cdot 37) \cdot (11 \cdot 91) = 2^p \cdot 5^p \cdot 3^3 \cdot 37 \cdot 11 \cdot 7 \cdot 13$.
Since $n$ is a divisor of $2^p \cdot 5^p \cdot 3^3 \cdot 37 \cdot 11 \cdot 7 \cdot 13$, then $n$ cannot contain any prime factors other than $2, 3, 5, 7, 11, 13, 37$.
Since $n$ is not divisible by the square of any positive integer, then it cannot be divisible by the

square of any prime number.
Thus, $n$ must be a divisor of $2 \cdot 5 \cdot 3 \cdot 37 \cdot 11 \cdot 7 \cdot 13 = 1111110$.

*Step 3: Consolidating current progress*
We now know that any fraction $\frac{m}{n}$ satisfying the properties

- $m$ and $n$ are positive integers with $m < n$,
- $\frac{m}{n}$ is in lowest terms,
- $n$ is not divisible by the square of any integer larger than 1, and
- the decimal equivalent of $\frac{m}{n}$ includes a sequence of consecutive digits of length 6 that repeats consecutively and indefinitely,

can be written in the form $\frac{m}{n} = \frac{s}{1111110}$ for some positive integer $s$ with $1 \leq s \leq 1111109$.
(We note that we have not yet determined whether the *shortest* sequence of consecutive digits that repeats consecutively and indefinitely has length 6.)

*Step 4: Every fraction $\frac{s}{1111110}$ with $1 \leq s \leq 1111109$ can be written as a fraction satisfying these four bullets*
Each $\frac{s}{1111110}$ is between 0 and 1, can be written in lowest terms and has denominator not divisible by the square of any positive integer larger than 1, so any equivalent fraction in lower (or lowest terms) shares this property as factors of the denominator will only be removed in reducing, not added.
Furthermore,

$$\frac{s}{1111110} = \frac{1}{10} \cdot \frac{9s}{999999} = \frac{1}{10}\left(y + \frac{z}{999999}\right)$$

for some non-negative integers $y$ and $z$ with $0 \leq y < 10$ and $0 \leq z < 999998$. ($y$ and $z$ are the quotient and remainder, respectively, when $9s$ is divided by 999999.)
Writing $z = r_1r_2r_3r_4r_5r_6$ for some digits $r_1, r_2, r_3, r_4, r_5, r_6$ (some or all possibly 0), then

$$\frac{s}{1111110} = \frac{1}{10}\left(y + \frac{r_1r_2r_3r_4r_5r_6}{999999}\right) = \frac{1}{10}\left(y.\overline{r_1r_2r_3r_4r_5r_6}\right) = 0.y\overline{r_1r_2r_3r_4r_5r_6}$$

so every $\frac{s}{1111110}$ can be written as a decimal with a repeating sequence of length 6.
Also, each $\frac{s}{1111110}$ is different and so will produce a different $\frac{m}{n}$.
Therefore, the number of such fractions $\frac{s}{1111110}$ (which is 1111109) will equal the number of fractions $\frac{m}{n}$ that satisfy the four bullets above.
Notice that we have not yet checked to see if the sequence of digits of length 6 is the *shortest* such sequence.

*Step 5: Considering shorter possible lengths*
Since 6 is to be the length of the shortest sequence of repeating digits after the decimal point, then there can be no sequence of repeating digits of length 1, 2, 3, 4, or 5.
Using a similar approach to the first derivation above, we see that $\frac{m}{n}$ thus cannot be written

in any of the forms $\dfrac{c}{10^p \cdot 9}$ or $\dfrac{c}{10^p \cdot 99}$ or $\dfrac{c}{10^p \cdot 999}$ or $\dfrac{c}{10^p \cdot 9999}$ or $\dfrac{c}{10^p \cdot 99999}$.

Using a similar approach to the analysis of prime factors above, we see that $\dfrac{m}{n}$ cannot be written in any of the forms $\dfrac{t}{2 \cdot 5 \cdot 3} = \dfrac{t}{30}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 11} = \dfrac{t}{330}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 37} = \dfrac{t}{1110}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 11 \cdot 101} = \dfrac{t}{33330}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 41 \cdot 271} = \dfrac{t}{333330}$.

It is possible that an $\dfrac{m}{n}$ with properties as above including a repeating sequence of length 6 in its decimal equivalent can also be written with a repeating sequence of length 1, 2 or 3.

This is because, for example,

$$0.\overline{r_1} = 0.\overline{r_1r_1r_1r_1r_1r_1} \qquad 0.\overline{r_1r_2} = 0.\overline{r_1r_2r_1r_2r_1r_2} \qquad 0.\overline{r_1r_2r_3} = 0.\overline{r_1r_2r_3r_1r_2r_3}$$

which all have sequences of length 6 that repeat.

It is not possible for an $\dfrac{m}{n}$ with a repeating sequence of length 6 in its decimal equivalent to also be written with a repeating sequence of length 4 or 5, without a repeating sequence of length 1 or 2.

This is because if, for example, there is a repeating sequence of length 4, then

$$\frac{s}{1111110} = \frac{s}{2 \cdot 5 \cdot 3 \cdot 37 \cdot 11 \cdot 7 \cdot 13} = \frac{t}{2 \cdot 5 \cdot 3 \cdot 11 \cdot 101}$$

for some positive integer $t$ and so $2 \cdot 5 \cdot 3 \cdot 11 \cdot 101 \cdot s = 2 \cdot 5 \cdot 3 \cdot 37 \cdot 11 \cdot 7 \cdot 13 \cdot t$ and so $101 \cdot s = 37 \cdot 7 \cdot 13 \cdot t$ which tells us that 101 is divisor of $t$ and so $\dfrac{s}{1111110} = \dfrac{t'}{2 \cdot 5 \cdot 3 \cdot 11} = \dfrac{t'}{330}$ for some positive integer $t'$.

Therefore, the decimal equivalent to $\dfrac{s}{1111110} = \dfrac{t'}{330}$ has a sequence of length 2 that repeats, so any fraction with a decimal equivalent that has a repeating sequence of length 4 will be dealt with among those with sequences of length 1, 2 or 3.

In a similar way, we can rule out decimal equivalents with sequences of length 5 that repeat.

Therefore, we need to (carefully) remove fractions from our count that have decimal equivalents with sequences of lengths 1, 2 or 3 that repeat.

*Step 6: Considering overlap*

We have 1111109 fractions $\dfrac{m}{n}$ to consider, each of which satisfies the four bullets above.

Since the decimal equivalent of $\dfrac{m}{n}$ cannot have sequences of lengths 1, 2 or 3 that repeat, then $\dfrac{m}{n}$ cannot be written in any of the forms $\dfrac{u}{1110}$ or $\dfrac{v}{330}$ or $\dfrac{w}{30}$ for positive integers $u$, $v$ and $w$ with $u < 1110$, $v < 330$, and $w < 30$.

Let $U$ be the set of the 1009 fractions of the form $\dfrac{u}{1110}$, $V$ be the set of the 329 fractions of the form $\dfrac{v}{330}$, and $W$ be the set of the 29 fractions of the form $\dfrac{w}{30}$.

Any fraction of the form $\dfrac{w}{30}$ is also of the form $\dfrac{u}{1110}$ (since $\dfrac{w}{30} = \dfrac{37w}{1110}$) and is also of the form $\dfrac{v}{330}$ (since $\dfrac{w}{30} = \dfrac{11w}{330}$).

Therefore, every fraction in $W$ is also in $U$ and in $V$. In set notation $W \subseteq U$ and $W \subseteq V$.

Furthermore, any fraction that is in both $U$ and $V$ is also in $W$:

Suppose that a fraction can be written in both forms $\frac{u}{1110}$ and $\frac{v}{330}$.
Then $\frac{u}{1110} = \frac{v}{330}$ or $\frac{u}{37} = \frac{v}{11}$ and so $11u = 37v$.
Since $37v$ is a thus a multiple of 11 and 37 is not divisible by the prime number 11, then $v$ is a multiple of 11.
This means that $\frac{v}{330} = \frac{11f}{330} = \frac{f}{30}$ for some positive integer $f$, and so is in $W$.

In set notation, $U \cap V \subseteq W$.
Since $W \subseteq U$ and $W \subseteq V$ and $U \cap V \subseteq W$, then $U \cap V = W$; in other words, the set of fractions in each of $U$ and $V$ is precisely the set of fractions $W$.

*Step 7: Final counting*
We start with 1111109 fractions, as above, and want to remove all of the fractions in $U$, $V$ and $W$.
Since each fraction in $W$ is in $U$ and $V$, it is enough to remove those $U$ and $V$ only.
The total number of fractions in $U$ and $V$ (that is, in $U \cup V$) equals the number of fractions in $U$ plus the number of fractions in $V$ minus the number of fractions in their overlap (that is, in $U \cap V = W$). This is because any fraction in the overlap is "counted twice" when include all fractions in $U$ and all fractions in $V$.
Therefore, we need to remove $1009 + 329 - 29$ fractions from the set of 1111109.
Therefore, $F$, the number of fractions having the desired properties, is

$$F = 1111109 - (1009 + 329 - 29) = 1109700$$

Since $F$ has 7 digits, then $G = F + 7 = 1109707$. The sum of the squares of the digits of $G$ is $1^2 + 1^2 + 0^2 + 9^2 + 7^2 + 0^2 + 7^2 = 1 + 1 + 81 + 49 + 49 = 181$.

*Step 8: General algebraic derivation from Step 1*
Consider a real number $x$ whose decimal equivalent is of the form $0.g_1g_2\dots g_p\overline{r_1r_2\dots r_q}$ for some integers $p \geq 0$ and $q > 0$ and digits $g_1, g_2, \dots, g_p, r_1, r_2, \dots, r_q$. Then

$$\begin{aligned}
x &= 0.g_1g_2\dots g_p\overline{r_1r_2\dots r_q} \\
10^p x &= g_1g_2\dots g_p.\overline{r_1r_2\dots r_q} \\
10^p x - g_1g_2\dots g_p &= 0.\overline{r_1r_2\dots r_q} \\
10^q(10^p x - g_1g_2\dots g_p) &= r_1r_2\dots r_q.\overline{r_1r_2\dots r_q} \\
10^q(10^p x - g_1g_2\dots g_p) - r_1r_2\dots r_q &= 0.\overline{r_1r_2\dots r_q} \\
10^q(10^p x - g_1g_2\dots g_p) - r_1r_2\dots r_q &= 10^p x - g_1g_2\dots g_p \\
10^q(10^p x - g_1g_2\dots g_p) - (10^p x - g_1g_2\dots g_p) &= r_1r_2\dots r_q \\
(10^q - 1)(10^p x - g_1g_2\dots g_p) &= r_1r_2\dots r_q \\
10^p x - g_1g_2\dots g_p &= \frac{r_1r_2\dots r_q}{10^q - 1} \\
10^p x &= g_1g_2\dots g_p + \frac{r_1r_2\dots r_q}{10^q - 1} \\
x &= \frac{g_1g_2\dots g_p}{10^p} + \frac{r_1r_2\dots r_q}{10^p(10^q - 1)} \\
x &= \frac{(10^q - 1)g_1g_2\dots g_p + r_1r_2\dots r_q}{10^p(10^q - 1)}
\end{aligned}$$

ANSWER: (E)

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2014 Fermat Contest*

**(Grade 11)**

**Thursday, February 20, 2014**
**(in North America and South America)**

**Friday, February 21, 2014**
**(outside of North America and South America)**

*Solutions*

1. Evaluating the numerator first, we obtain $\frac{15-3^2}{3} = \frac{15-9}{3} = \frac{6}{3} = 2$.

   Answer: (A)

2. Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10\,000$, and $10^5 = 100\,000$, then 2014 is between $10^3$ and $10^4$.

   Answer: (D)

3. When $x = 2$, we have $(x+2-x)(2-x-2) = (2+2-2)(2-2-2) = (2)(-2) = -4$.
   Alternatively, we could simplify $(x+2-x)(2-x-2)$ to obtain $(2)(-x)$ or $-2x$ and then substitute $x = 2$ to obtain a result of $-2(2)$ or $-4$.

   Answer: (E)

4. The positive integer divisors of 24 are $1, 2, 3, 4, 6, 8, 12, 24$.
   The pairs of divisors that give a product of 24 are $24 \times 1$, $12 \times 2$, $8 \times 3$, and $6 \times 4$.
   We want to find two positive integers $x$ and $y$ whose product is 24 and whose difference is 5.
   Since $8 \times 3 = 24$ and $8 - 3 = 5$, then $x = 8$ and $y = 3$ are the required integers.
   Here, $x + y = 8 + 3 = 11$.

   Answer: (B)

5. Since square $WXYZ$ has area 9, then its side length is $\sqrt{9} = 3$.
   Since $W$ is the centre of the circle and $X$ lies on the circumference of the circle, then $WX$ is a radius of the circle, so the radius of the circle has length 3.
   Therefore, the area of the circle is $\pi(3^2) = 9\pi$.

   Answer: (C)

6. The percentage 50% is equivalent to the fraction $\frac{1}{2}$, while 75% is equivalent to $\frac{3}{4}$.
   Since 50% of $N$ is 16, then $\frac{1}{2}N = 16$ or $N = 32$.
   Therefore, 75% of $N$ is $\frac{3}{4}N$ or $\frac{3}{4}(32)$, which equals 24.

   Answer: (D)

7. *Solution 1*
   $\angle SRP$ is an exterior angle for $\triangle PQR$.
   Therefore, $\angle SRP = \angle RPQ + \angle RQP$ or $(180 - x)^\circ = 30^\circ + 2x^\circ$.
   Thus, $180 - x = 30 + 2x$ or $3x = 150$ and so $x = 50$.

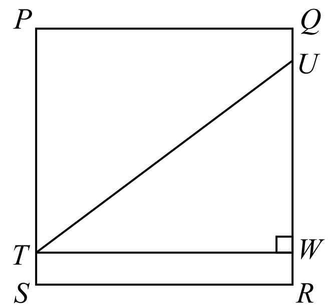
   *Solution 2*
   Since $QRS$ is a straight line segment and $\angle SRP = (180 - x)^\circ$, then $\angle PRQ$ is the supplement of $\angle SRP$ so $\angle PRQ = x^\circ$.
   Since the angles in a triangle add to $180^\circ$, then $\angle PRQ + \angle PQR + \angle RPQ = 180^\circ$, and so $x^\circ + 2x^\circ + 30^\circ = 180^\circ$.
   From this, we obtain $3x = 150$ and so $x = 50$.

   Answer: (E)

8. We use $A, B, C, D, E$ to represent Amy, Bob, Carl, Dan, and Eric, respectively.
   We use the greater than symbol ($>$) to represent "is taller than" and the less than symbol ($<$) to represent "is shorter than".
   From the first bullet, $A > C$.
   From the second bullet, $D < E$ and $D > B$ so $E > D > B$.
   From the third bullet, $E < C$ or $C > E$.
   Since $A > C$ and $C > E$ and $E > D > B$, then $A > C > E > D > B$, which means that Bob is the shortest.

   Answer: (B)

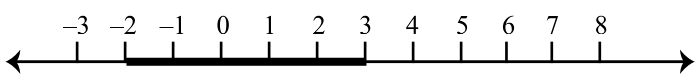
9. We draw a line through $T$ to point $W$ on $QR$ so that $TW$ is perpendicular to $QR$.



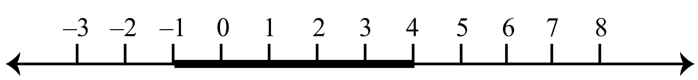

Since $TWRS$ has three right angles (at $W$, $R$ and $S$), then it must be a rectangle.
Therefore, $WR = TS = 1$ and $TW = SR = 8$.
Since $QU = 1$, then $UW = QR - QU - WR = 8 - 1 - 1 = 6$.
Now, $\triangle TWU$ is right-angled at $W$.
By the Pythagorean Theorem, we have $TU^2 = TW^2 + UW^2$.
Thus, $TU^2 = 8^2 + 6^2 = 64 + 36 = 100$.
Since $TU > 0$, then $TU = \sqrt{100} = 10$.

ANSWER: (C)

10. After the first rotation, the line segment lies between $-2$ and $3$.




The line segment is now to be rotated about the point at 1.
Since the right endpoint of the segment is 2 units to the right of 1 before the rotation, then the left endpoint of the segment will be 2 units to the left of 1 after the rotation.
Thus, the left endpoint will be at $-1$.
Since the line segment has length 5, then its right endpoint will be at $-1 + 5 = 4$.




Thus, the line segment lies between $-1$ and $4$.

ANSWER: (B)

11. Since $a = \frac{2}{3}b$, then $3a = 2b$. Since $b \neq 0$, then $a \neq 0$.
Thus, $\frac{9a+8b}{6a} = \frac{9a+4(2b)}{6a} = \frac{9a+4(3a)}{6a} = \frac{21a}{6a} = \frac{7}{2}$ (since $a \neq 0$).
Alternatively, $\frac{9a+8b}{6a} = \frac{3(3a)+8b}{2(3a)} = \frac{3(2b)+8b}{2(2b)} = \frac{14b}{4b} = \frac{7}{2}$ (since $b \neq 0$).

ANSWER: (A)

12. Since $100 = 10^2$, then $100^4 = (10^2)^4 = 10^8$.
Therefore, we must solve the equation $10^x \cdot 10^5 = 10^8$, which is equivalent to $10^{x+5} = 10^8$.
Thus, $x + 5 = 8$ or $x = 3$.

ANSWER: (E)

13. We note that the sum of the digits of 1000 is not 3. Every other positive integer in the given range has two or three digits.
For the sum of the digits of an integer to be 3, no digit can be greater than 3.
If a two-digit integer has sum of digits equal to 3, then its tens digit is 1, 2 or 3. The possible integers are 12, 21 and 30.
If a three-digit integer has sum of digits equal to 3, then its hundreds digit is 1, 2 or 3.
If the hundreds digit is 3, then the units and tens digits add to 0, so must be each 0. The integer must thus be 300.
If the hundreds digit is 2, then the units and tens digits add to 1, so must be 1 and 0 or 0 and 1. The possible integers are 210 and 201.
If the hundreds digit is 1, then the units and tens digits add to 2, so must be 2 and 0, or 1 and 1, or 0 and 2, giving possible integers 120, 111 and 102.
Overall, there are 9 such positive integers.

Answer: (D)

14. Let $x$ be the number of days on which Pat worked.
On each of these days, he earned \$100 and had no food costs, so he earned a total of $100x$ dollars.
Since Pat worked for $x$ of the 70 days, then he did not work on $70 - x$ days.
On each of these days, he earned no money and was charged \$20 for food, so was charged a total of $20(70 - x)$ dollars for food.
After 70 days, the money that he earned minus his food costs equalled \$5440.
Algebraically, we get $100x - 20(70 - x) = 5440$.
Thus, $100x - 1400 + 20x = 5440$ or $120x = 6840$, which gives $x = 57$.
Therefore, Pat worked on 57 of these 70 days.
(An alternative approach would be to test each of the five given choices to see how much money Pat earns after food costs are deducted.)

Answer: (D)

15. We make a chart that lists the possible results for the first spin down the left side, the possible results for the second spin across the top, and the product of the two results in the corresponding cells:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 |
| 2 | 2 | 4 | 6 | 8 |
| 3 | 3 | 6 | 9 | 12 |
| 4 | 4 | 8 | 12 | 16 |

Since each spin is equally likely to stop on 1, 2, 3, or 4, then each of the 16 products shown in the chart is equally likely.
Since the product 4 appears three times in the table and this is more than any of the other numbers, then it is the product that is most likely to occur.

Answer: (B)

16. Since Jill never drove faster than 80 km/h over her 5 hour drive, then she could not have driven more than $5 \times 80 = 400$ km.
Since the initial odometer reading was 13 831 km, then the final odometer reading is no more than $13\,831 + 400 = 14\,231$ km.
Determining her greatest possible average speed can be done by first determining the greatest possible distance that she could have travelled, which can be done by determining the greatest

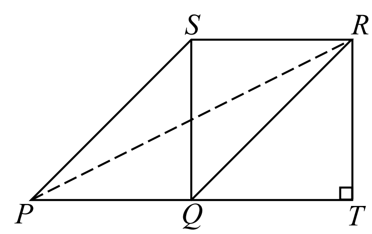
possible odometer reading.
Knowing that the final odometer reading was also a palindrome, we want to determine the greatest palindrome less than 14 231. This is 14 141. (To find this, we begin by trying to find palindromes that are at least 14 000. Such palindromes end with 41, so are of the form $14\,x41$. The greatest such integer less than 14 231 is 14 141.)
Since Jill's greatest possible final odometer reading was 14 141, then she would have travelled $14\,141 - 13\,831 = 310$ km, and so her greatest possible average speed was $\frac{310}{5} = 62$ km/h.

Answer: (A)

17. Suppose that there are $n$ employees at Sergio's store.
After his first average calculation, his $n$ employees had sold an average of 75 items each, which means that a total of $75n$ items had been sold.
The next day, one employee sold 6 items, one sold 5, one sold 4, and the remaining $(n-3)$ employees each sold 3 items.
After this day, the total number of items sold to date was $75n + (6+5+4+(n-3)3)$ or $75n + 15 + 3n - 9$ or $78n + 6$.
Since the new average number of items sold per employee was 78.3, then $\dfrac{78n+6}{n} = 78.3$ or $78n + 6 = 78.3n$.
Therefore, $0.3n = 6$ or $n = 20$.
Thus, there are 20 employees in the store.

Answer: (C)

18. Suppose that the original square had side length $x$ mm.
We extend $PQ$ and draw a line through $R$ perpendicular to $PQ$, meeting $PQ$ extended at $T$.




$SRTQ$ is a square, since it has three right angles at $S$, $Q$, $T$ (which makes it a rectangle) and since $SR = SQ$ (which makes the rectangle a square).
Now $RT = SQ = x$ mm and $PT = PQ + QT = 2x$ mm.
By the Pythagorean Theorem, $PR^2 = PT^2 + RT^2$ and so $90^2 = x^2 + (2x)^2$.
Therefore, $5x^2 = 8100$ or $x^2 = 1620$.
The area of the original square is $x^2$ mm$^2$, which equals 1620 mm$^2$.

Answer: (B)

19. Consider three three-digit numbers with digits $RST$, $UVW$ and $XYZ$.
The integer with digits $RST$ equals $100R + 10S + T$, the integer with digits $UVW$ equals $100U + 10V + W$, and the integer with digits $XYZ$ equals $100X + 10Y + Z$.
Therefore,

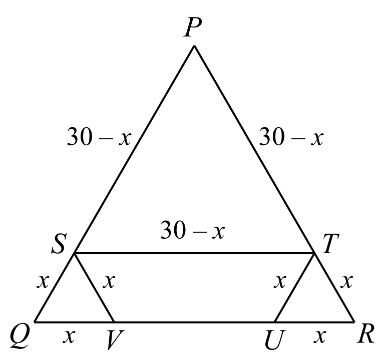
$$\begin{aligned} RST + UVW + XYZ &= 100R + 10S + T + 100U + 10V + W + 100X + 10Y + Z \\ &= 100(R+U+X) + 10(S+V+Y) + (T+W+Z) \end{aligned}$$

We note that each of $R, S, T, U, V, W, X, Y, Z$ can be any digit from 0 to 9, except that $R$, $U$ and $X$ cannot be 0.

Max wants to make $100(R+U+X)+10(S+V+Y)+(T+W+Z)$ as large as possible. He does this by placing the largest digits (9, 8 and 7) as hundreds digits, the next largest digits (6, 5 and 4) as tens digits, and the next largest digits (3, 2 and 1) as units digits. We note that no digits can be repeated, and that the placement of the digits assigned to any of the place values among the three different three-digit numbers is irrelevant as it does not affect the actual sum.
Max's sum is thus $100(9+8+7)+10(6+5+4)+(3+2+1)=2400+150+6=2556$.
Minnie wants to make $100(R+U+X)+10(S+V+Y)+(T+W+Z)$ as small as possible. She does this by placing the smallest allowable digits (1, 2 and 3) as hundreds digits, the next smallest remaining digits (0, 4 and 5) as tens digits, and the next smallest digits (6, 7 and 8) as units digits.
Minnie's sum is thus $100(1+2+3)+10(0+4+5)+(6+7+8)=600+90+21=711$.
The difference between their sums is $2556-711=1845$.

Answer: (C)

20. Since $\triangle PQR$ has $PQ=QR=RP$, then $\triangle PQR$ is equilateral and all of its angles equal $60^\circ$. Since $ST$ is parallel to $QR$, $SV$ is parallel to $PR$, and $TU$ is parallel to $PQ$, then all of the angles in $\triangle PST$, $\triangle SQV$ and $\triangle TUR$ equal $60^\circ$. In other words, each of these triangles is also equilateral.
Let $SQ=x$.
Since $\triangle SQV$ is equilateral, then $QV=VS=SQ=x$.
Since $PQ=30$, then $PS=30-x$.
Since $\triangle PST$ is equilateral, then $ST=TP=PS=30-x$.
Since $PR=30$, then $TR=30-(30-x)=x$.
Since $\triangle TUR$ is equilateral, then $TU=UR=TR=x$.




Since $VS+ST+TU=35$, then $x+(30-x)+x=35$ or $30+x=35$ and so $x=5$.
Therefore, $VU=QR-QV-UR=30-x-x=30-5-5=20$.

Answer: (D)

21. When 2 kg of the 10 kg of peanuts are removed, there are 8 kg of peanuts remaining.
Since 2 kg of raisins are added, then there are 2 kg of raisins in the bin.
The peanuts and raisins are thoroughly mixed.
Since 2 kg of this mixture is removed and this is one-fifth of the total mass, then one-fifth of the mass of peanuts (or $\frac{8}{5}$ kg) is removed and one-fifth of the mass of raisins (or $\frac{2}{5}$ kg) is removed.
This leaves $8-\frac{8}{5}=\frac{32}{5}$ kg of peanuts, and $2-\frac{2}{5}=\frac{8}{5}$ kg of raisins.
When 2 kg of raisins are added, the mass of raisins becomes $\frac{8}{5}+2=\frac{18}{5}$ kg.
There are $\frac{32}{5}$ kg of peanuts and $\frac{18}{5}$ kg of raisins in the bin.
Therefore, the ratio of the masses is $\frac{32}{5}:\frac{18}{5}=32:18=16:9$.

Answer: (E)

22. As Jillian drives from $J$ to $G$, suppose that she drives $x$ km uphill, $y$ km on flat ground, and $z$ km downhill.
This means that when she drives from $G$ to $J$, she will drive $z$ km uphill, $y$ km on flat ground, and $x$ km downhill. This is because downhill portions become uphill portions on the return trip, while uphill portions become downhill portions on the return trip.
We are told that Jillian drives at 77 km/h on flat ground, 63 km/h uphill, and 99 km/h downhill.
Since time equals distance divided by speed, then on her trip from $J$ to $G$, her time driving uphill is $\frac{x}{63}$ hours, her time driving on flat ground is $\frac{y}{77}$ hours, and her time driving downhill is $\frac{z}{99}$ hours.
Since it takes her 3 hours and 40 minutes (which is $3\frac{2}{3}$ or $\frac{11}{3}$ hours), then

$$\frac{x}{63}+\frac{y}{77}+\frac{z}{99}=\frac{11}{3}$$

A similar analysis of the return trip gives

$$\frac{x}{99}+\frac{y}{77}+\frac{z}{63}=\frac{13}{3}$$

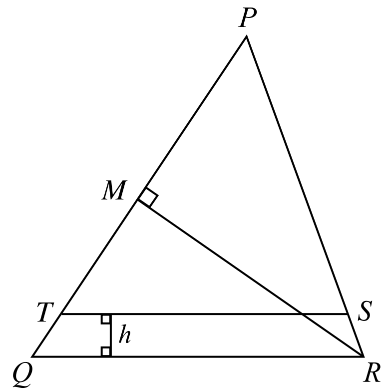
We are asked for the total distance from $J$ to $G$, which equals $x+y+z$ km. Therefore, we need to determine $x+y+z$.
We add the two equations above and simplify to obtain

$$\begin{aligned}
\frac{x}{63}+\frac{x}{99}+\frac{y}{77}+\frac{y}{77}+\frac{z}{99}+\frac{z}{63} &= \frac{24}{3}\\
x\left(\frac{1}{63}+\frac{1}{99}\right)+y\left(\frac{1}{77}+\frac{1}{77}\right)+z\left(\frac{1}{99}+\frac{1}{63}\right) &= 8\\
x\left(\frac{1}{7\cdot 9}+\frac{1}{9\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{1}{9\cdot 11}+\frac{1}{7\cdot 9}\right) &= 8\\
x\left(\frac{11}{7\cdot 9\cdot 11}+\frac{7}{7\cdot 9\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{7}{7\cdot 9\cdot 11}+\frac{11}{7\cdot 9\cdot 11}\right) &= 8\\
x\left(\frac{18}{7\cdot 9\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{18}{7\cdot 9\cdot 11}\right) &= 8\\
x\left(\frac{2}{7\cdot 11}\right)+\frac{2}{77}y+z\left(\frac{2}{7\cdot 11}\right) &= 8\\
\frac{2}{77}(x+y+z) &= 8
\end{aligned}$$

Thus, $x+y+z=\frac{77}{2}\cdot 8=77\cdot 4=308$.
Finally, the distance from $J$ to $G$ is 308 km.

Answer: (C)

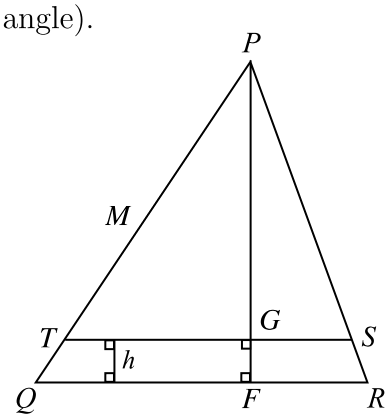
23. We label the other two vertices of the bottom section as $S$ (on $PR$) and $T$ (on $PQ$).
First, we calculate the area of $\triangle PQR$.
We do this by dropping a perpendicular from $R$ to $PQ$.
Since $PR = QR$, then $\triangle PQR$ is isosceles and the perpendicular from $R$ to $PQ$ meets $PQ$ at its midpoint $M$.
Thus, $PM = MQ = \frac{1}{2}(150) = 75$.




By the Pythagorean Theorem,

$$RM^2 = RQ^2 - MQ^2 = 125^2 - 75^2 = 15\,625 - 5625 = 10\,000$$

Since $RM > 0$, then $RM = \sqrt{10\,000} = 100$.
Therefore, the area of $\triangle PQR$ is $\frac{1}{2}(RM)(PQ) = \frac{1}{2}(100)(150) = 7500$.
Let $F$ be the point on $QR$ for which $PF$ is perpendicular to $QR$ and let $G$ be the point where $PF$ intersects $TS$ (also at a $90^\circ$ angle).

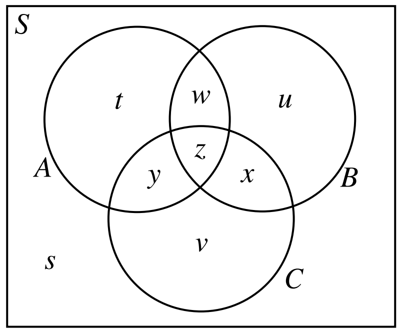



Then the area of $\triangle PQR$ is also equal to $\frac{1}{2}(QR)(PF)$.
Thus, $\frac{1}{2}(125)(PF) = 7500$ and so $PF = \frac{15\,000}{125} = 120$.
Since $TS$ is parallel to $QR$, then $\angle PTS = \angle PQR$ and $\angle PST = \angle PRQ$.
This means that $\triangle PTS$ is similar to $\triangle PQR$.
Since each of the four sections of $\triangle PQR$ is one-quarter of the total area, then $\triangle PTS$ is three-quarters of the total area of $\triangle PQR$.
This means that the dimensions of $\triangle PTS$ are $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$ of those of $\triangle PQR$. (In general, if two similar triangles have areas in the ratio $k : 1$, then their corresponding sides are in the ratio $\sqrt{k} : 1$.)
Therefore, $PG = \frac{\sqrt{3}}{2}PF = \frac{\sqrt{3}}{2}(120) = 60\sqrt{3}$.
Finally, since $TS$ is parallel to $QR$, then $h = GF = PF - PG = 120 - 60\sqrt{3} \approx 16.077$.
Of the given answers, the height is closest to 16.1.

Answer: (E)

24. If we have $n$ balls to be placed in $n$ boxes, one per box and without restriction, then there are $n! = n(n-1)(n-2)\cdots(3)(2)(1)$ ways to do this. (This is because there are $n$ choices for the ball to placed in box 1; for each of these, there are $n-1$ choices for the ball to be placed in box 2; for each of these pairs of choices, there are $n-2$ choices for the ball to be placed in box 3, and so on. In total, there are $n(n-1)(n-2)\cdots(3)(2)(1)$ ways to do this.)

We draw a Venn diagram where $S$ represents all of the ways of placing the 8 balls in 8 boxes without restriction, circle $A$ represents the ways in which the balls are placed with ball 1 going in box 1, circle $B$ represents the ways with ball 2 going in box 2, and circle $C$ represents the ways with ball 3 going in box 3.




Here, $s$ represents the number of ways of putting the balls in boxes so that ball 1 is not in box 1 ($s$ is outside circle $A$), ball 2 is not in box 2 ($s$ is outside circle $B$), and ball 3 is not in box 3 ($s$ is outside circle $C$). We want to calculate $s$.
The total number of ways in $S$ is $8!$.
Circle $A$ represents the ways when ball 1 is in box 1, and the other 7 balls are placed without restriction. There are $7!$ such ways.
Similarly, the number of ways inside each of circle $B$ and circle $C$ is $7!$.
In other words, $t+w+y+z = u+w+x+z = v+x+y+z = 7!$.
The overlap between circle $A$ and $B$ represents the ways with ball 1 in box 1 and ball 2 in box 2, with the other 6 balls placed without restriction. There are $6!$ such ways.
Similarly, there are $6!$ ways in the intersection of circles $A$ and $C$, and circles $B$ and $C$.
In other words, $w+z = y+z = x+z = 6!$.
Finally, the intersection of all three circles represents the ways in which ball 1 is in box 1, ball 2 is in box 2, ball 3 is in box 3, and the other 5 balls are placed without restriction. There are $5!$ such ways.
In other words, $z = 5!$.
Since $z = 5!$, then $w = x = y = 6! - 5!$.
Futhermore, $t = u = v = 7! - 2(6! - 5!) - 5! = 7! - 2(6!) + 5!$.
Finally,

$$\begin{aligned}
s &= 8! - (t+u+v+w+x+y+z) \\
&= 8! - 3(7! - 2(6!) + 5!) - 3(6! - 5!) - 5! \\
&= 8! - 3(7!) + 6(6!) - 3(5!) - 3(6!) + 3(5!) - 5! \\
&= 8! - 3(7!) + 3(6!) - 5!
\end{aligned}$$

Now, $5! = 5(4)(3)(2)(1) = 120$ so $6! = 6(5!) = 6(120) = 720$ and $7! = 7(6!) = 7(720) = 5040$ and $8! = 8(7!) = 8(5040) = 40\,320$.
Therefore, $s = 40\,320 - 3(5040) + 3(720) - 120 = 27\,240$.
Thus, the number of ways of putting the balls in the boxes with the given restrictions is $27\,240$.

We note that an alternative way of performing the algebraic steps is as follows:

$$\begin{aligned}
s &= 8! - (t+u+v+w+x+y+z) \\
&= 8! - (t+w+y+z) - (u+w+x+z) - (v+x+y+z) + w+x+y+2z \\
&= 8! - (t+w+y+z) - (u+w+x+z) - (v+x+y+z) + (w+z) + (x+z) + (y+z) - z \\
&= 8! - 7! - 7! - 7! + 6! + 6! + 6! - 5! \\
&= 8! - 3(7!) + 3(6!) - 5!
\end{aligned}$$

which leads to the same final answer.

ANSWER: (A)

25. We first examine the conditions on the length and slope of $PQ$, then simplify the condition that $r+s+t+u=27$, then finally incorporate the fact that points $P$ and $Q$ lie on the given parabola.

Since the slope of $PQ$ is positive, then one of the points is "up and to the right" from the other. Without loss of generality, we assume that $Q$ is "up and to the right" from $P$.
Thus, $u > s$ and $t > r$.
Since the slope of $PQ$ equals $\frac{12}{5}$, then $\dfrac{u-s}{t-r} = \dfrac{12}{5}$, which means that $u-s = 12k$ and $t-r = 5k$ for some real number $k > 0$.
Since $PQ = 13$, then $(u-s)^2 + (t-r)^2 = 13^2$ or $(12k)^2 + (5k)^2 = 169$.
Thus, $144k^2 + 25k^2 = 169$ and so $169k^2 = 169$ or $k^2 = 1$, which gives $k = 1$ (since $k > 0$).
Thus, $u - s = 12$ (or $u = s + 12$) and $t - r = 5$ (or $t = r + 5$).
Thus, $P$ has coordinates $(r, s)$ and $Q$ has coordinates $(r+5, s+12)$.
(This eliminates two of the variables from the problem.)

We eventually need to use the condition that $r+s+t+u = 27$.
Since $t = r+5$ and $u = s+12$, this is equivalent to $r+s+r+5+s+12 = 27$ or $2r+2s = 10$, which is equivalent to $r + s = 5$.
We will use this simplified condition shortly.

Since both $P(r,s)$ and $Q(r+5, s+12)$ are on the parabola with equation $y = x^2 - \frac{1}{5}mx + \frac{1}{5}n$, we obtain the following two equations:

$$\begin{aligned}
s &= r^2 - \tfrac{1}{5}mr + \tfrac{1}{5}n \\
s + 12 &= (r+5)^2 - \tfrac{1}{5}m(r+5) + \tfrac{1}{5}n
\end{aligned}$$

Here, $m$ and $n$ are treated as known constants and $r$ and $s$ are variables for which we solve.
We want to determine the number of pairs $(m,n)$ of positive integers with $n \leq 1000$ for which a solution $(r,s)$ to this equation satisfies $r+s=5$.
We solve this system of equations for $r$ and $s$.
Expanding the second equation, we obtain

$$s + 12 = r^2 + 10r + 25 - \tfrac{1}{5}mr - m + \tfrac{1}{5}n$$

Subtracting the first equation, we obtain $12 = 10r + 25 - m$ and so $r = \frac{1}{10}(m-13)$.
Substituting into the first equation, we obtain

$$\begin{aligned}
s &= \left(\tfrac{1}{10}(m-13)\right)^2 - \tfrac{1}{5}m\left(\tfrac{1}{10}(m-13)\right) + \tfrac{1}{5}n \\
&= \tfrac{1}{100}(m^2 - 26m + 169) - \tfrac{1}{50}(m^2 - 13m) + \tfrac{1}{5}n \\
&= \tfrac{1}{100}(m^2 - 26m + 169 - 2(m^2 - 13m) + 20n) \\
&= \tfrac{1}{100}(-m^2 + 169 + 20n)
\end{aligned}$$

Note that each pair $(m, n)$ gives a unique solution $(r, s)$.

We now need to determine the number of pairs $(m, n)$ of positive integers with $n \leq 1000$ that produce $r = \frac{1}{10}(m - 13)$ and $s = \frac{1}{100}(-m^2 + 169 + 20n)$ satisfying the equation $r + s = 5$.

Substituting, we obtain

$$\begin{aligned} r + s &= 5 \\ \tfrac{1}{10}(m-13) + \tfrac{1}{100}(-m^2 + 169 + 20n) &= 5 \\ 10m - 130 - m^2 + 169 + 20n &= 500 \\ 20n &= m^2 - 10m + 461 \\ 20n &= (m-5)^2 + 436 \qquad \text{(completing the square)} \end{aligned}$$

So we are left to determine the number of pairs $(m, n)$ of positive integers with $n \leq 1000$ that satisfy the equation $20n = (m - 5)^2 + 436$.
Since $20n$ and 436 are both even integers, then $(m - 5)^2$ is an even integer, which means that $m - 5$ is an even integer and thus $m$ is odd. (If $m - 5$ were odd, then $(m - 5)^2$ would also be odd.)
Therefore, we let $m = 2M - 1$ for some positive integer $M$.
Substituting, we obtain $20n = (2M - 6)^2 + 436$ or $20n = 4(M - 3)^2 + 436$, which is equivalent to $5n = (M - 3)^2 + 109$.
Therefore, we need to determine the number of pairs $(M, n)$ of positive integers with $n \leq 1000$ that satisfy the equation $5n = (M - 3)^2 + 109$.
This is the same as determining the number of positive integer values of $M$ for which the right side is a multiple of 5 that is at most 5000, because each such value of $M$ will give a corresponding positive integer value for $n$ that is at most 1000.
When $M = 1$, the right side equals 113, which is not a multiple of 5.
When $M = 2$, the right side equals 110, which is a multiple of 5. (This gives $n = 22$.)
When $M \geq 3$, we consider the units (ones) digits of $M - 3$, $(M - 3)^2$ and $(M - 3)^2 + 109$:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $M - 3$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| $(M - 3)^2$ | 0 | 1 | 4 | 9 | 6 | 5 | 6 | 9 | 4 | 1 |
| $(M - 3)^2 + 109$ | 9 | 0 | 3 | 8 | 5 | 4 | 5 | 8 | 3 | 0 |

Therefore, any positive integer $M$ for which $M - 3$ ends in a 1, 4, 6, or 9 produces a right side divisible by 5. This is because in each of these cases the expression $(M - 3)^2 + 109$ has a units digit of 0 or 5 and hence is divisible by 5.
For $(M - 3)^2 + 109 \leq 5000$, we need $(M - 3)^2 \leq 4891$.
Since $\sqrt{4891} \approx 69.94$ and $M - 3$ is a positive integer, then $M - 3 \leq 69$.
Therefore, the values of $M$ with the desired property are $M = 2$ and every positive integer $M$ with $0 \leq M - 3 \leq 69$ for which $M - 3$ has a units digit of 1, 4, 6, or 9.
There are 28 positive integers $M$ in the second list (four each with $M - 3$ between 0 and 9, 10 and 19, 20 and 29, 30 and 39, 40 and 49, 50 and 59, and 60 and 69). Thus, there are 29 such integers $M$ overall.
Finally, this means that there are 29 pairs $(m, n)$ which have the desired property.

Answer: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2013 Fermat Contest*

(Grade 11)

**Thursday, February 21, 2013**
**(in North America and South America)**

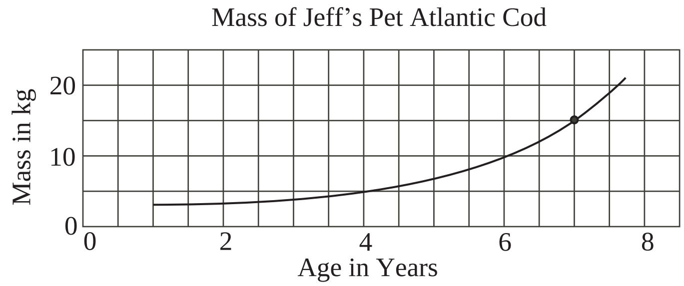
**Friday, February 22, 2013**
**(outside of North America and South America)**

*Solutions*

1. Simplifying, $\dfrac{10^2+6^2}{2} = \dfrac{100+36}{2} = \dfrac{136}{2} = 68$.

   Answer: (D)

2. A mass of 15 kg is halfway between 10 kg and 20 kg on the vertical axis.
   The point where the graph reaches 15 kg is halfway between 6 and 8 on the horizontal axis.




   Therefore, the cod is 7 years old when its mass is 15 kg.

   Answer: (B)

3. Each interior angle in a square is $90^\circ$. In particular, $\angle SPQ = 90^\circ$.
   Each interior angle in an equilateral triangle is $60^\circ$. In particular, $\angle TPQ = 60^\circ$.
   $PR$ is a diagonal of square $PQRS$. Thus, it bisects angle $\angle SPQ$, with $\angle SPR = \angle RPQ = 45^\circ$.
   Therefore, $\angle TPR = \angle TPQ + \angle QPR = 60^\circ + 45^\circ = 105^\circ$.

   Answer: (B)

4. Since the tick marks divide the cylinder into four parts of equal volume, then the level of the milk shown is a bit less than $\frac{3}{4}$ of the total volume of the cylinder.
   Three-quarters of the total volume of the cylinder is $\frac{3}{4} \times 50 = 37.5$ L.
   Of the five given choices, the one that is slightly less than 37.5 L is 36 L, or (D).

   Answer: (D)

5. Since $PQRS$ and $WXYZ$ are rectangles, then $SR = PQ = 30$ and $WX = ZY = 15$.
   Since $SX = 10$, then $WS = WX - SX = 15 - 10 = 5$.
   Thus, $WR = WS + SR = 5 + 30 = 35$.

   Answer: (E)

6. Since $x = 11$, $y = 8$ and $2x + 3z = 5y$, then $2 \times 11 + 3z = 5 \times 8$ or $3z = 40 - 22$.
   Therefore, $3z = 18$ and so $z = 6$.

   Answer: (A)

7. *Solution 1*
   Since $(x+a)(x+8) = x^2 + bx + 24$ for all $x$, then $x^2 + ax + 8x + 8a = x^2 + bx + 24$ or $x^2 + (a+8)x + 8a = x^2 + bx + 24$ for all $x$.
   Since the equation is true for all $x$, then the coefficients on the left side must match the coefficients on the right side.
   Therefore, $a + 8 = b$ and $8a = 24$.
   The second equation gives $a = 3$, from which the first equation gives $b = 3 + 8 = 11$.
   Finally, $a + b = 3 + 11 = 14$.

   *Solution 2*
   Since $(x+a)(x+8) = x^2 + bx + 24$ for all $x$, then the equation is true for $x = 0$ and $x = 1$.
   When $x = 0$, we obtain $(0+a)(0+8) = 0 + 0 + 24$ or $8a = 24$, which gives $a = 3$.
   When $x = 1$, we obtain $(1+3)(1+8) = 1 + b + 24$ or $36 = b + 25$, which gives $b = 11$.
   Finally, $a + b = 3 + 11 = 14$.

   Answer: (D)

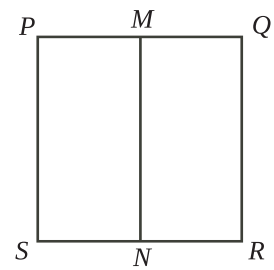
8. The original set contains 11 elements whose sum is 66.
When one number is removed, there will be 10 elements in the set.
For the average of these elements to be 6.1, their sum must be $10 \times 6.1 = 61$.
Since the sum of the original 11 elements is 66 and the sum of the remaining 10 elements is 61, then the element that has been removed is $66 - 61 = 5$.

ANSWER: (B)

9. Since the regular price for the bicycle is \$320 and the savings are 20%, then the amount of money that Sandra saves on the bicycle is $\$320 \times 20\% = \$320 \times 0.2 = \$64$.
Since the regular price for the helmet is \$80 and the savings are 10%, then the amount of money that Sandra saves on the helmet is $\$80 \times 10\% = \$80 \times 0.1 = \$8$.
The total of the original prices for the bicycle and helmet is $\$320 + \$80 = \$400$.
Sandra's total savings are $\$64 + \$8 = \$72$.
Therefore, her total percentage savings is $\dfrac{\$72}{\$400} \times 100\% = \dfrac{72}{4} \times 1\% = 18\%$.

ANSWER: (A)

10. Suppose that the side length of square $PQRS$ is $x$.
Then $PQ = QR = RS = SP = x$.
Since $M$ is the midpoint of $PQ$, then $PM = \frac{1}{2}x$.
In terms of $x$, the perimeter of rectangle $PMNS$ is




$$2(PM + PS) = 2(\tfrac{1}{2}x + x) = 3x$$

(Note that $SN = PM = \frac{1}{2}x$ since $N$ is the midpoint of $RS$. Also, $MN = PS = x$, since $MN$ is parallel to $PS$ and joins two parallel line segments.)
Since we are told that the perimeter of $PMNS$ is 36, then $3x = 36$ or $x = 12$.
Therefore, the area of square $PQRS$ is $x^2 = 144$.

ANSWER: (D)

11. On Monday, Ramya read $\frac{1}{5}$ of the 300 pages, which is $\frac{1}{5} \times 300 = 60$ pages in total.
After Monday, there were $300 - 60 = 240$ pages remaining to be read in the novel.
On Tuesday, Ramya read $\frac{4}{15}$ of these remaining 240 pages, or $\frac{4}{15} \times 240 = \frac{960}{15} = 64$ pages.
Therefore, she read $60 + 64 = 124$ pages in total over these two days.

ANSWER: (A)

12. There are 10 numbers in the list.
We note that

$$(-1)^4 = 1^4 = 1 \qquad (-3)^4 = 3^4 = 81 \qquad (-5)^4 = 5^4 = 625$$
$$(-7)^4 = 7^4 = 2401 \qquad (-9)^4 = 9^4 = 6561$$

Thus, if $m = -3, -1, 1, 3$, then $m^4 < 100$. If $m = -9, -7, -5, 5, 7, 9$, then $m^4 > 100$.
In other words, there are exactly six numbers in the list whose fourth power is larger than 100.
Thus, if $m$ is chosen at random from the list, the probability that $m^4 > 100$ is $\frac{6}{10}$ or $\frac{3}{5}$.

ANSWER: (E)

13. We note that $64 = 2^6$ and $512 = 2^9$.
Therefore, the equation $512^x = 64^{240}$ can be rewritten as $(2^9)^x = (2^6)^{240}$ or $2^{9x} = 2^{6(240)}$.
Since the bases in this last equation are equal, then the exponents are equal, so $9x = 6(240)$ or $x = \frac{1440}{9} = 160$.

ANSWER: (D)

14. Since 25% of the money donated came from parents, then the remaining $100\% - 25\% = 75\%$ came from the teachers and students.
Since the ratio of the amount donated by teachers to the amount donated by students is $2:3$, then the students donated $\frac{3}{2+3} = \frac{3}{5}$ of this remaining 75%.
This means that the students donated $\frac{3}{5} \times 75\% = 45\%$ of the total amount.
Therefore, the ratio of the amount donated by parents to the amount donated by students is $25\% : 45\% = 25 : 45 = 5 : 9$.

Answer: (C)

15. Let $n$ be the number of cookies in the cookie jar.
Let $r$ be the number of raisins in each of the $n-1$ smaller, identical cookies.
This means that there are $r+1$ raisins in the larger cookie.
If we removed one raisin from the larger cookie, it too would have $r$ raisins and so each of the $n$ cookies would have the same number of raisins ($r$), and the total number of raisins in the cookies would be $100 - 1 = 99$.
From this, we obtain $nr = 99$.
(We could also obtain this equation by noting that there are $n-1$ cookies containing $r$ raisins and 1 cookie containing $r+1$ raisins and 100 raisins in total, so $(n-1)r + (r+1) = 100$ or $nr - r + r + 1 = 100$ or $nr = 99$.)
Since $n$ and $r$ are positive integers whose product is 99, then the possibilities are:

$$99 = 99 \times 1 = 33 \times 3 = 11 \times 9 = 9 \times 11 = 3 \times 33 = 1 \times 99$$

Since $n$ is between 5 and 10, then we must have $99 = 9 \times 11$; that is, $n = 9$ and $r = 11$.
Since there are 11 raisins in each of the smaller cookies, then there are $11 + 1 = 12$ raisins in the larger cookie.

Answer: (E)

16. Let $s$ be the side length of each of the 60 identical squares.
Since the diagonal of each of the squares has length 2, then by the Pythagorean Theorem, $s^2 + s^2 = 2^2$ or $2s^2 = 4$, which gives $s^2 = 2$ or $s = \sqrt{2}$, since $s > 0$.
Now $PQ = 5s$ and $PS = 12s$, so since $QS > 0$, then by the Pythagorean Theorem,

$$QS = \sqrt{PQ^2 + PS^2} = \sqrt{(5s)^2 + (12s)^2} = \sqrt{25s^2 + 144s^2} = \sqrt{169s^2} = 13s$$

Since $QS = 13s$ and $s = \sqrt{2}$, then $QS = 13\sqrt{2} \approx 18.38$.
Of the given choices, this is closest to 18.

Answer: (A)

17. *Solution 1*
Suppose that the five consecutive integers represented by $p, q, r, s, t$ are $n, n+1, n+2, n+3, n+4$, for some integer $n$.
The sum of any two of these integers is at most $(n+3) + (n+4) = 2n + 7$; the sum of every other pair is smaller.
The sum of any two of these integers is at least $n + (n+1) = 2n + 1$; the sum of every other pair is larger.
Therefore, the maximum possible difference between the sums of two pairs is $(2n+7) - (2n+1)$ or 6; any other choice of pairs will give a smaller difference between the sums.
Since we are told that $p + q = 63$ and $s + t = 57$, which gives $(p+q) - (s+t) = 6$, then it must be the case that $p$ and $q$ are the two largest integers from the list while $s$ and $t$ are the

two smallest integers from the list.
In other words, $p+q=(n+3)+(n+4)=63$ and so $2n+7=63$ or $2n=56$ and so $n=28$.
Since $r$ must be the middle integer in the list, then $r=n+2=30$.

*Solution 2*
Suppose that the five consecutive integers represented by $p,q,r,s,t$ are $n, n+1, n+2, n+3, n+4$, for some integer $n$.
The sum of all five integers is $n+(n+1)+(n+2)+(n+3)+(n+4)=5n+10$.
We are told that $p+q=63$ and $s+t=57$.
Thus, the sum of the five integers is also $p+q+r+s+t=63+r+57=120+r$.
Comparing the two expressions for the sum of the integers, we obtain $5n+10=120+r$ or $r=5n-110$.
Since $r=5n-110=5(n-22)$, then $r$ is divisible by 5.
Of the five given answer choices, this means that we could have $r=20$ or $r=30$.
If $r=20$, then $20=5(n-22)$ or $n-22=4$ and so $n=26$. In this case, $r$ is not one of the integers between $n$ and $n+4$, inclusive, so $r$ cannot be 20.
If $r=30$, then $30=5(n-22)$ or $n-22=6$ and so $n=28$. Here, the integers in the list would be $28, 29, 30, 31, 32$, which can produce the given conditions if $p$ and $q$ are 31 and 32, and $t$ and $s$ are 28 and 29.
Therefore, $r=30$.

Answer: (E)

18. Since $p$ is a positive integer, then $p\geq 1$ and so $0<\dfrac{1}{p}\leq 1$.

Since $n$ is a positive integer, then $n\geq 1$ and so $n+\dfrac{1}{p}>1$, which tells us that $0<\dfrac{1}{n+\frac{1}{p}}<1$.

Therefore, $m<m+\dfrac{1}{n+\frac{1}{p}}<m+1$. Since $m+\dfrac{1}{n+\frac{1}{p}}=\dfrac{17}{3}$, which is between 5 and 6, and since $m$ is an integer, then $m=5$.

Since $m=5$, then $m+\dfrac{1}{n+\frac{1}{p}}=\dfrac{17}{3}$ gives $\dfrac{1}{n+\frac{1}{p}}=\dfrac{2}{3}$ or $n+\dfrac{1}{p}=\dfrac{3}{2}$.

Since $n<n+\dfrac{1}{p}\leq n+1$ and $n$ is an integer, then $n=1$.

Thus, $n+\dfrac{1}{p}=\dfrac{3}{2}$ gives $\dfrac{1}{p}=\dfrac{1}{2}$, which gives $p=2$.
Therefore, $n=1$.

Answer: (C)

19. We rewrite the integers from the list in terms of their prime factorizations:

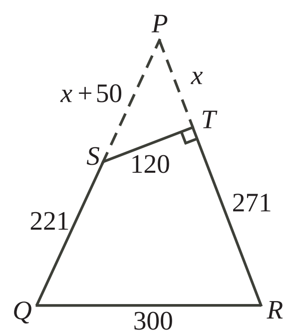
$$1, 2^1, 3^1, 2^2, 5^1, 2^1 3^1, 7^1, 2^3, 3^2$$

A positive integer larger than one is a perfect square if and only if each of its prime factors occurs an even number of times.
Since the integers in the list above contain in total only one factor of 5 and one factor of 7, then neither 5 nor 7 can be chosen to form a product that is a perfect square.
This leaves us with seven integers $1, 2^1, 3^1, 2^2, 2^1 3^1, 2^3, 3^2$, from which we need to choose six.

When the seven given integers are multiplied together, their product is $2^{1+2+1+3}3^{1+1+2} = 2^7 3^4$.
We can think of choosing six of the seven numbers and multiplying them together as choosing all seven and then dividing out the one we did not choose.
To divide the product $2^7 3^4$ by one of the integers to obtain a perfect square, the divisor must include an odd number of factors of 2 (since the product of all seven includes an odd number of factors of 2) and an even number of factors of 3 (since the product includes an even number of factors of 3). (Note that "an even number of factors of 3" includes the possibility of zero factors of 3.)
There are two such numbers in the list: $2^1$ and $2^3$.
(Alternatively, we could have divided the product by each of the seven numbers to determine which results in a perfect square.)
Therefore, the two sets of six numbers that satisfy the given conditions should be $1, 3^1, 2^2, 2^1 3^1, 2^3, 3^2$ (whose product is $2^6 3^4$) and $1, 2^1, 3^1, 2^2, 2^1 3^1, 3^2$ (whose product is $2^4 3^4$).
Thus, we can set $m^2 = 2^6 3^4$, which gives $m = 2^3 3^2 = 72$, and $n^2 = 2^4 3^4$, which gives $n = 2^2 3^2 = 36$.
Finally, $m + n = 72 + 36 = 108$.

Answer: (A)

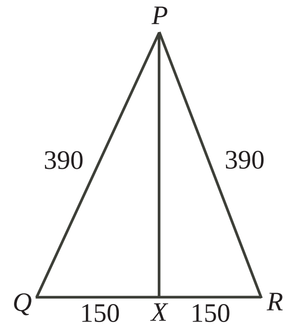
20. We calculate the area of quadrilateral $STRQ$ by subtracting the area of $\triangle PTS$ from the area of $\triangle PQR$.
Let $PT = x$.
Then $PR = PT + TR = x + 271$.
Since $PQ = PR = x + 271$ and $SQ = 221$, then $PS = PQ - SQ = (x + 271) - 221 = x + 50$.




By the Pythagorean Theorem in $\triangle PTS$, we have

$$\begin{aligned} PT^2 + TS^2 &= PS^2 \\ x^2 + 120^2 &= (x+50)^2 \\ x^2 + 14400 &= x^2 + 100x + 2500 \\ 11900 &= 100x \\ x &= 119 \end{aligned}$$

Therefore, $\triangle PTS$ has $PT = x = 119$, $TS = 120$, and $PS = x + 50 = 169$.
Since $\triangle PTS$ is right-angled at $T$, then its area is $\frac{1}{2}(PT)(TS) = \frac{1}{2}(119)(120) = 7140$.
Furthermore, in $\triangle PQR$, we have $PR = PQ = x + 271 = 390$.
Now, $\triangle PQR$ is isosceles, so when we draw a median $PX$ from $P$ to the midpoint $X$ of $QR$, it is perpendicular to $QR$.



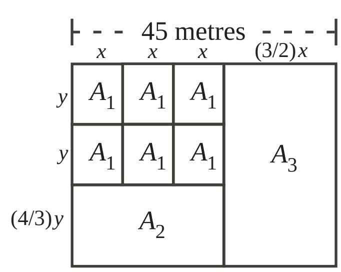

Since $X$ is the midpoint of $QR$ and $QR = 300$, then $QX = \frac{1}{2}QR = 150$.
We can use the Pythagorean Theorem in $\triangle PXQ$ to conclude that

$$PX = \sqrt{PQ^2 - QX^2} = \sqrt{390^2 - 150^2} = \sqrt{21600} = 360$$

since $PX > 0$.
Since $PX$ is a height in $\triangle PQR$, then the area of $\triangle PQR$ is $\frac{1}{2}(QR)(PX) = \frac{1}{2}(300)(360) = 54\,000$.
Finally, the area of $STRQ$ is the difference in the areas of these two triangles, or $54\,000 - 7140$, which equals $46\,860$.

Answer: (C)

21. We refer to distances in the horizontal direction as widths and distances in the vertical direction as lengths.
Suppose that each of the six enclosures labelled $A_1$ have width $x$ m and length $y$ m.
Then each of these has area $xy$ m$^2$.
We start by determining the dimensions of the remaining enclosures in terms of these two variables.
Enclosure $A_2$ has width $x + x + x = 3x$ m.
Since the area of enclosure $A_2$ is four times that of $A_1$, then its area is $4xy$ m$^2$.
Therefore, the length of the enclosure $A_2$ is its area divided by its width, or $\dfrac{4xy}{3x} = \dfrac{4}{3}y$ m. (We use the notation $(4/3)y$ in the diagram.)
Thus, the length of enclosure $A_3$ is $y + y + \frac{4}{3}y = \frac{10}{3}y$ m.
Since the area of enclosure $A_3$ is $5xy$ m$^2$, then its width is $\dfrac{5xy}{\frac{10}{3}y} = \dfrac{3}{2}x$ m. (We use the notation $(3/2)x$ in the diagram.)




The total width of the field is 45 m. This can also be expressed (using the top fence) as $x + x + x + \frac{3}{2}x = \frac{9}{2}x$ m.
Since $\frac{9}{2}x = 45$, then $x = \frac{2}{9}(45) = 10$.
In terms of $x$, the total length, in metres, of "horizontal" fencing is

$$(x + x + x + \tfrac{3}{2}x) + (x + x + x) + (x + x + x) + (x + x + x + \tfrac{3}{2}x) = 15x$$

which we calculate by going from left to right along each row from top to bottom.
In terms of $y$, the total length, in metres, of "vertical" fencing is

$$(y+y+\tfrac{4}{3}y)+(y+y)+(y+y)+(y+y+\tfrac{4}{3}y)+(y+y+\tfrac{4}{3}y)=14y$$

which we calculate by going from top to bottom along each column from left to right.
Since the total length of fencing is 360 m, then $15x+14y=360$.
Since $x=10$, then $150+14y=360$ or $14y=210$ and so $y=15$.
Therefore, the area of enclosure $A_1$ is $xy=(10)(15)=150\text{ m}^2$.
Of the given answers, this is closest to (in fact, equal to) 150.0.

ANSWER: (B)

22. Suppose that Megan and Shana competed in exactly $n$ races.
Since Shana won exactly 2 races, then Megan won exactly $n-2$ races.
Since Shana won 2 races and lost $n-2$ races, then she received $2x+(n-2)y$ coins.
Thus, $2x+(n-2)y=35$.
Since Megan won $n-2$ races and lost 2 races, then she received $(n-2)x+2y$ coins.
Thus, $(n-2)x+2y=42$.
If we add these two equations, we obtain $(2x+(n-2)y)+((n-2)x+2y)=35+42$ or $nx+ny=77$ or $n(x+y)=77$.
Since $n$, $x$ and $y$ are positive integers, then $n$ is a positive divisor of 77, so $n=1,7,11$ or 77.
Subtracting $2x+(n-2)y=35$ from $(n-2)x+2y=42$, we obtain

$$((n-2)x+2y)-(2x+(n-2)y)=42-35$$

or $(n-4)x+(4-n)y=7$ or $(n-4)(x-y)=7$.
Since $n$, $x$ and $y$ are positive integers and $x>y$, then $n-4$ is a positive divisor of 7, so $n-4=1$ or $n-4=7$, giving $n=5$ or $n=11$.
Comparing the two lists, we determine that $n$ must be 11.
Thus, we have $11(x+y)=77$ or $x+y=7$.
Also, $7(x-y)=7$ so $x-y=1$.
Adding these last two equations, we obtain $(x+y)+(x-y)=7+1$ or $2x=8$, and so $x=4$.
(Checking, if $x=4$, then $y=3$. Since $n=11$, then Megan won 9 races and Shana won 2 races. Megan should receive $9(4)+2(3)=42$ coins and Shana should receive $2(4)+9(3)=35$ coins, which agrees with the given information.)

ANSWER: (E)

23. First, we consider the first bag, which contains a total of $2 + 2 = 4$ marbles.
There are 4 possible marbles that can be drawn first, leaving 3 possible marbles that can be drawn second. This gives a total of $4 \times 3 = 12$ ways of drawing two marbles.
For both marbles to be red, there are 2 possible marbles (either red marble) that can be drawn first, and 1 marble that must be drawn second (the remaining red marble). This gives a total of $2 \times 1 = 2$ ways of drawing two red marbles.
For both marbles to be blue, there are 2 possible marbles that can be drawn first, and 1 marble that must be drawn second. This gives a total of $2 \times 1 = 2$ ways of drawing two blue marbles.
Therefore, the probability of drawing two marbles of the same colour from the first bag is the total number of ways of drawing two marbles of the same colour $(2 + 2 = 4)$ divided by the total number of ways of drawing two marbles (12), or $\frac{4}{12} = \frac{1}{3}$.
Second, we consider the second bag, which contains a total of $2 + 2 + g = g + 4$ marbles.
There are $g + 4$ possible marbles that can be drawn first, leaving $g + 3$ possible marbles that can be drawn second. This gives a total of $(g + 4)(g + 3)$ ways of drawing two marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two red marbles.
As with the first bag, there are $2 \times 1 = 2$ ways of drawing two blue marbles.
For both marbles to be green, there are $g$ possible marbles that can be drawn first, and $g - 1$ marbles that must be drawn second. This gives a total of $g(g - 1)$ ways of drawing two green marbles.
Therefore, the probability of drawing two marbles of the same colour from the second bag is the total number of ways of drawing two marbles of the same colour $(2+2+g(g-1) = g^2 - g + 4)$ divided by the total number of ways of drawing two marbles $((g+4)(g+3))$, or $\frac{g^2 - g + 4}{(g + 4)(g + 3)}$.

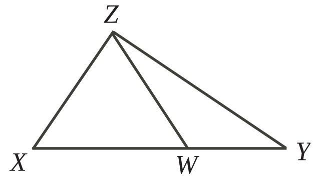
Since the two probabilities that we have calculated are to be equal and $g \neq 0$, then

$$\begin{aligned}
\frac{1}{3} &= \frac{g^2 - g + 4}{(g + 4)(g + 3)} \\
(g + 4)(g + 3) &= 3g^2 - 3g + 12 \\
g^2 + 7g + 12 &= 3g^2 - 3g + 12 \\
10g &= 2g^2 \\
0 &= 2g^2 - 10g \\
0 &= 2g(g - 5)
\end{aligned}$$

Therefore, $g = 0$ or $g = 5$. Since $g \neq 0$, then $g = 5$.

Answer: (B)

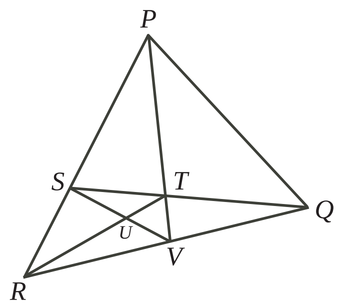
24. In this solution, we use the notation $|\triangle XYZ|$ to denote the area of $\triangle XYZ$.
In this solution, we also use a fact about a triangle (which we call $\triangle XYZ$) that is divided into two pieces by a line segment ($ZW$):




$$\frac{|\triangle ZXW|}{|\triangle ZWY|} = \frac{XW}{WY}$$

We label this fact ($*$). ($*$) is true because these triangles have a common height (the perpendicular distance, $h$, from $Z$ to $XY$), and so $\dfrac{|\triangle ZXW|}{|\triangle ZWY|} = \dfrac{\frac{1}{2}(XW)h}{\frac{1}{2}(WY)h} = \dfrac{XW}{WY}$.

We redraw the given diagram, removing line segments $PU$ and $QU$:

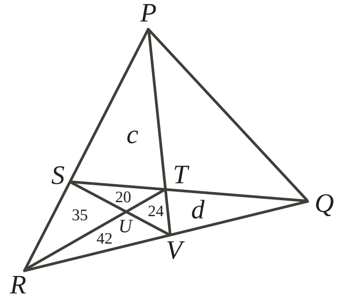



Suppose that $|\triangle SUT| = a$.
Since $|\triangle RST| = 55$, then $|\triangle RSU| = |\triangle RST| - |\triangle SUT| = 55 - a$.
Since $|\triangle RSV| = 77$, then $|\triangle RUV| = |\triangle RSV| - |\triangle RSU| = 77 - (55 - a) = 22 + a$.
Since $|\triangle RTV| = 66$, then $|\triangle TUV| = |\triangle RTV| - |\triangle RUV| = 66 - (22 + a) = 44 - a$.
By ($*$), $\dfrac{|\triangle SUT|}{|\triangle RSU|} = \dfrac{TU}{UR} = \dfrac{|\triangle TUV|}{|\triangle RUV|}$.
Therefore,

$$\begin{aligned} \frac{a}{55-a} &= \frac{44-a}{22+a} \\ a(22+a) &= (44-a)(55-a) \\ a^2 + 22a &= 2420 - 99a + a^2 \\ 121a &= 2420 \\ a &= 20 \end{aligned}$$

Thus, $|\triangle SUT| = 20$, $|\triangle RSU| = 35$, $|\triangle RUV| = 42$, and $|\triangle TUV| = 24$.

Let $|\triangle PST| = c$ and $|\triangle QTV| = d$. We have the following configuration:



By $(*)$, $\dfrac{|\triangle PST|}{|\triangle RST|} = \dfrac{PS}{SR} = \dfrac{|\triangle PSV|}{|\triangle RSV|}$.

Therefore, $\dfrac{c}{55} = \dfrac{c+44}{77}$ or $77c = 55c + 55(44)$, which gives $22c = 2420$ or $c = 110$.

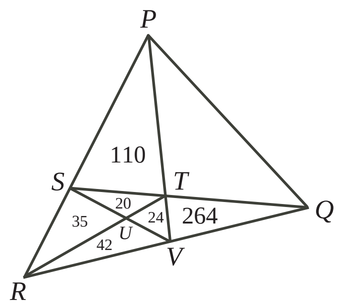
Thus, $\dfrac{PS}{SR} = \dfrac{|\triangle PST|}{|\triangle RST|} = \dfrac{110}{55} = 2$. (We'll use this later.)

Similarly, by $(*)$, $\dfrac{|\triangle QTV|}{|\triangle RTV|} = \dfrac{QV}{VR} = \dfrac{|\triangle SVQ|}{|\triangle RSV|}$.

Therefore, $\dfrac{d}{66} = \dfrac{d+44}{77}$ or $77d = 66d + 66(44)$, which gives $11d = 2904$ or $d = 264$.

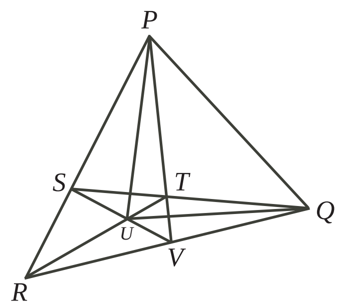
Thus, $\dfrac{QV}{VR} = \dfrac{|\triangle QTV|}{|\triangle RTV|} = \dfrac{264}{66} = 4$. (We'll use this later.)

This gives:




By $(*)$, $\dfrac{|\triangle PTQ|}{|\triangle PST|} = \dfrac{TQ}{ST} = \dfrac{|\triangle QTV|}{|\triangle STV|}$, and so $\dfrac{|\triangle PTQ|}{110} = \dfrac{264}{44}$ or $|\triangle PTQ| = \dfrac{110(264)}{44} = 660$.

We are now ready to calculate the area of $\triangle PQU$, so we add back in segments $PU$ and $QU$.




We calculate this area using

$$|\triangle PQU| = |\triangle PTQ| + |\triangle PST| + |\triangle QTV| + |\triangle STV| - |\triangle PSU| - |\triangle QVU|$$

Using $(*)$ and the facts that $\dfrac{PS}{SR} = 2$ and $\dfrac{QV}{VR} = 4$, we have

$$|\triangle PSU| = \frac{PS}{SR}|\triangle RSU| = 2(35) = 70$$

and

$$|\triangle QVU| = \frac{QV}{VR}|\triangle VRU| = 4(42) = 168$$

Therefore, $|\triangle PQU| = 660 + 110 + 264 + 44 - 70 - 168 = 840$.

Answer: (C)

25. Step 1: Using parity and properties of powers of 2 to simplify the equation

We note that if $2^x = 2^y$ for some real numbers $x$ and $y$, then $x = y$.
This is because $2^x = 2^y$ implies $\frac{2^x}{2^y} = 1$ or $2^{x-y} = 1$, and so $x - y = 0$ or $x = y$.
We examine equations of the form $2^a + 2^b = 2^c + 2^d$ where $a$, $b$, $c$, and $d$ are integers.
(This is more general than the given equation, but allows us to determine what is possible.)
We may assume without loss of generality that $a \leq b$ and $c \leq d$ and $a \leq c$. (We can always switch the variable names to make these true.)
We factor the equation as $2^a(1 + 2^{b-a}) = 2^c(1 + 2^{d-c})$, and then divide both sides by $2^a$ to obtain $1 + 2^{b-a} = 2^{c-a}(1 + 2^{d-c})$.
We show that $c = a$ by contradiction:

> If $c \neq a$, then $c \geq a$ gives $c > a$.
> If $c > a$, then $c - a > 0$, so $c - a \geq 1$, since $c - a$ is an integer.
> Therefore, the right side has a factor of $2^{c-a}$, so the right side is even.
> Thus, the left side is even too, which means that $2^{b-a}$ must be an odd integer.
> For $2^{b-a}$ to be an odd integer, we must have $2^{b-a} = 1$ and so $b - a = 0$ or $b = a$.
> In this case, the left side equals 2 and the right side is greater than 2, since $2^{c-a} \geq 2$ and $1 + 2^{d-c} > 1$. This is a contradiction.
> Therefore, $c = a$.

Since $a = c$, then $2^a + 2^b = 2^c + 2^d$ becomes $2^b = 2^d$ and so $b = d$.
Therefore, if $2^a + 2^b = 2^c + 2^d$ with $a, b, c, d$ integers, then either $a = b = c = d$ or $a = c$ and $b = d$ (with $a \neq b$) or $a = d$ and $b = c$ (with $a \neq b$).

We examine these three possibilities in the given equation, noting that $m$, $n$ and $k$ are all positive integers:

- *Case 1:* $4m^2 = m^2 - n^2 + 4 = k + 4 = 3m^2 + n^2 + k$
  From the last equality, we obtain $3m^2 + n^2 = 4$.
  Since $m, n$ are positive integers, then $m^2 \geq 1$ and $n^2 \geq 1$.
  Since $3m^2 + n^2 = 4$, then it must be that $m = n = 1$.
  Thus, $4m^2 = k + 4$ implies $4 = k + 4$ or $k = 0$.
  But $k > 0$, so this case is not possible.
- *Case 2:* $4m^2 = k + 4$ and $m^2 - n^2 + 4 = 3m^2 + n^2 + k$ and $4m^2 \neq m^2 - n^2 + 4$
  From the second equality, we obtain $2m^2 + 2n^2 + k = 4$, which is not possible since $m, n, k > 0$, and so $2m^2 + 2n^2 + k \geq 5$.
  Therefore, this case is not possible.
- *Case 3:* $4m^2 = 3m^2 + n^2 + k$ and $m^2 - n^2 + 4 = k + 4$ and $4m^2 \neq m^2 - n^2 + 4$
  The first equality rearranges to $m^2 - n^2 = k$.
  The second equality also rearranges to $m^2 - n^2 = k$.
  The last statement is equivalent to $3m^2 + n^2 \neq 4$. As we saw in Case 1, this means that $(m, n)$ cannot be the pair $(1, 1)$, which is consistent with $m^2 - n^2 = k$ and $k > 0$.

Therefore, having examined all of the cases, we have reduced the original problem to finding the number of odd integers $k$ between 0 and 100 for which the equation $m^2 - n^2 = k$ has exactly two pairs of positive integers $(m, n)$ that are solutions.

Step 2: Connecting solutions to $m^2 - n^2 = k$ with factorizations of $k$

We can factor the left side of this equation to give $(m+n)(m-n) = k$.
Since $m$, $n$ and $k$ are positive integers, then $m+n > 0$ and $k > 0$ so $m - n > 0$, or $m > n$.
Since $k$ is odd and each of $m+n$ and $m-n$ is an integer, then each of $m+n$ and $m-n$ is odd (since if either was even, then their product would be even).
Also, we note that $m+n > m-n$ since $n > 0$.
Suppose that $(m,n)$ is a solution of the equation $m^2 - n^2 = k$ with $m+n = a$ and $m-n = b$ for some odd positive integers $a$ and $b$ with $a > b$.
Then $ab = k$, so $ab$ is a factorization of $k$.
Therefore, the solution $(m,n)$ corresponds to a specific factorization of $k$.
Now suppose that we start with a factorization $k = AB$ where $A$ and $B$ are odd positive integers with $A \geq B$.
If we try setting $m+n = A$ and $m-n = B$, then we can add these equations to give $2m = A+B$ (or $m = \frac{1}{2}(A+B)$) and subtract them to give $2n = A - B$ (or $n = \frac{1}{2}(A-B)$). Note that since $n > 0$, then $A > B$.
Therefore, every factorization of $k$ as the product of two odd positive integers $A$ and $B$ with $A > B$ gives a solution to the equation $m^2 - n^2 = k$.
Since each solution gives a factorization and each factorization gives a solution, then the number of solutions equals the number of factorizations.
Therefore, we have reduced the original problem to finding the number of odd integers $k$ between 0 and 100 which have exactly two factorizations as the product of distinct odd integers $a$ and $b$ with $a > b$.

Step 3: Counting the values of $k$

Since $k$ is odd, then all of its prime factors are odd.
Since $k < 100$, then $k$ cannot have three or more distinct odd prime factors, because the smallest possible product of three distinct odd prime factors is $3 \cdot 5 \cdot 7 = 105$.
Thus, $k$ has two or fewer distinct prime factors.
If $k = pq$ for distinct primes $p < q$, then the divisors of $k$ are $1, p, q, pq$, so $k$ has exactly two factorizations of the desired type (namely $1 \cdot pq$ and $p \cdot q$).
Since $k < 100$ and $p \geq 3$, then $q < \frac{100}{3}$. Since $q$ is an integer, then $q \leq 33$.
The odd primes less than 33 are $3, 5, 7, 11, 13, 17, 19, 23, 29, 31$.
If $p \geq 11$, then $pq > 11^2 = 121$, which is larger than 100.
Therefore, $p$ can only be 3, 5 or 7.
If $p = 3$, there are 9 possible values for $q$ (primes from 5 to 31).
If $p = 5$, there are 5 possible values for $q$ (primes from 7 to 19).
If $p = 7$, there are 2 possible values for $q$ (11 and 13).
Thus, there are $9 + 5 + 2 = 16$ values of $k$ of this form that work.

If $k = p^r q^s$ with $r$ and $s$ positive integers and at least one of $r$ or $s$ is larger than 1, then $k$ will have at least three factorizations. (For example, if $r > 1$, then $k = 1 \cdot p^r q^s = p \cdot p^{r-1} q^s = p^r \cdot q^s$ and all of these are distinct.)

If $k = p$ or $k = p^2$ with $p$ an odd prime, then $k$ has only one factorization as the product of distinct factors ($1 \cdot p$ and $1 \cdot p^2$, respectively). Thus, $k$ cannot be of this form.

If $k = p^3$ with $p$ an odd prime, then the divisors of $k$ are $1, p, p^2, p^3$, so it has exactly two factorizations of the desired type (namely $1 \cdot p^3$ and $p \cdot p^2$).
Since $k < 100$, then $p$ can only equal 3 (because $5^3 > 100$).
Thus, there is 1 value of $k$ of this form that works.

If $k = p^4$ with $p$ an odd prime, then the divisors of $k$ are $1, p, p^2, p^3, p^4$, so it has exactly two factorizations of the desired type (namely $1 \cdot p^4$ and $p \cdot p^3$). In this case, $k$ has a third factorization, but it is of the wrong type since the two factors will be equal.
Since $k < 100$, then $p$ can only equal 3 (because $5^4 > 100$).
Thus, there is 1 value of $k$ of this form that works.

If $k$ has more than 4 factors of $p$, then $k$ will have at least three factorizations of the desired type, so $k$ cannot be of this form. (In particular, if $k = p^n$ and $n > 4$, then $k = 1 \cdot p^n = p \cdot p^{n-1} = p^2 \cdot p^{n-2}$ and these are all distinct since $n - 2 > 2$.)

Having examined all of the possible forms, we see that there are $16 + 1 + 1 = 18$ values of $k$ that work, and so there are 18 positive integer solutions to the original equation.

ANSWER: (D)

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*www.cemc.uwaterloo.ca*

# *2012 Fermat Contest*

(Grade 11)

**Thursday, February 23, 2012**
**(in North America and South America)**

**Friday, February 24, 2012**
**(outside of North America and South America)**

*Solutions*

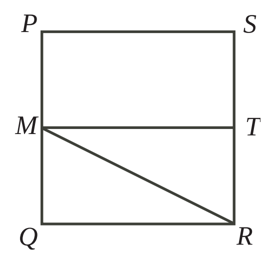
1. Since $\frac{60}{8} = 60 \div 8 = 7.5$, then this choice is not equal to a whole number.
Note as well that $\frac{60}{12} = 5$, $\frac{60}{5} = 12$, $\frac{60}{4} = 15$, and $\frac{60}{3} = 20$ are all whole numbers.
Answer: (B)

2. Simplifying the left side of the equation, we obtain $5 = 6 - x$.
Therefore, $x = 6 - 5 = 1$.
Answer: (C)

3. Since $JFG$ is a straight line, then $\angle HFG = 180^\circ - \angle HFJ = 180^\circ - 110^\circ = 70^\circ$.
Since $\triangle FGH$ is isosceles with $HF = HG$, then $\angle HGF = \angle HFG = 70^\circ$.
Since the sum of the angles in $\triangle FGH$ is $180^\circ$, then $70^\circ + 70^\circ + x^\circ = 180^\circ$, and so $140 + x = 180$ or $x = 40$.
Answer: (E)

4. Simplifying inside the brackets first, $(1 + \frac{1}{3})(1 + \frac{1}{4}) = (\frac{4}{3})(\frac{5}{4}) = \frac{20}{12} = \frac{5}{3}$.
Answer: (A)

5. *Solution 1*
Draw a line from $M$ to $T$ on $SR$ so that $MT$ is parallel to $QR$.
Then $MTRQ$ is a rectangle. This means that the area of $\triangle MQR$ is half of the area of rectangle $MTRQ$.
Thus, the area of $MTRQ$ is $2 \times 100 = 200$.
Since $M$ is the midpoint of $PQ$ and $PQRS$ is a square, then $T$ is the midpoint of $SR$.
This means that the area of $MTRQ$ is half of the area of $PQRS$.
Therefore, the area of $PQRS$ is $2 \times 200 = 400$.




*Solution 2*
Suppose that the side length of square $PQRS$ is $2x$.
Since $M$ is the midpoint of $PQ$, then $MQ = \frac{1}{2}(2x) = x$.
Since $PQRS$ is a square, then $\triangle MQR$ is right-angled at $Q$.
Therefore, the area of $\triangle MQR$ is $\frac{1}{2}(MQ)(QR) = \frac{1}{2}(x)(2x) = x^2$.
Since the area of $\triangle MQR$ is 100, then $x^2 = 100$, and so $x = 10$, since $x > 0$.
Thus, the side length of square $PQRS$ is $2x = 20$ and so the area of square $PQRS$ is $20^2 = 400$.
Answer: (D)

6. Suppose that John ate $x$ peanuts on the fourth night.
Since he ate 6 more peanuts each night than on the previous night, then he ate $x - 6$ peanuts on the third night, $(x - 6) - 6 = x - 12$ peanuts on the second night, and $(x - 12) - 6 = x - 18$ peanuts on the first night.
Since John ate 120 peanuts in total, then $x + (x - 6) + (x - 12) + (x - 18) = 120$, and so $4x - 36 = 120$ or $4x = 156$ or $x = 39$.
Therefore, John ate 39 peanuts on the fourth night.
Answer: (B)

7. Suppose that the side length of each of the five identical squares is $x$.
Then $PS = QR = x$ and $PQ = SR = 5x$.
Since the perimeter of rectangle $PQRS$ is 48, then $5x + x + 5x + x = 48$ or $12x = 48$ or $x = 4$.
Therefore, $PS = QR = 4$ and $PQ = SR = 5 \cdot 4 = 20$, and so the area of rectangle $PQRS$ is $20 \cdot 4 = 80$.
Answer: (C)

8. Since $v = 3x$ and $x = 2$, then $v = 3 \cdot 2 = 6$.
Therefore, $(2v - 5) - (2x - 5) = (2 \cdot 6 - 5) - (2 \cdot 2 - 5) = 7 - (-1) = 8$.

Answer: (B)

9. Suppose that Sally's original height was $s$ cm.
Since Sally grew 20% taller, her new height is $1.2s$ cm.
Since Sally is now 180 cm tall, then $1.2s = 180$ or $s = \frac{180}{1.2} = 150$.
Thus, Sally grew $180 - 150 = 30$ cm.
Since Mary grew half as many centimetres as Sally grew, then Mary grew $\frac{1}{2} \cdot 30 = 15$ cm.
Since Mary and Sally were originally the same height, then Mary was originally 150 cm tall, and so is now $150 + 15 = 165$ cm tall.

Answer: (B)

10. Since $(2^a)(2^b) = 64$, then $2^{a+b} = 64$, using an exponent law.
Since $64 = 2^6$, then $2^{a+b} = 2^6$ and so $a + b = 6$.
Therefore, the average of $a$ and $b$ is $\frac{1}{2}(a + b) = 3$.

Answer: (D)

11. If $N$ is divisible by both 5 and 11, then $N$ is divisible by $5 \times 11 = 55$.
This is because 5 and 11 have no common divisor larger than 1.
Therefore, we are looking for a multiple of 55 between 400 and 600 that is odd.
One way to find such a multiple is to start with a known multiple of 55, such as 550.
We can add or subtract 55 from this multiple and still obtain multiples of 55.
Note that $550 + 55 = 605$, which is too large.
Now $550 - 55 = 495$ which is in the correct range and is odd.
Since we are told that there is only such such integer, then it must be the case that $N = 495$.
The sum of the digits of $N$ is $4 + 9 + 5 = 18$.

Answer: (E)

12. Since $\triangle QUR$ and $\triangle SUR$ are equilateral, then $\angle QUR = \angle SUR = 60^\circ$.
Since $QU = PU = TU = SU$ and $QP = PT = TS$, then $\triangle QUP$, $\triangle PUT$ and $\triangle TUS$ are congruent.
Thus, $\angle QUP = \angle PUT = \angle TUS$.
The angles around point $U$ add to $360^\circ$.
Thus, $\angle SUR + \angle QUR + \angle QUP + \angle PUT + \angle TUS = 360^\circ$ and so $60^\circ + 60^\circ + 3\angle TUS = 360^\circ$ or $3\angle TUS = 240^\circ$ or $\angle TUS = 80^\circ$.
Since $\triangle TUS$ is isosceles with $TU = SU$, then $\angle UST = \angle UTS$.
Since the angles in $\triangle TUS$ add to $180^\circ$, then $\angle TUS + \angle UST + \angle UTS = 180^\circ$.
Therefore, $80^\circ + 2\angle UST = 180^\circ$ and so $2\angle UST = 100^\circ$ or $\angle UST = 50^\circ$.

Answer: (A)

13. The quilt consists of 25 identical squares.
Of the 25 squares, 4 are entirely shaded, 8 are shaded with a single triangle that covers half of the square, and 4 are shaded with two triangles that each cover a quarter of the square.
Therefore, the shading is equivalent to the area of $4 + 8 \times \frac{1}{2} + 4 \times 2 \times \frac{1}{4} = 10$ squares.
As a percentage, the shading is $\frac{10}{25} \times 100\% = 40\%$ of the total area of the quilt.

Answer: (B)

14. *Solution 1*
Since the two terms have a common factor, then we factor and obtain $(x-2)((x-4)+(x-6)) = 0$.
This gives $(x-2)(2x-10) = 0$.
Therefore, $x - 2 = 0$ (which gives $x = 2$) or $2x - 10 = 0$ (which gives $x = 5$).
Therefore, the two roots of the equation are $x = 2$ and $x = 5$. Their product is 10.

*Solution 2*
We expand and then simplify the left side:

$$\begin{aligned}(x-4)(x-2)+(x-2)(x-6) &= 0\\ (x^2-6x+8)+(x^2-8x+12) &= 0\\ 2x^2-14x+20 &= 0\end{aligned}$$

Since the product of the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$ is $\frac{c}{a}$, then the product of the roots of the equation $2x^2 - 14x + 20 = 0$ is $\frac{20}{2} = 10$.

Answer: (C)

15. Because of the way in which the oranges are stacked, each layer is a rectangle whose length is 1 orange less and whose width is 1 orange less than the layer below.
The bottom layer is 5 by 7 and so contains 35 oranges.
The next layer is 4 by 6 and so contains 24 oranges.
The next layer is 3 by 5 and so contains 15 oranges.
The next layer is 2 by 4 and so contains 8 oranges.
The next layer is 1 by 3 and so contains 3 oranges. This is the last layer, as it consists of a single row of oranges.
The total number of oranges in the stack is thus $35 + 24 + 15 + 8 + 3 = 85$.

Answer: (D)

16. Since there are 30 people in a room and 60% of them are men, then there are $\frac{6}{10} \times 30 = 18$ men in the room and 12 women.
Since no men enter or leave the room, then these 18 men represent 40% of the final number in the room.
Thus, 9 men represent 20% of the the final number in the room, and so the final number of people is $5 \times 9 = 45$.
Since 18 of these are men and 12 of these are the women originally in the room, then $45 - 18 - 12 = 15$ women entered the room.

Answer: (E)

17. Since $3^{2011} = 3^1 \cdot 3^{2010} = 3 \cdot 3^{2010}$ and $3^{2012} = 3^2 \cdot 3^{2010} = 9 \cdot 3^{2010}$, then

$$\frac{3^{2011}+3^{2011}}{3^{2010}+3^{2012}} = \frac{3\cdot 3^{2010}+3\cdot 3^{2010}}{3^{2010}+9\cdot 3^{2010}} = \frac{3^{2010}(3+3)}{3^{2010}(1+9)} = \frac{3+3}{1+9} = \frac{6}{10} = \frac{3}{5}$$

Answer: (A)

18. In order to find $N$, which is the smallest possible integer whose digits have a fixed product, we must first find the minimum possible number of digits with this product. (This is because if the integer $a$ has more digits than the integer $b$, then $a > b$.)
Once we have determined the digits that form $N$, then the integer $N$ itself is formed by writing the digits in increasing order. (Given a fixed set of digits, the leading digit of $N$ will contribute

to the largest place value, and so should be the smallest digit. The next largest place value should get the next smallest digit, and so on.)
Note that the digits of $N$ cannot include 0, else the product of its digits would be 0.
Also, the digits of $N$ cannot include 1, otherwise we could remove the 1s and obtain an integer with fewer digits (thus, a smaller integer) with the same product of digits.

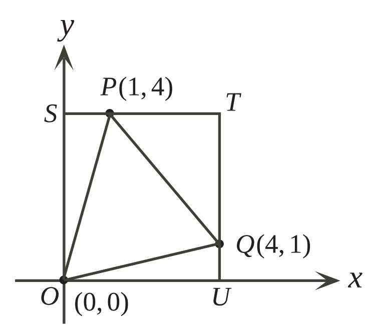
Since the product of the digits of $N$ is 1728, we find the prime factorization of 1728 to help us determine what the digits are:

$$1728 = 9 \times 192 = 3^2 \times 3 \times 64 = 3^3 \times 2^6$$

We must try to find a combination of the smallest number of possible digits whose product is 1728.
Note that we cannot have 3 digits with a product of 1728 since the maximum possible product of 3 digits is $9 \times 9 \times 9 = 729$.
Let us suppose that we can have 4 digits with a product of 1728.
In order for $N$ to be as small as possible, its leading digit (that is, its thousands digit) must be as small as possible.
From above, this digit cannot be 1.
This digit also cannot be 2, since otherwise the product of the remaining 3 digits would be 864, which is larger than the product of 3 digits can be.
Can the thousands digit be 3? If so, the remaining 3 digits have a product of 576.
Can 3 digits have a product of 576?
If one of these 3 digits were 7 or less, then the product of the 3 digits would be at most $7 \times 9 \times 9 = 567$, which is too small.
Therefore, if we have 3 digits with a product of 576, then each digit is 8 or 9.
Since the product is even, then at least one of the digits would have to be 8, leaving the remaining two digits to have a product of $576 \div 8 = 72$.
These two digits would then have to be 8 and 9.
Thus, we can have 3 digits with a product of 576, and so we can have 4 digits with a product of 1728 with smallest digit 3.
Therefore, the digits of $N$ must be 3, 8, 8, 9. The smallest possible number formed by these digits is when the digits are placed in increasing order, and so $N = 3889$.
The sum of the digits of $N$ is $3 + 8 + 8 + 9 = 28$.

Answer: (A)

19. We label the three points as $O(0, 0)$, $P(1, 4)$ and $Q(4, 1)$.
There are three possible locations for the fourth vertex $R$ of the parallelogram – between $O$ and $P$ (in the second quadrant), between $P$ and $Q$ (in the first quadrant), and between $Q$ and $O$ (in the fourth quadrant).
In each of these cases, $\triangle OPQ$ will make up half of the parallelogram, and so the area of the parallelogram is twice the area of $\triangle OPQ$.
There are many ways to calculate the area of $\triangle OPQ$.
We proceed by "completing the rectangle" which includes the $x$-axis, the $y$-axis, the line $y = 4$, and the line $x = 4$.
We label the point $(0, 4)$ as $S$, the point $(4, 4)$ as $T$, and the point $(4, 0)$ as $U$.
(Note that rectangle $OSTU$ is in fact a square, so we have "completed the square"!)
The area of $\triangle OPQ$ equals the area of rectangle $OSTU$ minus the combined areas of $\triangle OSP$, $\triangle PTQ$, and $\triangle QUO$.



The area of rectangle $OSTU$ is $4 \cdot 4 = 16$, since it is a square with side length 4.
Consider $\triangle OSP$. It is right-angled at $S$, with $OS = 4$ and $SP = 1$.
Thus, its area is $\frac{1}{2}(OS)(SP) = \frac{1}{2}(4)(1) = 2$.
Consider $\triangle PTQ$. It is right-angled at $T$, with $PT = TQ = 3$.
Thus, its area is $\frac{1}{2}(PT)(TQ) = \frac{1}{2}(3)(3) = \frac{9}{2}$.
Consider $\triangle QUO$. It is right-angled at $U$, with $OU = 4$ and $UQ = 1$.
Thus, its area is $\frac{1}{2}(OU)(UQ) = \frac{1}{2}(4)(1) = 2$.
Therefore, the area of $\triangle OPQ$ is $16 - 2 - \frac{9}{2} - 2 = \frac{15}{2}$.
Thus, the area of the parallelogram is $2 \cdot \frac{15}{2} = 15$.

Answer: (A)

20. In the first race, Katie ran 100 m in the same time that Sarah ran 95 m.
This means that the ratio of their speeds is $100 : 95 = 20 : 19$.
In other words, in the time that Sarah runs 1 m, Katie runs $\frac{20}{19} \approx 1.053$ m.
Put another way, in the time that Katie runs 1 m, Sarah runs $\frac{19}{20} = 0.95$ m.
In the second race, Katie must run 105 m and Sarah must run 100 m.
If Sarah finishes first, then Katie must not have completed 105 m in the time that it takes Sarah to complete 100 m.
But Katie runs 1.053 m for 1 m that Sarah runs, so Katie will in fact run more than 105 m in the time that Sarah runs 100 m.
Therefore, Katie must finish first.
In the time that Katie runs 105 m, Sarah will run $105 \times \frac{19}{20} = \frac{1995}{20} = \frac{399}{4} = 99\frac{3}{4}$.
Thus, Sarah was $100 - 99\frac{3}{4} = \frac{1}{4} = 0.25$ m behind.
Therefore, when Katie crossed the finish line, Sarah was 0.25 m behind.

Answer: (B)

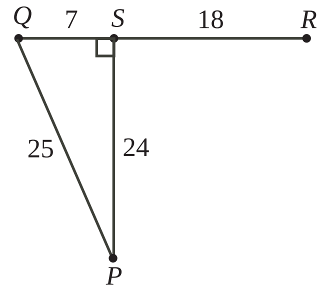
21. Since $x^2 = 8x + y$ and $y^2 = x + 8y$, then $x^2 - y^2 = (8x + y) - (x + 8y) = 7x - 7y$.
Factoring both sides, we obtain $(x + y)(x - y) = 7(x - y)$.
Since $x \neq y$, then $x - y \neq 0$, so we can divide both sides by $x - y$ to obtain $x + y = 7$.
Since $x^2 = 8x + y$ and $y^2 = x + 8y$, then

$$x^2 + y^2 = (8x + y) + (x + 8y) = 9x + 9y = 9(x + y) = 9 \cdot 7 = 63$$

Answer: (C)

22. From the chart, we see that $QR = 25$, $QS = 7$ and $SR = 18$.
Since $QR = QS + SR$ and $QR$ is the largest of these three lengths, then $S$ must be a point on line segment $QR$.
This gives the following configuration so far:


We have not yet used the fact that $PQ = 25$ or that $PS = 24$.
Note that $7^2 + 24^2 = 49 + 576 = 625 = 25^2$, so $QS^2 + PS^2 = PQ^2$.
Since these lengths satisfy this property, then the points $P$, $S$ and $Q$ form a triangle that is right-angled at $S$.
This gives the following configuration so far:




(We could have drawn $P$ "above" $QR$.)
Since $\angle PSQ = 90^\circ$, then $\angle PSR = 90^\circ$.
Therefore, $PR^2 = PS^2 + SR^2 = 24^2 + 18^2 = 576 + 324 = 900$.
Since $PR > 0$, then $PR = \sqrt{900} = 30$.
Thus, the distance between cities $P$ and $R$ is 30.

ANSWER: (A)

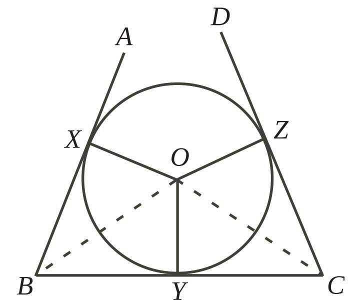
23. Initially, the bowl contains 320 g of white sugar and 0 g of brown sugar.
Mixture Y contains $(320 - x)$ g of white sugar and $x$ g of brown sugar.
When Mixture Z (the final mixture) is formed, there is still 320 g of sugar in the bowl.
Since we are told that the ratio of the mass of white sugar to the mass of brown sugar is 49 : 15, then the mass of white sugar in Mixture Z is $\frac{49}{49+15} \cdot 320 = \frac{49}{64} \cdot 320 = 49 \cdot 5 = 245$ g and the mass of brown sugar in Mixture Z is $320 - 245 = 75$ g.
In order to determine the value of $x$ (and hence determine the values of $w$ and $b$), we need to determine the mass of each kind of sugar in Mixture Z in terms of $x$.
Recall that Mixture Y consists of $(320 - x)$ g of white sugar and $x$ g of brown sugar, which are thoroughly mixed together.
Because Mixture Y is thoroughly mixed, then each gram of Mixture Y consists of $\dfrac{320 - x}{320}$ g of white sugar and $\dfrac{x}{320}$ g of brown sugar.
To form Mixture Z, $x$ g of Mixture Y are removed.
This amount of Mixture Y that is removed contains $x \cdot \dfrac{x}{320} = \dfrac{x^2}{320}$ g of brown sugar.
Mixture Z is made by removing $x$ g of Mixture Y (which contains $\dfrac{x^2}{320}$ g of brown sugar),then adding $x$ g of brown sugar.
Thus the mass of brown sugar, in g, in Mixture Z is $x - \dfrac{x^2}{320} + x$.
Since Mixture Z includes 75 g of brown sugar, then

$$\begin{aligned} 2x - \frac{x^2}{320} &= 75 \\ 0 &= x^2 - 2(320)x + 75(320) \\ 0 &= x^2 - 640x + 24000 \\ 0 &= (x-40)(x-600) \end{aligned}$$

Therefore, $x = 40$ or $x = 600$.
Since the initial mixture consists of 320 g of sugar, then $x < 320$, so $x = 40$.
This tells us that Mixture Y consists of $320 - 40 = 280$ g of white sugar and 40 g of brown sugar. The ratio of these masses is 280 : 40, which equals 7 : 1 in lowest terms. Thus, $w = 7$ and $b = 1$.
Therefore, $x + w + b = 40 + 7 + 1 = 48$.

ANSWER: (A)

24. We use without proof the fact that if a circle with centre $O$ and radius $r$ touches (that is, is tangent to) line segments $AB$, $BC$ and $CD$ at $X$, $Y$ and $Z$, respectively, then $\angle OBX = \angle OBY = \frac{1}{2}(\angle ABC)$ and $\angle OCY = \angle OCZ = \frac{1}{2}(\angle BCD)$.




Suppose that $\angle ABC = \theta$ and $\angle BCD = \alpha$.

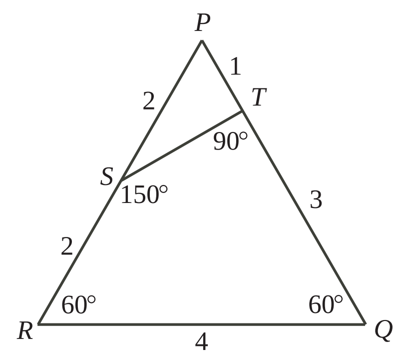
Since $OY$ is perpendicular to $BC$, then $\tan(\angle OBY) = \dfrac{OY}{BY}$ and $\tan(\angle OCY) = \dfrac{OY}{YC}$.

Thus, $BY = \dfrac{OY}{\tan(\angle OBY)} = \dfrac{r}{\tan(\theta/2)}$ and $YC = \dfrac{OY}{\tan(\angle OCY)} = \dfrac{r}{\tan(\alpha/2)}$.

Since $BC = BY + YC$, then

$$\begin{aligned} BC &= \frac{r}{\tan(\theta/2)} + \frac{r}{\tan(\alpha/2)} \\ BC &= r\left(\frac{1}{\tan(\theta/2)} + \frac{1}{\tan(\alpha/2)}\right) \\ BC &= r\left(\frac{\tan(\theta/2)+\tan(\alpha/2)}{\tan(\theta/2)\tan(\alpha/2)}\right) \\ r &= \frac{BC\tan(\theta/2)\tan(\alpha/2)}{\tan(\theta/2)+\tan(\alpha/2)} \end{aligned}$$

Consider now the given quadrilateral $QRST$. We know that $TQ = 3$. Since $\triangle PQR$ is equilateral, then $QR = PQ$ and so $QR = PT + TQ = 1 + 3 = 4$.




Since $PR = QR = 4$ and $S$ is the midpoint of $PR$, then $RS = 2$.

Since $\triangle PST$ has $PT = 1$ and $PS = \frac{1}{2}PR = 2$ and $\angle SPT = 60^\circ$, then it is a $30^\circ$-$60^\circ$-$90^\circ$ triangle, so we have $ST = \sqrt{3}$.

Also, we have $\angle PTS = 90^\circ$ and $\angle PST = 30^\circ$, which give $\angle STQ = 90^\circ$ and $\angle RST = 150^\circ$.

To summarize, quadrilateral $QRST$ has $QR = 4$, $RS = 2$, $ST = \sqrt{3}$, $TQ = 3$, $\angle TQR = 60^\circ$, $\angle QRS = 60^\circ$, $\angle RST = 150^\circ$, and $\angle STQ = 90^\circ$.

Next, we determine the radius of a circle tangent to each set of three consecutive sides, ignoring the fact that the fourth side might restrict the size of the circle. After determining these radii, we examine how the fourth side comes into play.

Suppose that we have a circle tangent to line segments $TQ$, $QR$ and $RS$.

By the formula above, the radius of this circle would equal

$$\frac{4\tan(60^\circ/2)\tan(60^\circ/2)}{\tan(60^\circ/2)+\tan(60^\circ/2)} = \frac{4\tan(30^\circ)\tan(30^\circ)}{\tan(30^\circ)+\tan(30^\circ)} \approx 1.1547$$

Suppose that we have a circle tangent to line segments $QR$, $RS$ and $ST$.

By the formula above, the radius of this circle would equal

$$\frac{2\tan(60^\circ/2)\tan(150^\circ/2)}{\tan(60^\circ/2)+\tan(150^\circ/2)} = \frac{2\tan(30^\circ)\tan(75^\circ)}{\tan(30^\circ)+\tan(75^\circ)} = 1$$

Suppose that we have a circle tangent to line segments $RS$, $ST$ and $TQ$.
By the formula above, the radius of this circle would equal

$$\frac{\sqrt{3}\tan(150^\circ/2)\tan(90^\circ/2)}{\tan(150^\circ/2)+\tan(90^\circ/2)} = \frac{\sqrt{3}\tan(75^\circ)\tan(45^\circ)}{\tan(75^\circ)+\tan(45^\circ)} \approx 1.3660$$

Suppose that we have a circle tangent to line segments $ST$, $TQ$ and $QR$.
By the formula above, the radius of this circle would equal

$$\frac{3\tan(90^\circ/2)\tan(60^\circ/2)}{\tan(90^\circ/2)+\tan(60^\circ/2)} = \frac{3\tan(45^\circ)\tan(30^\circ)}{\tan(45^\circ)+\tan(30^\circ)} \approx 1.0981$$

We need to determine the radius of the largest circle that can be drawn inside quadrilateral $QRST$.
The largest such circle will be touching at least two adjacent sides of $QRST$. Why is this? If a circle were touching zero sides or one side of $QRST$, it could be slid until it was touching two consecutive sides and then expanded a little bit and so cannot be the largest such circle. If it were touching two opposite sides of $QRST$, then it could be slid to touch a side adjacent to one of these two sides, perhaps losing contact with one of the two opposite sides.
Consider now a circle touching two adjacent sides of $QRST$. Such a circle can be expanded while maintaining contact to these two sides until it touches a third side. Once it touches the third side, it can't be expanded any further because its radius is fixed by the calculation that we did at the beginning of the solution.
Therefore, the largest circle will be touching three of the sides of $QRST$.

In order to complete the solution, we need to determine which of the circles that touch three sides actually lie completely inside $QRST$.
We do this by examining each of the four pairs of consecutive sides, and determining what the largest circle is that can be drawn touching these sides.

- We consider a circle tangent to $ST$ and $TQ$. We expand the circle, keeping it tangent to $ST$ and $TQ$, until it touches either $QR$ or $RS$. From the previous calculations, the circle that also touches $QR$ has radius about 1.0981, and the circle that also touches $RS$ has radius about 1.3660. Thus, the circle will first touch $QR$. In this case, the largest circle that is completely inside the quadrilateral has radius 1.0981.
- We consider a circle tangent to $TQ$ and $QR$. We expand the circle, keeping it tangent to $TQ$ and $QR$, until it touches either $ST$ or $RS$. From the previous calculations, the circle that also touches $ST$ has radius about 1.0981, and the circle that also touches $RS$ has radius about 1.1547. Thus, the circle will first touch $ST$. In this case, the largest circle that is completely inside the quadrilateral has radius 1.0981.
- We consider a circle tangent to $QR$ and $RS$. We expand the circle, keeping it tangent to $QR$ and $RS$, until it touches either $ST$ or $TQ$. From the previous calculations, the circle that also touches $ST$ has radius 1, and the circle that also touches $TQ$ has radius about 1.1547. Thus, the circle will first touch $ST$. In this case, the largest circle that is completely inside the quadrilateral has radius 1.
- We consider a circle tangent to $RS$ and $ST$. We expand the circle, keeping it tangent to $RS$ and $ST$, until it touches either $QR$ or $TQ$. From the previous calculations, the circle that also touches $QR$ has radius about 1, and the circle that also touches $TQ$ has radius about 1.3660. Thus, the circle will first touch $QR$. In this case, the largest circle that is completely inside the quadrilateral has radius 1.

Finally, comparing the four cases, we see that the largest circle that we can obtain has radius about 1.0981, which is closest to 1.10.

Answer: (B)

25. Let $N$ be an arbitrary positive integer with the desired properties.
Let $S(N)$ represent the sum of the digits of $N$ and let $S(2N)$ represent the sum of the digits of $2N$. In the table below, we make a claim about how each digit of $N$ contributes to $S(2N)$. We use the data in the table to answer the question, following which we justify the data in the table:

| Digit in $N$ | $2\times$Digit | Contribution to $S(2N)$ |
|---|---|---|
| 3 | 6 | 6 |
| 4 | 8 | 8 |
| 5 | 10 | $1+0=1$ |
| 6 | 12 | $1+2=3$ |

Suppose that the digits of $N$ include $w$ 3s, $x$ 4s, $y$ 5s, and $z$ 6s. Note that $w,x,y,z \geq 1$.

Step 1: Using information about $S(N)$ and $S(2N)$

Since $S(N)=900$, then $3w+4x+5y+6z=900$.
Since each 3 in $N$ contributes 6 to $S(2N)$, each 4 in $N$ contributes 8 to $S(2N)$, each 5 in $N$ contributes 1 to $S(2N)$, and each 6 in $N$ contributes 3 to $S(2N)$, then $S(2N)=900$ tells us that $6w+8x+y+3z=900$.

Step 2: Understanding which values of $N$ will be largest possible and smallest possible

The largest possible value of $N$ will be the integer $N^+$ that satisfies the given properties, has the largest number of digits (that is, the largest value of $w+x+y+z$), has the largest actual digits given this fixed number of digits, and has its digits in decreasing order from left to right (since the larger digits will correspond to the largest place values).
The smallest possible value of $N$ will be the integer $N^-$ that satisfies the given properties, has the smallest number of digits (that is, the smallest value of $w+x+y+z$), has the smallest actual digits given this fixed number of digits, and has its digits in increasing order from left to right.
Since we want to determine the number of digits in the product $N^+N^-$, we mostly care only about the number of digits in $N^+$ and $N^-$ (that is, the maximum and minimum values of $w+x+y+z$), and perhaps as well about the leading digits of each.

Step 3: Simplifying equations

We have that $w,x,y,z$ are positive integers that satisfy both $3w+4x+5y+6z=900$ and $6w+8x+y+3z=900$.
Multiplying the first of these equations by 2, we obtain $6w+8x+10y+12z=1800$.
When we subtract the second equation from this, we obtain $9y+9z=900$, or $y+z=100$.
Since $y+z=100$, then $3w+4x+5y+6z=900$ becomes $3w+4x+5(y+z)+z=900$ or $3w+4x+500+z=900$ or $3w+4x+z=400$.
Also since $y+z=100$, we obtain $w+x+y+z=w+x+100$, so to minimize and maximize $w+x+y+z$, we have to minimize and maximize $w+x$.

Step 4: Restating part of goal

We want to find the maximum and minimum possible values of $w + x + y + z$ subject to the conditions that $w, x, y, z$ are positive integers that satisfy $3w + 4x + 5y + 6z = 900$ and $6w + 8x + y + 3z = 900$.
From Step 3, these equations are true if and only if $y + z = 100$ and $3w + 4x + z = 400$. (This is because we can use each pair of equations to obtain the other pair.)
Therefore, we want to find the maximum and minimum possible values of $w+x+y+z$ subject to the conditions $y + z = 100$ and $3w + 4x + z = 400$.
Since $y + z$ is fixed, this is the same as finding the maximum and minimum possible values of $w + x$ subject to the conditions $y + z = 100$ and $3w + 4x + z = 400$.

Step 5: Determining information about $N^+$

We determine the maximum possible value of $w + x$.
We rewrite $3w + 4x + z = 400$ as $3(w + x) = 400 - x - z$.
To make $w + x$ as large as possible, we want the right side of this equation to be as large as possible, and so we want $x$ and $z$ to be as small as possible.
Note that $x \geq 1$ and $z \geq 1$ and so $400 - x - z \leq 398$.
Also, since the left side of $3(w + x) = 400 - x - z$ is divisible by 3, then the right side must be divisible by 3.
The largest multiple of 3 less than or equal to 398 is 396.
Therefore, $3(w + x) \leq 396$ and so $w + x \leq 132$.
Thus, the maximum possible value of $w + x + y + z$ is $132 + 100 = 232$.
To achieve this maximum, we need $400 - x - z = 396$ (that is, $x + z = 4$). The values $w = 129$, $x = 3$, $y = 99$, $z = 1$ achieve this maximum (and satisfy $y + z = 100$ and $3w + 4x + z = 400$).
Therefore, $N^+$, the largest possible value of $N$, consists of 232 digits (all 3s, 4s, 5s and 6s including at least one of each) arranged in descending order.
Thus, $N^+$ satisfies $6 \times 10^{231} < N^+ < 7 \times 10^{231}$.
(As it turns out, we will not actually have to determine the actual digits of $N^+$.)

Step 6: Determining information about $N^-$

We determine the minimum possible value of $w + x$.
We rewrite $3w + 4x + z = 400$ as $4(w + x) = 400 + w - z$.
To make $w + x$ as small as possible, we want the right side of this equation to be as small as possible, and so we want $w$ to be as small as possible and $z$ to be as large as possible.
Now $w \geq 1$ and since $y + z = 100$ and $y \geq 1$, then $z \leq 99$.
Thus, $400 + w - z \geq 302$.
Since the left side of the equation $4(w + x) = 400 + w - z$ is divisible by 4, then the right side must be divisible by 4.
The smallest multiple of 4 greater than or equal to 302 is 304.
Therefore, $4(w + x) \geq 304$ and so $w + x \geq 76$.
Thus, the minimum possible value of $w + x + y + z$ is $76 + 100 = 176$.
To achieve this minimum, we need $400 + w - z = 304$ (that is, $z - w = 96$). The values $w = 3$, $x = 73$, $y = 1$, $z = 99$ achieve this minimum (and satisfy $y + z = 100$ and $3w + 4x + z = 400$).
Therefore, $N^-$ consists of 176 digits (all 3s, 4s, 5s and 6s including at least one of each) arranged in ascending order.
Thus, $N^-$ satisfies $3 \times 10^{175} < N^- < 4 \times 10^{175}$, since this value of $N$ begins with a 3 and has 176 digits.
(Again, we will not actually have to determine the actual digits of $N^-$.)

Step 7: Determining the number of digits in $N^- \cdot N^+$

Finally, since $6 \times 10^{231} < N^+ < 7 \times 10^{231}$ and $3 \times 10^{175} < N^- < 4 \times 10^{175}$, then

$$18 \times 10^{406} = (3 \times 10^{175}) \cdot (6 \times 10^{231}) < N^- \cdot N^+ < (4 \times 10^{175}) \cdot (7 \times 10^{231}) = 28 \times 10^{406}$$

Therefore, $N^- \cdot N^+$ has 408 digits.

Justification of data in table

We must still justify the data in the table above.
Suppose that $N$ ends with the digits $abcd$. That is, $N = \cdots dcba$.
Then we can write $N = \cdots + 1000d + 100c + 10b + a$.
Then $2N = \cdots + 1000(2d) + 100(2c) + 10(2b) + (2a)$. The difficulty in determining the digits of $2N$ is that each of $2a$, $2b$, $2c$ and $2d$ may not be a single digit.
We use the notation $u(2a)$ and $t(2a)$ to represent the units digit and tens digit of $2a$, respectively.
Note that $u(2a)$ is one of 0, 2, 4, 6, or 8, and $t(2a)$ is 0 or 1.
We define $u(2b), t(2b), u(2c), t(2c), u(2d), t(2d)$ similarly.
Note that $2a = 10 \cdot t(2a) + u(2a)$ and $2b = 10 \cdot t(2b) + u(2b)$ and $2c = 10 \cdot t(2c) + u(2c)$ and $2d = 10 \cdot t(2d) + u(2d)$.
Thus,

$$\begin{aligned} 2N &= \cdots + 1000(10 \cdot t(2d) + u(2d)) + 100(10 \cdot t(2c) + u(2c)) \\ &\qquad +10(10 \cdot t(2b) + u(2b)) + (10 \cdot t(2a) + u(2a)) \\ &= \cdots + 1000(u(2d) + t(2c)) + 100(u(2c) + t(2b)) + 10(u(2b) + t(2a)) + u(2a) \end{aligned}$$

Since $u(2a), u(2b), u(2c), u(2d) \leq 8$ and $t(2a), t(2b), t(2c), t(2d) \leq 1$, then each of $u(2d) + t(2c)$ and $u(2c) + t(2b)$ and $u(2b) + t(2a)$ and $u(2a)$ is a single digit, so these are the thousands, hundreds, tens and units digits, respectively, of $2N$.
Thus, the sum of the digits of $2N$ is

$$\begin{aligned} u(2a) + (u(2b) + t(2a)) + (u(2c) + t(2b)) + (u(2d) + t(2c)) + \cdots &= \\ (t(2a) + u(2a)) + (t(2b) + u(2b)) + (t(2c) + u(2c)) + \cdots & \end{aligned}$$

The above argument extends to the left for the remaining digits of $N$.
In other words, if $m$ is a digit in $N$, then its contribution to the sum of the digits of $2N$ is the sum of the tens and units digits of $2m$.
Therefore, the digits of $N$ contribute to the sum of the digits of $2N$ as outlined in the table above.

Answer: (A)

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING**

# 2011 Fermat Contest

**(Grade 11)**

**Thursday, February 24, 2011**

*Solutions*

1. Evaluating, $\dfrac{2+3\times 6}{23+6} = \dfrac{2+18}{29} = \dfrac{20}{29}$.

   Answer: (D)

2. If $y = 77$, then $\dfrac{7y+77}{77} = \dfrac{7y}{77} + \dfrac{77}{77} = \dfrac{7(77)}{77} + 1 = 7 + 1 = 8$.

   Answer: (A)

3. Since the area of the rectangle is 192 and its length is 24, then its width is $192 \div 24 = 8$.
   Therefore, its perimeter is $2 \times 24 + 2 \times 8 = 64$.

   Answer: (A)

4. Since $\sqrt{n+9} = 25$, then $n + 9 = 25^2 = 625$.
   Thus, $n = 625 - 9 = 616$.

   Answer: (D)

5. Since $\triangle PRS$ is equilateral, then all three of its angles equal $60^\circ$.
   In particular, $\angle RSP = 60^\circ$.
   Since $QS = QT$, then $\triangle QST$ is isosceles and so $\angle TSQ = \angle STQ = 40^\circ$.
   Since $RST$ is a straight line segment, then $\angle RSP + \angle PSQ + \angle TSQ = 180^\circ$.
   Therefore, $60^\circ + x^\circ + 40^\circ = 180^\circ$ or $x = 180 - 60 - 40 = 80$.

   Answer: (C)

6. If the sum of three consecutive integers is 27, then the numbers must be 8, 9 and 10. (We could see this algebraically by calling the integers $x$, $x + 1$ and $x + 2$ and solving the equation $x + (x + 1) + (x + 2) = 27$.)
   Their product is $8 \times 9 \times 10 = 720$.

   Answer: (C)

7. The number halfway between two numbers is their average.
   Therefore, the number halfway between $\frac{1}{10}$ and $\frac{1}{12}$ is $\frac{1}{2}(\frac{1}{10} + \frac{1}{12}) = \frac{1}{2}(\frac{12}{120} + \frac{10}{120}) = \frac{1}{2}(\frac{22}{120}) = \frac{11}{120}$.

   Answer: (D)

8. Since the angle in the sector representing cookies is $90^\circ$, then this sector represents $\frac{1}{4}$ of the total circle.
   Therefore, 25% of the students chose cookies as their favourite food.
   Thus, the percentage of students who chose sandwiches was $100\% - 30\% - 25\% - 35\% = 10\%$.
   Since there are 200 students in total, then $200 \times \frac{10}{100} = 20$ students said that their favourite food was sandwiches.

   Answer: (B)

9. The set $S$ contains 25 multiples of 2 (that is, even numbers).
   When these are removed, the set $S$ is left with only the odd integers from 1 to 49.
   At this point, there are $50 - 25 = 25$ integers in $S$.
   We still need to remove the multiples of 3 from $S$.
   Since $S$ only contains odd integers at this point, then we must remove the odd multiples of 3 between 1 and 49.
   These are 3, 9, 15, 21, 27, 33, 39, 45, of which there are 8.
   Therefore, the number of integers remaining in the set $S$ is $25 - 8 = 17$.

   Answer: (D)

10. *Solution 1*

Since $PQRS$ is a square and $QR = 2 + 9 = 11$, then $PQ = QR = SR = PS = 11$.
The height of the shaded rectangle equals the height of the top left rectangle minus the height of the top right rectangle, or $6 - 2 = 4$.
The width of the shaded rectangle equals the width of the top right rectangle minus the width of the bottom right rectangle.
Since $SR = 11$, then the width of the bottom right rectangle is $11 - 10 = 1$.
Therefore, the width of the shaded rectangle is $8 - 1 = 7$.
Thus, the area of the shaded rectangle is $4 \times 7 = 28$.

*Solution 2*

Since $PQRS$ is a square and $QR = 2 + 9 = 11$, then $PQ = QR = SR = PS = 11$.
Since the side length of the square is 11, then its area is $11^2 = 121$.
Since $PQ = 11$, then the width of the top left rectangle is $11 - 8 = 3$, and so its area is $3 \times 6 = 18$.
Since $PS = 11$, then the height of the bottom left rectangle is $11 - 6 = 5$, and so its area is $5 \times 10 = 50$.
Since $SR = 11$, then the width of the bottom right rectangle is $11 - 10 = 1$, and so its area is $1 \times 9 = 9$.
The area of the top right rectangle is $8 \times 2 = 16$.
Thus, the area of the shaded rectangle equals the area of square $PQRS$ minus the combined areas of the four unshaded rectangles, or $121 - 18 - 50 - 9 - 16 = 28$.

ANSWER: (B)

11. It is possible that after buying 7 gumballs, Wally has received 2 red, 2 blue, 1 white, and 2 green gumballs.
This is the largest number of each colour that he could receive without having three gumballs of any one colour.
If Wally buys another gumball, he will receive a blue or a green or a red gumball.
In each of these cases, he will have at least 3 gumballs of one colour.
In summary, if Wally buys 7 gumballs, he is not guaranteed to have 3 of any one colour; if Wally buys 8 gumballs, he is guaranteed to have 3 of at least one colour.
Therefore, the least number that he must buy to guarantee receiving 3 of the same colour is 8.

ANSWER: (E)

12. *Solution 1*

A parabola is symmetric about its axis of symmetry.
Since the $x$-intercepts of the given parabola are $x = -1$ and $x = 4$, then the axis of symmetry of the parabola is $x = \frac{-1+4}{2} = \frac{3}{2}$.
Since the point $(3, w)$ is $\frac{3}{2}$ units to the right of the axis of symmetry, then its $y$-coordinate (namely $w$) equals the $y$-coordinate of the point $\frac{3}{2}$ units to the left of the axis of symmetry, which is the point with $x = 0$.
When $x = 0$, we know that $y = 8$.
Therefore, $w = 8$.
(We could also note that $x = 3$ is 1 unit to the left of the rightmost $x$-intercept so its $y$-coordinate is equal to that of the point 1 unit to the right of the leftmost $x$-intercept.)

*Solution 2*

Since the parabola has $x$-intercepts of $-1$ and 4, then its equation is of the form $y = a(x+1)(x-4)$ for some value of $a$.
Since the point $(0, 8)$ lies on the parabola, then $8 = a(1)(-4)$ or $a = -2$.
Therefore, the parabola has equation $y = -2(x+1)(x-4)$.
Since the point $(3, w)$ lies on the parabola, then $w = -2(4)(-1) = 8$.

Answer: (E)

13. Since Xavier, Yolanda and Zixuan have \$50 in total, and the ratio of the amount that Xavier has to the amount that the other two have is 3 : 2, then Xavier has $\frac{3}{5}$ of the total, or $\frac{3}{5} \times \$50 = \$30$.
Therefore, Yolanda and Zixuan together have $\$50 - \$30 = \$20$.
We know that Yolanda has \$4 more than Zixuan, so we must break \$20 into two parts, one of which is \$4 larger than the other.
If Yolanda has \$12 and Zixuan has \$8, this satisfies the requirements.
Therefore, Zixuan has \$8.

Answer: (B)

14. The average of two multiples of 4 must be even, since we can write these multiples of 4 as $4m$ and $4n$ for some integers $m$ and $n$, which means that their average is $\frac{1}{2}(4m + 4n)$ which equals $2m + 2n$ or $2(m + n)$, which is a multiple of 2, and so is even.
Each of the other four choices may be an odd integer in some cases. Here is an example for each:

(A) The average of 2 and 4 is 3, which is not even
(B) The average of 3 and 7 is 5, which is not even
(C) The average of 1 and 9 is 5, which is not even
(E) The average of 2, 3 and 4 is 3, which is not even

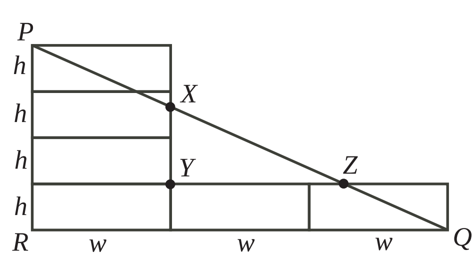
Therefore, the correct answer is (D).

Answer: (D)

15. Since $m$ and $n$ are consecutive positive integers with $n^2 - m^2 > 20$, then $n$ is greater than $m$.
Therefore, we can write $n = m + 1$.
Since $n^2 - m^2 > 20$, then $(m+1)^2 - m^2 > 20$ or $m^2 + 2m + 1 - m^2 > 20$ or $2m > 19$ or $m > \frac{19}{2}$.
Since $m$ is a positive integer, then $m \geq 10$.
Thus, we want to find the minimum value of $n^2 + m^2 = (m+1)^2 + m^2 = 2m^2 + 2m + 1$ when $m \geq 10$.
This minimum will occur when $m = 10$ (since $2m^2 + 2m + 1$ increases with $m$ when $m$ is a positive integer).
Therefore, the minimum possible value is $2(10^2) + 2(10) + 1 = 221$.

Answer: (E)

16. *Solution 1*
We label the bottom left corner as $R$ and label various side lengths as $h$ and $w$:



Since the diagram is made up of rectangles, then $XY$ is parallel to $PR$, which tells us that $\angle YXZ = \angle RPQ$. Also, $YZ$ is parallel to $RQ$, which tells us that $\angle XZY = \angle PQR$.
Therefore, $\triangle PRQ$ is similar to $\triangle XYZ$.
Thus, $\dfrac{RQ}{PR} = \dfrac{YZ}{XY}$.
But $YZ = 2XY$, $RQ = 3w$ and $PR = 4h$.
This tells us that $\dfrac{3w}{4h} = \dfrac{2XY}{XY}$ or $\dfrac{3}{4} \cdot \dfrac{w}{h} = 2$ or $\dfrac{h}{w} = \dfrac{3}{8}$.

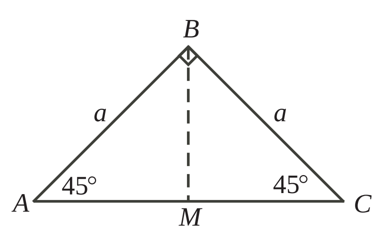
*Solution 2*
As in Solution 1, we label the bottom left corner as $R$ and label various side lengths as $h$ and $w$.
Since $YZ = 2XY$, then the slope of line segment $XZ$ is $\dfrac{XY}{YZ} = \dfrac{XY}{2XY} = \dfrac{1}{2}$.
Since $PR = 4h$ and $RQ = 3w$, then the slope of line segment $PQ$ is $\dfrac{PR}{RQ} = \dfrac{4h}{3w}$.
Since line segment $XZ$ is a portion of line segment $PQ$, then the slopes of these two line segments are equal, so $\dfrac{4h}{3w} = \dfrac{1}{2}$ and so $\dfrac{h}{w} = \dfrac{1}{2} \cdot \dfrac{3}{4} = \dfrac{3}{8}$.

Answer: (C)

17. Since $3^{2x} = 64$ and $3^{2x} = (3^x)^2$, then $(3^x)^2 = 64$ and so $3^x = \pm 8$.
Since $3^x > 0$, then $3^x = 8$.
Thus, $3^{-x} = \dfrac{1}{3^x} = \dfrac{1}{8}$.

Answer: (E)

18. We label the stages in this process as Stage 0 (a square), Stage 1 (2 triangles), Stage 2 (4 triangles), Stage 3 (8 triangles), and Stage 4 (16 triangles).
We want to determine the length of the longest edge of one of the 16 triangles in Stage 4.
At Stage 1, we have two right-angled isosceles triangles with legs of length 4.
Consider a general right-angled isosceles triangle $ABC$ with legs $AB$ and $BC$ of length $a$.
Since this is a 45°-45°-90° triangle, its hypotenuse $AC$ has length $\sqrt{2}$a.
We split the triangle into two equal pieces by bisecting the right-angle at $B$:




Since $\triangle ABC$ is isosceles, then this bisecting line is both an altitude and a median. In other words, it is perpendicular to $AC$ at $M$ and $M$ is the midpoint of $AC$.
Therefore, the two triangular pieces $\triangle AMB$ and $\triangle CMB$ are identical 45°-45°-90° triangles.
The longest edges of these triangles ($AB$ and $CB$) are the legs of the original triangle, and so have length $a$.
Since the longest edge of the original triangle was $\sqrt{2}a$, then the longest edge has been reduced by a factor of $\sqrt{2}$.
Since we have shown that this is the case for an arbitrary isosceles right-angled triangle, we can then apply this property to our problem.
In Stage 1, the longest edge has length $4\sqrt{2}$.

Since the longest edge in Stage 1 has length $4\sqrt{2}$, then the longest edge in Stage 2 has length $\frac{4\sqrt{2}}{\sqrt{2}} = 4$.
Since the longest edge in Stage 2 has length 4, then the longest edge in Stage 3 has length $\frac{4}{\sqrt{2}} = \frac{2\sqrt{2}\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}$.
Since the longest edge in Stage 3 has length $2\sqrt{2}$, then the longest edge in Stage 4 has length $\frac{2\sqrt{2}}{\sqrt{2}} = 2$.

ANSWER: (B)

19. Suppose that the radius of the larger circle is $r$.
Join $O$ to $P$. Then $OP = OS = r$.
Since $Q$ is the midpoint of $PR$, and $PR = 12$, then $PQ = \frac{1}{2}PR = 6$.
Since $OS = r$ and $QS = 4$, then $OQ = OS - QS = r - 4$.
Since $\triangle OPQ$ is right-angled at $Q$, then by the Pythagorean Theorem,

$$\begin{aligned} OQ^2 + PQ^2 &= OP^2 \\ (r-4)^2 + 6^2 &= r^2 \\ r^2 - 8r + 16 + 36 &= r^2 \\ 52 &= 8r \\ r &= \tfrac{52}{8} \end{aligned}$$

Therefore, the radius of the larger circle is $\frac{52}{8}$, or 6.5.

ANSWER: (C)

20. Since $b = ar$, $c = ar^2$, and the product of $a$, $b$ and $c$ is 46656, then $a(ar)(ar^2) = 46656$ or $a^3r^3 = 46656$ or $(ar)^3 = 46656$ or $ar = \sqrt[3]{46656} = 36$.
Therefore, $b = ar = 36$.
Since the sum of $a$, $b$ and $c$ is 114, then $a + c = 114 - b = 114 - 36 = 78$.

ANSWER: (A)

21. In the given pattern, the $r$th row contains $r$ integers.
Therefore, after $n$ rows, the total number of integers appearing in the pattern is

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$$

This expression is always equal to $\frac{1}{2}n(n+1)$.
(If you have never seen this formula before, try to prove it!)
Putting this another way, the largest number in the $n$th row is $\frac{1}{2}n(n+1)$.
To determine which row the number 400 is in, we want to determine the smallest value of $n$ for which $\frac{1}{2}n(n+1) \geq 400$ or $n(n+1) \geq 800$.
If $n = 27$, then $n(n+1) = 756$.
If $n = 28$, then $n(n+1) = 812$.
Therefore, 400 appears in the 28th row. Also, the largest integer in the 28th row is 406 and the largest integer in the 27th row is 378.
Thus, we want to determine the sum of the integers from 379 (the first integer in the 28th row) to 406, inclusive.
We can do this by calculating the sum of the integers from 1 to 406 and subtracting the sum of the integers from 1 to 378.
Since the sum of the integers from 1 to $m$ equals $\frac{1}{2}m(m+1)$, then the sum of the integers from 379 to 406 is equal to $\frac{1}{2}(406)(407) - \frac{1}{2}(378)(379) = 10990$.

ANSWER: (A)

22. Since $\dfrac{p+q^{-1}}{p^{-1}+q}=17$, then $\dfrac{p+\frac{1}{q}}{\frac{1}{p}+q}=17$ or $\dfrac{\frac{pq+1}{q}}{\frac{1+pq}{p}}=17$ or $\dfrac{p(pq+1)}{q(pq+1)}=17$.

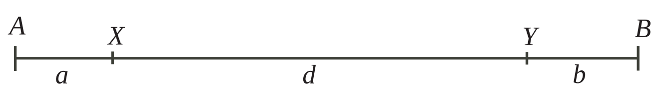
Since $p$ and $q$ are positive integers, then $pq+1>0$, so we can divide out the common factor in the numerator and denominator to obtain $\dfrac{p}{q}=17$ or $p=17q$.
Since $p$ and $q$ are positive integers, then $q\geq 1$.
Since $p+q\leq 100$, then $17q+q\leq 100$ or $18q\leq 100$ or $q\leq \frac{100}{18}=5\frac{5}{9}$.
Since $q$ is a positive integer, then $q\leq 5$.
Therefore, the combined restriction is $1\leq q\leq 5$, and so there are five pairs.
(We can check that these pairs are $(p,q)=(17,1),(34,2),(51,3),(68,4),(85,5)$.)

Answer: (E)

23. First, we note that the three people are interchangeable in this problem, so it does not matter who rides and who walks at any given moment. We abbreviate the three people as D, M and P.
We call their starting point $A$ and their ending point $B$.
Here is a strategy where all three people are moving at all times and all three arrive at $B$ at the same time:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to a point $Y$ before $B$.
D drops off M and rides back while P and M walk toward $B$.
D meets P at point $X$.
D picks up P and they drive back to $B$ meeting M at $B$.
Point $Y$ is chosen so that D, M and P arrive at $B$ at the same time.

Suppose that the distance from $A$ to $X$ is $a$ km, from $X$ to $Y$ is $d$ km, and the distance from $Y$ to $B$ is $b$ km.




In the time that it takes P to walk from $A$ to $X$ at 6 km/h, D rides from $A$ to $Y$ and back to $X$ at 90 km/h.
The distance from $A$ to $X$ is $a$ km.
The distance from $A$ to $Y$ and back to $X$ is $a+d+d=a+2d$ km.
Since the time taken by P and by D is equal, then $\dfrac{a}{6}=\dfrac{a+2d}{90}$ or $15a=a+2d$ or $7a=d$.
In the time that it takes M to walk from $Y$ to $B$ at 6 km/h, D rides from $Y$ to $X$ and back to $B$ at 90 km/h.
The distance from $Y$ to $B$ is $b$ km, and the distance from $Y$ to $X$ and back to $B$ is $d+d+b=b+2d$ km.
Since the time taken by M and by D is equal, then $\dfrac{b}{6}=\dfrac{b+2d}{90}$ or $15b=b+2d$ or $7b=d$.
Therefore, $d=7a=7b$, and so we can write $d=7a$ and $b=a$.
Thus, the total distance from $A$ to $B$ is $a+d+b=a+7a+a=9a$ km.
However, we know that this total distance is 135 km, so $9a=135$ or $a=15$.
Finally, D rides from $A$ to $Y$ to $X$ to $B$, a total distance of $(a+7a)+7a+(7a+a)=23a$ km.

Since $a = 15$ km and D rides at 90 km/h, then the total time taken for this strategy is $\frac{23\times 15}{90} = \frac{23}{6} \approx 3.83$ h.
Since we have a strategy that takes 3.83 h, then the smallest possible time is no more than 3.83 h. Can you explain why this is actually the smallest possible time?

If we didn't think of this strategy, another strategy that we might try would be:

D and M get on the motorcycle while P walks.
D and M ride the motorcycle to $B$.
D drops off M at $B$ and rides back to meet P, who is still walking.
D picks up P and they drive back to $B$. (M rests at $B$.)

This strategy actually takes 4.125 h, which is longer than the strategy shown above, since M is actually sitting still for some of the time.

Answer: (A)

24. The six possible sums are $w+x$, $w+y$, $w+z$, $x+y$, $x+z$, and $y+z$.
Since $x<y$, then $w+x<w+y$.
Since $w<x$, then $w+y<x+y$.
Since $y<z$, then $x+y<x+z$.
Since $x<y$, then $x+z<y+z$.
Therefore, we have $w+x<w+y<x+y<x+z<y+z$.
This list includes all of the sums except $w+z$.
Since $y<z$ and $w<x$, then $w+y<w+z<x+z$, but we cannot say for sure whether $x+y$ or $w+z$ is larger.
Thus, we know that $w+x$ is always the smallest sum and that $w+y$ is always the second smallest sum. Also, we know that the third and fourth smallest sums are $w+z$ and $x+y$ in some order.
We can conclude that $w+x=1$ and $w+y=2$, and $w+z$ and $x+y$ equal 3 and 4 in some order.
From the first and second equations, $(w+y)-(w+x)=2-1$ or $y-x=1$.

Case 1: $w+z=3$ and $x+y=4$
Since $y-x=1$ and $x+y=4$, we add these to obtain $2y=5$ or $y=\frac{5}{2}$.
Since $w+y=2$, then $w=2-y=2-\frac{5}{2}=-\frac{1}{2}$.
Since $w+z=3$, then $z=3-w=3-(-\frac{1}{2})=\frac{7}{2}$.
Since $x+y=4$, then $x=4-y=4-\frac{5}{2}=\frac{3}{2}$.
Therefore, we have $w=-\frac{1}{2}$, $x=\frac{3}{2}$, $y=\frac{5}{2}$, and $z=\frac{7}{2}$.
We can check that the six sums are $1,2,3,4,5,6$, which are all different as required.

Case 2: $w+z=4$ and $x+y=3$
Since $y-x=1$ and $x+y=3$, we add these to obtain $2y=4$ or $y=2$.
Since $w+y=2$, then $w=2-y=2-2=0$.
Since $w+z=4$, then $z=4-w=4-0=4$.
Since $x+y=3$, then $x=3-y=3-2=1$.
Therefore, we have $w=0$, $x=1$, $y=2$, and $z=4$.
We can check that the six sums are $1,2,3,4,5,6$, which are all different as required.

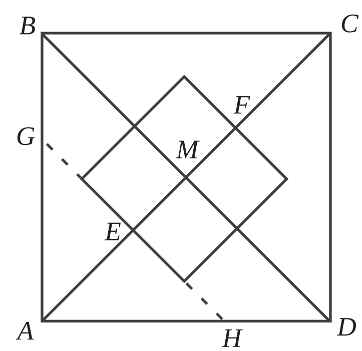
Therefore, the two possible values of $z$ are 4 and $\frac{7}{2}$.
The sum of these values is $4+\frac{7}{2}=\frac{15}{2}$.

Answer: (D)

25. The smallest possible height of the pyramid will occur when the four side faces are just touching the circumference of the end faces of the cylinder. To see this, consider starting with the top vertex of the pyramid much higher than in its position with minimum height. In this higher position, none of the lateral faces of the pyramid touch the cylinder. We gradually lower this top vertex towards the centre of the square base. Eventually, each of the lateral faces of the pyramid will touch the "rim" of one of the circular ends of the cylinder. We cannot lower the top vertex any further since otherwise part of the cylinder would be outside of the pyramid. This is our minimum height position. We calculate the height of this pyramid.

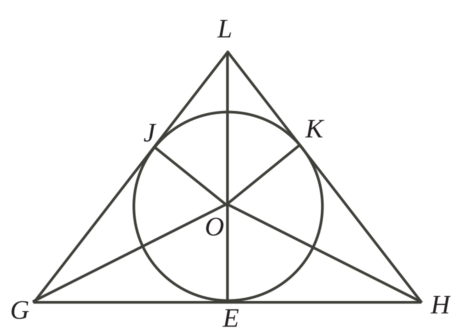
We label the square base of the pyramid as $ABCD$, and the top vertex of the pyramid as $T$. Join $AC$ and $BD$, the diagonals of the base. Label their point of intersection, which is also the centre of the base, as $M$.
Since the square base has side length 20, then $AC = BD = 20\sqrt{2}$.
Since the diagonals bisect each other, then $AM = BM = CM = DM = 10\sqrt{2}$.
Note that $T$ lies directly above $M$.
Let $t$ be the height of the pyramid; that is, let $t = TM$. We want to calculate $t$.

Suppose that the central axis of the cylinder lies above $AC$.
Since the midpoint of the central axis lies directly above $M$, then the central axis extends a distance of 5 to either side of $M$.
Label the points in contact with $AC$ at the two ends of the cylinder as $E$ and $F$. Since $EM = FM = 5$, then each end of the cylinder lies a distance of $10\sqrt{2} - 5$ from the corner of the base, as measured along the diagonal (that is, $AE = 10\sqrt{2} - 5$).




From above, the cylinder's "footprint" on the base of the pyramid is actually a square, since its diameter becomes its width.

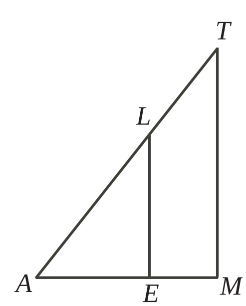
Consider a vertical cross section of the pyramid and cylinder through the end of the cylinder closest to $A$.



Let $L$ be the point where the cross-section intersects $AT$ and $G$ and $H$ be the points where the cross-section intersects $AB$ and $AD$, respectively. These points $G$ and $H$ are the same points shown in the first diagram.
Since $\angle BAM = 45^\circ$ and the cylinder's face is perpendicular to the diagonal of the square base, then $\triangle GEA$ is isosceles and right-angled (as is $\triangle HEA$) so $GE = HE = AE = 10\sqrt{2} - 5$.
Let $O$ be the centre of the end face of the cylinder.
Note that $GL$ and $HL$ are lines that lie along the faces $ABT$ and $ADT$ of the pyramid.
Since the faces $ABT$ and $ADT$ of the pyramid just touch the cylinder's rim, then $GL$ and $HL$ are tangent to the circular face, say at $J$ and $K$, respectively.
Join $O$ to points $G$, $H$ and $L$.
Also, join $O$ to points $J$, $K$ and $E$. Each of these segments is a radius of the circular face, so each has length 5.
Since the circle is tangent to faces of the pyramid (including the bottom face) at these points, then each of these segments is perpendicular to the corresponding side of $\triangle GHL$.
Our goal will be to calculate the length of $LE$.
Since $GE$ and $GJ$ are tangents to the circle from a common point, then $GJ = GE = 10\sqrt{2} - 5$.
Let $LE = h$. Then $LO = h - 5$. Also, let $LJ = x$.
Note that $\triangle LJO$ is similar to $\triangle LEG$, since they have a common angle at $L$ and each is right-angled.
Since these triangles are similar, then $\dfrac{LJ}{JO} = \dfrac{LE}{EG}$.
Therefore, $\dfrac{x}{5} = \dfrac{h}{10\sqrt{2} - 5}$ or $x = \dfrac{5h}{10\sqrt{2} - 5} = \dfrac{h}{2\sqrt{2} - 1}$.
Also, from the similarity, $\dfrac{LG}{GE} = \dfrac{LO}{OJ}$.
Therefore, $\dfrac{x + (10\sqrt{2} - 5)}{10\sqrt{2} - 5} = \dfrac{h - 5}{5}$ or $x + (10\sqrt{2} - 5) = (2\sqrt{2} - 1)(h - 5)$.
Substituting $x = \dfrac{h}{2\sqrt{2} - 1}$, we obtain

$$\begin{aligned}
\frac{h}{2\sqrt{2}-1} + (10\sqrt{2} - 5) &= (2\sqrt{2} - 1)(h - 5) \\
h + 5(2\sqrt{2} - 1)^2 &= (2\sqrt{2} - 1)^2(h - 5) \\
h + 5(2\sqrt{2} - 1)^2 &= (9 - 4\sqrt{2})h - 5(2\sqrt{2} - 1)^2 \\
10(2\sqrt{2} - 1)^2 &= (8 - 4\sqrt{2})h \\
h &= \frac{10(2\sqrt{2} - 1)^2}{8 - 4\sqrt{2}}
\end{aligned}$$

Finally, to calculate $t$, we extract $\triangle AMT$.



Note that $E$ lies on $AM$ and $L$ lies on $AT$.
Also, $TM$ is perpendicular to $AM$ and $LE$ is perpendicular to $AE$, which means that $\triangle AEL$ is similar to $\triangle AMT$.
Therefore, $\dfrac{TM}{AM} = \dfrac{LE}{AE}$, or $\dfrac{t}{10\sqrt{2}} = \dfrac{h}{10\sqrt{2}-5}$, and so

$$t = \frac{10\sqrt{2}}{5(2\sqrt{2}-1)} \cdot \frac{10(2\sqrt{2}-1)^2}{8-4\sqrt{2}} = \frac{20\sqrt{2}(2\sqrt{2}-1)}{8-4\sqrt{2}} = \frac{5\sqrt{2}(2\sqrt{2}-1)}{2-\sqrt{2}} \approx 22.07$$

Of the given answers, the smallest possible height is closest to 22.1.

Answer: (B)

**Canadian Mathematics Competition**

*An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario*

# *2010 Fermat Contest*

**(Grade 11)**

**Thursday, February 25, 2010**

*Solutions*

1. The sum consists of two halves and three thirds, each of which equals a whole. Therefore, the sum is 2.

   Answer: (A)

2. The quantity 2% is equivalent to the fraction $\frac{2}{100}$, so "2% of 1" is equal to $\frac{2}{100}$.

   Answer: (A)

3. *Solution 1*
   Since $PQ = 1$ and $QR = 2PQ$, then $QR = 2$.
   Since $QR = 2$ and $RS = 3QR$, then $RS = 3(2) = 6$.
   Therefore, $PS = PQ + QR + RS = 1 + 2 + 6 = 9$.

   *Solution 2*
   From the given information,

   $$PS = PQ + QR + RS = PQ + QR + 3QR = PQ + 4QR = PQ + 4(2PQ) = 9PQ$$

   Thus, $PS = 9(1) = 9$.

   Answer: (C)

4. Substituting, $x = \frac{1}{3}(3 - 4u) = \frac{1}{3}(3 - 4(-6)) = \frac{1}{3}(3 + 24) = \frac{1}{3}(27) = 9$.

   Answer: (C)

5. *Solution 1*
   Since $2^x = 16$, then $2^{x+3} = 2^3 2^x = 8(16) = 128$.

   *Solution 2*
   Since $2^x = 16$ and $2^4 = 16$, then $x = 4$.
   Since $x = 4$, then $2^{x+3} = 2^7 = 128$.

   Answer: (D)

6. A 12 by 12 grid of squares will have 11 interior vertical lines and 11 interior horizontal lines. (In the given 4 by 4 example, there are 3 interior vertical lines and 3 interior horizontal lines.)
   Each of the 11 interior vertical lines intersects each of the 11 interior horizontal lines and creates an interior intersection point.
   Thus, each interior vertical line accounts for 11 intersection points.
   Therefore, the number of interior intersection points is $11 \times 11 = 121$.

   Answer: (B)

7. Since $PQS$ is a straight line and $\angle PQR = 110^\circ$, then $\angle RQS = 180^\circ - \angle PQR = 70^\circ$.
   Since the sum of the angles in $\triangle QRS$ is $180^\circ$, then

   $$\begin{aligned} 70^\circ + (3x)^\circ + (x + 14)^\circ &= 180^\circ \\ 70 + 3x + x + 14 &= 180 \\ 4x &= 96 \\ x &= 24 \end{aligned}$$

   Answer: (D)

8. Each of the vertical strips accounts for $\frac{1}{2}$ of the total area of the rectangle.
The left strip is divided into three equal pieces, so $\frac{2}{3}$ of the left strip is shaded, accounting for $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$ of the large rectangle.
The right strip is divided into four equal pieces, so $\frac{2}{4} = \frac{1}{2}$ of the right strip is shaded, accounting for $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ of the large rectangle.
Therefore, the total fraction of the rectangle that is shaded is $\frac{1}{3} + \frac{1}{4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12}$.
Answer: (E)

9. From the definition, $(5\nabla 1) + (4\nabla 1) = 5(5-1) + 4(4-1) = 5(4) + 4(3) = 20 + 12 = 32$.
Answer: (D)

10. Since $2x^2 = 9x - 4$, then $2x^2 - 9x + 4 = 0$.
Factoring, we obtain $(2x-1)(x-4) = 0$.
Thus, $2x = 1$ or $x = 4$.
Since $x \neq 4$, then $2x = 1$.
Answer: (B)

11. Since the coins in the bag of loonies are worth \$400, then there are 400 coins in the bag.
Since 1 loonie has the same mass as 4 dimes, then 400 loonies have the same mass as 4(400) or 1600 dimes.
Therefore, the bag of dimes contains 1600 dimes, and so the coins in this bag are worth \$160.
Answer: (C)

12. Suppose that each of the 7 people received $q$ candies under the first distribution scheme.
Then the people received a total of $7q$ candies and 3 candies were left over. Since there were $k$ candies, then $k = 7q + 3$.
Multiplying both sides by 3, we obtain $3k = 21q + 9$.
When $21q + 9$ candies were distributed to 7 people, each person could have received $3q + 1$ candies, accounting for $21q + 7$ candies in total, with 2 candies left over. (The 7 people could not each receive more than $3q+1$ candies since this would account for at least $7(3q+2) = 21q+14$ candies, which is too many in total.)
Therefore, 2 candies would be left over.
Answer: (B)

13. If 50 numbers have an average of 76, then the sum of these 50 numbers is $50(76) = 3800$.
If 40 numbers have an average of 80, then the sum of these 40 numbers is $40(80) = 3200$.
Therefore, the sum of the 10 remaining numbers is $3800 - 3200 = 600$, and so the average of the 10 remaining numbers is $\frac{600}{10} = 60$.
Answer: (A)

14. Choice (A) is not necessarily true, since the four friends could have caught 2, 3, 3, and 3 fish.
Choice (B) is not necessarily true, since the four friends could have caught 1, 1, 1, and 8 fish.
Choice (C) is not necessarily true, since the four friends could have caught 2, 3, 3, and 3 fish.
Choice (E) is not necessarily true, since the four friends could have caught 1, 1, 1, and 8 fish.
Therefore, choice (D) must be the one that must be true.
We can confirm this by noting that it is impossible for each of the four friends to have caught at least 3 fish, since this would be at least 12 fish in total and they only caught 11 fish.
Answer: (D)

15. If $-1 < \sqrt{p} - \sqrt{100} < 1$, then $-1 < \sqrt{p} - 10 < 1$ or $9 < \sqrt{p} < 11$.
Since $\sqrt{p}$ is greater than 9, then $p$ is greater than $9^2 = 81$.
Since $\sqrt{p}$ is less than 11, then $p$ is less than $11^2 = 121$.
In other words, $81 < p < 121$.
Since $p$ is a positive integer, then $82 \leq p \leq 120$.
Therefore, there are $120 - 82 + 1 = 39$ such integers $p$.

Answer: (D)

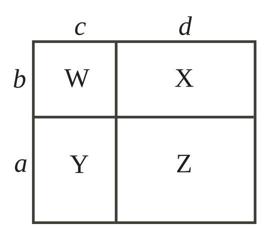
16. Note that $2010 = 10(201) = 2(5)(3)(67)$ and that 67 is prime.
Therefore, the positive divisors of 2010 are 1, 2, 3, 5, 6, 10, 15, 30, 67, 134, 201, 335, 402, 670, 1005, 2010.
Thus, the possible pairs $(a, b)$ with $ab = 2010$ and $a > b$ are $(2010, 1)$, $(1005, 2)$, $(670, 3)$, $(402, 5)$, $(335, 6)$, $(201, 10)$, $(134, 15)$, $(67, 30)$.
Of these pairs, the one with the smallest possible value of $a - b$ is $(a, b) = (67, 30)$, which gives $a - b = 37$.

Answer: (A)

17. Since $PQRS$ is a rectangle, then $PQ$ is perpendicular to $QR$.
Therefore, the area of $\triangle PQR$ is $\frac{1}{2}(PQ)(QR) = \frac{1}{2}(5)(3) = \frac{15}{2}$.
Since $PT = TU = UR$, then the areas of $\triangle PTQ$, $\triangle TUQ$ and $\triangle URQ$ are equal. (These triangles have bases $PT$, $TU$ and $UR$ of equal length and each has height equal to the distance between $Q$ and line segment $PR$.)
Therefore, the area of $\triangle TUQ$ is $\frac{1}{3}(\frac{15}{2}) = \frac{5}{2}$.
Similarly, the area of $\triangle TUS$ is $\frac{5}{2}$.
The area of quadrilateral $STQU$ is the sum of the areas of $\triangle TUQ$ and $\triangle TUS$, or $\frac{5}{2} + \frac{5}{2} = 5$.

Answer: (B)

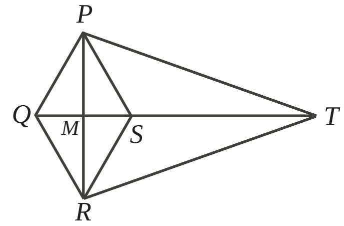
18. Label the lengths of the vertical and horizontal segments as $a, b, c, d$, as shown.




Rectangle W is $b$ by $c$, so its perimeter is $2b + 2c$, which equals 2.
Rectangle X is $b$ by $d$, so its perimeter is $2b + 2d$, which equals 3.
Rectangle Y is $a$ by $c$, so its perimeter is $2a + 2c$, which equals 5.
Rectangle Z is $a$ by $d$, so its perimeter is $2a + 2d$.
Therefore, $2a+2d = (2a+2b+2c+2d)-(2b+2c) = (2a+2c)+(2b+2d)-(2b+2c) = 5+3-2 = 6$.

Answer: (A)

19. *Solution 1*
First, we note that $\triangle PQS$ and $\triangle RQS$ are equilateral.
Join $P$ to $R$. Since $PQRS$ is a rhombus, then $PR$ and $QS$ bisect each other at their point of intersection, $M$, and are perpendicular.
Note that $QM = MS = \frac{1}{2}QS = 3$.

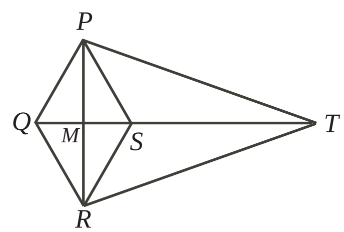



Since $\angle PSQ = 60^\circ$, then $PM = PS\sin(\angle PSM) = 6\sin(60^\circ) = 6(\frac{\sqrt{3}}{2}) = 3\sqrt{3}$.
Since $PT = TR$, then $\triangle PRT$ is isosceles.
Since $M$ is the midpoint of $PR$, then $TM$ is perpendicular to $PR$.
Since $SM$ is also perpendicular to $PR$, then $S$ lies on $TM$.
By the Pythagorean Theorem in $\triangle PMT$, since $MT > 0$, we have

$$MT = \sqrt{PT^2 - PM^2} = \sqrt{14^2 - (3\sqrt{3})^2} = \sqrt{196 - 27} = \sqrt{169} = 13$$

Therefore, $ST = MT - MS = 13 - 3 = 10$.

*Solution 2*
First, we note that $\triangle PQS$ and $\triangle RQS$ are equilateral.
Join $P$ to $R$. Since $PQRS$ is a rhombus, then $PR$ and $QS$ bisect each other at their point of intersection, $M$, and are perpendicular.
Since $PT = TR$, then $\triangle PRT$ is isosceles.
Since $M$ is the midpoint of $PR$, then $TM$ is perpendicular to $PR$.
Since $SM$ is also perpendicular to $PR$, then $S$ lies on $TM$.




Since $\angle PSQ = 60^\circ$, then $\angle PST = 180^\circ - 60^\circ = 120^\circ$.
Therefore, in $\triangle PST$, we know that $\angle PST = 120^\circ$, that $PS = 6$ and that $PT = 14$.
By the cosine law,

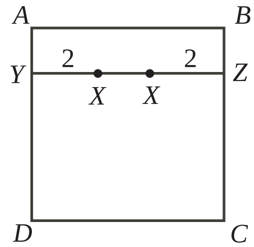
$$\begin{aligned}
PT^2 &= PS^2 + ST^2 - 2(PS)(ST)\cos(\angle PST) \\
14^2 &= 6^2 + ST^2 - 2(6)(ST)\cos(120^\circ) \\
196 &= 36 + ST^2 + 6ST \qquad (\text{since } \cos(120^\circ) = -\tfrac{1}{2}) \\
0 &= ST^2 + 6ST - 160 \\
0 &= (ST - 10)(ST + 16)
\end{aligned}$$

and so $ST = 10$ or $ST = -16$. Since $ST > 0$, then $ST = 10$.

ANSWER: (D)

20. Label the square as $ABCD$.
Suppose that the point $X$ is 1 unit from side $AB$.
Then $X$ lies on a line segment $YZ$ that is 1 unit below side $AB$.

Note that if $X$ lies on $YZ$, then it is automatically 4 units from side $DC$.
Since $X$ must be 2 units from either side $AD$ or side $BC$, then there are 2 possible locations for $X$ on this line segment:




Note that in either case, $X$ is 3 units from the fourth side, so the four distances are 1, 2, 3, 4 as required.
We can repeat the process with $X$ being 2, 3 or 4 units away from side $AB$. In each case, there will be 2 possible locations for $X$.
Overall, there are $4(2) = 8$ possible locations for $X$. These 8 locations are all different, since there are 2 different points on each of 4 parallel lines.

Answer: (D)

21. *Solution 1*
Since the problem asks us to find *the* value of $\dfrac{x-z}{y-z}$, then this value must be the same no matter what $x$, $y$ and $z$ we choose that satisfy $\dfrac{x-y}{z-y} = -10$.
Thus, if we can find numbers $x$, $y$ and $z$ that give $\dfrac{x-y}{z-y} = -10$, then these numbers must give the desired value for $\dfrac{x-z}{y-z}$.
If $x = 10$, $y = 0$ and $z = -1$, then $\dfrac{x-y}{z-y} = -10$.
In this case, $\dfrac{x-z}{y-z} = \dfrac{10-(-1)}{0-(-1)} = \dfrac{11}{1} = 11$.

*Solution 2*
Manipulating,

$$\frac{x-z}{y-z} = \frac{(x-y)+(y-z)}{y-z} = \frac{x-y}{y-z} + \frac{y-z}{y-z} = -\frac{x-y}{z-y} + 1$$

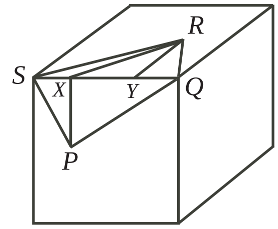
Since $\dfrac{x-y}{z-y} = -10$, then $\dfrac{x-z}{y-z} = -(-10) + 1 = 11$.

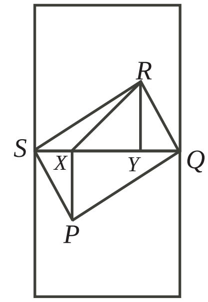
Answer: (A)

22. Since $PQRS$ is rectangular, then $\angle SRQ = \angle SPQ = 90^\circ$.
Also, $SR = PQ = 20$ and $SP = QR = 15$.
By the Pythagorean Theorem in $\triangle SPQ$, since $QS > 0$, we have

$$QS = \sqrt{SP^2 + PQ^2} = \sqrt{15^2 + 20^2} = \sqrt{225 + 400} = \sqrt{625} = 25$$

Draw perpendiculars from $P$ and $R$ to $X$ and $Y$, respectively, on $SQ$. Also, join $R$ to $X$.




We want to determine the length of $RP$.
Now, since $\triangle SPQ$ is right-angled at $P$, then

$$\sin(\angle PSQ) = \frac{PQ}{SQ} = \frac{20}{25} = \frac{4}{5} \qquad \cos(\angle PSQ) = \frac{SP}{SQ} = \frac{15}{25} = \frac{3}{5}$$

Therefore, $XP = PS\sin(\angle PSQ) = 15(\frac{4}{5}) = 12$ and $SX = PS\cos(\angle PSQ) = 15(\frac{3}{5}) = 9$.
Since $\triangle QRS$ is congruent to $\triangle SPQ$ (three equal side lengths), then $QY = SX = 9$ and $YR = XP = 12$.
Since $SQ = 25$, then $XY = SQ - SX - QY = 25 - 9 - 9 = 7$.
Consider $\triangle RYX$, which is right-angled at $Y$. By the Pythagorean Theorem,

$$RX^2 = YR^2 + XY^2 = 12^2 + 7^2 = 193$$

Next, consider $\triangle PXR$. Since $RX$ lies in the top face of the cube and $PX$ is perpendicular to this face, then $\triangle PXR$ is right-angled at $X$.
By the Pythagorean Theorem, since $PR > 0$, we have

$$PR = \sqrt{PX^2 + RX^2} = \sqrt{12^2 + 193} = \sqrt{144 + 193} = \sqrt{337} \approx 18.36$$

Of the given answers, this is closest to 18.4.

Answer: (E)

23. First, we try a few values of $n$ to see if we can find a pattern in the values of $t_n$:

| $n$ | $\sqrt{n}$ | $t_n$ | $n$ | $\sqrt{n}$ | $t_n$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 11 | 3.32 | 3 |
| 2 | 1.41 | 1 | 12 | 3.46 | 3 |
| 3 | 1.73 | 2 | 13 | 3.61 | 4 |
| 4 | 2 | 2 | 14 | 3.74 | 4 |
| 5 | 2.24 | 2 | 15 | 3.87 | 4 |
| 6 | 2.45 | 2 | 16 | 4 | 4 |
| 7 | 2.65 | 3 | 17 | 4.12 | 4 |
| 8 | 2.83 | 3 | 18 | 4.24 | 4 |
| 9 | 3 | 3 | 19 | 4.36 | 4 |
| 10 | 3.16 | 3 | 20 | 4.47 | 4 |
| | | | 21 | 4.58 | 5 |

(In each case, the $\sqrt{n}$ column contains an approximation of $\sqrt{n}$ to 2 decimal places.)
So $t_n = 1$ for 2 values of $n$, 2 for 4 values of $n$, 3 for 6 values of $n$, 4 for 8 values of $n$.
We conjecture that $t_n = k$ for $2k$ values of $n$. We will prove this fact at the end of the solution.
Next, we note that $\sqrt{2010} \approx 44.83$ and so $t_{2010} = 45$.

This means that before this point in the sequence, we have included all terms with $t_n \le 44$. According to our conjecture, the number of terms with $t_n \le 44$ should be

$$2+4+6+\cdots+86+88 = 2(1+2+3+\cdots+43+44) = 2\left(\tfrac{1}{2}(44)(45)\right) = 44(45) = 1980$$

Note that $\sqrt{1980} \approx 44.497$ and $\sqrt{1981} \approx 44.508$ so $t_{1980} = 44$ and $t_{1981} = 45$.
Since $t_{1981} = t_{2010} = 45$, then each of the terms from $t_{1981}$ to $t_{2010}$ equals 45. Therefore, there are 30 terms that equal 45.
Thus, the required sum equals

$$2\left(\tfrac{1}{1}\right)+4\left(\tfrac{1}{2}\right)+6\left(\tfrac{1}{3}\right)+\cdots+86\left(\tfrac{1}{43}\right)+88\left(\tfrac{1}{44}\right)+30\left(\tfrac{1}{45}\right) = 2+2+2+\cdots+2+2+\tfrac{2}{3}$$

where there are 44 copies of 2.
Therefore, the sum equals $88\frac{2}{3}$.

Lastly, we prove that for each positive integer $k$, there are $2k$ terms $t_n$ that equal $k$:

> In order to have $t_n = k$, we need $k - \frac{1}{2} \le \sqrt{n} < k + \frac{1}{2}$ (in other words, $\sqrt{n}$ needs to round to $k$).
> Since $n$ and $k$ are positive, then $k - \frac{1}{2} \le \sqrt{n}$ is equivalent to $(k-\frac{1}{2})^2 \le n$ and $\sqrt{n} < k + \frac{1}{2}$ is equivalent to $n < (k+\frac{1}{2})^2$.
> Therefore, we need $(k-\frac{1}{2})^2 \le n < (k+\frac{1}{2})^2$ or $k^2 - k + \frac{1}{4} \le n < k^2 + k + \frac{1}{4}$.
> Since $n$ is an integer, then $k^2 - k + 1 \le n \le k^2 + k$.
> There are thus $(k^2+k) - (k^2-k+1) + 1 = 2k$ such values of $n$, as required.

Answer: (C)

24. First, we fill in the numbers on the top four layers.
The top layer consists of only one sphere, labelled 1.
In the second layer, each sphere touches only one sphere in the layer above. This sphere is labelled 1, so each sphere in the second layer is labelled 1:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

In the third layer, each of the corner spheres touches only one sphere in the second layer and this sphere is labelled 1, so each of the corner spheres on the third layer is labelled 1. The other three spheres (the middle spheres on each edge) touch two spheres each labelled 1 in the layer above, so each is labelled 2. Therefore, the third layer is labelled

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 1 \end{array}$$

Similarly, we can complete the fourth layer as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 6 & & 3 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

We define an *external sphere* to be a sphere that is not an internal sphere. In the top four layers, only the sphere labelled 6 in the fourth layer is internal; the remaining spheres are all external. We also use the phrase "the sum of the spheres" to mean the "the sum of the numbers on the spheres".
We observe several patterns:

(i) The corner spheres in each layer are labelled 1.

(ii) The sum of the spheres along an outside edge in the first through fourth layers are 1, 2, 4, 8. It appears that the sum of the spheres along an outside edge in layer $k$ is $2^{k-1}$.

(iii) The sums of all of the spheres in the first through fourth layers are 1, 3, 9, 27. It appears that the sum of all of the spheres in layer $k$ is $3^{k-1}$.

We use these facts without proof to determine an answer, and then prove these facts.

To determine the sum of the internal spheres, we calculate the sum of all of the spheres and subtract the sum of the external spheres.
Based on fact (iii), the sum of all of the spheres in the 13 layers should be

$$3^0 + 3^1 + 3^2 + \cdots + 3^{11} + 3^{12} = \frac{1(3^{13}-1)}{3-1} = \tfrac{1}{2}(3^{13}-1)$$

(We could calculate the sum of the powers of 3 using a calculator or use the formula for the sum of a geometric series.)
To calculate the sum of all of the external spheres, we consider a fixed layer $k$, for $k \geq 2$. (The sum of the external spheres in the first layer is 1.)
The external spheres are along the three outside edges, each of which has sum $2^{k-1}$, by fact (ii). But using this argument we have included each corner sphere twice (each is included in two edges), so we must subtract 1 for each corner that is doubled. Thus, the sum of the external spheres in layer $k$ should be $3(2^{k-1}) - 3$. (We can check that this formula agrees in each of the first through fourth layers.)
Therefore, the sum of all of the external spheres should be

$$\begin{aligned} 1 + (3(2^1) - 3)) + (3(2^2) - 3) + \cdots + (3(2^{12}) - 3) &= 1 + 3(2^1 + 2^2 + \cdots + 2^{12}) - 36 \\ &= 3\left(\frac{2(2^{12}-1)}{2-1}\right) - 35 \\ &= 3(2^{13}-2) - 35 \\ &= 3(2^{13}) - 41 \end{aligned}$$

Therefore, the sum of all of the internal spheres should be

$$\tfrac{1}{2}(3^{13}-1) - 3(2^{13}) + 41 = 772\,626$$

Now we must justify the three facts above:

(i) Each corner sphere in layer $k$ touches only one sphere in layer $k-1$, which is itself a corner sphere.
Therefore, the number on a corner sphere in layer $k$ is equal to the number on the corresponding corner sphere in layer $k-1$.
Since the corner spheres are labelled 1 on each of the first four layers, then all corner spheres are labelled 1.

(ii) Consider a fixed edge of spheres in layer $k$ with $k \geq 2$, and consider as well its parallel edge in layer $k+1$.
Consider a sphere, numbered $x$, on the edge in layer $k$.
This sphere touches two edge spheres on the parallel edge in layer $k+1$.
Also, spheres from the fixed edge in layer $k+1$ do not touch spheres in layer $k$ that are not on the fixed edge.

The given sphere contributes $x$ to the sum of spheres in the fixed edge in layer $k$. It thus contributes $x$ to the number on each of the two spheres that it touches in the fixed edge in layer $k+1$.
Therefore, this sphere labelled $x$ in layer $k$ contributes $2x$ to the sum of spheres on the fixed edge in layer $k+1$.
Therefore, the sum of the spheres on the fixed edge in layer $k+1$ is two times the sum of the spheres on the corresponding edge of layer $k$.
Since the sum of the numbers on the first few layers are powers of 2, then this pattern continues by successively multiplying by 2.

(iii) Suppose a given sphere in layer $k$ is labelled $x$.
This sphere touches three spheres in layer $k+1$.
Therefore, the sphere contributes $x$ to the sum in layer $k$, and $3x$ to the sum in layer $k+1$ ($x$ to each of 3 spheres).
Therefore, the sum of the spheres in layer $k+1$ is three times the sum of the spheres in layer $k$, since each sphere from layer $k$ contributes three times in layer $k+1$.
Since the sum of the numbers on the first few layers are powers of 3, then this pattern continues by successively multiplying by 3.

Answer: (E)

25. Define $f(x)=(1-x)^a(1+x)^b(1-x+x^2)^c(1+x^2)^d(1+x+x^2)^e(1+x+x^2+x^3+x^4)^f$.
We note several algebraic identities, each of which can be checked by expanding and simplifying:

$$\begin{aligned}
1-x^5 &= (1-x)(1+x+x^2+x^3+x^4)\\
1-x^3 &= (1-x)(1+x+x^2)\\
1-x^4 &= (1-x^2)(1+x^2)=(1-x)(1+x)(1+x^2)\\
1+x^3 &= (1+x)(1-x+x^2)\\
1-x^6 &= (1-x^3)(1+x^3)=(1-x)(1+x)(1-x+x^2)(1+x+x^2)
\end{aligned}$$

This allows us to regroup the terms successively in the given expansion to create the simpler left sides in the equation above:

$$\begin{aligned}
&(1-x)^a(1+x)^b(1-x+x^2)^c(1+x^2)^d(1+x+x^2)^e(1+x+x^2+x^3+x^4)^f\\
&= (1-x)^{a-e-f}(1+x)^b(1-x+x^2)^c(1+x^2)^d(1+x+x^2)^e(1-x)^e(1+x+x^2+x^3+x^4)^f(1-x)^f\\
&= (1-x)^{a-e-f}(1+x)^b(1-x+x^2)^c(1+x^2)^d(1-x^3)^e(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1-x+x^2)^c(1+x)^c(1+x^2)^d(1-x^3)^e(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1+x^3)^c(1+x^2)^d(1-x^3)^e(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1+x^3)^c(1-x^3)^c(1+x^2)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-e-f}(1+x)^{b-c}(1-x^6)^c(1+x^2)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-e-f-d}(1+x)^{b-c-d}(1-x^6)^c(1+x^2)^d(1-x)^d(1+x)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-e-f-d}(1+x)^{b-c-d}(1-x^6)^c(1-x^4)^d(1-x^3)^{e-c}(1-x^5)^f\\
&= (1-x)^{a-d-e-f}(1+x)^{b-c-d}(1-x^3)^{e-c}(1-x^4)^d(1-x^5)^f(1-x^6)^c
\end{aligned}$$

Since $a>d+e+f$ and $e>c$ and $b>c+d$, then the exponents $a-d-e-f$ and $b-c-d$ and $e-c$ are positive integers.
Define $A=a-d-e-f$, $B=b-c-d$, $C=e-c$, $D=d$, $E=f$, and $F=c$.
We want the expansion of

$$f(x)=(1-x)^A(1+x)^B(1-x^3)^C(1-x^4)^D(1-x^5)^E(1-x^6)^F$$

to have only terms $1-2x$ when all terms involving $x^7$ or larger are removed.
We use the facts that

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{2}y^2 + \cdots \qquad (*)$$

and

$$(1-y)^n = 1 - ny + \frac{n(n-1)}{2}y^2 + \cdots \qquad (**)$$

which can be derived by multiplying out directly or by using the Binomial Theorem.

Since each factor contains a constant term of 1, then $f(x)$ will have a constant term of 1, regardless of the values of $A$, $B$, $C$, $D$, $E$, $F$.
Consider the first two factors. Only these factors can affect the coefficients of $x$ and $x^2$ in the final product. Any terms involving $x$ and $x^2$ in the final product will come from these factors multiplied by the constant 1 from each of the other factors.
We consider the product of the first two factors ignoring any terms of degree three or higher:

$$\begin{aligned}(1-x)^A(1+x)^B &= (1 - Ax + \frac{A(A-1)}{2}x^2 - \cdots)(1 + Bx + \frac{B(B-1)}{2}x^2 + \cdots) \\ &= 1 - Ax + \frac{A(A-1)}{2}x^2 + Bx - ABx^2 + \frac{B(B-1)}{2}x^2 + \cdots \\ &= 1 - (A-B)x + \left[\frac{A(A-1)}{2} + \frac{B(B-1)}{2} - AB\right]x^2 + \cdots\end{aligned}$$

These will be the terms involving 1, $x$ and $x^2$ in the final expansion of $f(x)$.
Since $f(x)$ has a term $-2x$ and no $x^2$ term, then $A-B=2$ and $\frac{A(A-1)}{2} + \frac{B(B-1)}{2} - AB = 0$.
The second equation becomes $A^2 - A + B^2 - B - 2AB = 0$ or $(A-B)^2 = A+B$.
Since $A-B=2$, then $A+B=4$, whence $2A = (A+B)+(A-B) = 6$, so $A=3$ and $B=1$.
Thus, the first two factors are $(1-x)^3(1+x)$.
Note that $(1-x)^3(1+x) = (1-3x+3x^2-x^3)(1+x) = 1-2x+2x^3-x^4$.
Therefore, $f(x) = (1-2x+2x^3-x^4)(1-x^3)^C(1-x^4)^D(1-x^5)^E(1-x^6)^F$.
The final result contains no $x^3$ term. Since the first factor contains a term "$+2x^3$" which will appear in the final product by multiplying by all of the constant terms in subsequent factors, then this "$+2x^3$" must be balanced by a "$-2x^3$". The only other factor possibly containing an $x^3$ is $(1-x^3)^C$. To balance the "$+2x^3$" term, the expansion of $(1-x^3)^C$ must include a term "$-2x^3$" which will be multiplied by the constant terms in the other factors to provide a "$-2x^3$" in the final expansion, balancing the "$+2x^3$".
For $(1-x^3)^C$ to include $-2x^3$, we must have $C=2$, from $(*)$.
Therefore,

$$\begin{aligned}f(x) &= (1-2x+2x^3-x^4)(1-x^3)^2(1-x^4)^D(1-x^5)^E(1-x^6)^F \\ &= (1-2x+2x^3-x^4)(1-2x^3+x^6)(1-x^4)^D(1-x^5)^E(1-x^6)^F \\ &= (1-2x+3x^4-3x^6+\cdots)(1-x^4)^D(1-x^5)^E(1-x^6)^F\end{aligned}$$

When we simplify at this stage, we can ignore any terms with exponent greater than 6, since we do not care about these terms and they do not affect terms with smaller exponents when we multiply out.
To balance the "$+3x^4$", the factor $(1-x^4)^D$ needs to include "$-3x^4$" and so $D=3$.

Therefore,

$$\begin{aligned} f(x) &= (1 - 2x + 3x^4 - 3x^6 + \cdots)(1 - x^4)^3(1 - x^5)^E(1 - x^6)^F \\ &= (1 - 2x + 3x^4 - 3x^6 + \cdots)(1 - 3x^4 + \cdots)(1 - x^5)^E(1 - x^6)^F \\ &= (1 - 2x + 6x^5 - 3x^6 + \cdots)(1 - x^5)^E(1 - x^6)^F \end{aligned}$$

To balance the "$+6x^5$", the factor $(1 - x^5)^E$ needs to include "$-6x^5$" and so $E = 6$.
Therefore,

$$\begin{aligned} f(x) &= (1 - 2x + 6x^5 - 3x^6 + \cdots)(1 - x^5)^6(1 - x^6)^F \\ &= (1 - 2x + 6x^5 - 3x^6 + \cdots)(1 - 6x^5 + \cdots)(1 - x^6)^F \\ &= (1 - 2x + 9x^6 + \cdots)(1 - x^6)^F \end{aligned}$$

To balance the "$+9x^6$", the factor $(1 - x^6)^F$ needs to include "$-9x^6$" and so $F = 9$.

We now know that $A = 3$, $B = 1$, $C = 2$, $D = 3$, $E = 6$, and $F = 9$.
Since $D = d$, $E = f$, and $F = c$, then $c = 9$, $f = 6$, and $d = 3$.
Since $C = e - c$, $C = 2$ and $c = 9$, then $e = 11$.
Since $A = a - d - e - f$, $d = 3$, $e = 11$, $f = 6$, and $A = 3$, then $a = 3 + 3 + 11 + 6$, or $a = 23$.

Answer: (E)

Canadian
Mathematics
Competition

*An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario*

# *2009 Fermat Contest*

**(Grade 11)**

**Wednesday, February 18, 2009**

*Solutions*

1. Calculating, $3 + 3^3 = 3 + 27 = 30$.

   Answer: (D)

2. Since $3 \times 2 + 8 = \nabla + 5$, then $\nabla = 6 + 8 - 5 = 9$.

   Answer: (E)

3. Since $\angle TQR = 125^\circ$, then $\angle TQP = 180^\circ - \angle TQR = 180^\circ - 125^\circ = 55^\circ$.
   Since the sum of the angles in a triangle is $180^\circ$, then

   $$\angle PSQ = 180^\circ - \angle SPQ - \angle SQP = 180^\circ - 30^\circ - 55^\circ = 95^\circ$$

   Since $\angle TSU$ and $\angle PSQ$ are opposite, then $\angle TSU = \angle PSQ = 95^\circ$, so $x = 95$.

   Answer: (C)

4. Since $w = 4$, $x = 9$ and $z = 25$, then

   $$\sqrt{\frac{w}{x}} + \sqrt{\frac{x}{z}} = \sqrt{\frac{2^2}{3^2}} + \sqrt{\frac{3^2}{5^2}} = \frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$$

   Answer: (B)

5. Since $a^{-1} = \dfrac{1}{a}$, then $1 - 4(3-1)^{-1} = 1 - 4(2^{-1}) = 1 - \frac{4}{2} = 1 - 2 = -1$.

   Answer: (A)

6. There are 64 cubes to start.
   If we look at the bottom layer of cubes, we see that there are 6 uncovered cubes, each of which is missing 3 cubes above it. These are the only cubes that are missing.
   Thus, there are $6(3) = 18$ missing cubes, so there are $64 - 18 = 46$ cubes remaining.

   Answer: (A)

7. *Solution 1*
   Since $\sqrt{n^2 + n^2 + n^2 + n^2} = 64$, then $\sqrt{4n^2} = 64$ or $2n = 64$, since $n > 0$.
   Thus, $n = 32$.

   *Solution 2*
   Since $\sqrt{n^2 + n^2 + n^2 + n^2} = 64$, then $\sqrt{4n^2} = 64$.
   Thus, $4n^2 = 64^2 = 4096$, and so $n^2 = 1024$.
   Since $n > 0$, then $n = \sqrt{1024} = 32$.

   Answer: (D)

8. To maximize the number of songs used, Gavin should use as many of the shortest length songs as possible. (This is because he can always trade a longer song for a shorter song and shorten the total length used.)
   If Gavin uses all 50 songs of 3 minutes in length, this takes 150 minutes.
   There are $180 - 150 = 30$ minutes left, so he can play an additional $30 \div 5 = 6$ songs that are 5 minutes in length.
   In total, he plays $50 + 6 = 56$ songs.

   Answer: (C)

9. Since there are 4 ♠'s in each of the first three columns, then at least 1 ♠ must be moved out of each of these columns to make sure that each column contains exactly three ♠'s.
Therefore, we need to move at least 3 ♠'s in total.
If we move the ♠ from the top left corner to the bottom right corner

| | | | | |
|---|---|---|---|---|
| | ♠ | ♠ | ♠ | |
| ♠ | ♠ | ♠ | | ♠ |
| ♠ | ♠ | | | |
| ♠ | ♠ | ♠ | ♠ | |
| | | ♠ | | ♠ |

and the ♠ from the fourth row, third column to the fifth row, fourth column

| | | | | |
|---|---|---|---|---|
| | ♠ | ♠ | ♠ | |
| ♠ | ♠ | ♠ | | ♠ |
| ♠ | ♠ | | | |
| ♠ | ♠ | | ♠ | |
| | | ♠ | ♠ | ♠ |

and the ♠ from the second row, second column to the third row, fifth column

| | | | | |
|---|---|---|---|---|
| | ♠ | ♠ | ♠ | |
| ♠ | | ♠ | | ♠ |
| ♠ | ♠ | | | ♠ |
| ♠ | ♠ | | ♠ | |
| | | ♠ | ♠ | ♠ |

then we have exactly three ♠'s in each row and each column.
Therefore, since we must move at least 3 ♠'s and we can achieve the configuration that we want by moving 3 ♠'s, then 3 is the smallest number.
(There are also other combinations of moves that will give the required result.)

ANSWER: (C)

10. Initially, the 25 m ladder has its bottom 7 m from the wall.
Suppose that its top is initially $h$ m above the ground.
By the Pythagorean Theorem, $h^2 + 7^2 = 25^2$ or $h^2 + 49 = 625$.
Thus, $h^2 = 625 - 49 = 576$, so $h = \sqrt{576} = 24$, since $h > 0$.
When the top slides 4 m down the wall, the top is now $24 - 4 = 20$ m above the ground.
Suppose that the bottom ends up $d$ m from the wall.
Again by the Pythagorean Theorem, $20^2 + d^2 = 25^2$, so $d^2 = 25^2 - 20^2 = 625 - 400 = 225$.
Since $d > 0$, then $d = 15$.
This means that the bottom is $15 - 7 = 8$ m farther from the wall than in its original position.

ANSWER: (E)

11. Since the result must work no matter what positive integers $m$ and $n$ we choose with $m < n$, we try $m = 1$ and $n = 2$.
In this case, $\frac{m}{n} = \frac{1}{2}$ and $\frac{m+3}{n+3} = \frac{4}{5}$.
Here, $\frac{m+3}{n+3} > \frac{m}{n}$, so the answer must be (D).

(We could also prove this algebraically by starting with $m < n$, which gives $3m < 3n$, which gives $mn + 3m < mn + 3n$, which gives $m(n + 3) < n(m + 3)$ which gives $\frac{m}{n} < \frac{m+3}{n+3}$.)

ANSWER: (D)

12. Between 5000 and 6000, every integer except for 6000 has thousands digit equal to 5.
Note that the number 6000 does not have the desired property.
Thus, we are looking for integers $5xyz$ with $x + y + z = 5$.
The possible combinations of three digits for $x$, $y$ and $z$ are: 5, 0, 0; 4, 1, 0; 3, 2, 0; 3, 1, 1; 2, 2, 1.
A combination of three different digits (like 4, 1, 0) can be arranged in 6 ways: 410, 401, 140, 104, 041, 014.
A combination of three digits with one repeated (like 5, 0, 0) can be arranged in 3 ways: 500, 050, 005.
Therefore, 5, 0, 0 and 3, 1, 1 and 2, 2, 1 each give 3 integers, and 4, 1, 0 and 3, 2, 0 each give 6 integers.
So the number of integers with the desired property is $3(3) + 2(6) = 21$.

ANSWER: (C)

13. Since $x$ is an integer, then $x + 1$ is an integer.
Since $\frac{-6}{x+1}$ is to be integer, then $x + 1$ must be a divisor of $-6$.
Thus, there are 8 possible values for $x + 1$, namely $-6$, $-3$, $-2$, $-1$, 1, 2, 3, and 6.
This gives 8 possible values for $x$, namely $-7$, $-4$, $-3$, $-2$, 0, 1, 2, and 5.

ANSWER: (A)

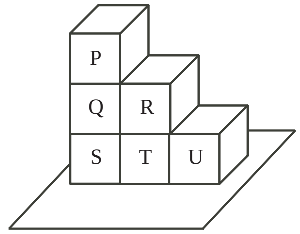
14. Since the three numbers in each straight line must have a product of 3240 and must include 45, then the other two numbers in each line must have a product of $\frac{3240}{45} = 72$.
The possible pairs of positive integers are 1 and 72, 2 and 36, 3 and 24, 4 and 18, 6 and 12, and 8 and 9.
The sums of the numbers in these pairs are 73, 38, 27, 22, 18, and 17.
To maximize the sum of the eight numbers, we want to choose the pairs with the largest possible sums, so we choose the first four pairs.
Thus, the largest possible sum of the eight numbers is $73 + 38 + 27 + 22 = 160$.

ANSWER: (E)

15. Suppose that there are 1000 students at Dunkley S.S.
On Monday, there were thus 100 students absent and 900 students present.
On Tuesday, 10% of the 900 students who were present on Monday, or $0.1(900) = 90$ students, were absent. The remaining $900 - 90 = 810$ students who were present on Monday were still present on Tuesday.
Similarly, 10% of the 100 students who were absent on Monday, or $0.1(100) = 10$ students, were present on Tuesday. The remaining $100 - 10 = 90$ students who were absent on Monday were still absent on Tuesday.
Thus, there were $810 + 10 = 820$ students present on Tuesday, or $\frac{820}{1000} \times 100\% = 82\%$ of the whole student population.

ANSWER: (B)

16. Label the six dice as shown:




The maximum overall exposed sum occurs when the sum of the exposed faces on each die is maximized.
Die P has 5 exposed faces. The sum of these faces is a maximum when the 1 is hidden, so the maximum exposed sum on die P is $2 + 3 + 4 + 5 + 6 = 20$.
Dice Q and S each have 3 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 as the unpaired exposed face. (This is on the left face of the stack.) Each of these dice has a maximum exposed sum of $6 + 7 = 13$.
Dice R and U each have 4 exposed faces. Two of these are opposite to each other, so have a sum of 7. Thus, to maximize the exposed sum on these dice, we position them with the 6 and the 5 as the unpaired exposed faces (on the top and right of the stack). Each of these dice have a maximum exposed sum of $5 + 6 + 7 = 18$.
Die T has 2 exposed faces, which are opposite each other, so have a sum of 7.
Therefore, the maximum possible sum of the exposed faces is $20 + 13 + 13 + 18 + 18 + 7 = 89$.

Answer: (C)

17. Suppose that the radius of the region is $r$.
The length of the semi-circle is half of the circumference, or $\frac{1}{2}(2\pi r) = \pi r$.
Thus, the perimeter of the shaded region is $\pi r + 2r$.
Since the perimeter is 20, then $\pi r + 2r = 20$ or $r(\pi + 2) = 20$ or $r = \dfrac{20}{\pi + 2}$.
Thus, the area of the semi-circle is half of the area of a circle with this radius, or $\frac{1}{2}\pi \left(\dfrac{20}{\pi + 2}\right)^2$ which is approximately 23.768, which of the given choices is closest to 23.8.

Answer: (B)

18. Let the length of his route be $d$ km.
Since he arrives 1 minute early when travelling at 75 km/h and 1 minute late when travelling at 70 km/h, then the difference between these times is 2 minutes, or $\frac{1}{30}$ of an hour.
The time that his trip takes while travelling at 75 km/h is $\dfrac{d}{75}$ hours, and at 70 km/h is $\dfrac{d}{70}$ hours. Therefore,

$$\begin{aligned} \frac{d}{70} - \frac{d}{75} &= \frac{1}{30} \\ 75d - 70d &= \frac{75(70)}{30} \\ 5d &= 25(7) \\ d &= 35 \end{aligned}$$

Therefore, the route is 35 km long.

Answer: (B)

19. Since $2^x = 15$ and $15^y = 32$, then $(2^x)^y = 32$ or $2^{xy} = 32$.
Since $2^5 = 32$, then $xy = 5$.

Answer: (A)

20. Since the circle has radius 1, then its area is $\pi(1^2) = \pi$.
Since the square and the circle have the same area, then the side length of the square is $\sqrt{\pi}$.
Let $M$ be the midpoint of line segment $PQ$.
Since $PQ$ is a chord of the circle, then $OM$ is perpendicular to line segment $PQ$.
Since $OM$ is perpendicular to $PQ$ and $O$ is the centre of the square, then $OM$ is half of the length of one of the sides of the square, so $OM = \frac{1}{2}\sqrt{\pi}$.
By the Pythagorean Theorem in $\triangle OPM$, we have $PM^2 = OP^2 - OM^2 = 1^2 - \left(\frac{1}{2}\sqrt{\pi}\right)^2 = 1 - \frac{1}{4}\pi$.
Therefore, $PQ = 2PM = 2\sqrt{1 - \frac{1}{4}\pi} = \sqrt{4(1 - \frac{1}{4}\pi)} = \sqrt{4 - \pi}$.

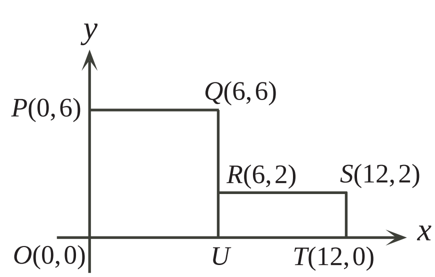
Answer: (A)

21. *Solution 1*
Since we want the maximum number of people, we start with the largest of the given choices and see if we can make it work.
Is it possible that 80 people could have eaten both ice cream and cake?
If so, there would be at least 80 people who ate cake and at least $\frac{3}{2}(80) = 120$ people who ate ice cream. (Note that there can be overlap between these two sets of people.)
Is this possible?
It is if we say that exactly 120 people ate ice cream and exactly 80 people ate cake, with all of these 80 eating ice cream as well.
So an overlap of 80 is possible, which means that the answer is (D).

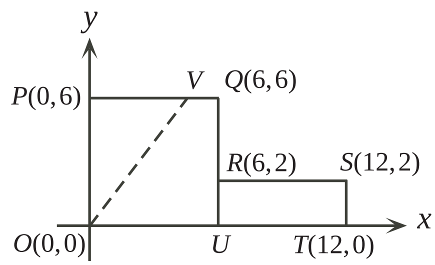
*Solution 2*
Suppose that the there were $x$ people who ate cake only, $y$ people who ate ice cream only, $b$ people who ate both cake and ice cream, and $n$ people who ate neither.
We know that $x + y + b + n = 120$ and so $x + y + b = 120 - n$.
We also know that the total number who ate cake was $x + b$ and the total number who ate ice cream was $y + b$, so $\dfrac{y+b}{x+b} = \dfrac{3}{2}$ or $2(y + b) = 3(x + b)$ or $2y = 3x + b$ or $y = \frac{3}{2}x + \frac{1}{2}b$.
Therefore, $\frac{3}{2}x + \frac{1}{2}b + x + b = 120 - n$ and so $\frac{5}{2}x + \frac{3}{2}b = 120 - n$.
Multiplying both sides by 2, we obtain $5x + 3b = 240 - 2n$.
Since $x$, $b$ and $n$ are non-negative, then the left side is at most 240 and so $b$ can be at most $\frac{1}{3}(240) = 80$, if $y = n = 0$.
We saw in Solution 1 that $b = 80$ is possible.

Answer: (D)

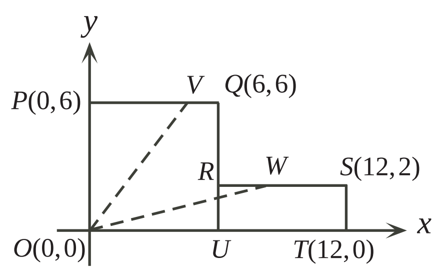
22. Extend $QR$ downwards to meet the $x$-axis at $U(6, 0)$.



The area of figure $OPQRST$ equals the sum of the areas of square $OPQU$ (which has side length 6, so area 36) and rectangle $RSTU$ (which has height 2 and width 6, so area 12).
Thus, the area of figure $OPQRST$ is 48. If we are to divide the figure into three pieces of equal area, then each piece has area 16.
Let $V$ be the first point on the perimeter (measured clockwise from $P$) so that the line through $O$ and $V$ cuts off an area of 16.
Note that the area of $\triangle OPQ$ is half of the area of square $OPQU$, or 18, so $V$ is to the left of $Q$ on $PQ$.
Thus, $V$ has coordinates $(v, 6)$ for some number $v$.




Consider $\triangle OPV$ having base $OP$ of length 6 and height $PV$ of length $v$.
Since the area of $\triangle OPV$ is 16, then $\frac{1}{2}(6)(v) = 16$ or $3v = 16$ or $v = \frac{16}{3}$.
Therefore, the slope of $OV$ is $\dfrac{6}{\frac{16}{3}} = \dfrac{9}{8}$.
Let $W$ be the second desired point.
Since the area of $\triangle OTS$ is $\frac{1}{2}(OT)(TS) = \frac{1}{2}(12)(2) = 12$ (less than $\frac{1}{3}$ of the total area) and the area of trapezoid $ORST$ is $\frac{1}{2}(RS + OT)(ST) = \frac{1}{2}(6 + 12)(2) = 18$ (more than $\frac{1}{3}$ of the total area), then $W$ lies on $RS$.




Suppose that $W$ has coordinates $(w, 2)$ for some number $w$.
We want the area of trapezoid $WSTO$ to be 16.
Therefore,

$$\begin{aligned}
\tfrac{1}{2}(WS + OT)(ST) &= 16 \\
\tfrac{1}{2}(12 - w + 12)(2) &= 16 \\
24 - w &= 16 \\
w &= 8
\end{aligned}$$

Thus, the coordinates of $W$ are $(8, 2)$, and so the slope of $OW$ is $\frac{2}{8} = \frac{1}{4}$.
Thus, the sum of the two required slopes is $\frac{9}{8} + \frac{1}{4} = \frac{11}{8}$.

Answer: (E)

23. Adding the second and third equations, we obtain

$$\begin{aligned} ac + bd + ad + bc &= 77 \\ ac + ad + bc + bd &= 77 \\ a(c + d) + b(c + d) &= 77 \\ (a + b)(c + d) &= 77 \end{aligned}$$

Since each of $a$, $b$, $c$ and $d$ is a positive integer, then $a + b$ and $c + d$ are each positive integers and are each at least 2.
Since the product of $a + b$ and $c + d$ is $77 = 7 \times 11$ (with 7 and 11 both prime), then one must equal 7 and the other must equal 11.
Therefore, $a + b + c + d = 7 + 11 = 18$.
(We can check with some work that $(a, b, c, d) = (5, 2, 4, 7)$ is a solution to the system.)

Answer: (D)

24. The three machines operate in a way such that if the two numbers in the output have a common factor larger than 1, then the two numbers in the input would have to have a common factor larger than 1.
To see this, let us look at each machine separately. We use the fact that if two numbers are each multiples of $d$, then their sum and difference are also multiples of $d$.
Suppose that $(m, n)$ is input into Machine A. The output is $(n, m)$. If $n$ and $m$ have a common factor larger than 1, then $m$ and $n$ do as well.
Suppose that $(m, n)$ is input into Machine B. The output is $(m + 3n, n)$. If $m + 3n$ and $n$ have a common factor $d$, then $(m + 3n) - n - n - n = m$ has a factor of $d$ as each part of the subtraction is a multiple of $d$. Therefore, $m$ and $n$ have a common factor of $d$.
Suppose that $(m, n)$ is input into Machine C. The output is $(m - 2n, n)$. If $m - 2n$ and $n$ have a common factor $d$, then $(m - 2n) + n + n = m$ has a factor of $d$ as each part of the addition is a multiple of $d$. Therefore, $m$ and $n$ have a common factor of $d$.
In each case, any common factor that exists in the output is present in the input.

Let us look at the numbers in the five candidates.
After some work, we can find the prime factorizations of the six integers:

$$\begin{aligned} 2009 &= 7(287) = 7(7)(41) \\ 1016 &= 8(127) = 2(2)(2)(127) \\ 1004 &= 4(251) = 2(2)(251) \\ 1002 &= 2(501) = 2(3)(167) \\ 1008 &= 8(126) = 8(3)(42) = 16(3)(3)(7) = 2(2)(2)(2)(3)(3)(7) \\ 1032 &= 8(129) = 8(3)(43) = 2(2)(2)(3)(43) \end{aligned}$$

Therefore, the only one of 1002, 1004, 1008, 1016, 1032 that has a common factor larger than 1 with 2009 is 1008, which has a common factor of 7 with 2009.
How does this help? Since 2009 and 1008 have a common factor of 7, then whatever pair was input to produce $(2009, 1008)$ must have also had a common factor of 7. Also, the pair that was input to create this pair also had a common factor of 7. This can be traced back through every step to say that the initial pair that produces the eventual output of $(2009, 1008)$ must have a common factor of 7.
Thus, $(2009, 1008)$ cannot have come from $(0, 1)$.

Notes:

- This does not tell us that the other four pairs necessarily work. It does tell us, though, that $(2009, 1008)$ cannot work.
- We can trace the other four outputs back to $(0, 1)$ with some effort. (This process is easier to do than it is to describe!)
  To do this, we notice that if the output of Machine A was $(a, b)$, then its input was $(b, a)$, since Machine A switches the two entries.
  Also, if the output of Machine B was $(a, b)$, then its input was $(a - 3b, b)$, since Machine B adds three times the second number to the first.
  Lastly, if the output of Machine C was $(a, b)$, then its input was $(a + 2b, b)$, since Machine C subtracts two times the second number from the first.
  Consider $(2009, 1016)$ for example. We try to find a way from $(2009, 1016)$ back to $(0, 1)$. We only need to find one way that works, rather than looking for a specific way.

  We note before doing this that starting with an input of $(m, n)$ and then applying Machine B then Machine C gives an output of $((m + 3n) - 2n, n) = (m + n, n)$. Thus, if applying Machine B then Machine C (we call this combination "Machine BC") gives an output of $(a, b)$, then its input must have been $(a - b, b)$. We can use this combined machine to try to work backwards and arrive at $(0, 1)$. This will simplify the process and help us avoid negative numbers.
  We do this by making a chart and by attempting to make the larger number smaller wherever possible:

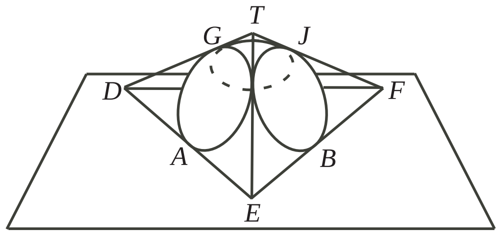
| Output | Machine | Input |
|---|---|---|
| $(2009, 1016)$ | BC | $(993, 1016)$ |
| $(993, 1016)$ | A | $(1016, 993)$ |
| $(1016, 993)$ | BC | $(23, 993)$ |
| $(23, 993)$ | A | $(993, 23)$ |
| $(993, 23)$ | BC, 43 times | $(4, 23)$ |
| $(4, 23)$ | A | $(23, 4)$ |
| $(23, 4)$ | BC, 5 times | $(3, 4)$ |
| $(3, 4)$ | A | $(4, 3)$ |
| $(4, 3)$ | BC | $(1, 3)$ |
| $(1, 3)$ | A | $(3, 1)$ |
| $(3, 1)$ | B | $(0, 1)$ |

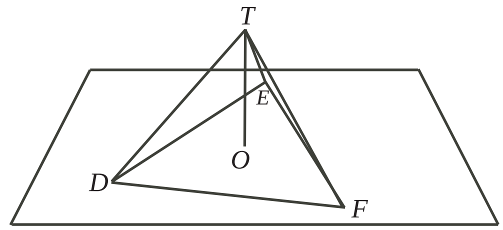
  Therefore, by going up through this table, we can see a way to get from an initial input of $(0, 1)$ to a final output of $(2009, 1016)$.
  In a similar way, we can show that we can obtain final outputs of each of $(2009, 1004)$, $(2009, 1002)$, and $(2009, 1032)$.

ANSWER: (D)

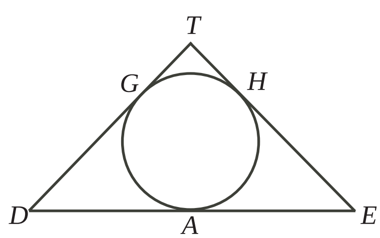
25. Let the three points at which the circles are tangent to the plane be $A$, $B$ and $C$.
Each of the three circles is contained in a plane. This plane will intersect the original plane along a line that passes through $A$, $B$ or $C$, and is tangent to the circle at one of these points.
Suppose that these three lines intersect at points $D$, $E$ and $F$, with $A$ on $DE$, $B$ on $EF$, and $C$ on $FD$.
By symmetry, $DE = EF = FD$. Also, $A$, $B$ and $C$ are the midpoints of these three segments.

(More formally, the configuration is not changed by rotating it through 120° or by reflecting it horizontally through the three vertical planes that pass through $T$ (defined below) and the vertices of $\triangle DEF$, so these facts are true.)
Thus, $\triangle DEF$ is equilateral. Let $O$ be its incentre. That is, $O$ is the point of intersection of the three angle bisectors of $\triangle DEF$. Since $\triangle DEF$ is equilateral, then its angle bisectors, medians and altitudes are all the same.
Let the points where the three circles touch be $G$, $H$ and $J$, with $G$ the point of contact between the circles containing $C$ and $A$, $H$ the point of contact between the circles containing $A$ and $B$, and $J$ the point of contact between the circles containing $B$ and $C$. Note that $GH = HJ = JG$ by symmetry. We want to determine the radius of the circle that passes through $G$, $H$ and $J$.
Draw the lines through $D$, $E$ and $F$ passing through $G$, $H$ and $J$, respectively. These lines will be tangent to the circles at points $G$, $H$ and $J$, and will meet at a common point, which we call $T$. $T$ is directly above point $O$, the centre of $\triangle DEF$. (Each of these facts is again true by the symmetry of rotation and reflection.) Now, $TDEF$ is a tetrahedron with its base ($\triangle DEF$) equilateral, and three congruent side faces.



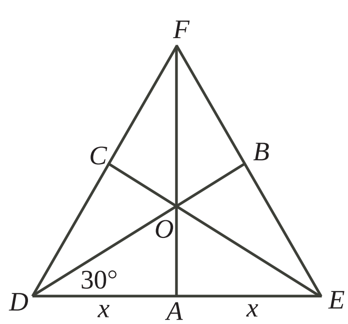




Consider the side face $\triangle DET$. The circle (of radius 10) containing $A$ is tangent to the three sides of $\triangle DET$ at $A$, $G$ and $H$.




Also, $\triangle DET$ is inclined at 45° to the horizontal.
Since $A$ is the midpoint of $DE$, then $OA$ is perpendicular to $DE$, as is $TA$.
Since $\triangle DET$ is inclined at 45° to the horizontal, then $\angle TAO = 45°$.
Note that $\triangle TAO$ is right-angled at $O$ (because $T$ is directly above $O$), so the fact that $\angle TAO = 45°$ tells us that $\triangle TAO$ is a 45°-45°-90° triangle, and so $TA = \sqrt{2}AO$.

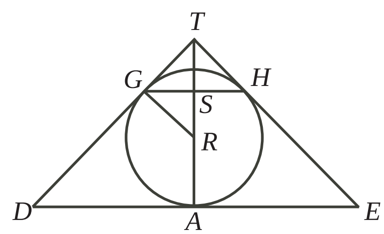
Next, we look at the dimensions of $\triangle DEF$.
Suppose that $DE = 2x$. Thus, $DA = AE = x$ and $\triangle DEF$ is equilateral with side length $2x$.



Now $DO$ is the angle bisector of $\angle EDF$, so $\angle ODA = 30^\circ$, and $OA$ is perpendicular to $DE$, so $\triangle DOA$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle. Thus, $OA = \frac{1}{\sqrt{3}}DA = \frac{1}{\sqrt{3}}x$, and $OD = \frac{2}{\sqrt{3}}DA = \frac{2}{\sqrt{3}}x$.
Since $TA = \sqrt{2}AO$, then $TA = \frac{\sqrt{2}}{\sqrt{3}}x$.
In $\triangle TAD$, the Pythagorean Theorem gives $TD = \sqrt{TA^2 + DA^2} = \sqrt{x^2 + \frac{2}{3}x^2} = \frac{\sqrt{5}}{\sqrt{3}}x$, since $x > 0$.

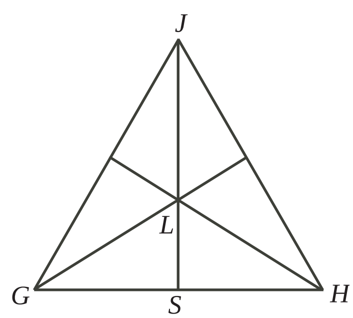
So, to summarize so far, $\triangle DET$ has $DE = 2x$, $DA = AE = x$, $TD = \frac{\sqrt{5}}{\sqrt{3}}x$ and $TA = \frac{\sqrt{2}}{\sqrt{3}}x$.

Let the centre of the circle contained in $\triangle DET$ be $R$ (which will lie on $TA$ by symmetry) and join $R$ to $G$. Since $RG$ is a radius of the circle, then $RG = 10$ and $RG$ is perpendicular to $DT$.
Also, join $GH$ and let the point of intersection of $GH$ with $TA$ be $S$. By symmetry, $GH$ is perpendicular to $TA$.




Now $\triangle TSG$, $\triangle TGR$ and $\triangle TAD$ are all similar, since each is right-angled and shares a common angle at $T$.
We want to determine the length of $SG$.
By similar triangles, $\dfrac{SG}{TG} = \dfrac{AD}{TD} = \dfrac{x}{\frac{\sqrt{5}}{\sqrt{3}}x}$, so $SG = \frac{\sqrt{3}}{\sqrt{5}}TG$.
Also by similar triangles, $\dfrac{TG}{GR} = \dfrac{TA}{AD} = \dfrac{\frac{\sqrt{2}}{\sqrt{3}}x}{x}$, so $TG = \frac{\sqrt{2}}{\sqrt{3}}GR = 10\frac{\sqrt{2}}{\sqrt{3}}$.
Thus, $SG = \frac{\sqrt{3}}{\sqrt{5}}\left(10\frac{\sqrt{2}}{\sqrt{3}}\right) = 10\frac{\sqrt{2}}{\sqrt{5}} = 2\sqrt{10}$.

Finally, consider $\triangle GHJ$. This triangle is equilateral. $S$ is the midpoint of $GH$ and $SG = 2\sqrt{10}$. Let $L$ be the incentre of this triangle.




Note that $L$ is also the centre of the circle that passes through $G$, $H$ and $J$ because $LG = LJ = LH$, so $LG$ is the radius of this circle.

Since above we saw that $OD = \frac{2}{\sqrt{3}}AD$, then $LG = \frac{2}{\sqrt{3}}SG$, since the configuration is the same in $\triangle DEF$ and $\triangle GHJ$. Thus, $LG = \frac{4\sqrt{10}}{\sqrt{3}} = \frac{4}{3}\sqrt{30}$, and this is the radius of the circle.
Since $\frac{4}{3}\sqrt{30} \approx 7.303$, the answer is closest to 7.3, of the given choices.

Answer: (C)

**Canadian Mathematics Competition**

*An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario*

# *2008 Fermat Contest*

**(Grade 11)**

**Tuesday, February 19, 2008**

*Solutions*

1. Calculating, $\dfrac{1^2+2^2+3^2+4^2}{1\times 2\times 3} = \dfrac{1+4+9+16}{6} = \dfrac{30}{6} = 5$.

   Answer: (D)

2. *Solution 1*
   Calculating, $6\left(\frac{3}{2}+\frac{2}{3}\right) = 6\left(\frac{3}{2}\right) + 6\left(\frac{2}{3}\right) = 9+4 = 13$.

   *Solution 2*
   Simplifying first insides the brackets, $6\left(\frac{3}{2}+\frac{2}{3}\right) = 6\left(\frac{9}{6}+\frac{4}{6}\right) = 6\left(\frac{13}{6}\right) = 13$.

   Answer: (A)

3. Since $1+2+3+4+5+x = 21+22+23+24+25$, then

   $$x = 21-1+22-2+23-3+24-4+25-5 = 5(20) = 100$$

   Answer: (C)

4. Since an empty truck weighs 9600 kg and when the 40 crates are added the weight is 38 000 kg, then the total weight of the crates is $38\,000 - 9600 = 28\,400$ kg.
   Since there are 40 identical crates that weigh 28 400 kg, then the weight of each crate is $28\,400 \div 40 = 710$ kg.

   Answer: (E)

5. Since $\dfrac{18}{\sqrt{x}} = 2$, then $\sqrt{x} = 9$, because the number by which we must divide 18 to get 2 is 9.
   Since $\sqrt{x} = 9$, then $x = 9^2 = 81$.

   Answer: (A)

6. Since $RQ = RS$, then $\angle RSQ = \angle RQS$.
   In $\triangle QRS$, we have $\angle RQS + \angle QRS + \angle RSQ = 180^\circ$, so $2(\angle RQS) + 60^\circ = 180^\circ$.
   Thus, $\angle RQS = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
   Since $PQ = PS$, then $\angle PSQ = \angle PQS$.
   In $\triangle QPS$, we have $\angle PQS + \angle QPS + \angle PSQ = 180^\circ$, so $2(\angle PQS) + 30^\circ = 180^\circ$.
   Thus, $\angle PQS = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.
   Therefore, $\angle PQR = \angle PQS - \angle RQS = 75^\circ - 60^\circ = 15^\circ$.

   Answer: (E)

7. *Solution 1*
   Since $p$ is odd and $q$ is even, then $3p$ is odd times odd (so is odd) and $2q$ is even times even (so is even).
   Therefore, $3p + 2q$ is odd plus even, which is odd.
   (Since we have found one possibility that is odd, we do not need to look at the others. We could check, though, that each of the others is always even.)

   *Solution 2*
   We check for a particular case of $p$ and $q$, since the problem implies that the result is the same no matter what odd and even integers $p$ and $q$ we choose.
   We test the five choices with $p = 1$ and $q = 2$, which are odd and even, respectively.
   In this case, $2p + 3q = 8$ and $3p + 2q = 7$ and $4p + q = 6$ and $2(p + 3q) = 14$ and $pq = 2$.
   Thus, the only possibility that is odd is $3p + 2q$.

   Answer: (B)

8. *Solution 1*

The wording of the problem tells us that $a+b+c+d+e+f$ must be the same no matter what numbers $abc$ and $def$ are chosen that satisfy the conditions.
An example that works is $889+111=1000$.
In this case, $a+b+c+d+e+f=8+8+9+1+1+1=28$, so this must always be the value.

*Solution 2*

Consider performing this "long addition" by hand.
Consider first the units column.
Since $c+f$ ends in a 0, then $c+f=0$ or $c+f=10$. The value of $c+f$ cannot be 20 or more, as $c$ and $f$ are digits.
Since none of the digits is 0, we cannot have $c+f=0+0$ so $c+f=10$. (This means that we "carry" a 1 to the tens column.)
Since the result in the tens column is 0 and there is a 1 carried into this column, then $b+e$ ends in a 9, so we must have $b+e=9$. (Since $b$ and $e$ are digits, $b+e$ cannot be 19 or more.)
In the tens column, we thus have $b+e=9$ plus the carry of 1, so the resulting digit in the tens column is 0, with a 1 carried to the hundreds column.
Using a similar analysis in the hundreds column to that in the tens column, we must have $a+d=9$.
Therefore, $a+b+c+d+e+f=(a+d)+(b+e)+(c+f)=9+9+10=28$.

Answer: (D)

9. *Solution 1*

Since $\frac{1}{5}$ is equivalent to 20%, then Beshmi invests a total of $20\%+42\%=62\%$ of her savings in Companies X and Y, leaving $100\%-62\%=38\%$ for Company Z.
Since 42% of her savings is \$10 500, then 38% should be just slightly less than this amount, so of the given choices, must be \$9500.

*Solution 2*

Since $\frac{1}{5}$ is equivalent to 20%, then Beshmi invests a total of $20\%+42\%=62\%$ of her savings in Companies X and Y, leaving $100\%-62\%=38\%$ for Company Z.
Since 42% of her savings is \$10 500, then 1% of her savings is $\$10\,000 \div 42 = \$250$.
But $38\% = 38\times 1\%$, which is $38\times \$250 = \$9500$ here.
Therefore, she invests \$9500 in Company Z.

Answer: (D)

10. The bottom left vertex of the triangle has coordinates $(0,0)$, since $y=x$ (the line with positive slope) passes through the origin.
The bottom right vertex of the triangle corresponds with the $x$-intercept of the line $y=-2x+3$, which we find by setting $y=0$ to obtain $-2x+3=0$ or $x=\frac{3}{2}$. Thus, the bottom right vertex is $\left(\frac{3}{2},0\right)$.
The top vertex is the point of intersection of the two lines, which we find by combining the equations of the two lines to get $x=-2x+3$ or $3x=3$ or $x=1$.
Thus, this point of intersection is $(1,1)$.
Therefore, the triangle has a base along the $x$-axis of length $\frac{3}{2}$ and a height of length 1 (the $y$-coordinate of the top vertex).
Thus, the area of the triangle is $\frac{1}{2}\left(\frac{3}{2}\right)(1)=\frac{3}{4}$.

Answer: (A)

11. Since $\dfrac{1}{x} = 2$, then $x = \frac{1}{2}$. Since $\dfrac{1}{x} = 2$ and $\dfrac{1}{x} + \dfrac{3}{y} = 3$, then $\dfrac{3}{y} = 1$, so $y = 3$.
Therefore, $x + y = \frac{1}{2} + 3 = \frac{7}{2}$.

Answer: (D)

12. Since Siobhan's average on the seven tests is 66, then the sum of the marks on the seven tests is $7 \times 66 = 462$.
From the given marks, $69+53+69+71+78+x+y = 462$ or $340+x+y = 462$ so $x+y = 122$.
Since the sum of $x$ and $y$ is constant, then for the value of $x$ to be minimum, we need the value of $y$ to be maximum, so $y = 100$.
Therefore, the minimum possible value of $x$ is $122 - 100 = 22$.

Answer: (A)

13. Since $P$ and $Q$ are the centres of their respective circles, then line segment $PQ$ passes through the point of tangency between these two circles. Therefore, the length $PQ$ is the sum of the radii of these two circles, or $PQ = 3 + 2 = 5$.
Similarly, $PR = 3 + 1 = 4$ and $QR = 2 + 1 = 3$.
Therefore, $\triangle PQR$ has side lengths 3, 4 and 5, so is right-angled since $3^2 + 4^2 = 5^2$. In fact, the right-angle is between the sides of length 3 and 4, so the area of $\triangle PQR$ is $\frac{1}{2}(3)(4) = 6$.

Answer: (B)

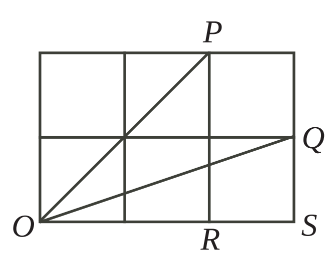
14. The circle with diameter $XZ = 12$ has radius $\frac{1}{2}(12) = 6$ so has area $\pi(6^2) = 36\pi$.
The circle with diameter $ZY = 8$ has radius $\frac{1}{2}(8) = 4$ so has area $\pi(4^2) = 16\pi$.
Thus, the total unshaded area is $36\pi + 16\pi = 52\pi$.
Since $XZY$ is a straight line, then $XY = XZ + ZY = 12 + 8 = 20$.
The circle with diameter $XY = 20$ has radius $\frac{1}{2}(20) = 10$, so has area $\pi(10^2) = 100\pi$.
The shaded area equals the area of the circle with diameter $XY$ minus the unshaded area, or $100\pi - 52\pi = 48\pi$.
Therefore, the ratio of the area of the shaded region to the area of the unshaded region is $48\pi : 52\pi$ or $48 : 52$ or $12 : 13$.

Answer: (B)

15. Since Bridget runs the second lap at $\frac{9}{10}$ of Ainslee's speed, then it takes her $\frac{10}{9}$ as long to run the lap, or $\frac{10}{9}(72) = 10(8) = 80$ seconds.
(If the lap length is $d$ and Ainslee's speed is $v$, then the amount of time that Ainslee takes is $t = \dfrac{d}{v}$ and so the amount of time that Bridget takes is $\dfrac{d}{\frac{9}{10}v} = \dfrac{10}{9}\dfrac{d}{v} = \dfrac{10}{9}t$.)
Similarly, Cecilia's time for the third lap is $\frac{3}{4}(80) = 3(20) = 60$ seconds and Dana's time for the fourth lap is $\frac{5}{6}(60) = 5(10) = 50$ seconds.
Therefore, the total time is $72 + 80 + 60 + 50$ seconds or 262 seconds or 4 minutes, 22 seconds.

Answer: (B)

16. We add label $R$ and $S$ in the diagram.



Since the side length of each small square is 2, then $OR = RP = 2(2) = 4$ and $\angle ORP = 90°$.
Since $\triangle ORP$ is isosceles and right-angled, then $\angle ROP = 45°$.
In $\triangle OSQ$, we have $QS = 2$, $OS = 3(2) = 6$ and $\angle OSQ = 90°$.
Therefore, $\tan(\angle QOS) = \frac{2}{6} = \frac{1}{3}$, so $\angle QOS \approx 18.43°$.
Thus, $\angle POQ = \angle POR - \angle QOS \approx 45° - 18.43° = 26.57°$, which, to the nearest tenth of a degree is $26.6°$.

Answer: (C)

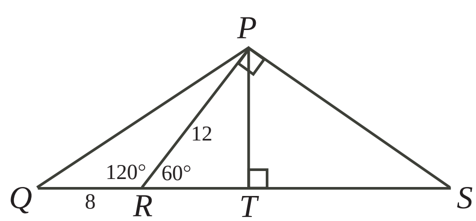
17. Suppose that these two integers are $x$ and $x+1$, since they are consecutive.
Then $(x+1)^2 - x^2 = 199$ or $(x^2+2x+1) - x^2 = 199$ or $2x+1 = 199$ or $x = 99$.
Therefore, the two integers are 99 and 100, and the sum of their squares is $99^2 + 100^2$ or $9801 + 10\,000 = 19\,801$.

Answer: (A)

18. Since each term is obtained by adding the same number to the previous term, then the differences between pairs of consecutive terms are equal.
Looking at the first three terms, we thus have $2a - a = b - 2a$ or $b = 3a$.
Therefore, in terms of $a$, the first four terms are $a$, $2a$, $3a$, and $a - 6 - 3a = -6 - 2a$.
Since the constant difference between the terms equals $a$ (as $2a - a = a$), then the fourth term should be $4a$, so $4a = -6 - 2a$ or $6a = -6$ or $a = -1$.
Thus, the sequence begins $-1$, $-2$, $-3$, $-4$.
The 100th term is thus $-100$ (which we can get by inspection or by saying that we must add the common difference 99 times to the first term, to get $-1 + 99(-1) = -100$).

Answer: (A)

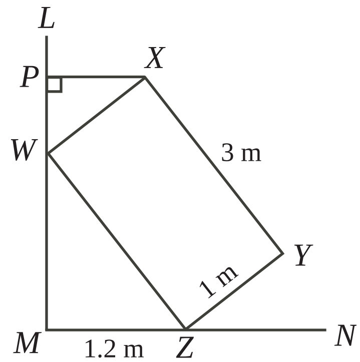
19. Since $\angle QRP = 120°$ and $QRS$ is a straight line, then $\angle PRS = 180° - 120° = 60°$.
Since $\angle RPS = 90°$, then $\triangle SRP$ is a $30°$-$60°$-$90°$ triangle.
Therefore, $RS = 2PR = 2(12) = 24$.
Drop a perpendicular from $P$ to $T$ on $RS$.




Since $\angle PRT = 60°$ and $\angle PTR = 90°$, then $\triangle PRT$ is also a $30°$-$60°$-$90°$ triangle.
Therefore, $PT = \frac{\sqrt{3}}{2}PR = 6\sqrt{3}$.
Consider $\triangle QPS$. We may consider $QS$ as its base with height $PT$.
Thus, its area is $\frac{1}{2}(6\sqrt{3})(8+24) = 96\sqrt{3}$.

Answer: (E)

20. Draw a line from $X$ to $P$ on $LM$, so that $XP$ is perpendicular to $LM$.




Since $\angle XPM = \angle PMN = 90^\circ$, then $PX$ is parallel to $MN$, so the distance from $X$ to line $MN$ equals the length of $PM$.
Since $WXYZ$ is a rectangle, then $WZ = XY = 3$ m and $WX = ZY = 1$ m.
By the Pythagorean Theorem, $WM = \sqrt{WZ^2 - MZ^2} = \sqrt{3^2 - 1.2^2} = \sqrt{9 - 1.44} = \sqrt{7.56}$ m.
Since $\angle XWZ = 90^\circ$ and $PWM$ is a straight line, then $\angle PWX + \angle XWZ + \angle ZWM = 180^\circ$, so $\angle PWX + \angle ZWM = 180^\circ - 90^\circ = 90^\circ$.
But since $\triangle XPW$ is right-angled, then

$$\angle PXW = 90^\circ - \angle PWX = 90^\circ - (90^\circ - \angle ZWM) = \angle ZWM$$

Therefore, $\triangle XPW$ is similar to $\triangle WMZ$.
This tells us that $\dfrac{PW}{MZ} = \dfrac{XW}{WZ}$, so $PW = \dfrac{MZ(XW)}{WZ} = \dfrac{1.2(1)}{3} = 0.4$ m.
Thus, $PM = PW + WM = 0.4 + \sqrt{7.56} \approx 3.1495$ m, which, to the nearest hundredth of a metre, equals 3.15 m.

ANSWER: (C)

21. There are 52 terms in the sum: the number 1, the number 11, and the 50 numbers starting with a 1, ending with a 1 and with 1 to 50 zeroes in between. The longest of these terms thus has 52 digits (50 zeroes and 2 ones).
When the units digits of all 52 terms are added up, their sum is 52, so the units digit of $N$ is 2, and a 5 carried to the tens digit.
In the tens digit, there is only 1 non-zero digit: the 1 in the number 11. Therefore, using the carry, the tens digit of $N$ is $1 + 5 = 6$.
In each of positions 3 to 52 from the right-hand end, there is only one non-zero digit, which is a 1.
Therefore, the digit in each of these positions in $N$ is also a 1. (There is no carrying to worry about.)
Therefore, $N = 11\cdots1162$, where $N$ has $52 - 2 = 50$ digits equal to 1.
This tells us that the sum of the digits of $N$ is $50(1) + 6 + 2 = 58$.

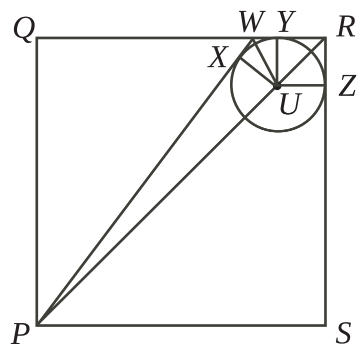
ANSWER: (A)

22. If the two parabolas $y = -\frac{1}{8}x^2 + 4$ and $y = x^2 - k$ do intersect, then they do so where $x$ satisfies the equation $-\frac{1}{8}x^2 + 4 = x^2 - k$ or $\frac{9}{8}x^2 = 4 + k$.
Since $x^2 \geq 0$, then $4 + k \geq 0$, so $k \geq -4$.
(This is the condition for these two parabolae to actually intersect.)

We also want the point of intersection to be on or above the $x$-axis, so $y \geq 0$.
Since we know that $\frac{9}{8}x^2 = 4 + k$, then $x^2 = \frac{8}{9}(4 + k)$, so at the point(s) of intersection, $y = x^2 - k = \frac{8}{9}(4 + k) - k = \frac{32}{9} - \frac{1}{9}k$.
Since we want $y \geq 0$, then $\frac{32}{9} - \frac{1}{9}k \geq 0$, so $k \leq 32$.
Therefore, the two parabolae do intersect and intersect on or above the $x$-axis precisely when $-4 \leq k \leq 32$.
There are $32 - (-4) + 1 = 37$ integer values of $k$ in this range.

ANSWER: (E)

23. Throughout this solution, we suppress the units (metres) until the very end. All lengths until then are given in metres.
Since square $PQRS$ has side length 4, then its diagonal $PR$ has length $4\sqrt{2}$.
Since $PR = 4UR$, then $PU = \frac{3}{4}PR = \frac{3}{4}(4\sqrt{2}) = 3\sqrt{2}$ and $UR = \frac{1}{4}PR = \sqrt{2}$.
Suppose that the circle touches $WR$ at $Y$, $RS$ at $Z$, and $PW$ at $X$.




Since $RS$ is tangent to the circle at $Z$, then $\angle UZR = 90^\circ$.
Since $\angle PRS = 45^\circ$ (because $PR$ is the diagonal of a square), then $\triangle UZR$ is isosceles and right-angled.
Thus, $UZ = \frac{1}{\sqrt{2}}UR = \frac{1}{\sqrt{2}}(\sqrt{2}) = 1$. That is, the radius of the circle is 1.
Therefore, $UY = UX = UZ = 1$.
Now since $PW$ is tangent to the circle at $X$, then $\angle PXU = 90^\circ$.
By the Pythagorean Theorem, $PX = \sqrt{PU^2 - UX^2} = \sqrt{(3\sqrt{2})^2 - 1^2} = \sqrt{18 - 1} = \sqrt{17}$.
Also, $\sin(\angle UPX) = \dfrac{UX}{UP} = \dfrac{1}{3\sqrt{2}}$ so $\angle UPX \approx 13.63^\circ$.
Since we know the length of $PX$, then to determine the length of $PW$, we must determine the length of $XW$.
Since $WX$ and $WY$ are tangents to the circle from the same point $W$, then $WX = WY$, which tells us that $\triangle UWX$ and $\triangle UWY$ are congruent, so $\angle UWX = \angle UWY$.
Looking at the angles in $\triangle PWR$, we have

$$\begin{aligned} \angle WPR + \angle PWR + \angle WRP &= 180^\circ \\ 2(\angle UWX) &\approx 180^\circ - 45^\circ - 13.63^\circ \\ 2(\angle UWX) &\approx 121.37^\circ \\ \angle UWX &\approx 60.68^\circ \end{aligned}$$

In $\triangle UWX$, we have $\tan(\angle UWX) = \dfrac{UX}{XW}$ so $XW \approx \dfrac{1}{\tan(60.68^\circ)} \approx 0.5616$.
Therefore, $PW = PX + XW \approx \sqrt{17} + 0.562 \approx 4.6847$ m.
To the thousandth of a metre, this equals 4.685 m.

ANSWER: (C)

24. We first suppose that $a \leq b \leq c$ and consider the other cases at the end.
Since $a$, $b$ and $c$ are positive integers, then $a \geq 1$.
Can $a = 1$? If $a = 1$, then $\frac{1}{a} = 1$, so $\frac{1}{b} + \frac{1}{c} = -\frac{1}{4}$, which is not possible, since $b$ and $c$ are positive.
Therefore, $a > 1$.
Since $a \leq b \leq c$, then $\frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c}$, so $\frac{3}{a} = \frac{1}{a} + \frac{1}{a} + \frac{1}{a} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{3}{4}$, and so $a \leq 4$.
Thus, $a = 2$, 3 or 4.

If $a = 4$, then $\frac{1}{b} + \frac{1}{c} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.
Since $b \leq c$, then $\frac{1}{b} \geq \frac{1}{c}$, so $\frac{1}{b} \geq \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$, so $b \leq 4$.
Since $a \leq b$, then $b \geq 4$, so $b = 4$.

If $a = 3$, then $\frac{1}{b} + \frac{1}{c} = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$.
Since $b \leq c$, then $\frac{1}{b} \geq \frac{1}{c}$, so $\frac{1}{b} \geq \frac{1}{2}\left(\frac{5}{12}\right) = \frac{5}{24}$, so $b \leq \frac{24}{5}$, so $b \leq 4$, since $b$ is an integer.
Since $a \leq b$, then $b \geq 3$, so $b = 3$ or $b = 4$.

If $a = 2$, then $\frac{1}{b} + \frac{1}{c} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$.
Since $b \leq c$, then $\frac{1}{b} \geq \frac{1}{c}$, so $\frac{1}{b} \geq \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$, so $b \leq 8$.
Since $\frac{1}{b} < \frac{1}{4}$ as well (because $c > 0$), then $b > 4$.
Thus, $b = 5$, 6, 7, or 8.

We now make a table of the possible values:

| $a$ | $\frac{1}{b} + \frac{1}{c}$ | $b$ | $\frac{1}{c}$ | $c$ |
|---|---|---|---|---|
| 4 | $\frac{1}{2}$ | 4 | $\frac{1}{4}$ | 4 |
| 3 | $\frac{5}{12}$ | 3 | $\frac{1}{12}$ | 12 |
| 3 | $\frac{5}{12}$ | 4 | $\frac{1}{6}$ | 6 |
| 2 | $\frac{1}{4}$ | 5 | $\frac{1}{20}$ | 20 |
| 2 | $\frac{1}{4}$ | 6 | $\frac{1}{12}$ | 12 |
| 2 | $\frac{1}{4}$ | 7 | $\frac{3}{28}$ | $\frac{28}{3}$ |
| 2 | $\frac{1}{4}$ | 8 | $\frac{1}{8}$ | 8 |

Thus, the triples with $a \leq b \leq c$ are $(4, 4, 4)$, $(3, 3, 12)$, $(3, 4, 6)$, $(2, 5, 20)$, $(2, 6, 12)$ and $(2, 8, 8)$.
Removing the condition $a \leq b \leq c$, we can see that any triple that solves this equation is a permutation of one of the 6 triples above, as it can be relabelled with $a$ its smallest number, $b$ its middle number and $c$ its largest number.
A triple of the form $(x, x, x)$ has only one permutation.
A triple of the form $(x, x, y)$ (with $x \neq y$) has 3 permutations (the other two being $(x, y, x)$ and $(y, x, x)$).
A triple of the form $(x, y, z)$ (with all three different) has 6 permutations. (Try listing these out.)
Therefore, permuting the 6 possible triples above in all possible ways, the total number of

triples that solve the equation is $1 + 3 + 6 + 6 + 6 + 3 = 25$.

Answer: (B)

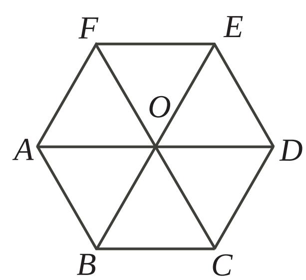
25. First, some preliminary information is needed.
Consider the base $ABCDEF$ of the sliced solid. This base is a regular hexagon. Thus, its six sides have equal length and each of its six interior angles equals $120^\circ$. (The sum of the angles of an $n$-gon is $(n-2)180^\circ$, which equals $720^\circ$ or $6(120^\circ)$ when $n = 6$.)
Let $O$ be the centre of the hexagon.
Join each vertex to $O$.

Fact #1: The 6 triangles formed are equilateral

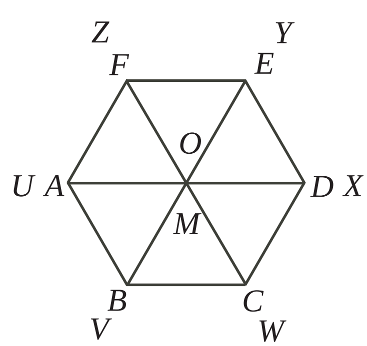
By symmetry, each of these segments bisects the angle at its vertex, creating two $60^\circ$ angles. Thus, each of the six triangles formed has two $60^\circ$ angles, so must have three $60^\circ$ angles, so is equilateral. Therefore, the six sides and six new line segments are equal in length.

Fact #2: $AOD$, $BOE$ and $COF$ are straight lines and parallel to sides of the hexagon

Since each of the six angles at $O$ equals $60^\circ$, then three of these angles form a straight line, so $AOD$, $BOE$ and $COF$ are straight lines, which are in addition parallel to $BC$ and $EF$, $CD$ and $FA$, and $DE$ and $AB$, respectively. This is because of the alternate angles between pairs of lines. For example, $\angle AOF = \angle OFE = 60^\circ$, so $FE$ and $AOD$ are parallel.




Consider the top face $UVWXYZ$.
Let $M$ be the point in this face directly above $O$.
Define $s = AU + BV + CW + DX + EY + FZ$.
Define $h(U)$ to be the height of $U$ above $A$, $h(V)$ to be the height of $V$ above $B$, and so on. That is, $h(U) = AU$, $h(V) = BV$, and so on.



Fact #3: $h(V) - h(U) = h(X) - h(Y)$

Note that the segments $UV$ and $YX$ lie directly above the segments $AB$ and $ED$, and so on.
Since $AB$ and $ED$ are parallel and equal, then $h(V) - h(U) = h(X) - h(Y)$. This is because parallel lines in a plane have the same slope. Try visualizing a piece of paper held above a table at an angle and slice this paper with two parallel vertical planes. The lines in the paper created by these slices will have the same slope, so will have the same height change over segments of equal length.

Since $AO$ and $BC$ are also parallel and equal, then $h(M) - h(U) = h(W) - h(V)$. Similar equations also hold.

Fact #4: $h(M) = \frac{1}{2}(h(U) + h(X))$

We know that $AO$ and $OD$ are parallel and equal
Thus, $h(M) - h(U) = h(X) - h(M)$ or $h(M) = \frac{1}{2}(h(U) + h(X))$.

Similarly, $h(M) = \frac{1}{2}(h(V) + h(Y)) = \frac{1}{2}(h(W) + h(Z))$.

Fact #5: $s = 6h(M)$

Adding these last three equations,

$$3h(M) = \frac{1}{2}(h(U) + h(V) + h(W) + h(X) + h(Y) + h(Z))$$

so $s = 2(3h(M)) = 6h(M)$.

So if we can determine $h(M)$, then we can determine the sum of the lengths of the vertical segments easily.

We are now ready to solve the problem. There are a number of cases to consider. Since we can rotate the prism, it is only the relative position of the known heights that is important.

Case 1: $h(U) = 7$, $h(V) = 4$, $h(W) = 10$
Since $AB$ and $OC$ are parallel and equal, then $h(M) - h(W) = h(U) - h(V) = 3$, so $h(M) = 10 + 3 = 13$, so $s = 6h(M) = 6(13) = 78$.
This will turn out to be the maximum value of $s$.

Case 2: The heights above two opposite vertices are two of 4, 7 and 10
In this case, $h(M)$ will be the average of two of 4, 7 and 10, and so $h(M)$ is certainly less than 10, so $s = 6h(M) < 6(10) = 60$. This does not give a maximum.

Case 3: The heights 4, 7 and 10 are above consecutive vertices
To avoid duplicating Case 1, we have either $h(U) = 4$, $h(V) = 7$, $h(W) = 10$, or $h(U) = 4$, $h(V) = 10$, $h(W) = 7$.
From the analysis in Case 1, $h(M) = h(U) + h(W) - h(V)$.
In these two cases, $h(M) = 7$ or $h(M) = 1$, giving $s = 42$ or $s = 6$, neither of which is a maximum.

Case 4: None of 4, 7, 10 are adjacent
Suppose $h(U) = 4$, $h(W) = 7$ and $h(Y) = 10$. (There are no other different such configurations to consider.)
Suppose that $h(M) = x$.
Since $h(M)$ is the average of the heights above opposite vertices, then $h(V) = 2h(M) - h(Y)$ so $h(V) = 2x - 10$.
But $AB$ and $OC$ are parallel and equal, so $h(V) - h(U) = h(W) - h(M)$ or $2x - 10 - 4 = 7 - x$ or $3x = 21$ or $x = 7$.
Thus, $s = 6h(M) = 6x = 42$.

Having considered all possible cases, the maximum value of $s$ (that is, the sum of the six vertical lengths) is 78.

Answer: (D)

**Canadian Mathematics Competition**

***An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario***

# *2007 Fermat Contest*

**(Grade 11)**

**Tuesday, February 20, 2007**

*Solutions*

1. Calculating, $\dfrac{36-12}{12-4} = \dfrac{24}{8} = 3$.

ANSWER: (E)

2. Since $7x = 28$, then $x = 4$.
Since $x + w = 9$ and $x = 4$, then $w = 5$.
Since $x = 4$ and $w = 5$, then $xw = 20$.

ANSWER: (B)

3. To determine which of the fractions are the largest and smallest, we write each using a common denominator of 16 as $\frac{12}{16}$, $\frac{14}{16}$, $\frac{13}{16}$, and $\frac{8}{16}$.
Therefore, the largest is $\frac{7}{8} = \frac{14}{16}$ and the smallest is $\frac{1}{2} = \frac{8}{16}$.
The difference between these two fractions is $\frac{14}{16} - \frac{8}{16} = \frac{6}{16} = \frac{3}{8}$.

ANSWER: (A)

4. When $x = -5$, we have $-2x^2 + \dfrac{5}{x} = -2(-5)^2 + \dfrac{5}{-5} = -2(25) + (-1) = -50 - 1 = -51$.

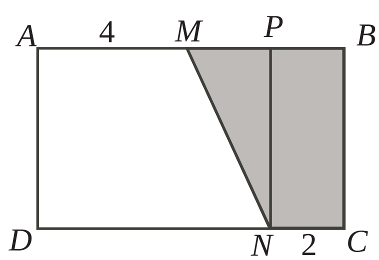
ANSWER: (C)

5. By definition, $1^{-2} + 2^{-1} = \dfrac{1}{1^2} + \dfrac{1}{2^1} = \dfrac{1}{1} + \dfrac{1}{2} = \dfrac{3}{2}$.

ANSWER: (A)

6. *Solution 1*
Since the area of rectangle $ABCD$ is 40 and $AB = 8$, then $BC = 5$.
Therefore, $MBCN$ is a trapezoid with height 5 and parallel bases of lengths 4 and 2, so has area $\frac{1}{2}(5)(4+2) = 15$.

*Solution 2*
Since the area of rectangle $ABCD$ is 40 and $AB = 8$, then $BC = 5$.
Draw a line from $N$ to $AB$ parallel to $BC$ (and so perpendicular to $AB$) meeting $AB$ at $P$.




This line divides $MBCN$ into a rectangle $PBCN$ of width 2 and height 5, and a triangle $MPN$ with base $MP$ of length 2 and height $PN$ of height 5.
The area of $MBCN$ is sum of the areas of these two parts, or $2(5) + \frac{1}{2}(2)(5) = 10 + 5 = 15$.

ANSWER: (A)

7. *Solution 1*
If the sum of two positive integers is 9, the possible pairs are 1 and 8, 2 and 7, 3 and 6, and 4 and 5.
Of these pairs, the only one in which each number is a divisor of 42 is 2 and 7.
Since the three positive integers have a product of 42, then two of them must be 2 and 7, so

the third is $42 \div (2 \times 7) = 42 \div 14 = 3$.

*Solution 2*

The possible sets of three positive integers which multiply to give 42 are $\{1, 1, 42\}$, $\{1, 2, 21\}$, $\{1, 3, 14\}$, $\{1, 6, 7\}$, and $\{2, 3, 7\}$.
The only one of these sets that contains two integers which add to 9 is $\{2, 3, 7\}$.
Therefore, the third number must be 3.

ANSWER: (D)

8. Suppose that Ivan ran a distance of $x$ km on Monday.
Then on Tuesday, he ran $2x$ km, on Wednesday, he ran $x$ km, on Thursday, he ran $\frac{1}{2}x$ km, and on Friday he ran $x$ km.
The shortest of any of his runs was on Thursday, so $\frac{1}{2}x = 5$ or $x = 10$.
Therefore, his runs were 10 km, 20 km, 10 km, 5 km, and 10 km, for a total of 55 km.

ANSWER: (A)

9. Since $\dfrac{1}{x+3} = 2$, then, taking reciprocals of both sides, $x + 3 = \frac{1}{2}$.
Since $x + 3 = \frac{1}{2}$, then $x + 5 = \frac{1}{2} + 2 = \frac{5}{2}$.
Since $x + 5 = \frac{5}{2}$, then $\dfrac{1}{x+5} = \dfrac{2}{5}$.
(Notice that we did not need to actually find the value of $x$.)

ANSWER: (C)

10. Phyllis pays \$20 for each of two DVDs and \$10 for the third DVD, so pays \$50 in total for 3 DVDs.
Since \$50 is the price of $2\frac{1}{2}$ DVDs, then she gets 3 DVDs for the price of $2\frac{1}{2}$, which is the same as getting 6 DVDs for the price of 5.

ANSWER: (E)

11. When a set of five numbers is listed in ascending order, the median of the set is the middle number in the list, or the third number in this case.
Since the median is 7 and $x$ is the middle number, then $x = 7$.
Therefore, the list is 2, 5, 7, 10, $y$.
Since the mean of the five numbers is 8, then the sum of the numbers is $5 \times 8 = 40$.
Therefore, $2 + 5 + 7 + 10 + y = 40$ or $24 + y = 40$ or $y = 16$.

ANSWER: (A)

12. Since $\angle QSR = \angle QRS$, then $\triangle QSR$ is isosceles with $QS = QR$, so $QS = x$.
Since $\angle SPQ = 90^\circ$ and $\angle PQS = 60^\circ$, then $\angle PSQ = 30^\circ$, so $\triangle PQS$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Therefore, $QS = 2PQ$ using the ratios of sides in such a triangle.
Thus, $x = QS = 2(10) = 20$.

ANSWER: (B)

13. *Solution 1*
Accounting for all of the missing numbers, $M + N + P + Q + R = 1 + 4 + 5 + 6 + 7 = 23$.
To determine the sum $M + N + P + Q$, we can determine the value of $R$ and subtract this from 23.
$R$ cannot be 1, since $0 + 1 = 1$ is not prime. (If $R$ was 1, we would thus have the sum of the

numbers at the ends of one of the edges not equal to a prime number.)
$R$ cannot be 4, since $0+4=4$ is not prime.
$R$ cannot be 6, since $0+6=6$ is not prime.
$R$ cannot be 7, since $2+7=9$ is not prime.
By process of elimination, $R=5$, so $M+N+P+Q=23-5=18$.
(We will see in Solution 2 that we can fill in the rest of the numbers in a way that satisfies the requirements.)

*Solution 2*
The missing numbers are 1, 4, 5, 6, 7.
Since $Q+3$ must be a prime number, then $Q$ must be 4 (since $1+3$, $5+3$, $6+3$, and $7+3$ are not prime).
Since $M+0$ and $M+4$ must both be prime numbers, then $M$ must be 7 (since $1+0$, $5+4$ and $6+4$ are not prime).
Since $P+2$ and $P+4$ must both be prime numbers, then $P$ must be 1 (since $5+4$ and $6+2$ are not prime).
Since $N+7$ and $N+1$ must both be prime numbers, then $N$ must be 6 (since $5+7$ is not prime).
We can check that if $R=5$, then the requirement that the sum of the two numbers at the ends of each edge be a prime number is met.
Thus, $M+N+P+Q=7+6+1+4=18$.

ANSWER: (C)

14. When $a$ is increased by 25%, the result is $1.25a$ or $\frac{5}{4}a$.
Thus, we would like $\frac{5}{4}a>5b$ or $5a>20b$ or $a>4b$.
We would like to find the smallest possible positive integers $a$ and $b$ that satisfy this inequality.
(If $a$ and $b$ are as small as possible, then their sum $a+b$ will be as small as possible.)
Since $b$ is a positive integer, $4b\geq 4$, so since $a>4b$, then $a>4$.
Thus, the smallest possible value of $a$ is 5 when $b=1$.
Since $b$ cannot be smaller, then $a$ cannot be smaller.
Therefore, the minimum possible value for $a+b$ is $5+1=6$.

ANSWER: (B)

15. *Solution 1*
Suppose $x$ has digits $pqr$.
Since $x$ can only have even digits, then $p$ is 2, 4, 6 or 8 and each of $q$ and $r$ can be 0, 2, 4, 6 or 8.
When $x$ is multiplied by 2, each digit is multiplied by 2, and some "carrying" may occur.
Note that $2\times 0=0$, $2\times 2=4$, $2\times 4=8$, $2\times 6=12$, and $2\times 8=16$.
So when $2x$ is calculated, each digit from $x$ will initially produce a corresponding even digit in $2x$ and a carry of 0 or 1.
If the carry is 0, whether the next digit to the left in $2x$ is even or odd is not affected.
If the carry is 1, the next digit to the left in $2x$ will be changed from even to odd. (Note that the carry can never be made larger than 1 by extra carrying into a given digit.)
Therefore, no digit of 6 or 8 can appear in $x$, since if a 6 or 8 appears in $x$, an odd digit in $2x$ is guaranteed.
Also, digits of 0, 2 and 4 can occur in any position (except no 0 in the leading position) as they will always produce an even digit in $2x$.
Therefore, there are 2 possible values for $p$ and 3 each for $q$ and $r$, giving $2\times 3\times 3=18$ possible values for $x$.

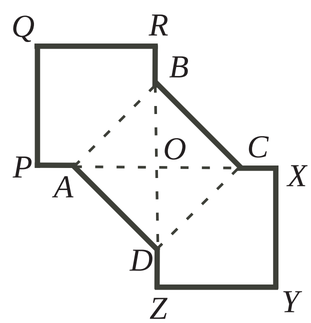
*Solution 2*
Suppose $x$ has digits $pqr$, so $x = 100p + 10q + r$.
Since $x$ can only have even digits, then $p$ is 2, 4, 6 or 8 and each of $q$ and $r$ can be 0, 2, 4, 6 or 8.
When $x$ is multiplied by 2, each digit is multiplied by 2, and some "carrying" may occur.
Note that $2 \times 0 = 0$, $2 \times 2 = 4$, $2 \times 4 = 8$, $2 \times 6 = 12$, and $2 \times 8 = 16$.
Suppose $2p = 10A + a$, $2q = 10B + b$, $2r = 10C + c$, where $A, a, B, b, C, c$ are all digits with $A, B, C$ each 0 or 1 and $a, b, c$ each even.
Then

$$\begin{aligned} 2x &= 2(100p + 10q + r) \\ &= 100(2p) + 10(2q) + 2r \\ &= 100(10A + a) + 10(10B + b) + 10C + c \\ &= 1000A + 100a + 100B + 10b + 10C + c \\ &= 1000A + 100(a + B) + 10(b + C) + c \end{aligned}$$

Since $a, b, c$ are each at most 8 and $A, B, C$ are each at most 1, then $a + B$ and $b + C$ are each at most 9, so each of $A$, $a + B$, $b + C$ and $c$ are indeed digits.
For $2x$ to have even digits only, then $A$ must be even (so must be 0), $c$ must be even (which it is), and $a + B$ and $b + C$ must both be even.
Since $a$ and $b$ are both even, this means that $B$ and $C$ must both be even, so must both be 0.
Since $A$, $B$ and $C$ are all 0, then none of $p$, $q$ or $r$ can be 6 or 8, and each can be 0, 2 or 4 (except $p \neq 0$).
Therefore, there are 2 possible values for $p$ and 3 each for $q$ and $r$, giving $2 \times 3 \times 3 = 18$ possible values for $x$.

ANSWER: (B)

16. We label the remaining vertices in the figure.




Since each of the squares has a side length of 3, then $PQ = QR = BC = XY = YZ = DA = 3$, so the perimeter of the figure equals $18 + AP + RB + CX + ZD$.
Since $O$ is the centre of square $ABCD$, then $OA = OB = OC = OD$.
Since $OP = OR = OX = OZ$, then $AP = RB = CX = ZD$.
Therefore, the perimeter equals $18 + 4AP$.
Since $O$ is centre of square $ABCD$, then $OA = OB$ and $\angle AOB = 90^\circ$, so $\triangle AOB$ is an isosceles right-angled triangle, so has angles $45^\circ$, $45^\circ$ and $90^\circ$.
Therefore, $AO = \dfrac{AB}{\sqrt{2}} = \dfrac{3}{\sqrt{2}} = \dfrac{3\sqrt{2}}{2}$ and so $AP = OP - AO = 3 - \dfrac{3\sqrt{2}}{2}$.
Thus, the perimeter is $18 + 4\left(3 - \dfrac{3\sqrt{2}}{2}\right) = 18 + 12 - 6\sqrt{2} = 30 - 6\sqrt{2} \approx 21.515$.

ANSWER: (A)

17. *Solution 1*

Since $AB = BC$, then $B$ lies on the perpendicular bisector of $AC$.
Since $A$ has coordinates $(2, 2)$ and $C$ has coordinates $(8, 4)$, then the midpoint of $AC$ is $\left(\frac{1}{2}(2+8), \frac{1}{2}(4+2)\right) = (5, 3)$ and the slope of $AC$ is $\dfrac{4-2}{8-2} = \dfrac{1}{3}$.
Therefore, the slope of the perpendicular bisector is $-3$ (the negative reciprocal of $\frac{1}{3}$) and it passes through $(5, 3)$, so has equation $y - 3 = -3(x - 5)$ or $y = -3x + 18$.
The $x$-intercept of this line comes when $y$ is set to 0; here, we obtain $x = 6$.
Therefore, since $B$ is the point where the perpendicular bisector of $AC$ crosses the $x$-axis, then the $x$-coordinate of $B$ is 6.
(We can check that indeed if $B$ has coordinates $(6, 0)$, then $AB$ and $BC$ are perpendicular.)

*Solution 2*

Since $\triangle ABC$ is an isosceles right-angled triangle, then $\angle ABC = 90°$, and so $AB$ is perpendicular to $BC$.
Suppose $B$ has coordinates $(b, 0)$.
The slope of $AB$ is $\dfrac{2-0}{2-b}$ and the slope of $BC$ is $\dfrac{4-0}{8-b}$.
Since $AB$ and $BC$ are perpendicular, their slopes are negative reciprocals, so

$$\begin{aligned} \frac{2}{2-b} &= -\frac{8-b}{4} \\ -8 &= (2-b)(8-b) \\ -8 &= b^2 - 10b + 16 \\ b^2 - 10b + 24 &= 0 \\ (b-4)(b-6) &= 0 \end{aligned}$$

and so $b = 4$ or $b = 6$.
We must determine which value of $b$ gives $AB = BC$ (since we have already used the perpendicularity).
If $b = 4$, then $AB = \sqrt{(4-2)^2 + (0-2)^2} = \sqrt{8}$ and $BC = \sqrt{(8-4)^2 + (4-0)^2} = \sqrt{32}$ and so $AB \neq BC$.
Therefore, the $x$-coordinate of $B$ must be 6.
(We can check that, in this case, $AB$ does equal $BC$.)

*Solution 3*

To go from $A$ to $C$, we go 6 units right and 2 units up.
Suppose that to go from $A$ to $B$, we go $p$ units right and $q$ units down, were $p, q > 0$.
Since $BC$ is equal and perpendicular to $AB$, then to go from $B$ to $C$, we must go $q$ units right and $p$ units up.
(We can see this by looking at the slopes of segments $AB$ and $BC$.)
Therefore, to get from $A$ to $C$ through $B$, we go $p+q$ units right and $q-p$ units up, so $p+q = 6$ and $q - p = 2$, as the result is the same as from going directly to $C$ from $A$.
Since $p + q = 6$ and $q - p = 2$, then $2q = 8$ (adding the equations), so $q = 4$, and so $p = 2$.
Since $A$ has coordinates $(2, 2)$, then $B$ has coordinates $(6, 0)$ which lies on the $x$-axis as required.

Answer: (D)

18. Suppose that Alphonso and Karen each start with $n$ apples.
After Karen gives 12 apples to Alphonso, Karen has $n - 12$ apples and Alphonso has $n + 12$

apples.
After Karen gives half of her remaining apples (that is, $\frac{1}{2}(n-12)$ apples) to Alphonso, she has $\frac{1}{2}(n-12) = \frac{1}{2}n - 6$ apples and Alphonso has $n + 12 + \frac{1}{2}(n-12) = \frac{3}{2}n + 6$ apples.
Since Alphonso now has four times as many as Karen, $4(\frac{1}{2}n - 6) = \frac{3}{2}n + 6$ or $2n - 24 = \frac{3}{2}n + 6$ so $\frac{1}{2}n = 30$ or $n = 60$.
This means that Karen now has $\frac{1}{2}(60-12) = 24$ apples.

Answer: (B)

19. In this problem, we use the *Triangle Inequality*, which says that in any triangle, the length of one side is less than the sum of the lengths of the other two sides. (For example, in $\triangle ABC$, $AC < AB + BC = 19$.) The Triangle Inequality is true because the shortest distance between any two points is a straight line, so any other route (such as travelling along the other two sides) must be longer.
In $\triangle ABC$, $AC < AB + BC = 19$.
In $\triangle ACD$, $DC < DA + AC$ or $19 < 5 + AC$ or $AC > 14$.
Of the given choices, only 15 lies between 14 and 19, so 15 must be the answer. (We can check that if $AC = 15$, each of these two triangles can be constructed.)

Answer: (D)

20. *Solution 1*
A specific parabola that has this shape (that is, opening downwards and with a negative $x$-intercept that is more negative than the positive $x$-intercept is positive) is
$y = -(x+2)(x-1) = -x^2 - x + 2$.
In this parabola, $a = -1$, $b = -1$, $c = 2$.
Using these values for $a$, $b$ and $c$, only $c - a$ is positive.
Since we are looking for the possibility which is positive no matter which parabola is used, then $c - a$ must be the answer.

*Solution 2*
Since the parabola in the diagram opens downwards, then $a < 0$. This tells us that (A) is not correct and that $ab^2$ is negative so (C) is not correct.
The $y$-intercept of the parabola $y = ax^2 + bx + c$ is $y = c$. (This comes from setting $x = 0$.)
Since the $y$-intercept is positive, then $c > 0$.
This tells us that $c - a$ is positive (since $a$ is negative) so must be the correct answer, since there is only one correct answer.
(We could check that $b$ is negative using the fact that the $x$-coordinate of the vertex is negative and so both $bc$ and $b - c$ must be negative.)

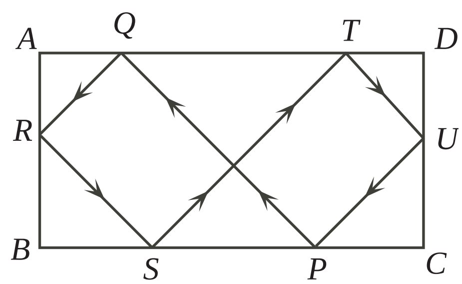
Answer: (E)

21. Since $m$ is the third of the five integers, then the five integers are $m-2$, $m-1$, $m$, $m+1$, and $m+2$.
The sum of all five is thus $(m-2) + (m-1) + m + (m+1) + (m+2) = 5m$ and the sum of the middle three is $(m-1) + m + (m+1) = 3m$.
Therefore, we want to find the smallest integer $m$ for which $3m$ is a perfect square and $5m$ is a perfect cube.
Consider writing $m$, $3m$ and $5m$ each as a product of prime numbers.
For $3m$ to be a perfect square, each prime must occur an even number of times in the product.
Thus, the prime 3 must occur an odd number of times in the product that represents $m$.
For $5m$ to be a perfect cube, the number of times that each prime occurs in the product must

be a multiple of 3. Thus, the prime 5 must occur a number of times which is one less than a multiple of 3 in the product that represents $m$.
Since both of the primes 3 and 5 are factors of $m$, then to minimize $m$, no other prime should occur.
For $3m$ to be a perfect square, 5 must occur an even number of times in $3m$.
For $5m$ to be a perfect cube, the number of times that 3 occurs in $5m$ is a multiple of 3.
Therefore, $m$ is a number which contains an odd number of 3s and an even number of 5s (since $3m$ does), and at the same time contains 3 a number of times which is a multiple of 3 (since $5m$ does) and 5 a number of times that is 1 less than a multiple of 3.
To minimize $m$, $m$ should contain as few 3s and 5s as possible, so should contain three 3s and two 5s, so $m = 3^3 5^2 = 675$.

Answer: (D)

22. Label the vertices of the rectangle $ABCD$ and the points of contact of the ball with the edges in order as $P$, $Q$, $R$, $S$, $T$, and $U$.



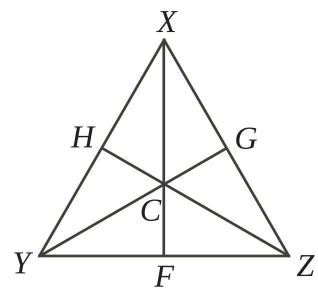

Note that the angle that each segment of the path makes with each side of the table is 45°.
Therefore, $PQ = \sqrt{2}AB$. Also, $QR = \sqrt{2}AR$ and $RS = \sqrt{2}RB$, so $QR+RS = \sqrt{2}(AR+RB) = \sqrt{2}AB$.
Similarly, $ST = \sqrt{2}CD$ and $TU + UP = \sqrt{2}CD$.
Since $AB = CD$, then $PQ + QR + RS + ST + TU + UP = 4\sqrt{2}AB$.
Since we know that the total length of the path is 7 m, then $AB = \frac{7}{4\sqrt{2}}$ m.

Also, $PQ = \sqrt{2} \times$ horizontal distance from $P$ to $Q$, $QR = \sqrt{2}QA$ and $PU = \sqrt{2}PC$.
Therefore, $PQ + QR + PU = \sqrt{2}AD$ (since $QA$ plus $PC$ plus the horizontal distance from $P$ to $Q$ equals the length of the rectangle).
Similarly, $RS + ST + TU = \sqrt{2}BC = \sqrt{2}AD$.
So $PQ + QR + RS + ST + TU + UP = 2\sqrt{2}AD$ so $AD = \frac{7}{2\sqrt{2}}$.
Therefore, the perimeter of the table is $2AD + 2AB = \frac{7}{\sqrt{2}} + \frac{7}{2\sqrt{2}} \approx 7.425$, which is closest to 7.5 m.

Answer: (B)

23. Since the vertical distance from $O$ to each of $M$, $N$ and $P$ is the same, then $OX = OY = OZ$ since the length of each wire is equal.
Let $x = OX = OY = OZ$.
Since each wire has total length 100, then $XM = YN = ZP = 100 - x$.
Therefore, $M$ is a distance $100 - x$ below $X$. Since the total distance of $M$ below the ceiling is 90, then the vertical distance from the ceiling to $X$ (and thus to the plane of the triangle) is $90 - (100 - x) = x - 10$.
Let $C$ be the centre of $\triangle XYZ$.

By symmetry, $O$ is directly above $C$ with $OC = x - 10$.
Also, $OX = x$.
By the Pythagorean Theorem, $OX^2 = OC^2 + XC^2$.
Therefore, we need to find the length of $XC$.
Draw altitudes from $X$, $Y$ and $Z$ to points $F$, $G$ and $H$ on the opposite sides. Note that $C$ is the point of intersection of $XF$, $YG$ and $ZH$.




Since $\triangle XYZ$ is equilateral, $H$ is the midpoint of $XY$ and $XF$ is the angle bisector of $\angle ZXY$.
Therefore, $XH = 30$ (since $XY = 60$) and $\angle CXH = 30^\circ$.
$\triangle CXH$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle, so $CX = \frac{2}{\sqrt{3}}XH = \frac{2}{\sqrt{3}}(30) = \frac{60\sqrt{3}}{3} = 20\sqrt{3}$.
Since $OX^2 = OC^2 + XC^2$, then

$$\begin{aligned} x^2 &= (x-10)^2 + (20\sqrt{3})^2 \\ x^2 &= x^2 - 20x + 100 + 1200 \\ 20x &= 1300 \\ x &= 65 \end{aligned}$$

Therefore, the distance between the triangle and the ceiling is $x - 10 = 55$ cm.

ANSWER: (D)

24. Suppose that the $y$-intercept of $PR$ is $b$ with $b > 0$.
Since $PR$ has slope 1, then $PR$ has $x$-intercept $-b$, so the coordinates of $P$ are $(-b, 0)$.
Since $P$, $Q$ and $R$ lie on a line with $PQ = QR$, then the horizontal distance from $P$ to $Q$ equals the horizontal distance to $Q$ to $R$. In other words, the difference between the $x$-coordinates of $Q$ and $P$ equals that of $R$ and $Q$.
Since $PR$ has slope 1 and $y$-intercept $b$, it has equation $y = x + b$.
We can determine the $x$-coordinates of $Q$ and $R$ by determining the points of intersection of $y = x^2$ and $y = x + b$, which we get by solving

$$\begin{aligned} x^2 &= x + b \\ x^2 - x - b &= 0 \\ x &= \frac{1 \pm \sqrt{1 - 4(1)(-b)}}{2} \qquad \text{(by the quadratic formula)} \\ x &= \frac{1 \pm \sqrt{1+4b}}{2} \end{aligned}$$

From the diagram, the $x$-coordinate of $Q$ is $\dfrac{1 - \sqrt{1+4b}}{2}$ and the $x$-coordinate of $R$ is $\dfrac{1 + \sqrt{1+4b}}{2}$.

From the given information,

$$\begin{aligned}
\frac{1-\sqrt{1+4b}}{2}-(-b) &= \frac{1+\sqrt{1+4b}}{2}-\frac{1-\sqrt{1+4b}}{2} \\
\frac{1-\sqrt{1+4b}}{2}+b &= \sqrt{1+4b} \\
1-\sqrt{1+4b}+2b &= 2\sqrt{1+4b} \\
1+2b &= 3\sqrt{1+4b} \\
(1+2b)^2 &= (3\sqrt{1+4b})^2 \\
1+4b+4b^2 &= 9(1+4b) \\
4b^2-32b-8 &= 0 \\
b^2-8b-2 &= 0 \\
b &= \frac{8\pm\sqrt{8^2-4(1)(-2)}}{2} \\
b &= \frac{8\pm\sqrt{72}}{2} \\
b &= 4\pm 3\sqrt{2}
\end{aligned}$$

Since $b>0$, then $b=4+3\sqrt{2}\approx 8.243$.
(In this problem, it is in fact possible to determine which of the given answers is correct using a carefully drawn scale diagram.)

Answer: (C)

25. Consider arranging the $b+g$ balls in a row by first randomly choosing one ball placing it in the leftmost position, then randomly choosing another ball and placing it in the rightmost position, and then the choosing and placing the rest of the balls.
For both end balls to be black, the first ball must be black (the probability of this $\frac{b}{b+g}$) and then the second ball must be black (there are $b+g-1$ balls remaining of which $b-1$ is black so the probability is $\frac{b-1}{b+g-1}$), and the remaining balls can be placed in any way. Thus, the probability that both end balls are black is $\frac{b}{b+g}\cdot\frac{b-1}{b+g-1}$.
Similarly, the probability that both end balls are gold is $\frac{g}{b+g}\cdot\frac{g-1}{b+g-1}$.
Thus, the probability that both end balls are the same colour is

$$\frac{b}{b+g}\cdot\frac{b-1}{b+g-1}+\frac{g}{b+g}\cdot\frac{g-1}{b+g-1}$$

which we know should be equal to $\frac{1}{2}$.
Thus,

$$\begin{aligned}
\frac{b}{b+g}\cdot\frac{b-1}{b+g-1}+\frac{g}{b+g}\cdot\frac{g-1}{b+g-1} &= \frac{1}{2} \\
b(b-1)+g(g-1) &= \frac{1}{2}(b+g)(b+g-1) \\
2b^2-2b+2g^2-2g &= b^2+g^2+2bg-b-g \\
b^2-2bg+g^2 &= b+g \\
(g-b)^2 &= b+g
\end{aligned}$$

Set $g - b = k$. (Since $g \geq b$, then $k \geq 0$.)
Therefore, $b + g = k^2$.
Adding these two equations, we obtain $2g = k^2 + k$, so $g = \frac{1}{2}(k^2 + k) = \frac{1}{2}k(k + 1)$.
Subtracting these two equations, we obtain $2b = k^2 - k$, so $b = \frac{1}{2}k(k - 1)$. (This tells us that $g$ and $b$ are consecutive triangular numbers.)
Since $b \geq 4$, then $k(k - 1) \geq 8$ so $k \geq 4$.
(If $k = 3$, the left side, which increases with $k$, equals 6; if $k = 4$, the left side equals 12.)
Since $g \leq 2007$, then $k(k + 1) \leq 4014$ so $k \leq 62$.
(If $k = 63$, the left side, which increases with $k$, equals 4032; if $k = 62$, the left side equals 3906.)
Therefore, $4 \leq k \leq 62$, so there are $62 - 4 + 1 = 59$ possible values for $k$, and so 59 possible pairs $(b, g)$.

ANSWER: (E)

Canadian
Mathematics
Competition
An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2006 Fermat Contest

(Grade 11)

**Wednesday, February 22, 2006**

*Solutions*

1. Calculating, $\dfrac{1}{4\times 5}=\dfrac{1}{20}=0.05$.

   Answer: (B)

2. Since $2x+3x+4x=12+9+6$, then $9x=27$ or $x=3$.

   Answer: (B)

3. Calculating each of the numerator and denominator first, $\dfrac{4^3}{10^2-6^2}=\dfrac{64}{100-36}=\dfrac{64}{64}=1$.

   Answer: (A)

4. Calculating from the inside out, $\left(\sqrt{\sqrt{9}+\sqrt{1}}\right)^4=\left(\sqrt{3+1}\right)^4=\left(\sqrt{4}\right)^4=2^4=16$.

   Answer: (C)

5. Since the edge lengths of three cubes are 4, 5 and 6, then their volumes are $4^3=64$, $5^3=125$, and $6^3=216$, respectively.
   Thus, their mean volume is $\dfrac{64+125+216}{3}=\dfrac{405}{3}=135$.

   Answer: (E)

6. The discount on the T-shirt is 30% of \$25, or $0.3\times \$25=\$7.50$.
   The discount on the jeans is 10% of \$75, or $0.1\times \$75=\$7.50$.
   So the total discount is $\$7.50+\$7.50=\$15$.

   Answer: (A)

7. If $\sqrt{2^3\times 5\times p}$ is an integer, then $2^3\times 5\times p$ is a perfect square.
   For $2^3\times 5\times p$ to be a perfect square, each prime factor must occur an even number of times.
   For $p$ to be as small as possible, $p$ must have at least one factor of 2 and at least one factor of 5.
   Thus, the smallest possible value of $p$ is 10.
   (Checking, $2^3\times 5\times 10=400$, which is a perfect square.)

   Answer: (C)

8. From the given information, $P+Q=16$ and $P-Q=4$.
   Adding these two equations, we obtain $P+Q+P-Q=16+4$ or $2P=20$ or $P=10$.

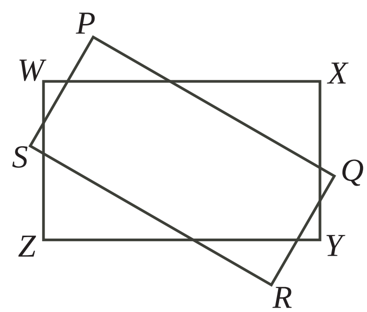
   Answer: (D)

9. *Solution 1*
   Since $\angle FAB$ is an external angle of $\triangle ABC$, then $\angle FAB=\angle ABC+\angle ACB$ or $70^\circ=(x^\circ+20^\circ)+(20^\circ+x^\circ)$ or $70=2x+40$ or $x=15$.

   *Solution 2*
   Since $\angle FAB=70^\circ$, then $\angle BAC=110^\circ$.
   Looking at $\triangle ABC$, we have $\angle ABC+\angle BCA+\angle CAB=180^\circ$, or $(x^\circ+20^\circ)+(20^\circ+x^\circ)+110^\circ=180^\circ$ or $150+2x=180$ or $x=15$.

   Answer: (A)

10. Consider rectangles $WXYZ$ and $PQRS$.
Each of the four sides of $PQRS$ can intersect at most 2 of the sides of $WXYZ$, as any straight line can intersect at most two sides of a rectangle.
Therefore, the maximum possible number of points of intersection between the two rectangles is 8.
8 points of intersection is possible, as we can see in the diagram:




So the maximum possible number of points of intersection is 8.

ANSWER: (D)

11. *Solution 1*
Since $\dfrac{a}{b} = 3$, then $a = 3b$. Since $\dfrac{b}{c} = 2$, then $b = 2c$.
Since $a = 3b$ and $b = 2c$, then $a = 6c$.
Therefore, $\dfrac{a-b}{c-b} = \dfrac{6c-2c}{c-2c} = \dfrac{4c}{-c} = -4$.

*Solution 2*
We divide the numerator and denominator of the given expression by $b$ to obtain $\dfrac{a-b}{c-b} = \dfrac{\frac{a}{b}-1}{\frac{c}{b}-1}$.

Since $\dfrac{b}{c} = 2$, then $\dfrac{c}{b} = \dfrac{1}{2}$.
Therefore, $\dfrac{a-b}{c-b} = \dfrac{3-1}{\frac{1}{2}-1} = \dfrac{2}{-\frac{1}{2}} = -4$.

*Solution 3*
Try $c = 1$. Since $\dfrac{b}{c} = 2$, then $b = 2$. Since $\dfrac{a}{b} = 3$, then $a = 6$.
Therefore, $\dfrac{a-b}{c-b} = \dfrac{6-2}{1-2} = \dfrac{4}{-1} = -4$.

ANSWER: (A)

12. *Solution 1*
The left side of the given equation equals $(2^4)(3^6) = 16(729) = 11\,664$.
Thus, $9(6^x) = 11\,664$ or $6^x = 1296$.
Since $6^4 = 1296$, then $x = 4$.

*Solution 2*
Rearranging the left side, $(2^4)(3^6) = (2^4)(3^4)(3^2) = (2 \times 3)^4(3^2) = 6^4(9) = 9(6^4)$.
Comparing with $9(6^x)$, we see that $x = 4$.

ANSWER: (C)

13. Let $c$ cents be the cost of downloading 1 song in 2005.
Then the cost of downloading 1 song in 2004 was $c + 32$ cents.
The total cost in 2005 was $360c$ and the total cost in 2004 was $200(c + 32)$.
Thus, $360c = 200(c + 32)$ or $160c = 6400$ or $c = 40$ cents, and so the total cost in 2005 was $360(40) = 14400$ cents, or \$144.00.

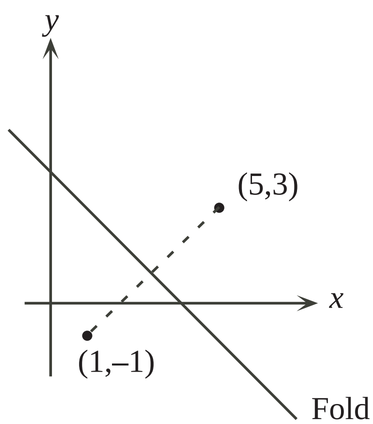
Answer: (A)

14. *Solution 1*
Adding the two given equations, we get $px + 3x = 46$ or $(p + 3)x = 46$.
Since $(x, y) = (2, -4)$ is a solution to both equations, then $x = 2$ satisfies the equation $(p + 3)x = 46$.
Thus, $2(p + 3) = 46$ or $p + 3 = 23$ or $p = 20$.

*Solution 2*
Substituting $(x, y) = (2, -4)$ into the second equation, we obtain $3(2) - q(-4) = 38$ or $6 + 4q = 38$ or $q = 8$.
The first equation thus becomes $px + 8y = 8$.
Substituting $(x, y) = (2, -4)$ into this new first equation, we obtain $p(2) + 8(-4) = 8$ or $2p - 32 = 8$ or $p = 20$.

Answer: (B)

15. Since the point $(5, 3)$ lands on the point $(1, -1)$ when folded, then the fold line must pass through the midpoint of these two points, namely $\left(\frac{1}{2}(5 + 1), \frac{1}{2}(3 + (-1))\right) = (3, 1)$.




Of the given possibilities, $(3, 1)$ lies only on the line $y = -x + 4$, so (D) is the answer.
(In fact, the fold line must be the perpendicular bisector of the line segment through $(5, 3)$ and $(1, -1)$. The slope of the line segment through $(5, 3)$ and $(1, -1)$ is $\dfrac{3 - (-1)}{5 - 1} = 1$, so the perpendicular bisector has slope $-1$.
Since the perpendicular bisector has slope $-1$ and passes through $(3, 1)$, then it has equation $y = -x + 4$.)

Answer: (D)

16. Since the areas of the circle and the shaded region are equal, then each is equal to half of the area of the entire rectangle, so each is equal to $\frac{1}{2}(8 \times 9) = 36$.
If the radius of the circle is $r$, then $\pi r^2 = 36$, so $r^2 = \frac{36}{\pi}$ or $r = \sqrt{\frac{36}{\pi}} = \frac{\sqrt{36}}{\sqrt{\pi}} = \frac{6}{\sqrt{\pi}}$.

Answer: (C)

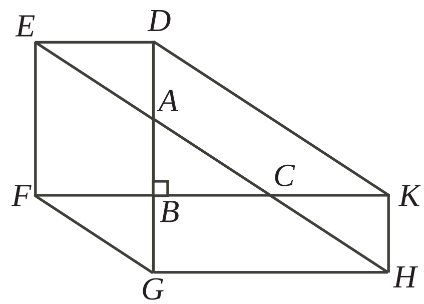
17. Since each term after the third is the sum of the preceding three terms, then, looking at the fourth term, $13 = 5 + p + q$ or $p + q = 8$.
Looking at the fifth term, $r = p + q + 13 = 8 + 13 = 21$.
Looking at the seventh term, $x = 13 + r + 40 = 13 + 21 + 40 = 74$.

Answer: (D)

18. If Georgina cycles for 6 minutes, then she cycles for $\frac{1}{10}$ of an hour.
If Georgina cycles for $\frac{1}{10}$ of an hour at 24 km/h, then she cycles a distance of 2.4 km or 2400 m.
Since the diameter of Georgina's front wheel is 0.75 m, then its circumference is $\pi d = 0.75\pi$ m.
With each rotation of the front wheel, Georgina will move $0.75\pi$ m.
So if Georgina travels 2400 m, then her front wheel rotates $\dfrac{2400}{0.75\pi} \approx 1018.59$ times, which is closest to 1020 times.

Answer: (B)

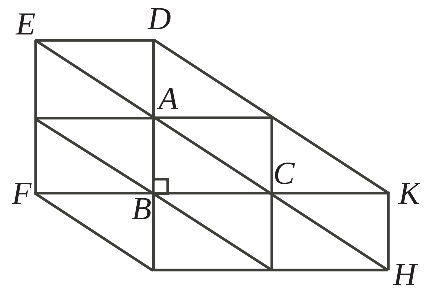
19. *Solution 1*




Let the area of $\triangle ABC$ be $x$.
We break up hexagon $DEFGHK$ into a number of pieces and calculate the area of each piece in terms of $x$.
Consider $\triangle ADE$. Since $AD = AB$, $AE = AC$ and $\angle DAE = \angle CAB$, then $\triangle ADE$ is congruent to $\triangle ABC$, so the area of $\triangle ADE$ is $x$.
Similarly, the area of each of $\triangle BGF$ and $\triangle CKH$ is equal to $x$.
Consider quadrilateral $AEFB$.
If we join this quadrilateral to $\triangle ABC$, we form $\triangle CFE$.
Since $AE = AC$, then $CE = 2CA$; similarly, $CF = 2CB$.
Since $\triangle CFE$ and $\triangle CBA$ share an angle at $C$ and have two pairs of corresponding sides enclosing this angle in the same ratio, then $\triangle CFE$ is similar to $\triangle CBA$.
Now, the side lengths of $\triangle CFE$ are twice those of $\triangle CBA$, so the area of $\triangle CFE$ is $2^2 = 4$ times that of $\triangle CBA$, so is $4x$.
Thus, the area of quadrilateral $AEFB$ is $3x$.
Similarly, the areas of quadrilaterals $ADKC$ and $BCHG$ are $3x$.
Therefore, the area of hexagon $DEFGHK$ equals the sum of the areas of triangles $ABC$, $ADE$, $BGF$, and $CKH$, and of quadrilaterals $AEFB$, $ADKC$ and $BCHG$, so equals
$4x + 3(3x) = 13x$.
Hence, the ratio of the ratio of hexagon $DEFGHK$ to the area of $\triangle ABC$ is $13 : 1$.

*Solution 2*

We can triangulate hexagon $DEFGHK$ by drawing vertical line segments of length equal to that of $AB$, horizontal line segments of length equal to that of $BC$, and slanting line segments of length equal to that of $AC$.




Thus, we have triangulated $DEFGHK$ into 13 congruent triangles. (We can argue that each of these triangles is congruent to $\triangle ABC$ by observing that each has two perpendicular sides and noting that each has at least two sides easily seen to be equal in length to the corresponding sides in $\triangle ABC$.)
Therefore, the area of $DEFGHI$ is 13 times that of the area of $\triangle ABC$, so the ratio of the areas is 13 : 1.

Answer: (E)

20. *Solution 1*
Suppose that Igor has removed some balls from the bag, and the remaining balls do not satisfy the required condition. What is the maximum number of balls that can remain? In order to not satisfy the required condition, either there are not 4 balls of any colour (so the maximum number is 9 balls, ie. 3 of each colour) or there are at least 4 balls of one colour, but there are not 3 of either of the other colours.
In this second case, we could have 2 balls of each of two colours, and as many as possible of the third colour. The maximum number of balls of any colour that can be in the bag is 8 (the number of yellow balls with which Igor starts). So the maximum number of balls still in the bag in this case is 12.
Therefore, if Igor removes 8 or more balls, then the remaining balls might not satisfy the required condition.
However, if Igor removes 7 or fewer balls, then the remaining balls will satisfy the required condition, since the maximum number of balls in any case which does not satisfy the condition is 12.
Therefore, the maximum possible value of $N$ is 7.

*Solution 2*
Since we want to determine the maximum possible value of $N$, we start with the largest of the answers and rule out answers until we come to the correct answer.
If Igor removed 10 marbles, he might remove 5 red and 5 black marbles, leaving 8 yellow, 2 red, and 0 black marbles, which does not meet the required condition.
Thus, 10 is not the answer.
If Igor removed 9 marbles, he might remove 5 red and 4 black marbles, leaving 8 yellow marbles, 2 red marbles, and 1 black marble, which does not meet the required condition.
Thus, 9 is not the answer.
If Igor removed 8 marbles, he might remove 5 red and 3 black marbles, leaving 8 yellow, 2 red, and 2 black marbles, which does not meet the required condition.
Thus, 8 is not the answer.

Is 7 the answer?
There are $8 + 7 + 5 = 20$ marbles to begin with. If 7 are removed, there are 13 marbles left.

Since there are 13 marbles left, then it is not possible to have 4 or fewer marbles of each of the three colours (otherwise there would be at most 12 marbles). Thus, there are at least 5 marbles of one colour.
Could there be 2 or fewer marbles of each of the other two colours? If so, then since there are 13 marbles in total, there must be at least 9 marbles of the first colour. But there cannot be 9 or more marbles of any colour, as there were at most 8 of each colour to begin with. Therefore, there must be at least 3 of one of the other two colours of marbles.
This tells us that if 7 marbles are removed, there are at least 5 marbles of one colour and 3 of another colour, so choosing $N = 7$ marbles guarantees us the required condition.
Therefore, 7 is the maximum possible value of $N$.

ANSWER: (B)

21. If $n$ is an odd integer, then each of $n-1$ and $n+1$ is even.
In fact, $n-1$ and $n+1$ are consecutive even integers, so one is a multiple of 4 and the other is divisible is 2 (since it is even).
Thus, $(n-1)(n+1)$ contains at least 3 factors of 2, which tells us that $(n-1)(n)(n+1)$ does as well, ie. is divisible by 8.
So if $n$ is an odd integer, then $\dfrac{(n-1)(n)(n+1)}{8}$ is an integer.
(There are 39 odd integers between 2 and 80, inclusive.)
If $n$ is an even integer, then each of $n-1$ and $n+1$ is odd.
Thus, $(n-1)(n)(n+1)$ is divisible by 8 only when $n$ is divisible by 8.
(There are 10 multiples of 8 between 2 and 80, inclusive.)
Therefore, there are $39 + 10 = 49$ integers $n$, with $2 \leq n \leq 80$, such that $\dfrac{(n-1)(n)(n+1)}{8}$ is an integer.

ANSWER: (E)

22. First, we do some experimentation.
Since Celine moves small boxes faster and Quincy moves large boxes faster, then suppose Celine moves all 16 small boxes (taking 32 minutes) and Quincy moves all 10 large boxes (taking 50 minutes). Thus, they would finish in 50 minutes.
We could transfer two large boxes from Quincy to Celine, who now moves 16 small and 2 large boxes, taking 44 minutes. Quincy would then move 8 large boxes, taking 40 minutes. So they would finish in 44 minutes. (So (E) is not the answer.)
If we transfer one small box from Celine to Quincy, then Quincy moves 8 large boxes and 1 small box, taking 43 minutes, and Celine moves 15 small and 2 large boxes, taking 42 minutes. So they would finish in 43 minutes. (So (D) is not the answer.)

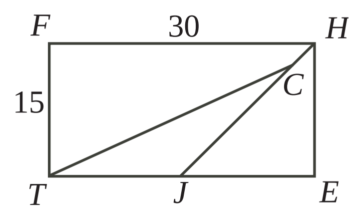
Why is 43 minutes the smallest possible total time?
Suppose that it took them at most 42 minutes to finish the job. Then the total amount of working time would be at most 84 minutes.
Suppose that Celine moves $x$ small boxes and $y$ large boxes, which would take $2x+6y$ minutes. Then Quincy moves $16-x$ small boxes and $10-y$ large boxes, which would take $3(16-x) + 5(10-y) = 98 - 3x - 5y$ minutes.
Since the total working time is at most 84 minutes, then $(2x + 6y) + (98 - 3x - 5y) \leq 84$ or $14 \leq x - y$.
Since $0 \leq x \leq 16$ and $0 \leq y \leq 10$, then the possible pairs of $x$ and $y$ are $(16, 0)$, $(16, 1)$, $(16, 2)$, $(15, 0)$, $(15, 1)$, $(14, 0)$, which produce working times as follows:

| $x$ | $y$ | Celine Small | Celine Large | Celine Time | Quincy Small | Quincy Large | Quincy Time |
|---|---|---|---|---|---|---|---|
| 16 | 0 | 16 | 0 | 32 | 0 | 10 | 50 |
| 16 | 1 | 16 | 1 | 38 | 0 | 9 | 45 |
| 16 | 2 | 16 | 2 | 44 | 0 | 8 | 40 |
| 15 | 0 | 15 | 0 | 30 | 1 | 10 | 53 |
| 15 | 1 | 15 | 1 | 36 | 1 | 9 | 48 |
| 14 | 0 | 14 | 0 | 28 | 2 | 10 | 56 |

In each of these cases, while the total working time is no more than 84 minutes, it takes longer than 43 minutes to finish.
Therefore, it is impossible for them to finish in 42 minutes or less, so the earliest possible finishing time is 9:43 a.m.

ANSWER: (C)

23. *Solution 1*
Let $t$ be the amount of time in seconds that it takes Tom to catch Jerry.




Then $TC = 5t$, since Tom runs at 5 m/s for $t$ seconds.
Also, $JC = 3t$, since Jerry runs at 3 m/s.
We know as well that $J$ is the midpoint of $TE$, so $TJ = 15 = JE$.
Since $HE = 15$, then $\angle HJE = 45°$, so $\angle TJC = 135°$.
By the cosine law,

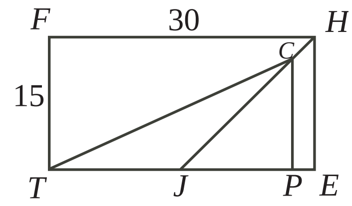
$$\begin{aligned}
TC^2 &= TJ^2 + JC^2 - 2(TJ)(JC)\cos(\angle TJC) \\
(5t)^2 &= 15^2 + (3t)^2 - 2(15)(3t)\cos(135°) \\
25t^2 &= 225 + 9t^2 - 90t\left(-\tfrac{1}{\sqrt{2}}\right) \\
16t^2 - 45\sqrt{2}t - 225 &= 0
\end{aligned}$$

Using the quadratic formula, since $t$ must be positive, we obtain

$$t = \frac{45\sqrt{2} + \sqrt{(45\sqrt{2})^2 - 4(16)(-225)}}{2(16)} = \frac{45\sqrt{2} + \sqrt{18450}}{32} \approx 6.23 \text{ seconds}$$

so it takes Tom about 6.2 seconds to catch Jerry.

*Solution 2*
Let $t$ be the amount of time in seconds that it takes Tom to catch Jerry.
Then $TC = 5t$, since Tom runs at 5 m/s for $t$ seconds.
Also, $JC = 3t$, since Jerry runs at 3 m/s.
We know as well that $J$ is the midpoint of $TE$, so $TJ = 15 = JE$.
Since $HE = 15$, then $\angle HJE = 45°$.
Drop a perpendicular from $C$ to $P$ on $JE$.




Since $CJ = 3t$, $\angle CJE = 45^\circ$ and $\angle CPJ = 90^\circ$, then $JP = CP = \frac{1}{\sqrt{2}}(3t)$.
By the Pythagorean Theorem in $\triangle CPT$, we obtain

$$\begin{aligned} TC^2 &= (TJ + JP)^2 + CP^2 \\ (5t)^2 &= \left(15 + \tfrac{1}{\sqrt{2}}(3t)\right)^2 + \left(\tfrac{1}{\sqrt{2}}(3t)\right)^2 \\ 25t^2 &= 225 + 2(15)\left(\tfrac{1}{\sqrt{2}}(3t)\right) + \tfrac{1}{2}(9t^2) + \tfrac{1}{2}(9t^2) \\ 25t^2 &= 225 + 45\sqrt{2}t + 9t^2 \\ 16t^2 - 45\sqrt{2}t - 225 &= 0 \end{aligned}$$

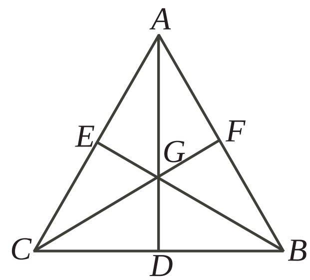
Proceeding exactly as in Solution 1, we obtain that the time is closest to 6.2 seconds.

Answer: (E)

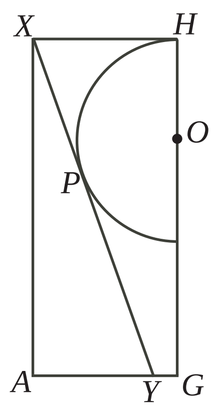
24. Since $\frac{1}{a} + \frac{1}{2a} + \frac{1}{3a} = \frac{1}{b^2 - 2b}$, then $\frac{6}{6a} + \frac{3}{6a} + \frac{2}{6a} = \frac{1}{b^2 - 2b}$ or $\frac{11}{6a} = \frac{1}{b^2 - 2b}$.
Cross-multiplying, we obtain $11(b^2 - 2b) = 6a$.
Since 11 is a divisor of the left side, then 11 must be a divisor of the right side, that is, $a$ must be divisible by 11.
Thus, let $a = 11A$, with $A$ a positive integer.
So we get $11(b^2 - 2b) = 6(11A)$ or $b^2 - 2b = 6A$.
Since 6 is a divisor of the right side, then 6 must be a divisor of the left side.
What is the smallest positive integer $b$ for which 6 is a divisor of $b^2 - 2b$? We can quickly check that if $b$ equals 1, 2, 3, 4, or 5, then $b^2 - 2b$ is not divisible by 6, but if $b = 6$, then $b^2 - 2b$ is divisible by 6.
Therefore, for the smallest values of $a$ and $b$, we must have $b = 6$, so $6A = 6^2 - 2(6) = 24$, whence $A = 4$ and so $a = 11A = 44$.
Thus, the smallest possible value of $a + b$ is $44 + 6 = 50$.

Answer: (E)

25. First, join the centres of the bases of the three cones together.
Since the radius of the base of each cone is 50, then the distance between the centre of one base and the centre of each of the other bases is $2(50) = 100$ since the three circular bases are mutually tangent and the lines connecting their centres will pass through these points of tangency.
Thus, the triangle formed by the centres of the three bases is equilateral, with side length 100.
Now, by symmetry, the sphere will sit "in the centre" of the three cones, so the centre of the sphere will lie directly above the centroid of this triangle.
How far is the centroid from each of the vertices?
Consider equilateral triangle $ABC$ with side length 100.
Draw in the three medians $AD$, $BE$, $CF$ (each of which is also an altitude and an angle bisector), which intersect at $G$.




Since $BD = \frac{1}{2}BC = 50$ and $\angle GBD = \frac{1}{2}\angle ABC = 30^\circ$, then $\triangle BGD$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle, so $BG = \frac{2}{\sqrt{3}}BD = \frac{100}{\sqrt{3}}$.
So the distance between each vertex of the triangle and the centroid is $\frac{100}{\sqrt{3}}$.
This tells us that the distance between the axis of each cone and the vertical line through the centre of the sphere is also $\frac{100}{\sqrt{3}}$.
Draw a vertical cross-section through the axis of one of the cones and the centre of the sphere, including half of the cone and half of the sphere.




Let $A$ be the centre of the base of the cone (the bottom left in the diagram), $G$ the centroid of the triangle formed by the centres, $O$ the centre of the sphere, $P$ the point where the sphere touches the cone, $X$ the vertex of the cone, $Y$ the point where the cone meets $AG$, and $H$ the point where the sphere touches the plane through the vertices of the three cones.
Let $r$ be the radius of the sphere.

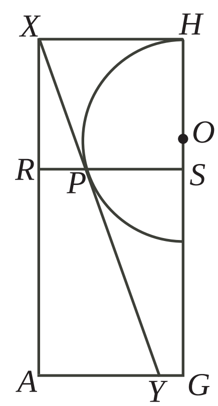
<u>Method 1</u>
Then we know $OH = OP = r$, $XH = AG = \frac{100}{\sqrt{3}}$ and $XA = HG = 120$.
Since the sphere is tangent to the cone at $P$, then $OP$ is perpendicular to $XY$.
Since both $XH$ and $XP$ are tangent to the sphere, then $XP = XH = \frac{100}{\sqrt{3}}$.
By the Pythagorean Theorem, $XY^2 = AY^2 + AX^2 = 50^2 + 120^2 = 16900 = 130^2$, so $XY = 130$.
Thus, $PY = XY - XP = 130 - \frac{100}{\sqrt{3}}$.
Also, $OG = 120 - r$ and $GY = AG - AY = \frac{100}{\sqrt{3}} - 50$.

By the Pythagorean Theorem twice, $OP^2 + PY^2 = OY^2 = GY^2 + GO^2$, so

$$\begin{aligned} r^2 + \left(130 - \tfrac{100}{\sqrt{3}}\right)^2 &= \left(\tfrac{100}{\sqrt{3}} - 50\right)^2 + (120 - r)^2 \\ r^2 + 130^2 - \frac{2(130)(100)}{\sqrt{3}} + \frac{10000}{3} &= \frac{10000}{3} - \frac{2(50)(100)}{\sqrt{3}} + 50^2 + 120^2 - 240r + r^2 \\ 240r &= \frac{2(130-50)(100)}{\sqrt{3}} \qquad (\text{since } 130^2 = 50^2 + 120^2) \\ r &= \frac{200}{3\sqrt{3}} = \frac{200\sqrt{3}}{9} \end{aligned}$$

Evaluating, $r = \dfrac{200\sqrt{3}}{9} \approx 38.49$, so the radius is closest to 38.5.
(We could have instead calculated the radius using trigonometry, since we can calculate $\angle YXA$ using the ratio of $AY$ to $AX$, and thus can calculate $\angle PXH$.
So $\tan\left(\frac{1}{2}\angle PXH\right) = \tan(\angle OXH) = \dfrac{OH}{XH} = \dfrac{r}{\frac{100}{\sqrt{3}}}$.)

Method 2
We know that $XH = AG = \frac{100}{\sqrt{3}}$.
Since $AX = 120$ and $AY = 50$, then the slope of $XY$ is $\frac{-120}{50} = -\frac{12}{5}$.
Since the sphere is tangent to the cone at $P$, then $OP$ is perpendicular to $XY$, so has slope $\frac{5}{12}$.
Draw a horizontal line through $P$ cutting $AX$ at $R$ and $GH$ at $S$.




Since $OP$ has slope $\frac{5}{12}$, then we can let $OS = 5t$ and $SP = 12t$ for some real number $t$.
But $\triangle OSP$ is right-angled at $S$, so $OP = 13t$ by the Pythagorean Theorem.
Also, $OH$ is a radius of the circle, so $OH = OP = 13t$, so $XR = HS = HO + OS = 18t$.
But the slope of $XP$ is $-\frac{12}{5}$, so if $XR = 18t$, then $RP = \frac{5}{12}(18t) = \frac{15}{2}t$.
Therefore, $RS = RP + PS = \frac{15}{2}t + 12t = \frac{39}{2}t$ and is equal to $AG$ which equals $\frac{100}{\sqrt{3}}$.
So $\frac{39}{2}t = \frac{100}{\sqrt{3}}$ and so the radius of the sphere, which is $13t$, equals $\dfrac{2}{3}\left(\dfrac{100}{\sqrt{3}}\right) = \dfrac{200\sqrt{3}}{9} \approx 38.49$, so the radius is closest to 38.5.

Answer: (D)

**Canadian Mathematics Competition**

*An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario*

# 2005 Fermat Contest

**(Grade 11)**

**Wednesday, February 23, 2005**

*Solutions*

1. Calculating, $\dfrac{150+(150\div 10)}{15-5}=\dfrac{150+15}{10}=\dfrac{165}{10}=16.5$.

   Answer: (E)

2. Since $\frac{3}{9}=\frac{1}{3}$, then $\frac{1}{2}-\frac{1}{3}+\frac{3}{9}=\frac{1}{2}$.

   Answer: (B)

3. *Solution 1*

   Since $a=\frac{1}{2}$ and $b=\frac{2}{3}$, then $\dfrac{6a+18b}{12a+6b}=\dfrac{6(\frac{1}{2})+18(\frac{2}{3})}{12(\frac{1}{2})+6(\frac{2}{3})}=\dfrac{3+12}{6+4}=\dfrac{15}{10}=\dfrac{3}{2}$.

   *Solution 2*

   Since $a=\frac{1}{2}$ and $b=\frac{2}{3}$, then $\dfrac{6a+18b}{12a+6b}=\dfrac{6(a+3b)}{6(2a+b)}=\dfrac{a+3b}{2a+b}=\dfrac{\frac{1}{2}+3(\frac{2}{3})}{2(\frac{1}{2})+\frac{2}{3}}=\dfrac{2\frac{1}{2}}{1\frac{2}{3}}=\dfrac{\frac{5}{2}}{\frac{5}{3}}=\dfrac{3}{2}$.

   Answer: (E)

4. Since $\sqrt{4+9+x^2}=7$, then $4+9+x^2=7^2$ or $13+x^2=49$ or $x^2=36$.
   Therefore, the possible values for $x$ are $x=\pm 6$. Therefore, the answer is (A).

   Answer: (A)

5. After the coin has rolled from $P$ to $Q$, the F on the face of the coin has rotated $270^\circ$ clockwise. Therefore, since the distance from $Q$ to $R$ equals the distance from $P$ to $Q$, then the F will rotate another $270^\circ$ clockwise, and so the orientation of the coin will be .

   Answer: (C)

6. The sequence repeats every 4 terms.
   How many times will the pattern $1,2,3,4$ occur in the first 2005 terms?
   Since 2005 divided by 4 gives a quotient of 501 and a remainder of 1, then the first 2005 terms contain the pattern $1,2,3,4$ a total of 501 times (ending at the 2004th term).
   Also, the 2005th term is a 1.
   Therefore, the sum of the first 2005 terms is $501(1+2+3+4)+1=501(10)+1=5011$.

   Answer: (A)

7. We are told that $\angle A=\angle B+21^\circ$ and $\angle C=\angle B+36^\circ$.
   Since the sum of the angles in a triangle is $180^\circ$, then

$$\begin{aligned}\angle A+\angle B+\angle C &= 180^\circ\\ \angle B+21^\circ+\angle B+\angle B+36^\circ &= 180^\circ\\ 3(\angle B)+57^\circ &= 180^\circ\\ 3(\angle B) &= 123^\circ\\ \angle B &= 41^\circ\end{aligned}$$

   Answer: (B)

8. *Solution 1*

   Since the seven children were born in seven consecutive years, then the oldest child is 4 years older than the oldest of the three youngest children, the second oldest child is 4 years older than the second oldest of the three youngest children, and the third oldest child is 4 years older than the youngest child.
   Since the sum of the ages of the three youngest children is 42, then the sum of the ages of the

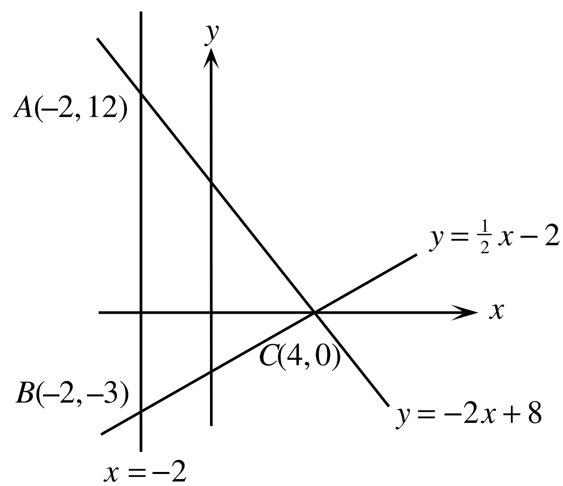
three oldest children is $42 + 4 + 4 + 4 = 54$.

*Solution 2*

Since the ages of the seven children are seven consecutive integers, let the ages of the youngest three children be $x$, $x + 1$ and $x + 2$.
Then $x + x + 1 + x + 2 = 42$ or $3x + 3 = 42$ or $x = 13$.
So the ages of the seven children are 13, 14, 15, 16, 17, 18, and 19.
Therefore, the sum of the ages of the oldest three children is $17 + 18 + 19 = 54$.

ANSWER: (B)

9. We first determine where the lines $y = -2x + 8$ and $y = \frac{1}{2}x - 2$ cross the line $x = -2$.
For the line $y = -2x + 8$, when $x = -2$, $y = -2(-2) + 8 = 12$, so the point of intersection is $(-2, 12)$.
For the line $y = \frac{1}{2}x - 2$, when $x = -2$, $y = \frac{1}{2}(-2) - 2 = -3$, so the point of intersection is $(-2, -3)$.



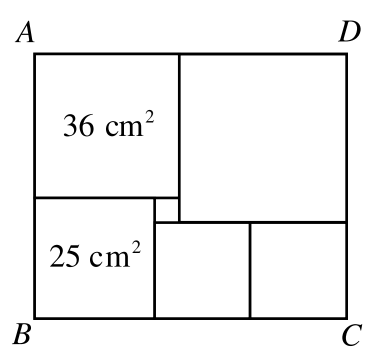

Therefore, we can think of $\triangle ABC$ as having base $AB$ of length $12 - (-3) = 15$ and height being the distance from $C$ to the line segment $AB$, or $4 - (-2) = 6$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(15)(6) = 45$.

ANSWER: (E)

10. Since 50% of $P$ equals 20% of $Q$, then $\frac{1}{2}P = \frac{1}{5}Q$ or $P = \frac{2}{5}Q$.
Therefore, $P$ is 40% of $Q$.

ANSWER: (C)

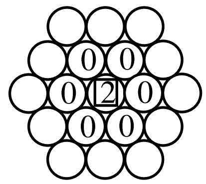
11. Since the area of the top left square is 36 cm$^2$, then its side length is 6 cm.
Since the area of the bottom left square is 25 cm$^2$, then its side length is 5 cm.
Therefore, the height $AB$ of rectangle $ABCD$ is $5 + 6 = 11$ cm.



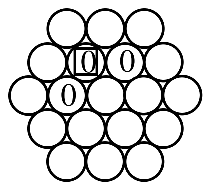
Since the side length of the bottom left square is 5 cm and the side length of the upper left square is 6 cm, then the side length of the small square in the middle must be 1 cm.
This tells us that the side length of the top right square is $6 + 1 = 7$ cm.
Therefore, the width $AD$ of rectangle $ABCD$ is $6 + 7 = 13$ cm.
Thus, the perimeter of rectangle $ABCD$ is $2(13) + 2(11) = 48$ cm.

Answer: (E)

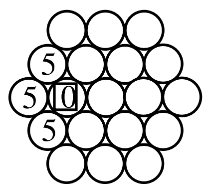
12. From the 2 in the centre, there are 6 possible 0s to which we can move.




From any 0, there are 2 possible 0s to which we can move.




From any 0, there are 3 possible 5s to which we can move.

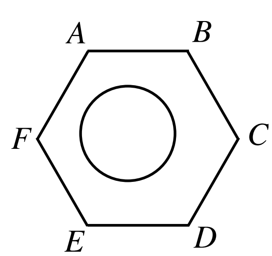



For each of the 6 choices of the first 0, we can choose either of the 2 choices for the second 0, and from whichever second 0 is chosen we can choose any of the 3 possible 5s.
Therefore, there are $6 \times 2 \times 3 = 36$ possible paths that can be followed.

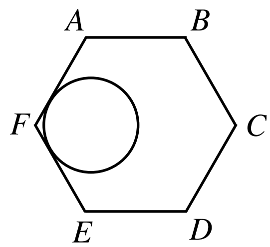
Answer: (A)

13. After fiddling around for a few minutes, we can see that the answer should be 2 sides. While we can convince ourselves that the answer is 2, how can we justify this mathematically? This turns out to be tricky.

Consider hexagon $ABCDEF$ and a circle contained entirely inside it.



Clearly, it is possible for the circle to touch 1 or 2 sides only.



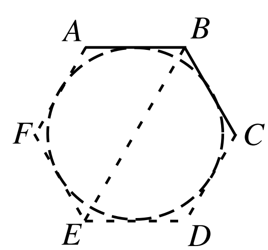

Next, we make a few notes:

- If the circle touches all 6 sides, then its radius is half the distance between opposite sides (eg. $AB$ and $DE$).
- If the circle actually touches one pair of opposite sides, then its radius is half the distance between opposite sides, so in order to be contained entirely inside the hexagon, it must touch all 6 sides.
- If a circle inside the hexagon touches at least 4 sides, then it must touch at least one pair of opposite sides, so must touch all 6 sides. So if the circle touches fewer than six sides, it must touch 1, 2 or 3 sides.

Is it possible for the circle to touch 3 sides and not all 6?
If so, then we need to make sure that the circle does not touch opposite sides.
There are two ways in which this can be done – if the circle touches three consecutive sides (for example, $AB$, $BC$, $CD$) or three sides no two of which are consecutive (for example, $AB$, $CD$, $EF$).

In order to complete our justification, we need to examine these two cases and use the fact that if a circle is tangent to two lines, then its centre must lie on the angle bisector of the angle formed at the point of intersection of these two lines.

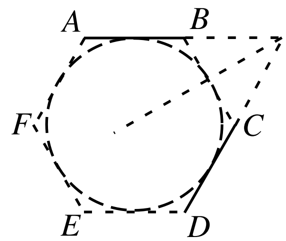
Case 1: Circle touches $AB$, $BC$, $CD$
Since the circle is tangent to $AB$ and $BC$, then its centre must lie on the angle bisector of $\angle ABC$, which is diagonal $BE$ of the hexagon.




Since the circle is tangent to $BC$ and $CD$, then its centre must lie on the angle bisector of $\angle BCD$, which is diagonal $CF$ of the hexagon.
Since the centre lies on $BE$ and on $CF$, then the centre of the circle lies at the centre of the hexagon, and so the circle must touch all 6 sides of the hexagon.

Case 2: Circle touches $AB$, $CD$, $EF$

Since the circle is tangent to $AB$ and $CD$, then its centre must lie on the angle bisector of the angle formed by extending $AB$ and $DC$ to their point of intersection. This angle bisector will be, by symmetry, the perpendicular bisector of $BC$.




Since the circle is tangent to $CD$ and $EF$, then its centre must lie on the angle bisector of the angle formed by extending $CD$ and $EF$ to their point of intersection. This angle bisector will be, by symmetry, the perpendicular bisector of $DE$.
Since the centre lies on the perpendicular bisectors of $BC$ and $DE$, then the centre of the circle lies at the centre of the hexagon, and so the circle must touch all 6 sides of the hexagon.

Therefore, if the circle touches at least 3 sides of the hexagon, then it touches all 6 sides.
Thus, the maximum number of sides it can touch without touching all 6 sides is 2 sides.

(This problem is an example of one where the answer can be obtained relatively quickly, but the justification is quite difficult. We have included this justification for completeness.)

ANSWER: (B)

14. *Solution 1*

Let the weight of the lioness be $L$.
Then the weight of the female cub is $\frac{1}{6}L$ and the weight of the male cub is $\frac{1}{4}L$.
Thus, $\frac{1}{4}L - \frac{1}{6}L = 14$ or $\frac{3}{12}L - \frac{2}{12}L = 14$ or $\frac{1}{12}L = 14$ or $L = 168$.
Therefore, the weight of the lioness is 168 kg.

*Solution 2*

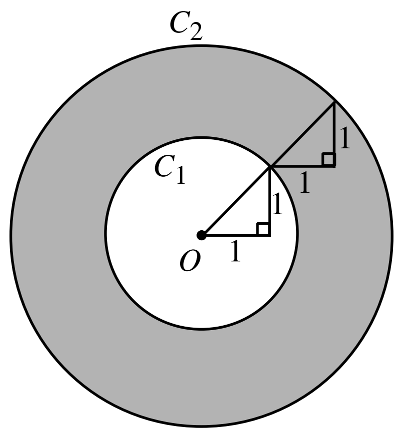
Let the weight of the female cub be $F$.
Then the weight of the male cub is $F + 14$.
Also, the weight of the lioness is equal to both of $6F$ and $4(F + 14)$.
Therefore, $6F = 4F + 56$ or $2F = 56$ or $F = 28$.
Thus, the weight of the lioness is $6(28) = 168$ kg.

ANSWER: (C)

15. Since $(x - 4)(5x + 2) = 0$, then $x - 4 = 0$ or $5x + 2 = 0$.
If $x - 4 = 0$, then $x = 4$, and so $5x + 2 = 22$.
If $5x + 2 = 0$, then $5x + 2 = 0$. (Here, we don't have to actually figure out $x$, which is $x = -\frac{2}{5}$.)
Therefore, the two possible values of $5x + 2$ are 0 and 22.

ANSWER: (C)

16. Each of the two right angled triangles has hypotenuse $\sqrt{1^2+1^2}=\sqrt{2}$, so the radius of $C_1$ is $\sqrt{2}$ and the radius of $C_2$ is $2\sqrt{2}$.




The area of the shaded region is the difference between the areas of $C_2$ and $C_1$, or

$$\pi(2\sqrt{2})^2 - \pi(\sqrt{2})^2 = 8\pi - 2\pi = 6\pi$$

ANSWER: (D)

17. The volume of a cylinder with radius $r$ and height $h$ is $\pi r^2 h$.
When the cylinder with radius 2 cm and height 8 cm is full of water, it contains $\pi \times 2^2 \times 8$ or $32\pi$ cm$^3$ of water.
When this water is poured into the second cylinder, suppose it fills this second cylinder to a depth of $h$.
Then $32\pi = \pi(4^2)h$ or $16\pi h = 32\pi$ or $h = 2$ cm.
Therefore, the depth of water in the second cylinder will be 2 cm.

ANSWER: (B)

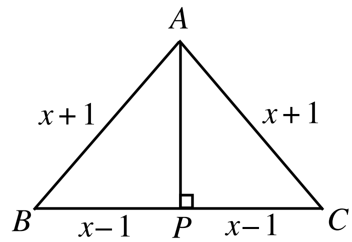
18. A score of 11 points can be obtained with 3 correct, 2 unanswered and 5 wrong.
A score of 13 points can be obtained with 4 correct, 1 unanswered and 5 wrong.
A score of 17 points can be obtained with 5 correct, 2 unanswered and 3 wrong.
A score of 23 points can be obtained with 7 correct, 2 unanswered and 1 wrong.
Therefore, by process of elimination, 29 is the total score which is not possible.
(Why is 29 not possible? If all ten questions are correct, the total score would be 30 points. If 9 or fewer questions are correct, at least 2 points will be lost from the maximum possible, ie. the maximum possible score is 28. Therefore, 29 is not possible.)

ANSWER: (E)

19. Since Chris bicycles at 24 km/h and Sam bicycles at 16 km/h, then Chris gains 8 km/h on Sam.
Since Sam starts 1 km north of Chris, then it takes Chris $\frac{1}{8}$ of an hour, or $\frac{1}{8}\times 60 = \frac{60}{8} = \frac{15}{2}$ or $7\frac{1}{2}$ minutes to catch Sam.

ANSWER: (D)

20. Draw the altitude from $A$ to $P$ on $BC$.
Since $\triangle ABC$ is isosceles, then $P$ is the midpoint of $BC$, so $BP = PC = x - 1$.




By the Pythagorean Theorem,

$$AP = \sqrt{AB^2 - BP^2} = \sqrt{(x+1)^2 - (x-1)^2} = \sqrt{(x^2+2x+1) - (x^2-2x+1)} = \sqrt{4x} = 2\sqrt{x}$$

Therefore, the area of $\triangle ABC$ is equal to

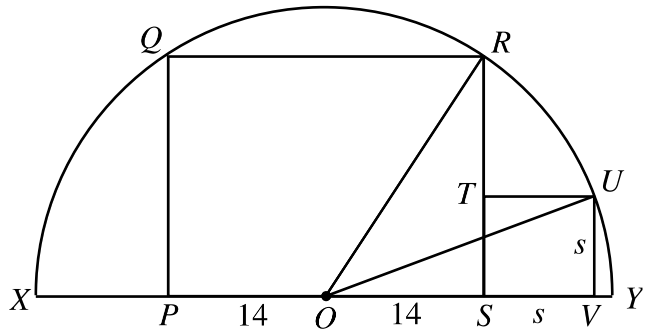
$$\frac{1}{2}(BC)(AP) = \frac{1}{2}(2x-2)(2\sqrt{x}) = 2(x-1)\sqrt{x}$$

Answer: (E)

21. Consider $a^b$ and choose $a$ and $b$ to be two different numbers from $-1$, $-2$, $-3$, $-4$, and $-5$.
What is the largest possible value for $a^b$?
Since $b$ will be negative, we write $a^b = \dfrac{1}{a^{-b}}$ and here $-b > 0$.
If $b$ is odd, then since $a$ is negative, $a^b$ will be negative.
If $b$ is even, then $a^b$ will be positive.
So to make $a^b$ as big as possible, we make $b$ even (ie. $-2$ or $-4$).
Also, in order to make $a^b = \dfrac{1}{a^{-b}}$ as big as possible, we want to make $a^{-b}$ as small as possible, so $a$ should be as small as possible in absolute value.
Therefore, the largest possible value of $a^b$ will be when $a = -1$ and $b$ is either $-2$ or $-4$, giving in either case 1 (ie. $(-1)^{-2} = (-1)^{-4} = 1$).
What is the second largest possible value for $a^b$?
Again, we need $b$ to be even to make $a^b$ positive, and here we can assume that $a \neq -1$.
To make $a^b$ as large as possible, then using similar logic, we choose $b = -2$ and $a = -3$, giving $a^b = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$.
Therefore, the two largest possible values for $a^b$ are 1 and $\frac{1}{9}$.
Thus, looking at $a^b + c^d$, since $-1$ can only be chosen for one of these four numbers, then the largest possible value for this expression is the sum of the largest two possible values for $a^b$, ie. $1 + \frac{1}{9} = \frac{10}{9}$, which is obtained by calculating $(-1)^{-4} + (-3)^{-2}$.

Answer: (D)

22. Let $O$ be the centre of the circle, let $r$ be the radius of the circle, and let $s$ be the side length of the square. We want to calculate $s^2$.
By symmetry, $O$ is the midpoint of $PS$, so $OP = OS = \frac{1}{2}QR = 14$.
Join $O$ to $R$ and $O$ to $U$. We see that $OR = OU = r$, the radius of the circle.



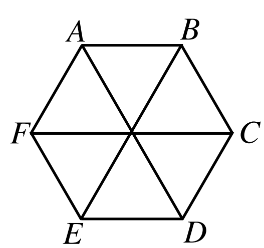

Since $\triangle OSR$ is right-angled at $S$, then $OR^2 = OS^2 + SR^2$ by the Pythagorean Theorem, or $r^2 = 14^2 + 12^2 = 196 + 144 = 340$.
Since $\triangle OVU$ is right-angled at $S$, then $OU^2 = OV^2 + VU^2$ or $r^2 = (14 + s)^2 + s^2$.
But $r^2 = 340$, so

$$\begin{aligned} 340 &= s^2 + 28s + 196 + s^2 \\ 0 &= 2s^2 + 28s - 144 \\ 0 &= s^2 + 14s - 72 \\ 0 &= (s + 18)(s - 4) \end{aligned}$$

Since $s$ must be positive, then $s = 4$, so the area of $STUV$ is $s^2 = 16$.

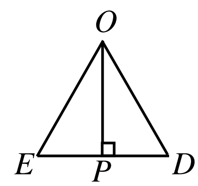
Answer: (C)

23. When the cube is sliced in half in this manner, each half cube will have 7 faces: one hexagonal face from the slice, and 6 faces each of which is part of a face of the original cube.
By symmetry, the total area of these last 6 faces will be half of the surface area of the original cube, or $\frac{1}{2} \times 6 \times 4^2 = 48$ cm$^2$.
Lastly, we must calculate the area of the hexagonal face. By symmetry, the face is a regular hexagon. The side length is equal to the length of the line segment joining the midpoints of adjacent sides of a square of side length 4, or $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.
So we need to calculate the area of a regular hexagon of side length $2\sqrt{2}$.
Consider a regular hexagon $ABCDEF$. Each interior angle is $120^\circ$.
Draw diagonals $AD$, $BE$ and $CF$.




By symmetry, these diagonals divide the hexagon into 6 equilateral triangles, each of side length $2\sqrt{2}$.

Consider one of these equilateral triangles.
Draw altitude from vertex $O$ to $P$ on side $DE$.




Since $\triangle ODE$ is equilateral, $P$ is the midpoint of $DE$, so $EP = \sqrt{2}$.
But $\triangle OPE$ is 30°-60°-90°, so $OP = \sqrt{3}EP = \sqrt{6}$.
Therefore, the area of $\triangle ODE = \frac{1}{2}(ED)(OP) = \frac{1}{2}(2\sqrt{2})(\sqrt{6}) = \sqrt{12} = 2\sqrt{3}$ cm$^2$.
Therefore, the area of hexagon $ABCDEF$ is $6(2\sqrt{3}) = 12\sqrt{3}$ cm$^2$.
Therefore, the total surface area of each half cube is $48 + 12\sqrt{3} \approx 69$ cm$^2$.

ANSWER: (A)

24. When we start at the first term and look at every other term from there, we will look at the last term. Therefore, the total number of terms after the first is a multiple of 2.
When we start at the first term and look at every third term from there, we will look at the last term. Therefore, the total number of terms after the first is a multiple of 3.
Thus, the total number of terms after the first must be a multiple of 6, so the total number of terms in the sequence can be written as $6k + 1$.
Now we look at the given sums, knowing that $n = 6k + 1$.
When we add up every other term, including the first and the last, we are adding up $3k + 1$ terms in total.
Therefore, $\frac{1}{2}(3k + 1)(a + a + 6kd) = 320$ or $(3k + 1)(2a + 6kd) = 640$.
(The sum of an arithmetic sequence is half of the product of the number of terms and the sum of the first and last terms. The sequence we get by looking at every other term (or every third term) of an arithmetic sequence is again arithmetic.)
When we add up every third term, including the first and the last, we are adding up $2k + 1$ terms in total.
Therefore, $\frac{1}{2}(2k + 1)(a + a + 6kd) = 224$, or $(2k + 1)(2a + 6kd) = 448$.
Dividing the first equation by the second, we obtain $\dfrac{3k+1}{2k+1} = \dfrac{640}{448} = \dfrac{10}{7}$, so $k = 3$.
Thus, $(3(3) + 1)(2a + 6kd) = 640$ or $2a + 6kd = 64$.
We want to determine the sum of the entire arithmetic sequence, which is

$$\frac{1}{2}(6k+1)(a + a + 6kd) = \frac{1}{2}(19)(2a + 6kd) = \frac{1}{2}(19)(64) = 608$$

ANSWER: (C)

25. Unfortunately, there was a problem with this question that we did not discover until after the Contests had been written. Our thanks go to Dr. Yongmoo Kim for pointing this out.

If the problem had been posed as

> A *triline* is a line with the property that three times its slope is equal to the sum of its
> $x$-intercept and its $y$-intercept. For how many integers $q$ with $1 \leq q \leq 10\,000$ is there at least one positive integer $p$ so that there is exactly one triline through $(p, q)$?

(ie. with the underlined word positive added), then the following solution would have been correct:

Consider a line through the point $(p, q)$ with slope $m$.
The equation of this line is $y = m(x - p) + q = mx + (q - mp)$.
Thus, the $y$-intercept of this line is $y = q - mp$ and the $x$-intercept comes from setting $y = 0$, which gives $x = \dfrac{mp - q}{m}$.
For this line to be a triline, we need $3m = (q - mp) + \dfrac{mp - q}{m}$ or $3m^2 = qm - pm^2 + mp - q$ or $(3 + p)m^2 - (p + q)m + q = 0$.
Given a fixed point $(p, q)$, for there to be only one triline through $(p, q)$, there can be only one slope $m$ satisfying $(3 + p)m^2 - (p + q)m + q = 0$, ie. this quadratic equation has exactly one real root. (The leading coefficient $3 + p$ is non-zero since $p$ is positive.)
So for a fixed point $(p, q)$, the condition that there be only one triline through $(p, q)$ is that the discriminant of $(3 + p)m^2 - (p + q)m + q = 0$ equals 0, or

$$\begin{aligned} (p+q)^2 - 4(3+p)q &= 0 \\ p^2 + 2pq + q^2 - 12q - 4pq &= 0 \\ p^2 - 2pq + q^2 - 12q &= 0 \\ (p-q)^2 &= 12q \end{aligned}$$

So we must determine the number of integers $q$ with $1 \leq q \leq 10\,000$ such that there is an integer $p$ such that $(p - q)^2 = 12q$.
In order for this to be true, $12q$ needs to be a perfect square, so $3q$ needs to be a perfect square.
In order for $3q$ to be a perfect square, $q$ needs to be 3 times a perfect square (since $q$ must contain an odd number of factors of 3 and even number of every other prime factor).
If $q = 3k^2$, then we can solve $(p - q)^2 = 12q$ since then $(p - 3k^2)^2 = 36k^2$ or $p - 3k^2 = \pm 6k$ or $p = 3k^2 \pm 6k$.
So how many integers $q$ between 1 and 10 000 are of the form $q = 3k^2$? The minimum value of $k$ that works is $k = 1$ and the maximum is $k = 57$ (since $3(58)^2 = 10\,092$ is too large).
Therefore, there are 57 such values of $q$, ie. the answer would be (B).

However, the problem was posed as

> A *triline* is a line with the property that three times its slope is equal to the sum of its
> $x$-intercept and its $y$-intercept. For how many integers $q$ with $1 \leq q \leq 10\,000$ is there at least one integer $p$ so that there is exactly one triline through $(p, q)$?

Following the above solution, we arrive at needing one slope $m$ to satisfy the equation

$$(3+p)m^2-(p+q)m+q=0$$

There are two ways for this "quadratic" equation to have a single root – either if the discriminant is 0, or if the leading coefficient $p+3$ is 0 (ie. the "quadratic" is actually linear).
So if $p=-3$, then we need the equation $(q-3)m+q=0$ to have exactly one solution for $m$, which it does *for every* $q$, as long as $q \neq 3$.
If $q=3$, then the value of $p=9$ makes the discriminant of the quadratic equation 0.
In other words, every value of $q$ between 1 and 10 000 has at least one integer $p$ so that there is exactly one triline through $(p,q)$, so there are 10 000 values of $q$ that work.

Our apologies for any confusion that this may have caused.

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

# 2004 Solutions
# Fermat Contest (Grade 11)

for

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING**

Awards

# 2004 Fermat Contest Solutions

1. Calculating,

$$\frac{10}{10(11)-10^2}=\frac{10}{110-100}=1$$

Answer: (D)

2. $\sqrt{4^0+4^2+4^3}=\sqrt{1+16+64}=\sqrt{81}=9$

Answer: (A)

3. First, we determine that $x=4$ and $y=3$. Therefore, $x-y=1$.

Answer: (B)

4. Since the loaf is cut into 25 pieces of equal volume, the volume of each piece is

$$\frac{(20\text{ cm})\times(18\text{ cm})\times(5\text{ cm})}{25}=(4\text{ cm})\times(18\text{ cm})\times(1\text{ cm})=72\text{ cm}^3.$$

Since the density of the loaf is $2\text{ g/cm}^3$, then the mass of each piece is volume times density, or $\left(2\text{ g/cm}^3\right)\times\left(72\text{ cm}^3\right)=144\text{ g}$.

Answer: (D)

5. *Solution 1*

Taking reciprocals of both sides,

$$\begin{aligned}\left(\frac{1}{2+3}\right)\left(\frac{1}{3+4}\right)&=\frac{1}{x+5}\\(2+3)(3+4)&=x+5\\35&=x+5\\x&=30\end{aligned}$$

*Solution 2*

Simplifying the left side,

$$\begin{aligned}\left(\frac{1}{2+3}\right)\left(\frac{1}{3+4}\right)&=\frac{1}{x+5}\\\left(\frac{1}{5}\right)\left(\frac{1}{7}\right)&=\frac{1}{x+5}\\\frac{1}{35}&=\frac{1}{x+5}\end{aligned}$$

Comparing denominators gives $x+5=35$ or $x=30$.

Answer: (C)

6. If three cans fill $\frac{2}{3}$ L, then one can will fill $\frac{2}{9}$ L.

   Therefore, the total number of cans required to fill 8 L is $\dfrac{8}{\left(\frac{2}{9}\right)} = 8 \times \dfrac{9}{2} = 36$.

   Answer: (A)

7. *Solution 1*

   We first simplify the expression:

   $$\frac{x^2-4}{x^2-2x} = \frac{(x+2)(x-2)}{x(x-2)} = \frac{x+2}{x} = \frac{x}{x} + \frac{2}{x} = 1 + \frac{2}{x}$$

   (We can cancel the factor of $x-2$ since $x$ is not equal to 2.)

   Substituting $x = \frac{1}{5}$, we get $1 + \dfrac{2}{\left(\frac{1}{5}\right)} = 1 + 10 = 11$.

   *Solution 2*

   Substituting $x = \frac{1}{5}$,

   $$\frac{x^2-4}{x^2-2x} = \frac{\frac{1}{25}-4}{\frac{1}{25}-\frac{2}{5}} = \frac{\frac{1}{25}-\frac{100}{25}}{\frac{1}{25}-\frac{10}{25}} = \frac{-\frac{99}{25}}{-\frac{9}{25}} = 11.$$

   Answer: (E)

8. From the graph, Jane arrives with 10 L of gas in her gas tank, and leaves with 50 L of gas, having paid \$36.60. So she buys 40 L of gas for \$36.60, so the cost per litre is $\dfrac{\$36.60}{40} = \$0.915$, or 91.5 cents.

   Answer: (A)

9. 4% of 10 000 is $\frac{4}{100} \times 10000 = 400$, so the population of Cayleyville in 2004 is 10 400.

   12% of 25 000 is $\frac{12}{100} \times 25000 = 3000$, so the population of Pascalberg in 2004 is 22 000.

   So the population difference in 2004 is $22000 - 10400 = 11600$.

   Answer: (B)

10. From the two given balances, 3 ▲'s balance 5 ●'s, and 1 ▲ balances 2 ■'s and 1 ●.

    Tripling the quantities on the second balance implies that 3 ▲'s will balance 6 ■'s and 3 ●'s.

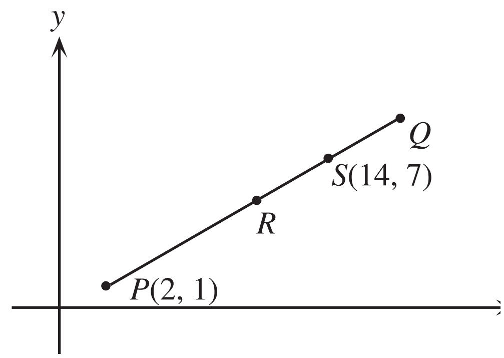
    Therefore, 5 ●'s will balance 6 ■'s and 3 ●'s, and removing 3 ●'s from each side implies that 2 ●'s will balance 6 ■'s, or 1 ● will balance 3 ■'s.

    Answer: (C)

11. Since $x$ is between –1 and 0, then $x^2$ is between 0 and 1, and so $-x^2$ is between –1 and 0. Therefore, the best letter is either $b$ or $c$. When a number between –1 and 0 is squared, it becomes closer to 0 than it was before, so the best choice must be $c$, not $b$.

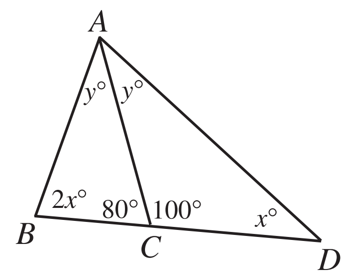
    Answer: (C)

12. Since $R$ is the midpoint of $PQ$ and $S$ is the midpoint of $QR$, then $S$ is $\frac{3}{4}$ of the way from $P$ to $Q$.
Since $S$ is 12 units to the right of $P$, then $Q$ is $\frac{4}{3}\times 12 = 16$ units to the right of $P$.
Since $S$ is 6 units up from $P$, then $Q$ is $\frac{4}{3}\times 6 = 8$ units up from $P$.
Therefore, the coordinates of $S$ are $(2+16,\ 1+8) = (18,9)$.




Answer: (D)

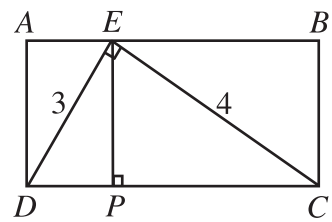
13. In triangle $ACD$, $x^\circ + y^\circ + 100^\circ = 180^\circ$, so $x + y = 80$ (*).
Since $\angle ACB$ and $\angle ACD$ are supplementary, then $\angle ACB = 180^\circ - \angle ACD = 80^\circ$.
Thus, in triangle $ACB$, $2x^\circ + y^\circ + 80^\circ = 180^\circ$, so $2x + y = 100$ (**).
Subtracting (*) from (**), we obtain $x = 20$.



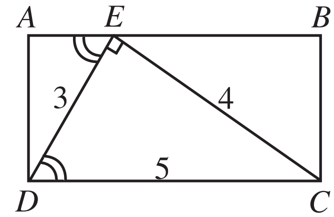

Answer: (E)

14. *Solution 1*
By the Pythagorean Theorem, $DC^2 = DE^2 + EC^2$ so $DC = 5$.
Draw a line from $E$ to point $P$ on $DC$ so that $EP$ is perpendicular to $DC$. Since $ABCD$ is a rectangle, then $AD = EP$.
Then the area of triangle $DEC$ is equal to $\frac{1}{2}(DE)(EC)$ and also to $\frac{1}{2}(DC)(EP)$.
So

$$\frac{1}{2}(3)(4) = \frac{1}{2}(5)(EP)$$
$$12 = 5(AD)$$
$$AD = \frac{12}{5} = 2.4$$



*Solution 2*




By the Pythagorean Theorem, $DC^2 = DE^2 + EC^2$ so $DC = 5$.

Then $\sin(\angle EDC) = \dfrac{EC}{DC} = \dfrac{4}{5}$.

But $\angle AED = \angle EDC$ since $AB$ and $DC$ are parallel,

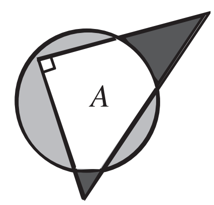
so $\dfrac{4}{5} = \sin(\angle AED) = \dfrac{AD}{ED} = \dfrac{AD}{3}$, and so $AD = \dfrac{12}{5} = 2.4$.

Answer: (B)

15. If $x^2 - y^2 = 0$, then $(x-y)(x+y) = 0$, so $y = x$ or $y = -x$. These are the equations of two straight lines (each of which passes through $(0,0)$).

Answer: (E)

16. Let the area that is inside the triangle and inside the circle be $A$, and let the area outside the triangle but inside the circle be $B$. Then $B$ is also equal to the area outside the circle but inside the triangle.




We can then see that $A + B$ is equal to the area of the circle and is also equal to the area of the
triangle, so the circle and triangle have the same area.

Thus, if $r$ is the radius of the circle, then $\pi r^2 = \frac{1}{2}(6)(8)$ or $\pi r^2 = 24$ or $r = \sqrt{\dfrac{24}{\pi}} \approx 2.8$.

Answer: (B)

17. Since the difference between consecutive terms is constant, then the difference between the third and fourth terms is equal to the difference between the first and second terms, or

$$\begin{aligned}(x+2y+2)-(3x+y) &= y-x\\ y-2x+2 &= y-x\\ 2 &= x\end{aligned}$$

Thus, we can rewrite the sequence as 2, $y$, $y+6$, and $2y+4$.

Again, since the difference is constant and the difference between the second and third terms is 6, then the difference between the first and second terms is 6, ie. $y = 8$.

Thus, $y - x = 6$.

Answer: (E)

18. *Solution 1*

We expand the two expressions.

First, $y = a(x-2)^2 + c = a\left(x^2 - 4x + 4\right) + c = ax^2 - 4ax + (4a + c)$.

Second, $y = (2x-5)(x-b) = 2x^2 - (5+2b)x + 5b$.

We can then equate the coefficients.

From the leading coefficients, $a = 2$.

From the coefficients of $x$, $4a = 8 = 5 + 2b$ or $b = \frac{3}{2}$.

*Solution 2*

The $x$-coordinate of the vertex of a parabola is the average value of the roots of the parabola.
From the first of the given forms, the vertex has $x$ coordinate 2.
From the second of the given forms, the roots are $x = \frac{5}{2}$ and $x = b$.
Therefore, $\frac{1}{2}\left(\frac{5}{2} + b\right) = 2$ or $\frac{5}{2} + b = 4$ or $b = \frac{3}{2}$.

Answer: (B)

19. Let $P$ be the original price the retailer sets.
Then one-half of the initial 1200, or 600 copies, will sell for a price of $P$, giving revenue of $600P$.
Two-thirds of the remaining 600, or 400 copies, will sell for a price of $0.6P$ (ie. 40% off of $P$), giving revenue of $400(0.6P) = 240P$.
The remaining 200 copies will sell for a price of $0.25P$ (ie. 75% off of $P$), giving revenue of $200(0.25P) = 50P$.
To make a reasonable profit, her revenue must be \$72 000, or

$$\begin{aligned} 600P + 240P + 50P &= 72000 \\ 890P &= 72000 \\ P &\approx 80.90 \end{aligned}$$

Thus, she should set an original price of \$80.90.

Answer: (D)

20. The ball is rolling towards Marcos at 4 m/s and he is running towards it at 8 m/s, so he gains 12 metres per second on the ball. Since he starts 30 m from the ball, it will take him $\frac{30}{12} = 2.5$ s to reach the ball.
The ball is rolling away from Michael at 4 m/s and he is running at 9 m/s, so he is gaining 5 m/s on the ball. Since he starts 15 m behind the ball, he would catch up to the ball in 3 s if it continued to roll.
Thus, Marcos gets to the ball first. After 2.5 s, the Michael has gained $5(2.5) = 12.5$ m on the ball, so is 2.5 m from the ball when Marcos touches it first.

Answer: (C)

21. *Solution 1*

In one hour, Bill paints $\frac{1}{B}$ of the line and Jill paints $\frac{1}{J}$ of the line.
Let $t$ be number of hours during which both Bill and Jill paint the line. Since Bill paints for one hour before Jill starts to paint and since the painting of the line is completely finished after the $t$ hours that they both work, then $\frac{1}{B} + t\left(\frac{1}{B} + \frac{1}{J}\right) = 1$ or

$$t = \frac{1 - \frac{1}{B}}{\frac{1}{B} + \frac{1}{J}} = \frac{\left[\frac{B-1}{B}\right]}{\left[\frac{B+J}{BJ}\right]} = \frac{J(B-1)}{B+J}.$$

So Bill works for $t+1=\dfrac{J(B-1)}{B+J}+1=\dfrac{BJ-J}{B+J}+\dfrac{B+J}{B+J}=\dfrac{BJ+B}{B+J}=\dfrac{B(J+1)}{B+J}$ hours.

*Solution 2*
Suppose that Bill could paint the line in 1 hour only, ie. $B=1$.
Then when Jill joins the painting after 1 hour, Bill would actually be finished the painting. In other words, if $B=1$, then the total time that Bill spends painting the line, regardless of the value of $J$, is 1 hour. If we substitute $B=1$ into the five choices, we obtain

(A) $\dfrac{J+1}{J+1}=1$ (B) $J+1$ (C) $\dfrac{J}{J+1}+1$
(D) $\dfrac{J}{2}$ (E) $\dfrac{J-1}{J+1}$

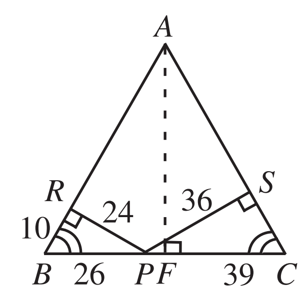
The only choice which is equal to 1, regardless of the value of $J$, is the first one.

Answer: (A)

22. Since we would like the product to have 303 digits, then we would like this product to be greater than $10^{302}$ but less than $10^{303}$.
We will start by trying $k=300$. In this case, $(2^k)(5^{300})=(2^{300})(5^{300})=10^{300}$, so we want $k$ to be bigger than 300.
Each time we increase $k$ by 1, the existing product is multiplied by 2. For the final product to have 303 digits, we need to multiply $10^{300}$ by a power of 2 between 100 and 1000. The smallest power of 2 that satisfies this is $2^7=128$.
Therefore, we would like $k=307$. In this case,
$$(2^{307})(5^{300})=(2^7)(2^{300})(5^{300})=(2^7)(10^{300})=128\times10^{300}.$$
When this number is expanded, the digits are 128 followed by 300 zeros. Therefore, the sum of the digits is 11.

Answer: (A)

23. Since triangle $ABC$ is isosceles, $\angle ABC=\angle ACB$, and so triangle $BRP$ is similar to triangle $CSP$ (equal angle, right angle).




Therefore, $\dfrac{BP}{RP}=\dfrac{CP}{SP}$ or $\dfrac{BP}{24}=\dfrac{CP}{36}$ or $\dfrac{BP}{CP}=\dfrac{2}{3}$.
But since $BC$ has length 65 cm, then $BP+CP=65$ cm, and so $BP=26$ cm and $BP=39$ cm. Since $BP=26$ cm and $RP=24$ cm, then by the Pythagorean Theorem, $BR=10$ cm.
Next, we drop a perpendicular from $A$ to $F$ on $BC$. Since triangle $ABC$ is isosceles, $BF=\frac{1}{2}(BC)=\frac{65}{2}$ cm.
Also, triangle $BFA$ is similar to triangle $BRP$ (common angle, right angle).
Thus, $\dfrac{BR}{RP}=\dfrac{BF}{FA}$, so $FA=\dfrac{\left(\frac{65}{2}\right)(24)}{10}=78$ cm.
The area of triangle $ABC$ is then $\frac{1}{2}(BC)(FA)=\frac{1}{2}(65)(78)=2535\text{ cm}^2$.

Answer: (D)

24. Since the difference between $f(x)$ and $f(x-2)$ has degree 2, then the degree of $f(x)$ is at least 2. If the degree of $f(x)$ was equal to 2, then the $x^2$ terms of $f(x)$ and $f(x-2)$ would cancel when we subtracted. (Try this.) Therefore, the degree of $f(x)$ must be at least 3.

So we try $f(x) = ax^3 + px^2 + qx + r$.

Thus,

$$\begin{aligned} f(x-2) &= a(x-2)^3 + p(x-2)^2 + q(x-2) + r \\ &= a\left(x^3 - 6x^2 + 12x - 8\right) + p\left(x^2 - 4x + 4\right) + q(x-2) + r \\ &= ax^3 + (-6a+p)x^2 + (12a - 4p + q)x + (-8a + 4p - 2q + r) \end{aligned}$$

We are told that $f(x) - f(x-2) = (2x-1)^2$, so

$$\left[ax^3 + px^2 + qx + r\right] - \left[ax^3 + (-6a+p)x^2 + (12a-4p+q)x + (-8a+4p-2q+r)\right] = 4x^2 - 4x + 1$$

$$6ax^2 + (-12a + 4p)x + (8a - 4p + 2q) = 4x^2 - 4x + 1$$

Comparing coefficients,

$$\begin{aligned} 6a &= 4 \\ -12a + 4p &= -4 \\ 8a - 4p + 2q &= 1 \end{aligned}$$

From the first equation, $a = \frac{2}{3}$.

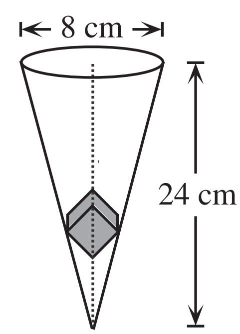
Substituting $6a = 4$ into the second equation, we obtain $-8 + 4p = -4$ or $p = 1$.

Substituting these two values into the third equation, we obtain $8\left(\frac{2}{3}\right) - 4(1) + 2q = 1$ or $q = -\frac{1}{6}$.

Therefore, $p + q = 1 + \left(-\frac{1}{6}\right) = \frac{5}{6}$.

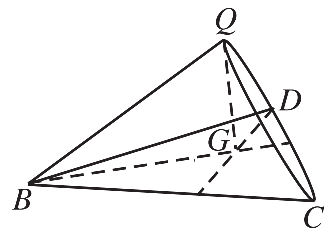
Answer: (B)

25. The first difficult thing about this problem is visualizing the position of the cube inside the cone. From there, we must determine how to calculate the required distance. Since the cube is balanced in an upright position with the axis of the cone coincides with one of the internal diagonals of the cube, then the cube is oriented with one vertex, $A$, pointing downwards, three vertices, $B$, $C$ and $D$, touching the walls of the cone, and one vertex, $Q$, pointing upwards.




(We do not need to consider the remaining three vertices.) By symmetry, $B$, $C$ and $D$ lie in a plane parallel to the base of the cone and form an equilateral triangle.

We draw a diagram to illustrate the configuration of $B$, $C$, $D$, and $Q$. In the diagram, $QB$ is the length of the diagonal of the face of a cube (this requires careful visualization of where the faces of the cube are positioned) and $G$ is the intersection of the three medians. By symmetry, vertex $A$ will lie directly beneath $G$ in a line from $Q$. Thus, the line $QGA$ lies along the axis of the cone.



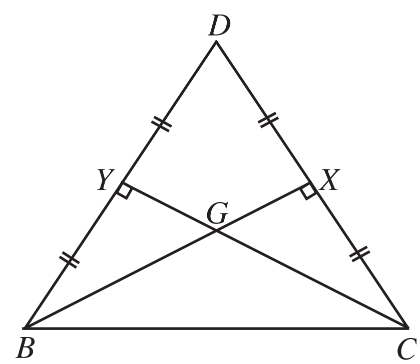

We can now determine the required distance by determining

- the vertical distance from the tip of the cone, $T$, at which $B$, $C$ and $D$ touch the cone, and
- the distance between $A$ and the plane formed by $B$, $C$ and $D$,

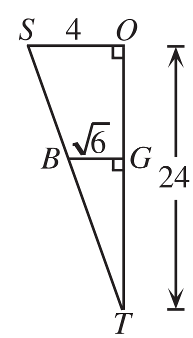
and then subtracting these two distances.

Now for some calculations.
Since each edge of the tetrahedron is a diagonal of a face of the cube, then each has length $\sqrt{3^2+3^2}=3\sqrt{2}$. In particular, $BQ$, $BC$, $BD$ and $CD$ all have length $3\sqrt{2}$.

Next, if we extract triangle $BCD$, then and draw the medians $BX$ and $CY$ which intersect at $G$, then we see that triangle $BGY$ is a 30-60-90 triangle, so $BG=\frac{2}{\sqrt{3}}(BY)=\frac{2}{\sqrt{3}}\left(\frac{1}{2}(BA)\right)=\frac{1}{\sqrt{3}}\left(3\sqrt{2}\right)=\sqrt{6}$. This tells us that the distance from the axis of the cone to the points where the cube touches the cone is $\sqrt{6}$.




We can now calculate the distance from $T$ to the plane containing $B$, $C$ and $D$. To do, this we look at a partial cross-section of the cone labelling $B$, $T$, $G$, as well as $O$, the centre of the base of the cone, and $S$, the remaining vertex of the cross-section.
We can see that triangle $TGB$ is similar to triangle $TOS$, so $\frac{TG}{TO}=\frac{GB}{OS}$ or $TG=24\left(\frac{\sqrt{6}}{4}\right)=6\sqrt{6}$, (since the radius of the base of the cone is 4).



Lastly, we need to calculate the distance $AG$. To do this, we can calculate $AQ$ and $QG$ and subtract these two distances.

$AQ$ is the main diagonal of the cube, so $AQ = \sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3}$.
Looking at triangle $BGQ$, which is right-angled, we see that
$QG = \sqrt{BQ^2 - BG^2} = \sqrt{\left(3\sqrt{2}\right)^2 - \left(\sqrt{6}\right)^2} = 2\sqrt{3}$.
Therefore, $AG = AQ - QG = \sqrt{3}$.
Therefore, the required distance is $TA = TG - AG = 6\sqrt{6} - \sqrt{3}$.

Answer: (A)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2003 Solutions
# Fermat Contest (Grade 11)

for

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING**
*Awards*

# 2003 Fermat Contest Solutions

1. Evaluating,
$3^3 - 3^2 + 3^1 - 3^0 = 27 - 9 + 3 - 1 = 20.$ Answer: (E)

2. Substituting the given value of $a$, we obtain

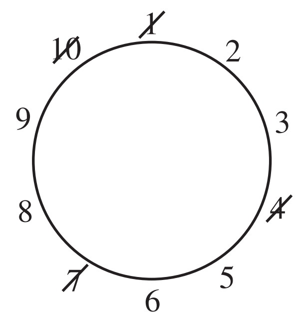
$$\begin{aligned} a^2 + ab &= 60 \\ 25 + 5b &= 60 \\ 5b &= 35 \\ b &= 7 \end{aligned}$$

Answer: (A)

3. Since the two angles below the line sum to $180^\circ$, then the given line must be a straight line. This tells us that $4x^\circ + x^\circ = 90^\circ$ or $5x^\circ = 90^\circ$ or $x = 18$.

Answer: (E)

4. On her first pass around the circle, Sandy crosses out 1, 4, 7, and 10. This leaves her with the numbers 2, 3, 5, 6, 8, and 9 remaining.
Starting from the 10, the third of the remaining numbers is 5, and the third of the remaining numbers after the 5 is 9.
Thus Sandy crosses off 5, 9, 6, and 3, leaving 2 and 8 as the last two remaining numbers.




Answer: (B)

5. *Solution 1*
Since the bear has lost 20% of its original mass, then the 220 kg represents 80% of its original mass. Therefore, 20% of its original mass is 55 kg (one quarter of 220 kg), and so its mass before hibernation was $220 + 55 = 275$ kg. (Notice that this just amounts to $\frac{5}{4} \times 220 = 275$.)

*Solution 2*
Let $x$ be the bear's mass just prior to hibernation.
Since the bear loses 20% of its mass during hibernation, then

$$\begin{aligned} \frac{80}{100}x &= 220 \\ x &= \frac{100}{80}(220) \\ x &= 275 \end{aligned}$$

Therefore, the bear's mass just before hibernation was 275 kg.

Answer: (D)

6. *Solution 1*
When $\frac{5}{8}$ of the players are girls then $\frac{3}{8}$ of the players will be boys. Since the number of boys playing is 6 (and does not change), then after the additional girls join, there must be 16 players in total for $\frac{3}{8}$ of the players to be boys. Since there were 8 players initially, then 8 additional girls must have joined the game.

*Solution 2*
Let the number of additional girls be $g$.
Then

$$\begin{aligned} \frac{2+g}{8+g} &= \frac{5}{8} \\ 16+8g &= 40+5g \\ 3g &= 24 \\ g &= 8 \end{aligned}$$

ANSWER: (D)

7. Since the height of the fish tank is 30 cm, and it is half full of water, then the depth of the water is 15 cm.
Now the area of the base of the tank is $(20 \text{ cm})(40 \text{ cm}) = 800 \text{ cm}^2$, and so $4000 \text{ cm}^3$ of water will cover this base to a depth of $\frac{4000 \text{ cm}^3}{800 \text{ cm}^2} = 5 \text{ cm}$.
Therefore, the new depth of water is $15 \text{ cm} + 5 \text{ cm} = 20 \text{ cm}$.

ANSWER: (C)

8. From the diagram, we see that the line segment $AD$ is perpendicular to the line segment $BC$, and so the product of the slopes of these two line segments is $-1$.
The slope of segment $BC$ is $\frac{7-(-4)}{6-9} = -\frac{11}{3}$, and so the slope of $AD$ is $\frac{3}{11}$.
(We notice that we did not use the coordinates of $A$!)

ANSWER: (A)

9. *Solution 1*
The average of two numbers is half-way between the two numbers.
So what number is half-way between $\frac{1}{5}$ and $\frac{1}{10}$?
We can write $\frac{1}{5} = \frac{4}{20}$ and $\frac{1}{10} = \frac{2}{20}$, so the number half-way in between is $\frac{3}{20}$ or $\frac{1}{\left(\frac{20}{3}\right)}$
Therefore, $x = \frac{20}{3}$.

*Solution 2*
The average of two numbers $X$ and $Y$ is $\frac{1}{2}(X+Y)$, so

$$\frac{\frac{1}{5}+\frac{1}{10}}{2}=\frac{1}{x}$$
$$\frac{\frac{3}{10}}{2}=\frac{1}{x}$$
$$x=\frac{2}{\left(\frac{3}{10}\right)}$$
$$x=\frac{20}{3}$$

ANSWER: (A)

10. Since the distance covered by Jim in 4 steps is the same as the distance covered by Carly in 3 steps, then the distance covered by Jim in 24 steps is the same as the distance covered by Carly in 18 steps.
Since each of Carly's steps covers 0.5 m, then she covers 9 m in 18 steps, ie. Jim covers 9 m in 24 steps.

ANSWER: (B)

11. We determine all of the possible routes:
Travelling $A$ to $X$ to $B$, there are 2 routes, since there are 2 paths $A$ to $X$.
Travelling $A$ to $X$ to $Y$ to $B$, there are 6 routes since there are two paths from $A$ to $X$ and 3 paths from $Y$ to $B$.
Travelling $A$ to $Y$ to $B$, there are 3 routes, since there are 3 paths $Y$ to $B$.
So there are 11 routes from $A$ to $B$, of which 8 pass through $X$.
Therefore, the probability that Hazel chooses a route that passes through $X$ is $\frac{8}{11}$.

ANSWER: (A)

12. Since $\Delta ABC$ is right-angled, then we can use Pythagoras' Theorem to say $AC^2=10^2+10^2=200$, or $AC=\sqrt{200}=10\sqrt{2}$. Therefore, $AD=AC-DC=10\sqrt{2}-10\approx 14.1-10=4.1$.
Thus, the length of $AD$ is closest to 4.

ANSWER: (E)

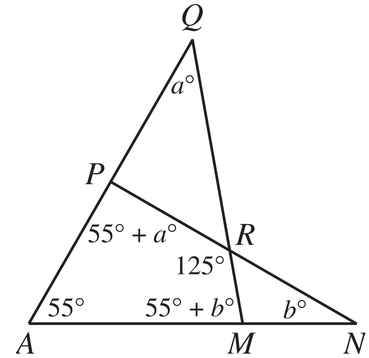
13. *Solution 1*
Since $x+y=1$ and $x-y=3$, then $x^2-y^2=(x-y)(x+y)=3$, and so $2^{x^2-y^2}=2^3=8$.

*Solution 2*
Since $x+y=1$ and $x-y=3$, then adding these two equations, we obtain $2x=4$ or $x=2$. Substituting this value for $x$ back into the first equation we see that $y=1$.
Therefore, $2^{x^2-y^2}=2^{2^2-1^2}=2^3=8$.

ANSWER: (B)

14. Since $\angle PRM = 125^\circ$, then $\angle QRP = \angle NRM = 55^\circ$.
Then $\angle APR = 180^\circ - \angle QPR = 180^\circ - \left[180^\circ - a^\circ - 55^\circ\right] = 55^\circ + a^\circ$.
Similarly, $\angle QPR = 55^\circ + b^\circ$. (This is an external angle in $\Delta QPR$.)
Since $AQ$ is a straight line,

$$\left(55^\circ + a^\circ\right) + \left(55^\circ + b^\circ\right) = 180^\circ$$
$$a^\circ + b^\circ + 110^\circ = 180^\circ$$
$$a + b = 70$$




ANSWER: (A)

15. Suppose that $T$ is the side length of the equilateral triangle and $S$ is the side length of the square. (Both $S$ and $T$ are integers.) Then, since the perimeters of the triangle and the square are equal, we have $3T = 4S$.
Since $3T = 4S$ and each side of the equation is an integer, then $T$ must be divisible by 4 because 4 must divide into $3T$ evenly and it does not divide into 3.
The only one of the five possibilities which is divisible by 4 is 20.
(We should check that $T = 20$ does indeed yield an integer for $S$, which it does ($S = 15$).)

ANSWER: (D)

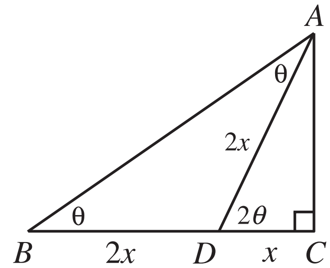
16. Suppose that the four digit number has digits $a$, $b$, $c$, and $d$, ie. the product $abcd = 810$.
We must determine how to write 810 as the product of 4 *different* digits, none of which can be 0. So we must start by factoring 810, as $810 = 81 \times 10 = 3^4 \times 2 \times 5$.
So one of the digits must have a factor of 5. But the only non-zero digit having a factor of 5 is 5 itself, so 5 is one of the required digits.
Now we need to find 3 different digits whose product is $3^4 \times 2$.
The only digits with a factor of 3 are 3, 6, and 9, and since we need 4 factors of 3, we must use each of these digits (the 9 contributes 2 factors of 3; the others contribute 1 each). In fact, $3 \times 6 \times 9 = 3^4 \times 2 = 162$.
Therefore, the digits of the number are 3, 5, 6, and 9, and so the sum of the digits is 23.

ANSWER: (C)

17. *Solution 1*

Let $\angle ABC = \theta$. Then $\angle ADC = 2\theta$, and so $\angle ADB = 180^\circ - 2\theta$ and $\angle BAD = \theta$.
Thus $\Delta ADB$ is isosceles with $BD = DA$, and so $DA = 2x$.

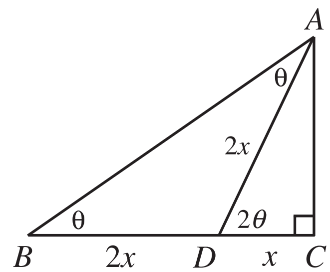
Since $AD$ is twice the length of $DC$ and $\Delta ADC$ is right-angled, then $\Delta ADC$ is a 30-60-90 triangle, that is, $\angle ADC = 60^\circ$ and so $\angle ABC = 30^\circ$.
Therefore, $\Delta ABC$ is also a 30-60-90 triangle, and so $\angle BAC = 60^\circ$ (which is opposite side $BC$ of length $3x$).
Thus, $AB = \frac{2}{\sqrt{3}}BC = \frac{2}{\sqrt{3}}(3x) = 2\sqrt{3}x$.




*Solution 2*

Let $\angle ABC = \theta$. Then $\angle ADC = 2\theta$, and so $\angle ADB = 180^\circ - 2\theta$ and $\angle BAD = \theta$.
Thus $\Delta ADB$ is isosceles with $BD = DA$, and so $DA = 2x$.
Since $\Delta ADC$ and $\Delta ABC$ are both right-angled, then

$$\begin{aligned} AB^2 &= BC^2 + AC^2 \\ &= BC^2 + \left(AD^2 - DC^2\right) \\ &= (3x)^2 + \left((2x)^2 - x^2\right) \\ &= 9x^2 + 3x^2 \\ &= 12x^2 \end{aligned}$$



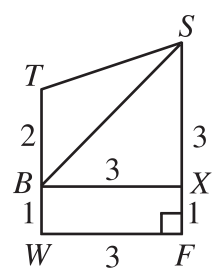

and so $AB = \sqrt{12}x = 2\sqrt{3}x$.

ANSWER: (C)

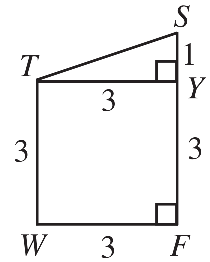
18. The cost to modify the car's engine (\$400) is the equivalent of the cost of $\dfrac{400}{0.80} = 500$ litres of gas. So the car would have to be driven a distance that would save 500 L of gas in order to make up the cost of the modifications.
Originally, the car consumes 8.4 L of gas per 100 km, and after the modifications the car consumes 6.3 L of gas per 100 km, a savings of 2.1 L per 100 km.
Thus, in order to save 500 L of gas, the car would have to be driven $\dfrac{500}{2.1} \times 100 = 23809.52$ km.

ANSWER: (D)

19. Let $X$ be the point on $SF$ so that $BX$ is perpendicular to $SF$.
Then $BX = 3$, $XF = 1$ and $XS = 3$.
Therefore, $\Delta BXS$ is an isosceles right-angled triangle, and so $\angle BSX = 45^\circ$.
Let $Y$ be the point on $SF$ so that $TY$ is perpendicular to $SF$.
Then $TY = 3$ and $SY = 1$, and $\Delta STY$ is right-angled.



Therefore, $\tan(\angle TSY) = \frac{3}{1} = 3$ and so $\angle TSY \approx 71.6^\circ$.
Thus, $\angle TSB = \angle TSY - \angle BSF \approx 71.6^\circ - 45^\circ = 26.6^\circ$, which is closest to $27^\circ$.




ANSWER: (A)

20. Since $a$, $b$ and $c$ are the consecutive terms of a geometric sequence, then $\frac{c}{b} = \frac{b}{a}$ or $b^2 = ac$.
Therefore, the discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac = ac - 4ac = -3ac < 0$ since $a$ and $c$ are both positive.
Since the discriminant is negative, the parabola does not intersect the $x$-axis. Since the leading coefficient, $a$, is positive, then the parabola is entirely above the $x$-axis.

ANSWER: (C)

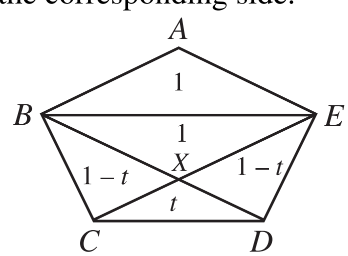
21. In order to get a better feel for this sequence we should write out the first few terms (and hope we spot a pattern!). We'll use the notation $t_n$ for the $n$th term in the sequence; that is, $t_1$ is the first term, $t_2$ is the second term, and so on.
So we have:

$$t_1 = 6,\ t_2 = \tfrac{1}{2}t_1 = 3,\ t_3 = 3t_2 + 1 = 10,\ t_4 = \tfrac{1}{2}t_3 = 5,\ t_5 = 16,\ t_6 = 8,\ t_7 = 4,\ t_8 = 2,$$
$$t_9 = 1,\ t_{10} = 4,\ t_{11} = 2,\ t_{12} = 1,\ t_{13} = 4,\ \ldots$$

Since each term in the sequence depends only on the previous term, then if a term repeats (as it does here), the sequence will then cycle. Here, the terms (starting with the 7th) will cycle 4, 2, 1, 4, 2, 1 etc.
We notice that the cycle has length 3, and $t_9 = 1$, so $1 = t_9 = t_{12} = t_{15} = t_{18} = \cdots$, and so every term whose subscript is a multiple of 3 will be equal to 1.
Thus, $t_{99} = 1$ and so $t_{100} = 4$.

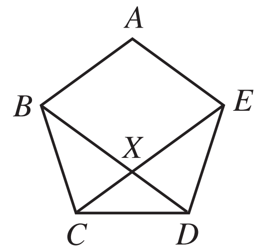
ANSWER: (D)

22. *Solution 1*
We will consider an arbitrary pentagon satisfying the given condition.
First, we will show that each of the diagonals is parallel to the corresponding side.
Since Area $\Delta BCD$ = Area $\Delta CDE$ and these triangles have a common base, then their heights must be equal, that is, the points $B$ and $E$ must be the same perpendicular distance from the line $CD$, that is, the line $BE$ is parallel to the line $CD$. The same can be said for all of the other diagonals.




Now join $BD$, $CE$ and $BE$. Since $BD$ is parallel to $AE$ and $CE$ is parallel to $AB$, then $BAEX$ is a parallelogram, and so the area of $\Delta BEX$ is equal to the area of $\Delta EAB$, which is 1.

Suppose the area of $\Delta CXD$ is $t$. Then the areas of $\Delta CXB$ and $\Delta EXD$ are both $1-t$.




Since $\Delta CXD$ and $\Delta CXB$ have a common height, then the ratio of their bases is the ratio of their areas, ie. $\frac{DX}{BX}=\frac{t}{1-t}$. Similarly, $\frac{CX}{EX}=\frac{t}{1-t}$.

But since $BE$ is parallel to $CD$, then $\Delta CXD$ is similar to $\Delta EXB$, and so the ratio of their areas is the square of the ratio of their side lengths, ie. $\frac{\text{Area } \Delta EXB}{\text{Area } \Delta CXD}=\frac{1}{t}=\left(\frac{1-t}{t}\right)^2$.

Simplifying, $t^2-3t+1=0$ or $t=\frac{3-\sqrt{5}}{2}$ (since $t$ is smaller than 1).

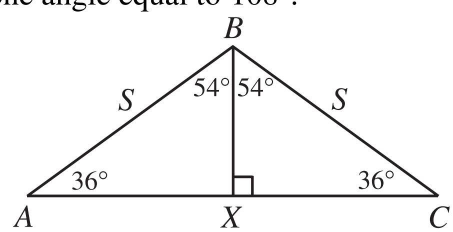
Therefore, the area of the entire pentagon is

$$1+1+t+(1-t)+(1-t)=4-t=\tfrac{5+\sqrt{5}}{2}\approx 3.62.$$

*Solution 2*

Since the problem implies that the area of the pentagon is independent of the configuration (assuming the appropriate area conditions hold), let us assume that the pentagon is a regular pentagon, with side length $s$.

We first determine the value of $s$. Consider $\Delta ABC$. Since a regular pentagon has each interior angle equal to $108^\circ$, then $\Delta ABC$ is isosceles with one angle equal to $108^\circ$.




Let $X$ be the midpoint of $AC$, and join $B$ to $X$. Then $BX$ is perpendicular to $AC$ since $\Delta ABC$ is isosceles, and $\angle ABX=\angle CBX=54^\circ$ ($BX$ bisects $\angle ABC$).

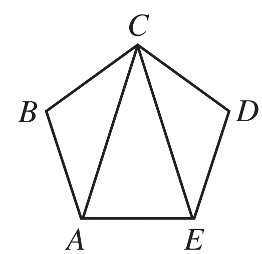
Therefore, $BX=s(\cos 54^\circ)$ and $AX=CX=s(\sin 54^\circ)$.

Therefore, the area of $\Delta ABC$ is

$$1=\tfrac{1}{2}\left[s(\cos 54^\circ)\right]\left[2s(\sin 54^\circ)\right]=s^2\sin 54^\circ\cos 54^\circ$$

which enables us to determine that $s^2=\frac{1}{\sin 54^\circ\cos 54^\circ}$.

$$\begin{aligned}\text{Area of pentagon} &= \text{Area } \Delta ABC+\text{Area } \Delta ACE+\text{Area } \Delta CDE\\ &= 2+\text{Area } \Delta ACE\end{aligned}$$

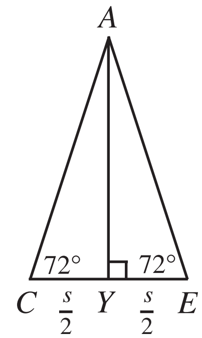



So we lastly consider $\Delta ACE$.

Joining $A$ to $Y$, the midpoint of $CE$, we see that $AY$ is perpendicular to $CE$, $CY = YE = \frac{1}{2}s$ and $\angle ACE = \angle AEC = 72^\circ$. Therefore, $AY = \frac{1}{2}s\left(\tan 72^\circ\right)$, and thus

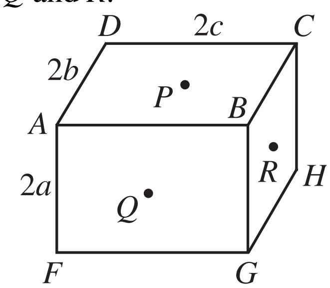
$$\text{Area } \Delta ACE = \tfrac{1}{2}s\left[\tfrac{1}{2}s\left(\tan 72^\circ\right)\right] = \tfrac{1}{4}s^2\left(\tan 72^\circ\right) = \frac{\tan 72^\circ}{4\sin 54^\circ \cos 54^\circ} \approx$$

and so the area of the pentagon is closest to $2 + 1.618$ or 3.62.

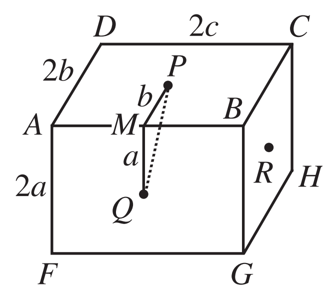



ANSWER: (A)

23. Let the edge lengths of the box be $2a$, $2b$ and $2c$. We label the vertices of the box $ABCDEFGH$.
Let the centres of the three faces meeting at a corner be $P$, $Q$ and $R$.
Then we can say that $PQ = 4$, $QR = 5$, and $PR = 6$.
We would like to try to express these distances in terms of the edge lengths of the box.
We will start by letting $M$ be the midpoint of edge $AB$, and joining $P$ to $M$, $Q$ to $M$ and $P$ to $Q$.
Since $P$ is the centre of face $ABCD$ and $PM$ is perpendicular to $AB$, then $PM = \frac{1}{2}AD = b$. Similarly, $MQ = a$.




Since $MP$ and $MQ$ are both perpendicular to $AB$ and they lie in perpendicular planes $ABCD$ and $ABGF$, then $MP$ is indeed perpendicular to $MQ$, and so $\Delta QMP$ is right-angled.
By Pythagoras, $PQ^2 = MP^2 + MQ^2$ or $16 = a^2 + b^2$.
Similarly, looking at $PR$, $36 = a^2 + c^2$, and, looking at $QR$, $25 = b^2 + c^2$.
So we now have $16 = a^2 + b^2$, $36 = a^2 + c^2$, and $25 = b^2 + c^2$, and we want to determine the volume of the box, which is $V = (2a)(2b)(2c) = 8abc$.




Adding the three equations, we obtain $77 = 2\left(a^2 + b^2 + c^2\right)$, or $a^2 + b^2 + c^2 = \frac{77}{2}$.
Subtracting the above equations one by one from this last equation yields

$$c^2 = \frac{45}{2}$$

$$b^2 = \frac{5}{2}$$

$$a^2 = \frac{27}{2}$$

Therefore, $a^2b^2c^2 = \frac{(27)(5)(45)}{8} = \frac{6075}{8}$ and so $abc = \sqrt{\frac{6075}{8}} = \frac{45\sqrt{3}}{2\sqrt{2}}$.

Thus, the volume is $V = 8\left(\frac{45\sqrt{3}}{2\sqrt{2}}\right) = \left(2\sqrt{2}\right)\left(45\sqrt{3}\right) = 90\sqrt{6}\ \text{cm}^3$.

ANSWER: (C)

24. We first of all rewrite the given expression as

$$\left[(1+x)(1+2x^3)(1+4x^9)(1+8x^{27})(1+16x^{81})(1+32x^{243})(1+64x^{729})\right]^2$$

$$=\left(1+2^0x^{3^0}\right)^2\left(1+2^1x^{3^1}\right)^2\left(1+2^2x^{3^2}\right)^2\left(1+2^3x^{3^3}\right)^2\left(1+2^4x^{3^4}\right)^2\left(1+2^5x^{3^5}\right)^2\left(1+2^6x^{3^6}\right)^2$$

We note that $\left(1+2^rx^{3^r}\right)^2=1+2^rx^{3^r}+2^rx^{3^r}+2^{2r}x^{2\cdot 3^r}$, so each of the four terms in this expansion has the form $2^{mr}x^{m3^r}$ where $m$ is 0, 1 or 2.

When we multiply the expansions of the seven factors, each of the four terms in each expansion will be multiplied by each of the four terms in each other expansion.

So when this complicated-looking expression is multiplied out, every term is of the form

$$2^{0a+1b+2c+3d+4e+5f+6g}x^{a3^0+b3^1+c3^2+d3^3+e3^4+f3^5+g3^6}$$

where the values of $a$, $b$, $c$, $d$, $e$, $f$, $g$ can each be 0, 1, 2, telling us to take the first term, one of the middle terms, or the last term, respectively.

Since we are looking for the coefficient of $x^{2003}$, we need to look at the equation

$a3^0+b3^1+c3^2+d3^3+e3^4+f3^5+g3^6=2003$

where each of the coefficients can be 0, 1 or 2.

Let's look first at the value of $g$. Could $g$ be 0 or 1? If this is the case then the largest possible value of the left side $2\cdot 3^0+2\cdot 3^1+2\cdot 3^2+2\cdot 3^3+2\cdot 3^4+2\cdot 3^5+3^6=1457$.

But this is too small, so $g=2$.

Substituting and simplifying, we get $a3^0+b3^1+c3^2+d3^3+e3^4+f3^5=545$.

Next, we look at the value of $f$. If $f$ was 0 or 1, then

$a3^0+b3^1+c3^2+d3^3+e3^4+f3^5\le 2\cdot 3^0+2\cdot 3^1+2\cdot 3^2+2\cdot 3^3+2\cdot 3^4+3^5=485$

So $f=2$.

In a similar fashion, we can determine that $e=0$, $d=2$, $c=0$, $b=1$, and $a=2$.

These values are uniquely determined.

[Alternatively, we could notice that the equation

$$a3^0+b3^1+c3^2+d3^3+e3^4+f3^5+g3^6=2003$$

is asking for the base 3 expansion of 2003, which we could calculate as 2202012, thus obtaining the same result.]

So each term containing $x^{2003}$ is of the form

$$2^{0(2)+1(1)+2(0)+3(2)+4(0)+5(2)+6(2)}x^{2003}=2^{29}x^{2003}.$$

To determine the coefficient of $x^{2003}$, we need to determine the number of terms of the form $2^{29}x^{2003}$ that occur before we simplify the expansion.

We note that each term of the form $2^{29}x^{2003}$ has a contribution of one term from each of the seven factors of the form $\left(1+2^rx^{3^r}+2^rx^{3^r}+2^{2r}x^{2\cdot 3^r}\right)$.

Since $a=2$, the expansion $1+x+x+x^2$ of $(1+x)^2$ contributes the term $x^2$ to the product. Similarly, since $d=2$, $f=2$, and $g=2$, the expansions of $\left(1+2^3x^{3^3}\right)^2$, $\left(1+2^5x^{3^5}\right)^2$ and $\left(1+2^6x^{3^6}\right)^2$ each contribute their last term.

Since $c=0$ and $e=0$, the expansions of $\left(1+2^2x^{3^2}\right)^2$ and $\left(1+2^4x^{3^4}\right)^2$ each contribute their first term, namely 1.

Since $b=1$, the expansion of $\left(1+2^1x^{3^1}\right)^2$ contributes the two middle terms $2^1x^{3^1}$ and $2^1x^{3^1}$. There are thus two terms of the form $2^{29}x^{2003}$ in the product, and so their sum is $2\cdot 2^{29}x^{2003}=2^{30}x^{2003}$.

The coefficient of $x^{2003}$ is $2^{30}$.

ANSWER: (C)

25. According to the requirements, the three quantities $4+2112=2116$, $n+2112$, and $4n+2112$ must be perfect squares.

Now $2116=46^2$ (a perfect square), so the conditions say that we must have $n+2112=x^2$ and $4n+2112=y^2$ for some positive integers $x$ and $y$. Since $n$ is a positive integer, then each of $x$ and $y$ must be greater than 46.

We now have 3 variables, but can easily eliminate the $n$ by manipulating the two equations to obtain $4n+4(2112)=4x^2$ and $4n+2112=y^2$, and thus $4x^2-y^2=3(2112)$.

We can factor both sides of this equation to get $(2x+y)(2x-y)=3(11)(192)=2^6 3^2 11$.

So we must now determine the possibilities for $x$ and $y$, and hence the possibilities for $n$. We could do this by direct trial and error, but we will try to reduce the work we have to do somewhat by analyzing the situation.

Suppose $2x+y=A$ and $2x-y=B$, with $AB=2^6 3^2 11$. Then solving for $x$ and $y$, we obtain $x=\dfrac{A+B}{4}$ and $y=\dfrac{A-B}{2}$.

Since $AB$ is even, then at least one of $A$ and $B$ is even.

Since $y=\dfrac{A-B}{2}$ is an integer and one of $A$ and $B$ is even, then both of $A$ and $B$ are even.

Since $A$ and $B$ contain in total 6 powers of 2, then at least one of them contains 3 powers of 2, ie. is divisible by 8 (and thus 4).

Since $x=\dfrac{A+B}{4}$ is an integer and one of $A$ and $B$ is divisible by 4, then both $A$ and $B$ are divisible by 4.

So let $A=4a$ and $B=4b$, and so $x=a+b$ and $y=2a-2b$ and $ab=2^2 3^2 11$.

[We may assume that $y$ is positive, because if $y$ is negative this reverses the role of $a$ and $b$.]

Since $y$ is positive, then $a>b$. How many possibilities for $a$ and $b$ are there given that $ab=2^2 3^2 11$? The integer $2^2 3^2 11$ has $(2+1)(2+1)(1+1)=18$ factors, so there are 9 possibilities for the pair $(a,b)$, namely

$$(396,1),\ (198,2),\ (132,3),\ (99,4),\ (66,6),\ (44,9),\ (36,11),\ (33,12),\ (22,18)$$

Of these, only the first 7 give values for $x$ and $y$ bigger than 46 (ie. positive values for $n$). Therefore, there are 7 possible values for $n$.

[For completeness, it is worth determining these values:

$$n = x^2 - 2112 = (a+b)^2 - 2112$$

so the corresponding values of $n$ are 155497, 37888, 16113, 8497, 3072, 697, and 97.]

ANSWER: (B)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2002 Solutions
# Fermat Contest (Grade 11)

for

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING**
*Awards*

1. If $x = 3$, $5 - 2x^2 = 5 - 2(3)^2 = 5 - 18 = -13$.

ANSWER: (C)

2. Evaluating,
$$\frac{3^3 + 3}{2^2 + 2} = \frac{27 + 3}{4 + 2} = \frac{30}{6} = 5.$$

ANSWER: (E)

3. Since there are 24 hours in a day, then 56 hours is two full days (48 hours) plus an additional 8 hours. So to find the correct time, we add 8 hours to 9:04 a.m., and obtain 5:04 p.m.

ANSWER: (B)

4. We look at each of the five statements.
25 is a perfect square, since $25 = 5^2$.
31 is a prime number, since it has no positive factors other than 1 and 31.
3 is not the smallest prime number, since 2 is a prime number.
8 is a perfect cube, since $8 = 2^3$.
15 is the product of two prime numbers, since $15 = 3 \times 5$.

ANSWER: (C)

5. The area of the entire poster is $(50 \text{ cm})(100 \text{ cm}) = 5000 \text{ cm}^2$.
The area of the picture of Pierre de Fermat is $(20 \text{ cm})(40 \text{ cm}) = 800 \text{ cm}^2$.
Thus, the percentage of the poster covered by the picture is
$$\frac{800 \text{ cm}^2}{5000 \text{ cm}^2} \times 100\% = \frac{8}{50} \times 100\% = 16\%$$

ANSWER: (B)

6. Let the heights of Gisa, Henry, Ivan, Justina and Katie be $G$, $H$, $I$, $J$, and $K$, respectively. From the first sentence, $H < G < J$. From the second sentence, $K < I < G$. So $J$ is bigger than all of $G$, $H$, $I$, and $K$, so Justina is the tallest.

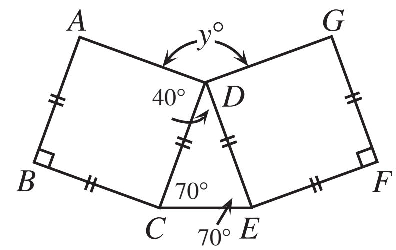
ANSWER: (D)

7. We can determine the area of the shaded region by guessing the side lengths of the various rectangles. Let us suppose that the top left rectangle has a width of 2 and a height of 3. Then the top right rectangle has a width of 5, since its height is also 3. Thus, we can conclude that the height of the bottom right rectangle is 5. This tells us that the shaded rectangle is 2 by 5, or has an area of 10. This problem can also be solved with a more algebraic approach, but this is the most straightforward way.

ANSWER: (E)

8. Since squares $ABCD$ and $DEFG$ have equal side lengths, then $DC = DE$, ie. $\Delta CDE$ is isosceles. Therefore, $\angle DEC = \angle DCE = 70^\circ$ and so

$$\angle CDE = 180^\circ - 70^\circ - 70^\circ = 40^\circ$$




and

$$y^\circ = 360^\circ - \angle ADC - \angle CDE - \angle GDE$$
$$y^\circ = 360^\circ - 90^\circ - 40^\circ - 90^\circ$$
$$y^\circ = 140^\circ$$
$$y = 140$$

ANSWER: (E)

9. There are 20 possible golf balls that can be drawn, and 6 of them have multiples of 3 written on them (ie. 3, 6, 9, 12, 15, 18). Therefore, the probability that the number on the golf ball drawn is a multiple of 3 is $\frac{6}{20}$. ANSWER: (B)

10. Since $ABCD$ is a square, then $AB = BC$, or

$$x + 16 = 3x$$
$$16 = 2x$$
$$x = 8$$

Thus the side length of the square is $x + 16 = 3x = 24$, and the perimeter is $4(24) = 96$.

ANSWER: (C)

11. The slope of the line is

$$\frac{-2-0}{0-1} = 2$$

and its $y$-intercept is –2 (from the first point given), so it has equation $y = 2x - 2$. Substituting in the point $(7, b)$, we obtain $b = 2(7) - 2 = 12$. ANSWER: (A)

12. We determine the smallest and largest three-digit perfect squares first.
Clearly, the smallest three-digit perfect square is $100 = 10^2$.
Now $\sqrt{1000} \approx 31.6$, so the largest perfect square less than 1000 is $31^2 = 961$ (since $31^2 < 1000$ and $32^2 > 1000$).
Therefore, only the perfect squares between $10^2 = 100$ and $31^2 = 961$ have three digits, which means that exactly 22 three-digit positive integers are perfect squares.

ANSWER: (B)

13. A “double-single” number is of the form *aab*, where *a* and *b* are different digits. There are 9 possibilities for *a* (since *a* cannot be 0). For each of these possibilities, there are 9

possibilities for *b* (since *b* can be any digit from 0 to 9, provided that it doesn't equal *a*). So there are $9 \times 9 = 81$ possibilities in total. ANSWER: (A)

14. If we divide 2002 by 7, we see that $2002 = 7(286)$. Since there are 7 natural numbers in each row, and the last entry in each row is the multiple of 7 corresponding to the row number, then 2002 must lie in the $7^{\text{th}}$ column of the $286^{\text{th}}$ row. So $m = 7$, $n = 286$, and $m + n = 293$. ANSWER: (D)

15. Using the rules for forming the sequence, the third term is $a + 2$ and the fourth term is $a + 2 + (a + 2) = 2(a + 2)$. Similarly, the fifth term is $4(a + 2)$ and the sixth term is $8(a + 2)$. But the sixth term is 56, so $8(a + 2) = 56$ or $a + 2 = 7$ or $a = 5$.

ANSWER: (E)

16. *Solution 1*

We factor out a common factor of *a* from the first two terms and a common factor of *b* from the last two terms, and so

$$\begin{aligned} ac + ad + bc + bd &= 68 \\ a(c + d) + b(c + d) &= 68 \\ a(4) + b(4) &= 68 \\ 4(a + b) &= 68 \\ a + b &= 17 \end{aligned}$$

using the fact that $c + d = 4$.

Then $a + b + c + d = (a + b) + (c + d) = 17 + 4 = 21$.

*Solution 2*

We let $c = d = 2$, since $c + d = 4$, and then substitute these values to find that

$$\begin{aligned} ac + ad + bc + bd &= 68 \\ 2a + 2a + 2b + 2b &= 68 \\ a + b &= 17 \end{aligned}$$

Thus $(a + b) + (c + d) = 17 + 4 = 21$.

ANSWER: (D)

17. Let the number of females in the group be *F*.

Then since the average age of the females is 28, then the sum of the ages of the females is $28F$.

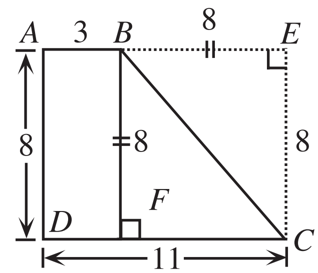
There are also $140 - F$ males in the group, and the sum of there ages is $21(140 - F)$.

So since the average of all of the ages is 24, then

$$\frac{\text{Sum of all ages}}{140} = 24$$
$$\frac{28F + 21(140 - F)}{140} = 24$$
$$28F + 21(140) - 21F = 24(140)$$
$$7F = 3(140)$$
$$F = 60$$

Therefore, there are 60 females in the group. ANSWER: (D)

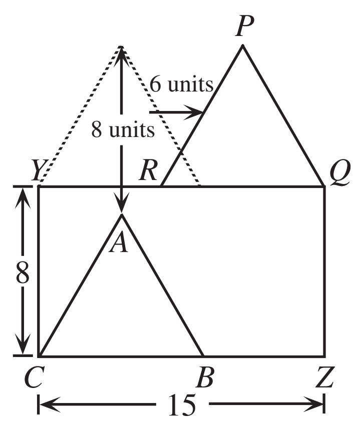
18. Since $E$ is folded over onto $F$, then $BE$ is equal to $BF$ and $\angle BFC = \angle BEC = 90^\circ$.
So $BECF$ is a rectangle, and since $EC = 8$ cm, then $BF = 8$ cm and so $BE = FC = 8$ cm. Thus, $AB = AE - BE = 11 - 8 = 3$ cm and $BC = \sqrt{BF^2 + FC^2} = \sqrt{64 + 64} = \sqrt{128} \approx 11.31$ cm
Therefore, the perimeter of the trapezoid is approximately $3 + 11.31 + 11 + 8 = 33.31$ cm, which is closest to 33.3 cm.




ANSWER: (A)

19. We rewrite the right side using exponent laws:
$$2^a 3^b = 8(6^{10})$$
$$= 2^3([(2)(3)]^{10})$$
$$= 2^3(2^{10}3^{10})$$
$$= 2^{13}3^{10}$$
and so $a = 13$ and $b = 10$, which tells us that $b - a = -3$. ANSWER: (E)

20. Since $\Delta ABC$ and $\Delta PQR$ are both equilateral triangles with side length 9, they are congruent triangles.
So we can think of translating $\Delta ABC$ until it lies on top of $\Delta PQR$ by shifting it up 8 units, until $CB$ lies on $YQ$ ($C$ will coincide with $Y$) and then shifting it to the right $15 - 9 = 6$ units until $B$ coincides with $Q$. (This last shift is 6 units to the right since $YQ = 15$ and $CB = 9$, so $B$ will be a distance of 6 from $Q$ before this second shift.)
So we move from $A$ to $P$ by moving 8 units up and 6 to the right, or a distance of $\sqrt{6^2 + 8^2} = 10$ units.




ANSWER: (A)

21. Since

$$\sqrt{\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdot \cdots \cdot\frac{2n+1}{2n-1}}=9$$

then

$$\frac{3}{1}\cdot\frac{5}{3}\cdot\frac{7}{5}\cdot \cdots \cdot\frac{2n+1}{2n-1}=81$$

In the product on the left-hand side, the numerator in each fraction but the last appears as the denominator in the next fraction and so can be divided out. (This is called a "telescoping product".) After doing this cancellation, all that is left is the denominator of 1 from the first fraction and the numerator of $2n+1$ from the last fraction. Therefore, $\frac{2n+1}{1}=2n+1=81$ or $n=40$.

ANSWER: (C)

22. Using the equation and information given,

$$f(2)=f(1+1)=f(1)+f(1)+2(1)(1)=4+4+2=10.$$

(Notice that there is nothing else we could have actually calculated at this stage.)

Continuing,

$$f(4)=f(2+2)=f(2)+f(2)+2(2)(2)=10+10+8=28$$
$$f(8)=f(4+4)=f(4)+f(4)+2(4)(4)=28+28+32=88$$

Alternatively, we could have used the functional equation to calculate the values of $f(2)$, $f(3)=f(2+1)$, $f(4)=f(3+1)$, etc., all the way up to $f(8)$.

ANSWER: (C)

23. Once the $m$ eights and $k$ nines are added to the list, we have $9+m+k$ numbers in total and the sum of these numbers is

$$1+2+3+4+5+6+7+8+9+8m+9k=45+8m+9k.$$

So we calculate the average and equate with the given value

$$\begin{aligned}\frac{\text{Sum of numbers in list}}{\text{Number of numbers in list}}&=7.3\\ \frac{45+8m+9k}{9+m+k}&=\frac{73}{10}\\ 450+80m+90k&=657+73m+73k\\ 7m+17k&=207\end{aligned}$$

Next we note that $m$ and $k$ are both positive integers, and that $k<13$, since $13(17)=221>207$. So we try the possibilities for $k$ from 1 to 12, and see that for $m$ to be a positive integer, the only possibility is $k=6$, which gives $m=15$.

Thus, $k+m=21$.

ANSWER: (B)

24. First, we calculate the volumes of the two cylindrical containers:

$$V_{\text{large}} = \pi(6)^2(20) = 720\pi \text{ cm}^3$$

$$V_{\text{small}} = \pi(5)^2(18) = 450\pi \text{ cm}^3$$

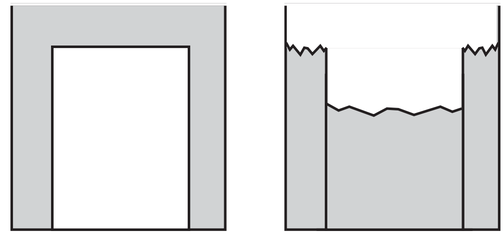


Figure 3 Figure 4

The volume of water initially contained in the large cylinder is

$$V_{\text{water, initial}} = \pi(6)^2(17) = 612\pi \text{ cm}^3$$

The easiest way to determine the final depth of water in the small cylinder is as follows. Imagine putting a lid on the smaller container and lowering it all the way to the bottom of the larger container, as shown in Figure 3. So there will be water beside and above the smaller container. Note that the larger container will be filled to the brim (since the combined volume of the small container and the initial water is greater than the volume of the large container) and some water will have spilled out of the larger container.

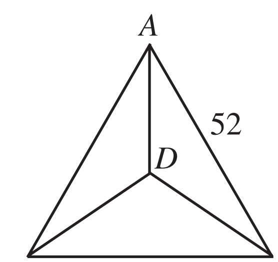
Now if the lid on the small container is removed, all of the water in the large container above the level of the brim of the small container will spill into the small container, as shown in Figure 4. This water occupies a cylindrical region of radius 6 cm and height 2 cm, and so has a volume of $\pi(6)^2(2) = 72\pi \text{ cm}^3$. This is the volume of water that is finally in the small container. Since the radius of the small container is 5 cm, then the depth of water is

$$\text{Depth} = \frac{72\pi \text{ cm}^3}{\pi(5 \text{ cm})^2} = \frac{72}{25} \text{ cm} = 2.88 \text{ cm}$$

ANSWER: (D)

25. The important fact to remember to solve this problem is that if *a*, *b*, *c* are the side lengths of a triangle, then $a+b>c$, $a+c>b$, and $b+c>a$. This is called the "Triangle Inequality". We can simplify this slightly by saying that if *a*, *b*, *c* are the side lengths of a triangle with $a \le b \le c$, then we need $a+b>c$ to be true. (The other two inequalities are automatically satisfied since *c* is the longest side.)

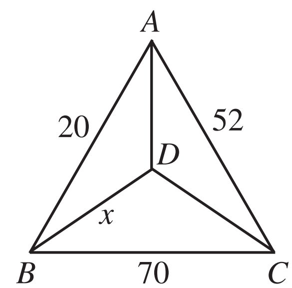
Let *x* represent the length of the unknown edge of the tetrahedron.
We represent the tetrahedron schematically as in the diagram. We can think here of triangle *ABC* as being the "base" of the tetrahedron, and vertex *D* as being the "top" vertex of the tetrahedron. This diagram also emphasizes that the tetrahedron is formed by 4 triangles *ABC*, *ABD*, *ADC*, and *BDC*.



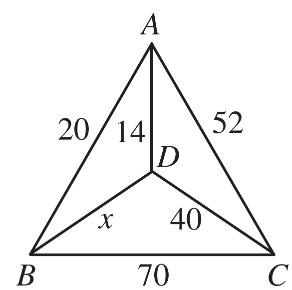

Let $BC = 70$. Now *BC* is an edge in two triangles (*ABC* and *BDC*) and so one of these triangles must be formed entirely from known edge lengths. Of the known edge lengths, only 40 and 52, or 20 and 52 can form a triangle with a third side of 70, by the Triangle Inequality. So let $AC = 52$ (the common edge length of these two triangles).

If $AB = 40$, then the three remaining edge lengths are 14, 20 and $x$. 14 and 20 cannot together make a triangle with any of 40, 52, or 70. So there are no possibilities for $x$ in this case.
If $AB = 20$, the three remaining edge lengths are 14, 40 and $x$. 14 and 40 cannot together make a triangle with 70, so either $BD = x$ or $DC = x$.

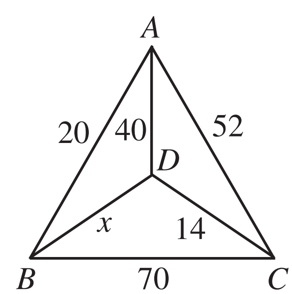
If $DC = x$, then triangle $ABD$ must have side lengths 20, 14 and 40, which is impossible, so $BD = x$.




Case 1: $AD = 14$ and $DC = 40$
From $\Delta ABD$, $20 + 14 > x$ and $14 + x > 20$, so since $x$ is an integer, $7 \le x \le 33$.
From $\Delta BDC$, $40 + 70 > x$ and $x + 40 > 70$, so since $x$ is an integer, $31 \le x \le 109$.
Combining these conditions, $31 \le x \le 33$.




Case 1: $DC = 14$ and $AD = 40$
From $\Delta ABD$, $20 + 40 > x$ and $20 + x > 40$, so since $x$ is an integer, $21 \le x \le 59$.
From $\Delta BDC$, $14 + 70 > x$ and $x + 14 > 70$, so since $x$ is an integer, $57 \le x \le 83$.
Combining these conditions, $57 \le x \le 59$.




So there are 6 possibilities for $x$.
[Note: In fact, all six of these possibilities do in fact form tetrahedra with the other 5 edge lengths. As we discovered as we were creating this question, it is possible to come up with side lengths that do form 4 triangles, but which cannot be put together to form a tetrahedron. This is a very interesting problem to work on.] ANSWER: (E)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 2001 Solutions
# Fermat Contest (Grade 11)

for

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING**
*Awards*

## Part A

1. If $x+2x+3x+4x=5$, then $x$ equals

**(A)** 10 **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{4}$ **(D)** 2 **(E)** $\frac{5}{9}$

*Solution*

$$\begin{aligned} x+2x+3x+4x&=5 \\ 10x&=5 \\ x&=\tfrac{1}{2} \end{aligned}$$

ANSWER: (B)

2. If $x=\frac{1}{4}$, which of the following has the largest value?

**(A)** $x$ **(B)** $x^2$ **(C)** $\frac{1}{2}x$ **(D)** $\dfrac{1}{x}$ **(E)** $\sqrt{x}$

*Solution*

If we calculate the value of the given expressions, we get

**(A)** $\frac{1}{4}$ **(B)** $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$ **(C)** $\frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$ **(D)** $\dfrac{1}{\frac{1}{4}} = 1\times 4 = 4$ **(E)** $\sqrt{\frac{1}{4}} = \frac{1}{2}$

ANSWER: (D)

3. In a school, 30 boys and 20 girls entered the Fermat competition. Certificates were awarded to 10% of the boys and 20% of the girls. Of the students who participated, the percentage that received certificates was

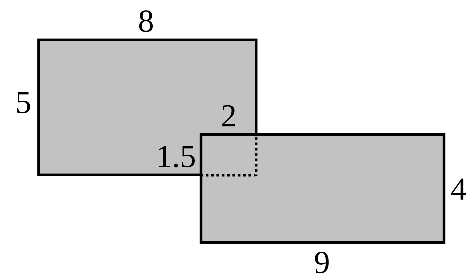
**(A)** 14 **(B)** 15 **(C)** 16 **(D)** 30 **(E)** 50

*Solution*

If 30 boys entered the Fermat competition and 10% of them received certificates, this implies that $(0.1)(30)$ or 3 boys received certificates. Of the 20 girls who entered the competition $(0.2)(20)$ or 4 girls received certificates. This implies that 7 students in total out of 50 received certificates.

Thus 14% of the students in total received certificates. ANSWER: (A)

4. Two rectangles overlap with their common region being a smaller rectangle, as shown. The total area of the shaded region is

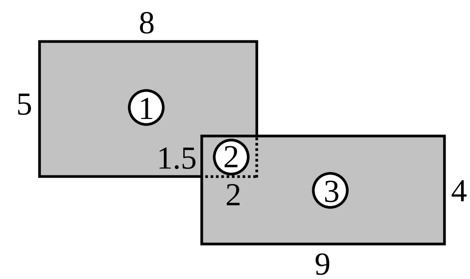



(**A**) 45 (**B**) 70 (**C**) 52
(**D**) 76 (**E**) 73

*Solution 1*

Area ① $= 8 \times 5 -$ Area ② $= 40 - 3 = 37$
Area ③ $= 4 \times 9 -$ Area ② $= 36 - 3 = 33$
Therefore, the shaded area equals,

① + ② + ③ $= 37 + 3 + 33 = 73$.




*Solution 2*

Shaded area = (Area of $5 \times 8$ rectangle) + (Area of $4 \times 9$ rectangle) – Overlap

$= 40 + 36 - 3$

$= 73$

ANSWER: (E)

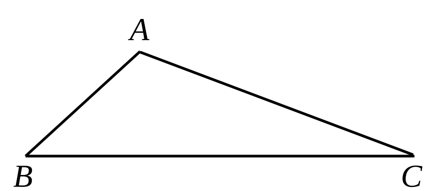
5. In $\Delta ABC$, $\angle A = 3\angle B$ and $\angle B = 2\angle C$. The measure of $\angle B$ is

(**A**) $10°$ (**B**) $20°$ (**C**) $30°$ (**D**) $40°$ (**E**) $60°$

*Solution*

Since we have a triangle,

$$\angle A + \angle B + \angle C = 180°$$
$$3(\angle B) + \angle B + \tfrac{1}{2}(\angle B) = 180°$$
$$\tfrac{9}{2}(\angle B) = 180°$$
$$\angle B = 40°.$$




ANSWER: (D)

6. Pat gives half of his marbles to his best friend and then a third of those remaining to his sister. If his sister receives 9 marbles, then the number Pat keeps is

(**A**) 27 (**B**) 54 (**C**) 18 (**D**) 36 (**E**) 9

*Solution*

Let $x$ be the total number of marbles that Pat has initially.

Then he gives $\frac{1}{2}x$ to his best friend, and $\frac{1}{3} \times \frac{1}{2}x = \frac{1}{6}x$ marbles to his sister.

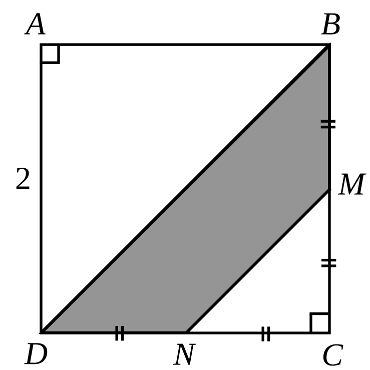
So $\frac{1}{6}x = 9$

$x = 54.$

Pat keeps $x - \frac{1}{2}x - \frac{1}{6}x = \frac{1}{3}x = 18$ marbles.

ANSWER: (C)

7. In the diagram, square $ABCD$ has side length 2, with $M$ the midpoint of $BC$ and $N$ the midpoint of $CD$. The area of the shaded region $BMND$ is




(**A**) 1 (**B**) $2\sqrt{2}$ (**C**) $\frac{4}{3}$

(**D**) $\frac{3}{2}$ (**E**) $4 - \frac{3}{2}\sqrt{2}$

*Solution*

The area of $\Delta MNC$ is $\frac{1}{2}(1)(1) = \frac{1}{2}$. Since $\Delta BDC$ is half the square, it will have an area of 2.

Since the shaded region has an area equal to that of $\Delta BDC$ minus the area of $\Delta MNC$, its area will be $2 - \frac{1}{2} = \frac{3}{2}$.

ANSWER: (D)

8. If $\sqrt{5+11-7} = \sqrt{5+11} - \sqrt{x}$, then the value of $x$ is

(**A**) 1 (**B**) 7 (**C**) $-7$ (**D**) 49 (**E**) 4

*Solution*

$$\begin{aligned}
\sqrt{5+11-7} &= \sqrt{5+11} - \sqrt{x} \\
\sqrt{9} &= \sqrt{16} - \sqrt{x} \\
3 &= 4 - \sqrt{x} \\
\sqrt{x} &= 1 \\
x &= 1
\end{aligned}$$

ANSWER: (A)

9. A bag contains 20 candies: 4 chocolate, 6 mint and 10 butterscotch. Candies are removed randomly from the bag and eaten. What is the minimum number of candies that must be removed to be *certain* that at least two candies of each flavour have been eaten?

(**A**) 6 (**B**) 10 (**C**) 12 (**D**) 16 (**E**) 18

*Solution*
At most, 17 candies could be removed before the second chocolate candy is removed, that is all 10 butterscotch, all 6 mint, and 1 chocolate.
So we need to remove 18 candies to ensure that 2 of each flavour have been eaten.

ANSWER: (E)

10. When a positive integer $N$ is divided by 60, the remainder is 49. When $N$ is divided by 15, the remainder is

(**A**) 0 (**B**) 3 (**C**) 4 (**D**) 5 (**E**) 8

*Solution*
This problem can be done in a number of ways. The easiest way is to consider that if $N$ is divided by 60 to achieve a remainder of 49, it must be a number of the form, $60k+49$, $k=0,1,2,\ldots$.
This implies that the smallest number to meet the requirements is 49 itself. If we divide 49 by 15 we get a remainder of 4. Or, if $k=1$ in our formula then the next number to satisfy the requirements is 109 which when divided by 15 gives 4 as the remainder. ANSWER: (C)

## Part B

11. The fourth root of 2001200120012001 (that is, $\sqrt[4]{2001200120012001}$) is closest to

(**A**) 2 001 (**B**) 6 700 (**C**) 21 000 (**D**) 12 000 (**E**) 2 100

*Solution*
2001200120012001 is roughly $2\times10^{15}=2000\times10^{12}$.
So the quantity desired is roughly $\sqrt[4]{2000\times10^{12}}=\sqrt[4]{2000}\times10^{3}\approx7\times10^{3}$ which is closest to 6700.

ANSWER: (B)

12. How many integer values of $x$ satisfy $\frac{x-1}{3}<\frac{5}{7}<\frac{x+4}{5}$?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 4

*Solution*
If we multiply all three fractions by $3(5)(7)$ we have,

$$(\cancel{3})(5)(7)\frac{(x-1)}{\cancel{3}}<(3)(5)(\cancel{7})\frac{5}{\cancel{7}}<(3)(\cancel{5})(7)\frac{x+4}{\cancel{5}}$$

$$35(x-1)<75<21(x+4)$$

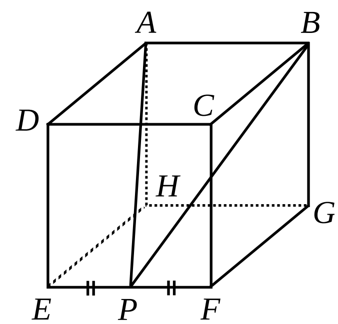
In order to satisfy this inequality then,

$$35(x-1)<75 \quad \text{and} \quad 21(x+4)>75$$
$$35x-35<75 \quad \text{and} \quad 21x+84>75$$

$$\begin{array}{ll} 35x < 110 & 21x > -9 \\ x < 3\frac{1}{7} & x > -\frac{9}{21} \end{array}$$

The only integers to satisfy both conditions are then in the set $\{0, 1, 2, 3\}$. ANSWER: (E)

13. $ABCDEFGH$ is a cube having a side length of 2. $P$ is the midpoint of $EF$, as shown. The area of $\Delta APB$ is




(**A**) $\sqrt{8}$ (**B**) 3 (**C**) $\sqrt{32}$
(**D**) $\sqrt{2}$ (**E**) 6

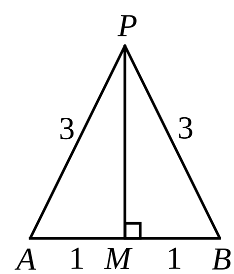
*Solution*

By symmetry, the lengths of $AP$ and $BP$ will be equal, and $AP = \sqrt{AD^2 + DE^2 + EP^2} = \sqrt{2^2 + 2^2 + 1^2} = 3$.

If $M$ is the midpoint of $AB$, then $PM$ is perpendicular to $AB$. By Pythagoras, $MP = \sqrt{3^2 - 1^2} = \sqrt{8}$.

So the area of $\Delta APB$ is

$$\text{Area} = \tfrac{1}{2}(2)\left(\sqrt{8}\right) = \sqrt{8}.$$




ANSWER: (A)

14. The last digit (that is, the units digit) of $(2002)^{2002}$ is

(**A**) 4 (**B**) 2 (**C**) 8 (**D**) 0 (**E**) 6

*Solution*

The units digit of $(2002)^{2002}$ is the same as the units digit of $2^{2002}$, since the first three digits of 2002 do not affect the units digit.

We write out the first few powers of 2 and check for the units digit.

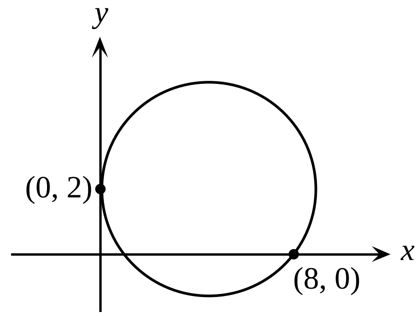
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $2^n$ | 2 | 4 | 8 | 16 | 32 | 64 | 128 | 256 | 512 |

From this table, we see that the units digits repeat every 4 powers. So the units digit of $2^{2000}$ will be 6, and thus the units digit of $2^{2002}$ (and so also of $(2002)^{2002}$) will be 4.

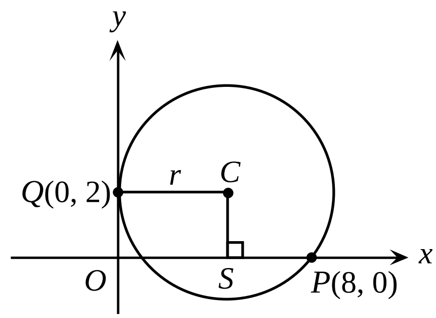
ANSWER: (A)

15. A circle is tangent to the $y$-axis at $(0, 2)$, and the larger of its $x$-intercepts is 8. The radius of the circle is

(**A**) $\frac{9}{2}$ (**B**) $\sqrt{17}$ (**C**) $\frac{17}{4}$
(**D**) $\frac{15}{4}$ (**E**) $\frac{\sqrt{17}}{2}$




*Solution*
Let the centre be $C$ and the radius $r$.
Then $CQ$ is perpendicular to the $y$-axis, and has length $r$.
So the $x$-coordinate of $C$ is $r$ and the $y$-coordinate is 2, ie. $C(r, 2)$.
Drop a perpendicular $CS$ to the $x$-axis and consider right triangle $CSP$.
$CP = r$, $CS = 2$ and $SP = 8 - r$.

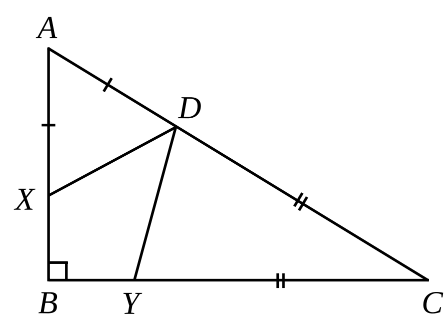



By Pythagoras, $r^2 = (8-r)^2 + 2^2 \Rightarrow r^2 = 64 - 16r + r^2 + 4 \Rightarrow 16r = 68 \Rightarrow r = \frac{17}{4}$.

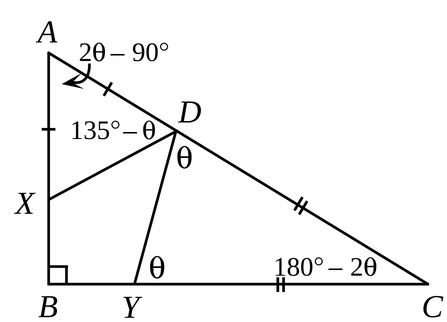
ANSWER: (C)

16. In right triangle $ABC$, $AX = AD$ and $CY = CD$, as shown. The measure of $\angle XDY$ is

(**A**) 35° (**B**) 40° (**C**) 45°
(**D**) 50° (**E**) not determined by this information




*Solution*
Let $\angle DYC = \theta$. Then $\angle YDC = \theta$ ($\Delta YDC$ is isosceles).
Then $\angle YCD = 180° - 2\theta$ (sum of angles in $\Delta YDC$).
Then $\angle CAB = 2\theta - 90°$ (sum of angles in $\Delta ABC$).
In $\Delta AXD$, $\angle AXD = \angle ADX$ so $2\theta - 90° + 2\angle AXD = 180°$.
So $\angle AXD = 135° - \theta$ and $\angle ADX = 135 - \theta$.
But $\angle ADX + \angle XDY + \angle YDC = 180°$.
Therefore, $135 - \theta + \angle XDY + \theta = 180°$.
Thus, $\angle XDY = 45°$.




ANSWER: (C)

17. Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69 and 76 result. The average of the three original numbers is

(**A**) 34 (**B**) 35 (**C**) 36 (**D**) 37 (**E**) 38

*Solution*
Let the three numbers be $a$, $b$ and $c$.
We construct the first equation to be,

$$a+\frac{b+c}{2}=65.$$

Or, $2a+b+c=130$.
Similarly we construct the two other equations to be,

$$a+2b+c=138$$

and $$a+b+2c=152.$$

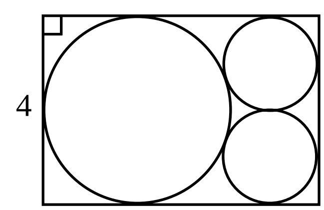
If we add the three equations we obtain,

$$4a+4b+4c=420.$$

The average is $\dfrac{4(a+b+c)}{12}=\dfrac{420}{12}$.

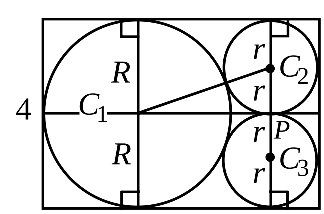
Or, $\dfrac{a+b+c}{3}=35$. ANSWER: (B)

18. In the diagram, the two smaller circles have equal radii. Each of the three circles is tangent to the other two circles, and each is also tangent to sides of the rectangle. If the width of the rectangle is 4, then its length is




(**A**) $2+\sqrt{8}$ (**B**) $3+\sqrt{8}$ (**C**) $3+\sqrt{10}$
(**D**) $\sqrt{32}$ (**E**) $4+\sqrt{3}$

*Solution*
Let the radius of the larger circle be $R$.
Let the radii of the smaller circle be $r$.
From the diagram, $2R=4$ so $R=2$ and $4r=4$ so $r=1$, as the radii are perpendicular to the sides of the rectangle since the circles are tangent to the sides of the rectangle.




Join $C_1$ to $P$ and extend until it hits both vertical sides of the rectangle.
Therefore the length of the rectangle is $R+C_1P+r=3+C_1P$.
Now $C_2P\perp C_1P$ and $C_2C_1$ goes through the point of tangency between the larger circle and the upper circle, so $C_1C_2=R+r=3$.
By Pythagoras $3^2=(C_1P)^2+1^2 \Rightarrow (C_1P)^2=8 \Rightarrow C_1P=\sqrt{8}$.
Thus the length is $3+\sqrt{8}$. ANSWER: (B)

19. Cindy leaves school at the same time every day. If she cycles at 20 km/h, she arrives home at 4:30 in the afternoon. If she cycles at 10 km/h, she arrives home at 5:15 in the afternoon. At what speed, in km/h, must she travel to arrive home at 5:00 in the afternoon?

(**A**) $16\frac{2}{3}$ (**B**) 15 (**C**) $13\frac{1}{3}$ (**D**) 12 (**E**) $18\frac{3}{4}$

*Solution*

Since the distance from Cindy's home to school is unknown, represent this distance by *d*, in kilometres. We will consider the problem in two separate cases, the first in which she travels at 20 km/h and the second when she travels at 10 km/h.

Distance travelled at 20 km/h = Distance travelled at 10 km/h

Let the time that Cindy takes travelling home at 20 km/h be *t* hours.
If Cindy arrives home $\frac{3}{4}$ h later when travelling at 10 km/h, then the length of time travelling is $\left(t+\frac{3}{4}\right)$ hours. The previous equation becomes

$$20t = 10\left(t+\frac{3}{4}\right)$$
$$20t = 10t + \frac{30}{4}$$
$$10t = \frac{15}{2}$$
$$t = \frac{15}{20} \text{ or } \frac{3}{4}.$$

Therefore the distance from school to home is $d = 20 \times \frac{3}{4}$, or $d = 15$ km.

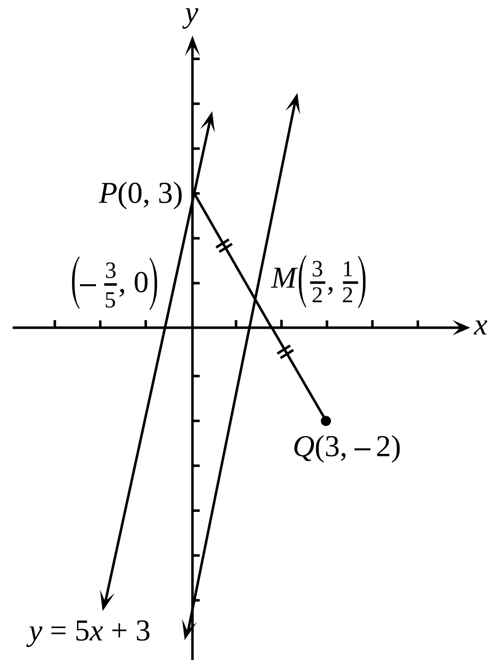
If Cindy arrives home at 5:00 in the afternoon, she would have travelled home in $\frac{3}{4}+\frac{1}{2}=\frac{5}{4}$ hours over a distance of 15 kilometres.
Therefore, $s = \frac{d}{t} = \frac{15}{\frac{5}{4}} = 15 \times \frac{4}{5} = 12$ km/h.
Therefore, Cindy would have had to travel at 12 km/h to arrive home at 5:00 p.m.

ANSWER: (D)

20. Point $P$ is on the line $y = 5x+3$. The coordinates of point $Q$ are $(3,-2)$. If $M$ is the midpoint of $PQ$, then $M$ must lie on the line

(**A**) $y=\frac{5}{2}x-\frac{7}{2}$ (**B**) $y=5x+1$ (**C**) $y=-\frac{1}{5}x-\frac{7}{5}$ (**D**) $y=\frac{5}{2}x+\frac{1}{2}$ (**E**) $y=5x-7$

We start by drawing a diagram and labelling the intercepts.




*Solution 1*
Since the point $P$ is on the line $y = 5x + 3$, select $P(0, 3)$ as a point on this line.
The midpoint of $PQ$ is $M\left(\frac{3+0}{2}, \frac{-2+3}{2}\right) = M\left(\frac{3}{2}, \frac{1}{2}\right)$.
The required line must contain $M$ and be midway between the given point and $y = 5x + 3$. The only possible line meeting this requirement is the line containing $M\left(\frac{3}{2}, \frac{1}{2}\right)$ and which has a slope of 5. The required line will this have as its equation

$$y - \tfrac{1}{2} = 5\left(x - \tfrac{3}{2}\right)$$

or, $$y = 5x - 7.$$

*Solution 2*
Let a general point on the line $y = 5x + 3$ be represented by $(a, 5a + 3)$. Also, let a point on the required line be $M(x, y)$. Since $M(x, y)$ is the midpoint of $PQ$ then

(1) $x = \dfrac{a+3}{2}$ and (2) $y = \dfrac{(5a+3)+(-2)}{2}$

$$y = \frac{5a+1}{2}$$

Solving (1) for $a$, we have $a = 2x - 3$ and solving (2) for $a$, we have $\frac{2y-1}{5} = a$.

Equating gives, $2x - 3 = \dfrac{2y-1}{5}$

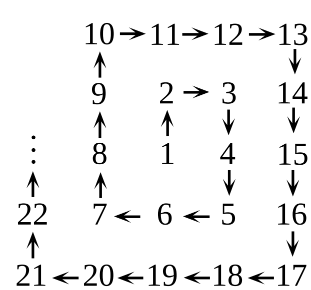
$$10x - 15 = 2y - 1$$

or, $$y = 5x - 7.$$

ANSWER: (E)

## Part C

21. A spiral of numbers is created, as shown, starting with 1. If the pattern of the spiral continues, in what configuration will the numbers 399, 400 and 401 appear?




(**A**) 399 → 400 → 401 (**B**) 401 ← 400 ← 399

**(C)** $\begin{array}{c} 401 \\ \uparrow \\ 400 \\ \uparrow \\ 399 \end{array}$ **(D)** $\begin{array}{c} 399 \\ \downarrow \\ 400 \\ \downarrow \\ 401 \end{array}$ **(E)** $\begin{array}{cc} 400 & \rightarrow 401 \\ \uparrow & \\ 399 & \end{array}$

*Solution*

We notice the following configurations:

| | |
|---|---|
| 2 | 3 |
| 1 | 4 |
| | 5 |

and

| | | | |
|---|---|---|---|
| 10 | 11 | 12 | 13 |
| 9 | 2 | 3 | 14 |
| 8 | 1 | 4 | 15 |
| 7 | 6 | 5 | 16 |
| | | | 17 |

And continuing the pattern of the spiral

| | | | | | |
|---|---|---|---|---|---|
| 26 | 27 | 28 | 29 | 30 | 31 |
| 25 | 10 | 11 | 12 | 13 | 32 |
| 24 | 9 | 2 | 3 | 14 | 33 |
| 23 | 8 | 1 | 4 | 15 | 34 |
| 22 | 7 | 6 | 5 | 11 | 35 |
| 21 | 20 | 19 | 18 | 17 | 36 |
| | | | | | 37 |

So we see that when the pattern continues, all even perfect squares will occur in the configuration

$$\begin{array}{c} (2k)^2 - 1 \\ \downarrow \\ (2k)^2 \\ \downarrow \\ (2k)^2 + 1 \end{array}$$

so we have $\begin{array}{c} 399 \\ \downarrow \\ 400 \\ \downarrow \\ 401 \end{array}$ as $400 = (20)^2$.

ANSWER: (D)

22. A sealed bottle, which contains water, has been constructed by attaching a cylinder of radius 1 cm to a cylinder of radius 3 cm, as shown in Figure A. When the bottle is right side up, the height of the water inside is 20 cm, as shown in the cross-section of the bottle in Figure B. When the bottle is upside down, the height of the liquid is 28 cm, as shown in Figure C. What is the total height, in cm, of the bottle?

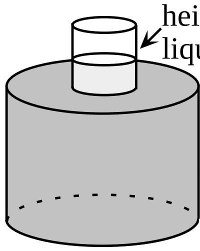


Figure A

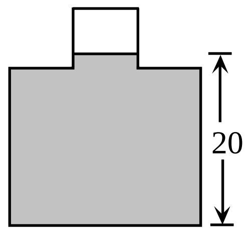


Figure B

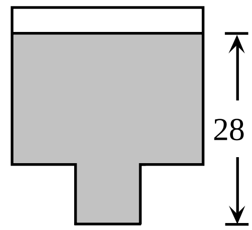


Figure C

(**A**) 29 (**B**) 30 (**C**) 31 (**D**) 32 (**E**) 48

*Solution*

We'll start by representing the height of the large cylinder as $h_1$ and the height of the small cylinder as $h_2$. For simplicity, we'll let $x = h_1 + h_2$.

If the bottom cylinder is completely filled and the top cylinder is only partially filled the top cylinder will have a cylindrical space that is not filled. This cylindrical space will have a height equal to $x - 20$ and a volume equal to, $\pi(1)^2(x-20)$.

Similarly, if we turn the cylinder upside down there will be a cylindrical space unfilled that will have a height equal to $x - 28$ and a volume equal to, $\pi(3)^2(x-28)$.

Since these two unoccupied spaces must be equal, we then have,

$$\begin{aligned}\pi(1)^2(x-20) &= \pi(3)^2(x-28)\\ x-20 &= 9x-252\\ 8x &= 272\\ x &= 29.\end{aligned}$$

Therefore, the total height is 29. ANSWER: (A)

23. A sequence $t_1, t_2, ..., t_n, ...$ is defined as follows:

$t_1 = 14$

$t_k = 24 - 5t_{k-1}$, for each $k \geq 2$.

For every positive integer $n$, $t_n$ can be expressed as $t_n = p \cdot q^n + r$, where $p$, $q$ and $r$ are constants. The value of $p + q + r$ is

(**A**) $-5$ (**B**) $-3$ (**C**) 3 (**D**) 17 (**E**) 31

*Solution 1*

Since $t_n = p \cdot q^n + r$ for every $n \geq 1$, then

$$t_1 = pq + r$$

$$t_2 = pq^2 + r$$

$$t_3 = pq^3 + r.$$

However, $t_1 = 14$, $t_2 = 24 - 5(t_1) = 24 - 5(14) = -46$, and $t_3 = 24 - 5t_2 = 24 - 5(-46) = 254$.

So $pq + r = 14$ (1)

$pq^2 + r = -46$ (2)

$pq^3 + r = 254$ (3)

Subtracting (2) – (1) yields $pq^2 - pq = pq(q-1) = -60$ (4)

Subtracting (3) – (2) yields $pq^3 - pq^2 = pq^2(q-1) = 300$ (5)

Dividing (5) by (4) gives $q = -5$.

Substituting back into (1) and (2)

$-5p + r = 14$ (1)

$25p + r = -46$ (2)

Adding $5\times$ (1) to (2) yields $6r = 24$ so $r = 4$ and thus $p = -2$.

Therefore $t_n = -2(-5)^n + 4$.

So $p + q + r = -2 - 5 + 4 = -3$.

*Solution 2*

Substituting $t_n = p \cdot q^n + r$ and $t_{n-1} = p \cdot q^{n-1} + r$ into the difference equation $t_n = 24 - 5t_{n-1}$, we get

$$p \cdot q^n + r = 24 - 5\left(p \cdot q^{n-1} + r\right)$$

$$p \cdot q^n + 5pq^{n-1} = 24 - 5r - r$$

$$p \cdot q^{n-1}(q+5) = 24 - 6r.$$

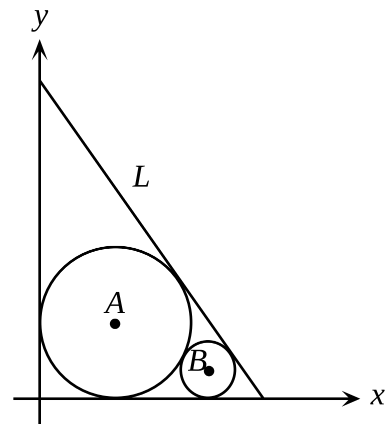
Now the right side is independent of *n*, so the left side must be as well, thus $q + 5 = 0$ or $q = -5$. (Clearly, $p \neq 0$ or else $t_n$ is constant).

So $24 - 6r = 0 \Rightarrow r = 4$.

Therefore $t_n = p(-5)^n + 4$, so $t_1 = 14 = -5p + 4 \Rightarrow p = -2$.

Therefore $p + q + r = -3$. ANSWER: (B)

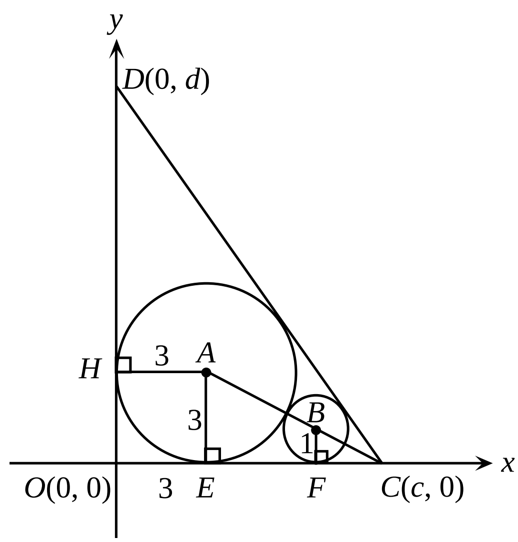
24. The circle with centre $A$ has radius 3 and is tangent to both the positive $x$-axis and positive $y$-axis, as shown. Also, the circle with centre $B$ has radius 1 and is tangent to both the positive $x$-axis and the circle with centre $A$. The line $L$ is tangent to both circles. The $y$-intercept of line $L$ is



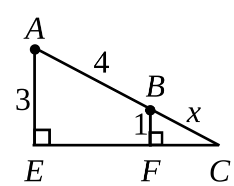

(**A**) $3+6\sqrt{3}$ (**B**) $10+3\sqrt{2}$ (**C**) $8\sqrt{3}$
(**D**) $10+2\sqrt{3}$ (**E**) $9+3\sqrt{3}$

*Solution*

We start by drawing a line from point $C$ that will pass through $A$ and $B$. From $A$ and $B$, we drop perpendiculars to the points of tangency on the $x$-axis and label these points as $E$ and $F$ as shown. We also drop a perpendicular from $A$ to the $y$-axis which makes $AH = AE = 3$.




Extracting $\Delta CAE$ from the diagram and labelling with the given information we would have the following noted in the diagram.




If we represent the distance from $C$ to $B$ as $x$ and recognize that $\Delta CBF$ is similar to $\Delta CAE$,

$$\frac{x}{1} = \frac{x+4}{3}$$

$$x = 2.$$

In $\Delta CBF$, $FC^2 = 2^2 - 1^2 = 3$

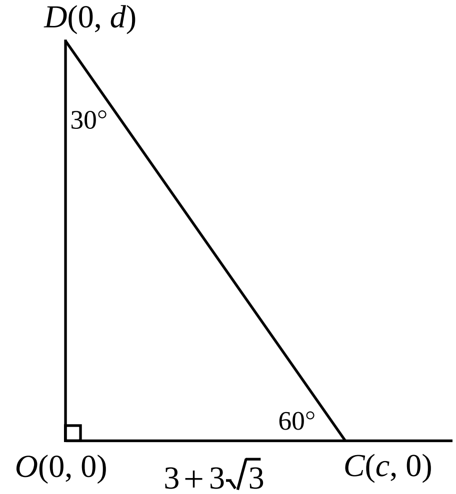
$$FC = \sqrt{3},\ (FC > 0).$$

This implies that $\angle BCF = 30°$ and $\angle OCD = 60°$. Therefore $EF = 2\sqrt{3}$, from similar triangles again.

This now gives us the diagram shown.

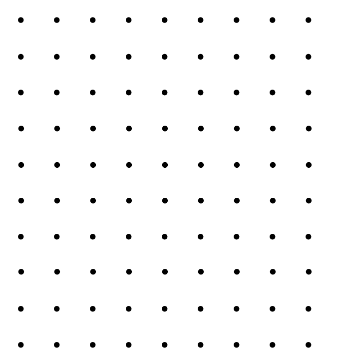
Thus, $d = \sqrt{3}\left(3 + 3\sqrt{3}\right)$

$= 3\sqrt{3} + 9.$




ANSWER: (E)

25. A square array of dots with 10 rows and 10 columns is given. Each dot is coloured either blue or red. Whenever two dots of the same colour are adjacent in the same row or column, they are joined by a line segment of the same colour as the dots. If they are adjacent but of different colours, they are then joined by a green line segment. In total, there are 52 red dots. There are 2 red dots at corners with an additional 16 red dots on the edges of the array. The remainder of the red dots are inside the array. There are 98 green line segments. The number of blue line segments is



(**A**) 36 (**B**) 37 (**C**) 38
(**D**) 39 (**E**) 40

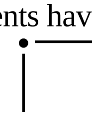
*Solution*

First, we note that there are 9 line segments in each row and in each column, so there are $9(10) + 9(10) = 180$ line segments in total.

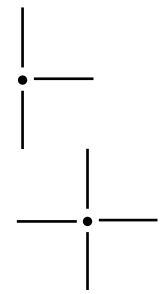
Let $B$ be the number of blue segments and $R$ the number of red segments. Then $B + R + 98 = 180$, so $B + R = 82$, as there are 98 green line segments.

Coming out of a red dot, there can only be a green line segment or a red line segment. We count the *total number* of line segments starting from red dots. Note that in this total, the green segments are counted once and the red segments twice, as the red segments have both ends at red dots.

From a corner dot, there are 2 segments



From an edge dot (not on corner), there are 3 segments



From an interior dot, there are 4 segments

So the total number of segments coming from red dots is

$$2(2)+3(16)+4(34)=188$$

and so, since 98 segments from red dots are accounted for by green segments the remaining $188-98=90$ segments from red dots are accounted for by red segments, for a total of 45 red segments, or $R=45$.

Therefore $B=82-R=37$. ANSWER: (B)

Canadian
Mathematics
Competition

*An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario*

# *2000 Solutions*
# *Fermat Contest* (Grade 11)

for

**The CENTRE for EDUCATION in MATHEMATICS and COMPUTING**
*Awards*

## Part A:

1. The sum $29+12+23$ is equal to

**(A)** $6^2$ **(B)** $4^4$ **(C)** $8^8$ **(D)** $64^0$ **(E)** $2^6$

*Solution*

$$\begin{aligned}29+12+23 &= 64\\ &= 2^6\end{aligned}$$

ANSWER: (E)

2. If the following sequence of five arrows repeats itself continuously, what arrow will be in the 48th position?

$\longrightarrow$, $\nearrow$, $\nwarrow$, $\longleftarrow$, $\downarrow$

**(A)** $\longrightarrow$ **(B)** $\nearrow$ **(C)** $\nwarrow$ **(D)** $\longleftarrow$ **(E)** $\downarrow$

*Solution*

Since this sequence repeats itself, once it has completed nine cycles it will be the same as starting at the beginning. Thus the 48th arrow will be the same as the third one.

ANSWER: (C)

3. A farmer has 7 cows, 8 sheep and 6 goats. How many more goats should be bought so that half of her animals will be goats?

**(A)** 18 **(B)** 15 **(C)** 21 **(D)** 9 **(E)** 6

*Solution 1*

If the cows and sheep were themselves goats we would have 15 goats. This means that she would need nine extra goats.

*Solution 2*

Let the number of goats added be $x$.

Therefore, $\dfrac{6+x}{21+x}=\dfrac{1}{2}$

Cross multiplying gives,
$$2(6+x)=21+x$$
$$12+2x=21+x$$
$$x=9.$$

As in solution 1, she would add 9 goats.

ANSWER: (D)

4. The square of 9 is divided by the cube root of 125. What is the remainder?

(**A**) 6 (**B**) 3 (**C**) 16 (**D**) 2 (**E**) 1

*Solution*

The square of 9 is 81 and the cube root of 125 is 5. When 81 is divided by 5, the quotient is 16 and the remainder is 1.

Since $81=5\times 16+1$, the remainder is 1.

ANSWER: (E)

5. The product of 2, 3, 5, and $y$ is equal to its sum. What is the value of $y$?

(**A**) $\frac{1}{3}$ (**B**) $\frac{10}{31}$ (**C**) $\frac{10}{29}$ (**D**) $\frac{3}{10}$ (**E**) $\frac{10}{3}$

*Solution*

Since
$$(2)(3)(5)(y)=2+3+5+y.$$
$$30y=10+y$$
$$29y=10$$
and
$$y=\tfrac{10}{29}.$$

ANSWER: (C)

6. A student uses a calculator to find an answer but instead of pressing the $\boxed{x^2}$ key presses the $\boxed{\sqrt{x}}$ key by mistake. The student's answer was 9. What should the answer have been?

(**A**) 243 (**B**) 81 (**C**) 729 (**D**) 3 (**E**) 6561

*Solution 1*

Since $\sqrt{x}=9$, $x=81$.

Thus, $x^2=81^2=6561$.

*Solution 2*
Since the square root of the number entered is 9, the number must have been 81. The answer desired is $81^2 = 6561$.

ANSWER: (E)

7. The sum of the arithmetic series $(-300)+(-297)+(-294)+ ... +306+309$ is

(**A**) 309 (**B**) 927 (**C**) 615 (**D**) 918 (**E**) 18

*Solution*
The given series is
$(-300)+(-297)+(-294)+...+(-3)+0+3+...+294+297+300+303+306+309.$
The sum of the terms from –300 to 300 is 0. The sum of all the terms is thus
$303+306+309=918$.

ANSWER: (D)

8. In a school referendum, $\frac{3}{5}$ of a student body voted 'yes' and 28% voted 'no'. If there were no spoiled ballots, what percentage of the students did not vote?

(**A**) 72% (**B**) 40% (**C**) 32% (**D**) 12% (**E**) 88%

*Solution*
Since $\frac{3}{5}$ or 60% of the students voted 'yes' and 28% voted 'no', 88% of the student body voted. Hence, $100\%-88\%$ or $12\%$, of the students did not vote.

ANSWER: (D)

9. The numbers 6, 14, $x$, 17, 9, $y$, 10 have a mean of 13. What is the value of $x+y$?

(**A**) 20 (**B**) 21 (**C**) 23 (**D**) 25 (**E**) 35

*Solution*
If the 7 numbers have a mean of 13, this implies that these numbers would have a sum of $7\times 13$ or 91. We now can calculate $x+y$ since, $6+14+x+17+9+y+10=91$ or, $x+y+56=91$
Therefore, $x+y=35$.

ANSWER: (E)

10. If $x(x(x+1)+2)+3=x^3+x^2+x-6$ then $x$ is equal to

(**A**) 11 (**B**) $-9$ (**C**) $-4$ or 3 (**D**) –1 or 0 (**E**) $-2$

*Solution*

$$x\left(x(x+1)+2\right)+3 = x\left(x^2+x+2\right)+3$$
$$= x^3+x^2+2x+3$$

Since $\quad x^3+x^2+2x+3 = x^3+x^2+x-6$

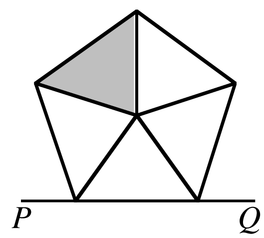
$$2x+3 = x-6$$
$$x = -9$$

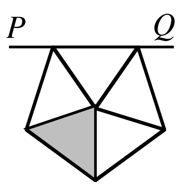
ANSWER: (B)

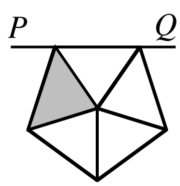
---

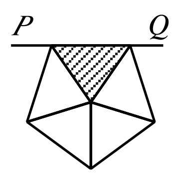
## Part B:

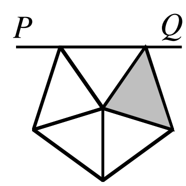
11. When the regular pentagon is reflected in the line $PQ$, and then rotated *clockwise* 144° about the centre of the pentagon, its position is

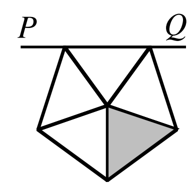



(**A**) 

(**B**) 

(**C**) 

(**D**) 

(**E**) 


*Solution*

After the reflection, the new pentagon is

Since each of the five central angles equals $72^\circ$, a clockwise rotation of $144^\circ\left(2\times 72^\circ\right)$ about its centre will place the pentagon in the shown position.

ANSWER: (C)

12. If the expression $15^6 \times 28^5 \times 55^7$ was evaluated, it would end with a string of consecutive zeros. How many zeros are in this string?

(**A**) 10 (**B**) 18 (**C**) 26 (**D**) 13 (**E**) 5

*Solution*

A zero at the end of a number results from the product of 2 and 5. The number of zeros at the end of a number equals the number of product pairs of 2 and 5 that can be formed from the prime factorization of that number.

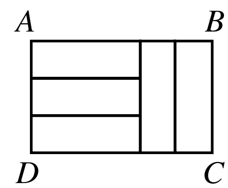
Since $15^6 \times 28^5 \times 55^7 = (3.5)^6\left(2^2.7\right)^5(5.11)^7$

$$= 3^6.5^6.2^{10}.7^5.5^7.11^7$$
$$= 3^6.5^3.7^5.11^7.10^{10}$$

There will be ten zeros in the string at the end of the number.

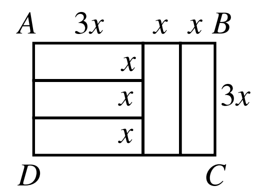
ANSWER: (A)

13. Rectangle $ABCD$ is divided into five congruent rectangles as shown. The ratio $AB : BC$ is




(**A**) 3:2 (**B**) 2:1 (**C**) 5:2
(**D**) 5:3 (**E**) 4:3

*Solution*

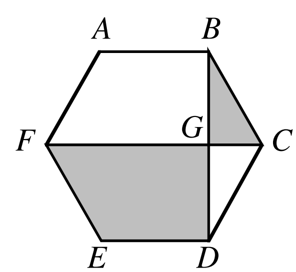



If we let the width of each rectangle be $x$ units then the length of each rectangle is $3x$ units. (This is illustrated in the diagram.) The length, $AB$, is now $3x + x + x$ or $5x$ units and $BC = 3x$. Thus $AB : BC = 5x : 3x$

$$= 5:3, \quad x \neq 0.$$

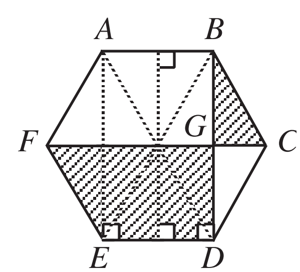
ANSWER: (D)

14. In the regular hexagon $ABCDEF$, two of the diagonals, $FC$ and $BD$, intersect at $G$. The ratio of the area of quadrilateral $FEDG$ to the area of $\Delta BCG$ is




(**A**) $3\sqrt{3}:1$ (**B**) 4:1 (**C**) 6:1
(**D**) $2\sqrt{3}:1$ (**E**) 5:1

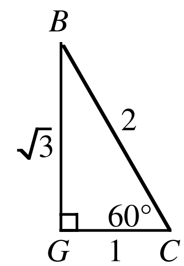
*Solution 1*




Join $E$ to $B$ and $D$ to $A$ as shown. Also join $E$ to $A$ and draw a line parallel to $AE$ through the point of intersection of $BE$ and $AD$. Quadrilateral $FEDG$ is now made up of five triangles each of which has the same area as $\Delta BCG$. The required ratio is 5:1.

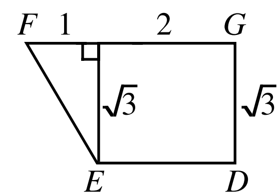
*Solution 2*

For convenience, assume that each side of the hexagon has a length of 2 units. Each angle in the hexagon equals $120^\circ$ so $\angle BCG = \frac{1}{2}(120^\circ) = 60^\circ$. Now label $\Delta BCG$ as shown. Using the standard ratios for a $30^\circ - 60^\circ - 90^\circ$ triangle we have $BG = \sqrt{3}$ and $GC = 1$.




The area of $\Delta BCG = \frac{1}{2}(1)\sqrt{3} = \frac{\sqrt{3}}{2}$. Dividing the quadrilateral $FGDE$ as illustrated, it will have an area of

$2(\sqrt{3}) + \frac{1}{2}(1)(\sqrt{3}) = \frac{5\sqrt{3}}{2}$.




The required ratio is $\frac{5\sqrt{3}}{2} : \frac{\sqrt{3}}{2}$ or 5:1, as in solution 1.

ANSWER: (E)

15. In a sequence, every term after the second term is twice the sum of the two preceding terms. The seventh term of the sequence is 8, and the ninth term is 24. What is the eleventh term of the sequence?

(**A**) 160 (**B**) 304 (**C**) 28 (**D**) 56 (**E**) 64

*Solution*

Let the seventh through eleventh terms of the sequence be $t_7, t_8, t_9, t_{10}$, and $t_{11}$.

Since
$$t_9 = 2(t_7 + t_8),$$
$$24 = 2(8 + t_8)$$
$$12 = 8 + t_8$$
$$t_8 = 4.$$
Hence
$$t_{10} = 2(t_8 + t_9) = 2(4 + 24) = 56$$
and
$$t_{11} = 2(24 + 56)$$
$$= 160.$$

ANSWER: (A)

16. The digits 2, 2, 3, and 5 are randomly arranged to form a four digit number. What is the probability that the sum of the first and last digits is even?

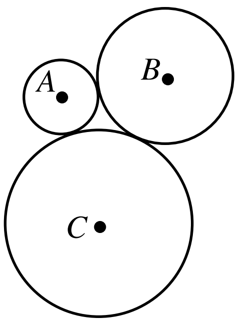
(**A**) $\frac{1}{4}$ (**B**) $\frac{1}{3}$ (**C**) $\frac{1}{6}$ (**D**) $\frac{1}{2}$ (**E**) $\frac{2}{3}$

*Solution*

The numbers that can be formed from the digits 2, 2, 3, and 5, in ascending order, are 2235, 2253, 2325, 2352, 2523, 2532, 3225, 3252, 3522, 5223, 5232, and 5322. These are twelve possibilities. For the sum of two digits to be even, both must be even or both must be odd. From the above list, the numbers for which the sum of the first and last digits is even, are 2352, 2532, 3225, and 5223. These are four possibilities. Thus, the probability of getting one of these numbers is $\frac{4}{12}$ or $\frac{1}{3}$.

ANSWER: (B)

17. Three circles have centres $A$, $B$ and $C$ with radii 2, 4 and 6 respectively. The circles are tangent to each other as shown. Triangle $ABC$ has

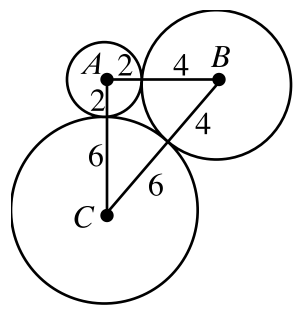



(**A**) $\angle A$ obtuse (**B**) $\angle B = 90°$ (**C**) $\angle A = 90°$

(**D**) all angles acute (**E**) $\angle B = \angle C$

*Solution*

Since the circles are mutually tangent, the lines joining their centres pass through the points of tangency. Thus, the sides of $\Delta ABC$ have lengths 6, 8 and 10 if we write in the radii as shown.
Since $10^2 = 6^2 + 8^2$, the triangle is right-angled at $A$.




ANSWER: (C)

18. If $P = 3^{2000} + 3^{-2000}$ and $Q = 3^{2000} - 3^{-2000}$ then the value of $P^2 - Q^2$ is

(**A**) $3^{4000}$ (**B**) $2 \times 3^{-4000}$ (**C**) 0 (**D**) $2 \times 3^{4000}$ (**E**) 4

*Solution 1*

$$\begin{aligned} P^2 - Q^2 &= (P+Q)(P-Q) = \left[\left(3^{2000} + 3^{-2000}\right) + \left(3^{2000} - 3^{-2000}\right)\right]\left[\left(3^{2000} + 3^{-2000}\right) - \left(3^{2000} - 3^{-2000}\right)\right] \\ &= \left(2.3^{2000}\right)\left(2.3^{-2000}\right) \\ &= 4.3^{\circ} \\ &= 4 \end{aligned}$$

*Solution 2*

$$P^2 - Q^2 = \left(3^{2000} + 3^{-2000}\right)^2 - \left(3^{2000} - 3^{-2000}\right)^2$$

$$= 3^{4000} + 2.3^{2000}.3^{-2000} + 3^{-4000} - 3^{4000} + 2.3^{2000}.3^{-2000} - 3^{-4000}$$
$$= 3^{4000} + 2.3^{\circ} + 3^{-4000} - 3^{4000} + 2.3^{\circ} - 3^{-4000}$$
$$= 4$$

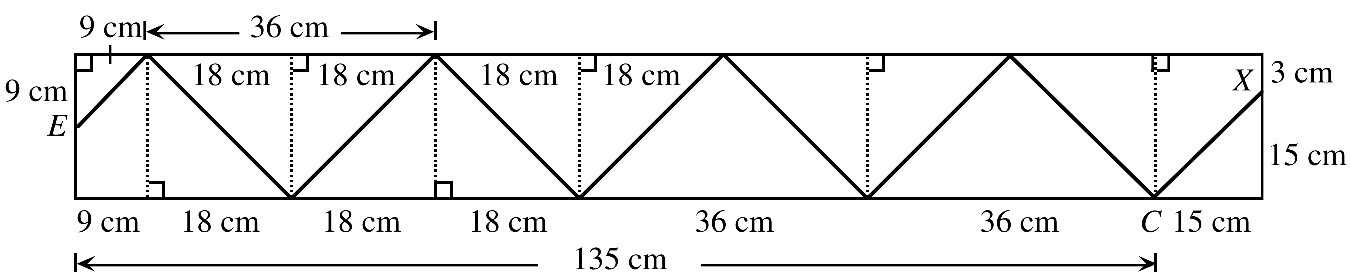
ANSWER: (E)

19. An ant walks inside a 18 cm by 150 cm rectangle. The ant's path follows straight lines which always make angles of $45^\circ$ to the sides of the rectangle. The ant starts from a point $X$ on one of the shorter sides. The first time the ant reaches the opposite side, it arrives at the mid-point. What is the distance, in centimetres, from $X$ to the nearest corner of the rectangle?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) 8 (**E**) 9

*Solution*

If we took a movie of the ant's path and then played it backwards, the ant would now start at the point $E$ and would then end up at point $X$. Since the ant now 'starts' at a point nine cm from the corner, the 'first' part of his journey is from $E$ to $B$. This amounts to nine cm along the length of the rectangle since $\Delta BAE$ is an isosceles right-angled triangle. This process continues as illustrated, until the ant reaches point $C$. By the time the ant has reached $C$, it has travelled $9 + 18 + 3 \times 36$ or 135 cm along the length of the rectangle. To travel from $C$ to $X$, the ant must travel 15 cm along the length of the rectangle which puts the ant 3 cm from the closest vertex.




ANSWER: (A)

20. Given $a + 2b + 3c + 4d + 5e = k$ and $5a = 4b = 3c = 2d = e$ find the smallest positive integer value for $k$ so that $a$, $b$, $c$, $d$, and $e$ are all positive integers.

(**A**) 87 (**B**) 522 (**C**) 10 (**D**) 120 (**E**) 60

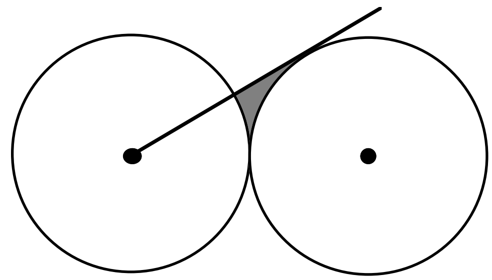
*Solution*

From the equalities $5a = 4b = 3c = 2d = e,$ we conclude that $e$ is the largest and $a$ is the smallest of the integers. Since $e$ is divisible by 5, 4, 3, and 2, the smallest possible value for $e$ is 60. The corresponding values for $a, b, c,$ and $d$, are 12, 15, 20, and 30, respectively. Thus, the smallest positive integer value for $k$ is $12 + 2(15) + 3(20) + 4(30) + 5(60) = 522.$

ANSWER: (B)

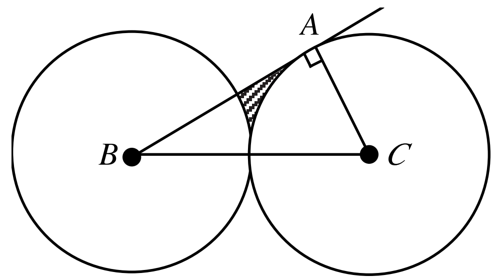
## Part C:

21. Two circles of radius 10 are tangent to each other. A tangent is drawn from the centre of one of the circles to the second circle. To the nearest integer, what is the area of the shaded region?



(**A**) 6 (**B**) 7 (**C**) 8
(**D**) 9 (**E**) 10

*Solution*




Let the centres of the circles be $B$ and $C$ and the point of contact of the tangent line from $B$ to the circle with centre $C$ be $A$. Thus, $\angle CAB = 90^\circ$ using properties of tangents to a circle.
Since the sides of right-angled triangle $ABC$ are in the ratio of $1:2:\sqrt{3}$, $\angle B = 30^\circ$ and $\angle C = 60^\circ$. The area of the shaded region is equal to the area of $\Delta ABC$ minus the area of the two sectors of the circles.
The area of the two sectors is equivalent to $\frac{1}{4}$ the area of a circle with radius 10.

The area of the shaded region $= \frac{1}{2}(10)\left(10\sqrt{3}\right) - \frac{1}{4}\pi(10)^2$
$= 50\sqrt{3} - 25\pi$
$\approx 8.063$

To the nearest integer, the area is 8. ANSWER: (C)

22. The left most digit of an integer of length 2000 digits is 3. In this integer, any two consecutive digits must be divisible by 17 or 23. The 2000th digit may be either '$a$' or '$b$'. What is the value of $a + b$?

(**A**) 3 (**B**) 7 (**C**) 4 (**D**) 10 (**E**) 17

*Solution*

We start by noting that the two-digit multiples of 17 are 17, 34, 51, 68, and 85. Similarly we note that the two-digit multiples of 23 are 23, 46, 69, and 92. The first digit is 3 and since the only two-digit number in the two lists starting with 3 is 34, the second digit is 4. Similarly the third digit must be 6. The fourth digit, however, can be either 8 or 9. Let's consider this in two cases.

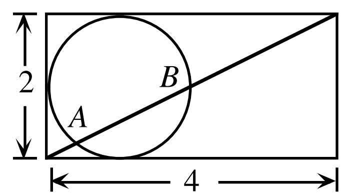
*Case 1*
If the fourth digit is 8, the number would be 3468517 and would stop here since there isn't a number in the two lists starting with 7.

*Case 2*
If the fourth digit is 9, the number would be 34692 34692 34 ... and the five digits '34692' would continue repeating indefinitely as long as we choose 9 to follow 6.

If we consider a 2000 digit number, its first 1995 digits must contain 399 groups of '34692'. The last groups of five digits could be either 34692 or 34685 which means that the 2000th digit may be either 2 or 5 so that $a+b=2+5=7$.

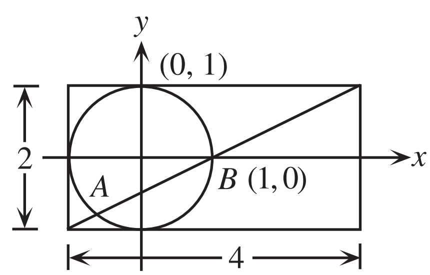
ANSWER: (B)

23. A circle is tangent to three sides of a rectangle having side lengths 2 and 4 as shown. A diagonal of the rectangle intersects the circle at points $A$ and $B$. The length of $AB$ is




**(A)** $\sqrt{5}$ **(B)** $\frac{4\sqrt{5}}{5}$ **(C)** $\sqrt{5}-\frac{1}{5}$ **(D)** $\sqrt{5}-\frac{1}{6}$ **(E)** $\frac{5\sqrt{5}}{6}$

*Solution 1*
Of the many ways to solve this problem perhaps the easiest is to use a Cartesian co-ordinate system. There are two choices for the origin, the centre of the circle and the bottom left vertex of the rectangle. The first solution presented uses the centre of the circle as the origin where the axes are lines drawn parallel to the sides of the rectangle through $(0,0)$. The equation of the circle is now $x^2+y^2=1$ and the equation of the line containing the diagonal is $y+1=\frac{1}{2}(x+1)$ or $x-2y=1$.




Equation of Line: $y+1=\frac{1}{2}(x+1) \Leftrightarrow x-2y=1$
Equation of Circle: $x^2+y^2=1$

Solving to find the intersection points of the line and circle gives, $(2y+1)^2+y^2=1$

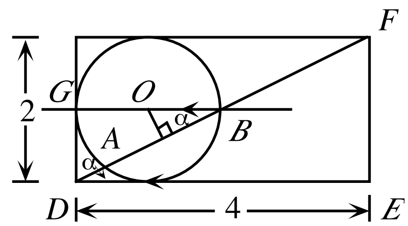
$$5y^2+4y=0$$
$$y(5y+4)=0$$
$$y=0 \text{ or } y=-\frac{4}{5}$$

If we substitute these values of $y$ into $x-2y=1$ we find,
The points of intersection are $A\left(\frac{-3}{5},-\frac{4}{5}\right)$ and $B(1,0)$.

The length of $AB$ is $\sqrt{\left(1+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2} = \sqrt{\frac{80}{25}} = \frac{4\sqrt{5}}{5}$.

**Note:** The other suggested choice for the origin would involve using the equation of the line $y=\frac{1}{2}x$ and the circle $(x-1)^2+(y-1)^2=1$.

*Solution 2*




The diameter, $BG$, is parallel to the sides of the rectangle and has its endpoint at $B$ which is the centre of the rectangle.
Draw $OC \perp AB$ where $O$ is the centre of the circle and by chord properties of a circle, $CB=\frac{1}{2}AB$.
Since $OB \parallel DE$, $\angle BDE = \angle CBO = \alpha$.
Thus, $\Delta OBC$ is similar to $\Delta FDE$.
Since $DF = 2\sqrt{5}$, $\frac{1}{2\sqrt{5}} = \frac{CB}{4}$ and $CB = \frac{2}{5}\sqrt{5}$.
The length of $AB$ is $2\left(\frac{2}{5}\sqrt{5}\right)$ or $\frac{4}{5}\sqrt{5}$.

ANSWER: (B)

24. For the system of equations $x^2+x^2y^2+x^2y^4=525$ and $x+xy+xy^2=35$, the sum of the real $y$ values that satisfy the equations is

(**A**) 20 (**B**) 2 (**C**) 5 (**D**) $\frac{55}{2}$ (**E**) $\frac{5}{2}$

*Solution*

Consider the system of equations $x^2+x^2y^2+x^2y^4=525$ (1)
and $x+xy+xy^2=35$ (2)

The expression on the left side of equation (1) can be rewritten as,

$$\begin{aligned} x^2+x^2y^2+x^2y^4 &= \left(x+xy^2\right)^2 - x^2y^2 \\ &= \left(x+xy^2-xy\right)\left(x+xy^2+xy\right) \end{aligned}$$

Thus, $\left(x+xy^2-xy\right)\left(x+xy^2+xy\right) = 525$

Substituting from (2) gives, $\left(x+xy^2-xy\right)(35) = 525$

or, $x+xy^2-xy=15$ (3)

Now subtracting (3) from (2), $2xy=20$, $x=\frac{10}{y}$

Substituting for $x$ in (3) gives,

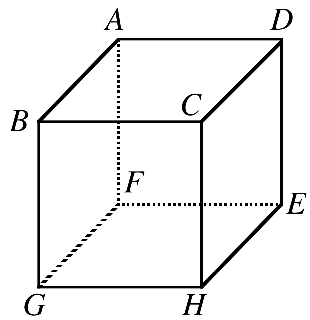
$$\frac{10}{y}+10y-10=15$$

$$10y^2 - 25y + 10 = 0$$
$$2y^2 - 5y + 2 = 0$$
$$(2y-1)(y-2) = 0$$
$$y = \tfrac{1}{2} \quad \text{or} \quad y = 2$$

The sum of the real $y$ values satisfying the system is $\frac{5}{2}$.

ANSWER: (E)

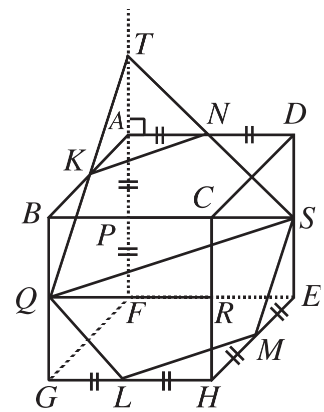
25. The given cube is cut into four pieces by two planes. The first plane is parallel to face $ABCD$ and passes through the midpoint of edge $BG$. The second plane passes through the midpoints of edges $AB$, $AD$, $HE$, and $GH$. Determine the ratio of the volumes of the smallest and largest of the four pieces.




(**A**) 3:8 (**B**) 7:24 (**C**) 7:25
(**D**) 7:17 (**E**) 5:11

*Solution*

For convenience, let each edge of the cube have length 2. The plane *PQRS* through the mid-point of *BG* and parallel to face *ABCD* bisects the volume of the cube. The plane through *K, L, M,* and *N* also bisects the volume of the cube and contains the line segment *QS*. Hence the two planes divide the cube into four pieces, two equal 'smaller' pieces and two equal 'larger' pieces. Extend *QK* and *SN* to meet at *T*.




We note that $\Delta TAN$ is similar to $\Delta TPS$. Since $\frac{TA}{TP} = \frac{AN}{PS}$ and $AN = 1$, $PS = 2$ and $PA = 1$ it is easy to calculate to find $TA = 1$.

The volume of the upper 'smallest' piece is equal to the volume of the tetradedron *TQSP* - volume of the tetradedron *TKNA*. This volume is, $\frac{1}{3}\left[\left(\frac{1}{2}(2)(2)(2)\right)\right] - \frac{1}{3}\left[\left(\frac{1}{2}\right)(1)(1)(1)\right]$

$$= \frac{4}{3} - \frac{1}{6} = \frac{7}{6}.$$

The volume of a 'largest' piece is, $2 \times 2 \times 1 - \frac{7}{6} = \frac{17}{6}$.

The ratio of the volume of the 'smallest' piece to that of the 'largest' piece is 7:17.

ANSWER: (D)

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

# 1999 Solutions
# Fermat Contest (Grade 11)

for the

NATIONAL BANK OF CANADA

Awards

## Part A

1. The value of $\left(\sqrt{25}-\sqrt{9}\right)^2$ is

(**A**) 26 (**B**) 16 (**C**) 34 (**D**) 8 (**E**) 4

*Solution*

$\left(\sqrt{25}-\sqrt{9}\right)^2=(5-3)^2=4$ ANSWER: (E)

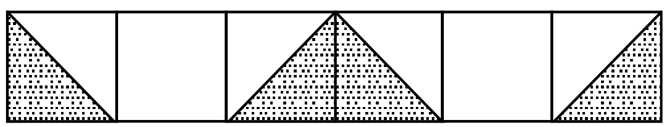
2. Today is Wednesday. What day of the week will it be 100 days from now?

(**A**) Monday (**B**) Tuesday (**C**) Thursday (**D**) Friday (**E**) Saturday

*Solution*

Since there are 7 days in a week it will be Wednesday in 98 days.
In 100 days it will be Friday. ANSWER: (D)

3. Six squares are drawn and shaded as shown. What fraction of the total area is shaded?



(**A**) $\frac{1}{2}$ (**B**) $\frac{1}{3}$ (**C**) $\frac{1}{4}$ (**D**) $\frac{2}{5}$ (**E**) $\frac{2}{3}$

*Solution*

Out of a possible six squares, there is the equivalent of two shaded squares.
Thus $\frac{1}{3}$rd of the figure is shaded. ANSWER: (B)

4. Turning a screwdriver $90°$ will drive a screw 3 mm deeper into a piece of wood. How many complete revolutions are needed to drive the screw 36 mm into the wood?

(**A**) 3 (**B**) 4 (**C**) 6 (**D**) 9 (**E**) 12

*Solution*

One complete revolution of the screw driver, $360°$, will drive it 12 mm deeper into the wood.
In order for the screw to go 36 mm into the wood it will take three revolutions.

ANSWER: (A)

5. A value of $x$ such that $(5-3x)^5=-1$ is

(**A**) $\frac{4}{3}$ (**B**) 0 (**C**) $\frac{10}{3}$ (**D**) $\frac{5}{3}$ (**E**) 2

*Solution*
Since $(-1)^5 = -1$, $5 - 3x = -1$ ***or*** $x = 2$. ANWSER: (E)

6. The number which is 6 less than twice the square of 4 is

(**A**) $-26$ (**B**) 10 (**C**) 26 (**D**) 38 (**E**) 58

*Solution*
$2(4)^2 - 6 = 26$ ANSWER: (C)

7. The Partridge family pays each of their five children a weekly allowance. The average allowance for each of the three younger children is \$8. The two older children each receive an average allowance of \$13. The total amount of allowance money paid per week is

(**A**) \$50 (**B**) \$52.50 (**C**) \$105 (**D**) \$21 (**E**) \$55

*Solutiion*
The total paid out was, $3 \times \$8 + 2 \times \$13 = \$50$. ANSWER: (A)

8. The time on a digital clock is 5:55. How many minutes will pass before the clock next shows a time with all digits identical?

(**A**) 71 (**B**) 72 (**C**) 255 (**D**) 316 (**E**) 436

*Solution*
The digits on the clock will next be identical at 11:11. This represents a time difference of 316 minutes. (Notice that times like 6:66, 7:77 etc. are not possible.) ANSWER: (D)

9. In an election, Harold received 60% of the votes and Jacquie received all the rest. If Harold won by 24 votes, how many people voted?

(**A**) 40 (**B**) 60 (**C**) 72 (**D**) 100 (**E**) 120

*Solution*
If Harold received 60% of the votes this implies that Jacquie received 40% of the total number of votes. The difference between them, 20%, represents 24 votes.
Therefore, the total number of votes cast was $5 \times 24 = 120$. ANSWER: (E)

10. If $x$ and $y$ are each chosen from the set $\{1, 2, 3, 5, 10\}$, the largest possible value of $\frac{x}{y} + \frac{y}{x}$ is

(**A**) 2 (**B**) $12\frac{1}{2}$ (**C**) $10\frac{1}{10}$ (**D**) $2\frac{1}{2}$ (**E**) 20

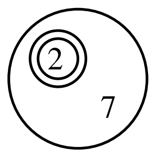
*Solution*
The best strategy is to choose the largest value and the smallest so that, $\frac{x}{y} > 1$, is as large as possible.

When we consider the reciprocal, $\frac{y}{x}$, this will always produce a number less than 1 and will be of little consequence in our final consideration. The best choices, then, are $x = 10$ and $y = 1$ and $\frac{x}{y} + \frac{y}{x}$ becomes $\frac{10}{1} + \frac{1}{10} = 10\frac{1}{10}$.

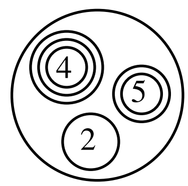
ANSWER: (C)

## Part B

11. In *Circle Land*, the numbers 207 and 4520 are shown in the following way:

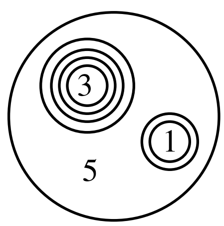


207



4520

In *Circle Land*, what number does the following diagram represent?




(**A**) 30 105 (**B**) 30 150 (**C**) 3105 (**D**) 3015 (**E**) 315

*Solution 1*

$= 3 \times 10^4 = 30\,000$

$= 1 \times 10^2 = 100$

5 $= 5 \times 10^0 = 5$

The required number is $30\,000 + 100 + 5 = 30\,105$.

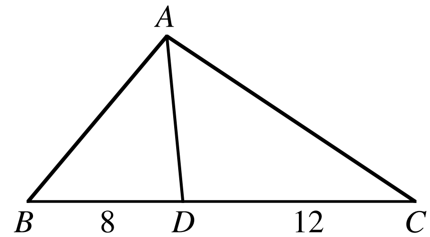
*Solution 2*

Since there are four circles around the ‘3’ this corresponds to $3 \times 10^4 = 30\,000$.

The ‘5’ corresponds to a 5 in the units digit which leads to $30\,105$ as the only correct possibility.

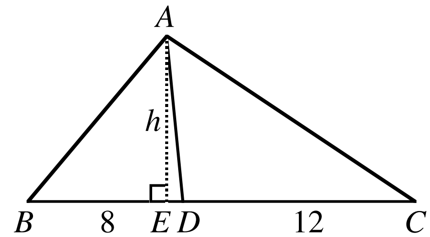
ANSWER: (A)

12. The area of $\Delta ABC$ is 60 square units. If $BD = 8$ units and $DC = 12$ units, the area (in square units) of $\Delta ABD$ is




(**A**) 24 (**B**) 40 (**C**) 48
(**D**) 36 (**E**) 6

*Solution*
From $A$, draw a line perpendicular to $BC$ to meet $BC$ at $E$. Thus the line segment $AE$ which is labelled as $h$ is the height of $\Delta ABD$ and $\Delta ABC$. Since the heights of the two triangles are equal, their areas are then proportionate to their bases. If the area of $\Delta ABC$ is 60, then the area of $\Delta ABD$ is $\frac{8}{20} \times 60 = 24$.




ANSWER: (A)

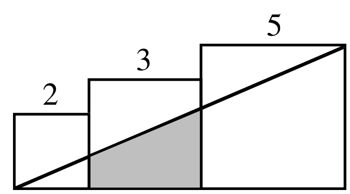
13. Crippin wrote four tests each with a maximum possible mark of 100. The average mark he obtained on these tests was 88. What is the lowest score he could have achieved on one of these tests?

(**A**) 88 (**B**) 22 (**C**) 52 (**D**) 0 (**E**) 50

*Solution*
If the average score of four tests was 88, this implies that a total of $4 \times 88$ or 352 marks were obtained. The lowest mark would be obtained if Crippin had three marks of 100 and one mark of 52.

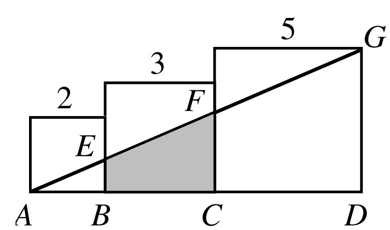
ANSWER: (C)

14. Three squares have dimensions as indicated in the diagram. What is the area of the shaded quadrilateral?




(**A**) $\frac{21}{4}$ (**B**) $\frac{9}{2}$ (**C**) 5
(**D**) $\frac{15}{4}$ (**E**) $\frac{25}{4}$

*Solution 1*
In the first solution, we use similar triangles. We start by labelling the diagram as shown. The objective in this question is to calculate the lengths $EB$ and $FC$ which will allow us to calculate the area of $\Delta AEB$ and $\Delta AFC$. We first note that $\Delta AFC$ and $\Delta AGD$ are similar and that,
$$\frac{AC}{AD} = \frac{FC}{GD} = \frac{5}{10} = \frac{1}{2}.$$
Therefore, $FC = \frac{1}{2} GD = \frac{1}{2}(5) = \frac{5}{2}$.



Using the same reasoning, $\Delta AEB$ and $\Delta AFC$ are also similar triangles meaning that, $\dfrac{EB}{FC}=\dfrac{2}{5}$.

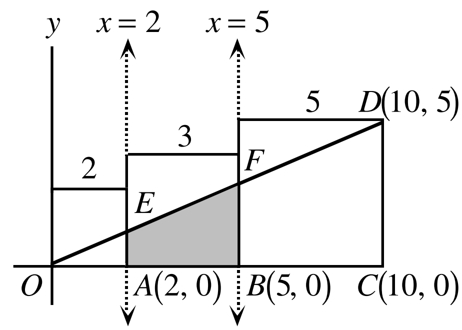
Thus, $EB=\frac{2}{5}\left(\frac{5}{2}\right)=1$.

We find the required area to be

$$\begin{aligned}\text{area } \Delta AFC - \text{area } \Delta AEB &= \tfrac{1}{2}(5)\left(\tfrac{5}{2}\right)-\tfrac{1}{2}(2)(1)\\ &= \tfrac{21}{4}.\end{aligned}$$

*Solution 2*

We start by putting the information on a coordinate axes and labelling as shown. The line containing $OD$ has equation $y=\frac{1}{2}x$ while $x=2$ and $x=5$ contains $AE$ and $BF$. Solving the systems $y=\frac{1}{2}x$, $x=2$ and $y=\frac{1}{2}x$, $x=5$ gives the coordinates of $E$ to be $(2,1)$ and $F$ to be $\left(5,\frac{5}{2}\right)$. This makes $AE=1$ and $BF=\frac{5}{2}$ which now leads to exactly the same answer as in solution 1.




ANSWER: (A)

15. If $(a+b+c+d+e+f+g+h+i)^2$ is expanded and simplified, how many different terms are in the final answer?

(**A**) 36 (**B**) 9 (**C**) 45 (**D**) 81 (**E**) 72

*Solution*

$$\overset{\text{Bracket 1}}{(a+b+c+d+e+f+g+h+i)}\overset{\text{Bracket 2}}{(a+b+c+d+e+f+g+h+i)}$$

If we wish to determine how many different terms can be produced we begin by multiplying the '$a$' in bracket 1 by each term in bracket 2. This calculation gives 9 different terms. We continue this process by now multiplying the '$b$' in bracket 1 by the elements from $b$ to $i$ in bracket 2 to give 8 different terms. We continue this process until we finally multiply the '$i$' in the first bracket by the '$i$' in the second bracket. Altogether we have, $9+8+7+6+5+4+3+2+1=45$ different terms.

ANSWER: (C)

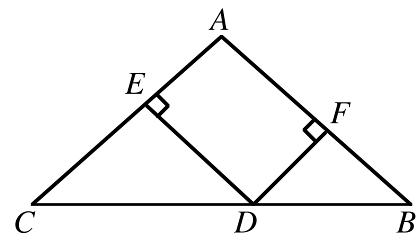
16. If $px+2y=7$ and $3x+qy=5$ represent the same straight line, then $p$ equals

(**A**) 5 (**B**) 7 (**C**) 21 (**D**) $\frac{21}{5}$ (**E**) $\frac{10}{7}$

*Solution*

If we multiply the equation of the first line by 5 and the second by 7 we obtain, $5px+10y=35$ and $21x+7qy=35$. Comparing coefficients gives, $5p=21$ or $p=\frac{21}{5}$. ANSWER: (D)

17. In $\Delta ABC$, $AC = AB = 25$ and $BC = 40$. $D$ is a point chosen on $BC$. From $D$, perpendiculars are drawn to meet $AC$ at $E$ and $AB$ at $F$. $DE + DF$ equals

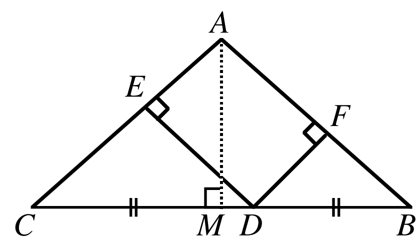



(**A**) 12 (**B**) 35 (**C**) 24

(**D**) 25 (**E**) $\frac{35}{2}\sqrt{2}$

*Solution*

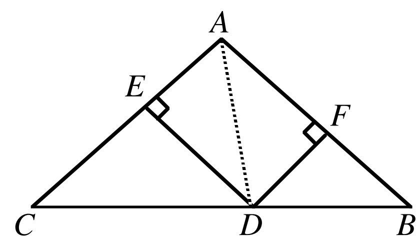
We start by drawing a line from $A$ that is perpendicular to the base $CB$. Since $\Delta ABC$ is isosceles, $M$ is the midpoint of $CB$ thus making $CM = MB = 20$. Using pythagoras in $\Delta ACM$ we find $AM$ to be $\sqrt{25^2 - 20^2} = 15$.




Join $A$ to $D$. The area of $\Delta ABC$ is $\frac{1}{2}(40)(15) = 300$ but it is also, $\frac{1}{2}(ED)(25) + \frac{1}{2}(DF)(25)$

$$= \frac{25}{2}(ED + DF).$$

Therefore, $ED + DF = \frac{2}{25}(300) = 24$.




ANSWER: (C)

18. The number of solutions $(P, Q)$ of the equation $\dfrac{P}{Q} - \dfrac{Q}{P} = \dfrac{P+Q}{PQ}$, where $P$ and $Q$ are integers from 1 to 9 inclusive, is

(**A**) 1 (**B**) 8 (**C**) 16 (**D**) 72 (**E**) 81

*Solution*

If we simplify the rational expression on the left side of the equation and then factor the resulting numerator as a difference of squares we obtain,

$$\frac{(P-Q)(P+Q)}{PQ}.$$

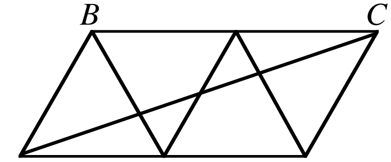
The equation can now be written as,

$$\frac{(P-Q)(P+Q)}{PQ} = \frac{P+Q}{PQ} \quad \text{or} \quad P - Q = 1 \quad (PQ \neq 0 \text{ and } P + Q \neq 0).$$

The only integers that satisfy this are: $(2,1)$, $(3,2)$, $(4,3)$, ..., $(9,8)$.

Thus there are 8 possibilities. ANSWER: (B)

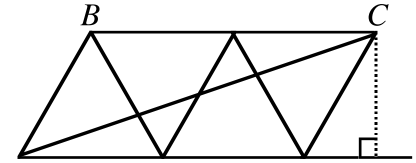
19. Parallelogram $ABCD$ is made up of four equilateral triangles of side length 1. The length of diagonal $AC$ is




(**A**) $\sqrt{5}$ (**B**) $\sqrt{7}$ (**C**) 3
(**D**) $\sqrt{3}$ (**E**) $\sqrt{10}$

*Solution*

From $C$, we draw a line perpendicular to $AD$ extended so that they meet at point $E$ as shown in the diagram.




This construction makes $\Delta CDE$ a $30° - 60° - 90°$ triangle with $\angle CDE = 60°$ and $CD = 1$. Thus $CE = \frac{\sqrt{3}}{2}$ and $DE = \frac{1}{2}$. Using pythagoras in $\Delta ACE$, we have $AE = \frac{5}{2}$ and $CE = \frac{\sqrt{3}}{2}$, $AC = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \sqrt{7}$.

ANSWER: (B)

20. If $a_1 = \frac{1}{1-x}$, $a_2 = \frac{1}{1-a_1}$, and $a_n = \frac{1}{1-a_{n-1}}$, for $n \geq 2$, $x \neq 1$ and $x \neq 0$, then $a_{107}$ is

(**A**) $\frac{1}{1-x}$ (**B**) $x$ (**C**) $-x$ (**D**) $\frac{x-1}{x}$ (**E**) $\frac{1}{x}$

*Solution*

$$a_1 = \frac{1}{1-x}$$

$$a_2 = \frac{1}{1-\frac{1}{1-x}} = \frac{(1-x)(1)}{(1-x)\left(1-\frac{1}{1-x}\right)} = \frac{1-x}{1-x-1} = \frac{1-x}{-x} = \frac{x-1}{x}$$

$$a_3 = \frac{1}{1-\frac{x-1}{x}} = \frac{x(1)}{x\left(1-\frac{x-1}{x}\right)} = \frac{x}{x-(x-1)} = x$$

$$a_4 = \frac{1}{1-x}$$

Since $a_1 = a_4$, we conclude $a_1 = a_4 = a_7 = \ldots = a_{3n-2} = a_{106}$.
Also, $a_2 = a_5 = a_8 = \ldots = a_{3n-1} = a_{107}$ for $n = 36$.
Since $a_2 = \frac{x-1}{x}$ then $a_{107} = \frac{x-1}{x}$.

ANSWER: (D)

## Part C

21. How many integers can be expressed as a sum of three distinct numbers if chosen from the set $\{4, 7, 10, 13, ..., 46\}$?

(**A**) 45 (**B**) 37 (**C**) 36 (**D**) 43 (**E**) 42

*Solution*
Since each number is of the form $1+3n$, $n = 1, 2, 3, ..., 15$, the sum of the three numbers will be of the form $3+3k+3l+3m$ where $k$, $l$ and $m$ are chosen from $\{1, 2, 3, ..., 15\}$. So the question is equivalent to the easier question of, 'How many distinct integers can be formed by adding three numbers from, $\{1, 2, 3, ..., 15\}$?'
The smallest is $1+2+3=6$ and the largest is $13+14+15=42$.
It is clearly possible to get every sum between 6 and 42 by:
(a) increasing the sum by one replacing a number with one that is 1 larger or,
(b) decreasing the sum by one by decreasing one of the addends by 1.
Thus all the integers from 6 to 42 inclusive can be formed.
This is the same as asking, 'How many integers are there between 1 and 37 inclusive?' The answer, of course, is 37. ANSWER: (B)

22. If $x^2+ax+48=(x+y)(x+z)$ and $x^2-8x+c=(x+m)(x+n)$, where $y$, $z$, $m$, and $n$ are integers between $-50$ and 50, then the maximum value of $ac$ is

(**A**) 343 (**B**) 126 (**C**) 52 234 (**D**) 784 (**E**) 98 441

*Solution*
For the equation, $x^2+ax+48=(x+y)(x+z)$ we consider the possible factorizations of 48 which produce different values for $a$. The factorizations and possible values for $a$ are listed in the table that follows:

| Possible Factorizations of 48 | Possible Values for $a$ |
|---|---|
| $1\times 48$, $-1\times -48$ | 49, $-49$ |
| $2\times 24$, $-2\times -24$ | 26, $-26$ |
| $3\times 16$, $-3\times -16$ | 19, $-19$ |
| $4\times 12$, $-4\times -12$ | 16, $-16$ |
| $6\times 8$, $-6\times -8$ 14, $-14$ | |

For the equation, $x^2-8x+c=(x+m)(x+n)$, we list some of its possible factorizations and the related possible values of $c$.

| Possible Factorizations | Related Values of $c$ |
|---|---|
| $(x-49)(x+41)$ | $-49\times 41=-2009$ |
| $(x-48)(x+40)$ | $-48\times 40=-1920$ |
| $\vdots$ | $\vdots$ |

| | |
|---|---|
| $(x-9)(x+1)$ | $-9\times 1=-9$ |
| $(x-8)(x+0)$ | $0$ |

From these tables, we can see that the maximum value of $ac$ is $-49\times -2009=98\,441$.

ANSWER: (E)

23. The sum of all values of $x$ that satisfy the equation $\left(x^2-5x+5\right)^{x^2+4x-60}=1$ is

(**A**) $-4$ (**B**) 3 (**C**) 1 (**D**) 5 (**E**) 6

*Solution*

We consider the solution in three cases.

*Case 1* It is possible for the base to be 1.

Therefore, $x^2-5x+5=1$

$$x^2-5x+4=0$$

$$(x-1)(x-4)=0$$

Therefore $x=1$ or $x=4$.

Both these values are acceptable for $x$.

*Case 2* It is possible that the exponent be 0.

Therefore, $x^2+4x-60=0$

$$(x+10)(x-6)=0$$

$$x=-10 \text{ or } x=6$$

Note: It is easy to verify that neither $x=-10$ nor $x=6$ is a zero of $x^2-5x+5$, so that the indeterminate form $0°$ does not occur.

*Case 3* It is possible that the base is –1 and the exponent is even.

Therefore, $x^2-5x+5=-1$ but $x^2+4x-60$ must also be even.

$$x^2-5x+5=-1$$

$$x^2-5x+6=0$$

$$(x-2)(x-3)=0$$

$$x=2 \text{ or } x=3$$

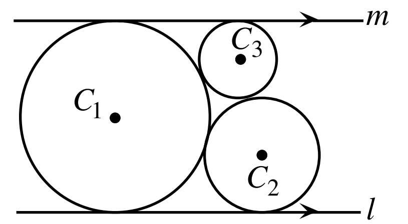
If $x=2$, then $x^2-4x-60$ is even, so $x=2$ is a solution.

If $x=3$, then $x^2-4x-60$ is odd, so $x=3$ is ***not*** a solution.

Therefore the sum of the solutions is $1+4-10+6+2=3$.

ANSWER: (B)

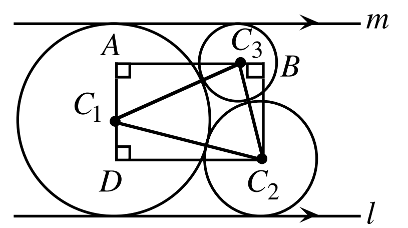
24. Two circles $C_1$ and $C_2$ touch each other externally and the line $l$ is a common tangent. The line $m$ is parallel to $l$ and touches the two circles $C_1$ and $C_3$. The three circles are mutually tangent. If the radius of $C_2$ is 9 and the radius of $C_3$ is 4, what is the radius of $C_1$?



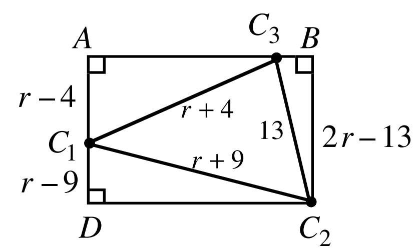

(**A**) 10.4 (**B**) 11 (**C**) $8\sqrt{2}$
(**D**) 12 (**E**) $7\sqrt{3}$

*Solution*

We start by joining the centres of the circles to form $\Delta C_1C_2C_3$. (The lines joining the centres pass through the corresponding points of tangency.)
Secondly, we construct the rectangle $ABC_2D$ as shown in the diagram. If the radius of the circle with centre $C_1$ is $r$ we see that: $C_1C_2 = r+9$, $C_1C_3 = r+4$ and $C_2C_3 = 13$.




We now label lengths on the rectangle in the way noted in the diagram.




To understand this labelling, look for example at $C_1D$. The radius of the large circle is $r$ and the radius of the circle with centre $C_2$ is 9. The length $C_1D$ is then $r-9$.
This same kind of reasoning can be applied to both $C_1A$ and $BC_2$.

Using Pythagoras we can now derive the following:

In $\Delta AC_3C_1$, $\quad C_3A^2 = (r+4)^2 - (r-4)^2$
$\qquad = 16r$.

Therefore $C_3A = 4\sqrt{r}$.

In $\Delta DC_1C_2$, $\quad (DC_2)^2 = (r+9)^2 - (r-9)^2$
$\qquad = 36r$.

Therefore $DC_2 = 6\sqrt{r}$.

In $\Delta BC_3C_2$, $\quad (C_3B)^2 = 13^2 - (2r-13)^2$
$\qquad = -4r^2 + 52r$.

Therefore $C_3B = \sqrt{-4r^2+52r}$.

In a rectangle opposite sides are equal, so:

$$DC_2 = C_3A + C_3B$$

or, $6\sqrt{r} = 4\sqrt{r} + \sqrt{-4r^2 + 52r}$

$$2\sqrt{r} = \sqrt{-4r^2 + 52r}\,.$$

Squaring gives, $4r = -4r^2 + 52r$

$$4r^2 - 48r = 0$$

$$4r(r - 12) = 0$$

Therefore $r = 0$ or $r = 12$.

Since $r > 0$, $r = 12$. ANSWER: (D)

25. Given that $n$ is an integer, for how many values of $n$ is $\dfrac{2n^2 - 10n - 4}{n^2 - 4n + 3}$ an integer?

(**A**) 9 (**B**) 7 (**C**) 6 (**D**) 4 (**E**) 5

*Solution*

We start by dividing $n^2 - 4n + 3$ into $2n^2 - 10n - 4$.

$$\begin{array}{r|l} & 2 \\ n^2 - 4n + 3 & 2n^2 - 10n - 4 \\ & \underline{2n^2 - 8n + 6} \\ & \quad -2n - 10 \end{array}$$

This allows us to write the original expression in the following way,

$$\frac{2n^2 - 10n - 4}{n^2 - 4n + 3} = 2 + \frac{-2n - 10}{n^2 - 4n + 3} = 2 - \frac{2n + 10}{n^2 - 4n + 3}.$$

The original question comes down to the consideration of $\dfrac{2n+10}{n^2 - 4n + 3}$ and when this expression is an integer. This rational expression can only assume integer values when, $2n + 10 \geq n^2 - 4n + 3$ (the numerator must be greater than the denominator) and when $2n + 10 = 0$.

Or, $n^2 - 6n - 7 \leq 0$ and $n = -5$

or, $(n - 7)(n + 1) \leq 0$

$$-1 \leq n \leq 7.$$

This means that we only have to consider values of $n$, $-1 \leq n \leq 7$, $n \in Z$ and $n = -5$. Also note that since $n^2 - 4n + 3 = (n - 1)(n - 3)$ we can remove $n = 1$ and $n = 3$ from consideration. We construct a table and check each value.

| $n$ | $-5$ | $-1$ | $0$ | $2$ | $4$ | $5$ | $6$ | $7$ |
|---|---|---|---|---|---|---|---|---|
| $\dfrac{2n+10}{(n-3)(n-1)}$ | $0$ | $+1$ | $\frac{10}{3}$ | $-14$ | $6$ | $\frac{5}{2}$ | $\frac{22}{15}$ | $1$ |

From this table we can see that there are just four acceptable values of $n$ that produce an integer.

Note also that $\frac{2n+10}{n^2-4n+3}$ would also be an integer if $2n+10=0$ and $n^2-4n+3\neq 0$. Thus $n=-5$ is a fifth value since the denominator $\neq 0$. ANSWER: (E)

35th Anniversary 1963 – 1998

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

# 1998 Solutions
# Fermat Contest (Grade 11)

for the

**NATIONAL BANK OF CANADA**

Awards

## PART A:

1. The value of $\frac{1+2+3+4+5}{2+4+6+8+10}$ is

(**A**) $\frac{1}{3}$ (**B**) 2.5 (**C**) $\frac{1}{2}$ (**D**) $\frac{11}{26}$ (**E**) $\frac{3}{8}$

***Solution 1***

$$\frac{1+2+3+4+5}{2+4+6+8+10} = \frac{15}{30} = \frac{1}{2}$$

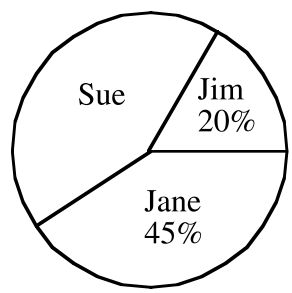
***Solution 2***

$$\frac{1+2+3+4+5}{2+4+6+8+10} = \frac{(1+2+3+4+5)}{2(1+2+3+4+5)} = \frac{1}{2}$$

ANSWER: (C)

2. The pie chart shows a percentage breakdown of 1000 votes in a student election. How many votes did Sue receive?

(**A**) 550 (**B**) 350 (**C**) 330
(**D**) 450 (**E**) 935

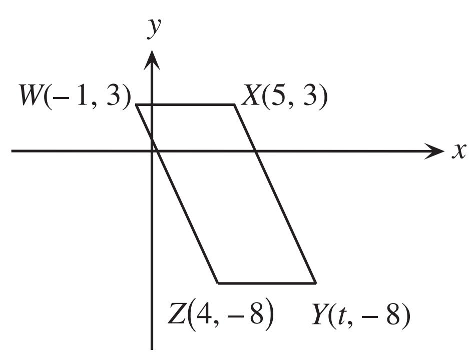



***Solution***

Sue received $100-(20+45)=35$ percent of the total number of votes. Since there was a total of 1000 votes, Sue received $0.35(1000)=350$ votes. ANSWER: (B)

3. If $WXYZ$ is a parallelogram, then $t$ equals

(**A**) 8 (**B**) 9 (**C**) 10
(**D**) 11 (**E**) 12




***Solution***

Since $WXYZ$ is a parallelogram, opposite sides are equal in length. The length of $WX$ is $5-(-1)=6$. Since $WX=ZY$, then $t-4=6$ or $t=10$. ANSWER: (C)

4. The product of two positive integers $p$ and $q$ is 100. What is the largest possible value of $p+q$?
(**A**) 52 (**B**) 101 (**C**) 20 (**D**) 29 (**E**) 25

***Solution***
The pairs of positive integers whose product is 100 are: 1 and 100, 2 and 50, 4 and 25, 5 and 20, 10 and 10. The pair with the largest sum is 1 and 100. The sum is 101.

ANSWER: (B)

5. If $\otimes$ is a new operation defined as $p \otimes q = p^2 - 2q$, what is the value of $7 \otimes 3$?

***Solution***
Using the definition of the new operation $\otimes$,

$$\begin{aligned} 7 \otimes 3 &= 7^2 - 2(3) \\ &= 49 - 6 \\ &= 43 \end{aligned}$$

ANSWER: (A)

6. The value of $\frac{1}{3}$ of $6^{30}$ is
(**A**) $6^{10}$ (**B**) $2^{30}$ (**C**) $2^{10}$ (**D**) $2 \times 6^{29}$ (**E**) $2 \times 6^{10}$

***Solution***

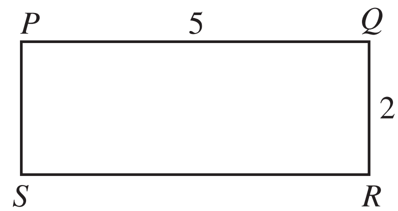
$$\begin{aligned} \frac{1}{3} \times 6^{30} &= \frac{1}{3} \times 6 \times 6^{29} \\ &= 2 \times 6^{29} \end{aligned}$$

ANSWER: (D)

7. The average (mean) of a list of 10 numbers is 0. If 72 and $-12$ are added to the list, the new average will be
(**A**) 30 (**B**) 6 (**C**) 0 (**D**) 60 (**E**) 5

***Solution***
If the average (mean) of a list of 10 numbers is 0, then the sum of the numbers is 10(0) = 0. When 72 and –12 are added to the list, the sum of these 12 numbers is $0 + 72 - 12 = 60$.
Thus, the average of the 12 numbers is $60 \div 12 = 5$. ANSWER: (E)

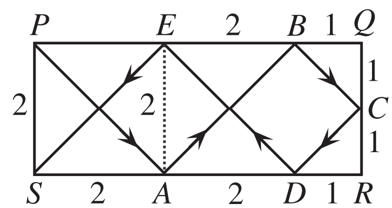
8. On a rectangular table 5 units long and 2 units wide, a ball is rolled from point $P$ at an angle of $45°$ to $PQ$ and bounces off $SR$. The ball continues to bounce off the sides at $45°$ until it reaches $S$. How many bounces of the ball are required?




(**A**) 9 (**B**) 8 (**C**) 7
(**D**) 5 (**E**) 4

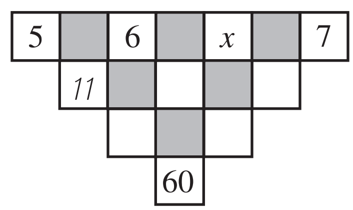
***Solution***

Since the ball bounces off the sides of the rectangular table at $45°$, right-angled isosceles triangles are created as shown. The ball begins at point $P$ then bounces at points $A, B, C, D,$ and $E$ before reaching $S$, for a total of 5 bounces.




ANSWER: (D)

9. The number in an unshaded square is obtained by adding the numbers connected to it from the row above. (The '*11*' is one such number.) The value of $x$ must be




(**A**) 4 (**B**) 6 (**C**) 9
(**D**) 15 (**E**) 10

***Solution***

The three entries in row two, from left to right, are $11$, $6+x$, and $x+7$. The two entries in row three, from left to right, are $11+(6+x)=17+x$ and $(6+x)+(x+7)=2x+13$. The single entry in row four is $(17+x)+(2x+13)=3x+30$.

Thus, $3x+30=60$

$$3x=30$$

$$x=10$$

ANSWER: (E)

10. Four points are on a line segment, as shown.
If $AB:BC=1:2$ and $BC:CD=8:5$, then $AB:BD$ equals


(**A**) 4:13 (**B**) 1:13 (**C**) 1:7
(**D**) 3:13 (**E**) 4:17

***Solution***

In order to compare the given ratios, we must rewrite the ratio $AB:BC=1:2$ as $AB:BC=4:8$. Now both ratios express $BC$ as 8 units and we can write $AB:BC:CD=4:8:5$.

$$\begin{aligned}\text{Thus, } AB : BD &= 4:(8+5)\\ &= 4:13\end{aligned}$$

ANSWER: (A)

## PART B:

11. The number of solutions $(x, y)$ of the equation $3x + y = 100$, where $x$ and $y$ are positive integers, is

(**A**) 33 (**B**) 35 (**C**) 100 (**D**) 101 (**E**) 97

***Solution***

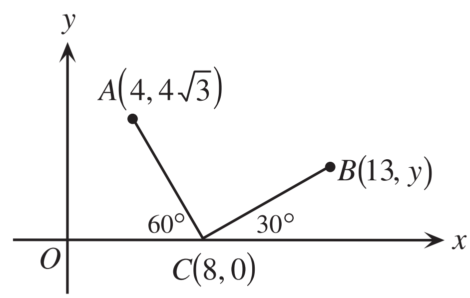
Rewrite the given equation as $x = \frac{100 - y}{3}$. Since $x$ must be an integer, $100 - y$ must be divisible by 3. Since both $x$ and $y$ must be positive integers, the only possible values of $y$ are 1, 4, 7, 10, 13, ... , 94, and 97. Thus, there are 33 possible values for $y$ and 33 solutions $(x, y)$ that meet the given conditions.

ANSWER: (A)

12. In the diagram, the value of $y$ is

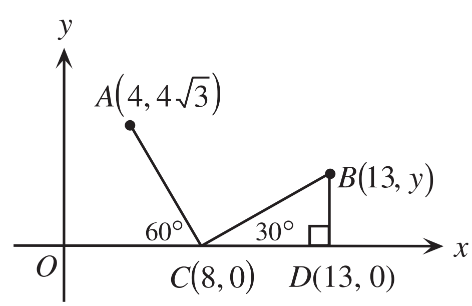
(**A**) $\frac{13}{2\sqrt{3}}$ (**B**) $\frac{5}{\sqrt{3}}$ (**C**) 2

(**D**) 12 (**E**) $\frac{\sqrt{3}}{5}$



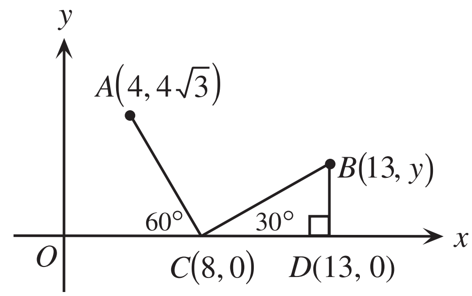

***Solution 1***

Label point $D$ (13,0), where $\Delta BDC$ is a right-angled triangle. The slope of $AC$ is $\frac{4\sqrt{3} - 0}{4 - 8} = -\sqrt{3}$. Since $\angle ACB$ is a right angle, $AC$ is perpendicular to $CB$ and the slope of $CB$ is $\frac{1}{\sqrt{3}}$. The length of $CD$ is $13 - 8 = 5$ and the length of $DB$ is $\frac{1}{\sqrt{3}}(5) = \frac{5}{\sqrt{3}}$. Thus, $y = \frac{5}{\sqrt{3}}$.



***Solution 2***

Label point $D$ (13,0), where $\Delta BDC$ is a right-angled triangle. The length of $BD$ is $y$. The length of $CD$ is $13-8=5$. Since $\Delta CBD$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle, its sides are in the ratio $1:\sqrt{3}:2$.




Thus, $\frac{BD}{CD}=\frac{1}{\sqrt{3}}$

$$\frac{y}{5}=\frac{1}{\sqrt{3}}$$

$$y=\frac{5}{\sqrt{3}}$$

ANSWER: (B)

13. Three-digit integers are formed using only the digits 1 and/or 2. The sum of all such integers formed is

(**A**) 1332 (**B**) 333 (**C**) 999 (**D**) 666 (**E**) 1665

***Solution***

The only three-digit integers that can be formed are 111, 112, 121, 122, 211, 212, 221, 222. The sum of these integers is 1332. ANSWER: (A)

14. Three straight lines, $l_1$, $l_2$ and $l_3$, have slopes $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively. All three lines have the same $y$-intercept. If the sum of the $x$-intercepts of the three lines is 36, then the $y$-intercept is

(**A**) $\frac{-13}{12}$ (**B**) $\frac{-12}{13}$ (**C**) $-4$ (**D**) 4 (**E**) $-9$

***Solution***

Let $b$ represent the common $y$-intercept of the three lines. The first line, $l_1$, has equation $y=\frac{1}{2}x+b$.

Set $y=0$ in this equation to find the $x$-intercept of the first line.

$$0=\frac{1}{2}x+b$$

$$-\frac{1}{2}x=b$$

$$x=-2b$$

Similarly the second line, $l_2$, has equation $y=\frac{1}{3}x+b$ and $x$ - intercept $-3b$. The third line, $l_3$, has equation $y=\frac{1}{4}x+b$ and $x$ - intercept $-4b$.

We know that $-2b-3b-4b=36$

$$-9b=36$$

$$b=-4$$

Thus, the common $y$-intercept of the three lines is $-4$. ANSWER: (C)

15. If $-2 \leq x \leq 5$, $-3 \leq y \leq 7$, $4 \leq z \leq 8$, and $w = xy - z$, then the smallest value $w$ may have is
(**A**) $-14$ (**B**) $-18$ (**C**) $-19$ (**D**) $-22$ (**E**) $-23$

***Solution***
We obtain the smallest value of $w = xy - z$ by finding the smallest value of the product $xy$ and then subtracting the largest value of $z$.
Since both $x$ and $y$ can take on positive or negative values, the smallest product $xy$ will be negative with one of $x$ and $y$ positive and the other negative. The smallest such product $xy$ is $(5)(-3) = -15$.
Thus, the smallest possible value of $w$ is $-15 - 8 = -23$. ANSWER: (E)

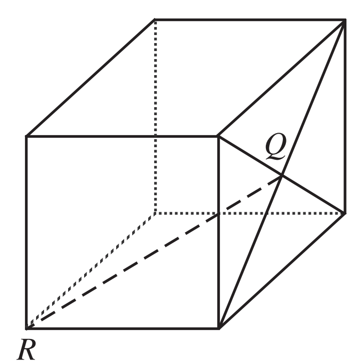
16. If $N = \left(7^{p+4}\right)\left(5^{q}\right)\left(2^{3}\right)$ is a perfect cube, where $p$ and $q$ are positive integers, the smallest possible value of $p + q$ is
(**A**) 5 (**B**) 2 (**C**) 8 (**D**) 6 (**E**) 12

***Solution***
In order for $N$ to be a perfect cube, each prime factor of $N$ must have an exponent that is divisible by 3. Since $p$ and $q$ must be positive integers, the smallest value of $p$ is 2 and the smallest value of $q$ is 3. Thus, the smallest value of $p + q$ is $5$. ANSWER: (A)

17. Using only digits 1, 2, 3, 4, and 5, a sequence is created as follows: one 1, two 2's, three 3's, four 4's, five 5's, six 1's, seven 2's, and so on.
The sequence appears as: 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 2, 2, ... .
The 100th digit in the sequence is
(**A**) 1 (**B**) 2 (**C**) 3 (**D**) 4 (**E**) 5

***Solution***
The total number of digits in $n$ groups of the sequence is given by $1 + 2 + 3 + ... + n$ . In order to determine the group containing the 100th digit in the sequence, we must find the positive integer $n$ such that $1 + 2 + 3 + ... + (n-1) < 100$ and $1 + 2 + 3 + ... + n > 100$. By examining a few of these sums we find that $1 + 2 + 3 + ... + 13 = 91$ and $1 + 2 + 3 + ... + 13 + 14 = 105$. Thus the 100th digit in the sequence is in the 14th group. The 100th digit is a 4. ANSWER: (D)

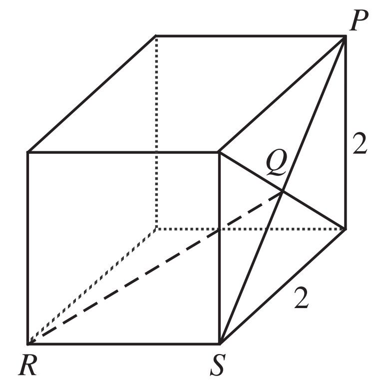
18. $Q$ is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of $QR$ is
(**A**) 2 (**B**) $\sqrt{8}$ (**C**) $\sqrt{5}$
(**D**) (**E**) $\sqrt{6}$



***Solution***

Label points $P$ and $S$ as shown. Since each face of the cube is a square with sides of length 2, use the Pythagorean Theorem to find the length of diagonal $PS$.

$$PS^2 = 2^2 + 2^2$$
$$= 8$$
$$PS = 2\sqrt{2}$$



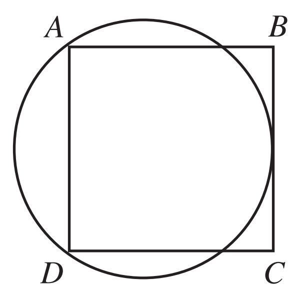

Then $QS$ has length $\sqrt{2}$, as $Q$ is the midpoint of diagonal $PS$.
Because we are working with a cube, $\angle QSR = 90^\circ$ and $\Delta QRS$ is a right-angled triangle. Use the Pythagorean Theorem in $\Delta QRS$ to get

$$QR^2 = 2^2 + \left(\sqrt{2}\right)^2$$
$$= 6$$
$$QR = \sqrt{6}$$

ANSWER: (E)

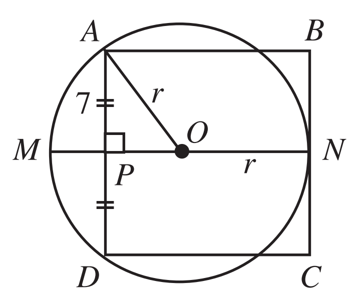
19. Square $ABCD$ has sides of length 14. A circle is drawn through $A$ and $D$ so that it is tangent to $BC$, as shown. What is the radius of the circle?

(**A**) 8.5 (**B**) 8.75 (**C**) 9
(**D**) 9.25 (**E**) 9.5




***Solution***

Let $r$ represent the length of the radius and let $O$ represent the centre of the circle. Draw diameter $MN$ that bisects chord $AD$ perpendicularly at $P$. Join $OA$.
$\Delta OAP$ is a right-angled triangle with $\angle APO = 90^\circ$. The length of $AP$ is 7, since it is half of a side of the square. The length of $OA$ is $r$, and the length of $PO$ is $PN - ON = 14 - r$.




Using the Pythagorean Theorem we get

$$r^2 = 7^2 + (14 - r)^2$$
$$r^2 = 49 + 196 - 28r + r^2$$
$$28r = 245$$
$$r = 8.75$$

Thus, the radius of the circle is 8.75.

ANSWER: (B)

20. A deck of 100 cards is numbered from 1 to 100. Each card has the same number printed on both sides. One side of each card is red and the other side is yellow. Barsby places all the cards, red side up, on a table. He first turns over every card that has a number divisible by 2. He then examines all the cards, and turns over every card that has a number divisible by 3. How many cards have the red side up when Barsby is finished?

(**A**) 83 (**B**) 17 (**C**) 66 (**D**) 50 (**E**) 49

***Solution***

Initially, all 100 cards have the red side up. After Barsby's first pass only the 50 odd-numbered cards have the red side up, since he has just turned all the even-numbered cards from red to yellow.

During Barsby's second pass he turns over all cards whose number is divisible by 3. On this pass Barsby will turn any odd-numbered card divisible by 3 from red to yellow. Between 1 and 100, there are 17 odd numbers that are divisible by 3, namely 3, 9, 15, 21, ... , 93, and 99. Also on this pass, Barsby will turn any even-numbered card divisible by 3 from yellow to red. Between 1 and 100, there are 16 even numbers that are divisible by 3, namely 6, 12, 18, 24, ... , 90, and 96.

When Barsby is finished, the cards that have the red side up are the 50 odd-numbered cards from the first pass, minus the 17 odd-numbered cards divisible by 3 from the second pass, plus the 16 even-numbered cards divisible by 3, also from the second pass.

Thus, $50 - 17 + 16 = 49$ cards have the red side up. ANSWER: (E)

**PART C:**

21. The numbers $123\,456\,789$ and $999\,999\,999$ are multiplied. How many of the digits in the final result are 9's?

(**A**) 0 (**B**) 1 (**C**) 2 (**D**) 3 (**E**) 17

***Solution***

Rewrite the product as follows:

$$\begin{aligned}(123\ \ 456\ \ 789)(999\ \ 999\ \ 999) &= (123\ \ 456\ \ 789)\left(10^9 - 1\right)\\ &= (123\ \ 456\ \ 789)\times 10^9 - (123\ \ 456\ \ 789)\end{aligned}$$

When 123 456 789 is subtracted from $(123\ \ 456\ \ 789)\times 10^9$ the result is 123 456 788 876 543 211. None of the digits are 9's. ANSWER: (A)

22. There are four unequal, positive integers $a$, $b$, $c$, and $N$ such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and $N$ is between 131 and 150. What is the value of $a + b + c$?

(**A**) 13 (**B**) 17 (**C**) 22 (**D**) 33 (**E**) 36

***Solution***

We are told that $N = 5a + 3b + 5c$ (1) and $N = 4a + 5b + 4c$ (2). Multiply equation (1) by 4 to get $4N = 20a + 12b + 20c$ (3). Similarly, multiply equation (2) by 5 to get $5N = 20a + 25b + 20c$ (4). Subtract equation (3) from equation (4) to get $N = 13b$.

Since $N$ and $b$ are both positive integers with $131 < N < 150$, $N$ must be a multiple of 13. The only possible value for $N$ is 143, when $b = 11$.

Substitute $N = 143$ and $b = 11$ into equation (1) to get

$$143 = 5a + 3(11) + 5c$$
$$110 = 5a + 5c$$
$$22 = a + c$$

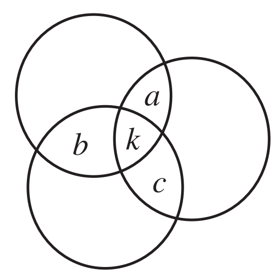
Thus, the value of $a$+$b$+$c$ is 22+11=33. ANSWER: (D)

23. Three rugs have a combined area of $200 \text{ m}^2$. By overlapping the rugs to cover a floor area of $140 \text{ m}^2$, the area which is covered by exactly two layers of rug is $24 \text{ m}^2$. What area of floor is covered by three layers of rug?

(**A**) $12 \text{ m}^2$ (**B**) $18 \text{ m}^2$ (**C**) $24 \text{ m}^2$ (**D**) $36 \text{ m}^2$ (**E**) $42 \text{ m}^2$

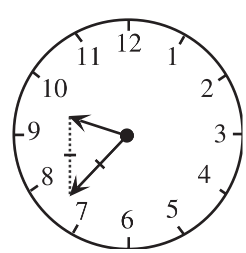
***Solution***

Draw the rugs in the following manner, where $a + b + c$ represents the amount of floor covered by exactly two rugs and $k$ represents the amount of floor covered by exactly three rugs. We are told that $a + b + c = 24$ (1).




Since the total amount of floor covered when the rugs do not overlap is $200 \text{ m}^2$ and the total covered when they do overlap is $140 \text{ m}^2$, then $60 \text{ m}^2$ of rug is wasted on double or triple layers. Thus, $a + b + c + 2k = 60$ (2). Subtract equation (1) from equation (2) to get $2k = 36$ and solve for $k = 18$. Thus, the area of floor covered by exactly three layers of rug is $18 \text{ m}^2$. ANSWER: (B)

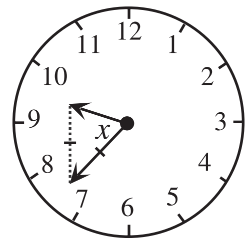
24. At some time between 9:30 and 10 o'clock the triangle determined by the minute hand and the hour hand is an isosceles triangle (see diagram). If the two equal angles in this triangle are each twice as large as the third angle, what is the time?




(**A**) 9:35 (**B**) 9:36 (**C**) 9:37
(**D**) 9:38 (**E**) 9:39

***Solution***

Let $x$ represent the angle, in degrees, between the hour and the minute hands. We are told that the triangle in the diagram is isosceles, with the two equal angles each twice as large as the third angle.




Thus, $x + x + \frac{1}{2}x = 180$

$$\frac{5}{2}x = 180$$

$$x = 72$$

For each minute that passes, the minute hand moves through an angle of $360^\circ \div 60 = 6^\circ$, and the hour hand moves through an angle of $\left(360^\circ \div 12\right) \div 60 = \frac{1}{2}^\circ$.

At 9:00 there is an angle of $270^\circ$ between the hour and the minute hands. At the time shown in the diagram there is an angle of $72^\circ$ between the hour and the minute hands. Since the minute hand gains $5\frac{1}{2}^\circ$ on the hour hand every minute, it takes $\dfrac{270-72}{5\frac{1}{2}} = 36$ minutes from 9:00 for the hour and minute hands to reach the given position. Thus, the time is 9:36.

ANSWER: (B)

25. For each value of $x$, $f(x)$ is defined to be the minimum value of the three numbers $2x+2$, $\frac{1}{2}x+1$ and $-\frac{3}{4}x+7$. What is the maximum value of $f(x)$?

(**A**) $\frac{2}{3}$ (**B**) 2 (**C**) $\frac{17}{5}$ (**D**) $\frac{62}{11}$ (**E**) 7

***Solution***

The three numbers $2x+2$, $\frac{1}{2}x+1$ and $\frac{-3}{4}x+7$ can be viewed as the $y$-coordinates of points lying on the lines $y = 2x+2$ (1), $y = \frac{1}{2}x+1$ (2), and $y = \frac{-3}{4}x+7$ (3), respectively.

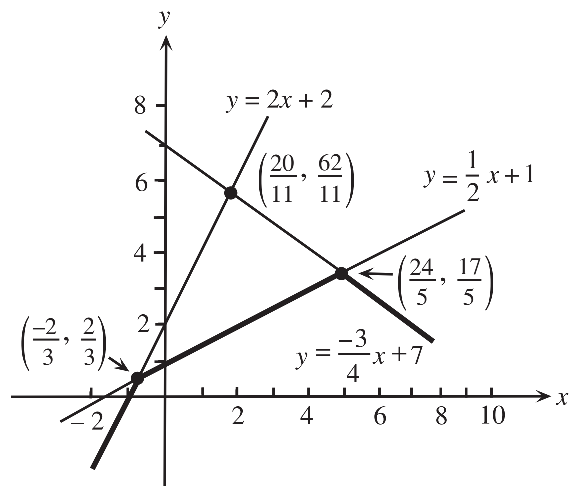
Draw all three lines on the same set of axes and find the points of intersection.

Subtract equation (2) from equation (1) to get

$$0 = \frac{3}{2}x + 1$$

$$\frac{-3}{2}x = 1$$

$$x = \frac{-2}{3}$$




Substitute $x = \frac{-2}{3}$ into equation (1) and get

$$y = 2\left(\frac{-2}{3}\right) + 2$$

$$y = \frac{2}{3}$$

Thus, the point of intersection of the lines $y = 2x + 2$ and $y = \frac{1}{2}x + 1$ is $\left(\frac{-2}{3}, \frac{2}{3}\right)$.

Similarly, we find $\left(\frac{20}{11}, \frac{62}{11}\right)$ as the point of intersection of the lines $y = 2x + 2$ and $y = \frac{-3}{4}x + 7$, and $\left(\frac{24}{5}, \frac{17}{5}\right)$ as the point of intersection of the lines $y = \frac{1}{2}x + 1$ and $y = \frac{-3}{4}x + 7$.

The minimum value of the three numbers $2x + 2$, $\frac{1}{2}x + 1$, and $\frac{-3}{4}x + 7$ is shown in the diagram as the smallest of the $y$-coordinates of points on the three lines for a given value of $x$. The maximum of these $y$-coordinates is $\frac{17}{5}$.

ANSWER: (C)

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

# 1997 Solutions
# Fermat Contest (Grade 11)

for the
NATIONAL BANK OF CANADA
Awards

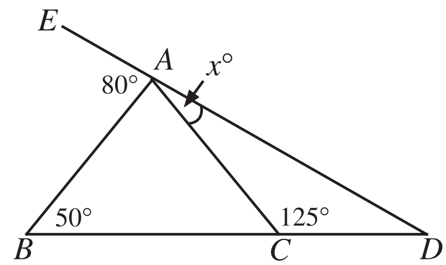
## PART A:

1. ***Solution***

$$(1)^{10}+(-1)^{8}+(-1)^{7}+(1)^{5}=1+1-1+1$$
$$=2$$

ANSWER: (E)

2. ***Solution***




Since $B$, $C$ and $D$ lie on a straight line,

$$\angle BCA = 180-125$$
$$=55$$

In $\Delta ABC$, $\angle BAC = 180-(50+55)$

$$=75$$

Since $E$, $A$ and $D$ lie on a straight line,

$$x = 180-(80+75)$$
$$=25$$

The value of $x$ is 25. ANSWER: (C)

3. ***Solution***

If the first day is a Monday, then every seventh day is also a Monday, and Monday falls on the following numbered days: 1, 8, 15, 22, 29, 36, 43.
In 45 consecutive days, the maximum number of Mondays is seven. ANSWER: (C)

4. ***Solution***

Let $x$ represent the number.

Then $x\left(x^2\right)\left(\frac{1}{x}\right)=\frac{100}{81}$

$$x^2=\frac{100}{81}$$
$$x=\pm\frac{10}{9}$$

Since $x$ is positive, $x=\frac{10}{9}$. ANSWER: (D)

5. ***Solution***

Since the sum of the seven integers is 77, their average is $\frac{77}{7}=11$.
Because there is an odd number of consecutive integers, 11 is the middle integer and 14 is the largest.

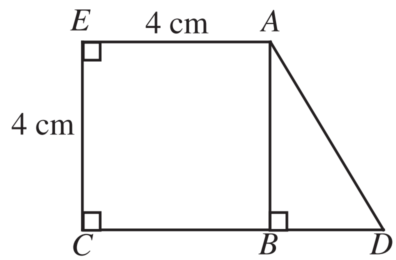
ANSWER: (C)

6. ***Solution***

The product of $2E3$ and $3E2$ is $\left(2\times10^3\right)\left(3\times10^2\right)=6\times10^5$, which can be written as $6E5$.

ANSWER: (B)

7. ***Solution***



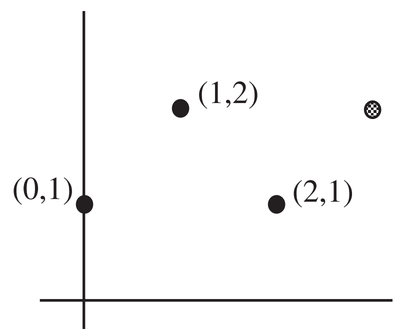

Draw in $AB$, perpendicular to $CD$ at $B$.
Since $ABCE$ is a square, each side has length 4 cm. Thus, $BD=3$ cm.
Using Pythagoras in $\Delta ABD$ gives

$$AD^2=4^2+3^2$$
$$AD=5.$$

The perimeter of the figure is $4+4+5+7=20$ cm.

ANSWER: (E)

8. ***Solution***




There are three possible locations for the fourth vertex, but each resulting parallelogram has the same area.
Locating the fourth vertex at $(3,2)$ yields a parallelogram with base 2 and height 1. Its area is $(2)(1)=2$.

ANSWER: (B)

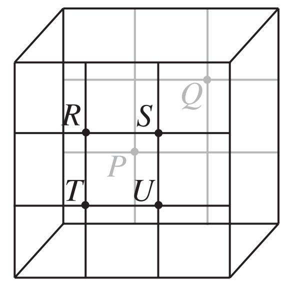
9. ***Solution***

The largest value of $\dfrac{x^2}{2y}$ will occur when the numerator is largest and the denominator is smallest.
Since $10\le x\le 20$, the largest value of $x^2$ is $(20)^2=400$. Similarly, since $40\le y\le 60$, the smallest value of $2y$ is $2(40)=80$.
Thus, the largest value of $\dfrac{x^2}{2y}$ is $\frac{400}{80}=5$.

ANSWER: (A)

10. ***Solution***

When the cube is folded together, the points are located as shown in the diagram. The point $S$ is diametrically opposite $P$.




ANSWER: (C)

## PART B:

11. ***Solution***

Since the middle integer (the median) is 83, the two larger integers must both be 85 (the mode). Since the range of the integers is 70, the smallest integer is $85 - 70 = 15$.
The sum of all five integers is $5(69) = 345$.
Thus, the second smallest integer is $345 - (85 + 85 + 83 + 15) = 77$. ANSWER: (A)

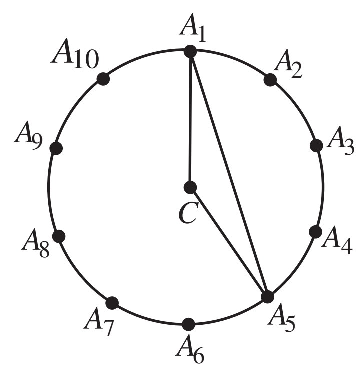
12. ***Solution***

Join $A_1A_5$, $A_1C$, and $A_5C$, as shown.
Since the points $A_1, A_2, A_3, \ldots, A_{10}$ are equally spaced, they generate equal angles at $C$, each of measure $\frac{360}{10} = 36$.

Thus, $\angle A_1CA_5 = 4(36)$
$= 144$

Since $A_1C = A_5C$ (radii), then $\Delta A_1CA_5$ is isosceles and

$$\angle A_1A_5C = \frac{(180 - 144)}{2}$$
$$= 18$$




Thus, the value of $\angle A_1A_5C$ is 18 degrees. ANSWER: (A)

13. ***Solution***

The first twelve numbers in the list begin with either the digit 1 or 2. The next six begin with the digit 3. In order, these six numbers are 3124, 3142, 3214, 3241, 3412, 3421.
We see that the number 3142 is in the fourteenth position. ANSWER: (B)

14. ***Solution***

In the diagram, extend $TP$ to meet $RS$ at $A$. Since $AT \perp RS$, then $\angle SPA = 180° - 90° - 26°$

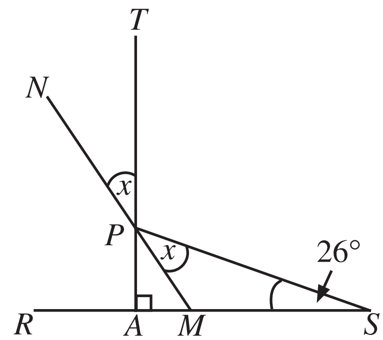
$$= 64°$$

Label points $M$ and $N$. Since $\angle TPN$ and $\angle MPA$ are vertically opposite angles, they are equal in size, so $\angle MPA = x$.

But $\angle SPA = 2x$, so

$$2x = 64°$$

$$x = 32°.$$




Thus, the value of $x$ is $32°$. ANSWER: (D)

15. ***Solution***

Multiplying the given equations together yields

$$\left(x^2yz^3\right)\left(xy^2\right) = \left(7^3\right)\left(7^9\right)$$

$$x^3y^3z^3 = 7^{12}$$

Taking the cube root of each side of the equation gives $xyz = 7^4$. ANSWER: (E)

16. ***Solution***

An even integer is found by taking the previous odd integer and adding 1.

Thus, the sum of the first 50 positive even numbers is obtained by finding the sum of the first 50 positive odd integers then adding 50 1's.

The sum of the first 50 positive even integers is $50^2 + 50$. ANSWER: (D)

17. ***Solution***

Let $s$ represent the population of Sudbury at the beginning of 1996, and $v$ represent the population of Victoria at the beginning of 1996.

At the end of 1996, Sudbury's population was $0.94s$ and Victoria's population was $1.14v$, where $0.94s = 1.14v \quad (1)$.

To find $s$:$v$, rearrange (1) to obtain

$$\frac{s}{v} = \frac{1.14}{0.94} = \frac{57}{47}$$

ANSWER: (B)

18. ***Solution***

First count the number of integers between 3 and 89 that can be written as the sum of exactly two elements. Since each element in the set is the sum of the two previous elements, 55 can be added to each of the seven smallest elements to form seven unique integers smaller than 89.

In the same way, 34 can be added to each of the seven smaller elements, 21 can be added to each of the six smaller elements, and so on.

The number of integers between 3 and 89 that can be written as the sum of two elements of the set is $7+7+6+5+4+3+2=34$. Since there are 85 integers between 3 and 89, then $85-34=51$ integers cannot be written as the sum of exactly two elements in the set.

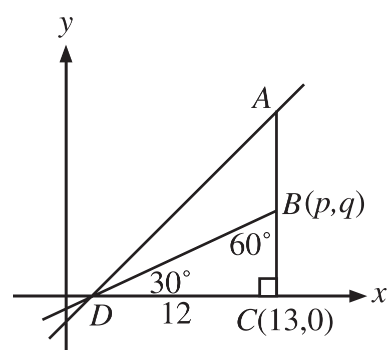
ANSWER: (E)

19. ***Solution***

Since $D$ is the $x$-intercept of line $AD$, the coordinates of $D$ are $(1,0)$. Thus, $DC=12$.

The slope of $AD$ is $\sqrt{3}$, so $AC=12\sqrt{3}$ and $\angle ADC=60°$. Since $BD$ bisects $\angle ADC$, then $\angle BDC=30°$ and $\Delta DBC$ is a $30°-60°-90°$ triangle.

Thus, $BC=\frac{12}{\sqrt{3}}$ and this is the value of $q$.



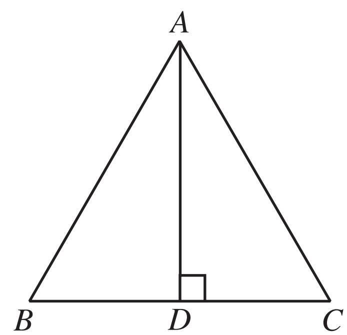

ANSWER: (D)

20. ***Solution***

In the diagram, there are 27 black triangles. If the entire diagram was divided into the smallest size equilateral triangles, there would be

$8+2(7)+2(6)+2(5)+2(4)+2(3)+2(2)+2(1)=64$

(counting by rows). Thus, $\frac{27}{64}$ of $\Delta ABC$ is coloured black.




Drop a perpendicular from $A$, meeting $BC$ at $D$. Since $\Delta ABC$ is equilateral and $AB=16$, then $BD=DC=8$. Using either the Pythagorean Theorem or the fact that $\Delta ABD$ is a $30°-60°-90°$ triangle, we find that $AD=8\sqrt{3}$.

Thus, the area of $\Delta ABC$ is $\frac{1}{2}\left(8\sqrt{3}\right)(16)=64\sqrt{3}$, and the area of all the black triangles is $\frac{27}{64}\left(64\sqrt{3}\right)=27\sqrt{3}$.

ANSWER: (C)

## PART C:

21. ***Solution***

Simplify the expression $\dfrac{\left(\frac{a}{c}+\frac{a}{b}+1\right)}{\left(\frac{b}{a}+\frac{b}{c}+1\right)}=11$ as follows:

$$\frac{\left(\dfrac{ab+ac+bc}{bc}\right)}{\left(\dfrac{bc+ab+ac}{ac}\right)}=11$$

$$\frac{ac}{bc}=11$$

$$\frac{a}{b}=11 \quad (\text{since } c \neq 0)$$

$$a=11b$$

By substitution, the condition $a+2b+c \le 40$ becomes $13b+c \le 40$.

Since $b$ and $c$ are positive integers, then $b$ can only take on the values 1, 2, or 3. The values of $a$ correspond directly to the values of $b$, since $a=11b$.

If $b=3$, there is one corresponding value of $c$. When $b=2$, there are 14 possible values of $c$. Finally if $b=1$, there are 27 possible values of $c$.

Therefore, the number of different ordered triples satisfying the given conditions is $1+14+27=42$.

ANSWER: (D)

22. ***Solution***

Rewrite the expression $2x^2-2xy+y^2=289$ as $x^2+(x-y)^2=289$. Using Pythagorean triples, the possible values for $x$ are 0, 8, 15, and 17.

Substituting each value of $x$ and solving for the corresponding values of $y$ yields seven different ordered pairs. ANSWER: (B)

23. ***Solution***

Using $f(x)=px+q$, calculate the following:

$$\begin{aligned} f(f(x)) &= f(px+q) \\ &= p(px+q)+q \\ &= p^2x+pq+q \end{aligned}$$

and

$$\begin{aligned} f\big(f\big(f(x)\big)\big) &= f\big(p^2x + pq + q\big) \\ &= p\big(p^2x + pq + q\big) + q \\ &= p^3x + p^2q + pq + q \end{aligned}$$

Equate this to $8x + 21$ to find $p = 2$ and $q = 3$.
Thus, $p + q = 5$. ANSWER: (C)

24. ***Solution***

This is an example of a telescoping series.
Substitute $n = 2, 3, 4, ..., 50$ into the given expression to obtain

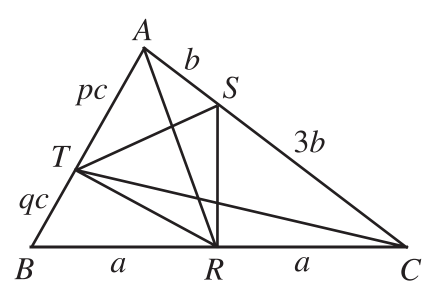
$$\begin{aligned} t_2 - t_1 &= 7 \\ t_3 - t_2 &= 9 \\ t_4 - t_3 &= 11 \\ &\vdots \\ t_{49} - t_{48} &= 101 \\ t_{50} - t_{49} &= 103 \end{aligned}$$

Adding the left and right sides separately gives

$$\begin{aligned} t_2 - t_1 + t_3 - t_2 + \cdots + t_{50} - t_{49} &= 7 + 9 + 11 + \cdots 101 + 103 \\ t_{50} - t_1 &= (7 + 103) + (9 + 101) + \cdots + (53 + 57) + 55 \\ &= 24(110) + 55 \\ &= 2695 \end{aligned}$$

Since $t_1 = 5$, then $t_{50} = 2700$. ANSWER: (A)

25. ***Solution***




Join $A$ to $R$ and $C$ to $T$. Label the diagram as shown. Let the area of $\Delta ABC$ be $k$.

Since $w$ is the area of $\Delta CRS$, and triangles $CRS$ and $ARC$ have equal heights and bases that are in the ratio $3b : 4b = 3 : 4$, then

$$w = \frac{3}{4} \text{ (area of } \Delta ARC)$$

However, triangles $ARC$ and $ABR$ also have equal heights and bases that are in the ratio $a : a = 1 : 1$, so

$$\begin{aligned} w &= \tfrac{3}{8} \text{ (area of } \Delta ABC) \\ &= \tfrac{3}{8}k \end{aligned}$$

Similarly, $x = \dfrac{q}{(p+q)}$ (area of $\Delta ABR$)

$$= \frac{q}{2(p+q)} \text{ (area of } \Delta ABC)$$

$$= \frac{q}{2(p+q)} k$$

and $z = \frac{1}{4}$ (area of $\Delta ATC$)

$$= \frac{p}{4(p+q)} \text{ (area of } \Delta ABC)$$

$$= \frac{p}{4(p+q)} k$$

Since $x^2 = wz$, then $\frac{q^2 k^2}{4(p+q)^2} = \frac{3pk^2}{32(p+q)}$ which simplifies to $3p^2 + 3pq - 8q^2 = 0$.

Dividing through by $q^2$ yields a quadratic in $\frac{p}{q}$, the desired ratio.

Solving this quadratic gives $\frac{p}{q} = \frac{\sqrt{105} - 3}{6}$. ANSWER: (E)