The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2024 Euclid Contest

Wednesday, April 3, 2024
(in North America and South America)

Thursday, April 4, 2024
(outside of North America and South America)

Solutions

1. (a) For every $x \neq 0$, we note that $\dfrac{x^4+3x^2}{x^2} = x^2+3$.

Therefore, when $x=2$, we have $\dfrac{x^4+3x^2}{x^2} = x^2+3 = 2^2+3 = 7$.

Alternatively, when $x=2$, we have $\dfrac{x^4+3x^2}{x^2} = \dfrac{2^4+3\cdot 2^2}{2^2} = \dfrac{28}{4} = 7$.

(b) By the Pythagorean Theorem in $\triangle ABC$, we have

$$\begin{aligned} AC^2 &= AB^2+BC^2 \\ (t+1)^2 &= 10^2+(t-1)^2 \\ t^2+2t+1 &= 100+t^2-2t+1 \\ 4t &= 100 \end{aligned}$$

and so $t=25$.

Alternatively, we could remember the Pythagorean triple 5-12-13 and scale this triple by a factor of 2 to obtain the Pythagorean triple 10-24-26, noting that the difference between $t+1$ and $t-1$ is 2 as is the difference between 26 and 24, which gives $t+1=26$ and so $t=25$.

(c) Since $\dfrac{2}{y}+\dfrac{3}{2y}=14$, then $\dfrac{4}{2y}+\dfrac{3}{2y}=14$ or $\dfrac{7}{2y}=14$.

Therefore, $2y=\dfrac{7}{14}=\dfrac{1}{2}$ and so $y=\dfrac{1}{4}$.

2. (a) Suppose that the sequence is $a, b, c, 13, e, 36$.
Since $36 = 13+e$, then $e = 36-13 = 23$.
Since $e = c+13$, then $23 = c+13$ and so $c=10$.
Since $13 = b+c$ and $c=10$, then $b=3$.
Since $c = a+b$ and $c=10$ and $b=3$, then $a = 10-3 = 7$.
Therefore, the first term is 7.

(b) From the given information, we obtain $5r^2+5r^3=(5r)^2$ and so $5r^2+5r^3=25r^2$.
Since $r\neq 0$, we can divide by $5r^2$ to obtain $1+r=5$, which gives $r=4$.

(c) Suppose that Jimmy's marks on his first, second, third, and fourth tests were w, x, y, and z, respectively.

Since Jimmy's average on his first, second and third tests was 65, then $\dfrac{w+x+y}{3}=65$ or $w+x+y=195$.

Since Jimmy's average on his second, third and fourth tests was 80, then $\dfrac{x+y+z}{3}=80$ or $x+y+z=240$.

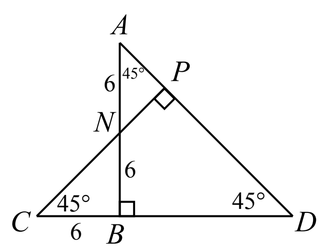
Since his mark on the fourth test was 2 times his mark on the first test, then $z=2w$.
Thus, $w+x+y=195$ and $x+y+2w=240$.
Subtracting the first equation from the second equation, we obtain $w=45$ and so his mark on the fourth test was $z=2w=90$.

3. (a) Since $y = r(x-3)(x-r)$ passes through $(0, 48)$, then $48 = r(0-3)(0-r)$.
Thus, $48 = 3r^2$ and so $r^2 = 16$ or $r = \pm 4$.

(b) With 13% sales tax on an item whose price is \$$B$, the total cost is \$$(1.13B)$.
With 5% sales tax on an item whose price is \$$B$, the total cost is \$$(1.05B)$.
From the given information \$$(1.13B)$ − \$$(1.05B)$ = \$24 or $1.13B - 1.05B = 24$.
Therefore, $0.08B = 24$, which gives $B = 300$.
Alternatively, we could note that the difference in total prices is the difference in the amount of tax paid. This is the difference between 13% of the original price and 5% of the original price; this difference is equal to 8% of the original price. If 8% of the original price is equal to \$24, then 1% of the original price is equal to \$3 and so the original price is \$3 × 100 = \$300.

(c) When $n = 1$, $f(2n) = (f(n))^2$ becomes $f(2) = (f(1))^2$.
Since $f(1) = 3$, then $f(2) = 3^2 = 9$.
When $m = 1$, $f(2m+1) = 3f(2m)$ becomes $f(3) = 3f(2)$.
Since $f(2) = 9$, then $f(3) = 3 \cdot 9 = 27$.
When $n = 2$, $f(2n) = (f(n))^2$ becomes $f(4) = (f(2))^2$.
Since $f(2) = 9$, then $f(4) = 9^2 = 81$.
Therefore, $f(2) + f(3) + f(4) = 9 + 27 + 81 = 117$.

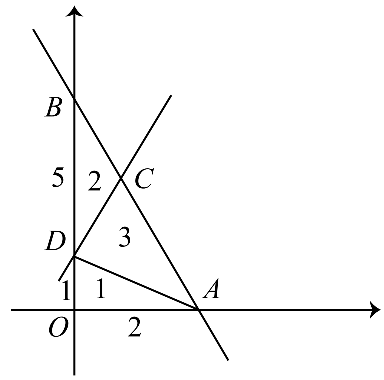
4. (a) Since $\triangle ABD$ is right-angled at B and has $\angle ADB = 45^\circ$, then $\angle BAD = 45^\circ$.
Similarly, $\triangle CPD$ is right-angled and isosceles with $\angle PCD = 45^\circ$.
Further, $\triangle APN$ and $\triangle CBN$ are also both right-angled and isosceles.
Since $CB = 6$ and $NB = CB$, then $NB = 6$.
Since $AB = 12$ and $NB = 6$, then $AN = AB - NB = 6$.



Since $\triangle APN$ is right-angled and isoceles, then its sides are in the ratio $1 : 1 : \sqrt{2}$.
Thus, $AP = PN = \frac{1}{\sqrt{2}}AN = \frac{6}{\sqrt{2}} = 3\sqrt{2}$.
Alternatively, if $AP = PN = x$, then the Pythagorean Theorem gives $AN^2 = AP^2 + PN^2$ and so $6^2 = 2x^2$ which gives $AP^2 = x^2 = 18$.
Thus, the area of $\triangle APN$ is $\frac{1}{2} \cdot AP \cdot PN = \frac{1}{2} \cdot 3\sqrt{2} \cdot 3\sqrt{2} = 9$.

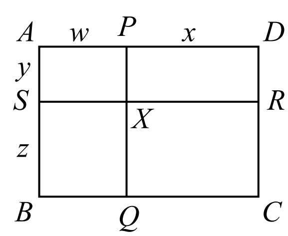
(b) The line with equation $y = -3x + 6$ has y-intercept 6, which means that $OB = 6$.
To find the x-intercept of this line, we set $y = 0$ and obtain the equation $-3x + 6 = 0$ which gives $3x = 6$ or $x = 2$. This means that $OA = 2$.
Since $\triangle ABO$ is right-angled at O, its area is $\frac{1}{2} \cdot OB \cdot OA = \frac{1}{2} \cdot 6 \cdot 2 = 6$.
Since the area of $\triangle ACD$ is $\frac{1}{2}$ of the area of $\triangle ABO$, then the area of $\triangle ACD$ is 3.
Next, we note that the line with equation $y = mx + 1$ has y-intercept 1; thus, $OD = 1$.
This means that the area of $\triangle ADO$ is $\frac{1}{2} \cdot OD \cdot OA = \frac{1}{2} \cdot 1 \cdot 2 = 1$.

We can determine the area of $\triangle BCD$ by subtracting the areas of $\triangle ACD$ and $\triangle ADO$ from that of $\triangle ABO$, which tells us that the area of $\triangle BCD$ is $6 - 3 - 1 = 2$.



Now, we can consider BD, which has length $6 - 1 = 5$, as the base of $\triangle BCD$; the corresponding height of $\triangle BCD$ is the distance from C to the y-axis, which we call h.
Thus, $\frac{1}{2} \cdot 5 \cdot h = 2$ and so $h = \frac{4}{5}$.
This means that C has x-coordinate $\frac{4}{5}$.
Since C is on the line with equation $y = -3x + 6$, we have $y = -3 \cdot \frac{4}{5} + 6 = \frac{18}{5}$.
Therefore, the coordinates of C are $(\frac{4}{5}, \frac{18}{5})$.

5. (a) Suppose that $AP = w$, $PD = x$, $AS = y$, and $SB = z$.



We use the notation $|APXS|$ to represent the area of $APXS$, and so on.
Thus, $|APXS| = wy$, $|PDRX| = xy$, $|SXBQ| = wz$, and $|XRCQ| = xz$.
Then,

$$|APXS| \cdot |XRCQ| = wy \cdot xz = xy \cdot wz = |PDRX| \cdot |SXQB|$$

If $|APXS| = 2$, $|PDRX| = 3$, and $|SWQB| = 6$, then $a = |XRQC| = \dfrac{2 \cdot 6}{3} = 4$.

If $|APXS| = 2$, $|PDRX| = 6$, and $|SWQB| = 3$, then $a = |XRQC| = \dfrac{2 \cdot 3}{6} = 1$.

If $|APXS| = 6$, $|PDRX| = 2$, and $|SWQB| = 3$, then $a = |XRQC| = \dfrac{6 \cdot 3}{2} = 9$.

Since we are told that there are three possible values for a, then these are 1, 4 and 9.
(Can you explain why there are exactly three such values?)

(b) The x-intercepts of the parabola with equation $y = x^2 - 4tx + 5t^2 - 6t$ are

$$x = \frac{4t \pm \sqrt{(-4t)^2 - 4(5t^2 - 6t)}}{2}$$

The distance, d, between these intercepts is their difference, which is

$$d = \frac{4t + \sqrt{(-4t)^2 - 4(5t^2 - 6t)}}{2} - \frac{4t - \sqrt{(-4t)^2 - 4(5t^2 - 6t)}}{2} = \sqrt{(-4t)^2 - 4(5t^2 - 6t)}$$

From this we see that d is as large as possible exactly when the discriminant is as large as possible. Here, the discriminant, Δ, is

$$\Delta = (-4t)^2 - 4(5t^2 - 6t) = 16t^2 - 20t^2 + 24t = -4t^2 + 24t$$

Completing the square,

$$\Delta = -4(t^2 - 6t) = -4(t^2 - 6t + 9 - 9) = -4(t^2 - 6t + 9) + 36 = -4(t-3)^2 + 36$$

Since $(t-3)^2 \geq 0$, then $\Delta \leq 36$ and $\Delta = 36$ exactly when $(t-3)^2 = 0$ or $t = 3$.
Therefore, the discriminant is maximized when $t = 3$, which means that the distance between the x-intercepts is as large as possible when $t = 3$.

6. (a) Every multiple of 21 is of the form $21k$ for some integer k.
For such a multiple to be between 10 000 and 100 000, we need $10\,000 < 21k < 100\,000$ or $\frac{10\,000}{21} < k < \frac{100\,000}{21}$.
Since $\frac{10\,000}{21} \approx 476.2$ and $\frac{100\,000}{21} \approx 4761.9$ and k is an integer, then $477 \leq k \leq 4761$. (Note that k is greater than 476.2 and is an integer, so must be at least 477; similarly, k is at most 4761.)
We also want the units digit of $21k$ to be 1.
This means that the units digit of k itself is 1, since the units digit of the product of 21 and k is equal to units digit of k because the units digit of 21 is 1.
Therefore, the possible values of k are 481, 491, 501, ..., 4751, 4761.
There are 429 such values. To see this, we can see that counting the integers in this list is the same as counting the integers in the list 48, 49, 50, ..., 475, 476. This list is equivalent to removing the integers from 1 to 47 from the list of integers from 1 to 476, giving $476 - 47 = 429$ integers.
Thus, $M = 429$.

(b) *Solution 1*
We can partition the N students at Strickland S.S. into four groups:

- a students who are in the physics club and are in the math club
- b students who are in the physics club and are not in the math club
- c students who are not in the physics club but are in the math club
- d students who are not in the physics club and are not in the math club

	In Math Club	Not in Math Club
In Physics Club	a	b
Not in Physics Club	c	d

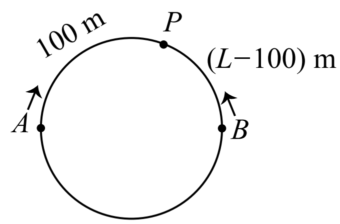
From the given information, there are $\frac{2}{5}N$ students in the physics club. In other words, $a + b = \frac{2}{5}N$.
Among the students in the physics club, twice as many are not in the math club as are in the math club. This means that $b = \frac{2}{3} \cdot \frac{2}{5}N = \frac{4}{15}N$ and $a = \frac{1}{3} \cdot \frac{2}{5}N = \frac{2}{15}N$.
From the given information, there are $\frac{1}{4}N$ students in the math club. In other words, $a + c = \frac{1}{4}N$.
Since $a = \frac{2}{15}N$, then $c = \frac{1}{4}N - \frac{2}{15}N = \frac{7}{60}N$.
Since $a + b + c + d = N$, then $d = N - a - b - c = N - \frac{4}{15}N - \frac{2}{15}N - \frac{7}{60}N = \frac{29}{60}N$.
Lastly, we know that $500 < N < 600$.
Since each of a, b, c, and d is an integer, then N must be divisible by 60.
Therefore, $N = 540$ and so the number of students not in either club is $d = \frac{29}{60} \cdot 540 = 261$.

Solution 2
Since there are N students at Strickland S.S., then $\frac{2}{5}N$ are in the physics club and $\frac{1}{4}N$ are in the math club.
Since each of $\frac{2}{5}N$ and $\frac{1}{4}N$ must be an integer, then N must be divisible by 5 and must be divisible by 4.
Since 5 and 4 share no common divisor larger than 1, then N must be divisible by $5 \cdot 4 = 20$.
Thus, we let $N = 20m$ for some positive integer m.
In this case, $\frac{2}{5}N = 8m$ students are in the physics club and $\frac{1}{4}N = 5m$ students are in the math club.
Now, among the $8m$ students in the physics club, twice as many are not in the math club as are in the math club.
In other words, $\frac{1}{3}$ of the $8m$ students in the physics club are in the math club. This means that m must be divisible by 3, since 3 is a prime number and 8 is not divisible by 3.
Therefore, $m = 3k$ for some positive integer k, which means that $N = 20m = 60k$ and $\frac{2}{5}N = 8m = 24k$ and $\frac{1}{4}N = 5m = 15k$.
Since $500 < N < 600$ and N is a multiple of 60, then $N = 540$, which means that $k = 9$.
Thus, the number of students in the physics club is $24k = 216$, of whom $\frac{1}{3} \cdot 216 = 72$ are in the math club and $\frac{2}{3} \cdot 216 = 144$ are not in the math club.
Also, the number of students in the math club is $15k = 135$.
Finally, we know that

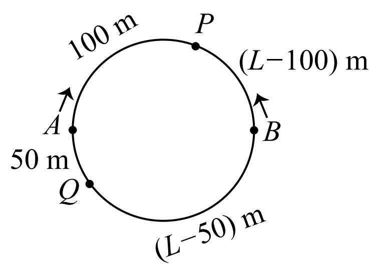
- there are 540 students at the school,
- 72 of whom are in both the physics club and the math club,
- 144 of whom are in the physics club and not in the math club, and
- $135 - 72 = 63$ are in the math club and not in the physics club.

Therefore, the number of students in neither club is $540 - 72 - 144 - 63 = 261$.

7. (a) Suppose that the length of the track is $2L$ m, that Arun's constant speed is a m/s, and that Bella's constant speed is b m/s.
When Arun and Bella run over the same interval of time, the ratio of the distances that they run is equal to the ratio of their speeds.
Consider the interval of time from the start to when they first meet. In the diagram, A is Arun's starting point, B is Bella's starting point, and P is this first meeting point.



Since Arun has run 100 m and together they have covered half of the length of the track, then Bella has run $(L - 100)$ m.
Thus, $\dfrac{a}{b} = \dfrac{100}{L-100}$.
From their first meeting point P to their second meeting point, which we label Q, Bella runs 150 m.



Over this time, Arun runs from P to B to Q.
Since Bella runs $150 - 100 = 50$ m past A, then $QB = (L-50)$ m (because $AB = L$ m and $AQ = 50$ m) and so Arun runs $(L - 100)$ m $+ (L - 50)$ m which is equal to $(2L - 150)$ m.
Thus, over this second interval of time, $\dfrac{a}{b} = \dfrac{2L-150}{150}$.
Equating expressions for $\dfrac{a}{b}$ and solving,

$$\begin{aligned}
\frac{100}{L-100} &= \frac{2L-150}{150} \\
100 \cdot 150 &= (L-100)(2L-150) \\
15\,000 &= 2L^2 - 350L + 15\,000 \\
350L &= 2L^2
\end{aligned}$$

Since $L \neq 0$, then $2L = 350$, and so the total length of the track is 350 m.

Checking, if the length of the track is 350 m, then half of the length is 175 m.
This means that from the start to P, Arun runs 100 m and Bella runs 75 m.
Also, from P to Q, Bella runs 150 m and Arun runs 200 m.
Note that $\dfrac{100}{75} = \dfrac{200}{150}$ so these numbers are consistent with the given information.

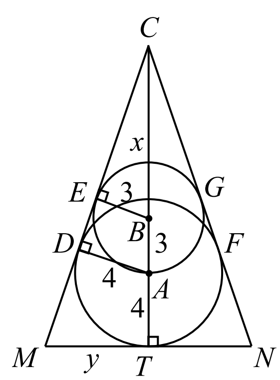
(b) Using exponent laws, the following equations are equivalent:

$$\begin{aligned}
4^{1+\cos^3\theta} &= 2^{2-\cos\theta}\cdot 8^{\cos^2\theta} \\
(2^2)^{1+\cos^3\theta} &= 2^{2-\cos\theta}\cdot (2^3)^{\cos^2\theta} \\
2^{2+2\cos^3\theta} &= 2^{2-\cos\theta}\cdot 2^{3\cos^2\theta} \\
2^{2+2\cos^3\theta} &= 2^{2-\cos\theta+3\cos^2\theta} \\
2+2\cos^3\theta &= 2-\cos\theta+3\cos^2\theta \\
2\cos^3\theta-3\cos^2\theta+\cos\theta &= 0 \\
\cos\theta(2\cos^2\theta-3\cos\theta+1) &= 0 \\
\cos\theta(2\cos\theta-1)(\cos\theta-1) &= 0
\end{aligned}$$

and so $\cos\theta = 0$ or $\cos\theta = 1$ or $\cos\theta = \frac{1}{2}$.
Since $0^\circ \leq \theta \leq 360^\circ$, the solutions are $\theta = 90^\circ, 270^\circ, 0^\circ, 360^\circ, 60^\circ, 300^\circ$.
Listing these in increasing order, the solutions to the original equation are

$$\theta = 0^\circ, 60^\circ, 90^\circ, 270^\circ, 300^\circ, 360^\circ$$

8. (a) We join B to E and A to D.
Since MC is tangent to the circles with centres A and B at D and E, respectively, then AD and BE are perpendicular to MC.
Since the radius of the circle with centre B is 3, then $AB = 3$ and $BE = 3$.
Since the radius of the circle with centre A is 4, then $AD = 4$ and $AT = 4$.
Let $CB = x$ and $MT = y$.



We note that $\triangle CEB$, $\triangle CDA$ and $\triangle CTM$ are all similar, since they are right-angled at E, D and T, respectively, and share a common angle at C.
Since $\triangle CEB$ and $\triangle CDA$ are similar, then $\dfrac{CB}{CA} = \dfrac{BE}{AD}$ and so $\dfrac{x}{x+3} = \dfrac{3}{4}$ which gives $4x = 3x + 9$ and so $x = 9$.
By the Pythagorean Theorem, $CE = \sqrt{CB^2 - BE^2} = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$.
Since $\triangle CEB$ and $\triangle CTM$ are similar, then $\dfrac{BE}{CE} = \dfrac{MT}{CT}$ and so $\dfrac{3}{6\sqrt{2}} = \dfrac{y}{9+3+4}$ which gives $y = \dfrac{16\cdot 3}{6\sqrt{2}} = \dfrac{8}{\sqrt{2}} = 4\sqrt{2}$.

Finally, the area of $\triangle MNC$ is equal to $\frac{1}{2} \cdot MN \cdot CT$.

If we joined B to G, we would see that $\triangle CEB$ is congruent to $\triangle CGB$ (each is right-angled, they have a common hypotenuse, and $BE = BG$). This means that $\angle BCE = \angle BCG$, which in turn means that $MT = TN$.

Since $MT = TN$, then $MN = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$ and so the area of $\triangle MNC$ is $\frac{1}{2} \cdot 8\sqrt{2} \cdot 16$ or $64\sqrt{2}$.

(b) First, we note that

$$\log_3 z = \frac{\log_{10} z}{\log_{10} 3} = \frac{2\log_{10} z}{2\log_{10} 3} = \frac{\log_{10}(z^2)}{\log_{10}(3^2)} = \frac{\log_{10}(z^2)}{\log_{10} 9} = \log_9(z^2)$$

Similarly, $\log_4 y = \log_{16}(y^2)$ and $\log_5 x = \log_{25}(x^2)$.

We also note from the original system of equations that $x > 0$ and $y > 0$ and $z > 0$.

Therefore, we can re-write the original system of equations as

$$\begin{aligned} \log_9 x + \log_9 y + \log_9(z^2) &= 2 \\ \log_{16} x + \log_{16}(y^2) + \log_{16} z &= 1 \\ \log_{25}(x^2) + \log_{25} y + \log_{25} z &= 0 \end{aligned}$$

Using logarithm rules, this is equivalent to the system

$$\begin{aligned} \log_9(xyz^2) &= 2 \\ \log_{16}(xy^2z) &= 1 \\ \log_{25}(x^2yz) &= 0 \end{aligned}$$

and to the system

$$\begin{aligned} xyz^2 &= 9^2 = 81 \\ xy^2z &= 16^1 = 16 \\ x^2yz &= 25^0 = 1 \end{aligned}$$

Multiplying these three equations together, we obtain $x^4y^4z^4 = 1296$ and so $(xyz)^4 = 6^4$.

Thus, $xyz = 6$.

Since $xyz^2 = 81$ and $xyz = 6$, then $z = \frac{xyz^2}{xyz} = \frac{81}{6} = \frac{27}{2}$.

Similarly, $y = \frac{xy^2z}{xyz} = \frac{16}{6} = \frac{8}{3}$ and $x = \frac{x^2yz}{xyz} = \frac{1}{6}$.

Therefore, $(x, y, z) = \left(\frac{1}{6}, \frac{8}{3}, \frac{27}{2}\right)$.

We can check by substitution that this triple does satisfy the original system of equations.

9. (a) Suppose that a sequence of n steps includes p steps in the positive direction and m steps in the negative direction.
This means that $n = p + m$ (total number of steps) and $d = p - m$ (final position).
Since $n = 9$ and $d = 5$, then $p + m = 9$ and $p - m = 5$ which give $p = 7$ and $m = 2$.
Thus, we need to count the number of sequences of 9 steps that include 7 steps in the positive direction and 2 steps in the negative direction.
This is the same as creating a sequence of 9 letters, 7 of which are R and 2 of which are L.
There are $\binom{9}{2} = \frac{9 \cdot 8}{2} = 36$ such sequences.
(Alternatively, we could see that there are 36 sequences by counting directly: there are 8 sequences where the first L appears in the first position, 7 where the first L appears appears in the second position, and so on.)

(b) *Solution 1*
Since $n = 9$ and $d = 3$, then $p + m = 9$ and $p - m = 3$ which gives $p = 6$ and $m = 3$.
This means that we can think about sequences of 9 letters that include 6 R's and 3 L's.
In this context, a direction change happens when the sequence changes from a block of one letter to a block of another letter (that is, when the sequence has an occurrence of "RL" or "LR").
For the sequence to change directions an even number of times, the number of blocks of letters in the sequence is odd.
For the number of blocks to be odd, the sequence must begin and end with the same letter.
If the sequence of 9 letters begins and ends with an R, the 7 letters in between include 4 R's and 3 L's; there are $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$ such sequences.
If the sequence of 9 letters begins and ends with an L, the 7 letters in between include 6 R's and 1 L; there are $\binom{7}{1} = 7$ such sequences.
In total, there are $35 + 7 = 42$ such sequences.

Solution 2
As in Solution 1, we want to count the number of sequences of 9 letters that include 6 R's and 3 L's and that have an odd number of blocks.
Since there are 3 L's, there can be at most 3 blocks of L's, and so at most 7 blocks.

<u>Case 1: There are 3 blocks</u>
Suppose that the block order is R-L-R.
Here, there are 3 L's in the block of L's and 6 R's distributed between two blocks of R's.
There are 5 ways to distribute the R's: $1 + 5$, $2 + 4$, $3 + 3$, $4 + 2$, $5 + 1$.
This means that there are 5 such sequences.
Suppose that the block order is L-R-L.
The block of R's includes 6 R's and there are 3 L's to distribute between two blocks.
There are 2 ways to do this: $1 + 2$ or $2 + 1$.
This means that there are 2 such sequences.
In total, there are $5 + 2 = 7$ sequences in this case.

<u>Case 2: There are 7 blocks</u>
Since there cannot be 4 blocks of L's, the block order must be R-L-R-L-R-L-R.
Each block of L's includes exactly 1 L, and there are 2 additional R's to distribute after one R is placed in each block.
There are 4 ways to put these two R's in the same block, and 6 ways to put them in

separate blocks (1st and 2nd, 1st and 3rd, 1st and 4th, 2nd and 3rd, 2nd and 4th, 3rd and 4th).
Therefore, there are 10 different sequences in this case.

Case 3: There are 5 blocks
Suppose that the block order is R-L-R-L-R.
The two blocks of L's include 3 L's in total; there are 2 ways to distribute these ($1+2$ or $2+1$).
The three blocks of L's include 6 R's in total.
Starting with one R in each block, there are 3 additional R's to distribute.
There are 3 ways in which the 3 R's can go in the same block: $3+0+0$, $0+3+0$, $0+0+3$.
There are 6 ways in which the 3 R's can go in two blocks: $2+1+0$, $2+0+1$, $1+2+0$, $0+2+1$, $1+0+2$, $0+1+2$.
There is 1 way in which the 3 R's can go in three different blocks: $1+1+1$.
Thus, there are 2 ways to distribute the L's and $3+6+1=10$ ways to distribute the R's and so $2 \cdot 10 = 20$ such sequences.
Suppose that the block order is L-R-L-R-L.
Each block of L's includes exactly 1 L, and the two blocks of R's include exactly 6 R's.
There are 5 ways to distribute these R's, as we saw in a similar situation in Case 1.
This means that there are 5 such sequences.
In total, there are $20+5=25$ sequences in this case.

Combining the cases, there are $10+7+25=42$ such sequences.

(c) Consider the sequences of length n that end at $x=d$ with $d \geq 0$.
Suppose that such a sequence includes p steps in the positive direction and m steps in the negative direction.
Since $p+m=n$ (total number of steps) and $p-m=d$ (ending position), then $2p=n+d$ (which gives $p=\dfrac{n+d}{2}$) and $2m=n-d$ (which gives $m=\dfrac{n-d}{2}$).
This means that all sequences of length n that end at $x=d$ with $d \geq 0$ correspond to the same values of p and m, and so we treat p and m as fixed in what follows.
Before proceeding to the general case, we deal with some specific small values.

When $n=1$, there is exactly 1 sequence that ends with $d \geq 0$. This sequence moves 1 step to the right. Since there is an odd number of sequences when $n=1$, it cannot be the case that half of the sequences have an even number of changes of direction.

When $n=2$, we can either have $p=2$ and $m=0$ (giving $d=2$) or $p=1$ and $m=1$ (giving $d=0$).
When $n=2$ and $d=2$, there is only 1 sequence, and so it cannot be the case that half of the sequences in this category have an even number of changes of direction.
When $n=2$ and $d=0$, there are 2 sequences (RL or LR), each of which has 1 change of direction, so it is not the case that half of the sequences in this category have an even number of changes of direction.

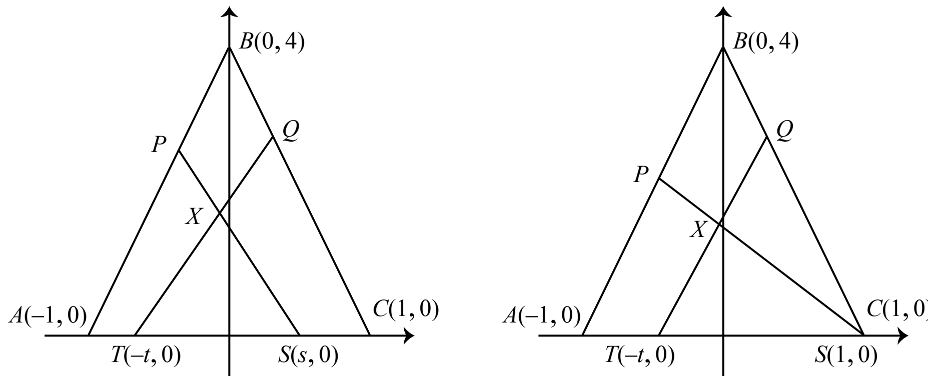
Therefore, we assume that $n \geq 3$. Since $n=p+m$ and $p \geq m$, then $p \geq 2$ as well.
We think about sequences of n letters that include p R's and m L's.
As in (b) Solution 1, the number of direction changes is even exactly when the number of blocks of letters is odd, which is exactly when the sequence begins and ends with the same letter.
For such a sequence to begin and end with R, the $n-2$ letters between include $(p-2)$ R's and m L's. (Note that $n-2 \geq 0$ and $p-2 \geq 0$ since $n \geq 2$ and $p \geq 0$.)

For such a sequence to begin and end with L, the $n-2$ letters between include p R's and $(m-2)$ L's. (Note that $n-2 \geq 0$. It is possible, though, that $m-2<0$, in which case we adopt the convention that there are 0 such sequences. In this case, we have $p=n-m=(n-2)-(m-2)>n-2$ and so $\binom{n-2}{p}=0$, making the calculations that follow consistent with this convention.)
Thus, there are $\binom{n-2}{p-2}+\binom{n-2}{p}$ such sequences.
This is true for exactly half of all sequences when the following equivalent equations are true:

$$\begin{aligned}
\binom{n-2}{p-2}+\binom{n-2}{p} &= \frac{1}{2}\binom{n}{p} \\
\frac{(n-2)!}{(p-2)!m!}+\frac{(n-2)!}{p!(m-2)!} &= \frac{n!}{2\cdot p!m!} \\
\frac{(n-2)!}{(p-2)!m!}+\frac{(n-2)!}{p!(m-2)!} &= \frac{n\cdot(n-1)\cdot(n-2)!}{2\cdot p!m!} \\
\frac{1}{(p-2)!m!}+\frac{1}{p!(m-2)!} &= \frac{n(n-1)}{2\cdot p!m!} \\
\frac{p!}{(p-2)!}+\frac{m!}{(m-2)!} &= \frac{n(n-1)}{2} \\
2p(p-1)+2m(m-1) &= n(n-1) \\
2p^2+2m^2 &= n^2-n+2p+2m \\
(p^2+2mp+m^2)+(p^2-2mp+m^2) &= n^2-(p+m)+2p+2m \\
(p+m)^2+(p-m)^2 &= n^2+(p+m) \\
n^2+d^2 &= n^2+n \\
d^2 &= n
\end{aligned}$$

Therefore, exactly half of these sequences have an even number of changes of direction exactly when $n=d^2$.
To finish the problem, we now need to count the number of perfect squares (that is, possible values for n) in the correct range.
Since $2 \leq n \leq 2024$ and $d \geq 0$ and n is a perfect square, then the facts that $1^2=1$, $2^2=4$, $44^2=1936$, and $45^2=2025$ tell us that there are 43 perfect squares from 2 to 2024, inclusive, and so 43 such pairs (d,n). These are the pairs (d,d^2) for $d=2,3,\ldots,43,44$.

10. (a) Here is the general set-up for this problem along with the specific instance in (a) where $s = 1$:



Since SP and TQ divide $\triangle ABC$ into four regions of equal area, then $\triangle APS$, which is made up of two of these regions, has area equal to one-half of the area of $\triangle ABC$.
Since S and C coincide, then P is the midpoint of AB, which means that P has coordinates $(-\frac{1}{2}, 2)$.

Alternatively, we could note that the area of $\triangle ABC$ is $\frac{1}{2} \cdot 2 \cdot 4 = 4$ and so the area of $\triangle APS$ must be 2.
If P has y-coordinate p, then $\frac{1}{2} \cdot 2 \cdot p = 2$ which gives $p = 2$.
Since AB has slope 4 and y-intercept 4, its equation is $y = 4x + 4$.
Since P lies on AB and has y-coordinate 2, its x-coordinate satisfies $2 = 4x + 4$ and so $4x = -2$ or $x = -\frac{1}{2}$.
Thus, the coordinates of P are $(-\frac{1}{2}, 2)$.

(b) We note that the area of $\triangle ABC$ is $\frac{1}{2} \cdot 2 \cdot 4 = 4$. When the triangle is divided into four equal areas, each of these areas must be 1.
Suppose that (s, t) is a balancing pair.
This is true exactly when

- the area of $\triangle SXT$ is 1, and
- the area of $\triangle APS$ is 2, and
- the area of $\triangle CQT$ is 2.

(Bullets 1 and 2 tell us that the area of quadrilateral $APXT$ is 1.
Bullets 1 and 3 tell us that the area of quadrilateral $CQXS$ is 1.
Since three of the areas are 1, then the fourth must be 1.)

$\triangle SXT$ has base $TS = s + t$. If its height is h, then $\dfrac{1}{2}(s + t)h = 1$ and so $h = \dfrac{2}{s + t}$.
$\triangle APS$ has base $AS = 1 + s$. If its height is p, then $\dfrac{1}{2}(s + 1)p = 2$ and so $p = \dfrac{4}{s + 1}$.
$\triangle CQT$ has base $TC = 1 + t$. If its height is q, then $\dfrac{1}{2}(t + 1)q = 2$ and so $q = \dfrac{4}{t + 1}$.
The line through A and B has slope 4 and y-intercept 4, so its equation is $y = 4x + 4$.
The line through C and B has slope -4 and y-intercept 4, so its equation is $y = -4x + 4$.
Since P lies on the line with equation $y = 4x + 4$ and the y-coordinate of P is $\dfrac{4}{s + 1}$, then the x-coordinate of P satisfies $\dfrac{4}{s + 1} = 4x + 4$ which gives $\dfrac{1}{s + 1} = x + 1$, from which we obtain $x = \dfrac{1}{s + 1} - 1 = \dfrac{1 - s - 1}{s + 1} = -\dfrac{s}{s + 1}$.

Therefore, P has coordinates $\left(-\frac{s}{s+1}, \frac{4}{s+1}\right)$.

Since Q lies on the line with equation $y = -4x + 4$ and the y-coordinate of Q is $\frac{4}{t+1}$, then the x-coordinate of Q satisfies $\frac{4}{t+1} = -4x + 4$ which gives $\frac{1}{t+1} = -x + 1$ or $x = 1 - \frac{1}{t+1} = \frac{t+1-1}{t+1} = \frac{t}{t+1}$.

Therefore, Q has coordinates $\left(\frac{t}{t+1}, \frac{4}{t+1}\right)$.

Next, we find the coordinates of X by finding the equations of the lines through P and S, and through Q and T.

The slope of the line through S and P is

$$\frac{\frac{4}{s+1} - 0}{-\frac{s}{s+1} - s} = \frac{4}{-s - s(s+1)} = -\frac{4}{s^2 + 2s}$$

Since this line passes through $S(s, 0)$, its equation is $y = -\frac{4}{s^2+2s}(x - s)$.

The slope of the line through T and Q is

$$\frac{\frac{4}{t+1} - 0}{\frac{t}{t+1} - (-t)} = \frac{4}{t + t(t+1)} = \frac{4}{t^2 + 2t}$$

Since this line passes through $S(-t, 0)$, its equation is $y = \frac{4}{t^2+2t}(x + t)$.

To find the x-coordinate of X, we find the point of intersection of the line through S and P and the line through T and Q; thus, we solve

$$\begin{aligned}
-\frac{4}{s^2+2s}(x-s) &= \frac{4}{t^2+2t}(x+t) \\
\frac{4s}{s^2+2s} - \frac{4t}{t^2+2t} &= \left(\frac{4}{s^2+2s} + \frac{4}{t^2+2t}\right)x \\
4s(t^2+2t) - 4t(s^2+2s) &= (4(t^2+2t) + 4(s^2+2s))x \\
x &= \frac{st^2 - s^2t}{t^2+2t+s^2+2s}
\end{aligned}$$

Therefore, the y-coordinate of X is obtained from

$$\begin{aligned}
y &= \frac{4}{t^2+2t}\left(\frac{st^2 - s^2t}{t^2+2t+s^2+2s} + t\right) \\
&= \frac{4}{t^2+2t} \cdot \frac{st^2 - s^2t + t^3 + 2t^2 + ts^2 + 2st}{t^2+2t+s^2+2s} \\
&= \frac{4}{t^2+2t} \cdot \frac{t^3 + 2t^2 + st^2 + 2st}{t^2+2t+s^2+2s} \\
&= \frac{4}{t^2+2t} \cdot \frac{t(t^2+2t) + s(t^2+2t)}{t^2+2t+s^2+2s} \\
&= \frac{4(s+t)}{t^2+2t+s^2+2s}
\end{aligned}$$

Finally, the y-coordinate of X is equal to the height h from earlier, so

$$\begin{aligned}
\frac{2}{s+t} &= \frac{4(s+t)}{t^2+2t+s^2+2s} \\
\frac{1}{s+t} &= \frac{2(s+t)}{t^2+2t+s^2+2s} \\
t^2+2t+s^2+2s &= 2(s+t)^2 \\
t^2+2t+s^2+2s &= 2s^2+4st+2t^2 \\
-4st+2s+2t &= s^2+t^2
\end{aligned}$$

Therefore, the desired relationship is true, with $d=-4$, $e=f=2$ and $g=0$.

(c) We look for pairs of the form $(s,t)=(s,ks)$ where s and k are rational numbers.
From the relationship from (b),

$$\begin{aligned}
s^2+k^2s^2 &= -4ks^2+2s+2ks \\
s^2(k^2+4k+1) &= (2k+2)s \\
s(k^2+4k+1) &= 2k+2 \qquad \text{(since } s>0\text{)} \\
s &= \frac{2k+2}{k^2+4k+1}
\end{aligned}$$

Thus, $t=ks=\dfrac{2k^2+2k}{k^2+4k+1}$.
Since $k>0$, then $s>0$.
We need to have $s \leq t$. This is equivalent to

$$\begin{aligned}
\frac{2k+2}{k^2+4k+1} &\leq \frac{2k^2+2k}{k^2+4k+1} \\
2k+2 &\leq 2k^2+2k \qquad \text{(since } k^2+4k+1>0\text{)} \\
2 &\leq 2k^2 \\
1 &\leq k \qquad \text{(since } k>0\text{)}
\end{aligned}$$

Thus, when $k \geq 1$, we have $0<s\leq t$.
Finally, we need $t \leq 1$. This is equivalent to

$$\begin{aligned}
\frac{2k^2+2k}{k^2+4k+1} &\leq 1 \\
2k^2+2k &\leq k^2+4k+1 \qquad \text{(since } k^2+4k+1>0\text{)} \\
k^2-2k-1 &\leq 0
\end{aligned}$$

The roots of $k^2-2k-1=0$ are $k=\dfrac{2\pm\sqrt{(-2)^2-4(1)(-1)}}{2}=1\pm\sqrt{2}$.
Thus, $k^2-2k-1\leq 0$ when $1-\sqrt{2}\leq k\leq 1+\sqrt{2}$.
Since $k>0$, then $t\leq 1$ when $0<k\leq 1+\sqrt{2}$.
Therefore, when $1\leq k\leq 1+\sqrt{2}$, we have $0<s\leq t\leq 1$.
We note that there are infinitely many rational numbers in any interval of non-zero length; in particular, there are infinitely many rational numbers k that satisfy $1\leq k\leq 1+\sqrt{2}$.
We do need to confirm that the values of $\dfrac{2k+2}{k^2+4k+1}$ are different for different values of k, which will confirm that there are infinitely many different values of s as k assumes these

infinitely many values itself.

To do this, we show that if $\dfrac{2k+2}{k^2+4k+1} = \dfrac{2j+2}{j^2+4j+1}$, then $k=j$; this fact will allow us to conclude that if $k \neq j$, then $\dfrac{2k+2}{k^2+4k+1} \neq \dfrac{2j+2}{j^2+4j+1}$.

When $k>0$ and $j>0$, then following equations are equivalent:

$$\begin{aligned}
\frac{2k+2}{k^2+4k+1} &= \frac{2j+2}{j^2+4j+1} \\
(2k+2)(j^2+4j+1) &= (2j+2)(k^2+4k+1) \\
(k+1)(j^2+4j+1) &= (j+1)(k^2+4k+1) \\
j^2k+4jk+k+j^2+4j+1 &= jk^2+4jk+j+k^2+4k+1 \\
j^2k-jk^2+3j-3k+j^2-k^2 &= 0 \\
(j-k)(jk+j+k+3) &= 0
\end{aligned}$$

Since $jk+j+k+3>0$, then it must be the case that $j-k=0$ and so $j=k$.

This means that if $k \neq j$, then $\dfrac{2k+2}{k^2+4k+1} \neq \dfrac{2j+2}{j^2+4j+1}$ and so there are infinitely many different values of s.

Therefore, the pairs $(s,t) = \left(\dfrac{2k+2}{k^2+4k+1}, \dfrac{2k^2+2k}{k^2+4k+1}\right)$ where k is a rational number with $1 \leq k \leq 1+\sqrt{2}$ is an infinite family of solutions to the relationship from (b).

The CENTRE for EDUCATION
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2023 Euclid Contest

Tuesday, April 4, 2023
(in North America and South America)

Wednesday, April 5, 2023
(outside of North America and South America)

Solutions

1. (a) Since the average of the 5 numbers n, $2n$, $3n$, $4n$, and $5n$ is 18, we obtain the equation $\dfrac{n+2n+3n+4n+5n}{5} = 18$.
Therefore, $\dfrac{15n}{5} = 18$ and so $3n = 18$ or $n = 6$.

(b) *Solution 1*
Adding the equations $2x + y = 5$ and $x + 2y = 7$, we obtain $(2x + y) + (x + 2y) = 5 + 7$ and so $3x + 3y = 12$.
Therefore, the average of x and y is $\dfrac{x+y}{2} = \dfrac{3x+3y}{6} = \dfrac{12}{6} = 2$.

Solution 2
Since $2x + y = 5$, then $4x + 2y = 10$.
Subtracting the second equation, we obtain $(4x + 2y) - (x + 2y) = 10 - 7$ which gives $3x = 3$ and so $x = 1$.
Thus, $y = 5 - 2x = 3$.
The average of x and y is thus $\dfrac{1+3}{2} = 2$.

(c) Since the average of the three numbers t^2, $2t$ and 3 is 9, then $\dfrac{t^2+2t+3}{3} = 9$.
Therefore, $t^2 + 2t + 3 = 27$ and so $t^2 + 2t - 24 = 0$ which gives $(t + 6)(t - 4) = 0$.
Since $t < 0$, then $t = -6$.

2. (a) Since $Q(5, 3)$ is the midpoint of $P(1, p)$ and $R(r, 5)$, then $\dfrac{1+r}{2} = 5$ and $\dfrac{p+5}{2} = 3$.
Thus, $1 + r = 10$ which gives $r = 9$, and $p + 5 = 6$ which gives $p = 1$.
Therefore, $p = 1$ and $r = 9$.

(b) *Solution 1*
The point with coordinates $P(3, 6)$ is 6 units above the x-axis.
A line with slope 3 moves 2 units to the right as it moves 6 units up. Therefore, to move from $P(3, 6)$ to the x-axis along a line with slope 3 results in a move of 6 units down and 2 units left. Thus, its x-intercept is $3 - 2 = 1$.
A line with slope -1 moves 6 units to the left as it moves 6 units up. Therefore, to move from $P(3, 6)$ to the x-axis along a line with slope -1 results in a move of 6 units down and 6 units right. Thus, its x-intercept is $3 + 6 = 9$.
The distance between these x-intercepts is $9 - 1 = 8$.

Solution 2
The line with slope 3 that passes through $P(3, 6)$ has equation $y - 6 = 3(x - 3)$ or $y = 3x - 3$.
The x-intercept of this line has $y = 0$ and so $0 = 3x - 3$ or $3x = 3$, which gives $x = 1$.
The line with slope -1 that passes through $P(3, 6)$ has equation $y - 6 = (-1)(x - 3)$ or $y = -x + 9$.
The x-intercept of this line has $y = 0$ and so $0 = -x + 9$ or $x = 9$.
The distance between these x-intercepts is $9 - 1 = 8$.

(c) The line with equation $y = 2x + 7$ has slope 2.
The line with equation $y = tx + t$ has slope t.
Since these lines are perpendicular, the product of their slopes is -1 and so $2t = -1$ which gives $t = -\frac{1}{2}$.
We now need to find the point of intersection of the lines with equations $y = 2x + 7$ and $y = -\frac{1}{2}x - \frac{1}{2}$.
Equating expressions for y, we obtain $2x+7 = -\frac{1}{2}x - \frac{1}{2}$ or $\frac{5}{2}x = -\frac{15}{2}$, which gives $x = -3$.
Therefore, $y = 2x + 7 = 2(-3) + 7 = 1$, and so the point of intersection of these lines is $(-3, 1)$.

3. (a) Since $64 = 2^6$, its positive divisors are 1, 2, 4, 8, 16, 32, and 64.
The sum of these divisors is $1 + 2 + 4 + 8 + 16 + 32 + 64 = 127$.

(b) Suppose that the four consecutive integers that Fionn originally wrote on the blackboard were x, $x + 1$, $x + 2$, and $x + 3$.
When Lexi erases one of these integers, the sum of the remaining three integers is equal to one of the following:

$$\begin{aligned}(x+1)+(x+2)+(x+3) &= 3x+6\\ x+(x+2)+(x+3) &= 3x+5\\ x+(x+1)+(x+3) &= 3x+4\\ x+(x+1)+(x+2) &= 3x+3\end{aligned}$$

We are told that the sum of these integers is 847.
We note that $847 = 3 \cdot 282 + 1$, which is one more than a multiple of 3. Since $3x + 3$ and $3x + 6$ are always multiples of 3 and $3x + 5$ is 2 more than a multiple of 3, then we must have $3x + 4 = 847$ and so $3x = 843$ or $x = 281$. (Alternatively, we could have set each of the four sums above equal to 847 to determine in which case or cases we obtained an integer solution for x.)
Therefore, the original integers were 281, 282, 283, 284 and Lexi erased $x + 2 = 283$.

(c) From the given information, the 7 terms in the arithmetic sequence are

$$d^2, \quad d^2+d, \quad d^2+2d, \quad d^2+3d, \quad d^2+4d, \quad d^2+5d, \quad d^2+6d$$

Since the sum of these 7 terms is 756, we obtain the following equivalent equations:

$$\begin{aligned}d^2+(d^2+d)+(d^2+2d)+(d^2+3d)+(d^2+4d)+(d^2+5d)+(d^2+6d) &= 756\\ 7d^2+21d &= 756\\ d^2+3d &= 108\\ d^2+3d-108 &= 0\\ (d+12)(d-9) &= 0\end{aligned}$$

and so $d = -12$ or $d = 9$.
The corresponding arithmetic sequences are

$$144, 132, 120, 108, 96, 84, 72 \qquad \text{and} \qquad 81, 90, 99, 108, 117, 126, 135$$

4. (a) In 1 hour, Liang paints $\frac{1}{3}$ of the room.

Thus, in 2 hours, Liang paints $\frac{2}{3}$ of the room.

Edmundo needs to paint $1 - \frac{2}{3} = \frac{1}{3}$ of the room.

In 1 hour, Edmundo paints $\frac{1}{4}$ of the room.

Since $\frac{1}{4} = \frac{3}{12}$ and $\frac{1}{3} = \frac{4}{12}$, this means that Edmundo paints for $\frac{1}{3} \div \frac{1}{4} = \frac{4}{12} \div \frac{3}{12} = \frac{4}{3}$ of an hour.

Therefore, Edmundo paints for 80 minutes.

(b) When converted to a fraction, $A\%$ is equal to $\dfrac{A}{100}$.

When an amount is increased by $A\%$, we can find its new value by multiplying by $1+\dfrac{A}{100}$.

When an amount is decreased by $A\%$, we can find its new value by multiplying by $1-\dfrac{A}{100}$.

When \$400 is increased by $A\%$, the amount becomes $\$400\left(1+\dfrac{A}{100}\right)$.

When this value is decreased by $A\%$, the amount becomes $\$400\left(1+\dfrac{A}{100}\right)\left(1-\dfrac{A}{100}\right)$.

Therefore,

$$\begin{aligned}
\$400\left(1+\frac{A}{100}\right)\left(1-\frac{A}{100}\right) &= \$391 \\
\left(1+\frac{A}{100}\right)\left(1-\frac{A}{100}\right) &= \frac{391}{400} \\
1-\frac{A^2}{100^2} &= 1-\frac{9}{400} \\
\frac{A^2}{100^2} &= \frac{9}{400} \\
\frac{A^2}{100^2} &= \frac{3^2}{20^2} \\
\frac{A}{100} &= \frac{3}{20} \qquad \text{(since } A > 0\text{)} \\
A &= 100 \cdot \frac{3}{20} = 15
\end{aligned}$$

Therefore, $A = 15$.

5. (a) The quadratic function $f(x) = x^2 + (2n - 1)x + (n^2 - 22)$ has no real roots exactly when its discriminant, Δ, is negative.
The discriminant of this function is

$$\begin{aligned}\Delta &= (2n-1)^2 - 4(1)(n^2 - 22) \\ &= (4n^2 - 4n + 1) - (4n^2 - 88) \\ &= -4n + 89\end{aligned}$$

We have $\Delta < 0$ exactly when $-4n + 89 < 0$ or $4n > 89$.
This final inequality is equivalent to $n > \frac{89}{4} = 22\frac{1}{4}$.
Therefore, the smallest positive integer that satisfies this inequality, and hence for which $f(x)$ has no real roots, is $n = 23$.

(b) Using the cosine law in $\triangle PQR$,

$$\begin{aligned}PR^2 &= PQ^2 + QR^2 - 2 \cdot PQ \cdot QR \cdot \cos(\angle PQR) \\ 21^2 &= a^2 + b^2 - 2ab\cos(60^\circ) \\ 441 &= a^2 + b^2 - 2ab \cdot \tfrac{1}{2} \\ 441 &= a^2 + b^2 - ab\end{aligned}$$

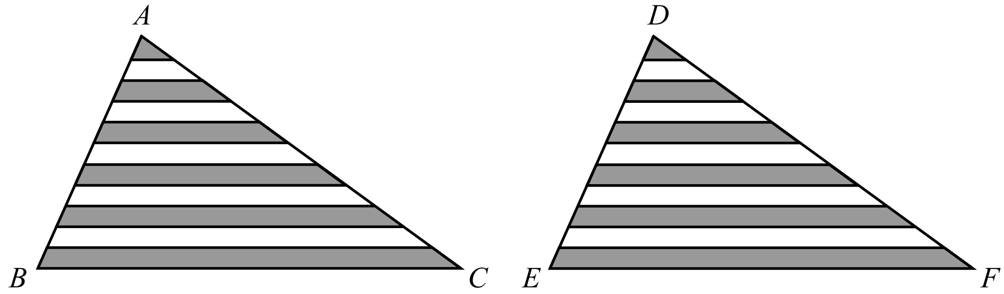
Using the sine law in $\triangle STU$, we obtain $\dfrac{ST}{\sin(\angle TUS)} = \dfrac{TU}{\sin(\angle TSU)}$ and so $\dfrac{a}{4/5} = \dfrac{b}{\sin(30^\circ)}$.
Therefore, $\dfrac{a}{4/5} = \dfrac{b}{1/2}$ and so $a = \frac{4}{5} \cdot 2b = \frac{8}{5}b$.
Substituting into the previous equation,

$$\begin{aligned}441 &= \left(\tfrac{8}{5}b\right)^2 + b^2 - \left(\tfrac{8}{5}b\right)b \\ 441 &= \tfrac{64}{25}b^2 + b^2 - \tfrac{8}{5}b^2 \\ 441 &= \tfrac{64}{25}b^2 + \tfrac{25}{25}b^2 - \tfrac{40}{25}b^2 \\ 441 &= \tfrac{49}{25}b^2 \\ 225 &= b^2\end{aligned}$$

Since $b > 0$, then $b = 15$ and so $a = \frac{8}{5}b = \frac{8}{5} \cdot 15 = 24$.

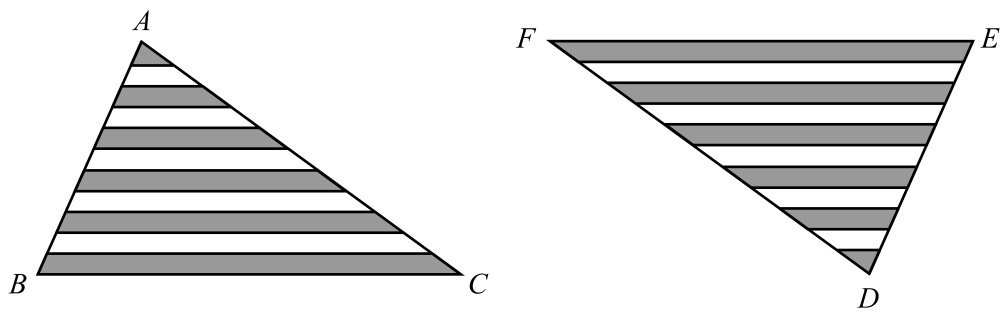
6. (a) *Solution 1*

We make two copies of the given triangle, labelling them $\triangle ABC$ and $\triangle DEF$, as shown:

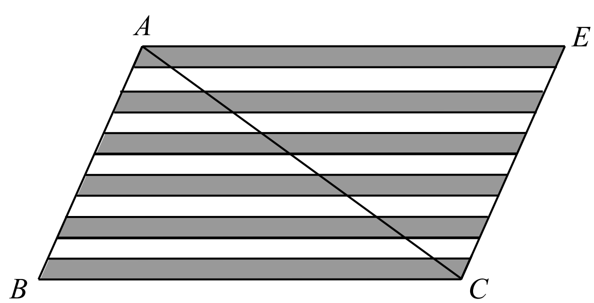


The combined area of these two triangles is $2 \cdot 770 \text{ cm}^2 = 1540 \text{ cm}^2$, and the shaded area in each triangle is the same.

Next, we rotate $\triangle DEF$ by 180°:



and join the two triangles together:



We note that BC and AE (which was FE) are equal in length (since they were copies of each other) and parallel (since they are 180° rotations of each other). The same is true for AB and EC.

Therefore, $ABCE$ is a parallelogram.

Further, $ABCE$ is divided into 11 identical parallelograms (6 shaded and 5 unshaded) by the horizontal lines. (Since the sections of the two triangles are equal in height, the horizontal lines on both sides of AC align.)

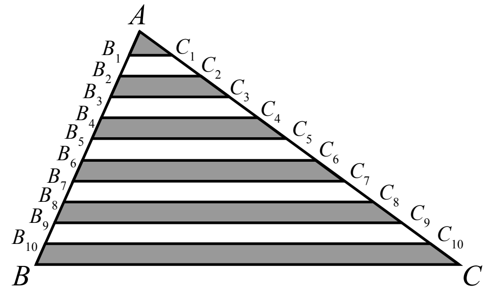
The total area of parallelogram $ABCE$ is 1540 cm^2.

Thus, the shaded area of $ABCE$ is $\frac{6}{11} \cdot 1540 \text{ cm}^2 = 840 \text{ cm}^2$.

Since this shaded area is equally divided between the two halves of the parallelogram, then the combined area of the shaded regions of $\triangle ABC$ is $\frac{1}{2} \cdot 840 \text{ cm}^2 = 420 \text{ cm}^2$.

Solution 2

We label the points where the horizontal lines touch AB and AC as shown:



We use the notation $|\triangle ABC|$ to represent the area of $\triangle ABC$ and use similar notation for the area of other triangles and quadrilaterals.

Let $\mathcal{A}$ be equal to the total area of the shaded regions.

Thus,

$$\mathcal{A} = |\triangle AB_1C_1| + |B_2B_3C_3C_2| + |B_4B_5C_5C_4| + |B_6B_7C_7C_6| + |B_8B_9C_9C_8| + |B_{10}BCC_{10}|$$

The area of each of these quadrilaterals is equal to the difference of the area of two triangles. For example,

$$|B_2B_3C_3C_2| = |\triangle AB_3C_3| - |\triangle AB_2C_2| = -|\triangle AB_2C_2| + |\triangle AB_3C_3|$$

Therefore,

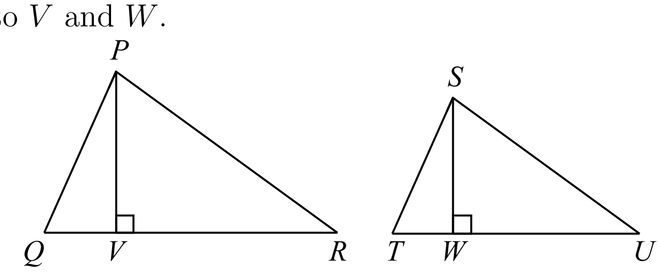
$$\begin{aligned}\mathcal{A} = {} & |\triangle AB_1C_1| - |\triangle AB_2C_2| + |\triangle AB_3C_3| - |\triangle AB_4C_4| + |\triangle AB_5C_5| \\ & - |\triangle AB_6C_6| + |\triangle AB_7C_7| - |\triangle AB_8C_8| + |\triangle AB_9C_9| - |\triangle AB_{10}C_{10}| + |\triangle ABC|\end{aligned}$$

Each of $\triangle AB_1C_1$, $\triangle AB_2C_2$, $\ldots$, $\triangle AB_{10}C_{10}$ is similar to $\triangle ABC$ because their two base angles are equal due.

Suppose that the height of $\triangle ABC$ from A to BC is h.

Since the height of each of the 11 regions is equal in height, then the height of $\triangle AB_1C_1$ is $\frac{1}{11}h$, the height of $\triangle AB_2C_2$ is $\frac{2}{11}h$, and so on.

When two triangles are similar, their heights are in the same ratio as their side lengths:

> To see this, suppose that $\triangle PQR$ is similar to $\triangle STU$ and that altitudes are drawn from P and S to V and W.
>
> 
>
>
> Since $\angle PQR = \angle STU$, then $\triangle PQV$ is similar to $\triangle STW$ (equal angle; right angle), which means that $\dfrac{PQ}{ST} = \dfrac{PV}{SW}$. In other words, the ratio of sides is equal to the ratio of heights.

Since the height of $\triangle AB_1C_1$ is $\frac{1}{11}h$, then $B_1C_1 = \frac{1}{11}BC$.

Therefore, $|\triangle AB_1C_1| = \frac{1}{2} \cdot B_1C_1 \cdot \frac{1}{11}h = \frac{1}{2} \cdot \frac{1}{11}BC \cdot \frac{1}{11}h = \frac{1^2}{11^2} \cdot \frac{1}{2} \cdot BC \cdot h = \frac{1^2}{11^2}|\triangle ABC|$.

Similarly, since the height of $\triangle AB_2C_2$ is $\frac{2}{11}h$, then $B_2C_2 = \frac{2}{11}BC$.

Therefore, $|\triangle AB_2C_2| = \frac{1}{2} \cdot B_2C_2 \cdot \frac{2}{11}h = \frac{1}{2} \cdot \frac{2}{11}BC \cdot \frac{2}{11}h = \frac{2^2}{11^2} \cdot \frac{1}{2} \cdot BC \cdot h = \frac{2^2}{11^2}|\triangle ABC|$.

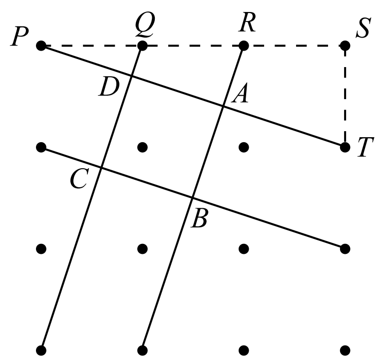
This result continues for each of the triangles.
Therefore,

$$\begin{aligned}
\mathcal{A} &= \tfrac{1^2}{11^2}|\triangle ABC| - \tfrac{2^2}{11^2}|\triangle ABC| + \tfrac{3^2}{11^2}|\triangle ABC| - \tfrac{4^2}{11^2}|\triangle ABC| + \tfrac{5^2}{11^2}|\triangle ABC| \\
&\qquad - \tfrac{6^2}{11^2}|\triangle ABC| + \tfrac{7^2}{11^2}|\triangle ABC| - \tfrac{8^2}{11^2}|\triangle ABC| + \tfrac{9^2}{11^2}|\triangle ABC| - \tfrac{10^2}{11^2}|\triangle ABC| + \tfrac{11^2}{11^2}|\triangle ABC| \\
&= \tfrac{1}{11^2}|\triangle ABC|(11^2 - 10^2 + 9^2 - 8^2 + 7^2 - 6^2 + 5^2 - 4^2 + 3^2 - 2^2 + 1) \\
&= \tfrac{1}{11^2}(770 \text{ cm}^2)((11+10)(11-10) + (9+8)(9-8) + \cdots + (3+2)(3-2) + 1) \\
&= \tfrac{1}{11^2}(770 \text{ cm}^2)(11+10+9+8+7+6+5+4+3+2+1) \\
&= \tfrac{1}{11}(70 \text{ cm}^2) \cdot 66 \\
&= 420 \text{ cm}^2
\end{aligned}$$

Therefore, the combined area of the shaded regions of $\triangle ABC$ is 420 cm^2.

(b) *Solution 1*

We label five additional points in the diagram:

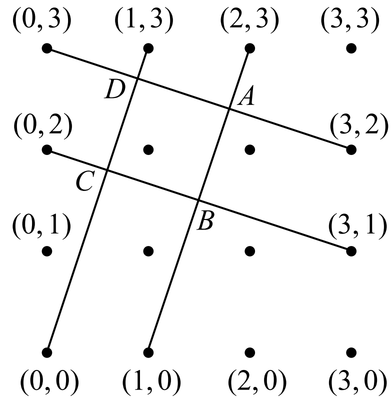


Since $PQ = QR = RS = 1$, then $PS = 3$ and $PR = 2$.
Since $\angle PST = 90^\circ$, then $PT = \sqrt{PS^2 + ST^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$ by the Pythagorean Theorem.
We are told that $ABCD$ is a square.
Thus, PT is perpendicular to QC and to RB.
Thus, $\triangle PDQ$ is right-angled at D and $\triangle PAR$ is right-angled at A.
Since $\triangle PDQ$, $\triangle PAR$ and $\triangle PST$ are all right-angled and all share an angle at P, then these three triangles are similar.
This tells us that $\dfrac{PA}{PS} = \dfrac{PR}{PT}$ and so $PA = \dfrac{3 \cdot 2}{\sqrt{10}}$. Also, $\dfrac{PD}{PS} = \dfrac{PQ}{PT}$ and so $PD = \dfrac{1 \cdot 3}{\sqrt{10}}$.
Therefore,

$$DA = PA - PD = \frac{6}{\sqrt{10}} - \frac{3}{\sqrt{10}} = \frac{3}{\sqrt{10}}$$

This means that the area of square $ABCD$ is equal to $DA^2 = \left(\dfrac{3}{\sqrt{10}}\right)^2 = \dfrac{9}{10}$.

Solution 2
We add coordinates to the diagram as shown:



We determine the side length of square $ABCD$ by determining the coordinates of D and A and then calculating the distance between these points.
The slope of the line through $(0,3)$ and $(3,2)$ is $\dfrac{3-2}{0-3} = -\dfrac{1}{3}$.
This equation of this line can be written as $y = -\dfrac{1}{3}x + 3$.
The slope of the line through $(0,0)$ and $(1,3)$ is 3.
The equation of this line can be written as $y = 3x$.
The slope of the line through $(1,0)$ and $(2,3)$ is also 3.
The equation of this line can be written as $y = 3(x-1) = 3x - 3$.
Point D is the intersection point of the lines with equations $y = -\dfrac{1}{3}x + 3$ and $y = 3x$.
Equating expressions for y, we obtain $-\dfrac{1}{3}x + 3 = 3x$ and so $\dfrac{10}{3}x = 3$ which gives $x = \dfrac{9}{10}$.
Since $y = 3x$, we get $y = \dfrac{27}{10}$ and so the coordinates of D are $\left(\dfrac{9}{10}, \dfrac{27}{10}\right)$.
Point A is the intersection point of the lines with equations $y = -\dfrac{1}{3}x + 3$ and $y = 3x - 3$.
Equating expressions for y, we obtain $-\dfrac{1}{3}x + 3 = 3x - 3$ and so $\dfrac{10}{3}x = 6$ which gives $x = \dfrac{18}{10}$.
Since $y = 3x - 3$, we get $y = \dfrac{24}{10}$ and so the coordinates of A are $\left(\dfrac{18}{10}, \dfrac{24}{10}\right)$. (It is easier to not reduce these fractions.)
Therefore,

$$DA = \sqrt{\left(\frac{9}{10} - \frac{18}{10}\right)^2 + \left(\frac{27}{10} - \frac{24}{10}\right)^2} = \sqrt{\left(-\frac{9}{10}\right)^2 + \left(\frac{3}{10}\right)^2} = \sqrt{\frac{90}{100}} = \sqrt{\frac{9}{10}}$$

This means that the area of square $ABCD$ is equal to $DA^2 = \left(\sqrt{\dfrac{9}{10}}\right)^2 = \dfrac{9}{10}$.

7. (a) Each possible order in which Akshan removes the marbles corresponds to a sequence of 9 colours, 3 of which are red and 6 of which are blue.
We write these as sequences of 3 R's and 6 B's.
Since are told that the first marble is red and the third is blue, we would like to consider all sequences of the form

$$R \_\_\_ \; B \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_$$

The 7 blanks must be filled with the remaining 2 R's and 5 B's.
There are $\binom{7}{2} = \dfrac{7 \cdot 6}{2} = 21$ ways of doing this, because 2 of the 7 blanks must be chosen in which to place the R's. (We could count these 21 ways directly by working systematically through the possible pairs of blanks.)
Of these 21 ways, some have the last two marbles being blue.
These correspond to the sequences of the form

$$R \_\_\_ \; B \_\_\_ \; \_\_\_ \; \_\_\_ \; \_\_\_ \; B \; B$$

In these sequences, the 5 blanks must be filled with the remaining 2 R's and 3 B's.
There are $\binom{5}{2} = \dfrac{5 \cdot 4}{2} = 10$ ways of doing this, because 2 of the 5 blanks must be chosen in which to place the R's.
Therefore, 10 of the 21 possible sequences end in two B's, and so the probability that the last two marbles removed are blue is $\dfrac{10}{21}$.

(b) Factoring the first equation, we obtain

$$ac + ad + bc + bd = a(c + d) + b(c + d) = (a + b)(c + d)$$

We now have the equations

$$\begin{aligned}(a + b)(c + d) &= 2023 \\ (a + b) + (c + d) &= 296\end{aligned}$$

If we let $s = a + b$ and $t = c + d$, we obtain the equations

$$\begin{aligned}st &= 2023 \\ s + t &= 296\end{aligned}$$

Noting that s and t are integers since a, b, c, and d are integers, we look for divisor pairs of 2023 whose sum is 296.
To find the divisors of 2023, we first find its prime factorization:

$$2023 = 7 \cdot 289 = 7 \cdot 17^2$$

Therefore, the divisors of 2023 are 1, 7, 17, 119, 289, 2023.
This means that the divisor pairs of 2023 are

$$2023 = 1 \cdot 2023 = 7 \cdot 289 = 17 \cdot 119$$

The one divisor pair with a sum of 296 is 7 and 289. (Alternatively, we could have found these by substituting $t = 206 - s$ into $st = 2023$ and using the quadratic formula.)

Since $a < b < c < d$, then $a + b < c + d$ and so $s = a + b = 7$ and $t = c + d = 289$.
Since a and b are positive integers with $a < b$ and $a + b = 7$, then the possible pairs (a, b) are

$$(a, b) = (1, 6), (2, 5), (3, 4)$$

We know that c and d are positive integers with $c < d$ and $c + d = 289$, but also with $b < c < d$.
When $(a, b) = (1, 6)$, this means that the possibilities are

$$(c, d) = (7, 282), (8, 281), (9, 280), \ldots, (143, 146), (144, 145)$$

There are $144 - 7 + 1 = 138$ such pairs.
When $(a, b) = (2, 5)$, the possibilities are

$$(c, d) = (6, 283), (7, 282), (8, 281), (9, 280), \ldots, (143, 146), (144, 145)$$

There are $138 + 1 = 139$ such pairs.
When $(a, b) = (3, 4)$, the possibilities are

$$(c, d) = (5, 284), (6, 283), (7, 282), (8, 281), (9, 280), \ldots, (143, 146), (144, 145)$$

There are $139 + 1 = 140$ such pairs.
In total, there are $138 + 139 + 140 = 417$ possible quadruples (a, b, c, d).

8. (a) Since $\triangle ABC$ is right-angled at B, then

$$\begin{aligned} BC^2 &= AC^2 - AB^2 \\ &= ((n+1)(n+4))^2 - (n(n+1))^2 \\ &= (n+1)^2(n+4)^2 - n^2(n+1)^2 \\ &= (n+1)^2\left((n+4)^2 - n^2\right) \\ &= (n+1)^2\left(n^2 + 8n + 16 - n^2\right) \\ &= (n+1)^2(8n+16) \\ &= 4(n+1)^2(2n+4) \end{aligned}$$

The length of BC is an integer exactly when $4(n+1)^2(2n+4)$ is a perfect square.
Since $4(n+1)^2$ is a perfect square, then BC is an integer exactly when $2n+4$ is a perfect square.
We note that $2n + 4 \geq 6$ (since $n \geq 1$) and that $2n + 4$ is even.
Since $n < 100\,000$, then $6 \leq 2n + 4 < 200\,004$, and so we need to count the number of even perfect squares between 6 and 200 004.
The smallest even perfect square in this range is $4^2 = 16$.
Since $\sqrt{200\,004} \approx 447.2$, the largest even perfect square in this range is 446^2.
Therefore, the number of even perfect squares in this range is $\dfrac{446}{2} - 1 = 222$.
Thus, there are 222 positive integers n for which the length of BC is an integer.

(b) Let $f(x) = \sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9}$.

Using logarithm laws,

$$\begin{aligned}\log_2 x \cdot \log_2(4x) + 1 &= \log_2 x(\log_2 4 + \log_2 x) + 1 \\ &= \log_2 x(2 + \log_2 x) + 1 \qquad (\text{since } 2^2 = 4) \\ &= (\log_2 x)^2 + 2 \cdot \log_2 x + 1 \\ &= (\log_2 x + 1)^2\end{aligned}$$

and

$$\begin{aligned}\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9 &= \log_2 x(\log_2 x - \log_2 64) + 9 \\ &= \log_2 x(\log_2 x - 6) + 9 \qquad (\text{since } 2^6 = 64) \\ &= (\log_2 x)^2 - 6\log_2 x + 9 \\ &= (\log_2 x - 3)^2\end{aligned}$$

Therefore,

$$f(x) = \sqrt{\log_2 x \cdot \log_2(4x) + 1} + \sqrt{\log_2 x \cdot \log_2\left(\frac{x}{64}\right) + 9} = \sqrt{(\log_2 x + 1)^2} + \sqrt{(\log_2 x - 3)^2}$$

Before proceeding, we recall that if $a \leq 0$, then $\sqrt{a^2} = -a$ and if $a > 0$, then $\sqrt{a^2} = a$.

When $\log_2 x \leq -1$, we know that $\log_2 x + 1 \leq 0$ and $\log_2 x - 3 < 0$, and so

$$f(x) = -(\log_2 x + 1) - (\log_2 x - 3) = 2 - 2\log_2 x$$

When $-1 < \log_2 x \leq 3$, we know that $\log_2 x + 1 > 0$ and $\log_2 x - 3 \leq 0$, and so

$$f(x) = (\log_2 x + 1) - (\log_2 x - 3) = 4$$

When $\log_2 x > 3$, we know that $\log_2 x + 1 \geq 0$ and $\log_2 x - 3 > 0$, and so

$$f(x) = (\log_2 x + 1) + (\log_2 x - 3) = 2\log_2 x - 2$$

We want to find all values of x for which $f(x) = 4$.

When $\log_2 x \leq -1$, $f(x) = 2 - 2\log_2 x = 4$ exactly when $\log_2 x = -1$.

When $-1 < \log_2 x \leq 3$, $f(x)$ is always equal to 4.

When $\log_2 x > 3$, $f(x) = 2\log_2 x - 2 = 4$ exactly when $\log_2 x = 3$.

Therefore, $f(x) = 4$ exactly when $-1 \leq \log_2 x \leq 3$, which is true exactly when $\frac{1}{2} \leq x \leq 8$.

(It seems surprising that the solution to this equation is actually an interval of values, rather than a finite number of specific values.)

9. (a) If there are 5 or more people seated around a table with 8 chairs, then there are at most 3 empty chairs. But there must be an empty chair between each pair of people, and this is not possible with 5 people and 3 empty chairs.
Therefore, there are at most 4 people seated.
If there were only 2 people seated, then there would be 6 empty chairs which would mean that at least one of the two "gaps" around the circular table had at least 3 empty chairs, and so another person could be seated, meaning that the table wasn't full.
Therefore, there are at least 3 people seated.
This means that a full table with 8 chairs has either 3 or 4 people.
If there are 4 people, there are 4 empty chairs, and so there is exactly 1 empty chair between each pair of people.
Thus, people are seated in chairs $\{1,3,5,7\}$ or in chairs $\{2,4,6,8\}$.
If there are 3 people, there are 5 empty chairs.
With 3 people, there are 3 gaps totalling 5 chairs, and each gap has at most 2 chairs in it.
Therefore, the gaps must be 1, 2, 2 in some order. This is the only list of three positive integers, each equal to 1 or 2, that adds to 5.
The gap of 1 can be between any pair of seats. In other words, the gap of 1 could be between $\{1,3\}$, $\{2,4\}$, and so on. In each case, the position of the third person is completely determined because the remaining two gaps have 2 chairs each.
Thus, with 3 people, they are seated in chairs

$$\{1,3,6\},\{2,4,7\},\{3,5,8\},\{4,6,1\},\{5,7,2\},\{6,8,3\},\{7,1,4\},\{8,2,5\}$$

In total, there are thus 10 ways to seat people at a table with 8 chairs:

$$\{1,3,5,7\},\{2,4,6,8\},\{1,3,6\},\{2,4,7\},\{3,5,8\},\{4,6,1\},\{5,7,2\},\{6,8,3\},\{7,1,4\},\{8,2,5\}$$

(b) Suppose that k is a positive integer.
Suppose that t people are seated at a table with $6k+5$ chairs so that the table is full.
When t people are seated, there are t gaps. Each gap consists of either 1 or 2 chairs. (A gap with 3 or more chairs can have an additional person seated in it, so the table is not full.)
Therefore, there are between t and $2t$ empty chairs.
This means that the total number of chairs is between $t+t$ and $t+2t$.
In other words, $2t \leq 6k+5 \leq 3t$.
Since $2t \leq 6k+5$, then $t \leq 3k+\frac{5}{2}$. Since k and t are integers, then $t \leq 3k+2$.
We note that it is possible to seat $3k+2$ people around the table in seats

$$\{2,4,6,\ldots,6k+2,6k+4\}$$

This table is full becase $3k+1$ of the gaps consist of 1 chair and 1 gap consists of 2 chairs.
Since $3t \geq 6k+5$, then $t \geq 2k+\frac{5}{3}$. Since k and t are integers, then $t \geq 2k+2$.
We note that it is possible to seat $2k+2$ people around the table in seats

$$\{3,6,9,\ldots,6k,6k+3,6k+5\}$$

This table is full becase $2k+1$ of the gaps consist of 2 chairs and 1 gap consists of 1 chair.

We now know that, if there are t people seated at a full table with $6k+5$ chairs, then $2k+2 \leq t \leq 3k+2$.
To confirm that every such value of t is possible, consider a table with t people, $3t-(6k+5)$

gaps of 1 chair, and $(6k+5)-2t$ gaps of 2 chairs.
From the work above, we know that $3t \geq 6k+5$ and so $3t-(6k+5) \geq 0$, and that $2t \leq 6k+5$ and so $(6k+5)-2t \geq 0$.
The total number of gaps is $3t-(6k+5)+(6k+5)-2t = t$, since there are t people seated.
Finally, the total number of chairs is

$$t+1\cdot(3t-(6k+5))+2\cdot((6k+5)-2t) = t+3t-4t-(6k+5)+2(6k+5) = 6k+5$$

as expected.
This shows that every t with $2k+2 \leq t \leq 3k+2$ can produce a full table.
Therefore, the possible values of t are those integers that satisfy $2k+2 \leq t \leq 3k+2$.
There are $(3k+2)-(2k+2)+1 = k+1$ possible values of t.

(c) *Solution 1*

For each integer $n \geq 3$, we define $f(n)$ to be the number of different full tables of size n.
We can check that

- $f(3) = 3$ because the full tables when $n=3$ have people in chairs $\{1\}$, $\{2\}$, $\{3\}$,
- $f(4) = 2$ because the full tables when $n=4$ have people in chairs $\{1,3\}$, $\{2,4\}$, and
- $f(5) = 5$ because the full tables when $n=4$ have people in chairs $\{1,3\}$, $\{2,4\}$, $\{3,5\}$, $\{4,1\}$, $\{5,2\}$.

In the problem, we are told that $f(6) = 5$ and in part (a), we determined that $f(8) = 10$.
This gives us the following table:

n	$f(n)$
3	3
4	2
5	5
6	5
7	?
8	10

Based on this information, we make the guess that for every integer $n \geq 6$, we have $f(n) = f(n-2)+f(n-3)$.
For example, this would mean that $f(7) = f(5)+f(4) = 5+2 = 7$ which we can verify is true.
Based on this recurrence relation (which we have yet to prove), we deduce the values of $f(n)$ up to and including $n = 19$:

n	$f(n)$
3	3
4	2
5	5
6	5
7	7
8	10
9	12
10	17

n	$f(n)$
11	22
12	29
13	39
14	51
15	68
16	90
17	119
18	158
19	209

We now need to prove that the equation $f(n) = f(n-2)+f(n-3)$ is true for all $n \geq 6$.

We think about each full table as a string of 0s and 1s, with 1 representing a chair that is occupied and 0 representing an empty chair.
Let $a(n)$ be the number of full tables with someone in seat 1 (and thus nobody in seat 2).
Let $b(n)$ be the number of full tables with someone in seat 2 (and thus nobody in seat 1).
Let $c(n)$ be the number of full tables with nobody in seat 1 or in seat 2.
Since every full table must be in one of these categories, then $f(n) = a(n) + b(n) + c(n)$.
A full table with n seats $n \geq 4$ must correspond to a string that starts with 10, 01 or 00.
Since there cannot be more than two consecutive 0s, we can further specify this, namely to say that a full table with n seats must correspond to a string that starts with 1010 or 1001 or 0100 or 0101 or 0010. In each case, these are the first 4 characters of the string and correspond to full (1) and empty (0) chairs.

Consider the full tables starting with 1010. Note that such strings end with 0 since the table is circular. Removing the 10 from positions 1 and 2 creates strings of length $n-2$ that begin 10. These strings will still correspond to a full table, and so there are $a(n-2)$ such strings. (We note that all possible strings starting 1010 of length n will lead to all possible strings starting with 1010 of length $n-2$.)
Consider the full tables starting with 1001. Note that such a string ends with 0 since the table is circular. Removing the 100 from positions 1, 2 and 3 creates strings of length $n-3$ that begin 10. (There must have been a 0 in position 5 after the 1 in position 4.) These strings will still correspond to full tables, and so there are $a(n-3)$ such strings.
Consider the full tables starting with 0100. Removing the 100 from positions 2, 3 and 4 creates strings of length $n-3$ that begin 01. (There must have been a 1 in position 5 after the 0 in position 4.) These strings will still correspond to full tables, and so there are $b(n-3)$ such strings.
Consider the full tables starting with 0101. Removing the 01 from positions 3 and 4 creates strings of length $n-2$ that begin 01. (The 1 in position 4 must have been followed by one or two 0s and so these strings maintains the desired properties.) These strings will still correspond to full tables, and so there are $b(n-2)$ such strings.
Consider the full tables starting with 0010. These strings must begin with either 00100 or 00101.
If strings start 00100, then they start 001001 and so we remove the 001 in positions 4, 5 and 6 and obtain strings of length $n-3$ that start 001 (and thus start 00). There are $c(n-3)$ such strings.
If strings start 00101, we remove the 01 in positions 4 and 5 and obtain strings of length $n-2$ that start 001 (and thus start 00). There are $c(n-2)$ such strings.
These 6 cases and subcases count all strings counted by $f(n)$.
Therefore,

$$\begin{aligned} f(n) &= a(n-2) + a(n-3) + b(n-3) + b(n-2) + c(n-3) + c(n-2) \\ &= a(n-2) + b(n-2) + c(n-2) + a(n-3) + b(n-3) + c(n-3) \\ &= f(n-2) + f(n-3) \end{aligned}$$

as required, which means that the number of different full tables when $n = 19$ is 209.

Solution 2
Extending our approach from (b), the number of people seated at a full table with 19 chairs is at least $\frac{19}{3} = 6\frac{1}{3}$ and at most $\frac{19}{2} = 9\frac{1}{2}$.
Since the number of people is an integer, there must be 7, 8 or 9 people at the table, which means that the number of empty chairs is 12, 11 or 10, respectively.

Suppose that there are 9 people and 9 gaps with a total of 10 empty chairs.
In this case, there is 1 gap with 2 empty chairs and 8 gaps with 1 empty chair.
There are 19 pairs of chairs in which we can put 2 people with a gap of 2 in between: $\{1,4\}$, $\{2,5\}$, ..., $\{19,3\}$.
Once we choose one of these pairs, the seat choice for the remaining 8 people is completely determined by placing people in every other chair.
Therefore, there are 19 different full tables with 9 people.

Suppose that there are 8 people and 8 gaps with a total of 11 empty chairs.
In this case, there are 3 gaps with 2 empty chairs and 5 gaps with 1 empty chair.
There are 7 different circular orderings in which these 8 gaps can be arranged:

22211111 22121111 22112111 22111211 22111121 21212111 21211211

We note that "22211111" would be the same as, for example, "11222111" since these gaps are arranged around a circle.
If the three gaps of length 2 are consecutive, there is only one configuration (22211111).
If there are exactly 2 consecutive gaps of length 2, there are 4 relative places in which the third gap of length 2 can be placed.
If there are no consecutive gaps of length 2, these gaps can either be separated by 1 gap each (21212111) with 3 gaps on the far side, or can be separated by 1 gap, 2 gaps, and 2 gaps (21211211). There is only one configuration for the gaps in this last situation.
There are 7 different circular orderings for these 8 gaps.
Each of these 7 different orderings can be placed around the circle of 19 chairs in 19 different ways, because each can be started in 19 different places. Because 19 is prime, none of these orderings overlap.
Therefore, there are $7 \cdot 19 = 133$ different full tables with 8 people.

Suppose that there are 7 people and 7 gaps with a total of 12 empty chairs.
In this case, there are 2 gaps with 1 empty chair and 5 gaps with 2 empty chairs.
The 2 gaps with 1 empty chair can be separated by 0 gaps with 2 empty chairs, 1 gap with 2 empty chairs, or 2 gaps with 2 empty chairs. Because the chairs are around a circle, if there were 3, 4 or 5 gaps with 2 empty chairs between them, there would be 2, 1 or 0 gaps going the other way around the circle.
This means that there are 3 different configurations for the gaps.
Each of these configurations can be placed in 19 different ways around the circle of chairs.
Therefore, there are $3 \cdot 19 = 57$ full tables with 7 people.

In total, there are $19 + 133 + 57 = 209$ full tables with 19 chairs.

Solution 3
As in Solution 2, there must be 7, 8 or 9 people in chairs, and so there are 7, 8 or 9 gaps.
If there are 7 gaps, there are 2 gaps of 1 chair and 5 gaps of 2 chairs.
If there are 8 gaps, there are 5 gaps of 1 chair and 3 gaps of 2 chairs.
If there are 9 gaps, there are 8 gaps of 1 chair and 1 gap of 2 chairs.
We consider three mutually exclusive cases: (i) there is a person in chair 1 and not in chair 2, (ii) there is a person in chair 2 and not in chair 1, and (iii) there is nobody in chair 1 or in chair 2. Every full table fits into exactly one of these three cases.

<u>Case (i): there is a person in chair 1 and not in chair 2</u>
We use the person in chair 1 to "anchor" the arrangement, by starting at chair 1 and

arranging the gaps (and thus the full chairs) clockwise around the table from chair 1.

If there are 7 gaps, we need to choose 2 of them to be of length 1, and so there are $\binom{7}{2}$ ways of arranging the gaps starting at chair 1.

If there are 8 gaps, we need to choose 3 of them to be of length 2, and so there are $\binom{8}{3}$ ways of arranging the gaps starting at chair 1.

If there are 9 gaps, we need to choose 1 of them to be of length 2, and so there are $\binom{9}{1}$ ways of arranging the gaps starting at chair 1.

In this case, there are a total of $\binom{7}{2}+\binom{8}{3}+\binom{9}{1} = 21+56+9 = 86$ full tables.

Case (ii): there is a person in chair 2 and not in chair 1

We use the same reasoning starting with the person in chair 2 as the anchor.
Again, there are 86 full tables in this case.

Case (iii): there is nobody in chair 1 or chair 2

Since there is nobody in chair 1 or chair 2, there must be a person in chair 3 and also in chair 19, which fixes one gap of 2 chairs.
Here, we use the person in chair 3 as the anchor.

If there are 7 gaps, there are 2 gaps of 1 chair and 4 gaps of 2 chairs left to place. There are $\binom{6}{2}$ ways of doing this.

If there are 8 gaps, there are 5 gaps of 1 chair and 2 gaps of 2 chairs left to place. There are $\binom{7}{2}$ ways of doing this.

If there are 9 gaps, there are 8 gaps of 1 chair and 0 gaps of 2 chairs left to place. There is 1 way to do this.

In this case, there are a total of $\binom{6}{2}+\binom{7}{2}+1 = 15+21+1 = 37$ full tables.

In total, there are $86+86+37 = 209$ full tables with 19 chairs.

10. (a) Since $0 < \frac{1}{3} < \frac{2}{3} < 1$, then $\left\lfloor \frac{1}{3} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor = 0$.

Since $1 \leq \frac{3}{3} < \frac{4}{3} < \frac{5}{3} < 2$, then $\left\lfloor \frac{3}{3} \right\rfloor = \left\lfloor \frac{4}{3} \right\rfloor = \left\lfloor \frac{5}{3} \right\rfloor = 1$.

These fractions can continue to be grouped in groups of 3 with the last full group of 3 satisfying $19 \leq \frac{57}{3} < \frac{58}{3} < \frac{59}{3} < 20$, which means that $\left\lfloor \frac{57}{3} \right\rfloor = \left\lfloor \frac{58}{3} \right\rfloor = \left\lfloor \frac{59}{3} \right\rfloor = 19$.

The last term is $\left\lfloor \frac{60}{3} \right\rfloor = \lfloor 20 \rfloor = 20$.

If the given sum is S, we obtain

$$\begin{aligned} S &= 2\cdot 0 + 3\cdot 1 + 3\cdot 2 + \cdots + 3\cdot 19 + 1\cdot 20 \\ &= 0 + 3(1 + 2 + \cdot + 19) + 20 \\ &= 3\cdot \tfrac{1}{2}\cdot 19\cdot 20 + 20 \\ &= 570 + 20 \\ &= 590 \end{aligned}$$

(b) For every positive integer $m > 4$, let

$$q(m) = \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{m-2}{3} \right\rfloor + \left\lfloor \frac{m-1}{3} \right\rfloor$$

Extending our work from (a), we know that $k - 1 \le \dfrac{3k-3}{3} < \dfrac{3k-2}{3} < \dfrac{3k-1}{3} < k$ for each positive integer k, and so $\left\lfloor \dfrac{3k-3}{3} \right\rfloor = \left\lfloor \dfrac{3k-2}{3} \right\rfloor = \left\lfloor \dfrac{3k-1}{3} \right\rfloor = k-1$.

Every positive integer $m > 4$ can be written as $m = 3s$ or $m = 3s+1$ or $m = 3s+2$, for some positive integer s, depending on its remainder when divided by 3.

We can thus write

$$\begin{aligned} q(3s) &= \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{3s-2}{3} \right\rfloor + \left\lfloor \frac{3s-1}{3} \right\rfloor \\ &= 2\cdot 0 + 3(1 + 2 + 3 + \cdots + (s-1)) \\ &= 3\cdot \frac{1}{2}\cdot (s-1)s \\ &= \frac{3s(s-1)}{2} \\ &= \frac{3s(3s-3)}{6} \end{aligned}$$

$$\begin{aligned} q(3s+1) &= \left\lfloor \frac{1}{3} \right\rfloor + \left\lfloor \frac{2}{3} \right\rfloor + \left\lfloor \frac{3}{3} \right\rfloor + \ldots + \left\lfloor \frac{3s-2}{3} \right\rfloor + \left\lfloor \frac{3s-1}{3} \right\rfloor + \left\lfloor \frac{3s}{3} \right\rfloor \\ &= q(3s) + s \\ &= \frac{3s(3s-3)}{6} + \frac{3s\cdot 2}{6} \\ &= \frac{3s(3s-1)}{6} \end{aligned}$$

$$\begin{aligned} q(3s+2) &= q(3s+1) + \left\lfloor \frac{3s+1}{3} \right\rfloor \\ &= \frac{3s(3s-1)}{6} + s \\ &= \frac{3s(3s-1)}{6} + \frac{3s\cdot 2}{6} \\ &= \frac{3s(3s+1)}{6} \end{aligned}$$

We want to find a polynomial $p(x)$ for which $q(m) = \lfloor p(m) \rfloor$ for every positive integer $m > 4$.

In other words, we want to find a polynomial $p(x)$ for which

$$\lfloor p(3s) \rfloor = \frac{3s(3s-3)}{6} \qquad \lfloor p(3s+1) \rfloor = \frac{3s(3s-1)}{6} \qquad \lfloor p(3s+2) \rfloor = \frac{3s(3s+1)}{6}$$

for every positive integer s.

We will show that the polynomial $p(x) = \dfrac{(x-1)(x-2)}{6}$ satisfies the desired conditions.

If $x = 3s+1$ for some positive integer s, then

$$\frac{(x-1)(x-2)}{6} = \frac{(3s+1-1)(3s+1-2)}{6} = \frac{3s(3s-1)}{6}$$

We note that $3s$ is a multiple of 3. Since $3s$ and $3s-1$ are consecutive integers, then one of these is a multiple of 2. Thus, $3s(3s-1)$ is a multiple of 6 and so $\dfrac{3s(3s-1)}{6}$ is an integer.

This means that $\left\lfloor \dfrac{3s(3s-1)}{6} \right\rfloor = \dfrac{3s(3s-1)}{6}$.

Therefore, $q(3s+1) = \dfrac{3s(3s-1)}{6} = \left\lfloor \dfrac{3s(3s-1)}{6} \right\rfloor = \lfloor p(3s+1) \rfloor$.

If $x = 3s+2$ for some positive integer s, then

$$\frac{(x-1)(x-2)}{6} = \frac{(3s+2-1)(3s+2-2)}{6} = \frac{3s(3s+1)}{6}$$

We note that $3s$ is a multiple of 3. Since $3s$ and $3s+1$ are consecutive integers, then one of these is a multiple of 2. Thus, $3s(3s+1)$ is a multiple of 6 and so $\dfrac{3s(3s+1)}{6}$ is an integer.

This means that $\left\lfloor \dfrac{3s(3s+1)}{6} \right\rfloor = \dfrac{3s(3s+1)}{6}$.

Therefore, $q(3s+2) = \dfrac{3s(3s+1)}{6} = \left\lfloor \dfrac{3s(3s+1)}{6} \right\rfloor = \lfloor p(3s+2) \rfloor$.

If $x = 3s$ for some positive integer s, then

$$\frac{(x-1)(x-2)}{6} = \frac{(3s-1)(3s-2)}{6} = \frac{9s^2-9s+2}{6}$$

Now, $\dfrac{9s^2-9s}{6} = \dfrac{9s(s-1)}{6}$ is an integer because $9s$ is a multiple of 3 and one of s and $s-1$ is even.

Since $\dfrac{9s^2-9s+2}{6} = \dfrac{9s^2-9s}{6} + \dfrac{1}{3}$, then $\dfrac{9s^2-9s+2}{6}$ is $\dfrac{1}{3}$ more than an integer which means that $\left\lfloor \dfrac{9s^2-9s+2}{6} \right\rfloor = \dfrac{9s^2-9s}{6} = \dfrac{3s(3s-3)}{6} = q(3s)$.

Therefore, $q(3s) = \dfrac{3s(3s-3)}{6} = \left\lfloor \dfrac{(3s-1)(3s-2)}{6} \right\rfloor = \lfloor p(3s) \rfloor$.

This means that the polynomial $p(x) = \dfrac{(x-1)(x-2)}{6}$ satisfies the required conditions.

(c) Before working on the specific question we have been asked, we simplify the given expression for $f(n)$ by noting that if $k \geq n$, then $kn \leq k^2 < k^2 + 1$.
This means that if $k \geq n$, we have $0 < \dfrac{kn}{n^2+1} < 1$ and so $\left\lfloor \dfrac{kn}{k^2+1} \right\rfloor = 0$.
This means that, for a fixed positive integer n, the apparently infinite sum that represents $f(n)$ can be stopped when $k = n-1$ because every subsequent term equals 0.
Thus,

$$f(n) = \left\lfloor \frac{n}{1^2+1} \right\rfloor + \left\lfloor \frac{2n}{2^2+1} \right\rfloor + \left\lfloor \frac{3n}{3^2+1} \right\rfloor + \cdots + \left\lfloor \frac{(n-2)n}{(n-2)^2+1} \right\rfloor + \left\lfloor \frac{(n-1)n}{(n-1)^2+1} \right\rfloor$$

We note that

$$\begin{aligned}
f(1) &= 0 \qquad \text{(since no terms are non-zero)} \\
f(2) &= \left\lfloor \frac{1\cdot 2}{1^2+1} \right\rfloor = 1 \\
f(3) &= \left\lfloor \frac{1\cdot 3}{1^2+1} \right\rfloor + \left\lfloor \frac{2\cdot 3}{2^2+1} \right\rfloor = \left\lfloor \frac{3}{2} \right\rfloor + \left\lfloor \frac{6}{5} \right\rfloor = 1+1 = 2 \\
f(4) &= \left\lfloor \frac{1\cdot 4}{1^2+1} \right\rfloor + \left\lfloor \frac{2\cdot 4}{2^2+1} \right\rfloor + \left\lfloor \frac{3\cdot 4}{3^2+1} \right\rfloor = \left\lfloor \frac{4}{2} \right\rfloor + \left\lfloor \frac{8}{5} \right\rfloor + \left\lfloor \frac{12}{10} \right\rfloor = 2+1+1 = 4
\end{aligned}$$

Suppose that t is an odd positive integer for which $f(t+1) - f(t) = 2$.
We will assume that t is not a prime number, and show that $f(t+1) - f(t) \neq 2$. This will show us that if $f(t+1) - f(t) = 2$, it must be the case that t is prime. Since t is odd and not prime, then $t = 1$ or t is composite.
We note that when $t = 1$, we obtain $f(2) - f(1) = 1 - 0 = 1 \neq 2$.
Next, suppose that t is odd and composite.
Since t is odd and composite, then t can be written as $t = rs$ for some odd positive integers $r \geq s > 1$. (t can be written in this form in at least one way, so we take one of these possibilities.)
In this case, consider $f(t+1) - f(t)$.
We can write this as

$$\begin{aligned}
f(t+1) - f(t) \quad = \quad & \left\lfloor \frac{t+1}{1^2+1} \right\rfloor + \left\lfloor \frac{2(t+1)}{2^2+1} \right\rfloor + \cdots + \left\lfloor \frac{(t-1)(t+1)}{(t-1)^2+1} \right\rfloor + \left\lfloor \frac{t(t+1)}{t^2+1} \right\rfloor \\
& - \left\lfloor \frac{t}{1^2+1} \right\rfloor - \left\lfloor \frac{2t}{2^2+1} \right\rfloor - \cdots - \left\lfloor \frac{(t-1)t}{(t-1)^2+1} \right\rfloor
\end{aligned}$$

We re-write this as

$$\begin{aligned}
f(t+1) - f(t) = & \left(\left\lfloor \frac{t+1}{1^2+1} \right\rfloor - \left\lfloor \frac{t}{1^2+1} \right\rfloor \right) + \left(\left\lfloor \frac{2(t+1)}{2^2+1} \right\rfloor - \left\lfloor \frac{2t}{2^2+1} \right\rfloor \right) + \cdots \\
& + \left(\left\lfloor \frac{(t-1)(t+1)}{(t-1)^2+1} \right\rfloor - \left\lfloor \frac{(t-1)t}{(t-1)^2+1} \right\rfloor \right) + \left\lfloor \frac{t(t+1)}{t^2+1} \right\rfloor
\end{aligned}$$

In the $t-1$ sets of parentheses, we have terms of the form $\left\lfloor \dfrac{k(t+1)}{k^2+1} \right\rfloor - \left\lfloor \dfrac{kt}{k^2+1} \right\rfloor$ for each integer k from 1 to $t-1$.
We know that $\dfrac{k(t+1)}{k^2+1} > \dfrac{kt}{k^2+1}$ because both k and t are positive, the denominators are

equal and $k(t+1) > kt$.

Thus, $\left\lfloor \dfrac{k(t+1)}{k^2+1} \right\rfloor \geq \left\lfloor \dfrac{kt}{k^2+1} \right\rfloor$. (The greatest integer less than or equal to the first fraction must be at least as large as the greatest integer less than or equal to the second fraction.)

This means that the $t-1$ differences in parentheses, each of which is an integer, is at least 0.

To show that $f(t+1)-f(t) \neq 2$, we show that there are at least 2 places where the difference is at least 1, and that the final term is at least 1. This will tell us that $f(t+1)-f(t) \geq 3$ and so $f(t+1)-f(t) \neq 2$, which will tell us that t cannot be composite, and so t must be prime, as required.

Consider $\left\lfloor \dfrac{t(t+1)}{t^2+1} \right\rfloor$.

Since $t(t+1) = t^2+t \geq t^2+1$, then $\dfrac{t(t+1)}{t^2+1} \geq 1$, which means that $\left\lfloor \dfrac{t(t+1)}{t^2+1} \right\rfloor \geq 1$.

Consider $\left\lfloor \dfrac{t+1}{1^2+1} \right\rfloor - \left\lfloor \dfrac{t}{1^2+1} \right\rfloor = \left\lfloor \dfrac{t+1}{2} \right\rfloor - \left\lfloor \dfrac{t}{2} \right\rfloor$.

Since t is odd, then we write $t = 2u+1$ for some positive integer u, which gives

$$\left\lfloor \frac{t+1}{2} \right\rfloor - \left\lfloor \frac{t}{2} \right\rfloor = \left\lfloor \frac{2u+2}{2} \right\rfloor - \left\lfloor \frac{2u+1}{2} \right\rfloor = \lfloor u+1 \rfloor - \left\lfloor u+\frac{1}{2} \right\rfloor = (u+1)-u = 1$$

Recall that $t = rs$ with $r \geq s > 1$.

Consider the term $\left\lfloor \dfrac{r(t+1)}{r^2+1} \right\rfloor - \left\lfloor \dfrac{rt}{r^2+1} \right\rfloor$.

We have

$$\left\lfloor \frac{r(t+1)}{r^2+1} \right\rfloor - \left\lfloor \frac{rt}{r^2+1} \right\rfloor = \left\lfloor \frac{r(rs+1)}{r^2+1} \right\rfloor - \left\lfloor \frac{r \cdot rs}{r^2+1} \right\rfloor = \left\lfloor \frac{r^2s+r}{r^2+1} \right\rfloor - \left\lfloor \frac{r^2s}{r^2+1} \right\rfloor$$

We note that $\dfrac{r^2s+r}{r^2+1} \geq \dfrac{r^2s+s}{r^2+1} = s$ and $\dfrac{r^2s}{r^2+1} < \dfrac{r^2s+s}{r^2+1} = s$.

Thus, $\left\lfloor \dfrac{r^2s+r}{r^2+1} \right\rfloor \geq s$.

Also, $\left\lfloor \dfrac{r^2s}{r^2+1} \right\rfloor < s$ which means $\left\lfloor \dfrac{r^2s}{r^2+1} \right\rfloor \leq s-1$ and so $\left\lfloor \dfrac{r(t+1)}{r^2+1} \right\rfloor - \left\lfloor \dfrac{rt}{r^2+1} \right\rfloor \geq 1$.

Therefore, if t is odd and not prime, then $f(t+1)-f(t) \neq 2$ because we have found three terms that are equal to at least 1 meaning that $f(t+1)-f(t) \geq 3$, and so if $f(t+1)-f(t)$, then t must be prime.

Here is an alternative approach so show that $f(t+1)-f(t) \geq 3$ when t is odd and composite.

As above, we look for at least 3 integers k for which $\left\lfloor \dfrac{k(t+1)}{k^2+1} \right\rfloor - \left\lfloor \dfrac{kt}{k^2+1} \right\rfloor \geq 1$. Here, we allow for the possibility that $k = t$ knowing that the second term in this difference will be 0 in this case.

The positive integer k has the property that $\left\lfloor \dfrac{k(t+1)}{k^2+1} \right\rfloor - \left\lfloor \dfrac{kt}{k^2+1} \right\rfloor \geq 1$ exactly when there is an integer N for which $\dfrac{k(t+1)}{k^2+1} \geq N > \dfrac{kt}{k^2+1}$.

This pair of inequalities is equivalent to the pair of inequalities $t+1 \geq N \cdot \dfrac{k^2+1}{k} > t$ which is in turn equivalent to $t+1 \geq Nk + \dfrac{N}{k} > t$.

The following three pairs (N,k) of integers satisfy this equation:

- $k=1$ and $N = \dfrac{t+1}{2}$ (noting that t is odd), which give $Nk + \dfrac{N}{k} = t+1$;
- $k=r$ and $N=s$, which give $Nk + \dfrac{N}{k} = rs + \dfrac{s}{r}$ (noting that $\dfrac{s}{r} < 1$);
- $k=t$ and $N=1$, which give $Nk + \dfrac{N}{k} = t + \dfrac{1}{t}$.

This shows that $f(t+1) - f(t) \geq 3$ when t is odd and composite, as required.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2022 Euclid Contest

Tuesday, April 5, 2022
(in North America and South America)

Wednesday, April 6, 2022
(outside of North America and South America)

Solutions

1. (a) Evaluating, $\dfrac{3^2-2^3}{2^3-3^2}=\dfrac{9-8}{8-9}=\dfrac{1}{-1}=-1$.

Alternatively, since $2^3-3^2=-(3^2-2^3)$, then $\dfrac{3^2-2^3}{2^3-3^2}=-1$.

(b) Evaluating, $\sqrt{\sqrt{81}+\sqrt{9}-\sqrt{64}}=\sqrt{9+3-8}=\sqrt{4}=2$.

(c) Since $\dfrac{1}{\sqrt{x^2+7}}=\dfrac{1}{4}$, then $\sqrt{x^2+7}=4$.
This means that $x^2+7=4^2=16$ and so $x^2=9$.
Since $x^2=9$, then $x=\pm 3$.
We can check by substitution that both of these values are solutions.

2. (a) Factoring, $2022=2\cdot 1011=2\cdot 3\cdot 337$. (It turns out that 337 is a prime number, though this fact is not needed here.)
Therefore, $2022=2\cdot 1011$ and $2022=3\cdot 674$ and $2022=6\cdot 337$.
Thus, the three ordered pairs are $(a,b)=(2,1011),(3,674),(6,337)$.

(b) Manipulating algebraically, the following equations are equivalent:

$$\begin{aligned}\frac{2c+1}{2d+1}&=\frac{1}{17}\\ 17(2c+1)&=2d+1\\ 34c+17&=2d+1\\ 34c+16&=2d\\ d&=17c+8\end{aligned}$$

Since c is an integer with $c>0$, then $c\geq 1$, which means that $17c+8\geq 25$.
Therefore, the smallest possible value of d is $d=25$.
Note that, when $d=25$, we obtain $c=1$ and so $\dfrac{2c+1}{2d+1}=\dfrac{3}{51}=\dfrac{1}{17}$.

(c) *Solution 1*
When $x=-5$, the left side of the equation equals 0.
This means that when $x=-5$, the right side of the equation must equal 0 as well.
Thus, $(-5)^2+3(-5)+t=0$ and so $25-15+t=0$ or $t=-10$.

Solution 2
Expanding the left side, we obtain

$$(px+r)(x+5)=px^2+rx+5px+5r$$

Since this is equal to x^2+3x+t for all real numbers, then the coefficients of the two quadratic expressions must be the same.
Comparing coefficients of x^2, we obtain $p=1$.
This means that

$$x^2+rx+5x+5r=x^2+3x+t$$

Comparing coefficients of x, we obtain $r+5=3$ and so $r=-2$.
This means that

$$x^2+3x-10=x^2+3x+t$$

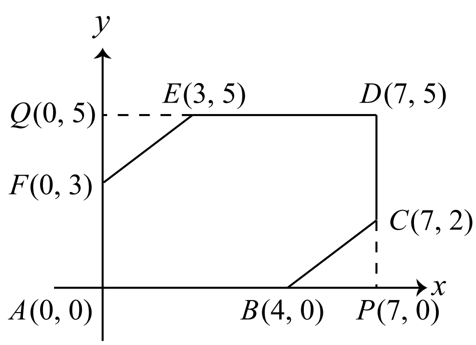
Comparing constant terms, we obtain $t=-10$.

3. (a) Suppose that the volume of the jug is V L.
Then $\frac{1}{4}V + 24 = \frac{5}{8}V$.
Multiplying by 8, we obtain $2V + 24 \cdot 8 = 5V$ which gives $3V = 192$ and so $V = 64$.
Therefore, the volume of the jug is 64 L.

(b) Suppose that Stephanie starts with n soccer balls.
Since Stephanie can divide the n balls into fifths and into elevenths, then n is a multiple of both 5 and 11.
Since 5 and 11 are both prime numbers, then n must be a multiple of $5 \cdot 11 = 55$.
Thus, $n = 55k$ for some positive integer k.
In this case, $\frac{2}{5}n = \frac{2}{5} \cdot 55k = 22k$ and $\frac{6}{11}n = \frac{6}{11} \cdot 55k = 30k$.
When Stephanie has given these balls away, she is left with $55k - 22k - 30k = 3k$ balls.
Since $3k$ is a multiple of 9, then k is a multiple of 3.
Therefore, the smallest possible number of balls is obtained when $k = 3$, which means that Stephanie started with $n = 55 \cdot 3 = 165$ soccer balls.

(c) Suppose that the number of students in the Junior section is j and the number of students in the Senior section is s.
The number of left-handed Junior students is 60% of j, or $0.6j$.
The number of right-handed Junior students is 40% of j, or $0.4j$.
The number of left-handed Senior students is 10% of s, or $0.1s$.
The number of right-handed Senior students is 90% of s, or $0.9s$.
Since the total numbers of left-handed and right-students are equal, we obtain the equation $0.6j + 0.1s = 0.4j + 0.9s$ which gives $0.2j = 0.8s$ or $j = 4s$.
This means that there are 4 times as many Junior students as Senior students, which means that $\frac{4}{5}$ of the students are Junior and $\frac{1}{5}$ are Senior.
Therefore, 80% of the students in the math club are in the Junior section.

4. (a) Let P be the point with coordinates $(7, 0)$ and let Q be the point with coordinates $(0, 5)$.



Then $APDQ$ is a rectangle with width 7 and height 5, and so it has area $7 \cdot 5 = 35$.
Hexagon $ABCDEF$ is formed by removing two triangles from rectangle $APDQ$, namely $\triangle BPC$ and $\triangle EQF$.
Each of $\triangle BPC$ and $\triangle EQF$ is right-angled, because each shares an angle with rectangle $APDQ$.
Each of $\triangle BPC$ and $\triangle EQF$ has a base of length 3 and a height of 2.
Thus, their combined area is $2 \cdot \frac{1}{2} \cdot 3 \cdot 2 = 6$.
This means that the area of hexagon $ABCDEF$ is $35 - 6 = 29$.

(b) Since $\triangle PQS$ is right-angled at P, then by the Pythagorean Theorem,

$$SQ^2 = SP^2 + PQ^2 = (x+3)^2 + x^2$$

Since $\triangle QRS$ is right-angled at Q, then by the Pythagorean Theorem, we obtain

$$\begin{aligned} RS^2 &= SQ^2 + QR^2 \\ (x+8)^2 &= ((x+3)^2 + x^2) + 8^2 \\ x^2 + 16x + 64 &= x^2 + 6x + 9 + x^2 + 64 \\ 0 &= x^2 - 10x + 9 \\ 0 &= (x-1)(x-9) \end{aligned}$$

and so $x = 1$ or $x = 9$.
(We can check that if $x = 1$, $\triangle PQS$ has sides of lengths 4, 1 and $\sqrt{17}$ and $\triangle QRS$ has sides of lengths $\sqrt{17}$, 8 and 9, both of which are right-angled, and if $x = 9$, $\triangle PQS$ has sides of lengths 12, 9 and 15 and $\triangle QRS$ has sides of lengths 15, 8 and 17, both of which are right-angled.)
In terms of x, the perimeter of $PQRS$ is $x + 8 + (x+8) + (x+3) = 3x + 19$.
Thus, the possible perimeters of $PQRS$ are 22 (when $x = 1$) and 46 (when $x = 9$).

5. (a) If r is a term in the sequence and s is the next term, then $s = 1 + \dfrac{1}{1+r}$.

This means that $s - 1 = \dfrac{1}{1+r}$ and so $\dfrac{1}{s-1} = 1 + r$ which gives $r = \dfrac{1}{s-1} - 1$.

Therefore, since $a_3 = \dfrac{41}{29}$, then

$$a_2 = \frac{1}{a_3 - 1} - 1 = \frac{1}{(41/29) - 1} - 1 = \frac{1}{12/29} - 1 = \frac{29}{12} - 1 = \frac{17}{12}$$

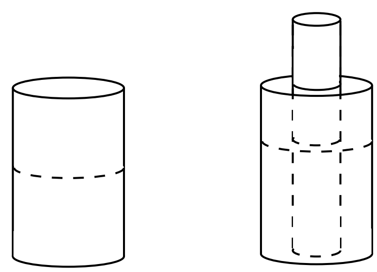
Further, since $a_2 = \dfrac{17}{12}$, then

$$a_1 = \frac{1}{a_2 - 1} - 1 = \frac{1}{(17/12) - 1} - 1 = \frac{1}{5/12} - 1 = \frac{12}{5} - 1 = \frac{7}{5}$$

(b) Initially, the water in the hollow tube forms a cylinder with radius 10 mm and height h mm. Thus, the volume of the water is $\pi(10\text{ mm})^2(h\text{ mm}) = 100\pi h\text{ mm}^3$.
After the rod is inserted, the level of the water rises to 64 mm. Note that this does not overflow the tube, since the tube's height is 100 mm.
Up to the height of the water, the tube is a cylinder with radius 10 mm and height 64 mm.
Thus, the volume of the tube up to the height of the water is

$$\pi(10\text{ mm})^2(64\text{ mm}) = 6400\pi\text{ mm}^3$$

This volume consists of the water that is in the tube (whose volume, which has not changed, is $100\pi h\text{ mm}^3$) and the rod up to a height of 64 mm.



Since the radius of the rod is 2.5 mm, the volume of the rod up to a height of 64 mm is $\pi(2.5\text{ mm})^2(64\text{ mm}) = 400\pi\text{ mm}^3$.
Comparing volumes, $6400\pi\text{ mm}^3 = 100\pi h\text{ mm}^3 + 400\pi\text{ mm}^3$ and so $100h = 6000$ which gives $h = 60$.

6. (a) We note that $\dfrac{2x+1}{x} = \dfrac{2x}{x} + \dfrac{1}{x} = 2 + \dfrac{1}{x}$.

Therefore, $\dfrac{2x+1}{x} = 4$ exactly when $2 + \dfrac{1}{x} = 4$ or $\dfrac{1}{x} = 2$ and so $x = \dfrac{1}{2}$.

Alternatively, we could solve $\dfrac{2x+1}{x} = 4$ directly to obtain $2x+1 = 4x$, which gives $2x = 1$ and so $x = \dfrac{1}{2}$.

Thus, to determine the value of $f(4)$, we substitute $x = \dfrac{1}{2}$ into the given equation $f\left(\dfrac{2x+1}{x}\right) = x + 6$ and obtain $f(4) = \dfrac{1}{2} + 6 = \dfrac{13}{2}$.

(b) Since the graph passes through $(3,5)$, $(5,4)$ and $(11,3)$, we can substitute these three points and obtain the following three equations:

$$\begin{aligned} 5 &= \log_a(3+b)+c \\ 4 &= \log_a(5+b)+c \\ 3 &= \log_a(11+b)+c \end{aligned}$$

Subtracting the second equation from the first and the third equation from the second, we obtain:

$$\begin{aligned} 1 &= \log_a(3+b)-\log_a(5+b) \\ 1 &= \log_a(5+b)-\log_a(11+b) \end{aligned}$$

Equating right sides and manipulating, we obtain the following equivalent equations:

$$\begin{aligned} \log_a(5+b)-\log_a(11+b) &= \log_a(3+b)-\log_a(5+b) \\ 2\log_a(5+b) &= \log_a(3+b)+\log_a(11+b) \\ \log_a\left((5+b)^2\right) &= \log_a\left((3+b)(11+b)\right) \qquad \text{(using log laws)} \\ (5+b)^2 &= (3+b)(11+b) \qquad \text{(raising both sides to the power of } a\text{)} \\ 25+10b+b^2 &= 33+14b+b^2 \\ -8 &= 4b \\ b &= -2 \end{aligned}$$

Since $b=-2$, the equation $1=\log_a(3+b)-\log_a(5+b)$ becomes $1=\log_a 1-\log_a 3$.
Since $\log_a 1 = 0$ for every admissible value of a, then $\log_a 3 = -1$ which gives $a = 3^{-1} = \frac{1}{3}$.
Finally, the equation $5=\log_a(3+b)+c$ becomes $5=\log_{1/3}(1)+c$ and so $c=5$.
Therefore, $a=\frac{1}{3}$, $b=-2$, and $c=5$, which gives $y=\log_{1/3}(x-2)+5$.

Checking:

- When $x=3$, we obtain $y=\log_{1/3}(3-2)+5=\log_{1/3}1+5=0+5=5$.
- When $x=5$, we obtain $y=\log_{1/3}(5-2)+5=\log_{1/3}3+5=-1+5=4$.
- When $x=11$, we obtain $y=\log_{1/3}(11-2)+5=\log_{1/3}9+5=-2+5=3$.

7. (a) The probability that the integer n is chosen is $\log_{100}\left(1+\frac{1}{n}\right)$.

The probability that an integer between 81 and 99, inclusive, is chosen equals the sum of the probabilities that the integers 81, 82, ..., 98, 99 are selected, which equals

$$\log_{100}\left(1+\frac{1}{81}\right)+\log_{100}\left(1+\frac{1}{82}\right)+\cdots+\log_{100}\left(1+\frac{1}{98}\right)+\log_{100}\left(1+\frac{1}{99}\right)$$

Since the second probability equals 2 times the first probability, the following equations are equivalent:

$$\log_{100}\left(1+\frac{1}{81}\right)+\log_{100}\left(1+\frac{1}{82}\right)+\cdots+\log_{100}\left(1+\frac{1}{98}\right)+\log_{100}\left(1+\frac{1}{99}\right)=2\log_{100}\left(1+\frac{1}{n}\right)$$
$$\log_{100}\left(\frac{82}{81}\right)+\log_{100}\left(\frac{83}{82}\right)+\cdots+\log_{100}\left(\frac{99}{98}\right)+\log_{100}\left(\frac{100}{99}\right)=2\log_{100}\left(1+\frac{1}{n}\right)$$

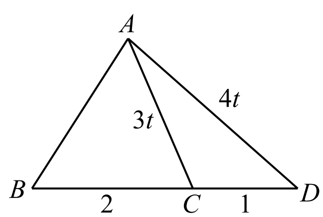
Using logarithm laws, these equations are further equivalent to

$$\log_{100}\left(\frac{82}{81}\cdot\frac{83}{82}\cdot\cdots\cdot\frac{99}{98}\cdot\frac{100}{99}\right)=\log_{100}\left(1+\frac{1}{n}\right)^2$$
$$\log_{100}\left(\frac{100}{81}\right)=\log_{100}\left(1+\frac{1}{n}\right)^2$$

Since logarithm functions are invertible, we obtain $\frac{100}{81}=\left(1+\frac{1}{n}\right)^2$.

Since $n>0$, then $1+\frac{1}{n}=\sqrt{\frac{100}{81}}=\frac{10}{9}$, and so $\frac{1}{n}=\frac{1}{9}$, which gives $n=9$.

(b) Since $\frac{AC}{AD}=\frac{3}{4}$, then we let $AC=3t$ and $AD=4t$ for some real number $t>0$.



Using the cosine law in $\triangle ACD$, the following equations are equivalent:

$$\begin{aligned}AD^2&=AC^2+CD^2-2\cdot AC\cdot CD\cdot\cos(\angle ACD)\\(4t)^2&=(3t)^2+1^2-2(3t)(1)(-\tfrac{3}{5})\\16t^2&=9t^2+1+\tfrac{18}{5}t\\80t^2&=45t^2+5+18t\\35t^2-18t-5&=0\\(7t-5)(5t+1)&=0\end{aligned}$$

Since $t>0$, then $t=\frac{5}{7}$.
Thus, $AC=3t=\frac{15}{7}$.
Using the cosine law in $\triangle ACB$ and noting that

$$\cos(\angle ACB)=\cos(180^\circ-\angle ACD)=-\cos(\angle ACD)=\tfrac{3}{5}$$

the following equations are equivalent:

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2 \cdot AC \cdot BC \cdot \cos(\angle ACB) \\ &= \left(\tfrac{15}{7}\right)^2 + 2^2 - 2(\tfrac{15}{7})(2)(\tfrac{3}{5}) \\ &= \tfrac{225}{49} + 4 - \tfrac{36}{7} \\ &= \tfrac{225}{49} + \tfrac{196}{49} - \tfrac{252}{49} \\ &= \tfrac{169}{49} \end{aligned}$$

Since $AB > 0$, then $AB = \frac{13}{7}$.

8. (a) The parabola with equation $y = ax^2 + 2$ is symmetric about the y-axis.
Thus, its vertex occurs when $x = 0$ (which gives $y = a \cdot 0^2 + 2 = 2$) and so V has coordinates $(0, 2)$.
To find the coordinates of B and C, we use the equations of the parabola and line to obtain

$$\begin{aligned} ax^2 + 2 &= -x + 4a \\ ax^2 + x + (2 - 4a) &= 0 \end{aligned}$$

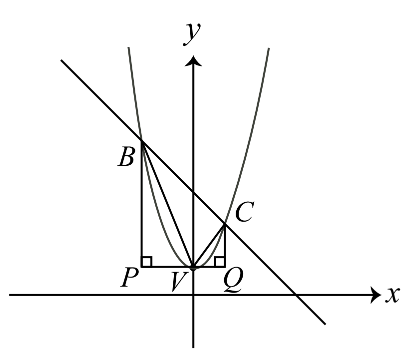
Using the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1^2 - 4a(2 - 4a)}}{2a} = \frac{-1 \pm \sqrt{1 - 8a + 16a^2}}{2a}$$

Since $1 - 8a + 16a^2 = (4a - 1)^2$ and $4a - 1 > 0$ (since $a > \frac{1}{2}$), then $\sqrt{1 - 8a + 16a^2} = 4a - 1$ and so

$$x = \frac{-1 \pm (4a - 1)}{2a}$$

which means that $x = \dfrac{4a - 2}{2a} = \dfrac{2a - 1}{a} = 2 - \dfrac{1}{a}$ or $x = \dfrac{-4a}{2a} = -2$.
We can use the equation of the line to obtain the y-coordinates of B and C.
When $x = -2$ (corresponding to point B), we obtain $y = -(-2) + 4a = 4a + 2$.
When $x = 2 - \dfrac{1}{a}$ (corresponding to point C), we obtain $y = -\left(2 - \dfrac{1}{a}\right) + 4a = 4a - 2 + \dfrac{1}{a}$.
Let P and Q be the points on the horizontal line through V so that BP and CQ are perpendicular to PQ.



Then the area of $\triangle VBC$ is equal to the area of trapezoid $PBCQ$ minus the areas of right-angled $\triangle BPV$ and right-angled $\triangle CQV$.
Since B has coordinates $(-2, 4a+2)$, P has coordinates $(-2, 2)$, V has coordiantes $(0, 2)$, Q has coordinates $\left(2 - \frac{1}{a}, 2\right)$, and C has coordinates $\left(2 - \frac{1}{a}, 4a - 2 + \frac{1}{a}\right)$, then

$$\begin{aligned}
BP &= (4a+2) - 2 = 4a \\
CQ &= \left(4a - 2 + \frac{1}{a}\right) - 2 = 4a - 4 + \frac{1}{a} \\
PV &= 0 - (-2) = 2 \\
QV &= 2 - \frac{1}{a} - 0 = 2 - \frac{1}{a} \\
PQ &= PV + QV = 2 + 2 - \frac{1}{a} = 4 - \frac{1}{a}
\end{aligned}$$

Therefore, the area of trapezoid $PBCQ$ is

$$\frac{1}{2}(BP + CQ)(PQ) = \frac{1}{2}\left(4a + 4a - 4 + \frac{1}{a}\right)\left(4 - \frac{1}{a}\right) = \left(4a - 2 + \frac{1}{2a}\right)\left(4 - \frac{1}{a}\right)$$

Also, the area of $\triangle BPV$ is $\frac{1}{2} \cdot BP \cdot PV = \frac{1}{2}(4a)(2) = 4a$.
Furthermore, the area of $\triangle CQV$ is

$$\frac{1}{2} \cdot CQ \cdot QV = \frac{1}{2}\left(4a - 4 + \frac{1}{a}\right)\left(2 - \frac{1}{a}\right) = \left(2a - 2 + \frac{1}{2a}\right)\left(2 - \frac{1}{a}\right)$$

From the given information,

$$\left(4a - 2 + \frac{1}{2a}\right)\left(4 - \frac{1}{a}\right) - 4a - \left(2a - 2 + \frac{1}{2a}\right)\left(2 - \frac{1}{a}\right) = \frac{72}{5}$$

Multiplying both sides by $2a^2$, which we distribute through the factors on the left side as $2a \cdot a$, we obtain

$$(8a^2 - 4a + 1)(4a - 1) - 8a^3 - (4a^2 - 4a + 1)(2a - 1) = \frac{144}{5}a^2$$

Multiplying both sides by 5, we obtain

$$5(8a^2 - 4a + 1)(4a - 1) - 40a^3 - 5(4a^2 - 4a + 1)(2a - 1) = 144a^2$$

Expanding and simplifying, we obtain

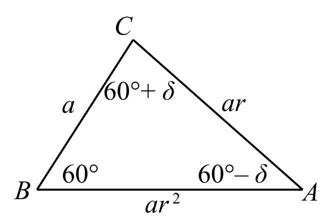
$$\begin{aligned}
(160a^3 - 120a^2 + 40a - 5) - 40a^3 - (40a^3 - 60a^2 + 30a - 5) &= 144a^2 \\
80a^3 - 204a^2 + 10a &= 0 \\
2a(40a^2 - 102a + 5) &= 0 \\
2a(20a - 1)(2a - 5) &= 0
\end{aligned}$$

and so $a = 0$ or $a = \frac{1}{20}$ or $a = \frac{5}{2}$. Since $a > \frac{1}{2}$, then $a = \frac{5}{2}$.

(b) We prove that there cannot be such a triangle.
We prove this by contradiction. That is, we suppose that there is such a triangle and prove that there is then a logical contradiction.
Suppose that $\triangle ABC$ is not equilateral, has side lengths that form a geometric sequence, and angles whose measures form an arithmetic sequence.
Suppose that $\triangle ABC$ has side lengths $BC = a$, $AC = ar$, and $AB = ar^2$, for some real numbers $a > 0$ and $r > 1$. (These lengths form a geometric sequence, and we can assume that this sequence is increasing, and that the sides are labelled in this particular order.)
Since $BC < AC < AB$, then the opposite angles have the same relationships, namely $\angle BAC < \angle ABC < \angle ACB$.
Suppose that $\angle BAC = \theta$, $\angle ABC = \theta + \delta$, and $\angle ACB = \theta + 2\delta$ for some angles θ and δ. (In other words, these angles form an arithmetic sequence.
Since these three angles are the angles in a triangle, then their sum is 180°, and so

$$\begin{aligned}\theta + (\theta+\delta) + (\theta + 2\delta) &= 180^\circ \\ 3\theta + 3\delta &= 180^\circ \\ \theta + \delta &= 60^\circ\end{aligned}$$

In other words, $\angle ABC = 60^\circ$.



We could proceed using the cosine law:

$$\begin{aligned}AC^2 &= BC^2 + AB^2 - 2\cdot BC\cdot AB\cdot \cos(\angle ABC) \\ (ar)^2 &= a^2 + (ar^2)^2 - 2a(ar^2)\cos(60^\circ) \\ a^2r^2 &= a^2 + a^2r^4 - 2a^2r^2\cdot \tfrac{1}{2} \\ a^2r^2 &= a^2 + a^2r^4 - a^2r^2 \\ 0 &= a^2r^4 - 2a^2r^2 + a^2 \\ 0 &= a^2(r^4 - 2r^2 + 1) \\ 0 &= a^2(r^2-1)^2\end{aligned}$$

This tells us that $a = 0$ (which is impossible) or $r^2 = 1$ (and thus $r = \pm 1$, which is impossible).
Therefore, we have reached a logical contradiction and so such a triangle cannot exist.

Alternatively, we could proceed using the sine law, noting that

$$\begin{aligned}\angle BAC &= \theta = (\theta+\delta) - \delta = 60^\circ - \delta \\ \angle ACB &= \theta + 2\delta = (\theta+\delta) + \delta = 60^\circ + \delta\end{aligned}$$

By the sine law,

$$\frac{BC}{\sin(\angle BAC)} = \frac{AC}{\sin(\angle ABC)} = \frac{AB}{\sin(\angle ACB)}$$

from which we obtain

$$\frac{a}{\sin(60^\circ - \delta)} = \frac{ar}{\sin(60^\circ)} = \frac{ar^2}{\sin(60^\circ + \delta)}$$

Since $a \neq 0$, from the first two parts,

$$r = \frac{ar}{a} = \frac{\sin 60^\circ}{\sin(60^\circ - \delta)}$$

Since $ar \neq 0$, from the second two parts,

$$r = \frac{ar^2}{ar} = \frac{\sin(60^\circ + \delta)}{\sin 60^\circ}$$

Equating expressions for r, we obtain successively

$$\begin{aligned}
\frac{\sin 60^\circ}{\sin(60^\circ - \delta)} &= \frac{\sin(60^\circ + \delta)}{\sin 60^\circ} \\
\sin^2 60^\circ &= \sin(60^\circ - \delta)\sin(60^\circ + \delta) \\
\left(\tfrac{\sqrt{3}}{2}\right)^2 &= (\sin 60^\circ \cos\delta - \cos 60^\circ \sin\delta)(\sin 60^\circ \cos\delta + \cos 60^\circ \sin\delta) \\
\tfrac{3}{4} &= \left(\tfrac{\sqrt{3}}{2}\cos\delta - \tfrac{1}{2}\sin\delta\right)\left(\tfrac{\sqrt{3}}{2}\cos\delta + \tfrac{1}{2}\sin\delta\right) \\
\tfrac{3}{4} &= \tfrac{3}{4}\cos^2\delta - \tfrac{1}{4}\sin^2\delta \\
\tfrac{3}{4} &= \tfrac{3}{4}\cos^2\delta + \tfrac{3}{4}\sin^2\delta - \sin^2\delta \\
\tfrac{3}{4} &= \tfrac{3}{4}(\cos^2\delta + \sin^2\delta) - \sin^2\delta \\
\tfrac{3}{4} &= \tfrac{3}{4} - \sin^2\delta \\
\sin^2\delta &= 0
\end{aligned}$$

which means that $\delta = 0^\circ$. (Any other angle δ with $\sin\delta = 0$ would not produce angles in a triangle.)
Therefore, all three angles in the triangle are 60°, which means that the triangle is equilateral, which it cannot be.
Therefore, we have reached a logical contradiction and so such a triangle cannot exist.

9. (a) The $(4,2)$-sawtooth sequence consists of the terms

$$1, \quad 2,3,4,3,2,1, \quad 2,3,4,3,2,1$$

whose sum is 31.

(b) *Solution 1*

Suppose that $m \geq 2$.
The $(m,3)$-sawtooth sequence consists of an initial 1 followed by 3 teeth, each of which goes from 2 to m to 1.
Consider one of these teeth whose terms are

$$2,3,4,\ldots,m-1,m,m-1,m-2,m-3,\ldots,2,1$$

When we write the ascending portion directly above the descending portion, we obtain

$$\begin{array}{cccccc} 2, & 3, & 4, & \ldots, & m-1, & m, \\ m-1, & m-2, & m-3, & \ldots, & 2, & 1 \end{array}$$

From this presentation, we can see $m-1$ pairs of terms, the sum of each of which is $m+1$. (Note that $2+(m-1)=3+(m-2)=4+(m-3)=\cdots=(m-1)+2=m+1$ and as we move from left to right, the terms on the top increase by 1 at each step and the terms on the bottom decrease by 1 at each step, so their sum is indeed constant.)
Therefore, the sum of the numbers in one of the teeth is $(m-1)(m+1)=m^2-1$.
This means that the sum of the terms in the $(m,3)$-sawtooth sequence is $1+3(m^2-1)$, which equals $3m^2-2$.

Solution 2

Suppose that $m \geq 2$.
The $(m,3)$-sawtooth sequence consists of an initial 1 followed by 3 teeth, each of which goes from 2 to m to 1.
Consider one of these teeth whose terms are

$$2,3,4,\ldots,m-1,m,m-1,m-2,m-3,\ldots,2,1$$

This tooth includes one 1, two 2s, two 3s, and so on, until we reach two $(m-1)$s, and one m.
The sum of these numbers is

$$1(1)+2(2)+2(3)+\cdots+2(m-1)+m$$

which can be rewritten as

$$2(1+2+3+\cdots+(m-1)+m)-1-m = 2\cdot\tfrac{1}{2}m(m+1)-m-1 = m^2+m-m-1 = m^2-1$$

Therefore, the sum of the numbers in one of the teeth is $(m-1)(m+1)=m^2-1$.
This means that the sum of the terms in the $(m,3)$-sawtooth sequence is $1+3(m^2-1)$, which equals $3m^2-2$.

(c) From (b), the sum of the terms in each tooth is $m^2 - 1$.
Thus, the sum of the terms in the (m, n)-sawtooth sequence is $1 + n(m^2 - 1)$.
For this to equal 145, we have $n(m^2 - 1) = 144$.
This means that n and $m^2 - 1$ form a divisor pair of 144.
As m ranges from 2 to 12, the values of $m^2 - 1$ are

$$3, 8, 15, 24, 35, 48, 63, 80, 99, 120, 143$$

(When $m = 13$, we get $m^2 - 1 = 168$ and so when $m \geq 13$, the value of $m^2 - 1$ is too large to be a divisor of 144.)
Of these, $3, 8, 24, 48$ are divisors of 144 (corresponding to $m = 2, 3, 5, 7$), and give corresponding divisors $48, 18, 6, 3$.
Therefore, the pairs (m, n) for which the sum of the terms is 145 are

$$(m, n) = (2, 48), (3, 18), (5, 6), (7, 3)$$

(d) In an (m, n)-sawtooth sequence, the sum of the terms is $n(m^2 - 1) + 1$.
In each tooth, there are $(m - 1) + (m - 1) = 2m - 2$ terms (from 2 to m, inclusive, and from $m - 1$ to 1, inclusive).
This means that there are $n(2m - 2) + 1$ terms in the sequence.
Thus, the average of the terms in the sequence is $\dfrac{n(m^2 - 1) + 1}{n(2m - 2) + 1}$.
We need to prove that this is not an integer for all pairs of positive integers (m, n) with $m \geq 2$.
Suppose that $\dfrac{n(m^2 - 1) + 1}{n(2m - 2) + 1} = k$ for some integer k. We will show, by contradiction, that this is not possible.
Since $\dfrac{n(m^2 - 1) + 1}{n(2m - 2) + 1} = k$, then

$$\begin{aligned}
\frac{m^2n - n + 1}{2mn - 2n + 1} &= k \\
m^2n - n + 1 &= 2mnk - 2nk + k \\
m^2n - 2mnk + (2nk - n - k + 1) &= 0
\end{aligned}$$

We treat this as a quadratic equation in m.
Since m is an integer, then this equation has integer roots, and so its discriminant must be a perfect square.
The discriminant of this quadratic equation is

$$\begin{aligned}
\Delta &= (-2nk)^2 - 4n(2nk - n - k + 1) \\
&= 4n^2k^2 - 8n^2k + 4n^2 + 4nk - 4n \\
&= 4n^2(k^2 - 2k + 1) + 4n(k - 1) \\
&= 4n^2(k - 1)^2 + 4n(k - 1) \\
&= (2n(k - 1))^2 + 2(2n(k - 1)) + 1 - 1 \\
&= (2n(k - 1) + 1)^2 - 1
\end{aligned}$$

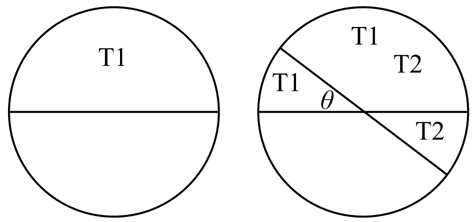
We note that $(2n(k-1)+1)^2$ is a perfect square and Δ is supposed to be a perfect square.
But these perfect squares differ by 1, and the only two perfect squares that differ by 1 are

1 and 0.
(To justify this last fact, we could look at the equation $a^2 - b^2 = 1$ where a and b are non-negative integers, and factor this to obtain $(a+b)(a-b) = 1$ which would give $a + b = a - b = 1$ from which we get $a = 1$ and $b = 0$.)
Since $(2n(k-1)+1)^2 = 1$ and $2n(k-1)+1$ is non-negative, then $2n(k-1)+1 = 1$ and so $2n(k-1) = 0$.
Since n is positive, then $k - 1 = 0$ or $k = 1$.
Therefore, the only possible way in which the average is an integer is if the average is 1.
In this case, we get

$$\begin{aligned} m^2n - 2mn + (2n - n - 1 + 1) &= 0 \\ m^2n - 2mn + n &= 0 \\ n(m^2 - 2m + 1) &= 0 \\ n(m-1)^2 &= 0 \end{aligned}$$

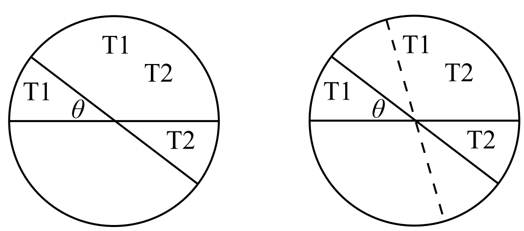
Since n and m are positive integers with $m \geq 2$, then $n(m-1)^2 \neq 0$, which is a contradiction.
Therefore, the average of the terms in an (m, n)-sawtooth sequence cannot be an integer.

10. (a) Assume that the first topping is placed on the top half of the pizza. (We can rotate the pizza so that this is the case.)
Assume that the second topping is placed on the half of the pizza that is above the horizontal diameter that makes an angle of θ clockwise with the horizontal as shown. In other words, the topping covers the pizza from θ to $\theta + 180^\circ$.



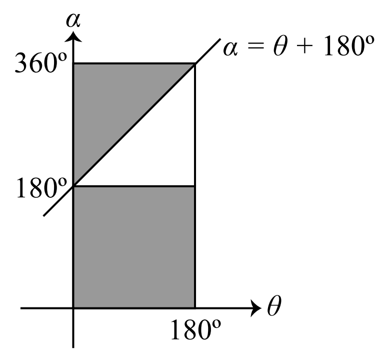
We may assume that $0^\circ \leq \theta \leq 360^\circ$.
When $0^\circ \leq \theta \leq 90^\circ$, the angle of the sector covered by both toppings is at least 90° (and so is at least a quarter of the circle).
When $90^\circ < \theta \leq 180^\circ$, the angle of the sector covered by both toppings is less than 90° (and so is less than a quarter of the circle).
When θ moves past 180°, the left-hand portion of the upper half circle starts to be covered with both toppings again. When $180^\circ \leq \theta < 270^\circ$, the angle of the sector covered by both toppings is less than 90° (and so is less than a quarter of the circle).
When $270^\circ \leq \theta \leq 360^\circ$, the angle of the sector covered by both toppings at least 90° (and so is at least a quarter of the circle).
Therefore, if θ is chosen randomly between 0° and 360°, the combined length of the intervals in which at least $\frac{1}{4}$ of the pizza is covered with both toppings is 180°.
Therefore, the probability is $\dfrac{180^\circ}{360^\circ}$, or $\dfrac{1}{2}$.

(b) Suppose that the first topping is placed on the top half of the pizza. (Again, we can rotate the pizza so that this is the case.)
Assume that the second topping is placed on the half of the pizza that is above the diameter that makes an angle of θ clockwise with the horizontal as shown. In other words, the topping covers the pizza from θ to $\theta + 180^\circ$.
We may assume that $0^\circ \leq \theta \leq 180^\circ$. If $180^\circ \leq \theta \leq 360^\circ$, the resulting pizza can be seen as a reflection of the one shown.



Consider the third diameter added, shown dotted in the diagram above. Suppose that its angle with the horizontal is α. (In the diagram, $\alpha < 90^\circ$.) We assume that the topping is added on the half pizza clockwise beginning at the angle of α, and that this topping stays in the same relative position as the diameter sweeps around the circle.
For what angles α will there be a portion of the pizza covered with all three toppings?
If $0^\circ \leq \alpha < 180^\circ$, there will be a portion covered with three toppings; this portion is above the right half of the horizontal diameter.
If $180^\circ \leq \alpha < 180^\circ + \theta$, the third diameter will pass through the two regions with angle θ and the third topping will be below this diameter, so there will not be a region covered

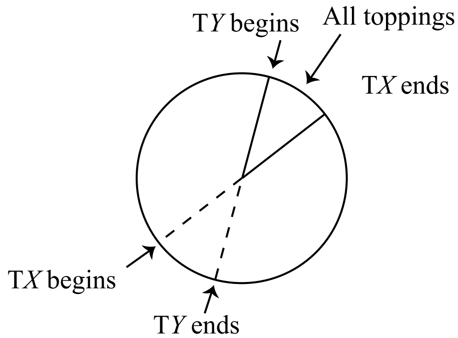
with three toppings.
If $180^\circ + \theta \le \alpha \le 360^\circ$, the third topping starts to cover the leftmost part of the region currently covered with two toppings, and so a region is covered with three toppings.
Therefore, for an angle θ with $0^\circ \le \theta \le 180^\circ$, a region of the pizza is covered with three toppings when $0^\circ \le \alpha < 180^\circ$ and when $180^\circ + \theta \le \alpha \le 360^\circ$.
To determine the desired probability, we graph points (θ, α). A particular choice of diameters corresponds to a choice of angles θ and α with $0^\circ \le \theta \le 180^\circ$ and $0^\circ \le \alpha \le 360^\circ$, which corresponds to a point on the graph below.
The probability that we are looking for then equals the area of the region of this graph where three toppings are in a portion of the pizza divided by the total allowable area of the graph.
The shaded region of the graph corresponds to instances where a portion of the pizza will be covered by three toppings.



This shaded region consists of the entire portion of the graph where $0^\circ \le \alpha \le 180^\circ$ (regardless of θ) as well as the region above the line with equation $\alpha = \theta + 180^\circ$ (that is, the region with $\theta + 180^\circ \le \alpha \le 360^\circ$).
Since the slope of the line is 1, it divides the upper half of the region, which is a square, into two pieces of equal area.
Therefore, $\frac{3}{4}$ of the graph is shaded, which means that the probability that a region of the pizza is covered by all three toppings is $\dfrac{3}{4}$.

(c) The main idea of this solution is that the toppings all overlap exactly when there is one topping with the property that all other toppings "begin" somewhere in that toppings semi-circle. In the rest of this solution, we determine the probability using this fact and then justify this fact.

Suppose that, for $1 \le j \le N$, topping j is put on the semi-circle that starts at an angle of θ_j clockwise from the horizontal left-hand radius and continues to an angle of $\theta_j + 180^\circ$, where $0^\circ \le \theta_j < 360^\circ$. By establishing these variables and this convention, we are fixing both the angle of the diameter and the semi-circle defined by this diameter on which the topping is placed.
Suppose that there is some region of the pizza with non-zero area that is covered by all N toppings.
This region will be a sector with two bounding radii, each of which must be half of a diameter that defines one of the toppings.
Suppose that the radius at the clockwise "end" of the sector is the end of the semi-circle where topping X is placed, and that the radius at the counter-clockwise "beginning" of the sector is the start of the semi-circle where topping Y is placed.



This means that each of the other $N-2$ toppings begins between (in the clockwise sense) the points where topping X begins and where topping Y begins.
Consider the beginning angle for topping X, θ_X.
To say that the other $N-1$ toppings begin at some point before topping X ends is the same as saying that each θ_j with $j \neq X$ is between θ_X and $\theta_X + 180^\circ$.
Here, we can allow for the possibility that $\theta_X + 180^\circ$ is greater than 360° by saying that an angle equivalent to θ_j (which is either θ_j or $\theta_j + 360^\circ$) is between θ_X and $\theta_X + 180^\circ$.
For each $j \neq X$, the angle θ_j is randomly, uniformly and independently chosen on the circle, so there is a probability of $\frac{1}{2}$ that this angle (or its equivalent) will be in the semi-circle between θ_X and $\theta_X + 180^\circ$.
Since there are $N-1$ such angles, the probability that all are between θ_X and $\theta_X + 180^\circ$ is $\frac{1}{2^{N-1}}$.
Since there are N possible selections for the first topping that can end the common sector, then the desired probability will be $\frac{N}{2^{N-1}}$ as long as we can show that no set of angles can give two different sectors that are both covered with all toppings.
To show this last fact, we suppose without loss of generality that

$$0^\circ = \theta_1 < \theta_2 < \theta_3 < \cdots < \theta_{N-1} < \theta_N < 180^\circ$$

(We can relabel the toppings if necessary to obtain this order and rotate the pizza so that topping 1 begins at 0°.)
We need to show that it is not possible to have a Z for which $\theta_Z, \theta_{Z+1}, \ldots, \theta_N, \theta_1, \theta_2, \ldots, \theta_{Z-1}$ all lie in a semi-circle starting with θ_Z.
Since $\theta_Z < 180^\circ$ and θ_1 can be thought of as 360°, then this is not possible as θ_1 and the angles after it are all not within 180° of θ_Z.
Therefore, it is not possible to have two such regions with the same set of angles, and so the desired probability is $\frac{N}{2^{N-1}}$.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Euclid Contest

Wednesday, April 7, 2021
(in North America and South America)

Thursday, April 8, 2021
(outside of North America and South America)

Solutions

1. (a) Since $(a-1)+(2a-3)=14$, then $3a=18$ and so $a=6$.

   (b) Since $(c^2-c)+(2c-3)=9$, then $c^2+c-3=9$ and so $c^2+c-12=0$.
   Factoring, we obtain $(c+4)(c-3)=0$ and so $c=3$ or $c=-4$.

   (c) *Solution 1*
   Manipulating algebraically, we obtain the following equivalent equations:

$$\begin{aligned}\frac{1}{x^2}+\frac{3}{2x^2} &= 10 \\ 2+3 &= 20x^2 \qquad \text{(multiplying through by } 2x^2\text{, given that } x\neq 0) \\ 5 &= 20x^2 \\ x^2 &= \frac{1}{4}\end{aligned}$$

   and so $x=\pm\frac{1}{2}$.

   Solution 2
   Manipulating algebraically, we obtain the following equivalent equations:

$$\begin{aligned}\frac{1}{x^2}+\frac{3}{2x^2} &= 10 \\ \frac{2}{2x^2}+\frac{3}{2x^2} &= 10 \\ \frac{5}{2x^2} &= 10 \\ 5 &= 20x^2 \qquad \text{(since } x\neq 0) \\ x^2 &= \frac{1}{4}\end{aligned}$$

   and so $x=\pm\frac{1}{2}$.

2. (a) Using a calculator, we see that

$$(10^3+1)^2 = 1001^2 = 1\,002\,001$$

   The sum of the digits of this integer is $1+2+1$ which equals 4.
   To determine this integer without using a calculator, we can let $x=10^3$.
   Then

$$\begin{aligned}(10^3+1)^2 &= (x+1)^2 \\ &= x^2+2x+1 \\ &= (10^3)^2+2(10^3)+1 \\ &= 1\,002\,001\end{aligned}$$

(b) Before the price increase, the total cost of 2 small cookies and 1 large cookie is $2 \cdot \$1.50 + \$2.00 = \$5.00$.
10% of $\$1.50$ is $0.1 \cdot \$1.50 = \$0.15$. After the price increase, 1 small cookie costs $\$1.50 + \$0.15 = \$1.65$.
5% of $\$2.00$ is $0.05 \cdot \$2.00 = \$0.10$. After the price increase, 1 large cookie costs $\$2.00 + \$0.10 = \$2.10$.
After the price increase, the total cost of 2 small cookies and 1 large cookie is $2 \cdot \$1.65 + \$2.10 = \$5.40$.
The percentage increase in the total cost is $\dfrac{\$5.40 - \$5.00}{\$5.00} \times 100\% = \dfrac{40}{500} \times 100\% = 8\%$.

(c) Suppose that Rayna's age is x years.
Since Qing is twice as old as Rayna, Qing's age is $2x$ years.
Since Qing is 4 years younger than Paolo, Paolo's age is $2x + 4$ years.
Since the average of their ages is 13 years, we obtain

$$\frac{x + (2x) + (2x + 4)}{3} = 13$$

This gives $5x + 4 = 39$ and so $5x = 35$ or $x = 7$.
Therefore, Rayna is 7 years old, Qing is 14 years old, and Paolo is 18 years old.
(Checking, the average of 7, 14 and 18 is $\dfrac{7 + 14 + 18}{3} = \dfrac{39}{3} = 13$.)

3. (a) The length of PQ is equal to $\sqrt{(0-5)^2 + (12-0)^2} = \sqrt{(-5)^2 + 12^2} = 13$.
In a similar way, we can see that $QR = RS = SP = 13$.
Therefore, the perimeter of $PQRS$ is $4 \cdot 13 = 52$.
(We can also see that if O is the origin, then $\triangle POQ$, $\triangle POS$, $\triangle ROQ$, and $\triangle ROS$ are congruent because $OQ = OS$ and $OP = OR$, which means that $PQ = QR = RS = SP$.)

(b) *Solution 1*
Suppose that B has coordinates (r, s) and C has coordinates (t, u).
Since $M(3, 9)$ is the midpoint of $A(0, 8)$ and $B(r, s)$, then 3 is the average of 0 and r (which gives $r = 6$) and 9 is the average of 8 and s (which gives $s = 10$).
Since $N(7, 6)$ is the midpoint of $B(6, 10)$ and $C(t, u)$, then 7 is the average of 6 and t (which gives $t = 8$) and 6 is the average of 10 and u (which gives $u = 2$).
The slope of the line segment joining $A(0, 8)$ and $C(8, 2)$ is $\dfrac{8-2}{0-8}$ which equals $-\dfrac{3}{4}$.

Solution 2
Since M is the midpoint of AB and N is the midpoint of BC, then MN is parallel to AC.
Therefore, the slope of AC equals the slope of the line segment joining $M(3, 9)$ to $N(7, 6)$, which is $\dfrac{9-6}{3-7}$ or $-\dfrac{3}{4}$.

(c) Since $V(1, 18)$ is on the parabola, then $18 = -2(1^2) + 4(1) + c$ and so $c = 18 + 2 - 4 = 16$.
Thus, the equation of the parabola is $y = -2x^2 + 4x + 16$.
The y-intercept occurs when $x = 0$, and so $y = 16$. Thus, D has coordinates $(0, 16)$.
The x-intercepts occur when $y = 0$. Here,

$$\begin{aligned} -2x^2 + 4x + 16 &= 0 \\ -2(x^2 - 2x - 8) &= 0 \\ -2(x-4)(x+2) &= 0 \end{aligned}$$

and so $x = 4$ and $x = -2$.
This means that E and F, in some order, have coordinates $(4, 0)$ and $(-2, 0)$.
Therefore, $\triangle DEF$ has base EF of length $4 - (-2) = 6$ and height 16 (vertical distance from the x-axis to the point D).
Finally, the area of $\triangle DEF$ is $\frac{1}{2} \cdot 6 \cdot 16 = 48$.

4. (a) We obtain successively

$$\begin{aligned} 3(8^x) + 5(8^x) &= 2^{61} \\ 8(8^x) &= 2^{61} \\ 8^{x+1} &= 2^{61} \\ (2^3)^{x+1} &= 2^{61} \\ 2^{3(x+1)} &= 2^{61} \end{aligned}$$

Thus, $3(x + 1) = 61$ and so $3x + 3 = 61$ which gives $3x = 58$ or $x = \frac{58}{3}$.

(b) Since the list $3n^2$, m^2, $2(n+1)^2$ consists of three consecutive integers written in increasing order, then

$$\begin{aligned} 2(n + 1)^2 - 3n^2 &= 2 \\ 2n^2 + 4n + 2 - 3n^2 &= 2 \\ -n^2 + 4n &= 0 \\ -n(n - 4) &= 0 \end{aligned}$$

and so $n = 0$ or $n = 4$.
If $n = 0$, the list becomes 0, m^2, 2. This means that $m^2 = 1$ and so $m = \pm 1$.
If $n = 4$, we have $3n^2 = 3 \cdot 16 = 48$ and $2(n + 1)^2 = 2 \cdot 25 = 50$ giving the list 48, m^2, 50.
This means that $m^2 = 49$ and so $m = \pm 7$.
Thus, the possible values for m are 1, -1, 7, -7.

5. (a) *Solution 1*

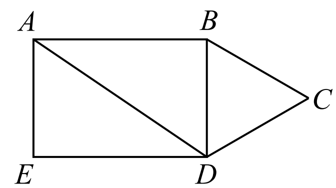
Suppose that S_0 has coordinates (a, b).
Step 1 moves (a, b) to $(a, -b)$.
Step 2 moves $(a, -b)$ to $(a, -b + 2)$.
Step 3 moves $(a, -b + 2)$ to $(-a, -b + 2)$.
Thus, S_1 has coordinates $(-a, -b + 2)$.
Step 1 moves $(-a, -b + 2)$ to $(-a, b - 2)$.
Step 2 moves $(-a, b - 2)$ to $(-a, b)$.
Step 3 moves $(-a, b)$ to (a, b).
Thus, S_2 has coordinates (a, b), which are the same coordinates as S_0.
Continuing this process, S_4 will have the same coordinates as S_2 (and thus as S_0) and S_6 will have the same coordinates as S_4, S_2 and S_0.
Since the coordinates of S_6 are $(-7, -1)$, the coordinates of S_0 are also $(-7, -1)$.

Solution 2

We work backwards from $S_6(-7, -1)$.
To do this, we undo the Steps of the process $\mathcal{P}$ by applying them in reverse order.

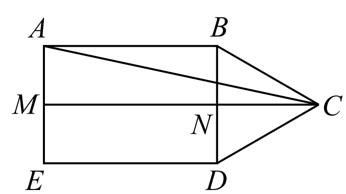
Since Step 3 reflects a point in the y-axis, its inverse does the same.
Since Step 2 translates a point 2 units upwards, its inverse translates a point 2 units downwards.
Since Step 1 reflects a point in the x-axis, its inverse does the same.
Applying these inverse steps to $S_6(-7,-1)$, we obtain $(7,-1)$, then $(7,-3)$, then $(7,3)$.
Thus, S_5 has coordinates $(7,3)$.
Applying the inverse steps to $S_5(7,3)$, we obtain $(-7,3)$, then $(-7,1)$, then $(-7,-1)$.
Thus, S_4 has coordinates $(-7,-1)$, which are the same coordinates as S_6.
If we apply these steps two more times, we will see that S_2 is the same point as S_4.
Two more applications tell us that S_0 is the same point as S_2.
Therefore, the coordinates of S_0 are the same as the coordinates of S_6, which are $(-7,-1)$.

(b) We begin by determining the length of AB in terms of x.
Since $ABDE$ is a rectangle, $BD = AE = 2x$.
Since $\triangle BCD$ is equilateral, $\angle DBC = 60^\circ$.
Join A to D.



Since AD and BC are parallel, $\angle ADB = \angle DBC = 60^\circ$.
Consider $\triangle ADB$. This is a 30°-60°-90° triangle since $\angle ABD$ is a right angle.
Using ratios of side lengths, $\dfrac{AB}{BD} = \dfrac{\sqrt{3}}{1}$ and so $AB = \sqrt{3}BD = 2\sqrt{3}x$, which is the answer to (i).

Next, we determine $\dfrac{AC}{AD}$.
Now, $\dfrac{AD}{BD} = \dfrac{2}{1}$ and so $AD = 2BD = 4x$.
Suppose that M is the midpoint of AE and N is the midpoint of BD.
Since $AE = BD = 2x$, then $AM = ME = BN = ND = x$.
Join M to N and N to C and A to C.



Since $ABDE$ is a rectangle, then MN is parallel to AB and so MN is perpendicular to both AE and BD.
Also, $MN = AB = 2\sqrt{3}x$.
Since $\triangle BCD$ is equilateral, its median CN is perpendicular to BD.
Since MN and NC are perpendicular to BD, MNC is actually a straight line segment and so $MC = MN + NC$.
Now $\triangle BNC$ is also a 30°-60°-90° triangle, and so $NC = \sqrt{3}BN = \sqrt{3}x$.
This means that $MC = 2\sqrt{3}x + \sqrt{3}x = 3\sqrt{3}x$.

Finally, $\triangle AMC$ is right-angled at M and so

$$AC = \sqrt{AM^2 + MC^2} = \sqrt{x^2 + (3\sqrt{3}x)^2} = \sqrt{x^2 + 27x^2} = \sqrt{28x^2} = 2\sqrt{7}x$$

since $x > 0$.

This means that $\dfrac{AC}{AD} = \dfrac{2\sqrt{7}x}{4x} = \dfrac{\sqrt{7}}{2} = \sqrt{\dfrac{7}{4}}$, which means that the integers $r = 7$ and $s = 4$ satisfy the conditions for (ii).

6. (a) *Solution 1*

Since the sequence $t_1, t_2, t_3, \ldots, t_{n-2}, t_{n-1}, t_n$ is arithmetic, then

$$t_1 + t_n = t_2 + t_{n-1} = t_3 + t_{n-2}$$

This is because, if d is the common difference, we have $t_2 = t_1 + d$ and $t_{n-1} = t_n - d$, as well as having $t_3 = t_1 + 2d$ and $t_{n-2} = t_n - 2d$.

Since the sum of all n terms is 1000, using one formula for the sum of an arithmetic sequence gives

$$\begin{aligned} \frac{n}{2}(t_1 + t_n) &= 1000 \\ n(t_1 + t_n) &= 2000 \\ n(t_3 + t_{n-2}) &= 2000 \\ n(5 + 95) &= 2000 \end{aligned}$$

and so $n = 20$.

Solution 2

Suppose that the arithmetic sequence with n terms has first term a and common difference d.

Then $t_3 = a + 2d = 5$ and $t_{n-2} = a + (n-3)d = 95$.

Since the sum of the n terms equals 1000, then

$$\frac{n}{2}(2a + (n-1)d) = 1000$$

Adding the equations $a + 2d = 5$ and $a + (n-3)d = 95$, we obtain $2a + (n-1)d = 100$.

Substituting, we get $\dfrac{n}{2}(100) = 1000$ from which we obtain $n = 20$.

(b) Since the sum of a geometric sequence with first term a, common ratio r and 4 terms is $6 + 6\sqrt{2}$, then

$$a + ar + ar^2 + ar^3 = 6 + 6\sqrt{2}$$

Since the sum of a geometric sequence with first term a, common ratio r and 8 terms is $30 + 30\sqrt{2}$, then

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + ar^7 = 30 + 30\sqrt{2}$$

But

$$\begin{aligned} &a + ar + ar^2 + ar^3 + ar^4 + ar^5 + ar^6 + ar^7 \\ &\qquad = (a + ar + ar^2 + ar^3) + r^4(a + ar + ar^2 + ar^3) \\ &\qquad = (1 + r^4)(a + ar + ar^2 + ar^3) \end{aligned}$$

Therefore,

$$\begin{aligned}
30+30\sqrt{2} &= (1+r^4)(6+6\sqrt{2}) \\
\frac{30+30\sqrt{2}}{6+6\sqrt{2}} &= 1+r^4 \\
5 &= 1+r^4 \\
r^4 &= 4 \\
r^2 &= 2 \qquad (\text{since } r^2>0) \\
r &= \pm\sqrt{2}
\end{aligned}$$

If $r=\sqrt{2}$,

$$a+ar+ar^2+ar^3 = a+\sqrt{2}a+a(\sqrt{2})^2+a(\sqrt{2})^3 = a+\sqrt{2}a+2a+2\sqrt{2}a = a(3+3\sqrt{2})$$

Since $a+ar+ar^2+ar^3 = 6+6\sqrt{2}$, then $a(3+3\sqrt{2}) = 6+6\sqrt{2}$ and so $a = \dfrac{6+6\sqrt{2}}{3+3\sqrt{2}} = 2$.

If $r=-\sqrt{2}$,

$$a+ar+ar^2+ar^3 = a-\sqrt{2}a+a(-\sqrt{2})^2+a(-\sqrt{2})^3 = a-\sqrt{2}a+2a-2\sqrt{2}a = a(3-3\sqrt{2})$$

Since $a+ar+ar^2+ar^3 = 6+6\sqrt{2}$, then $a(3-3\sqrt{2}) = 6+6\sqrt{2}$ and so

$$a = \frac{6+6\sqrt{2}}{3-3\sqrt{2}} = \frac{2+2\sqrt{2}}{1-\sqrt{2}} = \frac{(2+2\sqrt{2})(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{2+2\sqrt{2}+2\sqrt{2}+4}{1-2} = -6-4\sqrt{2}$$

Therefore, the possible values of a are $a=2$ and $a=-6-4\sqrt{2}$.

An alternate way of arriving at the equation $1+r^4=5$ is to use the formula for the sum of a geometric sequence twice to obtain

$$\frac{a(1-r^4)}{1-r} = 6+6\sqrt{2} \qquad \frac{a(1-r^8)}{1-r} = 30+30\sqrt{2}$$

assuming that $r \neq 1$. (Can you explain why $r\neq 1$ and $r^4 \neq 1$ without knowing already that $r=\pm\sqrt{2}$?)

Dividing the second equation by the first, we obtain

$$\frac{a(1-r^8)}{1-r}\cdot\frac{1-r}{a(1-r^4)} = \frac{30+30\sqrt{2}}{6+6\sqrt{2}}$$

which gives

$$\frac{1-r^8}{1-r^4} = 5$$

Since $1-r^8 = (1+r^4)(1-r^4)$, we obtain $1+r^4=5$. We then can proceed as above.

7. (a) Victor stops when there are either 2 green balls on the table or 2 red balls on the table.
If the first 2 balls that Victor removes are the same colour, Victor will stop.
If the first 2 balls that Victor removes are different colours, Victor does not yet stop, but when he removes a third ball, its colour must match the colour of one of the first 2 balls and so Victor does stop.
Therefore, the probability that he stops with at least 1 red ball and 1 green ball on the table is equal to the probability that the first 2 balls that he removes are different colours.
Also, the probability that the first 2 balls that he removes are different colours is equal to 1 minus the probability that the first 2 balls that he removes are the same colour.
The probability that the first two balls that Victor draws are both green is $\frac{3}{7} \cdot \frac{2}{6}$ because for the first ball there are 7 balls in the bag, 3 of which are green and for the second ball there are 6 balls in the bag, 2 of which are green.
The probability that the first two balls that Victor draws are both red is $\frac{4}{7} \cdot \frac{3}{6}$ because for the first ball there are 7 balls in the bag, 4 of which are red and for the second ball there are 6 balls in the bag, 3 of which are red.
Thus, the probability that the first two balls that Victor removes are the same colour is

$$\frac{3}{7} \cdot \frac{2}{6} + \frac{4}{7} \cdot \frac{3}{6} = \frac{1}{7} + \frac{2}{7} = \frac{3}{7}$$

This means that the desired probability is $1 - \frac{3}{7} = \frac{4}{7}$.

(b) Using the definition of f, the following equations are equivalent:

$$\begin{aligned} f(a) &= 0 \\ 2a^2 - 3a + 1 &= 0 \\ (a-1)(2a-1) &= 0 \end{aligned}$$

Therefore, $f(a) = 0$ exactly when $a = 1$ or $a = \frac{1}{2}$.
Thus, $f(g(\sin\theta)) = 0$ exactly when $g(\sin\theta) = 1$ or $g(\sin\theta) = \frac{1}{2}$.

Using the definition of g,

- $g(b) = 1$ exactly when $\log_{\frac{1}{2}} b = 1$, which gives $b = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$, and
- $g(b) = 1/2$ exactly when $\log_{\frac{1}{2}} b = 1/2$, which gives $b = \left(\frac{1}{2}\right)^{1/2} = \frac{1}{\sqrt{2}}$.

Therefore, $f(g(\sin\theta)) = 0$ exactly when $\sin\theta = \frac{1}{2}$ or $\sin\theta = \frac{1}{\sqrt{2}}$.
Since $0 \le \theta \le 2\pi$, the solutions are $\theta = \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{1}{4}\pi, \frac{3}{4}\pi$.

8. (a) Suppose that the integers in the first row are, in order, a, b, c, d, e.
Using these, we calculate the integer in each of the boxes below the top row in terms of these variables, using the rule that each integer is the product of the integers in the two boxes above:

$$\begin{array}{ccccccccc} a & & b & & c & & d & & e \\ & ab & & bc & & cd & & de & \\ & & ab^2c & & bc^2d & & cd^2e & & \\ & & & ab^3c^3d & & bc^3d^3e & & & \\ & & & & ab^4c^6d^4e & & & & \end{array}$$

Therefore, $ab^4c^6d^4e = 9\,953\,280\,000$.

Next, we determine the prime factorization of the integer 9 953 280 000:

$$\begin{aligned} 9\,953\,280\,000 &= 10^4 \cdot 995\,328 \\ &= 2^4 \cdot 5^4 \cdot 2^3 \cdot 124\,416 \\ &= 2^7 \cdot 5^4 \cdot 2^3 \cdot 15\,552 \\ &= 2^{10} \cdot 5^4 \cdot 2^3 \cdot 1944 \\ &= 2^{13} \cdot 5^4 \cdot 2^3 \cdot 243 \\ &= 2^{16} \cdot 5^4 \cdot 3^5 \\ &= 2^{16} \cdot 3^5 \cdot 5^4 \end{aligned}$$

Thus, $ab^4c^6d^4e = 2^{16} \cdot 3^5 \cdot 5^4$.
Since the right side is not divisible by 7, none of a, b, c, d, e can equal 7.
Thus, a, b, c, d, e are five distinct integers chosen from $\{1, 2, 3, 4, 5, 6, 8\}$.
The only one of these integers divisible by 5 is 5 itself.
Since $2^{16} \cdot 3^5 \cdot 5^4$ includes exactly 4 factors of 5, then either $b = 5$ or $d = 5$. No other placement of the 5 can give exactly 4 factors of 5.

Case 1: $b = 5$
Here, $ac^6d^4e = 2^{16} \cdot 3^5$ and a, c, d, e are four distinct integers chosen from $\{1, 2, 3, 4, 6, 8\}$. Since ac^6d^4e includes exactly 5 factors of 3 and the possible values of a, c, d, e that are divisible by 3 are 3 and 6, then either $d = 3$ and one of a and e is 6, or $d = 6$ and one of a and e is 3. No other placements of the multiples of 3 can give exactly 5 factors of 3.

Case 1a: $b = 5$, $d = 3$, $a = 6$
Here, $a \cdot c^6 \cdot d^4 \cdot e = 6 \cdot c^6 \cdot 3^4 \cdot e = 2 \cdot 3^5 \cdot c^6 \cdot e$.
This gives $c^6e = 2^{15}$ and c and e are distinct integers from $\{1, 2, 4, 8\}$.
Trying the four possible values of c shows that $c = 4$ and $e = 8$ is the only solution in this case. Here, $(a, b, c, d, e) = (6, 5, 4, 3, 8)$.

Case 1b: $b = 5$, $d = 3$, $e = 6$ We obtain $(a, b, c, d, e) = (8, 5, 4, 3, 6)$.

Case 1c: $b = 5$, $d = 6$, $a = 3$
Here, $a \cdot c^6 \cdot d^4 \cdot e = 3 \cdot c^6 \cdot 6^4 \cdot e = 2^4 \cdot 3^5 \cdot c^6 \cdot e$.
This gives $c^6e = 2^{12}$ and c and e are distinct integers from $\{1, 2, 4, 8\}$.
Trying the four possible values of c shows that $c = 4$ and $e = 1$ is the only solution in this case. Here, $(a, b, c, d, e) = (3, 5, 4, 6, 1)$.

Case 1d: $b = 5$, $d = 6$, $e = 3$ We obtain $(a, b, c, d, e) = (1, 5, 4, 6, 3)$.

Case 2: $d = 5$: A similar analysis leads to 4 further quintuples (a, b, c, d, e).

Therefore, there are 8 ways in which the integers can be chosen and placed in the top row to obtain the desired integer in the bottom box.

(b) Let $N = \dfrac{(1!)(2!)(3!)\cdots(398!)(399!)(400!)}{200!}$.

For each integer k from 1 to 200, inclusive, we rewrite $(2k)!$ as $2k \cdot (2k-1)!$.
Therefore, $(2k-1)!(2k)! = (2k-1)! \cdot 2k \cdot (2k-1)! = 2k((2k-1)!)^2$.
(In particular, $(1!)(2!) = 2(1!)^2$, $(3!)(4!) = 4(3!)^2$, and so on.)
Thus,

$$N = \frac{2(1!)^2 \cdot 4(3!)^2 \cdot \cdots \cdot 398(397!)^2 \cdot 400(399!)^2}{200!}$$

Re-arranging the numerator of the expression, we obtain

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2 \cdot (2 \cdot 4 \cdot \cdots \cdot 398 \cdot 400)}{200!}$$

We can now re-write $2 \cdot 4 \cdot \cdots \cdot 398 \cdot 400$ as $(2 \cdot 1) \cdot (2 \cdot 2) \cdot \cdots \cdot (2 \cdot 199) \cdot (2 \cdot 200)$.
Since there are 200 sets of parentheses, we obtain

$$N = \frac{(1!)^2(3!)^2\cdots(397!)^2(399!)^2 \cdot 2^{200} \cdot (1 \cdot 2 \cdot \cdots \cdot 199 \cdot 200)}{200!}$$

Since $1 \cdot 2 \cdot \cdots \cdot 199 \cdot 200 = 200!$, we can conclude that

$$N = 2^{200}(1!)^2(3!)^2\cdots(397!)^2(399!)^2$$

Therefore,

$$\sqrt{N} = 2^{100}(1!)(3!)\cdots(397!)(399!)$$

which is a product of integers and thus an integer itself.
Since $\sqrt{N}$ is an integer, N is a perfect square, as required.

9. (a) When $a = 5$ and $b = 4$, we obtain $a^2 + b^2 - ab = 5^2 + 4^2 - 5 \cdot 4 = 21$.
Therefore, we want to find all pairs of integers (K, L) with $K^2 + 3L^2 = 21$.
If $L = 0$, then $L^2 = 0$, which gives $K^2 = 21$ which has no integer solutions.
If $L = \pm 1$, then $L^2 = 1$, which gives $K^2 = 18$ which has no integer solutions.
If $L = \pm 2$, then $L^2 = 4$, which gives $K^2 = 9$ which gives $K = \pm 3$.
If $L = \pm 3$, then $L^2 = 9$. Since $3L^2 = 27 > 21$, then there are no real solutions for K.
Similarly, if $L^2 > 9$, there are no real solutions for K.
Therefore, the solutions are $(K, L) = (3, 2), (-3, 2), (3, -2), (-3, -2)$.

(b) Suppose that K and L are integers.
Then

$$\begin{aligned}(K+L)^2 &+ (K-L)^2 - (K+L)(K-L) \\ &= (K^2 + 2KL + L^2) + (K^2 - 2KL + L^2) - (K^2 - L^2) \\ &= K^2 + 3L^2\end{aligned}$$

Therefore, the integers $a = K+L$ and $b = K-L$ satisfy the equation $K^2+3L^2 = a^2+b^2-ab$, and so for all integers K and L, there is at least one pair of integers (a, b) that satisfy the equation.

How could we come up with this? One way to do this would be trying some small values of K and L, calculating $K^2 + 3L^2$ and using this to make a guess, which can then be proven algebraically as above. In particular, here are some values:

K	L	$K^2 + 3L^2 = a^2 + b^2 - ab$	a	b
1	1	4	2	0
2	1	7	3	1
3	1	12	4	2
1	2	13	3	-1
2	2	16	4	0
3	2	21	5	1

The columns for a and b might lead us to guess that $a = K + L$ and $b = K - L$, which we proved above does in fact work.

(c) Suppose that a and b are integers.
If a is even, then $\frac{a}{2}$ is an integer and so

$$\left(\frac{a}{2} - b\right)^2 + 3\left(\frac{a}{2}\right)^2 = \frac{a^2}{4} - 2 \cdot \frac{a}{2} \cdot b + b^2 + \frac{3a^2}{4} = a^2 + b^2 - ab$$

Thus, if $K = \frac{a}{2} - b$ and $L = \frac{a}{2}$, we have $K^2 + 3L^2 = a^2 + b^2 - ab$.
If b is even, then $\frac{b}{2}$ is an integer and so a similar algebraic argument shows that

$$\left(\frac{b}{2} - a\right)^2 + 3\left(\frac{b}{2}\right)^2 = a^2 + b^2 - ab$$

and so if $K = \frac{b}{2} - a$ and $L = \frac{b}{2}$, we have $K^2 + 3L^2 = a^2 + b^2 - ab$.

If a and b are both odd, then $a + b$ and $a - b$ are both even, which means that $\frac{a+b}{2}$ and $\frac{a-b}{2}$ are both integers, and so

$$\left(\frac{a+b}{2}\right)^2 + 3\left(\frac{a-b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4} + \frac{3a^2 - 6ab + 3b^2}{4} = \frac{4a^2 + 4b^2 - 4ab}{4} = a^2 + b^2 - ab$$

Thus, if $K = \frac{a+b}{2}$ and $L = \frac{a-b}{2}$, we have $K^2 + 3L^2 = a^2 + b^2 - ab$.
Therefore, in all cases, for all integers a and b, there is at least one pair of integers (K, L) with $K^2 + 3L^2 = a^2 + b^2 - ab$.

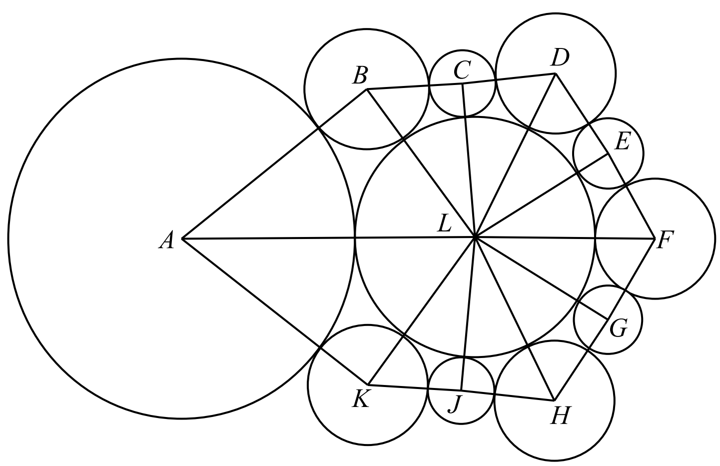
As in (b), trying some small cases might help us make a guess of possible expressions for K and L in terms of a and b:

a	b	$K^2 + 3L^2 = a^2 + b^2 - ab$	K	L
1	1	1	1	0
2	1	3	0	1
3	1	7	2	1
4	1	13	1	2
1	2	3	0	1
2	2	4	1	1
3	2	7	2	1
4	2	12	3	1
5	3	19	4	1

While there might not initially seem to be useful patterns here, re-arranging the rows and adding some duplicates might help show a pattern:

a	b	$K^2+3L^2=a^2+b^2-ab$	K	L
2	1	3	0	1
4	1	13	1	2
2	2	4	1	1
4	2	12	3	1
1	2	3	0	1
3	2	7	2	1
4	2	12	3	1
1	1	1	1	0
3	1	7	2	1
5	3	19	4	1

10. (a) We label the centres of the outer circles, starting with the circle labelled Z and proceeding clockwise, as A, B, C, D, E, F, G, H, J, and K, and the centre of the circle labelled Y as L.



Join L to each of A, B, C, D, E, F, G, H, J, and K. Join A to B, B to C, C to D, D to E, E to F, F to G, G to H, H to J, J to K, and K to A.

When two circles are tangent, the distance between their centres equals the sum of their radii.

Thus,

$$\begin{aligned} BC = CD = DE = EF = FG = GH = HJ = JK &= 2+1=3 \\ BL = DL = FL = HL = KL &= 2+4=6 \\ CL = EL = GL = JL &= 1+4=5 \\ AB = AK &= r+2 \\ AL &= r+4 \end{aligned}$$

By side-side-side congruence, the following triangles are congruent:

$$\triangle BLC, \triangle DLC, \triangle DLE, \triangle FLE, \triangle FLG, \triangle HLG, \triangle HLJ, \triangle KLJ$$

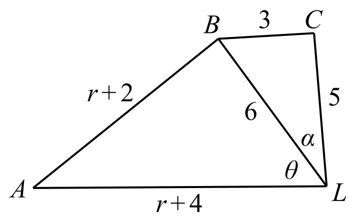
Similarly, $\triangle ALB$ and $\triangle ALK$ are congruent by side-side-side.

Let $\angle ALB = \theta$ and let $\angle BLC = \alpha$.

By congruent triangles, $\angle ALK = \theta$ and

$$\angle BLC = \angle DLC = \angle DLE = \angle FLE = \angle FLG = \angle HLG = \angle HLJ = \angle KLJ = \alpha$$

The angles around L add to 360° and so $2\theta + 8\alpha = 360^\circ$ which gives $\theta + 4\alpha = 180^\circ$ and so $\theta = 180^\circ - 4\alpha$.
Since $\theta = 180^\circ - 4\alpha$, then $\cos\theta = \cos(180^\circ - 4\alpha) = -\cos 4\alpha$.
Consider $\triangle ALB$ and $\triangle BLC$.



By the cosine law in $\triangle ALB$,

$$\begin{aligned} AB^2 &= AL^2 + BL^2 - 2 \cdot AL \cdot BL \cdot \cos\theta \\ (r+2)^2 &= (r+4)^2 + 6^2 - 2(r+4)(6)\cos\theta \\ 12(r+4)\cos\theta &= r^2 + 8r + 16 + 36 - r^2 - 4r - 4 \\ \cos\theta &= \frac{4r+48}{12(r+4)} \\ \cos\theta &= \frac{r+12}{3r+12} \end{aligned}$$

By the cosine law in $\triangle BLC$,

$$\begin{aligned} BC^2 &= BL^2 + CL^2 - 2 \cdot BL \cdot CL \cdot \cos\alpha \\ 3^2 &= 6^2 + 5^2 - 2(6)(5)\cos\alpha \\ 60\cos\alpha &= 36 + 25 - 9 \\ \cos\alpha &= \frac{52}{60} \\ \cos\alpha &= \frac{13}{15} \end{aligned}$$

Since $\cos\alpha = \dfrac{13}{15}$, then

$$\begin{aligned} \cos 2\alpha &= 2\cos^2\alpha - 1 \\ &= 2 \cdot \frac{169}{225} - 1 \\ &= \frac{338}{225} - \frac{225}{225} \\ &= \frac{113}{225} \end{aligned}$$

and

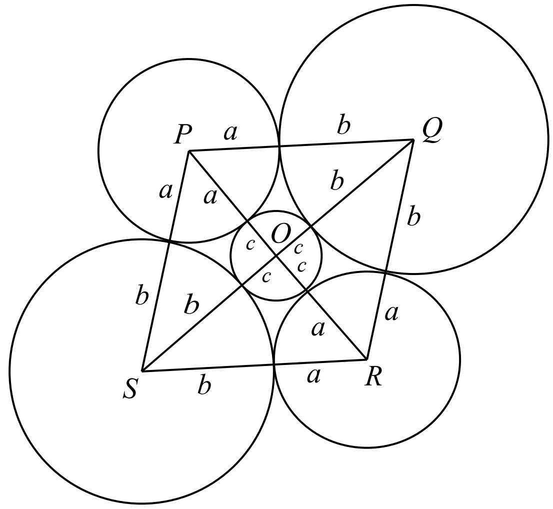
$$\begin{aligned}
\cos 4\alpha &= 2\cos^2 2\alpha - 1 \\
&= 2 \cdot \frac{113^2}{225^2} - 1 \\
&= \frac{25\,538}{50\,625} - \frac{50\,625}{50\,625} \\
&= -\frac{25\,087}{50\,625}
\end{aligned}$$

Finally,

$$\begin{aligned}
\cos\theta &= -\cos 4\alpha \\
\frac{r+12}{3r+12} &= \frac{25\,087}{50\,625} \\
\frac{r+12}{r+4} &= \frac{25\,087}{16\,875} \\
\frac{(r+4)+8}{r+4} &= \frac{25\,087}{16\,875} \\
1 + \frac{8}{r+4} &= \frac{25\,087}{16\,875} \\
\frac{8}{r+4} &= \frac{8212}{16\,875} \\
\frac{2}{r+4} &= \frac{2053}{16\,875} \\
\frac{r+4}{2} &= \frac{16\,875}{2053} \\
r+4 &= \frac{33\,750}{2053} \\
r &= \frac{25\,538}{2053}
\end{aligned}$$

Therefore, the positive integers $s = 25\,538$ and $t = 2053$ satisfy the required conditions.

(b) Let the centre of the middle circle be O, and the centres of the other circles be P, Q, R, and S, as shown.
Join O to P, Q, R, and S, and join P to Q, Q to R, R to S, and S to P.



Using a similar argument as in (a), we see that

$$\begin{aligned}OP = OR &= a+c \\ OQ = OS &= b+c \\ PQ = QR = RS = SP &= a+b\end{aligned}$$

By side-side-side congruence, $\triangle OPQ$, $\triangle OPS$, $\triangle ORQ$, and $\triangle ORS$ are congruent.
This means that $\angle POQ = \angle POS = \angle ROQ = \angle ROS$.
Since $\angle POQ + \angle POS + \angle ROQ + \angle ROS = 360^\circ$ (these angles surround O), then

$$\angle POQ = \frac{1}{4} \cdot 360^\circ = 90^\circ$$

This means that $\triangle OPQ$ is right-angled at O.
By the Pythagorean Theorem, $PQ^2 = OP^2 + OQ^2$ and so $(a+b)^2 = (a+c)^2 + (b+c)^2$.
Manipulating algebraically, the following equations are equivalent:

$$\begin{aligned}(a+b)^2 &= (a+c)^2 + (b+c)^2 \\ a^2 + 2ab + b^2 &= a^2 + 2ac + c^2 + b^2 + 2bc + c^2 \\ 2ab &= 2ac + 2bc + 2c^2 \\ ab &= ac + bc + c^2 \\ ab - ac - bc &= c^2 \\ ab - ac - bc + c^2 &= 2c^2 \\ a(b-c) - c(b-c) &= 2c^2 \\ (a-c)(b-c) &= 2c^2\end{aligned}$$

Therefore, if a, b and c are real numbers for which the diagram can be constructed, then a, b and c satisfy this last equation.
Also, if real numbers a, b and c satisfy the final equation, then $(a+b)^2 = (a+c)^2 + (b+c)^2$ (because these equations were equivalent) and so the triangle with side lengths $a+b$, $a+c$ and $b+c$ is right-angled with hypotenuse $a+b$ (because the Pythagorean Theorem works in both directions), which means that four such triangles can be assembled to form a rhombus $PQRS$ with side lengths $a+b$ and centre O, which means that the five circles can be drawn by marking off the appropriate lengths a, b and c and drawing the circles as in the original diagram.

In other words, the diagram can be drawn exactly when $(a-c)(b-c) = 2c^2$.

Suppose that c is a fixed positive integer.
Determining the value of $f(c)$ is thus equivalent to counting the number of pairs of positive integers (a, b) with $c < a < b$ and $(a-c)(b-c) = 2c^2$.
Since a and b are integers with $a > c$ and $b > c$, the integers $a-c$ and $b-c$ are positive and form a positive divisor pair of the integer $2c^2$.
Since $a < b$, we have $a-c < b-c$ and so $a-c$ and $b-c$ are distinct integers.
Also, since $c > 0$, $\sqrt{2c^2} = \sqrt{2}c$ which is not an integer since c is an integer, which means that $2c^2$ is not a perfect square.
Therefore, every pair (a, b) corresponds to a positive divisor pair of $2c^2$ (namely, $a-c$ and $b-c$).
Similarly, every divisor pair e and g of $2c^2$ with $e > g$ gives a pair of positive integers (a, b) with $a < b$ by setting $a = e + c$ and $b = g + c$.
In other words, $f(c)$ is exactly the number of positive divisor pairs of $2c^2$. (Again, we note that $2c^2$ is not a perfect square.)
Therefore, we want to determine all positive integers c for which the integer $2c^2$ has an even number of divisor pairs, which means that we want to determine all positive integers c for which the number of positive divisors of $2c^2$ is a multiple of 4 (because each positive divisor pair corresponds to 2 positive divisors and 2 times an even integer is a multiple of 4).

Suppose that the prime factorization of c is

$$c = 2^r p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

for some integer $k \geq 0$, integer $r \geq 0$, odd prime numbers $p_1, p_2, \ldots, p_k$, and positive integers $e_1, e_2, \ldots, e_k$.
Then

$$2c^2 = 2^{2r+1} p_1^{2e_1} p_2^{2e_2} \cdots p_k^{2e_k}$$

and so $2c^2$ has

$$(2r+2)(2e_1+1)(2e_2+1)\cdots(2e_k+1)$$

positive divisors.
The first factor in this product is even and each factor after the first is odd.
Therefore, this product is a multiple of 4 exactly when $2r+2$ is a multiple of 4.
This is true exactly when $2r+2 = 4s$ for some positive integer s and so $2r = 4s-2$ or $r = 2s-1$.
In other words, the number of positive divisors of $2c^2$ is a multiple of 4 exactly when r is an odd integer.

Finally, this means that the positive integers for which $f(c)$ are even are exactly those positive integers that have exactly an odd number of factors of 2 in their prime factorization.

The CENTRE for EDUCATION
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2020 Euclid Contest

Tuesday, April 7, 2020
(in North America and South America)

Wednesday, April 8, 2020
(outside of North America and South America)

Solutions

1. (a) *Solution 1*

If $x \neq -2$, then $\dfrac{3x+6}{x+2} = \dfrac{3(x+2)}{x+2} = 3$.

In other words, for every $x \neq -2$, the expression is equal to 3.

Therefore, when $x = 11$, we get $\dfrac{3x+6}{x+2} = 3$.

Solution 2

When $x = 11$, we obtain $\dfrac{3x+6}{x+2} = \dfrac{3(11)+6}{11+2} = \dfrac{39}{13} = 3$.

(b) *Solution 1*

The point at which a line crosses the y-axis has x-coordinate 0.
Because A has x-coordinate -1 and B has x-coordinate 1, then the midpoint of AB is on the y-axis and is on the line through A and B, so is the point at which this line crosses the x-axis.
The midpoint of $A(-1,5)$ and $B(1,7)$ is $\left(\frac{1}{2}(-1+1), \frac{1}{2}(5+7)\right)$ or $(0,6)$.
Therefore, the line that passes through $A(-1,5)$ and $B(1,7)$ has y-intercept 6.

Solution 2

The line through $A(-1,5)$ and $B(1,7)$ has slope $\dfrac{7-5}{1-(-1)} = \dfrac{2}{2} = 1$.
Since the line passes through $B(1,7)$, its equation can be written as $y - 7 = 1(x-1)$ or $y = x + 6$.
The line with equation $y = x + 6$ has y-intercept 6.

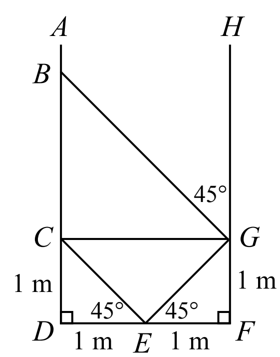
(c) First, we find the coordinates of the point at which the lines with equations $y = 3x + 7$ and $y = x + 9$ intersect.
Equating values of y, we obtain $3x + 7 = x + 9$ and so $2x = 2$ or $x = 1$.
When $x = 1$, we get $y = x + 9 = 10$.
Thus, these two lines intersect at $(1,10)$.
Since all three lines pass through the same point, the line with equation $y = mx + 17$ passes through $(1,10)$.
Therefore, $10 = m \cdot 1 + 17$ which gives $m = 10 - 17 = -7$.

2. (a) Suppose that m has hundreds digit a, tens digit b, and ones (units) digit c.
From the given information, a, b and c are distinct, each of a, b and c is less than 10, $a = bc$, and c is odd (since m is odd).
The integer $m = 623$ satisfies all of these conditions. Since we are told there is only one such number, then 623 must be the only answer.
Why is this the only possible value of m?
We note that we cannot have $b = 1$ or $c = 1$, otherwise $a = c$ or $a = b$.
Thus, $b \geq 2$ and $c \geq 2$.
Since $c \geq 2$ and c is odd, then c can equal 3, 5, 7, or 9.
Since $b \geq 2$ and $a = bc$, then if c equals 5, 7 or 9, a would be larger than 10, which is not possible.
Thus, $c = 3$.
Since $b \geq 2$ and $b \neq c$, then $b = 2$ or $b \geq 4$.
If $b \geq 4$ and $c = 3$, then $a > 10$, which is not possible.
Therefore, we must have $c = 3$ and $b = 2$, which gives $a = 6$.

(b) Since Eleanor has 100 marbles which are black and gold in the ratio $1:4$, then $\frac{1}{5}$ of her marbles are black, which means that she has $\frac{1}{5} \cdot 100 = 20$ black marbles.
When more gold marbles are added, the ratio of black to gold is $1:6$, which means that she has $6 \cdot 20 = 120$ gold marbles.
Eleanor now has $20 + 120 = 140$ marbles, which means that she added $140 - 100 = 40$ gold marbles.

(c) First, we see that $\dfrac{n^2+n+15}{n} = \dfrac{n^2}{n} + \dfrac{n}{n} + \dfrac{15}{n} = n + 1 + \dfrac{15}{n}$.
This means that $\dfrac{n^2+n+15}{n}$ is an integer exactly when $n+1+\dfrac{15}{n}$ is an integer.
Since $n+1$ is an integer, then $\dfrac{n^2+n+15}{n}$ is an integer exactly when $\dfrac{15}{n}$ is an integer.
The expression $\dfrac{15}{n}$ is an integer exactly when n is a divisor of 15.
Since n is a positive integer, then the possible values of n are 1, 3, 5, and 15.

3. (a) First, we note that a triangle with one right angle and one angle with measure 45° is isosceles.
This is because the measure of the third angle equals $180^\circ - 90^\circ - 45^\circ = 45^\circ$ which means that the triangle has two equal angles.
In particular, $\triangle CDE$ is isosceles with $CD = DE$ and $\triangle EFG$ is isosceles with $EF = FG$.
Since $DE = EF = 1$ m, then $CD = FG = 1$ m.
Join C to G.



Consider quadrilateral $CDFG$. Since the angles at D and F are right angles and since $CD = GF$, it must be the case that $CDFG$ is a rectangle.
This means that $CG = DF = 2$ m and that the angles at C and G are right angles.
Since $\angle CGF = 90^\circ$ and $\angle DCG = 90^\circ$, then $\angle BGC = 180^\circ - 90^\circ - 45^\circ = 45^\circ$ and $\angle BCG = 90^\circ$.
This means that $\triangle BCG$ is also isosceles with $BC = CG = 2$ m.
Finally, $BD = BC + CD = 2\text{ m} + 1\text{ m} = 3$ m.

(b) We apply the process two more times:

	x	y
Before Step 1	24	3
After Step 1	27	3
After Step 2	81	3
After Step 3	81	4

	x	y
Before Step 1	81	4
After Step 1	85	4
After Step 2	340	4
After Step 3	340	5

Therefore, the final value of x is 340.

(c) The parabola with equation $y = kx^2 + 6x + k$ has two distinct x-intercepts exactly when the discriminant of the quadratic equation $kx^2 + 6x + k = 0$ is positive.
Here, the disciminant equals $\Delta = 6^2 - 4 \cdot k \cdot k = 36 - 4k^2$.
The inequality $36 - 4k^2 > 0$ is equivalent to $k^2 < 9$.
Since k is an integer and $k \neq 0$, then k can equal $-2, -1, 1, 2$.
(If $k \geq 3$ or $k \leq -3$, we get $k^2 \geq 9$ so no values of k in these ranges give the desired result.)

4. (a) Since $\dfrac{a}{b} < \dfrac{4}{7}$ and $\dfrac{4}{7} < 1$, then $\dfrac{a}{b} < 1$.
Since a and b are positive integers, then $a < b$.
Since the difference between a and b is 15 and $a < b$, then $b = a + 15$.
Therefore, we have $\dfrac{5}{9} < \dfrac{a}{a+15} < \dfrac{4}{7}$.
We multiply both sides of the left inequality by $9(a + 15)$ (which is positive) to obtain $5(a + 15) < 9a$ from which we get $5a + 75 < 9a$ and so $4a > 75$.
From this, we see that $a > \dfrac{75}{4} = 18.75$.
Since a is an integer, then $a \geq 19$.
We multiply both sides of the right inequality by $7(a + 15)$ (which is positive) to obtain $7a < 4(a + 15)$ from which we get $7a < 4a + 60$ and so $3a < 60$.
From this, we see that $a < 20$.
Since a is an integer, then $a \leq 19$.
Since $a \geq 19$ and $a \leq 19$, then $a = 19$, which means that $\dfrac{a}{b} = \dfrac{19}{34}$.

(b) The first 6 terms of a geometric sequence with first term 10 and common ratio $\dfrac{1}{2}$ are $10, 5, \dfrac{5}{2}, \dfrac{5}{4}, \dfrac{5}{8}, \dfrac{5}{16}$.
Here, the ratio of its 6th term to its 4th term is $\dfrac{5/16}{5/4}$ which equals $\dfrac{1}{4}$. (We could have determined this without writing out the sequence, since moving from the 4th term to the 6th involves multiplying by $\dfrac{1}{2}$ twice.)
The first 6 terms of an arithmetic sequence with first term 10 and common difference d are $10, 10 + d, 10 + 2d, 10 + 3d, 10 + 4d, 10 + 5d$.
Here, the ratio of the 6th term to the 4th term is $\dfrac{10 + 5d}{10 + 3d}$.
Since these ratios are equal, then $\dfrac{10 + 5d}{10 + 3d} = \dfrac{1}{4}$, which gives $4(10 + 5d) = 10 + 3d$ and so $40 + 20d = 10 + 3d$ or $17d = -30$ and so $d = -\dfrac{30}{17}$.

5. (a) Let $a = f(20)$. Then $f(f(20)) = f(a)$.
To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$.
By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \leq p \leq 30$.
The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$.
Thus, $f(f(20)) = f(a) = f(2)$.
By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \leq p \leq 12$.
The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5.
Therefore, $f(f(20)) = 5$.

(b) Since $(x-1)(y-2) = 0$, then $x = 1$ or $y = 2$.
Suppose that $x = 1$. In this case, the remaining equations become:

$$\begin{aligned} (1-3)(z+2) &= 0 \\ 1 + yz &= 9 \end{aligned}$$

or

$$\begin{aligned} -2(z+2) &= 0 \\ yz &= 8 \end{aligned}$$

From the first of these equations, $z = -2$.
From the second of these equations, $y(-2) = 8$ and so $y = -4$.
Therefore, if $x = 1$, the only solution is $(x, y, z) = (1, -4, -2)$.
Suppose that $y = 2$. In this case, the remaining equations become:

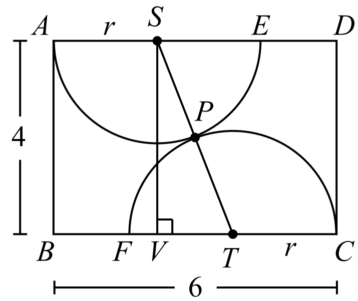
$$\begin{aligned} (x-3)(z+2) &= 0 \\ x + 2z &= 9 \end{aligned}$$

From the first equation $x = 3$ or $z = -2$.
If $x = 3$, then $3 + 2z = 9$ and so $z = 3$.
If $z = -2$, then $x + 2(-2) = 9$ and so $x = 13$.
Therefore, if $y = 2$, the solutions are $(x, y, z) = (3, 2, 3)$ and $(x, y, z) = (13, 2, -2)$.
In summary, the solutions to the system of equations are

$$(x, y, z) = (1, -4, -2), (3, 2, 3), (13, 2, -2)$$

We can check by substitution that each of these triples does indeed satisfy each of the equations.

6. (a) Draw a perpendicular from S to V on BC.
Since $ASVB$ is a quadrilateral with three right angles, then it has four right angles and so is a rectangle.
Therefore, $BV = AS = r$, since AS is a radius of the top semi-circle, and $SV = AB = 4$.
Join S and T to P. Since the two semi-circles are tangent at P, then SPT is a straight line, which means that $ST = SP + PT = r + r = 2r$.



Consider right-angled $\triangle SVT$. We have $SV = 4$ and $ST = 2r$.
Also, $VT = BC - BV - TC = 6 - r - r = 6 - 2r$.
By the Pythagorean Theorem,

$$\begin{aligned} SV^2 + VT^2 &= ST^2 \\ 4^2 + (6-2r)^2 &= (2r)^2 \\ 16 + 36 - 24r + 4r^2 &= 4r^2 \\ 52 &= 24r \end{aligned}$$

Thus, $r = \frac{52}{24} = \frac{13}{6}$.

(b) Since $\triangle ABE$ is right-angled at A and is isosceles with $AB = AE = 7\sqrt{2}$, then $\triangle ABE$ is a 45°-45°-90° triangle, which means that $\angle ABE = 45^\circ$ and $BE = \sqrt{2}AB = \sqrt{2}\cdot 7\sqrt{2} = 14$.
Since $\triangle BCD$ is right-angled at C with $\dfrac{DB}{DC} = \dfrac{8x}{4x} = 2$, then $\triangle BCD$ is a 30°-60°-90° triangle, which means that $\angle DBC = 30^\circ$.
Since $\angle ABC = 135^\circ$, then $\angle EBD = \angle ABC - \angle ABE - \angle DBC = 135^\circ - 45^\circ - 30^\circ = 60^\circ$.
Now consider $\triangle EBD$. We have $EB = 14$, $BD = 8x$, $DE = 8x - 6$, and $\angle EBD = 60^\circ$.
Using the cosine law, we obtain the following equivalent equations:

$$\begin{aligned} DE^2 &= EB^2 + BD^2 - 2\cdot EB\cdot BD\cdot\cos(\angle EBD) \\ (8x-6)^2 &= 14^2 + (8x)^2 - 2(14)(8x)\cos(60^\circ) \\ 64x^2 - 96x + 36 &= 196 + 64x^2 - 2(14)(8x)\cdot\tfrac{1}{2} \\ -96x &= 160 - 14(8x) \\ 112x - 96x &= 160 \\ 16x &= 160 \\ x &= 10 \end{aligned}$$

Therefore, the only possible value of x is $x = 10$.

7. (a) *Solution 1*

Since the function g is linear and has positive slope, then it is one-to-one and so invertible. This means that $g^{-1}(g(a)) = a$ for every real number a and $g(g^{-1}(b)) = b$ for every real number b.
Therefore, $g(f(g^{-1}(g(a)))) = g(f(a))$ for every real number a.
This means that

$$\begin{aligned} g(f(a)) &= g(f(g^{-1}(g(a)))) \\ &= 2(g(a))^2 + 16g(a) + 26 \\ &= 2(2a-4)^2 + 16(2a-4) + 26 \\ &= 2(4a^2 - 16a + 16) + 32a - 64 + 26 \\ &= 8a^2 - 6 \end{aligned}$$

Furthermore, if $b = f(a)$, then $g^{-1}(g(f(a))) = g^{-1}(g(b)) = b = f(a)$.
Therefore,

$$f(a) = g^{-1}(g(f(a))) = g^{-1}(8a^2 - 6)$$

Since $g(x) = 2x - 4$, then $y = 2g^{-1}(y) - 4$ and so $g^{-1}(y) = \frac{1}{2}y + 2$.
Therefore,

$$f(a) = \tfrac{1}{2}(8a^2 - 6) + 2 = 4a^2 - 1$$

and so $f(\pi) = 4\pi^2 - 1$.

Solution 2

Since the function g is linear and has positive slope, then it is one-to-one and so invertible.
To find a formula for $g^{-1}(y)$, we start with the equation $g(x) = 2x - 4$, convert to $y = 2g^{-1}(y) - 4$ and then solve for $g^{-1}(y)$ to obtain $2g^{-1}(y) = y + 4$ and so $g^{-1}(y) = \dfrac{y+4}{2}$.
We are given that $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$.
We can apply the function g^{-1} to both sides to obtain successively:

$$\begin{aligned} f(g^{-1}(x)) &= g^{-1}(2x^2 + 16x + 26) \\ f(g^{-1}(x)) &= \frac{(2x^2 + 16x + 26) + 4}{2} \qquad \text{(knowing a formula for } g^{-1}\text{)} \\ f(g^{-1}(x)) &= x^2 + 8x + 15 \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 15 \qquad \text{(knowing a formula for } g^{-1}\text{)} \\ f\left(\frac{x+4}{2}\right) &= x^2 + 8x + 16 - 1 \\ f\left(\frac{x+4}{2}\right) &= (x+4)^2 - 1 \end{aligned}$$

We want to determine the value of $f(\pi)$.
Thus, we can replace $\dfrac{x+4}{2}$ with π, which is equivalent to replacing $x + 4$ with 2π.
Thus, $f(\pi) = (2\pi)^2 - 1 = 4\pi^2 - 1$.

(b) *Solution 1*

Using logarithm laws, the given equations are equivalent to

$$\begin{aligned}\log_2(\sin x) + \log_2(\cos y) &= -\frac{3}{2} \\ \log_2(\sin x) - \log_2(\cos y) &= \frac{1}{2}\end{aligned}$$

Adding these two equations, we obtain $2\log_2(\sin x) = -1$ which gives $\log_2(\sin x) = -\frac{1}{2}$ and so $\sin x = 2^{-1/2} = \frac{1}{2^{1/2}} = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, then $x = 45^\circ$ or $x = 135^\circ$.

Since $\log_2(\sin x) + \log_2(\cos y) = -\frac{3}{2}$ and $\log_2(\sin x) = -\frac{1}{2}$, then $\log_2(\cos y) = -1$, which gives $\cos y = 2^{-1} = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

Solution 2

First, we note that $2^{1/2} = \sqrt{2}$ and $2^{-3/2} = \frac{1}{2^{3/2}} = \frac{1}{2^1 2^{1/2}} = \frac{1}{2\sqrt{2}}$.

From the given equations, we obtain

$$\begin{aligned}\sin x \cos y &= 2^{-3/2} = \frac{1}{2\sqrt{2}} \\ \frac{\sin x}{\cos y} &= 2^{1/2} = \sqrt{2}\end{aligned}$$

Multiplying these two equations together, we obtain $(\sin x)^2 = \frac{1}{2}$ which gives $\sin x = \pm\frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, it must be the case that $\sin x \geq 0$ and so $\sin x = \frac{1}{\sqrt{2}}$.

Since $0^\circ \leq x < 180^\circ$, we obtain $x = 45^\circ$ or $x = 135^\circ$.

Since $\sin x \cos y = \frac{1}{2\sqrt{2}}$ and $\sin x = \frac{1}{\sqrt{2}}$, we obtain $\cos y = \frac{1}{2}$.

Since $0^\circ \leq y < 180^\circ$, then $y = 60^\circ$.

Therefore, $(x, y) = (45^\circ, 60^\circ)$ or $(x, y) = (135^\circ, 60^\circ)$.

8. (a) *Solution 1*

Let x be the probability that Bianca wins the tournament.

Because Alain, Bianca and Chen are equally matched and because their roles in the tournament are identical, then the probability that each of them wins will be the same.

Thus, the probability that Alain wins the tournament is x and the probability that Chen wins the tournament is x.

Let y be the probability that Dave wins the tournament.

Since exactly one of Alain, Bianca, Chen, and Dave wins the tournament, then $3x+y=1$ and so $x=\dfrac{1-y}{3}$. We can calculate y in terms of p.

In order for Dave to win the tournament, he needs to win two matches.

No matter who Dave plays, his probability of winning each match is p.

Thus, the probability that he wins his two consecutive matches is p^2 and so the probability that he wins the tournament is $y=p^2$.

Thus, the probability that Bianca wins the tournament is $\dfrac{1-p^2}{3}$.

(We could rewrite this as $\dfrac{-p^2+0p+1}{3}$ to match the desired form.)

Solution 2

Let x be the probability that Bianca wins the tournament.

There are three possible pairings for the first two matches:

(i) Bianca versus Alain, and Chen versus Dave

(ii) Bianca versus Chen, and Alain versus Dave

(iii) Bianca versus Dave, and Alain versus Chen

Each of these three pairings occurs with probability $\frac{1}{3}$.

In (i), Bianca wins either if Bianca beats Alain, Chen beats Dave, and Bianca beats Chen, or if Bianca beats Alain, Dave beats Chen, and Bianca beats Dave.

Since Bianca beats Alain with probability $\frac{1}{2}$, Chen beats Dave with probability $1-p$, and Bianca beats Chen with probability $\frac{1}{2}$, then the first possibility has probability $\frac{1}{2}\cdot(1-p)\cdot\frac{1}{2}$.

Since Bianca beats Alain with probability $\frac{1}{2}$, Dave beats Chen with probability p, and Bianca beats Dave with probability $1-p$, then the second possibility has probability $\frac{1}{2}\cdot p\cdot(1-p)$.

Therefore, the probability of Bianca winning, given that possibility (i) occurs, is $\frac{1}{2}\cdot(1-p)\cdot\frac{1}{2}+\frac{1}{2}\cdot p\cdot(1-p)$.

In (ii), Bianca wins either if Bianca beats Chen, Alain beats Dave, and Bianca beats Alain, or if Bianca beats Alain, Dave beats Alain, and Bianca beats Dave.

The combined probability of these is $\frac{1}{2}\cdot(1-p)\cdot\frac{1}{2}+\frac{1}{2}\cdot p\cdot(1-p)$.

In (iii), Bianca wins either if Bianca beats Dave, Alain beats Chen, and Bianca beats Alain, or if Bianca beats Dave, Chen beats Alain, and Bianca beats Chen.

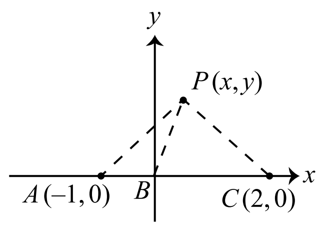
The combined probability of these is $(1-p)\cdot\frac{1}{2}\cdot\frac{1}{2}+(1-p)\cdot\frac{1}{2}\cdot\frac{1}{2}$.

Therefore,

$$\begin{aligned} x &= \tfrac{1}{3}\left(\tfrac{1}{4}(1-p)+\tfrac{1}{2}p(1-p)+\tfrac{1}{4}(1-p)+\tfrac{1}{2}p(1-p)+\tfrac{1}{4}(1-p)+\tfrac{1}{4}(1-p)\right) \\ &= \tfrac{1}{3}(p(1-p)+(1-p)) \\ &= \tfrac{1}{3}(p-p^2+1-p) \end{aligned}$$

Thus, the probability that Bianca wins the tournament is $\dfrac{1-p^2}{3}$.

(b) Throughout this solution, we will mostly not include units, but will assume that all lengths are in kilometres, all times are in seconds, and all speeds are in kilometres per second.
We place the points in the coordinate plane with B at $(0,0)$, A on the negative x-axis, and C on the positive x-axis.
We put A at $(-1,0)$ and C at $(2,0)$.
Suppose that P has coordinates (x,y) and that the distance from P to B is d km.



Since the sound arrives at A $\frac{1}{2}$ s after arriving at B and sound travels at $\frac{1}{3}$ km/s, then A is $(\frac{1}{2}\text{ s})\cdot(\frac{1}{3}\text{ km/s})=\frac{1}{6}$ km farther from P than B is.
Thus, the distance from P to A is $(d+\frac{1}{6})$ km.
Since the sound arrives at C an additional 1 second later, then C is an additional $\frac{1}{3}$ km farther, and so is $(d+\frac{1}{6})$ km $+$ $(\frac{1}{3}$ km$)=(d+\frac{1}{2})$ km from P.
Since the distance from P to B is d km, then $(x-0)^2+(y-0)^2=d^2$.
Since the distance from P to A is $(d+\frac{1}{6})$ km, then $(x+1)^2+(y-0)^2=(d+\frac{1}{6})^2$.
Since the distance from P to C is $(d+\frac{1}{2})$ km, then $(x-2)^2+(y-0)^2=(d+\frac{1}{2})^2$.
When these equations are expanded and simplified, we obtain

$$\begin{aligned} x^2+y^2 &= d^2 \\ x^2+2x+1+y^2 &= d^2+\tfrac{1}{3}d+\tfrac{1}{36} \\ x^2-4x+4+y^2 &= d^2+d+\tfrac{1}{4} \end{aligned}$$

Subtracting the first equation from the second, we obtain

$$2x+1=\tfrac{1}{3}d+\tfrac{1}{36}$$

Subtracting the first equation from the third, we obtain

$$-4x+4=d+\tfrac{1}{4}$$

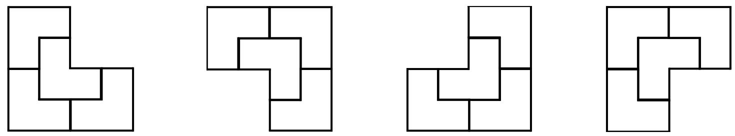
Therefore,

$$\begin{aligned} 2(2x+1)+(-4x+4) &= 2(\tfrac{1}{3}d+\tfrac{1}{36})+(d+\tfrac{1}{4}) \\ 6 &= \tfrac{2}{3}d+\tfrac{1}{18}+d+\tfrac{1}{4} \\ 216 &= 24d+2+36d+9 \qquad \text{(multiplying by 36)} \\ 205 &= 60d \\ d &= \tfrac{41}{12} \end{aligned}$$

Therefore, the distance from B to P is $\frac{41}{12}$ km.

9. (a) After each round, each L shape is divided into 4 smaller L shapes.
This means that the number of L shapes increases by a factor of 4 after each round.
After 1 round, there are 4 L shapes.
After 2 rounds, there are $4^2 = 16$ L's of the smallest size.
After 3 rounds, there are $4^3 = 64$ L's of the smallest size.

(b) There are four orientations of L shapes of a given size: , , , .
When an L of each orientation is subdivided, the following figures are obtained:



From these figures, we can see that after each subsequent round,

- Each produces 2 , 0 , 1 , and 1 of the smallest size.
- Each produces 0 , 2 , 1 , and 1 .
- Each produces 1 , 1 , 2 , and 0 .
- Each produces 1 , 1 , 0 , and 2 .

After 1 round, there are 2 , 0 , 1 , and 1 .
After 2 rounds, the number of L's of each orientation are as follows:

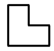
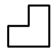
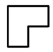
- : $2 \cdot 2 + 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 = 6$
- : $2 \cdot 0 + 0 \cdot 2 + 1 \cdot 1 + 1 \cdot 1 = 2$
- : $2 \cdot 1 + 0 \cdot 1 + 1 \cdot 2 + 1 \cdot 0 = 4$
- : $2 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 + 1 \cdot 2 = 4$

After 3 rounds, the number of L's of each orientation are as follows:

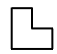
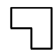
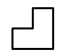
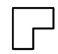
- : $6 \cdot 2 + 2 \cdot 0 + 4 \cdot 1 + 4 \cdot 1 = 20$
- : $6 \cdot 0 + 2 \cdot 2 + 4 \cdot 1 + 4 \cdot 1 = 12$
- : $6 \cdot 1 + 2 \cdot 1 + 4 \cdot 2 + 4 \cdot 0 = 16$
- : $6 \cdot 1 + 2 \cdot 1 + 4 \cdot 0 + 4 \cdot 2 = 16$

Where do these numbers come from?
For example, to determine the number of after 2 rounds, we look at the number of L's of each orientation after round 1 (2, 0, 1, 1) and ask how many each of these produces at the next level. Since the four types each produce 2, 0, 1, and 1 , then the total number of after 2 rounds equals $2 \cdot 2 + 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1$ which equals 6.
As a second example, to determine the number of after 3 rounds, we note that after 2 rounds the number of L's of the four different orientations are 6, 2, 4, 4 and that each L of each of the four types produces 0, 2, 1, 1 . This means that the total number of after 3 rounds is $6 \cdot 0 + 2 \cdot 2 + 4 \cdot 1 + 4 \cdot 1 = 12$.
Putting all of this together, the number of L's of the smallest size in the same orientation as the original L is 20.

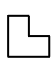
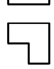
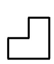
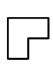
(c) In (b), we determined the number of L's of the smallest size in each orientation after 1, 2 and 3 rounds.
We continue to determine the number of L's of the smallest size after 4 rounds.
After 4 rounds, the number of L's of each orientation are as follows:

- : $20 \cdot 2 + 12 \cdot 0 + 16 \cdot 1 + 16 \cdot 1 = 72$
- : $20 \cdot 0 + 12 \cdot 2 + 16 \cdot 1 + 16 \cdot 1 = 56$
- : $20 \cdot 1 + 12 \cdot 1 + 16 \cdot 2 + 16 \cdot 0 = 64$
- : $20 \cdot 1 + 12 \cdot 1 + 16 \cdot 0 + 16 \cdot 2 = 64$

This gives us the following tables of the numbers of L's of the smallest size in each orientation after 1, 2, 3, and 4 rounds:

After Round				
1	2	0	1	1
2	6	2	4	4
3	20	12	16	16
4	72	56	64	64

We re-write these numbers in the third row as $16 + 4, 16 - 4, 16, 16$ and the numbers in the fourth row as $64 + 8, 64 - 8, 64, 64$.
Based on this, we might guess that the numbers of L's of the smallest size in each orientation after n rounds are $4^{n-1} + 2^{n-1}$, $4^{n-1} - 2^{n-1}$, 4^{n-1}, 4^{n-1}.
If this guess is correct, then, after 2020 rounds, the number of L's of the smallest size in the same orientation as the original L is $4^{2019} + 2^{2019}$.
We prove that these guesses are right by using an inductive process.
First, we note that the table above shows that our guess is correct when $n = 1, 2, 3, 4$.
Next, if we can show that our guess being correct after a given number of rounds implies that it is correct after the next round, then it will be correct after every round. This is because being correct after 4 rounds will mean that it is correct after 5 rounds, being correct after 5 rounds will mean that it is correct after 6 rounds, and so on to be correct after any number of rounds.
Suppose, then, that after k rounds the numbers of L's of the smallest size in each orientation are $4^{k-1} + 2^{k-1}$, $4^{k-1} - 2^{k-1}$, 4^{k-1}, 4^{k-1}.
After $k + 1$ rounds (that is, after the next round), the number of L's of each orientation is:

- : $(4^{k-1}+2^{k-1})\cdot 2+(4^{k-1}-2^{k-1})\cdot 0+4^{k-1}\cdot 1+4^{k-1}\cdot 1 = 4\cdot 4^{k-1}+2\cdot 2^{k-1} = 4^k+2^k$
- : $(4^{k-1}+2^{k-1})\cdot 0+(4^{k-1}-2^{k-1})\cdot 2+4^{k-1}\cdot 1+4^{k-1}\cdot 1 = 4\cdot 4^{k-1}-2\cdot 2^{k-1} = 4^k-2^k$
- : $(4^{k-1} + 2^{k-1}) \cdot 1 + (4^{k-1} - 2^{k-1}) \cdot 1 + 4^{k-1} \cdot 2 + 4^{k-1} \cdot 0 = 4 \cdot 4^{k-1} = 4^k$
- : $(4^{k-1} + 2^{k-1}) \cdot 1 + (4^{k-1} - 2^{k-1}) \cdot 1 + 4^{k-1} \cdot 0 + 4^{k-1} \cdot 2 = 4 \cdot 4^{k-1} = 4^k$

Since $k = (k + 1) - 1$, these expressions match our guess. This means that our guess is correct after every number of rounds.
Therefore, after 2020 rounds, the number of L's of the smallest size in the same orientation as the original L is $4^{2019} + 2^{2019}$.

10. (a) Here, the pairwise sums of the numbers $a_1 \leq a_2 \leq a_3 \leq a_4$ are $s_1 \leq s_2 \leq s_3 \leq s_4 \leq s_5 \leq s_6$. The six pairwise sums of the numbers in the list can be expressed as

$$a_1 + a_2, a_1 + a_3, a_1 + a_4, a_2 + a_3, a_2 + a_4, a_3 + a_4$$

Since $a_1 \leq a_2 \leq a_3 \leq a_4$, then the smallest sum must be the sum of the two smallest numbers. Thus, $s_1 = a_1 + a_2$.
Similarly, the largest sum must be the sum of the two largest numbers, and so $s_6 = a_3 + a_4$.
Since $a_1 \leq a_2 \leq a_3 \leq a_4$, then the second smallest sum is $a_1 + a_3$. This is because $a_1 + a_3$ is no greater than each of the four sums $a_1 + a_4$, $a_2 + a_3$, $a_2 + a_4$, and $a_3 + a_4$:

Since $a_3 \leq a_4$, then $a_1 + a_3 \leq a_1 + a_4$.
Since $a_1 \leq a_2$, then $a_1 + a_3 \leq a_2 + a_3$.
Since $a_1 \leq a_2$ and $a_3 \leq a_4$, then $a_1 + a_3 \leq a_2 + a_4$.
Since $a_1 \leq a_4$, then $a_1 + a_3 \leq a_3 + a_4$.

Thus, $s_2 = a_1 + a_3$.
Using a similar argument, $s_5 = a_2 + a_4$.
So far, we have $s_1 = a_1 + a_2$ and $s_2 = a_1 + a_3$ and $s_5 = a_2 + a_4$ and $s_6 = a_3 + a_4$.
This means that s_3 and s_4 equal $a_1 + a_4$ and $a_2 + a_3$ in some order.
It turns out that either order is possible.

Case 1: $s_3 = a_1 + a_4$ and $s_4 = a_2 + a_3$
Here, $a_1 + a_2 = 8$ and $a_1 + a_3 = 104$ and $a_2 + a_3 = 110$.
Adding these three equations gives

$$(a_1 + a_2) + (a_1 + a_3) + (a_2 + a_3) = 8 + 104 + 110$$

and so $2a_1 + 2a_2 + 2a_3 = 222$ or $a_1 + a_2 + a_3 = 111$.
Since $a_2 + a_3 = 110$, then $a_1 = (a_1 + a_2 + a_3) - (a_2 + a_3) = 111 - 110 = 1$.
Since $a_1 = 1$ and $a_1 + a_2 = 8$, then $a_2 = 7$.
Since $a_1 = 1$ and $a_1 + a_3 = 104$, then $a_3 = 103$.
Since $a_3 = 103$ and $a_3 + a_4 = 208$, then $a_4 = 105$.
Thus, $(a_1, a_2, a_3, a_4) = (1, 7, 103, 105)$.

Case 2: $s_3 = a_2 + a_3$ and $s_4 = a_1 + a_4$
Here, $a_1 + a_2 = 8$ and $a_1 + a_3 = 104$ and $a_2 + a_3 = 106$.
Using the same process, $a_1 + a_2 + a_3 = 109$.
From this, we obtain $(a_1, a_2, a_3, a_4) = (3, 5, 101, 107)$.

Therefore, Kerry's two possible lists are $1, 7, 103, 105$ and $3, 5, 101, 107$.

(b) Suppose that the values of $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}$ are fixed, but unknown.
In terms of the numbers $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$, the ten pairwise sums are

$$a_1 + a_2, a_1 + a_3, a_1 + a_4, a_1 + a_5, a_2 + a_3, a_2 + a_4, a_2 + a_5, a_3 + a_4, a_3 + a_5, a_4 + a_5$$

These will equal $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}$ in some order.
Using a similar analysis to that in (a), the smallest sum is $a_1 + a_2$ and the largest sum is $a_4 + a_5$. Thus, $s_1 = a_1 + a_2$ and $s_{10} = s_4 + s_5$.
Also, the second smallest sum will be $s_2 = a_1 + a_3$ and the second largest sum will be $s_9 = a_3 + a_5$.
We let

$$S = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8 + s_9 + s_{10}$$

Note that S has a fixed, but unknown, value.
Even though we do not know the order in which these pairwise sums are assigned to s_1 through s_{10}, the value of S will equal the sum of these ten pairwise expressions.
In other words, $S = 4a_1 + 4a_2 + 4a_3 + 4a_4 + 4a_5$, since each of the numbers in the list occurs in four sums.
Thus, $a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{4}S$ and so $(a_1 + a_2) + a_3 + (a_4 + a_5) = \frac{1}{4}S$.
This means that $s_1 + a_3 + s_{10} = \frac{1}{4}S$ and so $a_3 = \frac{1}{4}S - s_1 - s_{10}$.
Since the values of s_1, s_{10} and S are fixed, then we are able to determine the value of a_3 from the list of sums s_1 through s_{10}.
Using the value of a_3, the facts that $s_2 = a_1 + a_3$ and $s_9 = a_3 + a_5$, and that s_2 and s_9 are known, we can determine a_1 and a_5.
Finally, using $s_1 = a_1 + a_2$ and $s_{10} = a_4 + a_5$ and the values of a_1 and a_5, we can determine a_2 and a_4.
Therefore, given the ten sums s_1 through s_{10}, we can determine the values of a_3, a_1, a_5, a_2, a_4 and so there is only one possibility for the list a_1, a_2, a_3, a_4, a_5. (Can you write out expressions for each of a_1 through a_5 in terms of s_1 through s_{10} only?)

(c) Suppose that the lists a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 produce the same list of sums s_1, s_2, s_3, s_4, s_5, s_6. (Examples of such lists can be found in (a).)
Let x be a positive integer. Consider the following list with 8 entries:

$$a_1, a_2, a_3, a_4, b_1 + x, b_2 + x, b_3 + x, b_4 + x$$

From this list, there are three categories of pairwise sums:

(i) $a_i + a_j$, $1 \leq i < j \leq 4$: these give the sums s_1 through s_6

(ii) $(b_i + x) + (b_j + x)$, $1 \leq i < j \leq 4$: each of these is $2x$ greater than the six sums s_1 through s_6 because the pairwise sums $b_i + b_j$ give the six sums s_1 through s_6

(iii) $a_i + (b_j + x)$, $1 \leq i \leq 4$ and $1 \leq j \leq 4$

Consider also the list with 8 entries:

$$a_1 + x, a_2 + x, a_3 + x, a_4 + x, b_1, b_2, b_3, b_4$$

From this list, there are again three categories of pairwise sums:

(i) $b_i + b_j$, $1 \leq i < j \leq 4$: these give the sums s_1 through s_6

(ii) $(a_i + x) + (a_j + x)$, $1 \leq i < j \leq 4$: each of these is $2x$ greater than the six sums s_1 through s_6 because the pairwise sums $a_i + a_j$ give the six sums s_1 through s_6

(iii) $(a_i + x) + b_j$, $1 \leq i \leq 4$ and $1 \leq j \leq 4$

Thus, the 28 pairwise sums in each case are the same. In each case, there are 6 sums in (i), 6 sums in (ii), and 16 sums in (iii).
If we choose the initial lists to have the same pairwise sums and choose the value of x to be large enough so that $a_i + x$ is not equal to any b_j and $b_i + x$ is not equal to any a_j, we obtain two different lists of 8 numbers that each produce the same list of 28 sums.
For example, if we choose a_1, a_2, a_3, a_4 to be $1, 7, 103, 105$ and b_1, b_2, b_3, b_4 to be $3, 5, 101, 107$ and $x = 10\,000$, we get the lists

$$1, 7, 103, 105, 10\,003, 10\,005, 10\,101, 10\,107$$

and

$$3, 5, 101, 107, 10\,001, 10\,007, 10\,103, 10\,105$$

Using a similar analysis to that above, if the lists a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 and b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8 have the same set of pairwise sums, then the lists

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, b_1 + y, b_2 + y, b_3 + y, b_4 + y, b_5 + y, b_6 + y, b_7 + y, b_8 + y$$

and

$$a_1 + y, a_2 + y, a_3 + y, a_4 + y, a_5 + y, a_6 + y, a_7 + y, a_8 + y, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$$

will also have the same pairwise sums.
Therefore, setting $y = 1\,000\,000$, we see that the lists

$$1, 7, 103, 105, 10\,003, 10\,005, 10\,101, 10\,107, 1\,000\,003, 1\,000\,005, 1\,000\,101, 1\,000\,107,$$

$$1\,010\,001, 1\,010\,007, 1\,010\,103, 1\,010\,105$$

and

$$3, 5, 101, 107, 10\,001, 10\,007, 10\,103, 10\,105, 1\,000\,001, 1\,000\,007, 1\,000\,103, 1\,000\,105,$$

$$1\,010\,003, 1\,010\,005, 1\,010\,101, 1\,010\,107$$

have the same list of sums $s_1, s_2, \ldots, s_{120}$, as required.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2019 Euclid Contest

Wednesday, April 3, 2019
(in North America and South America)

Thursday, April 4, 2019
(outside of North America and South America)

Solutions

1. (a) *Solution 1*
Since $\frac{3}{4}$ of a jar has a volume of 300 mL, then $\frac{1}{4}$ of a jar has a volume of (300 mL) $\div$ 3 or 100 mL.

Solution 2
Since $\frac{3}{4}$ of a jar has a volume of 300 mL, then the volume of the entire jar is $\frac{4}{3}$(300 mL) or 400 mL.
In this case, the volume of $\frac{1}{4}$ of the jar is (400 mL) $\div 4 = 100$ mL.

(b) We note that since $\dfrac{24}{a} > 3 > 0$, then a is positive.
Since $3 < \dfrac{24}{a}$ and $a > 0$, then $a < \dfrac{24}{3} = 8$.
Since $\dfrac{24}{a} < 4$ and $a > 0$, then $a > \dfrac{24}{4} = 6$.
Since $6 < a < 8$ and a is an integer, then $a = 7$.
Note that it is indeed true that $3 < \dfrac{24}{7} < 4$.

(c) Since x and x^2 appear in the denominators of the equation, then $x \neq 0$.
Multiplying by x^2 and manipulating, we obtain successively

$$\begin{aligned} \frac{1}{x^2} - \frac{1}{x} &= 2 \\ 1 - x &= 2x^2 \\ 0 &= 2x^2 + x - 1 \\ 0 &= (2x-1)(x+1) \end{aligned}$$

and so $x = \dfrac{1}{2}$ or $x = -1$.
Checking in the original equation we obtain,

$$\frac{1}{(1/2)^2} - \frac{1}{1/2} = \frac{1}{1/4} - \frac{1}{1/2} = 4 - 2 = 2$$

and

$$\frac{1}{(-1)^2} - \frac{1}{-1} = \frac{1}{1} + 1 = 2$$

and so the solutions to the equation are $x = \dfrac{1}{2}$ and $x = -1$.

2. (a) Since the radius of the large circle is 2, its area is $\pi \cdot 2^2 = 4\pi$.
Since the radius of each small circle is 1, the area of each small circle is $\pi \cdot 1^2 = \pi$.
Since the two small circles are tangent to each other and to the large circle, then their areas do not overlap and are contained entirely within the large circle.
Since the shaded region consists of the part of the large circle that is outside the two small circles, then the shaded area is $4\pi - \pi - \pi = 2\pi$.

(b) Mo starts at 10:00 a.m. and finishes at 11:00 a.m. and so runs for 1 hour.
Mo runs at 6 km/h, and so runs 6 km in 1 hour.
Thus, Kari also runs 6 km.
Since Kari runs at 8 km/h, then Kari runs for $\dfrac{6 \text{ km}}{8 \text{ km/h}} = \dfrac{3}{4}$ h which is 45 minutes.
Since Kari finishes at 11:00 a.m., then Kari started at 10:15 a.m.

(c) The equation $x + 3y = 7$ can be rearranged to $3y = -x + 7$ and $y = -\frac{1}{3}x + \frac{7}{3}$.
Therefore, the line with this equation has slope $-\frac{1}{3}$.
Since the two lines are parallel and the line with equation $y = mx + b$ has slope m, then $m = -\frac{1}{3}$.
Thus, the equation of the second line can be re-written as $y = -\frac{1}{3}x + b$.
Since $(9, 2)$ lies on this line, then $2 = -\frac{1}{3} \cdot 9 + b$ and so $2 = -3 + b$, which gives $b = 5$.

3. (a) Michelle's list consists of 8 numbers and so its average is

$$\frac{5 + 10 + 15 + 16 + 24 + 28 + 33 + 37}{8} = \frac{168}{8} = 21$$

Daphne's list thus consists of 7 numbers (one fewer than in Michelle's list) with an average of 20 (1 less than that of Michelle).
The sum of 7 numbers whose average is 20 is $7 \cdot 20 = 140$.
Since the sum of Michelle's numbers was 168, then Daphne removed the number equal to $168 - 140$ which is 28.

(b) Since $16 = 2^4$ and $32 = 2^5$, then the given equation is equivalent to the following equations

$$\begin{aligned} (2^4)^{15/x} &= (2^5)^{4/3} \\ 2^{60/x} &= 2^{20/3} \end{aligned}$$

This means that $\dfrac{60}{x} = \dfrac{20}{3} = \dfrac{60}{9}$ and so $x = 9$.

(c) Using exponent laws, the following equations are equivalent:

$$\begin{aligned} \frac{2^{2022} + 2^a}{2^{2019}} &= 72 \\ 2^{2022-2019} + 2^{a-2019} &= 72 \\ 2^3 + 2^{a-2019} &= 72 \\ 8 + 2^{a-2019} &= 72 \\ 2^{a-2019} &= 64 \\ 2^{a-2019} &= 2^6 \end{aligned}$$

which means that $a - 2019 = 6$ and so $a = 2025$.

4. (a) *Solution 1*
Since $\triangle CDB$ is right-angled at B, then $\angle DCB = 90^\circ - \angle CDB = 30^\circ$.
This means that $\triangle CDB$ is a 30°-60°-90° triangle.
Using the ratios of side lengths in a 30°-60°-90° triangle, $CD : DB = 2 : 1$.
Since $DB = 10$, then $CD = 20$.
Since $\angle CDB = 60^\circ$, then $\angle ADC = 180^\circ - \angle CDB = 120^\circ$.
Since the angles in $\triangle ADC$ add to 180°, then $\angle DAC = 180^\circ - \angle ADC - \angle ACD = 30^\circ$.
This means that $\triangle ADC$ is isosceles with $AD = CD$.
Therefore, $AD = CD = 20$.

Solution 2

Since $\triangle CDB$ is right-angled at B, then $\angle DCB = 90^\circ - \angle CDB = 30^\circ$.
Since $\triangle ACB$ is right-angled at B, then $\angle CAB = 90^\circ - \angle ACB = 90^\circ - (30^\circ + 30^\circ) = 30^\circ$.
This means that each of $\triangle CDB$ and $\triangle ACB$ is a 30°-60°-90° triangle.
Using the ratios of side lengths in a 30°-60°-90° triangle, $CB : DB = \sqrt{3} : 1$.
Since $DB = 10$, then $CB = 10\sqrt{3}$.
Similarly, $AB : CB = \sqrt{3} : 1$.
Since $CB = 10\sqrt{3}$, then $AB = \sqrt{3} \cdot 10\sqrt{3} = 30$.
Finally, this means that $AD = AB - DB = 30 - 10 = 20$.

(b) Since the points $A(d, -d)$ and $B(-d + 12, 2d - 6)$ lie on the same circle centered at the origin, O, then $OA = OB$.
Since distances are always non-negative, the following equations are equivalent:

$$\begin{aligned}
\sqrt{(d-0)^2 + (-d-0)^2} &= \sqrt{((-d+12)-0)^2 + ((2d-6)-0)^2} \\
d^2 + (-d)^2 &= (-d+12)^2 + (2d-6)^2 \\
d^2 + d^2 &= d^2 - 24d + 144 + 4d^2 - 24d + 36 \\
2d^2 &= 5d^2 - 48d + 180 \\
0 &= 3d^2 - 48d + 180 \\
0 &= d^2 - 16d + 60 \\
0 &= (d-10)(d-6)
\end{aligned}$$

and so $d = 10$ or $d = 6$.
We can check that the points $A(10, -10)$ and $B(2, 14)$ are both of distance $\sqrt{200}$ from the origin and the points $A(6, -6)$ and $B(6, 6)$ are both of distance $\sqrt{72}$ from the origin.

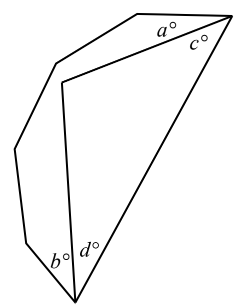
5. (a) First, we note that $\sqrt{50} = 5\sqrt{2}$.
Next, we note that $\sqrt{2} + 4\sqrt{2} = 5\sqrt{2}$ and $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$.
From the first of these, we obtain $\sqrt{2} + \sqrt{32} = \sqrt{50}$.
From the second of these, we obtain $\sqrt{8} + \sqrt{18} = \sqrt{50}$.
Thus, $(a, b) = (2, 32)$ and $(a, b) = (8, 18)$ are solutions to the original equation.
(We are not asked to justify why these are the only two solutions.)

(b) From the second equation, we note that $d \neq 0$.
Rearranging this second equation, we obtain $c = kd$.
Substituting into the first equation, we obtain $kd + d = 2000$ or $(k + 1)d = 2000$.
Since $k \geq 0$, note that $k + 1 \geq 1$.
This means that if (c, d) is a solution, then $k + 1$ is a divisor of 2000.
Also, if $k + 1$ is a divisor of 2000, then the equation $(k + 1)d = 2000$ gives us an integer value of d (which is non-zero) from which we can find an integer value of c using the first equation.
Therefore, the values of k that we want to count correspond to the positive divisors of 2000.
Since $2000 = 10 \cdot 10 \cdot 20 = 2^4 \cdot 5^3$, then 2000 has $(4 + 1)(3 + 1) = 20$ positive divisors.
This comes from the fact that if p and q are distinct prime numbers then the positive integer $p^a \cdot q^b$ has $(a + 1)(b + 1)$ positive divisors.
We could list these divisors as

$$1, 2, 4, 5, 8, 10, 16, 20, 25, 40, 50, 80, 100, 125, 200, 250, 400, 500, 1000, 2000$$

if we did not know the earlier formula.
Since 2000 has 20 positive divisors, then there are 20 values of k for which the system of equations has at least one integer solution.
For example, if $k+1=8$, then $k=7$. This gives the system $c+d=2000$ and $\dfrac{c}{d}=7$ which has solution $(c,d)=(1750,250)$.

6. (a) *Solution 1*
The angles in a polygon with n sides have a sum of $(n-2)\cdot 180^\circ$.
This means that the angles in a pentagon have a sum of $3\cdot 180^\circ$ or 540°, which means that each interior angle in a regular pentagon equals $\dfrac{1}{5}\cdot 540^\circ$ or 108°.
Also, each interior angle in a regular polygon with n sides equals $\dfrac{n-2}{n}\cdot 180^\circ$. (This is the general version of the statement in the previous sentence.)
Consider the portion of the regular polygon with n sides that lies outside the pentagon and join the points from which the angles that measure a° and b° emanate to form a hexagon.



This polygon has 6 sides, and so the sum of its 6 angles is $4\cdot 180^\circ$.
Four of its angles are the original angles from the n-sided polygon, so each equals $\dfrac{n-2}{n}\cdot 180^\circ$.
The remaining two angles have measures $a^\circ+c^\circ$ and $b^\circ+d^\circ$.
We are told that $a^\circ+b^\circ=88^\circ$.
Also, the angles that measure c° and d° are two angles in a triangle whose third angle is 108°.
Thus, $c^\circ+d^\circ=180^\circ-108^\circ=72^\circ$.
Therefore,

$$\begin{aligned}
4\cdot\frac{n-2}{n}\cdot 180^\circ+88^\circ+72^\circ &= 4\cdot 180^\circ \\
160^\circ &= \left(4-\frac{4(n-2)}{n}\right)\cdot 180^\circ \\
160^\circ &= \frac{4n-(4n-8)}{n}\cdot 180^\circ \\
\frac{160^\circ}{180^\circ} &= \frac{8}{n} \\
\frac{8}{9} &= \frac{8}{n}
\end{aligned}$$

and so the value of n is 9.

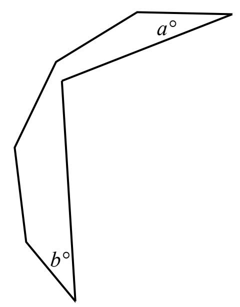
Solution 2

The angles in a polygon with n sides have a sum of $(n-2)\cdot 180^\circ$.

This means that the angles in a pentagon have a sum of $3\cdot 180^\circ$ or 540°, which means that each interior angle in a regular pentagon equals $\frac{1}{5}\cdot 540^\circ$ or 108°.

Also, each interior angle in a regular polygon with n sides equals $\frac{n-2}{n}\cdot 180^\circ$. (This is the general version of the statement in the previous sentence.)

Consider the portion of the regular polygon with n sides that lies outside the pentagon.



This polygon has 7 sides, and so the sum of its 7 angles is $5\cdot 180^\circ$.

Four of its angles are the original angles from the n-sided polygon, so each equals $\frac{n-2}{n}\cdot 180^\circ$.

Two of its angles are the angles equal to a° and b°, whose sum is 88°.

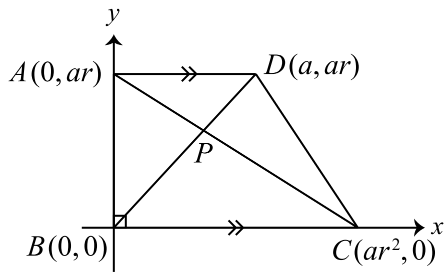
Its seventh angle is the reflex angle corresponding to the pentagon's angle of 108°, which equals $360^\circ - 108^\circ$ or 252°.

Therefore,

$$\begin{aligned}
4\cdot\frac{n-2}{n}\cdot 180^\circ + 88^\circ + 252^\circ &= 5\cdot 180^\circ \\
340^\circ &= \left(5-\frac{4(n-2)}{n}\right)\cdot 180^\circ \\
340^\circ &= \frac{5n-(4n-8)}{n}\cdot 180^\circ \\
\frac{340^\circ}{180^\circ} &= \frac{n+8}{n} \\
\frac{17}{9} &= \frac{n+8}{n} \\
17n &= 9(n+8) \\
17n &= 9n+72 \\
8n &= 72
\end{aligned}$$

and so the value of n is 9.

(b) Since the lengths of AD, AB and BC form a geometric sequence, we suppose that these lengths are a, ar and ar^2, respectively, for some real numbers $a > 0$ and $r > 0$.
Since the angles at A and B are both right angles, we assign coordinates to the diagram, putting B at the origin $(0, 0)$, C on the positive x-axis at $(ar^2, 0)$, A on the positive y-axis at $(0, ar)$, and D at (a, ar).



Therefore, the slope of the line segment joining $B(0, 0)$ and $D(a, ar)$ is $\dfrac{ar - 0}{a - 0} = r$.
Also, the slope of the line segment joining $A(0, ar)$ and $C(ar^2, 0)$ is $\dfrac{ar - 0}{0 - ar^2} = -\dfrac{1}{r}$.
Since the product of the slopes of these two line segments is -1, then the segments are perpendicular, as required.

7. (a) Using logarithm and exponent laws, we obtain the following equivalent equations:

$$\begin{aligned}
2\log_2(x-1) &= 1 - \log_2(x+2) \\
2\log_2(x-1) + \log_2(x+2) &= 1 \\
\log_2\left((x-1)^2\right) + \log_2(x+2) &= 1 \\
\log_2\left((x-1)^2(x+2)\right) &= 1 \\
(x-1)^2(x+2) &= 2^1 \\
(x^2 - 2x + 1)(x+2) &= 2 \\
x^3 - 3x + 2 &= 2 \\
x^3 - 3x &= 0 \\
x(x^2 - 3) &= 0
\end{aligned}$$

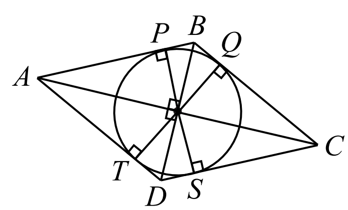
and so $x = 0$ or $x = \sqrt{3}$ or $x = -\sqrt{3}$.
Note that if $x = 0$, then $x - 1 = -1 < 0$ and so $\log_2(x - 1)$ is not defined. Thus, $x \neq 0$.
Note that if $x = -\sqrt{3}$, then $x - 1 = -\sqrt{3} - 1 < 0$ and so $\log_2(x - 1)$ is not defined. Thus, $x \neq -\sqrt{3}$.
If $x = \sqrt{3}$, we can verify that both logarithms in the original equation are defined and that the original equation is true. We could convince ourselves of this with a calculator or we could algebraically verify that raising 2 to the power of both sides gives the same number, so the expressions must actually be equal.
Therefore, $x = \sqrt{3}$ is the only solution.

(b) Let $a = f(f(x))$.
Thus, the equation $f(f(f(x))) = 3$ is equivalent to $f(a) = 3$.
Since $f(a) = a^2 - 2a$, then we obtain the equation $a^2 - 2a = 3$ which gives $a^2 - 2a - 3 = 0$ and $(a - 3)(a + 1) = 0$.
Thus, $a = 3$ or $a = -1$ which means that $f(f(x)) = 3$ or $f(f(x)) = -1$.
Let $b = f(x)$.
Thus, the equations $f(f(x)) = 3$ and $f(f(x)) = -1$ become $f(b) = 3$ and $f(b) = -1$.
If $f(b) = 3$, then $b = f(x) = 3$ or $b = f(x) = -1$ using similar reasoning to above when $f(a) = 3$.
If $f(b) = -1$, then $b^2 - 2b = -1$ and so $b^2 - 2b + 1 = 0$ or $(b - 1)^2 = 0$ which means that $b = f(x) = 1$.
Thus, $f(x) = 3$ or $f(x) = -1$ or $f(x) = 1$.
If $f(x) = 3$, then $x = 3$ or $x = -1$ as above.
If $f(x) = -1$, then $x = 1$ as above.
If $f(x) = 1$, then $x^2 - 2x = 1$ and so $x^2 - 2x - 1 = 0$.
By the quadratic formula,

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

Therefore, the solutions to the equation $f(f(f(x))) = 3$ are $x = 3, 1, -1, 1 + \sqrt{2}, 1 - \sqrt{2}$.

8. (a) Since $\angle AOB = \angle BOC = \angle COD = \angle DOA$ and these angles form a complete circle around O, then $\angle AOB = \angle BOC = \angle COD = \angle DOA = \frac{1}{4} \cdot 360° = 90°$.
Join point O to P, B, Q, C, S, D, T, and A.



Since P, Q, S, and T are points of tangency, then the radii meet the sides of $ABCD$ at right angles at these points.
Since $AO = 3$ and $OT = 1$ and $\angle OTA = 90°$, then by the Pythagorean Theorem, $AT = \sqrt{AO^2 - OT^2} = \sqrt{8} = 2\sqrt{2}$.
Since $\triangle OTA$ is right-angled at T, then $\angle TAO + \angle AOT = 90°$.
Since $\angle DOA = 90°$, then $\angle AOT + \angle DOT = 90°$.
Thus, $\angle TAO = \angle DOT$.
This means that $\triangle ATO$ is similar to $\triangle OTD$.
Thus, $\dfrac{DT}{OT} = \dfrac{OT}{AT}$ and so $DT = \dfrac{OT^2}{AT} = \dfrac{1}{2\sqrt{2}}$.
Since DS and DT are tangents to the circle from the same point, then $DS = DT = \dfrac{1}{2\sqrt{2}}$.

(b) Since $0 < x < \dfrac{\pi}{2}$, then $0 < \cos x < 1$ and $0 < \sin x < 1$.

This means that $0 < \dfrac{3}{2}\cos x < \dfrac{3}{2}$ and $0 < \dfrac{3}{2}\sin x < \dfrac{3}{2}$. Since $3 < \pi$, then $0 < \dfrac{3}{2}\cos x < \dfrac{\pi}{2}$ and $0 < \dfrac{3}{2}\sin x < \dfrac{\pi}{2}$.

If Y and Z are angles with $0 < Y < \dfrac{\pi}{2}$ and $0 < Z < \dfrac{\pi}{2}$, then $\cos Y = \sin Z$ exactly when $Y + Z = \dfrac{\pi}{2}$. To see this, we could picture points R and S on the unit circle corresponding to the angles Y and Z; the x-coordinate of R is equal to the y-coordinate of S exactly when the angles Y and Z are complementary.

Therefore, the following equations are equivalent:

$$\begin{aligned}
\cos\left(\frac{3}{2}\cos x\right) &= \sin\left(\frac{3}{2}\sin x\right) \\
\frac{3}{2}\cos x + \frac{3}{2}\sin x &= \frac{\pi}{2} \\
\cos x + \sin x &= \frac{\pi}{3} \\
(\sin x + \cos x)^2 &= \frac{\pi^2}{9} \\
\sin^2 x + 2\sin x\cos x + \sin^2 x &= \frac{\pi^2}{9} \\
2\sin x\cos x + (\sin^2 x + \cos^2 x) &= \frac{\pi^2}{9} \\
\sin 2x + 1 &= \frac{\pi^2}{9} \\
\sin 2x &= \frac{\pi^2 - 9}{9}
\end{aligned}$$

Therefore, the only possible value of $\sin 2x$ is $\dfrac{\pi^2 - 9}{9}$.

9. (a) By definition, $f(2,5) = \dfrac{2}{5} + \dfrac{5}{2} + \dfrac{1}{2\cdot 5} = \dfrac{2\cdot 2 + 5\cdot 5 + 1}{2\cdot 5} = \dfrac{4+25+1}{10} = \dfrac{30}{10} = 3$.

(b) By definition, $f(a,a) = \dfrac{a}{a} + \dfrac{a}{a} + \dfrac{1}{a^2} = 2 + \dfrac{1}{a^2}$.
For $2 + \dfrac{1}{a^2}$ to be an integer, it must be the case that $\dfrac{1}{a^2}$ is an integer.
For $\dfrac{1}{a^2}$ to be an integer and since a^2 is an integer, a^2 needs to be a divisor of 1.
Since a^2 is positive, then $a^2 = 1$.
Since a is a positive integer, then $a = 1$.
Thus, the only positive integer a for which $f(a,a)$ is an integer is $a = 1$.

(c) Suppose that a and b are positive integers for which $f(a,b)$ is an integer.
Assume that $k = f(a,b)$ is not a multiple of 3.
We will show that there must be a contradiction, which will lead to the conclusion that k must be a multiple of 3.
By definition, $k = f(a,b) = \dfrac{a}{b} + \dfrac{b}{a} + \dfrac{1}{ab}$.
Multiplying by ab, we obtain $kab = a^2+b^2+1$, which we re-write as $a^2-(kb)a+(b^2+1) = 0$.
We treat this as a quadratic equation in a with coefficients in terms of the variables b and k.
Solving for a in terms of b and k using the quadratic formula, we obtain

$$a = \frac{kb \pm \sqrt{(-kb)^2 - 4(1)(b^2+1)}}{2} = \frac{kb \pm \sqrt{k^2b^2 - 4b^2 - 4}}{2}$$

Since a is an integer, then the discriminant $k^2b^2 - 4b^2 - 4$ must be a perfect square.
Re-writing the discriminant, we obtain

$$k^2b^2 - 4b^2 - 4 = b^2(k^2-4) - 4 = b^2(k-2)(k+2) - 4$$

Since k is not a multiple of 3, then it is either 1 more than a multiple of 3 or it is 2 more than a multiple of 3.
If k is 1 more than a multiple of 3, then $k+2$ is a multiple of 3.
If k is 2 more than a multiple of 3, then $k-2$ is a multiple of 3.
In either case, $(k-2)(k+2)$ is a multiple of 3, say $(k-2)(k+2) = 3m$ for some integer m.
This means that the discriminant can be re-written again as

$$b^2(3m) - 4 = 3(b^2m - 2) + 2$$

In other words, the discriminant is itself 2 more than a multiple of 3.
However, every perfect square is either a multiple of 3 or one more than a multiple of 3:

Suppose that r is an integer and consider r^2.
The integer r can be written as one of $3q$, $3q+1$, $3q+2$, for some integer q.
These three cases give

$$\begin{aligned}(3q)^2 &= 9q^2 = 3(3q^2)\\ (3q+1)^2 &= 9q^2 + 6q + 1 = 3(3q^2+2q) + 1\\ (3q+2)^2 &= 9q^2 + 12q + 4 = 3(3q^2+4q+1) + 1\end{aligned}$$

and so r^2 is either a multiple of 3 or 1 more than a multiple of 3.

We have determined that the discriminant is a perfect square and is 2 more than a multiple of 3. This is a contradiction.
This means that our initial assumption must be incorrect, and so $k = f(a, b)$ must be a multiple of 3.

(d) *Solution 1*
We find additional pairs of positive integers (a, b) with $f(a, b) = 3$.
Suppose that $f(a, b) = 3$.
This is equivalent to the equations $\frac{a}{b} + \frac{b}{a} + \frac{1}{ab} = 3$ and $a^2 + b^2 - 3ab + 1 = 0$.
Then

$$\begin{aligned} f(b, 3b-a) - 3 &= \frac{b}{3b-a} + \frac{3b-a}{b} + \frac{1}{b(3b-a)} - 3 \\ &= \frac{b^2 + (3b-a)^2 + 1 - 3b(3b-a)}{b(3b-a)} \\ &= \frac{b^2 + (3b-a)(3b-a) + 1 - 3b(3b-a)}{b(3b-a)} \\ &= \frac{b^2 - a(3b-a) + 1}{b(3b-a)} \\ &= \frac{b^2 + a^2 - 3ab + 1}{b(3b-a)} \\ &= 0 \end{aligned}$$

Therefore, if $f(a, b) = 3$, then $f(b, 3b - a) = 3$.
The equation $f(1, 2) = 3$ gives $f(2, 3(2) - 1) = f(2, 5) = 3$.
The equation $f(2, 5) = 3$ gives $f(5, 3(5) - 2) = f(5, 13) = 3$.
The equation $f(5, 13) = 3$ gives $f(13, 3(13) - 5) = f(13, 34) = 3$.
The equation $f(13, 34) = 3$ gives $f(34, 3(34) - 13) = f(34, 89) = 3$.
The equation $f(34, 89) = 3$ gives $f(89, 3(89) - 34) = f(89, 233) = 3$.
Thus, the pairs $(a, b) = (5, 13), (13, 34), (34, 89), (89, 233)$ satisfy the requirements.

Solution 2
From (a), we know that $f(2, 5) = 3$.
Since the function $f(a, b)$ is symmetric in a and b (that is, a and b can be switched without changing the value of the function), then $f(5, 2) = 3$.
Consider the equation $f(5, b) = 3$. We know that $b = 2$ is a solution, but is there another solution?
By definition, $f(5, b) = \frac{5}{b} + \frac{b}{5} + \frac{1}{5b}$.
Thus, $f(5, b) = 3$ gives the following equivalent equations:

$$\begin{aligned} \frac{5}{b} + \frac{b}{5} + \frac{1}{5b} &= 3 \\ 25 + b^2 + 1 &= 15b \\ b^2 - 15b + 26 &= 0 \\ (b-2)(b-13) &= 0 \end{aligned}$$

and so $b = 2$ or $b = 13$. This means that $f(5, 13) = 3$ and so $(a, b) = (5, 13)$ has the property that $f(a, b)$ is an integer.

From $f(5,13)=3$, we get $f(13,5)=3$.
As above, we consider the equation $f(13,b)=3$, for which $b=5$ is a solution.
We obtain the equivalent equations

$$\begin{aligned}\frac{13}{b}+\frac{b}{13}+\frac{1}{13b}&=3\\ 169+b^2+1&=39b\\ b^2-39b+170&=0\\ (b-5)(b-34)&=0\end{aligned}$$

and so $b=5$ or $b=34$. This means that $f(13,34)=3$ and so $(a,b)=(13,34)$ has the property that $f(a,b)$ is an integer.
Continuing in a similar manner, we can also find that $f(34,89)$ and $f(89,233)$ are both integers.
Thus, the pairs $(a,b)=(5,13),(13,34),(34,89),(89,233)$ satisfy the requirements.

Solution 3
Note that

$$\begin{aligned}f(5,13)&=\frac{5}{13}+\frac{13}{5}+\frac{1}{5\cdot 13}=\frac{5^2+13^2+1}{65}=\frac{195}{65}=3\\ f(13,34)&=\frac{13}{34}+\frac{34}{13}+\frac{1}{13\cdot 34}=\frac{13^2+34^2+1}{442}=\frac{1326}{442}=3\\ f(34,89)&=\frac{34}{89}+\frac{89}{34}+\frac{1}{34\cdot 89}=\frac{34^2+89^2+1}{3026}=\frac{9078}{3026}=3\\ f(89,233)&=\frac{89}{233}+\frac{233}{89}+\frac{1}{89\cdot 233}=\frac{89^2+233^2+1}{20737}=\frac{62\,211}{20\,737}=3\end{aligned}$$

and so the pairs $(a,b)=(5,13),(13,34),(34,89),(89,233)$ satisfy the requirements.

Where do these pairs come from?
We define the Fibonacci sequence $F_1,F_2,F_3,F_4,\ldots$ by $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ when $n\geq 3$.
Thus, the Fibonacci sequence begins $1,1,2,3,5,8,13,21,34,55,89,\ldots$.
The pairs (a,b) found above are of the form (F_{2k-1},F_{2k+1}) for integers $k\geq 3$.
We note that

$$\begin{aligned}f(F_{2k-1},F_{2k+1})&=\frac{F_{2k-1}}{F_{2k+1}}+\frac{F_{2k+1}}{F_{2k-1}}+\frac{1}{F_{2k-1}F_{2k+1}}\\ &=\frac{(F_{2k-1})^2+(F_{2k+1})^2+1}{F_{2k-1}F_{2k+1}}\\ &=\frac{(F_{2k-1})^2+(F_{2k}+F_{2k-1})^2+1}{F_{2k-1}(F_{2k}+F_{2k-1})}\\ &=\frac{2(F_{2k-1})^2+2F_{2k}F_{2k-1}+(F_{2k})^2+1}{(F_{2k-1})^2+F_{2k}F_{2k-1}}\\ &=\frac{2(F_{2k-1})^2+2F_{2k}F_{2k-1}}{(F_{2k-1})^2+F_{2k}F_{2k-1}}+\frac{(F_{2k})^2+1}{(F_{2k-1})^2+F_{2k}F_{2k-1}}\\ &=2+\frac{(F_{2k})^2+1}{F_{2k-1}F_{2k+1}}\end{aligned}$$

This means that $f(F_{2k-1}, F_{2k+1}) = 3$ if and only if $\dfrac{(F_{2k})^2+1}{F_{2k-1}F_{2k+1}} = 1$, or equivalently if and only if $(F_{2k})^2 + 1 = F_{2k-1}F_{2k+1}$, or $(F_{2k})^2 - F_{2k-1}F_{2k+1} = -1$.
We note that $(F_2)^2 - F_1F_3 = 1^2 - 1 \cdot 2 = -1$ and $(F_4)^2 - F_3F_5 = 3^2 - 2 \cdot 5 = -1$ so this is true when $k = 1$ and $k = 2$.
Furthermore, we note that

$$\begin{aligned}
(F_{2k+2})^2 - F_{2k+1}F_{2k+3} &= (F_{2k+2})^2 - F_{2k+1}(F_{2k+2} + F_{2k+1}) \\
&= (F_{2k+2})^2 - F_{2k+1}F_{2k+2} - (F_{2k+1})^2 \\
&= F_{2k+2}(F_{2k+2} - F_{2k+1}) - (F_{2k+1})^2 \\
&= F_{2k+2}F_{2k} - (F_{2k+1})^2 \\
&= (F_{2k+1} + F_{2k})F_{2k} - (F_{2k+1})^2 \\
&= (F_{2k})^2 + F_{2k+1}F_{2k} - (F_{2k+1})^2 \\
&= (F_{2k})^2 + F_{2k+1}(F_{2k} - F_{2k+1}) \\
&= (F_{2k})^2 + F_{2k+1}(-F_{2k-1}) \\
&= (F_{2k})^2 - F_{2k+1}F_{2k-1}
\end{aligned}$$

which means that since $(F_4)^2 - F_3F_5 = -1$, then $(F_6)^2 - F_5F_7 = -1$, which means that $(F_8)^2 - F_7F_9 = -1$, and so on.
Continuing in this way, $(F_{2k})^2 - F_{2k-1}F_{2k+1} = -1$ for all positive integers $k \geq 1$, which in turn means that $f(F_{2k-1}, F_{2k+1}) = 3$, as required.

10. (a) On her first two turns, Brigitte either chooses two cards of the same colour or two cards of different colours. If she chooses two cards of different colours, then on her third turn, she must choose a card that matches one of the cards that she already has.
Therefore, the game ends on or before Brigitte's third turn.
Thus, if Amir wins, he wins on his second turn or on his third turn. (He cannot win on his first turn.)

For Amir to win on his second turn, the second card he chooses must match the first card that he chooses.
On this second turn, there will be 5 cards in his hand, of which 1 matches the colour of the first card that he chose.
Therefore, the probability that Amir wins on his second turn is $\frac{1}{5}$.
Note that there is no restriction on the first card that he chooses or the first card that Brigitte chooses.

For Amir to win on his third turn, the following conditions must be true: (i) the colour of the second card that he chooses is different from the colour of the first card that he chooses, (ii) the colour of the second card that Brigitte chooses is different from the colour of the first card that she chooses, and (iii) the colour of the third card that Amir chooses matches the colour of one of the first two cards.
The probability of (i) is $\frac{4}{5}$, since he must choose any card other than the one that matches the first one.
The probability of (ii) is $\frac{2}{3}$, since Brigitte must choose either of the cards that does not match her first card.
The probability of (iii) is $\frac{2}{4}$, since Amir can choose either of the 2 cards that matches one of the first two cards that he chose.
Again, the cards that Amir and Brigitte choose on their first turns do not matter.
Thus, the probability that Amir wins on his third turn is $\frac{4}{5} \cdot \frac{2}{3} \cdot \frac{2}{4}$ which equals $\frac{4}{15}$.

Finally, the probabilty that Amir wins the game is thus $\frac{1}{5} + \frac{4}{15}$ which equals $\frac{7}{15}$.

(b) Suppose that, after flipping the first 13 coins, the probability that there is an even number of heads is p.
Then the probability that there is an odd number of heads is $1 - p$.
When the 14th coin is flipped, the probability of heads is h_{14} and the probability of not heads is $1 - h_{14}$.
After the 14th coin is flipped, there can be an even number of heads if the first 13 include an even number of heads and the 14th is not heads, or if the first 13 include an odd number of heads and the 14th is heads.
The probability of this is $p(1 - h_{14}) + (1 - p)h_{14}$.
Therefore,

$$\begin{aligned} p(1 - h_{14}) + (1 - p)h_{14} &= \tfrac{1}{2} \\ 2p - 2ph_{14} + 2h_{14} - 2ph_{14} &= 1 \\ 0 &= 4ph_{14} - 2p - 2h_{14} + 1 \\ 0 &= 2p(2h_{14} - 1) - (2h14 - 1) \\ 0 &= (2p - 1)(2h_{14} - 1) \end{aligned}$$

Therefore, either $h_{14} = \frac{1}{2}$ or $p = \frac{1}{2}$.
If $h_{14} = \frac{1}{2}$, we have proven the result.
If $h_{14} \neq \frac{1}{2}$, then $p = \frac{1}{2}$.
This would mean that the probability of getting an even number of heads when the first 13 coins are flipped is $\frac{1}{2}$.
We could repeat the argument above to conclude that either $h_{13} = \frac{1}{2}$ or the probability of obtaining an even number of heads when the first 12 coins are flipped is $\frac{1}{2}$.
Continuing in this way, either one of $h_{14}, h_{13}, \ldots, h_3, h_2$ will equal $\frac{1}{2}$, or the probability of obtaining an even number of heads when 1 coin is flipped is $\frac{1}{2}$.
This last statement is equivalent to saying that the probability of obtaining a head with the first coin is $\frac{1}{2}$ (that is, $h_1 = \frac{1}{2}$).
Therefore, at least one of $h_1, h_2, \ldots, h_{13}, h_{14}$ must equal $\frac{1}{2}$.

(c) For the sum of the two digits printed to be 2, each digit must equal 1.
Thus, $S(2) = p_1q_1$.
For the sum of the two digits printed to be 12, each digit must equal 6.
Thus, $S(12) = p_6q_6$.
For the sum of the two digits printed to be 7, the digits must be 1 and 6, or 2 and 5, or 3 and 4, or 4 and 3, or 5 and 2, or 6 and 1.
Thus, $S(7) = p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1$.
Since $S(2) = S(12)$, then $p_1q_1 = p_6q_6$.
Since $S(2) > 0$ and $S(12) > 0$, then $p_1, q_1, p_6, q_6 > 0$.
If $p_1 = p_6$, then we can divide both sides of $p_1q_1 = p_6q_6$ by $p_1 = p_6$ to obtain $q_1 = q_6$.
If $q_1 = q_6$, then we can divide both sides of $p_1q_1 = p_6q_6$ by $q_1 = q_6$ to obtain $p_1 = p_6$.
Therefore, if we can show that either $p_1 = p_6$ or $q_1 = q_6$, our result will be true.
Suppose that $p_1 \neq p_6$ and $q_1 \neq q_6$.
Since $S(2) = \frac{1}{2}S(7)$ and $S(12) = \frac{1}{2}S(7)$, then

$$\begin{aligned}
S(7) - \tfrac{1}{2}S(7) - \tfrac{1}{2}S(7) &= 0 \\
S(7) - S(2) - S(12) &= 0 \\
p_1q_6 + p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 + p_6q_1 - p_1q_1 - p_6q_6 &= 0 \\
p_1q_6 + p_6q_1 - p_1q_1 - p_6q_6 + (p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2) &= 0 \\
(p_1 - p_6)(q_6 - q_1) + (p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2) &= 0 \\
p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 &= -(p_1 - p_6)(q_6 - q_1) \\
p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 &= (p_1 - p_6)(q_1 - q_6)
\end{aligned}$$

Since $p_2, p_3, p_4, p_5, q_2, q_3, q_4, q_5 \geq 0$, then $p_2q_5 + p_3q_4 + p_4q_3 + p_5q_2 \geq 0$.
From this, $(p_1 - p_6)(q_1 - q_6) \geq 0$.
Since $p_1 \neq p_6$, then either $p_1 > p_6$ or $p_1 < p_6$.
If $p_1 > p_6$, then $p_1 - p_6 > 0$ and so $(p_1 - p_6)(q_1 - q_6) \geq 0$ tells us that $q_1 - q_6 > 0$ which means $q_1 > q_6$.
But we know that $p_1q_1 = p_6q_6$ and $p_1, q_1, p_6, q_6 > 0$ so we cannot have $p_1 > p_6$ and $q_1 > q_6$.
If $p_1 < p_6$, then $p_1 - p_6 < 0$ and so $(p_1 - p_6)(q_1 - q_6) \geq 0$ tells us that $q_1 - q_6 < 0$ which means $q_1 < q_6$.
But we know that $p_1q_1 = p_6q_6$ and $p_1, q_1, p_6, q_6 > 0$ so we cannot have $p_1 < p_6$ and $q_1 < q_6$.
This is a contradiction.
Therefore, since we cannot have $p_1 > p_6$ or $p_1 < p_6$, it must be the case that $p_1 = p_6$ which means that $q_1 = q_6$, as required.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018 Euclid Contest

Wednesday, April 11, 2018
(in North America and South America)

Thursday, April 12, 2018
(outside of North America and South America)

Solutions

1. (a) When $x = 11$,

$$x + (x + 1) + (x + 2) + (x + 3) = 4x + 6 = 4(11) + 6 = 50$$

Alternatively,

$$x + (x + 1) + (x + 2) + (x + 3) = 11 + 12 + 13 + 14 = 50$$

(b) We multiply the equation $\frac{a}{6} + \frac{6}{18} = 1$ by 18 to obtain $3a + 6 = 18$.
Solving, we get $3a = 12$ and so $a = 4$.

(c) *Solution 1*
Since the cost of one chocolate bar is \$1.00 more than that of a pack of gum, then if we replace a pack of gum with a chocolate bar, then the price increases by \$1.00.
Starting with one chocolate bar and two packs of gum, we replace the two packs of gum with two chocolate bars.
This increases the price by \$2.00 from \$4.15 to \$6.15.
In other words, three chocolate bars cost \$6.15, and so one chocolate bar costs $\frac{1}{3}$(\$6.15) or \$2.05.

Solution 2
Let the cost of one chocolate bar be \$$x$.
Let the cost of one pack of gum be \$$y$.
Since the cost of one chocolate bar and two packs of gum is \$4.15, then $x + 2y = 4.15$.
Since one chocolate bar costs \$1.00 more than one pack of gum, then $x = y + 1$.
Since $x = y + 1$, then $y = x - 1$.
Since $x + 2y = 4.15$, then $x + 2(x - 1) = 4.15$.
Solving, we obtain $x + 2x - 2 = 4.15$ or $3x = 6.15$ and so $x = 2.05$.
In other words, the cost of one chocolate bar is \$2.05.

2. (a) Suppose that the five-digit integer has digits $abcde$.
The digits a, b, c, d, e are $1, 3, 5, 7, 9$ in some order.
Since $abcde$ is greater than 80 000, then $a \geq 8$, which means that $a = 9$.
Since $9bcde$ is less than 92 000, then $b < 2$, which means that $b = 1$.
Since $91cde$ has units (ones) digit 3, then $e = 3$.
So far, the integer is $91cd3$, which means that c and d are 5 and 7 in some order.
Since the two-digit integer cd is divisible by 5, then it must be 75.
This means that the the five-digit integer is 91753.

(b) By the Pythagorean Theorem in $\triangle ADB$,

$$AD^2 = AB^2 - BD^2 = 13^2 - 12^2 = 169 - 144 = 25$$

Since $AD > 0$, then $AD = \sqrt{25} = 5$.
By the Pythagorean Theorem in $\triangle CDB$,

$$CD^2 = BC^2 - BD^2 = (12\sqrt{2})^2 - 12^2 = 12^2(2) - 12^2 = 12^2$$

Since $CD > 0$, then $CD = 12$.
Therefore, $AC = AD + DC = 5 + 12 = 17$.

(c) *Solution 1*

The area of the shaded region equals the area of square $OABC$ minus the area of $\triangle OCD$.
Since square $OABC$ has side length 6, then its area is 6^2 or 36.
Also, $OC = 6$.
Since the equation of the line is $y = 2x$, then its slope is 2.
Since the slope of the line is 2, then $\dfrac{OC}{CD} = 2$.
Since $OC = 6$, then $CD = 3$.
Thus, the area of $\triangle OCD$ is $\frac{1}{2}(OC)(CD) = \frac{1}{2}(6)(3) = 9$.
Finally, the area of shaded region must be $36 - 9 = 27$.

Solution 2

Since square $OABC$ has side length 6, then $OA = AB = CB = OC = 6$.
Since the slope of the line is 2, then $\dfrac{OC}{CD} = 2$.
Since $OC = 6$, then $CD = 3$.
Since $CB = 6$ and $CD = 3$, then $DB = CB - CD = 3$.
The shaded region is a trapezoid with parallel sides $DB = 3$ and $OA = 6$ and height $AB = 6$.
Therefore, the area of the shaded region is $\frac{1}{2}(DB + OA)(AB) = \frac{1}{2}(3 + 6)(6) = 27$.

3. (a) Calculating, $\left(\sqrt{4+\sqrt{4}}\right)^4 = \left(\sqrt{4+2}\right)^4 = \left(\sqrt{6}\right)^4 = \left(\left(\sqrt{6}\right)^2\right)^2 = 6^2 = 36$.

(b) Since y is an integer, then $8 - y^2$ is an integer.
Therefore, $\sqrt{23 - x}$ is an integer which means that $23 - x$ is a perfect square.
Since x is a positive integer, then $23 - x < 23$ and so $23 - x$ must be a perfect square that is less than 23.
We make a table listing the possible values of $23 - x$ and the resulting values of x, $\sqrt{23 - x} = 8 - y^2$, y^2, and y:

$23 - x$	x	$\sqrt{23-x} = 8 - y^2$	y^2	y
16	7	4	4	± 2
9	14	3	5	$\pm\sqrt{5}$
4	19	2	6	$\pm\sqrt{6}$
1	22	1	7	$\pm\sqrt{7}$
0	23	0	8	$\pm\sqrt{8}$

Since x and y are positive integers, then we must have $(x, y) = (7, 2)$.
(We note that since we were told that there is only one such pair, we did not have to continue the table beyond the first row.)

(c) Since the line with equation $y = mx + 2$ passes through $(1, 5)$, then $5 = m + 2$ and so $m = 3$.
Since the parabola with equation $y = ax^2 + 5x - 2$ passes through $(1, 5)$, then $5 = a + 5 - 2$ and so $a = 2$.
To find the coordinates of Q, we determine the second point of intersection of $y = 3x + 2$ and $y = 2x^2 + 5x - 2$ by equating values of y:

$$\begin{aligned} 2x^2 + 5x - 2 &= 3x + 2 \\ 2x^2 + 2x - 4 &= 0 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \end{aligned}$$

Therefore, $x = 1$ or $x = -2$.
Since P has x-coordinate 1, then Q has x-coordinate -2.
Since Q lies on the line with equation $y = 3x + 2$, we have $y = 3(-2) + 2 = -4$.
In summary, (i) $m = 3$, (ii) $a = 2$, and (iii) the coordinates of Q are $(-2, -4)$.

4. (a) Since $80 = 2^4 \cdot 5$, its positive divisors are $1, 2, 4, 5, 8, 10, 16, 20, 40, 80$.
For an integer n to share exactly two positive common divisors with 80, these divisors must be either 1 and 2 or 1 and 5. (1 is a common divisor of any two integers. The second common divisor must be a prime number since any composite divisor will cause there to be at least one more common divisor which is prime.)
Since $1 \leq n \leq 30$ and n is a multiple of 2 or of 5, then the possible values of n come from the list

$$2, 4, 5, 6, 8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 25, 26, 28, 30$$

We remove the multiples of 4 from this list (since they would share at least the divisors $1, 2, 4$ with 80) and the multiples of 10 from this list (since they would share at least the divisors $1, 2, 5, 10$ with 80).
This leaves the list

$$2, 5, 6, 14, 15, 18, 22, 25, 26$$

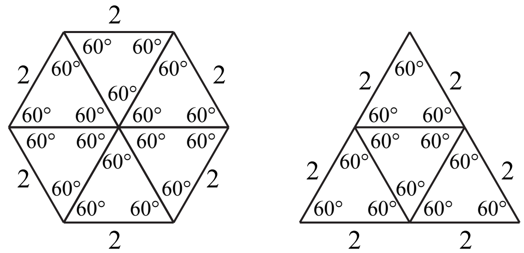
The common divisors of any number from this list and 80 are either 1 and 2 or 1 and 5.
There are 9 such integers.

(b) We start with $f(50)$ and apply the given rules for the function until we reach $f(1)$:

$$\begin{aligned} f(50) &= f(25) && \text{(since 50 is even and } \tfrac{1}{2}(50) = 25) \\ &= f(24) + 1 && \text{(since 25 is odd and } 25 - 1 = 24) \\ &= f(12) + 1 && (\tfrac{1}{2}(24) = 12) \\ &= f(6) + 1 && (\tfrac{1}{2}(12) = 6) \\ &= f(3) + 1 && (\tfrac{1}{2}(6) = 3) \\ &= (f(2) + 1) + 1 && (3 - 1 = 2) \\ &= f(1) + 1 + 1 && (\tfrac{1}{2}(2) = 1) \\ &= 1 + 1 + 1 && (f(1) = 1) \\ &= 3 \end{aligned}$$

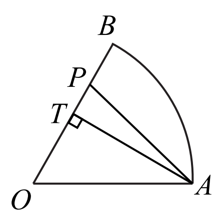
Therefore, $f(50) = 3$.

5. (a) Since the hexagon has perimeter 12 and has 6 sides, then each side has length 2.
Since equilateral $\triangle PQR$ has perimeter 12, then its side length is 4.
Consider equilateral triangles with side length 2.
Six of these triangles can be combined to form a regular hexagon with side length 2 and four of these can be combined to form an equilateral triangle with side length 4.



Note that the six equilateral triangles around the centre of the hexagon give a total central angle of $6 \cdot 60^\circ = 360^\circ$ (a complete circle) and the three equilateral triangles along each side of the large equilateral triangle make a straight angle of 180° (since $3 \cdot 60^\circ = 180^\circ$).
Also, the length of each side of the hexagon is 2 and the measure of each internal angle is 120°, which means that the hexagon is regular. Similarly, the triangle is equilateral.
Since the triangle is made from four identical smaller triangles and the hexagon is made from six of these smaller triangles, the ratio of the area of the triangle to the hexagon is $4 : 6$ which is equivalent to $2 : 3$.

(b) Since sector AOB is $\frac{1}{6}$ of a circle with radius 18, its area is $\frac{1}{6}(\pi \cdot 18^2)$ or 54π.
For the line AP to divide this sector into two pieces of equal area, each piece has area $\frac{1}{2}(54\pi)$ or 27π.
We determine the length of OP so that the area of $\triangle POA$ is 27π.
Since sector AOB is $\frac{1}{6}$ of a circle, then $\angle AOB = \frac{1}{6}(360^\circ) = 60^\circ$.
Drop a perpendicular from A to T on OB.



The area of $\triangle POA$ is $\frac{1}{2}(OP)(AT)$.
$\triangle AOT$ is a 30°-60°-90° triangle.
Since $AO = 18$, then $AT = \frac{\sqrt{3}}{2}(AO) = 9\sqrt{3}$.
For the area of $\triangle POA$ to equal 27π, we have $\frac{1}{2}(OP)(9\sqrt{3}) = 27\pi$ which gives $OP = \dfrac{54\pi}{9\sqrt{3}} = \dfrac{6\pi}{\sqrt{3}} = 2\sqrt{3}\pi$.
(Alternatively, we could have used the fact that the area of $\triangle POA$ is $\frac{1}{2}(OA)(OP)\sin(\angle POA)$.)

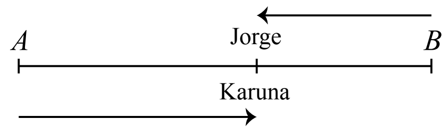
6. (a) Let $\theta = 10k^\circ$.

The given inequalities become $0^\circ < \theta < 180^\circ$ and $\dfrac{5\sin\theta - 2}{\sin^2\theta} \geq 2$.
When $0^\circ < \theta < 180^\circ$, $\sin\theta \neq 0$.
This means that we can can multiply both sides by $\sin^2\theta > 0$ and obtain the equivalent inequalities:

$$\begin{aligned} \frac{5\sin\theta - 2}{\sin^2\theta} &\geq 2 \\ 5\sin\theta - 2 &\geq 2\sin^2\theta \\ 0 &\geq 2\sin^2\theta - 5\sin\theta + 2 \\ 0 &\geq (2\sin\theta - 1)(\sin\theta - 2) \end{aligned}$$

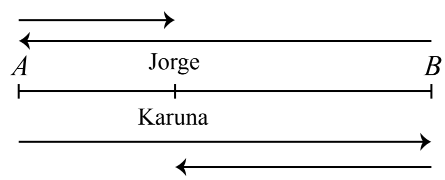
Since $\sin\theta \leq 1$, then $\sin\theta - 2 \leq -1 < 0$ for all θ.
Therefore, $(2\sin\theta - 1)(\sin\theta - 2) \leq 0$ exactly when $2\sin\theta - 1 \geq 0$.
Note that $2\sin\theta - 1 \geq 0$ exactly when $\sin\theta \geq \frac{1}{2}$.
Therefore, the original inequality is true exactly when $\frac{1}{2} \leq \sin\theta \leq 1$.
Note that $\sin 30^\circ = \sin 150^\circ = \frac{1}{2}$ and $0^\circ < \theta < 180^\circ$.
When $\theta = 0^\circ$, $\sin\theta = 0$.
From $\theta = 0^\circ$ to $\theta = 30^\circ$, $\sin\theta$ increases from 0 to $\frac{1}{2}$.
From $\theta = 30^\circ$ to $\theta = 150^\circ$, $\sin\theta$ increases from $\frac{1}{2}$ to 1 and then decreases to $\frac{1}{2}$.
From $\theta = 150^\circ$ to $\theta = 180^\circ$, $\sin\theta$ decreases from $\frac{1}{2}$ to 0.
Therefore, the original inequality is true exactly when $30^\circ \leq \theta \leq 150^\circ$ which is equivalent to $30^\circ \leq 10k^\circ \leq 150^\circ$ and to $3 \leq k \leq 15$.
The integers k in this range are $k = 3, 4, 5, 6, \ldots, 12, 13, 14, 15$, of which there are 13.

(b) Suppose that Karuna and Jorge meet for the first time after t_1 seconds and for the second time after t_2 seconds.
When they meet for the first time, Karuna has run partway from A to B and Jorge has run partway from B to A.



At this instant, the sum of the distances that they have run equals the total distance from A to B.
Since Karuna runs at 6 m/s for these t_1 seconds, she has run $6t_1$ m.
Since Jorge runs at 5 m/s for these t_1 seconds, he has run $5t_1$ m.
Therefore, $6t_1 + 5t_1 = 297$ and so $11t_1 = 297$ or $t_1 = 27$.

When they meet for the second time, Karuna has run from A to B and is running back to A and Jorge has run from B to A and is running back to B. This is because Jorge gets to A halfway through his run before Karuna gets back to A at the end of her run.



Since they each finish running after 99 seconds, then each has $99 - t_2$ seconds left to run.
At this instant, the sum of the distances that they have left to run equals the total distance from A to B.
Since Karuna runs at 6 m/s for these $(99 - t_2)$ seconds, she has to run $6(99 - t_2)$ m.
Since Jorge runs at 7.5 m/s for these $(99 - t_2)$ seconds, he has to run $7.5(99 - t_2)$ m.
Therefore, $6(99 - t_2) + 7.5(99 - t_2) = 297$ and so $13.5(99 - t_2) = 297$ or $99 - t_2 = 22$ and so $t_2 = 77$.

Alternatively, to calculate the value of t_2, we note that when Karuna and Jorge meet for the second time, they have each run the distance from A to B one full time and are on their return trips.
This means that they have each run the full distance from A to B once and the distances that they have run on their return trip add up to another full distance from A to B, for a total distance of $3 \cdot 297$ m $= 891$ m.
Karuna has run at 6 m/s for t_2 seconds, for a total distance of $6t_2$ m.
Jorge ran the first 297 m at 5 m/s, which took $\frac{297}{5}$ s and ran the remaining $(t_2 - \frac{297}{5})$ seconds at 7.5 m/s, for a total distance of $\left(297 + 7.5(t_2 - \frac{297}{5})\right)$ m.
Therefore,

$$\begin{aligned} 6t_2 + 297 + 7.5(t_2 - \tfrac{297}{5}) &= 891 \\ 13.5t_2 &= 891 - 297 + 7.5 \cdot \tfrac{297}{5} \\ 13.5t_2 &= 1039.5 \\ t_2 &= 77 \end{aligned}$$

Therefore, Karuna and Jorge meet after 27 seconds and after 77 seconds.

7. (a) *Solution 1*

Among a group of n people, there are $\dfrac{n(n-1)}{2}$ ways of choosing a pair of these people:

There are n people that can be chosen first.
For each of these n people, there are $n-1$ people that can be chosen second.
This gives $n(n-1)$ orderings of two people.
Each pair is counted twice (given two people A and B, we have counted both the pair AB and the pair BA), so the total number of pairs is $\dfrac{n(n-1)}{2}$.

We label the four canoes W, X, Y, and Z.
First, we determine the total number of ways to put the 8 people in the 4 canoes.
We choose 2 people to put in W. There are $\dfrac{8 \cdot 7}{2}$ pairs. This leaves 6 people for the remaining 3 canoes.
Next, we choose 2 people to put in X. There are $\dfrac{6 \cdot 5}{2}$ pairs. This leaves 4 people for the remaining 2 canoes.
Next, we choose 2 people to put in Y. There are $\dfrac{4 \cdot 3}{2}$ pairs. This leaves 2 people for the remaining canoe.
There is now 1 way to put the remaining people in Z.
Therefore, there are

$$\frac{8 \cdot 7}{2} \cdot \frac{6 \cdot 5}{2} \cdot \frac{4 \cdot 3}{2} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{2^3} = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$$

ways to put the 8 people in the 4 canoes.

Now, we determine the number of ways in which no two of Barry, Carrie and Mary will be in the same canoe.
There are 4 possible canoes in which Barry can go.
There are then 3 possible canoes in which Carrie can go, because she cannot go in the same canoe as Barry.
There are then 2 possible canoes in which Mary can go, because she cannot go in the same canoe as Barry or Carrie.
This leaves 5 people left to put in the canoes.
There are 5 choices of the person that can go with Barry, and then 4 choices of the person that can go with Carrie, and then 3 choices of the person that can go with Mary.
The remaining 2 people are put in the remaining empty canoe.
This means that there are $4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3$ ways in which the 8 people can be put in 4 canoes so that no two of Barry, Carrie and Mary are in the same canoe.

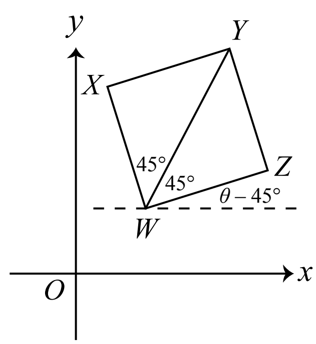
Therefore, the probability that no two of Barry, Carrie and Mary are in the same canoe is $\dfrac{4 \cdot 3 \cdot 2 \cdot 5 \cdot 4 \cdot 3}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3} = \dfrac{4 \cdot 3 \cdot 2}{7 \cdot 6} = \dfrac{24}{42} = \dfrac{4}{7}$.

Solution 2

Let p be the probability that two of Barry, Carrie and Mary are in the same canoe.
The answer to the original problem will be $1 - p$.
Let q be the probability that Barry and Carrie are in the same canoe.
By symmetry, the probability that Barry and Mary are in the same canoe also equals q as does the probability that Carrie and Mary are in the same canoe.
This means that $p = 3q$.
So we calculate q.
To do this, we put Barry in a canoe. Since there are 7 possible people who can go in the canoe with him, then the probability that Carrie is in the canoe with him equals $\frac{1}{7}$. The other 6 people can be put in the canoes in any way.
This means that the probability that Barry and Carrie are in the same canoe is $q = \frac{1}{7}$.
Therefore, the probability that no two of Barry, Carrie and Mary are in the same canoe is $1 - 3 \cdot \frac{1}{7}$ or $\frac{4}{7}$.

(b) *Solution 1*

Suppose that WY makes an angle of θ with the horizontal.



Since the slope of WY is 2, then $\tan\theta = 2$, since the tangent of an angle equals the slope of a line that makes this angle with the horizontal.
Since $\tan\theta = 2 > 1 = \tan 45^\circ$, then $\theta > 45^\circ$.
Now WY bisects $\angle ZWX$, which is a right-angle.
Therefore, $\angle ZWY = \angle YWX = 45^\circ$.
Therefore, WX makes an angle of $\theta + 45^\circ$ with the horizontal and WZ makes an angle of $\theta - 45^\circ$ with the horizontal. Since $\theta > 45^\circ$, then $\theta - 45^\circ > 0$ and $\theta + 45^\circ > 90^\circ$.
We note that since WZ and XY are parallel, then the slope of XY equals the slope of WZ.
To calculate the slopes of WX and WZ, we can calculate $\tan(\theta + 45^\circ)$ and $\tan(\theta - 45^\circ)$.
Using the facts that $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ and $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, we obtain:

$$\tan(\theta + 45^\circ) = \frac{\tan\theta + \tan 45^\circ}{1 - \tan\theta\tan 45^\circ} = \frac{2+1}{1-(2)(1)} = -3$$
$$\tan(\theta - 45^\circ) = \frac{\tan\theta - \tan 45^\circ}{1 - \tan\theta\tan 45^\circ} = \frac{2-1}{1+(2)(1)} = \frac{1}{3}$$

Therefore, the sum of the slopes of WX and XY is $-3 + \dfrac{1}{3} = -\dfrac{8}{3}$.

Solution 2

Consider a square $WXYZ$ whose diagonal WY has slope 2.

Translate this square so that W is at the origin $(0,0)$. Translating a shape in the plane does not affect the slopes of any line segments.

Let the coordinates of Y be $(2a,2b)$ for some non-zero numbers a and b.

Since the slope of WY is 2, then $\dfrac{2b-0}{2a-0}=2$ and so $2b=4a$ or $b=2a$.

Thus, the coordinates of Y can be written as $(2a,4a)$.

Let C be the centre of square $WXYZ$.

Then C is the midpoint of WY, so C has coordinates $(a,2a)$.

We find the slopes of WX and XY by finding the coordinates of X.

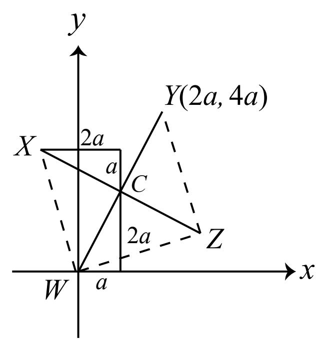
Consider the segment XC.

Since the diagonals of a square are perpendicular, then XC is perpendicular to WC.

Since the slope of WC is 2, then the slopes of XC and ZC are $-\frac{1}{2}$.

Since the diagonals of a square are equal in length and C is the midpoint of both diagonals, then $XC=WC$.

Since WC and XC are perpendicular and equal in length, then the "rise/run triangle" above XC will be a 90° rotation of the "rise/run triangle" below WC.



This is because these triangles are congruent (each is right-angled, their hypotenuses are of equal length, and their remaining angles are equal) and their hypotenuses are perpendicular.

In this diagram, we have assumed that X is to the left of W and Z is to the right of W. Since the slopes of parallel sides are equal, it does not matter which vertex is labelled X and which is labelled Z. We would obtain the same two slopes, but in a different order.

To get from $W(0,0)$ to $C(a,2a)$, we go up $2a$ and right a.

Thus, to get from $C(a,2a)$ to X, we go left $2a$ and up a.

Therefore, the coordinates of X are $(a-2a,2a+a)$ or $(-a,3a)$.

Thus, the slope of WX is $\dfrac{3a-0}{-a-0}=-3$.

Since XY is perpendicular to WX, then its slope is the negative reciprocal of -3, which is $\dfrac{1}{3}$.

The sum of the slopes of WX and XY is $-3+\dfrac{1}{3}=-\dfrac{8}{3}$.

8. (a) Since the base of a logarithm must be positive and cannot equal 1, then $x > 0$ and $x \neq \frac{1}{2}$ and $x \neq \frac{1}{3}$.
This tells us that $\log 2x$ and $\log 3x$ exist and do not equal 0, which we will need shortly when we apply the change of base formula.
We note further that $48 = 2^4 \cdot 3$ and $162 = 3^4 \cdot 2$ and $\sqrt[3]{3} = 3^{1/3}$ and $\sqrt[3]{2} = 2^{1/3}$.
Using logarithm rules, the following equations are equivalent:

$$\begin{aligned}
\log_{2x}(48\sqrt[3]{3}) &= \log_{3x}(162\sqrt[3]{2}) \\
\frac{\log(2^4 \cdot 3 \cdot 3^{1/3})}{\log 2x} &= \frac{\log(3^4 \cdot 2 \cdot 2^{1/3})}{\log 3x} && \text{(change of base formula)} \\
\frac{\log(2^4 \cdot 3^{4/3})}{\log 2 + \log x} &= \frac{\log(3^4 \cdot 2^{4/3})}{\log 3 + \log x} && (\log ab = \log a + \log b) \\
\frac{\log(2^4) + \log(3^{4/3})}{\log 2 + \log x} &= \frac{\log(3^4) + \log(2^{4/3})}{\log 3 + \log x} && (\log ab = \log a + \log b) \\
\frac{4\log 2 + \frac{4}{3}\log 3}{\log 2 + \log x} &= \frac{4\log 3 + \frac{4}{3}\log 2}{\log 3 + \log x} && (\log(a^c) = c\log a)
\end{aligned}$$

Cross-multiplying, we obtain

$$(4\log 2 + \tfrac{4}{3}\log 3)(\log 3 + \log x) = (4\log 3 + \tfrac{4}{3}\log 2)(\log 2 + \log x)$$

Expanding the left side, we obtain

$$4\log 2\log 3 + \tfrac{4}{3}(\log 3)^2 + (4\log 2 + \tfrac{4}{3}\log 3)\log x$$

Expanding the right side, we obtain

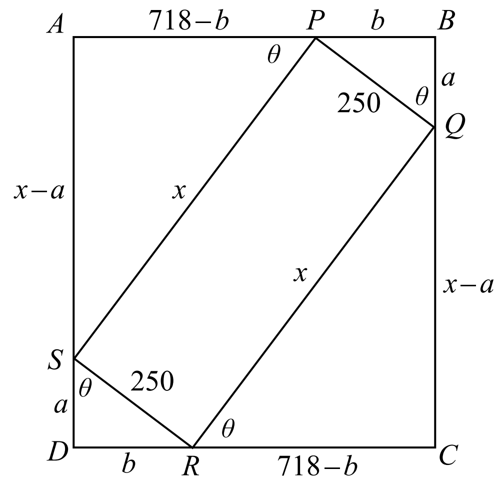
$$4\log 3\log 2 + \tfrac{4}{3}(\log 2)^2 + (4\log 3 + \tfrac{4}{3}\log 2)\log x$$

Simplifying and factoring, we obtain the following equivalent equations:

$$\begin{aligned}
\tfrac{4}{3}(\log 3)^2 - \tfrac{4}{3}(\log 2)^2 &= \log x(4\log 3 + \tfrac{4}{3}\log 2 - 4\log 2 - \tfrac{4}{3}\log 3) \\
\tfrac{4}{3}(\log 3)^2 - \tfrac{4}{3}(\log 2)^2 &= \log x\left(\tfrac{8}{3}\log 3 - \tfrac{8}{3}\log 2\right) \\
(\log 3)^2 - (\log 2)^2 &= 2\log x(\log 3 - \log 2) \\
\log x &= \frac{(\log 3)^2 - (\log 2)^2}{2(\log 3 - \log 2)} \\
\log x &= \frac{(\log 3 - \log 2)(\log 3 + \log 2)}{2(\log 3 - \log 2)} \\
\log x &= \frac{\log 3 + \log 2}{2} \\
\log x &= \tfrac{1}{2}\log 6 \\
\log x &= \log(\sqrt{6})
\end{aligned}$$

and so $x = \sqrt{6}$.

(b) Let $BC = x$, $PB = b$, and $BQ = a$.
Since $BC = x$, then $AD = PS = QR = x$.
Since $BC = x$ and $BQ = a$, then $QC = x - a$.
Since $AB = 718$ and $PB = b$, then $AP = 718 - b$.
Note that $PQ = SR = 250$.
Let $\angle BQP = \theta$.
Since $\triangle PBQ$ is right-angled at B, then $\angle BPQ = 90^\circ - \theta$.
Since BQC is a straight angle and $\angle PQR = 90^\circ$, then $\angle RQC = 180^\circ - 90^\circ - \theta = 90^\circ - \theta$.
Since APB is a straight angle and $\angle SPQ = 90^\circ$, then $\angle APS = 180^\circ - 90^\circ - (90^\circ - \theta) = \theta$.
Since $\triangle SAP$ and $\triangle QCR$ are each right-angled and have another angle in common with $\triangle PBQ$, then these three triangles are similar.



Continuing in the same way, we can show that $\triangle RDS$ is also similar to these three triangles.
Since $RS = PQ$, then $\triangle RDS$ is actually congruent to $\triangle PBQ$ (angle-side-angle).
Similarly, $\triangle SAP$ is congruent to $\triangle QCR$.
In particular, this means that $AS = x - a$, $SD = a$, $DR = b$, and $RC = 718 - b$.

Since $\triangle SAP$ and $\triangle PBQ$ are similar, then $\dfrac{SA}{PB} = \dfrac{AP}{BQ} = \dfrac{SP}{PQ}$.

Thus, $\dfrac{x-a}{b} = \dfrac{718-b}{a} = \dfrac{x}{250}$.

Also, by the Pythagorean Theorem in $\triangle PBQ$, we obtain $a^2 + b^2 = 250^2$.
By the Pythagorean Theorem in $\triangle SAP$,

$$\begin{aligned} x^2 &= (x-a)^2 + (718-b)^2 \\ x^2 &= x^2 - 2ax + a^2 + (718-b)^2 \\ 0 &= -2ax + a^2 + (718-b)^2 \qquad (*) \end{aligned}$$

Since $a^2 + b^2 = 250^2$, then $a^2 = 250^2 - b^2$.

Since $\dfrac{718-b}{a} = \dfrac{x}{250}$, then $ax = 250(718 - b)$.

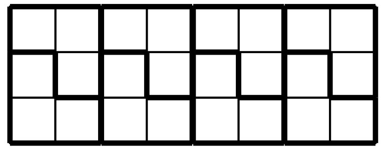
Therefore, substituting into $(*)$, we obtain

$$\begin{aligned} 0 &= -2(250)(718-b) + 250^2 - b^2 + (718-b)^2 \\ b^2 &= 250^2 - 2(250)(718-b) + (718-b)^2 \\ b^2 &= ((718-b) - 250)^2 \qquad (\text{since } y^2 - 2yz + z^2 = (y-z)^2) \\ b^2 &= (468-b)^2 \\ b &= 468 - b \qquad (\text{since } b \neq b - 468) \\ 2b &= 468 \\ b &= 234 \end{aligned}$$

Therefore, $a^2 = 250^2 - b^2 = 250^2 - 234^2 = (250+234)(250-234) = 484 \cdot 16 = 22^2 \cdot 4^2 = 88^2$ and so $a = 88$.

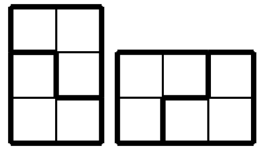
Finally, $x = \dfrac{250(718-b)}{a} = \dfrac{250 \cdot 484}{88} = 1375$. Therefore, $BC = 1375$.

9. (a) Here is a tiling of a 3×8 rectangle:



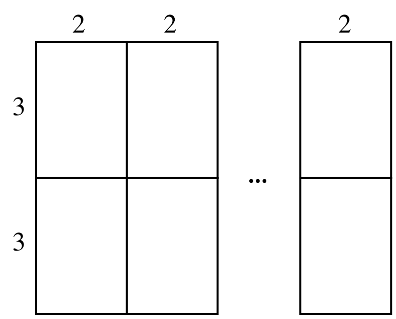
There are many other tilings.

(b) First, we note that it is possible to tile each of a 3×2 and a 2×3 rectangle:



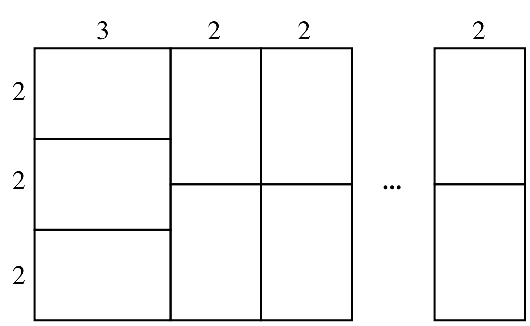
Next, we note that it is not possible to tile a 6×1 rectangle because each of the triominos needs a width of at least 2 to be placed.
Finally, we show that it is possible to tile a $6 \times W$ rectangle for every integer $W \geq 2$.
To do this, we show that such a $6 \times W$ rectangle can be made up from 3×2 and 2×3 rectangles. Since each of these types of rectangles can be tiled with triominos, then the larger rectangle can be tiled with triominos by combining these tilings.

Case 1: W is even
Suppose that $W = 2k$ for some positive integer k.
We build a $6 \times 2k$ rectangle by placing k 6×2 rectangles side by side.
Each 6×2 rectangle is built by stacking two 3×2 rectangles on top of each other.



Therefore, each such rectangle can be tiled.

Case 2: W is odd, $W \geq 3$
Suppose that $W = 2k + 1$ for some positive integer k.
We build a $6 \times (2k + 1)$ rectangle by building a 6×3 rectangle and then putting $k - 1$ 6×2 rectangles next to it. Note that $k - 1 \geq 0$ since $k \geq 1$ and that $2k + 1 = 3 + 2(k - 1)$.
The 6×3 rectangle is built by stacking three 2×3 rectangles on top of each other.
Each 6×2 rectangle is built by stacking two 3×2 rectangles on top of each other.



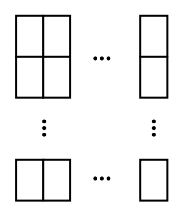
Therefore, each such rectangle can be tiled.

Thus, a $6 \times W$ rectangle can be tiled with triominos exactly when $W \geq 2$.

(c) Suppose that (H, W) is a pair of integers with $H \geq 4$ and $W \geq 4$.
Since the area of each triomino is 3, then the area of any rectangle that can be tiled must be a multiple of 3 since it is completely covered by triominos with area 3.
Since the area of an $H \times W$ rectangle is HW, then we need HW to be a multiple of 3, which means that at least one of H and W is a multiple of 3.
Since a rectangle that is $a \times b$ can be tiled if and only if a rectangle that is $b \times a$ can be tiled (we see this by rotating the tilings by 90° as we did with the 3×2 and 2×3 rectangles above), then we may assume without loss of generality that H is divisible by 3.
We show that if H is divisible by 3, then every $H \times W$ rectangle with $H \geq 4$ and $W \geq 4$ can be tiled.

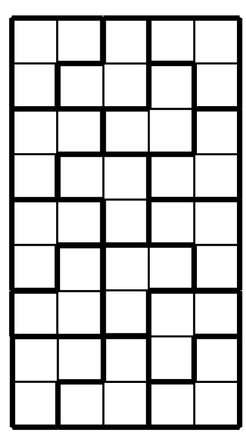
Case 1: H is divisible by 3, H is even
Here, H is a multiple of 6, say $H = 6m$ for some positive integer m.
Since $W \geq 4$, we know that a $6 \times W$ rectangle can be tiled.
By stacking m $6 \times W$ rectangles on top of each other, we obtain a $6m \times W$ rectangle.
Since each $6 \times W$ rectangle can be tiled, then the $6m \times W$ rectangle can be tiled.

Case 2: H is divisible by 3, H is odd, W is even
Suppose that $H = 3q$ for some odd positive integer q and $W = 2r$ for some positive integer r.
To tile a $3q \times 2r$ rectangle, we combine qr 3×2 rectangles in q rows and r columns:



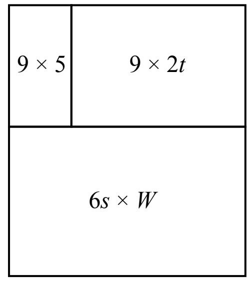
Therefore, every rectangle in this case can be tiled. (Note that in this case the fact that q was odd was not important.)

Case 3: H is divisible by 3, H is odd, W is odd
Since $H \geq 4$ and $W \geq 4$, the rectangle with the smallest values of H and W is 9×5 which can be tiled as shown:



(There are also other ways to tile this rectangle.)
Since H is an odd multiple of 3 and $H \geq 4$, we can write $H = 9 + 6s$ for some integer $s \geq 0$.

Since W is odd and $W \geq 5$, we can write $W = 5 + 2t$ for some integer $t \geq 0$.
Thus, the $H \times W$ rectangle is $(9 + 6s) \times (5 + 2t)$.
We break this rectangle into three rectangles – one that is 9×5, one that is $9 \times 2t$, and one that is $6s \times W$:



(If $s = 0$ or $t = 0$, there will be fewer than three rectangles.)
The 9×5 rectangle can be tiled as we showed earlier.
If $t > 0$, the $9 \times 2t$ rectangle can be tiled as seen in Case 2.
If $s > 0$, the $6s \times W$ rectangle can be tiled as seen in Case 1.
Therefore, the $H \times W$ rectangle can be tiled.

Through these three cases, we have shown that any $H \times W$ rectangle with $H \geq 4$ and $W \geq 4$ can be tiled when H is a multiple of 3.
Since the roles of H and W can be interchanged and since at least one of H and W must be a multiple of 3, then an $H \times W$ rectangle with $H \geq 4$ and $W \geq 4$ can be tiled exactly when at least one of H and W is a multiple of 3.

10. (a) We draw part of the array using the information that $A_0 = A_1 = A_2 = 0$ and $A_3 = 1$:

$\cdots$	A_0	A_1	A_2	A_3	A_4	A_5	$\cdots$
$\cdots$	B_0	B_1	B_2	B_3	B_4	B_5	$\cdots$

$\cdots$	0	0	0	1	A_4	A_5	$\cdots$
$\cdots$	B_0	B_1	B_2	B_3	B_4	B_5	$\cdots$

Since A_1 is the average of A_0, B_1 and A_2, then $A_1 = \dfrac{A_0 + B_1 + A_2}{3}$ or $3A_1 = A_0 + B_1 + A_2$.
Thus, $3(0) = 0 + B_1 + 0$ and so $B_1 = 0$.
Since A_2 is the average of A_1, B_2 and A_3, then $3A_2 = A_1 + B_2 + A_3$ and so $3(0) = 0 + B_2 + 1$ which gives $B_2 = -1$.
Since B_2 is the average of B_1, A_2 and B_3, then $3B_2 = B_1 + A_2 + B_3$ and so $3(-1) = 0 + 0 + B_3$ which gives $B_3 = -3$.
So far, this gives

$\cdots$	0	0	0	1	A_4	A_5	$\cdots$
$\cdots$	B_0	0	-1	-3	B_4	B_5	$\cdots$

Since A_3 is the average of A_2, B_3 and A_4, then $3A_3 = A_2 + B_3 + A_4$ and so $3(1) = 0 + (-3) + A_4$ which gives $A_4 = 6$.

(b) We draw part of the array:

$\cdots$	A_{k-1}	A_k	A_{k+1}	$\cdots$
$\cdots$	B_{k-1}	B_k	B_{k+1}	$\cdots$

Then

$$\begin{aligned}
3S_k &= 3A_k + 3B_k \\
&= 3\left(\frac{A_{k-1} + B_k + A_{k+1}}{3}\right) + 3\left(\frac{B_{k-1} + A_k + B_{k+1}}{3}\right) \\
&= A_{k-1} + B_k + A_{k+1} + B_{k-1} + A_k + B_{k+1} \\
&= (A_{k-1} + B_{k-1}) + (A_k + B_k) + (A_{k+1} + B_{k+1}) \\
&= S_{k-1} + S_k + S_{k+1}
\end{aligned}$$

Since $3S_k = S_{k-1} + S_k + S_{k+1}$, then $S_{k+1} = 2S_k - S_{k-1}$.

(c) *Proof of statement (P)*
Suppose that all of the entries in the array are positive integers.
Assume that not all of the entries in the array are equal.
Since all of the entries are positive integers, there must be a minimum entry. Let m be the minimum of all of the entries in the array.
Choose an entry in the array equal to m, say $A_r = m$ for some integer r. The same argument can be applied with $B_r = m$ if there are no entries equal to m in the top row.
If not all of the entries A_j are equal to m, then by moving one direction or the other along the row we will get to some point where $A_t = m$ for some integer t but one of its neighbours is not equal to m. (If this were not to happen, then all of the entries in both directions would be equal to m.)
If all of the entries A_j are equal to m, then since not all of the entries in the array are equal to m, then there will be an entry B_t which is not equal to m.
In other words, since not all of the entries in the array are equal, then there exists an integer t for which $A_t = m$ and not all of A_{t-1}, A_{t+1}, B_t are equal to m.
But $3m = 3A_t$ and $3A_t = A_{t-1} + B_t + A_{t+1}$ so $3m = A_{t-1} + B_t + A_{t+1}$.
Since not all of A_{t-1}, B_t and A_{t+1} are equal to m and each is at least m, then one of these entries will be greater than m.
This means that $A_{t-1} + B_t + A_{t+1} \geq m + m + (m+1) = 3m + 1 > 3m$, which is a contradiction.
Therefore our assumption that not all of the entries are equal must be false, which means that all of the entries are equal, which proves statement (P).

Proof of statement (Q)
Suppose that all of the entries are positive real numbers.
Assume that not all of the entries in the array are equal.
As in (b), define $S_k = A_k + B_k$ for each integer k.
Also, define $D_k = A_k - B_k$ for each integer k.

<u>Step 1: Prove that the numbers S_k form an arithmetic sequence</u>
From (b), $S_{k+1} = 2S_k - S_{k-1}$.
Re-arranging, we see $S_{k+1} - S_k = S_k - S_{k-1}$ for each integer k, which means that the differences between consecutive pairs of terms are equal.
Since this is true for all integers k, then the difference between each pair of consecutive

terms through the whole sequence is constant, which means that the sequence is an arithmetic sequence.

Step 2: Prove that S_k is constant

Suppose that $S_0 = c$. Since $A_0 > 0$ and $B_0 > 0$, then $S_0 = c > 0$.
Since the terms S_k form an arithmetic sequence, then the sequence is either constant, increasing or decreasing.
If the sequence of terms S_k is increasing, then the common difference $d = S_1 - S_0$ is positive.
Note that $S_{-1} = c - d$, $S_{-2} = c - 2d$, and so on.
Since c and d are constant, then if we move far enough back along the sequence, eventually S_t will be negative for some integer t. This is a contradiction since $A_t > 0$ and $B_t > 0$ and $S_t = A_t + B_t$.
Thus, the sequence cannot be increasing.
If the sequence of terms S_k is decreasing, then the common difference $d = S_1 - S_0$ is negative.
Note that $S_1 = c + d$, $S_2 = c + 2d$, and so on.
Since c and d are constant, then if we move far along the sequence, eventually S_t will be negative for some integer t. This is also a contradiction since $A_t > 0$ and $B_t > 0$ and $S_t = A_t + B_t$.
Thus, the sequence cannot be decreasing.
Therefore, since all of the entries are positive and the sequence S_k is arithmetic, then S_k is constant, say $S_k = c > 0$ for all integers k.

Step 3: Determine range of possible values for D_k

We note that $S_k = A_k + B_k = c$ for all integers k and $A_k > 0$ and $B_k > 0$.
Since $A_k > 0$, then $B_k = S_k - A_k = c - A_k < c$.
Similarly, $A_k < c$.
Therefore, $0 < A_k < c$ and $0 < B_k < c$.
Since $D_k = A_k - B_k$, then $D_k < c - 0 = c$ and $D_k > 0 - c = -c$.
In other words, $-c < D_k < c$.

Step 4: $D_{k+1} = 4D_k - D_{k-1}$

Using a similar approach to our solution to (b),

$$\begin{aligned} 3D_k &= 3A_k - 3B_k \\ 3D_k &= (A_{k-1} + B_k + A_{k+1}) - (B_{k-1} + A_k + B_{k+1}) \\ 3D_k &= (A_{k+1} - B_{k+1}) + (A_{k-1} - B_{k-1}) - (A_k - B_k) \\ 3D_k &= D_{k+1} + D_{k-1} - D_k \\ 4D_k - D_{k-1} &= D_{k+1} \end{aligned}$$

as required.

Step 5: Final contradiction

We want to show that $D_k = 0$ for all integers k.
This will show that $A_k = B_k$ for all integers k.
Since $S_k = A_k + B_k = c$ for all integers k, then this would show that $A_k = B_k = \frac{1}{2}c$ for all integers k, meaning that all entries in the array are equal.
Suppose that $D_k \neq 0$ for some integer k.
We may assume that $D_0 \neq 0$. (If $D_0 = 0$, then because the array is infinite in both directions, we can shift the numbering of the array so that a column where $D_k \neq 0$ is

labelled column 0.)
Thus, $D_0 > 0$ or $D_0 < 0$.
We may assume that $D_0 > 0$. (If $D_0 < 0$, then we can switch the bottom and top rows of the array so that D_0 becomes positive.)
Suppose that $D_1 \geq D_0 > 0$.
Then $D_2 = 4D_1 - D_0 \geq 4D_1 - D_1 = 3D_1$. Since $D_1 > 0$, this also means that $D_2 > D_1 > 0$.
Similarly, $D_3 = 4D_2 - D_1 \geq 4D_2 - D_2 = 3D_2 > D_2 > 0$. Since $D_2 \geq 3D_1$, then $D_3 \geq 9D_1$.
Continuing in this way, we see that $D_4 \geq 27D_1$ and $D_5 \geq 81D_1$ and so on, with $D_k \geq 3^{k-1}D_1$ for each positive integer $k \geq 2$. Since the value of D_1 is a fixed positive real number and $D_k < c$ for all integers k, this is a contradiction, because the sequence of values 3^{k-1} grows without bound.
The other possibility is that $D_1 < D_0$.
Here, we re-arrange $D_{k+1} = 4D_k - D_{k-1}$ to obtain $D_{k-1} = 4D_k - D_{k+1}$.
Thus, $D_{-1} = 4D_0 - D_1 > 4D_0 - D_0 = 3D_0 > D_0 > 0$.
Extending this using a similar method, we see that $D_{-j} > 3^j D_0$ for all positive integers j which will lead to the same contradiction as above.
Therefore, a contradiction is obtained in all cases and so it cannot be the case that $D_k \neq 0$ for some integer k.

Since $D_k = 0$ and $S_k = c$ for all integers k, then $A_k = B_k = \frac{1}{2}c$ for all integers k, which means that all entries in the array are equal.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2017 Euclid Contest

Thursday, April 6, 2017
(in North America and South America)

Friday, April 7, 2017
(outside of North America and South America)

Solutions

1. (a) Since $5(2) + 3(3) = 19$, then the pair of positive integers that satisfies $5a + 3b = 19$ is $(a, b) = (2, 3)$.

   (b) We list the first several powers of 2 in increasing order:

n	1	2	3	4	5	6	7	8	9	10	11
2^n	2	4	8	16	32	64	128	256	512	1024	2048

   Each power of 2 can be found by multiplying the previous power by 2.
   From the table, the smallest power of 2 greater than 5 is $2^3 = 8$ and the largest power of 2 less than 2017 is $2^{10} = 1024$. Since 2^n increases as n increases, there can be no further powers in this range.
   Therefore, the values of n for which $5 < 2^n < 2017$ are $n = 3, 4, 5, 6, 7, 8, 9, 10$.
   There are 8 such values of n.

   (c) Each of the 600 Euros that Jimmy bought cost \$1.50.
   Thus, buying 600 Euros cost $600 \times \$1.50 = \$900$.
   When Jimmy converted 600 Euros back into dollars, the rate was $\$1.00 = 0.75$ Euro.
   Therefore, Jimmy received $600 \text{ Euros} \times \dfrac{\$1.00}{0.75 \text{ Euros}} = \dfrac{\$600}{0.75} = \$800$.
   Thus, Jimmy had $\$900 - \$800 = \$100$ less than he had before these two transactions.

2. (a) Since $x \neq 0$ and $x \neq 1$, we can multiply both sides of the given equation by $x(x-1)$ to obtain $\dfrac{5x(x-1)}{x(x-1)} = \dfrac{x(x-1)}{x} + \dfrac{x(x-1)}{x-1}$ or $5 = (x-1) + x$.
   Thus, $5 = 2x - 1$ and so $2x = 6$ or $x = 3$. This means that $x = 3$ is the only solution.
   (We can substitute $x = 3$ into the original equation to verify that this is indeed a solution.)

   (b) The sum of the entries in the second column is $20 + 4 + (-12) = 12$.
   This means that the sum of the entries in each row, in each column, and on each diagonal is 12.
   In the first row, we have $0 + 20 + a = 12$ and so $a = -8$.
   On the "top left to bottom right" diagonal, we have $0 + 4 + b = 12$ and so $b = 8$.
   In the third column, we have entries $a = -8$ and $b = 8$ whose sum is 0. Thus, the third entry must be 12.
   Finally, in the second row, we have $c + 4 + 12 = 12$ and so $c = -4$.
   In summary, $a = -8$, $b = 8$, and $c = -4$.
   We can complete the magic square to obtain:

0	20	-8
-4	4	12
16	-12	8

(c) (i) If $100^2 - n^2 = 9559$, then $n^2 = 100^2 - 9559 = 10\,000 - 9559 = 441$.
Since $n > 0$, then $n = \sqrt{441} = 21$.

(ii) From (i), $9559 = 100^2 - 21^2$.
Factoring the right side as a difference of squares, we see that

$$9559 = (100 + 21)(100 - 21) = 121 \cdot 79$$

Therefore, $(a, b) = (121, 79)$ satisfies the conditions.
(In addition, the pairs $(a, b) = (79, 121), (869, 11), (11, 869)$ satisfy the conditions. The last two of these pairs cannot be obtained in the same way.)

3. (a) The area of quadrilateral $ABCD$ is the sum of the areas of $\triangle ABC$ and $\triangle ACD$.
Since $\triangle ABC$ is right-angled at B, its area equals $\frac{1}{2}(AB)(BC) = \frac{1}{2}(3)(4) = 6$.
Since $\triangle ABC$ is right-angled at B, then by the Pythagorean Theorem,

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

because $AC > 0$. (We could have also observed that $\triangle ABC$ must be a "3-4-5" triangle.)
Since $\triangle ACD$ is right-angled at A, then by the Pythagorean Theorem,

$$AD = \sqrt{CD^2 - AC^2} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$$

because $AD > 0$. (We could have also observed that $\triangle ACD$ must be a "5-12-13" triangle.)
Thus, the area of $\triangle ACD$ equals $\frac{1}{2}(AC)(AD) = \frac{1}{2}(5)(12) = 30$.
Finally, the area of quadrilateral $ABCD$ is thus $6 + 30 = 36$.

(b) Let the width of each of the identical rectangles be a.
In other words, $QP = RS = TW = WX = UV = VY = a$.
Let the height of each of the identical rectangles be b.
In other words, $QR = PS = TU = WV = XY = b$.
The perimeter of the whole shape equals

$$QP + PS + SX + XY + VY + UV + TU + TR + QR$$

Substituting for known lengths, we obtain

$$a + b + SX + b + a + a + b + TR + b$$

or $3a + 4b + (SX + TR)$.
But $SX + TR = (TR + RS + SX) - RS = (TW + WX) - RS = a + a - a = a$.
Therefore, the perimeter of the whole shape equals $4a + 4b$.
The perimeter of one rectangle is $2a + 2b$, which we are told equals 21 cm.
Finally, the perimeter of the whole shape is thus $2(2a + 2b)$ which equals 42 cm.

(c) *Solution 1*
Suppose that the rectangular prism has dimensions a cm by b cm by c cm.
Suppose further that one of the faces that is a cm by b cm is the face with area 27 cm^2 and that one of the faces that is a cm by c cm is the face with area 32 cm^2. (Since every pair of non-congruent faces shares exactly one side length, there is no loss of generality in picking these particular variables for these faces.)
Therefore, $ab = 27$ and $ac = 32$.
Further, we are told that the volume of the prism is 144 cm^3, and so $abc = 144$.

Thus, $bc = \dfrac{a^2b^2c^2}{a^2bc} = \dfrac{(abc)^2}{(ab)(ac)} = \dfrac{144^2}{(27)(32)} = 24$.
(We could also note that $abc = 144$ means $a^2b^2c^2 = 144^2$ or $(ab)(ac)(bc) = 144^2$ and so $bc = \dfrac{144^2}{(27)(32)}$.)
In other words, the third type of face of the prism has area 24 cm^2.
Thus, since the prism has two faces of each type, the surface area of the prism is equal to $2(27\text{ cm}^2 + 32\text{ cm}^2 + 24\text{ cm}^2)$ or 166 cm^2.

Solution 2
Suppose that the rectangular prism has dimensions a cm by b cm by c cm.
Suppose further that one of the faces that is a cm by b cm is the face with area 27 cm^2 and that one of the faces that is a cm by c cm is the face with area 32 cm^2. (Since every pair of non-congruent faces shares exactly one side length, there is no loss of generality in picking these particular variables for these faces.)
Therefore, $ab = 27$ and $ac = 32$.
Further, we are told that the volume of the prism is 144 cm^3, and so $abc = 144$.
Since $abc = 144$ and $ab = 27$, then $c = \frac{144}{27} = \frac{16}{3}$.
Since $abc = 144$ and $ac = 32$, then $b = \frac{144}{32} = \frac{9}{2}$.
This means that $bc = \frac{16}{3} \cdot \frac{9}{2} = 24$.
In cm^2, the surface area of the prism equals $2ab + 2ac + 2bc = 2(27) + 2(32) + 2(24) = 166$.
Thus, the surface area of the prism is 166 cm^2.

4. (a) *Solution 1*
We expand the right sides of the two equations, collecting like terms in each case:

$$y = a(x-2)(x+4) = a(x^2+2x-8) = ax^2+2ax-8a$$
$$y = 2(x-h)^2+k = 2(x^2-2hx+h^2)+k = 2x^2-4hx+(2h^2+k)$$

Since these two equations represent the same parabola, then the corresponding coefficients must be equal. That is, $a = 2$ and $2a = -4h$ and $-8a = 2h^2 + k$.
Since $a = 2$ and $2a = -4h$, then $4 = -4h$ and so $h = -1$.
Since $-8a = 2h^2 + k$ and $a = 2$ and $h = -1$, then $-16 = 2 + k$ and so $k = -18$.
Thus, $a = 2$, $h = -1$, and $k = -18$.

Solution 2
From the equation $y = a(x-2)(x+4)$, we can find the axis of symmetry by calculating the midpoint of the x-intercepts.
Since the x-intercepts are 2 and -4, the axis of symmetry is at $x = \frac{1}{2}(2 + (-4)) = -1$.
Since the vertex of the parabola lies on the axis of symmetry, then the x-coordinate of the vertex is -1.
To find the y-coordinate of the vertex, we substitute $x = -1$ back into the equation $y = a(x-2)(x+4)$ to obtain $y = a(-1-2)(-1+4) = -9a$.
Thus, the vertex of the parabola is $(-1, -9a)$.
Since the second equation for the same parabola is in vertex form, $y = 2(x-h)^2 + k$, we can see that the vertex is at (h, k) and $a = 2$.
Since $a = 2$, the vertex has coordinates $(-1, -18)$, which means that $h = -1$ and $k = -18$.
Thus, $a = 2$, $h = -1$ and $k = -18$.

(b) Let the common difference in this arithmetic sequence be d.
Since the first term in the sequence is 5, then the 5 terms are $5, 5+d, 5+2d, 5+3d, 5+4d$.
From the given information, $5^2 + (5+d)^2 + (5+2d)^2 = (5+3d)^2 + (5+4d)^2$.
Manipulating, we obtain the following equivalent equations:

$$\begin{aligned} 5^2 + (5+d)^2 + (5+2d)^2 &= (5+3d)^2 + (5+4d)^2 \\ 25 + (25+10d+d^2) + (25+20d+4d^2) &= (25+30d+9d^2) + (25+40d+16d^2) \\ 75+30d+5d^2 &= 50+70d+25d^2 \\ 0 &= 20d^2+40d-25 \\ 0 &= 4d^2+8d-5 \\ 0 &= (2d+5)(2d-1) \end{aligned}$$

Therefore, $d = -\frac{5}{2}$ or $d = \frac{1}{2}$.
These give possible fifth terms of $5+4d = 5+4(-\frac{5}{2}) = -5$ and $5+4d = 5+4(\frac{1}{2}) = 7$.
(We note that, for these two values of d, the sequences are $5, \frac{5}{2}, 0, -\frac{5}{2}, -5$ and $5, \frac{11}{2}, 6, \frac{13}{2}, 7$.)

5. (a) First, we determine the perfect squares between 1300 and 1400 and between 1400 and 1500.
Since $\sqrt{1300} \approx 36.06$, then the first perfect square larger than 1300 is $37^2 = 1369$.
The next perfect squares are $38^2 = 1444$ and $39^2 = 1521$.
Since Dan was born between 1300 and 1400 in a year that was a perfect square, then Dan was born in 1369.
Since Steve was born between 1400 and 1500 in a year that was a perfect square, then Steve was born in 1444.
Suppose that on April 7 in some year, Dan was m^2 years old and Steve was n^2 years old for some positive integers m and n. Thus, Dan was m^2 years old in the year $1369+m^2$ and Steve was n^2 years old in the year $1444+n^2$.
Since these represent the same years, then $1369+m^2 = 1444+n^2$, or $m^2 - n^2 = 1444 - 1369 = 75$.
In other words, we want to find two perfect squares less than 110 (since their ages are less than 110) whose difference is 75.
The perfect squares less than 110 are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$.
The two that differ by 75 are 100 and 25.
Thus, $m^2 = 100$ and $n^2 = 25$.
This means that the year in which the age of each of Dan and Steve was a perfect square was the year $1369 + 100 = 1469$.

(b) *Solution 1*
$\triangle ABC$ is right-angled exactly when one of the following statements is true:

- AB is perpendicular to BC, or
- AB is perpendicular to AC, or
- AC is perpendicular to BC.

Since $A(1,2)$ and $B(11,2)$ share a y-coordinate, then AB is horizontal.
For AB and BC to be perpendicular, BC must be vertical.
Thus, $B(11,2)$ and $C(k,6)$ must have the same x-coordinate, and so $k = 11$.
For AB and AC to be perpendicular, AC must be vertical.
Thus, $A(1,2)$ and $C(k,6)$ must have the same x-coordinate, and so $k = 1$.

For AC to be perpendicular to BC, their slopes must have a product of -1.
The slope of AC is $\dfrac{6-2}{k-1}$, which equals $\dfrac{4}{k-1}$.
The slope of BC is $\dfrac{6-2}{k-11}$, which equals $\dfrac{4}{k-11}$.
Thus, AC and BC are perpendicular when $\dfrac{4}{k-1}\cdot\dfrac{4}{k-11}=-1$.
Assuming that $k\neq 1$ and $k\neq 11$, we manipulate to obtain $16=-(k-1)(k-11)$ or $16=-k^2+12k-11$ or $k^2-12k+27=0$.
Factoring, we obtain $(k-3)(k-9)=0$ and so AC and BC are perpendicular when $k=3$ or $k=9$.

In summary, $\triangle ABC$ is right-angled when k equals one of $1,3,9,11$.

Solution 2
$\triangle ABC$ is right-angled exactly when its three side lengths satisfy the Pythagorean Theorem in some orientation. That is, $\triangle ABC$ is right-angled exactly when $AB^2+BC^2=AC^2$ or $AB^2+AC^2=BC^2$ or $AC^2+BC^2=AB^2$.
Using $A(1,2)$ and $B(11,2)$, we obtain $AB^2=(11-1)^2+(2-2)^2=100$.
Using $A(1,2)$ and $C(k,6)$, we obtain $AC^2=(k-1)^2+(6-2)^2=(k-1)^2+16$.
Using $B(11,2)$ and $C(k,6)$, we obtain $BC^2=(k-11)^2+(6-2)^2=(k-11)^2+16$.
Using the Pythagorean relationships above, $\triangle ABC$ is right-angled when one of the following is true:

(i)

$$\begin{aligned}100+((k-11)^2+16)&=(k-1)^2+16\\100+k^2-22k+121+16&=k^2-2k+1+16\\220&=20k\\k&=11\end{aligned}$$

(ii)

$$\begin{aligned}100+((k-1)^2+16)&=(k-11)^2+16\\100+k^2-2k+1+16&=k^2-22k+121+16\\20k&=20\\k&=1\end{aligned}$$

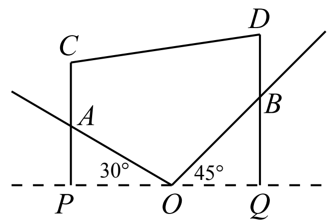
(iii)

$$\begin{aligned}((k-1)^2+16)+((k-11)^2+16)&=100\\k^2-2k+1+16+k^2-22k+121+16&=100\\2k^2-24k+54&=0\\k^2-12k+27&=0\\(k-3)(k-9)&=0\end{aligned}$$

and so $k=3$ or $k=9$.

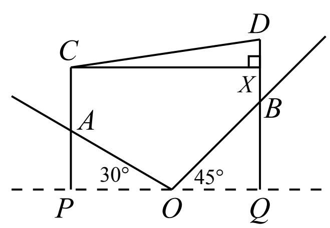
In summary, $\triangle ABC$ is right-angled when k equals one of $1,3,9,11$.

6. (a) Extend CA and DB downwards until they meet the horizontal through O at P and Q, respectively.



Since CA and DB are vertical, then $\angle CPO = \angle DQO = 90^\circ$.
Since $OA = 20$ m, then $AP = OA \sin 30^\circ = (20 \text{ m}) \cdot \frac{1}{2} = 10$ m.
Since $OB = 20$ m, then $BQ = OB \sin 45^\circ = (20 \text{ m}) \cdot \frac{1}{\sqrt{2}} = 10\sqrt{2}$ m.
Since $AC = 6$ m, then $CP = AC + AP = 16$ m.
For CD to be as short as possible and given that C is fixed, then it must be the case that CD is horizontal:

If CD were not horizontal, then suppose that X is on DQ, possibly extended, so that CX is horizontal.



Then $\angle CXD = 90^\circ$ and so $\triangle CXD$ is right-angled with hypotenuse CD.
In this case, CD is longer than CX or XD.
In particular, $CD > CX$, which means that if D were at X, then CD would be shorter.
In other words, a horizontal CD makes CD as short as possible.

When CD is horizontal, $CDQP$ is a rectangle, since it has two vertical and two horizontal sides. Thus, $DQ = CP = 16$ m.
Finally, this means that $BD = DQ - BQ = (16 - 10\sqrt{2})$ m.

(b) Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then we assume that $\cos\theta \neq 0$.
Therefore, we obtain the following equivalent equations:

$$\begin{aligned}
\cos\theta &= \tan\theta \\
\cos\theta &= \frac{\sin\theta}{\cos\theta} \\
\cos^2\theta &= \sin\theta \\
1 - \sin^2\theta &= \sin\theta \\
0 &= \sin^2\theta + \sin\theta - 1
\end{aligned}$$

Let $u = \sin\theta$. This quadratic equation becomes $u^2 + u - 1 = 0$.

By the quadratic formula, $u = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2}$.

Therefore, $\sin\theta = \dfrac{-1 + \sqrt{5}}{2} \approx 0.62$ or $\sin\theta = \dfrac{-1 - \sqrt{5}}{2} \approx -1.62$.

Since $-1 \leq \sin\theta \leq 1$, then the second solution is inadmissible. Thus, $\sin\theta = \dfrac{-1 + \sqrt{5}}{2}$.

7. (a) *Solution 1*

Suppose that the trains are travelling at v km/h.

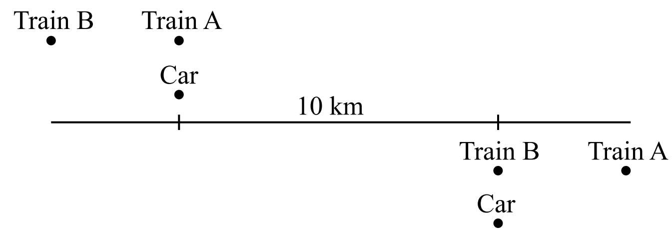
Consider two consecutive points in time at which the car is passed by a train.

Since these points are 10 minutes apart, and 10 minutes equals $\frac{1}{6}$ hour, and the car travels at 60 km/h, then the car travels $(60 \text{ km/h}) \cdot (\frac{1}{6} \text{ h}) = 10$ km.

During these 10 minutes, each train travels $\frac{1}{6}v$ km, since its speed is v km/h.

At the first instance, Train A and the car are next to each other.

At this time, Train B is "3 minutes" behind Train A.



Since 3 minutes is $\frac{1}{20}$ hour, then Train B is $\frac{1}{20}v$ km behind Train A and the car.

Therefore, the distance from the location of Train B at the first instance to the location where it passes the car is $(\frac{1}{20}v + 10)$ km.

But this distance also equals $\frac{1}{6}v$ km, since Train B travels for 10 minutes.

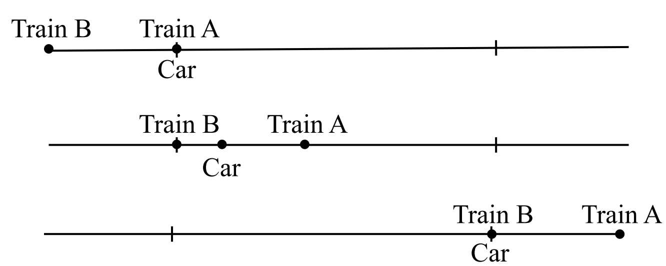
Thus, $\frac{1}{6}v = \frac{1}{20}v + 10$ or $\frac{10}{60}v - \frac{3}{60}v = 10$ and so $\frac{7}{60}v = 10$ or $v = \frac{600}{7}$.

Therefore, the trains are travelling at $\frac{600}{7}$ km/h.

Solution 2

Suppose that the trains are travelling at v km/h.

Consider the following three points in time: the instant when the car and Train A are next to each other, the instant when Train B is at the same location that the car and Train A were at in the previous instant, and the instant when the car and Train B are next to each other.



From the first instant to the second, Train B "catches up" to where Train A was, so this must take a total of 3 minutes, because the trains leave the station 3 minutes apart.

Since 3 minutes equals $\frac{3}{60}$ hour and the car travels at 60 km/h, then the car travels $(60 \text{ km/h}) \cdot (\frac{3}{60} \text{ h}) = 3$ km between these two instants.

From the first instant to the third, 10 minutes passes, since these are consecutive points at which the car is passed by trains. In 10 minutes, the car travels 10 km.

Therefore, between the second and third instants, $10 - 3 = 7$ minutes pass. During these 7 minutes, Train B travels 10 km.

Since 7 minutes equals $\frac{7}{60}$ hour, then $v \text{ km/h} = \frac{10 \text{ km}}{7/60 \text{ h}} = \frac{600}{7}$ km/h, and so the trains are travelling at $\frac{600}{7}$ km/h.

(b) From the first equation, we note that $a \geq 0$ and $b \geq 0$, since the argument of a square root must be non-negative.
From the second equation, we note that $a > 0$ and $b > 0$, since the argument of a logarithm must be positive.
Combining these restrictions, we see that $a > 0$ and $b > 0$.

From the equation $\log_{10} a + \log_{10} b = 2$, we obtain $\log_{10}(ab) = 2$ and so $ab = 10^2 = 100$.
From the first equation, obtain

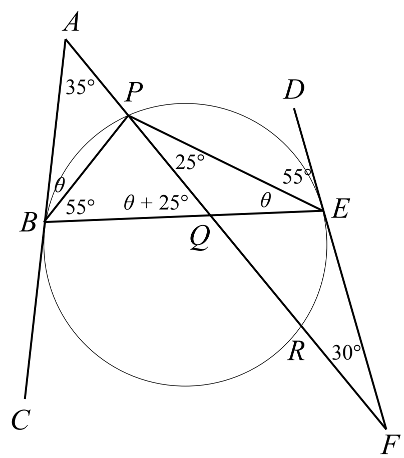
$$\begin{aligned}
(\sqrt{a} + \sqrt{b})^2 &= 8^2 \\
a + 2\sqrt{ab} + b &= 64 \\
a + 2\sqrt{100} + b &= 64 \\
a + b &= 64 - 2\sqrt{100} = 44
\end{aligned}$$

Since $a + b = 44$, then $b = 44 - a$.
Since $ab = 100$, then $a(44 - a) = 100$ or $44a - a^2 = 100$ and so $0 = a^2 - 44a + 100$.
By the quadratic formula,

$$a = \frac{44 \pm \sqrt{44^2 - 4(1)(100)}}{2 \cdot 1} = \frac{44 \pm \sqrt{1536}}{2} = \frac{44 \pm 16\sqrt{6}}{2} = 22 \pm 8\sqrt{6}$$

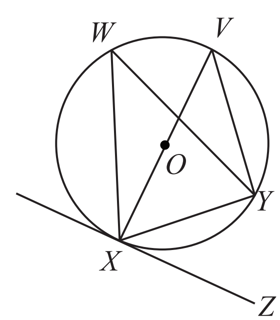
Since $b = 44 - a$, then $b = 44 - (22 \pm 8\sqrt{6}) = 22 \mp 8\sqrt{6}$.
Therefore, $(a, b) = (22 + 8\sqrt{6}, 22 - 8\sqrt{6})$ or $(a, b) = (22 - 8\sqrt{6}, 22 + 8\sqrt{6})$.
(We note that $22 + 8\sqrt{6} > 0$ and $22 - 8\sqrt{6} > 0$, so the initial restrictions on a and b are satisfied.)

8. (a) Let $\angle PEQ = \theta$.
Join P to B.
We use the fact that the angle between a tangent to a circle and a chord in that circle that passes through the point of tangency equals the angle inscribed by that chord. We prove this fact below.
More concretely, $\angle DEP = \angle PBE$ (using the chord EP and the tangent through E) and $\angle ABP = \angle PEQ = \theta$ (using the chord BP and the tangent through B).
Now $\angle DEP$ is exterior to $\triangle FEP$ and so $\angle DEP = \angle FPE + \angle EFP = 25^\circ + 30^\circ$, and so $\angle PBE = \angle DEP = 55^\circ$.
Furthermore, $\angle AQB$ is an exterior angle of $\triangle PQE$.
Thus, $\angle AQB = \angle QPE + \angle PEQ = 25^\circ + \theta$.



In $\triangle ABQ$, we have $\angle BAQ = 35^\circ$, $\angle ABQ = \theta + 55^\circ$, and $\angle AQB = 25^\circ + \theta$.
Thus, $35^\circ + (\theta + 55^\circ) + (25^\circ + \theta) = 180^\circ$ or $115^\circ + 2\theta = 180^\circ$, and so $2\theta = 65^\circ$.
Therefore $\angle PEQ = \theta = \frac{1}{2}(65^\circ) = 32.5^\circ$.

As an addendum, we prove that the angle between a tangent to a circle and a chord in that circle that passes through the point of tangency equals the angle inscribed by that chord.
Consider a circle with centre O and a chord XY, with tangent ZX meeting the circle at X. We prove that if ZX is tangent to the circle, then $\angle ZXY$ equals $\angle XWY$ whenever W is a point on the circle on the opposite side of XY as XZ (that is, the angle subtended by XY on the opposite side of the circle).

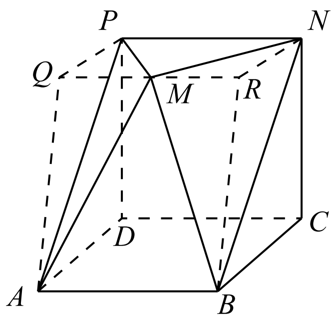


We prove this in the case that $\angle ZXY$ is acute. The cases where $\angle ZXY$ is a right angle or an obtuse angle are similar.
Draw diameter XOV and join VY.
Since $\angle ZXY$ is acute, points V and W are on the same arc of chord XY.
This means that $\angle XVY = \angle XWY$, since they are angles subtended by the same chord.
Since OX is a radius and XZ is a tangent, then $\angle OXZ = 90^\circ$.
Thus, $\angle OXY + \angle ZXY = 90^\circ$.
Since XV is a diameter, then $\angle XYV = 90^\circ$.
From $\triangle XYV$, we see that $\angle XVY + \angle VXY = 90^\circ$.
But $\angle OXY + \angle ZXY = 90^\circ$ and $\angle XVY + \angle VXY = 90^\circ$ and $\angle OXY = \angle VXY$ tells us that $\angle ZXY = \angle XVY$.
This gives us that $\angle ZXY = \angle XWY$, as required.

(b) *Solution 1*

Draw a line segment through M in the plane of $\triangle PMN$ parallel to PN and extend this line until it reaches the plane through P, A and D at Q on one side and the plane through N, B and C at R on the other side.

Join Q to P and A. Join R to N and B.



So the volume of solid $ABCDPMN$ equals the volume of solid $ABCDPQRN$ minus the volumes of solids $PMQA$ and $NMRB$.

Solid $ABCDPQRN$ is a trapezoidal prism. This is because NR and BC are parallel (since they lie in parallel planes), which makes $NRBC$ a trapezoid. Similarly, $PQAD$ is a trapezoid. Also, PN, QR, DC, and AB are all perpendicular to the planes of these trapezoids and equal in length, since they equal the side lengths of the squares.

Solids $PMQA$ and $NMRB$ are triangular-based pyramids. We can think of their bases as being $\triangle PMQ$ and $\triangle NMR$. Their heights are each equal to 2, the height of the original solid. (The volume of a triangular-based pyramid equals $\frac{1}{3}$ times the area of its base times its height.)

The volume of $ABCDPQRN$ equals the area of trapezoid $NRBC$ times the width of the prism, which is 2.

That is, this volume equals $\frac{1}{2}(NR+BC)(NC)(NP) = \frac{1}{2}(NR+2)(2)(2) = 2 \cdot NR + 4$.

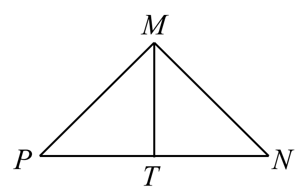
So we need to find the length of NR.

Consider quadrilateral $PNRQ$. This quadrilateral is a rectangle since PN and QR are perpendicular to the two side planes of the original solid.

Thus, NR equals the height of $\triangle PMN$.

Join M to the midpoint T of PN.

Since $\triangle PMN$ is isosceles, then MT is perpendicular to PN.



Since $NT = \frac{1}{2}PN = 1$ and $\angle PMN = 90^\circ$ and $\angle TNM = 45^\circ$, then $\triangle MTN$ is also right-angled and isosceles with $MT = TN = 1$.

Therefore, $NR = MT = 1$ and so the volume of $ABCDPQRN$ is $2 \cdot 1 + 4 = 6$.

The volumes of solids $PMQA$ and $NMRB$ are equal. Each has height 2 and their bases $\triangle PMQ$ and $\triangle NMR$ are congruent, because each is right-angled (at Q and at R) with $PQ = NR = 1$ and $QM = MR = 1$.

Thus, using the formula above, the volume of each is $\frac{1}{3}(\frac{1}{2}(1)(1))2 = \frac{1}{3}$.

Finally, the volume of the original solid equals $6 - 2 \cdot \frac{1}{3} = \frac{16}{3}$.

Solution 2

We determine the volume of $ABCDPMN$ by splitting it into two solids: $ABCDPN$ and $ABNPM$ by slicing along the plane of $ABNP$.

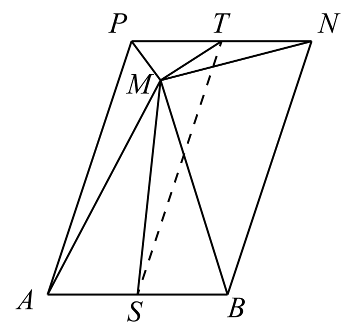
Solid $ABCDPN$ is a triangular prism, since $\triangle BCN$ and $\triangle ADP$ are each right-angled (at C and D), $BC = CN = AD = DP = 2$, and segments PN, DC and AB are perpendicular to each of the triangular faces and equal in length.

Thus, the volume of $ABCDPN$ equals the area of $\triangle BCN$ times the length of DC, or $\frac{1}{2}(BC)(CN)(DC) = \frac{1}{2}(2)(2)(2) = 4$. (This solid can also be viewed as "half" of a cube.)

Solid $ABNPM$ is a pyramid with rectangular base $ABNP$. (Note that PN and AB are perpendicular to the planes of both of the side triangular faces of the original solid, that $PN = AB = 2$ and $BN = AP = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, by the Pythagorean Theorem.)

Therefore, the volume of $ABNPM$ equals $\frac{1}{3}(AB)(BN)h = \frac{4\sqrt{2}}{3}h$, where h is the height of the pyramid (that is, the distance that M is above plane $ABNP$).

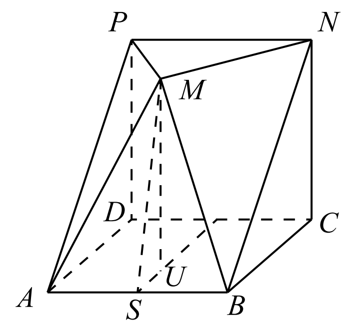
So we need to calculate h.



Join M to the midpoint, T, of PN and to the midpoint, S, of AB. Join S and T. By symmetry, M lies directly above ST.

Since $ABNP$ is a rectangle and S and T are the midpoints of opposite sides, then $ST = AP = 2\sqrt{2}$.

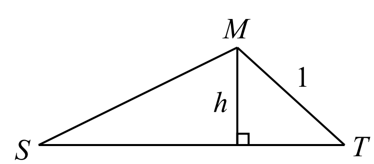
Since $\triangle PMN$ is right-angled and isosceles, then MT is perpendicular to PN. Since $NT = \frac{1}{2}PN = 1$ and $\angle TNM = 45^\circ$, then $\triangle MTN$ is also right-angled and isosceles with $MT = TN = 1$.



Also, MS is the hypotenuse of the triangle formed by dropping a perpendicular from M to U in the plane of $ABCD$ (a distance of 2) and joining U to S. Since M is 1 unit horizontally from PN, then $US = 1$.

Thus, $MS = \sqrt{2^2 + 1^2} = \sqrt{5}$ by the Pythagorean Theorem.

We can now consider $\triangle SMT$. h is the height of this triangle, from M to base ST.



Now $h = MT \sin(\angle MTS) = \sin(\angle MTS)$.

By the cosine law in $\triangle SMT$, we have

$$MS^2 = ST^2 + MT^2 - 2(ST)(MT)\cos(\angle MTS)$$

Therefore, $5 = 8 + 1 - 4\sqrt{2}\cos(\angle MTS)$ or $4\sqrt{2}\cos(\angle MTS) = 4$.

Thus, $\cos(\angle MTS) = \frac{1}{\sqrt{2}}$ and so $\angle MTS = 45^\circ$ which gives $h = \sin(\angle MTS) = \frac{1}{\sqrt{2}}$.

(Alternatively, we note that the plane of $ABCD$ is parallel to the plane of PMN, and so since the angle between plane $ABCD$ and plane $PNBA$ is 45°, then the angle between plane $PNBA$ and plane PMN is also 45°, and so $\angle MTS = 45^\circ$.)

Finally, this means that the volume of $ABNPM$ is $\frac{4\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4}{3}$, and so the volume of solid $ABCDPMN$ is $4 + \frac{4}{3} = \frac{16}{3}$.

9. (a) There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations of $1, 2, 3, 4$.
This is because there are 4 possible choices for a_1, and for each of these there are 3 possible choices for a_2, and for each of these there are 2 possible choices for a_3, and then 1 possible choice for a_4.
Consider the permutation $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$. (We write this as 1,2,3,4.)
Here, $|a_1 - a_2| + |a_3 - a_4| = |1 - 2| + |3 - 4| = 1 + 1 = 2$.
This value is the same as the value for each of 2,1,3,4 and 1,2,4,3 and 2,1,4,3 and 3,4,1,2 and 4,3,1,2 and 3,4,2,1 and 4,3,2,1.
Consider the permutation 1,3,2,4.
Here, $|a_1 - a_2| + |a_3 - a_4| = |1 - 3| + |2 - 4| = 2 + 2 = 4$.
This value is the same as the value for each of 3,1,2,4 and 1,3,4,2 and 3,1,4,2 and 2,4,1,3 and 4,2,1,3 and 2,4,3,1 and 4,2,3,1.
Consider the permutation 1,4,2,3.
Here, $|a_1 - a_2| + |a_3 - a_4| = |1 - 4| + |2 - 3| = 3 + 1 = 4$.
This value is the same as the value for each of 4,1,2,3 and 1,4,3,2 and 4,1,3,2 and 2,3,1,4 and 3,2,1,4 and 2,3,4,1 and 3,2,4,1.
This accounts for all 24 permutations.
Therefore, the average value is $\dfrac{2 \cdot 8 + 4 \cdot 8 + 4 \cdot 8}{24} = \dfrac{80}{24} = \dfrac{10}{3}$.

(b) There are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ permutations of $1, 2, 3, 4, 5, 6, 7$, because there are 7 choices for a_1, then 6 choices for a_2, and so on.
We determine the average value of $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7$ over all of these permutations by determining the sum of all 7! values of this expression and dividing by 7!.
To determine the sum of all 7! values, we determine the sum of the values of a_1 in each of these expressions and call this total s_1, the sum of the values of a_2 in each of these expressions and call this total s_2, and so on.
The sum of the 7! values of the original expression must equal $s_1 - s_2 + s_3 - s_4 + s_5 - s_6 + s_7$.
This uses the fact that, when adding, the order in which we add the same set of numbers does not matter.
By symmetry, the sums of the values of $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ will all be equal. That is, $s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7$.
This means that the desired average value equals

$$\frac{s_1 - s_2 + s_3 - s_4 + s_5 - s_6 + s_7}{7!} = \frac{(s_1 + s_3 + s_5 + s_7) - (s_2 + s_4 + s_6)}{7!} = \frac{4s_1 - 3s_1}{7!} = \frac{s_1}{7!}$$

So we need to determine the value of s_1.
Now a_1 can equal each of $1, 2, 3, 4, 5, 6, 7$.
If $a_1 = 1$, there are 6! combinations of values for $a_2, a_3, a_4, a_5, a_6, a_7$, since there are still 6 choices for a_2, 5 for a_3, and so on.
Similarly, there are 6! combinations with a_1 equal to each of $2, 3, 4, 5, 6, 7$.
Thus, $s_1 = 1 \cdot 6! + 2 \cdot 6! + 3 \cdot 6! + 4 \cdot 6! + 5 \cdot 6! + 6 \cdot 6! + 7 \cdot 6! = 6!(1 + 2 + 3 + 4 + 5 + 6 + 7) = 28(6!)$.
Therefore, the average value of the expression is $\dfrac{28(6!)}{7!} = \dfrac{28(6!)}{7(6!)} = \dfrac{28}{7} = 4$.

(c) There are 200! permutations of $1, 2, 3, \ldots, 198, 199, 200$.
We determine the average value of

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{197} - a_{198}| + |a_{199} - a_{200}| \qquad (*)$$

over all of these permutations by determining the sum of all 200! values of this expression and dividing by 200!.
As in (b), we let s_1 be the sum of the values of $|a_1 - a_2|$ in each of these expressions, s_2 be the sum of the values of $|a_3 - a_4|$, and so on.
The sum of the 200! values of $(*)$ equals $s_1 + s_2 + \cdots + s_{99} + s_{100}$.
By symmetry, $s_1 = s_2 = \cdots = s_{99} = s_{100}$.
Therefore, the average value of $(*)$ equals $\dfrac{100s_1}{200!}$. So we need to determine the value of s_1.
Suppose that $a_1 = i$ and $a_2 = j$ for some integers i and j between 1 and 200, inclusive.
There are 198! permutations with $a_1 = i$ and $a_2 = j$ because there are still 198 choices for a_3, 197 choices for a_4, and so on.
Similarly, there are 198! permutations with $a_1 = j$ and $a_2 = i$.
Since $|i - j| = |j - i|$, then there are 2(198!) permutations with $|a_1 - a_2| = |i - j|$ that come from a_1 and a_2 equalling i and j in some order.
Therefore, we may assume that $i > j$ and note that s_1 equals 2(198!) times the sum of $i - j$ over all possible pairs $i > j$.
(Note that there are $\dbinom{200}{2} = \dfrac{200(199)}{2}$ choices for the pair of integers (i, j) with $i > j$. For each of these choices, there are 2(198!) choices for the remaining entries in the permutation, which gives $\dfrac{200(199)}{2} \cdot 2(198!) = 200(199)(198!) = 200!$ permutations, as expected.)
So to determine s_1, we need to determine the sum of the values of $i - j$.
We calculate this sum, which we call D, by letting $j = 1, 2, 3, \ldots, 198, 199$ and for each of these, we let i be the possible integers with $j < i \leq 200$:

$$\begin{aligned}
D &= (2-1) + (3-1) + (4-1) + \cdots + (197-1) + (198-1) + (199-1) + (200-1) \\
&\quad + (3-2) + (4-2) + (5-2) + \cdots + (198-2) + (199-2) + (200-2) \\
&\quad + (4-3) + (5-3) + (6-3) + \cdots + (199-3) + (200-3) \\
&\quad \vdots \\
&\quad + (199-198) + (200-198) \\
&\quad + (200-199) \\
&= 199(1) + 198(2) + 197(3) + \cdots + 2(198) + 1(199) \qquad \text{(grouping by columns)} \\
&= 199(200-199) + 198(200-198) + 197(200-197) + \cdots + 2(200-2) + 1(200-1) \\
&= 200(199 + 198 + 197 + \cdots + 3 + 2 + 1) - (199^2 + 198^2 + 197^2 + \cdots + 3^2 + 2^2 + 1^2) \\
&= 200 \cdot \tfrac{1}{2}(199)(200) - \tfrac{1}{6}(199)(199+1)(2(199)+1) \\
&= 100(199)(200) - \tfrac{1}{6}(199)(200)(399) \\
&= 199(200)\left(100 - \tfrac{133}{2}\right) \\
&= 199(200)\tfrac{67}{2}
\end{aligned}$$

Therefore, $s_1 = 2(198!)D = 2(198!) \cdot \frac{199(200)(67)}{2} = 67(198!)(199)(200) = 67(200!)$.
Finally, this means that the average value of $(*)$ is $\dfrac{100s_1}{200!} = \dfrac{100(67)(200!)}{200!} = 6700$.

We note that we have used the facts that, if n is a positive integer, then

- $1+2+\cdots+(n-1)+n=\frac{1}{2}n(n+1)$
- $1^2+2^2+\cdots+(n-1)^2+n^2=\frac{1}{6}n(n+1)(2n+1)$

Using sigma notation, we could have calculated D as follows:

$$\begin{aligned}
D &= \sum_{i=2}^{200}\sum_{j=1}^{i-1}(i-j) \\
&= \left(\sum_{i=2}^{200}\sum_{j=1}^{i-1} i\right) - \left(\sum_{i=2}^{200}\sum_{j=1}^{i-1} j\right) \\
&= \left(\sum_{i=2}^{200} i(i-1)\right) - \left(\sum_{i=2}^{200} \tfrac{1}{2}(i-1)i\right) \\
&= \left(\sum_{i=2}^{200} i(i-1)\right) - \tfrac{1}{2}\left(\sum_{i=2}^{200} (i-1)i\right) \\
&= \tfrac{1}{2}\left(\sum_{i=2}^{200} (i-1)i\right) \\
&= \tfrac{1}{2}\left(\sum_{i=1}^{200} (i-1)i\right) \qquad \text{(since } (i-1)i=0 \text{ when } i=1\text{)} \\
&= \tfrac{1}{2}\left(\sum_{i=1}^{200} (i^2-i)\right) \\
&= \tfrac{1}{2}\left(\sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} i\right) \\
&= \tfrac{1}{2}\left(\tfrac{1}{6}(200)(200+1)(2(200)+1) - \tfrac{1}{2}(200)(200+1)\right) \\
&= \tfrac{1}{2}(200)(201)\left(\tfrac{1}{6}(401)-\tfrac{1}{2}\right) \\
&= 100(201)\cdot\tfrac{398}{6} \\
&= 100(201)\cdot\tfrac{199}{3} \\
&= 100(67)(199)
\end{aligned}$$

which equals $199(200)\frac{67}{2}$, as expected. (Can you determine a general formula when 200 is replaced with $2n$?)

10. (a) We start with the subset $\{1,2,3\}$.
The sums of pairs of elements are $1+2=3$ and $1+3=4$ and $2+3=5$, which are all different.
Thus, $\{1,2,3\}$ is exciting.
We proceed to include additional elements in $\{1,2,3\}$.
We cannot include 4 to create an exciting set, since if we did, we would have $1+4=2+3$, and so $\{1,2,3,4\}$ is boring.
Consider the subset $\{1,2,3,5\}$.
The sums of pairs of elements are

$$1+2=3 \qquad 1+3=4 \qquad 1+5=6 \qquad 2+3=5 \qquad 2+5=7 \qquad 3+5=8$$

which are all different.
Thus, $\{1,2,3,5\}$ is exciting.
We cannot include 6 or 7 since $2+5=1+6$ and $3+5=1+7$.
Consider the subset $\{1,2,3,5,8\}$.
In addition to the six sums above, we have the additional sums $1+8=9$ and $2+8=10$ and $3+8=11$ and $5+8=13$, so the 10 sums are all different.
Therefore, $\{1,2,3,5,8\}$ is an exciting subset of $\{1,2,3,4,5,6,7,8\}$ that contains exactly 5 elements.
(The subset $\{1,4,6,7,8\}$ is the only other exciting subset of $\{1,2,3,4,5,6,7,8\}$ that contains exactly 5 elements.)

(b) Suppose that S is an exciting set that contains exactly m elements.
There are $\binom{m}{2} = \dfrac{m(m-1)}{2}$ pairs of elements of S.
Since S is exciting, the sums of these pairs of elements are all distinct positive integers.
This means that the largest of these sums is greater than or equal to $\dfrac{m(m-1)}{2}$.
When two numbers add to $\dfrac{m(m-1)}{2}$ or greater, then at least one of them must be at least as large as $\dfrac{1}{2}\cdot\dfrac{m(m-1)}{2} = \dfrac{m^2-m}{4}$.
Therefore, there is an element of S that is greater than or equal to $\dfrac{m^2-m}{4}$.

(c) Let n be a positive integer with $n \geq 10$.
For each integer k with $1 \leq k \leq n$, define $x_k = 2n \cdot \text{rem}(k^2, n) + k$, where $\text{rem}(k^2, n)$ is the remainder when k^2 is divided by n.
Define $T = \{x_1, x_2, \ldots, x_{n-1}, x_n\}$.
We show that T is exciting exactly when n is prime.

Suppose that a, b, c, d are distinct integers between 1 and n with $x_a + x_b = x_c + x_d$.
This equation is equivalent to

$$(2n \cdot \text{rem}(a^2, n) + a) + (2n \cdot \text{rem}(b^2, n) + b) = (2n \cdot \text{rem}(c^2, n) + c) + (2n \cdot \text{rem}(d^2, n) + d)$$

and

$$2n \cdot (\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n)) = c + d - a - b$$

Since a, b, c, d are distinct integers between 1 and n, inclusive, then we have $1+2 \leq a+b \leq (n-1)+n$, or $3 \leq a+b \leq 2n-1$. Similarly, $3 \leq c+d \leq 2n-1$.

This means that $3-(2n-1) \leq c+d-a-b \leq (2n-1)-3$ or $-2n+4 \leq c+d-a-b \leq 2n-4$.
But the left side of the equation

$$2n \cdot (\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n)) = c + d - a - b$$

is an integer that is a multiple of $2n$, so the right side $(c + d - a - b)$ must be as well.
Since $-2n + 4 \leq c + d - a - b \leq 2n - 4$ and the only multiple of $2n$ between $-2n + 4$ and $2n - 4$ is $0 \cdot 2n = 0$, then $c + d - a - b = 0$ or $c + d = a + b$.
Thus, $2n \cdot (\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n)) = 0$.
Since $n \neq 0$, then $\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n) = 0$.
Therefore, $x_a + x_b = x_c + x_d$ exactly when

$$a + b = c + d \quad \text{and} \quad \text{rem}(a^2, n) + \text{rem}(b^2, n) = \text{rem}(c^2, n) + \text{rem}(d^2, n)$$

Suppose that n is composite. We show that T is boring.
We consider three cases: $n = p^2$ for some prime p, n is even, and all other n.
Suppose that $n = p^2$ for some prime p. Since $n \geq 10$, then $p \geq 5$.
Set $a = p$, $b = 4p$, $c = 2p$, and $d = 3p$.
Then $a + b = 5p = c + d$.
Also, since $p \geq 5$, then $0 < p < 2p < 3p < 4p < p^2$.
Furthermore, since each of a, b, c, d is divisible by p, then each of a^2, b^2, c^2, d^2 is divisible by $p^2 = n$, so $\text{rem}(a^2, n) = \text{rem}(b^2, n) = \text{rem}(c^2, n) = \text{rem}(d^2, n) = 0$.
This means that $a + b = c + d$ and $\text{rem}(a^2, n) + \text{rem}(b^2, n) = \text{rem}(c^2, n) + \text{rem}(d^2, n)$, and so $x_a + x_b = x_c + x_d$, which means that T is boring.

Next, suppose that n is even, say $n = 2t$ for some positive integer $t \geq 5$.
Set $a = 1$, $b = t + 2$, $c = 2$, and $d = t + 1$.
Since $t \geq 5$, then $1 \leq a < b < c < d < 2t$, so a, b, c, d are distinct positive integers in the correct range.
Also, $a + b = t + 3 = c + d$.
To show that $x_a + x_b = x_c + x_d$, it remains to show that

$$\text{rem}(1^2, 2t) + \text{rem}((t + 2)^2, 2t) = \text{rem}(2^2, 2t) + \text{rem}((t + 1)^2, 2t)$$

Now $\text{rem}(1^2, 2t) = \text{rem}(1, 2t) = 1$ and $\text{rem}(2^2, 2t) = \text{rem}(4, 2t) = 4$ since $2t > 4$.
Also, since $(t + 2)^2 = t^2 + 4t + 4$ and so $(t + 2)^2$ and $t^2 + 4$ differ by a multiple of $n = 2t$, then $\text{rem}((t + 2)^2, 2t) = \text{rem}(t^2 + 4, 2t)$.
Similarly, since $(t + 1)^2 = t^2 + 2t + 1$, then $\text{rem}((t + 1)^2, 2t) = \text{rem}(t^2 + 1, 2t)$.
Therefore, we need to show that $\text{rem}(t^2 + 4, 2t) - \text{rem}(t^2 + 1, 2t) = 4 - 1 = 3$.
Since $t \geq 5$, then $t^2 + t > t^2 + 4$.
This means that $t^2 < t^2 + 1 < t^2 + 2 < t^2 + 3 < t^2 + 4 < t^2 + t$; in other words, each of $t^2 + 1$, $t^2 + 2$, $t^2 + 3$, $t^2 + 4$ is strictly between two consecutive multiples of t, and so none of these four integers can be a multiple of t. This means that none of these is a multiple of $n = 2t$.
Therefore, $t^2 + 4$ and $t^2 + 1$ are bounded between the same two multiples of n, and so the difference between their remainders when dividing by n equals the difference between the integers, which is 3.
Thus, $x_a + x_b = x_c + x_d$, which means that T is boring.

Finally, we consider the case where n is odd and composite and can be written as $n = MN$ for some odd integers $M > N > 1$.

Set $a = \frac{1}{2}(M+N)$, $b = n - a$, $c = \frac{1}{2}(M-N)$, and $d = n - c$.
Since M and N are both odd, then $M+N$ and $M-N$ are even, and so a, b, c, d are integers.
Since $M > N > 0$, then $a > c > 0$.
Since $N \geq 3$, then $n = MN \geq 3M > 2M$ and so $M < \frac{1}{2}n$.
Since $M > N$, then $a = \frac{1}{2}(M+N) < \frac{1}{2}(M+M) = M < \frac{1}{2}n$.
Therefore, $0 < c < a < \frac{1}{2}n$.
Since $b = n - a$ and $d = n - c$, then $\frac{1}{2}n < b < d < n$ and so $0 < c < a < \frac{1}{2}n < b < d < n$.
This means that a, b, c, d are distinct integers in the correct range.
Also, note that $a + b = n = c + d$.
To show that $x_a + x_b = x_c + x_d$, it remains to show that

$$\text{rem}(a^2, n) + \text{rem}(b^2, n) = \text{rem}(c^2, n) + \text{rem}(d^2, n)$$

We show that $\text{rem}(a^2, n) = \text{rem}(b^2, n) = \text{rem}(c^2, n) = \text{rem}(d^2, n)$, which will provide the desired conclusion.
Since $b = n - a$, then $b^2 = n^2 - 2na + a^2$. Since b^2 and a^2 differ by a multiple of n, their remainders after division by n will be equal. Similarly, $\text{rem}(c^2, n) = \text{rem}(d^2, n)$.
Thus, it remains to show that $\text{rem}(a^2, n) = \text{rem}(c^2, n)$.
But

$$a^2 - c^2 = (a+c)(a-c) = \left(\tfrac{1}{2}(M+N) + \tfrac{1}{2}(M-N)\right)\left(\tfrac{1}{2}(M+N) - \tfrac{1}{2}(M-N)\right) = MN = n$$

Since a^2 and c^2 differ by a multiple of n, then $\text{rem}(a^2, n) = \text{rem}(c^2, n)$.
Thus, $x_a + x_b = x_c + x_d$, which means that T is boring.

Suppose that n is prime. We show that T is exciting.
Since $n \geq 10$, then n is odd.
Suppose that $x_a + x_b = x_c + x_d$. We will show that this is not possible.
Recall that $x_a + x_b = x_c + x_d$ is equivalent to the conditions $a + b = c + d$ and $\text{rem}(a^2, n) + \text{rem}(b^2, n) = \text{rem}(c^2, n) + \text{rem}(d^2, n)$.
We work with this second equation.
When a^2 is divided by n, we obtain a quotient that we call q_a and the remainder $\text{rem}(a^2, n)$.
Note that $a^2 = q_a n + \text{rem}(a^2, n)$ and $0 \leq \text{rem}(a^2, n) < n$.
We define q_b, q_c, q_d similarly and obtain

$$(a^2 - q_a n) + (b^2 - q_b n) = (c^2 - q_c n) + (d^2 - q_d n)$$

or

$$a^2 + b^2 - c^2 - d^2 = n(q_a + q_b - q_c - q_d)$$

Since $a + b = c + d$, then $a^2 + 2ab + b^2 = c^2 + 2cd + d^2$ or $a^2 + b^2 - c^2 - d^2 = 2cd - 2ab$.
Therefore, $x_a + x_b = x_c + x_d$ exactly when $a + b = c + d$ and $2cd - 2ab = n(q_a + q_b - q_c - q_d)$.
Since $d = a + b - c$, then this last equation becomes

$$\begin{aligned} 2c(a+b-c) - 2ab &= n(q_a + q_b - q_c - q_d) \\ -2(c^2 - ac - bc + ab) &= n(q_a + q_b - q_c - q_d) \\ -2(c(c-a) - b(c-a)) &= n(q_a + q_b - q_c - q_d) \\ -2(c-a)(c-b) &= n(q_a + q_b - q_c - q_d) \end{aligned}$$

Since $x_a + x_b = x_c + x_d$, then $a + b = c + d$ and $-2(c-a)(c-b) = n(q_a + q_b - q_c - q_d)$.
Therefore, $2(c-a)(c-b)$ is a multiple of n, which is an odd prime.

This means that either $c - a$ or $c - b$ is a multiple of n.
But a, b, c, d are between 1 and n inclusive and are distinct, so $1 - n \leq c - a \leq n - 1$ and $1 - n \leq c - b \leq n - 1$.
The only multiple of n in this range is 0, so $c - a = 0$ or $c - b = 0$, which contradicts the fact that a, b, c, d are distinct.
Therefore, if n is prime, there do not exist four distinct elements of T that make T boring, so T is exciting.

In summary, T is exciting exactly when $n \geq 10$ is prime.

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
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2016 Euclid Contest

Tuesday, April 12, 2016
(in North America and South America)

Wednesday, April 13, 2016
(outside of North America and South America)

Solutions

1. (a) The average is

$$\frac{5+15+25+35+45+55}{6} = \frac{(5+55)+(15+45)+(25+35)}{6} = \frac{60+60+60}{6} = 30$$

(b) Since $x^2 = 2016$, then $(x+2)(x-2) = x^2 - 4 = 2016 - 4 = 2012$.

(c) Since points P, Q and R lie on a straight line, then the slope of PQ equals the slope of PR.
The slope of PQ equals $\dfrac{2a-5}{a-7}$ and the slope of PR equals $\dfrac{30-5}{12-7} = \dfrac{25}{5} = 5$.
Therefore, $\dfrac{2a-5}{a-7} = 5$ and so $2a - 5 = 5(a-7)$.
This gives $2a - 5 = 5a - 35$ or $3a = 30$, and so $a = 10$.

2. (a) If $\dfrac{n}{9} = \dfrac{25}{n}$, then $n^2 = 25(9) = 225$. Therefore, $n = 15$ or $n = -15$.
(We can check by substitution that each of these values satisfies the original equation.)

(b) When we expand the left side of $(x-3)(x-2) = 6$, we obtain $x^2 - 5x + 6 = 6$.
Thus, $x^2 - 5x = 0$ or $x(x-5) = 0$, which gives $x = 0$ or $x = 5$.
(We can check by substitution that each of these values satisfies the original equation.)

(c) Let a be the cost, in dollars, of 1 apple and let b be the cost, in dollars, of 1 banana.
From the given information, $2a = 3b$ and $6a + 12b = 6.30$.
Since $3b = 2a$, then $12b = 4(3b) = 4(2a) = 8a$.
Therefore, $6a + 8a = 6.3$ or $14a = 6.3$, which gives $a = 0.45$.
In other words, the cost of 1 apple is \$0.45.

3. (a) *Solution 1*
Since the sum of the angles in any triangle is 180°, then the combined sum of the angles in $\triangle ABD$, $\triangle FBG$ and $\triangle CBE$ is $3 \cdot 180^\circ$ or 540°.
The nine angles in these triangles include those with measures, in degrees, of p, q, r, s, t, u as well as the three angles $\angle ABD$, $\angle FBG$ and $\angle CBE$.
These last three angles form a straight angle, and so their sum is 180°.
Therefore, the sum of the remaining six angles must be $540^\circ - 180^\circ = 360^\circ$.
In other words, $p + q + r + s + t + u = 360$.

Solution 2
We repeatedly use the fact that the sum of the angles in any triangle is 180°.
From $\triangle ABD$, $\angle ABD = 180^\circ - p^\circ - q^\circ$.
From $\triangle FBG$, $\angle FBG = 180^\circ - r^\circ - s^\circ$.
From $\triangle CBE$, $\angle CBE = 180^\circ - t^\circ - u^\circ$.
Since ABC forms a straight line segment, then

$$\angle ABD + \angle FBG + \angle CBE = 180^\circ$$

which gives

$$(180^\circ - p^\circ - q^\circ) + (180^\circ - r^\circ - s^\circ) + (180^\circ - t^\circ - u^\circ) = 180^\circ$$

Rearranging, we obtain $360 = p+q+r+s+t+u$ and so the value of $p+q+r+s+t+u$ is 360.

(b) *Solution 1*
The integer equal to 10^{20} consists of the digit 1 followed by 20 0s.
The integer equal to $10^{20} - 1$ thus consists of 20 9s.
Now, $n = 10^{20} - 20$ is 19 less than $10^{20} - 1$ which is the integer that consists of 20 9s.
So $n = 10^{20} - 20 = 99\cdots 980$ where this integer has 18 9s.
Therefore, the sum of the digits of n is $18(9) + 8 + 0 = 162 + 8 = 170$.

Solution 2
Since $10^{20} - 20 = 10(10^{19} - 2)$ and $10^{19} - 2 = 99\cdots 98$ (where this integer has 18 9s), then $10^{20} - 20 = 99\cdots 980$, where this integer has 18 9s.
Therefore, the sum of the digits of n is $18(9) + 8 + 0 = 162 + 8 = 170$.

(c) *Solution 1*
Since $P(2, 0)$ and $Q(8, 0)$, then $PQ = 8 - 2 = 6$.
Let h be the perpendicular distance from V to PQ.
Then the area of $\triangle VPQ$ equals $\frac{1}{2}(PQ)h$.
Since the area of $\triangle VPQ$ is 12, then $\frac{1}{2}(PQ)h = 12$ and so $\frac{1}{2}(6)h = 12$ or $h = 4$.
Since V is below the x-axis, then the y-coordinate of V is -4.
Since V is the vertex of a parabola and P and Q are points where the parabola intersects the x-axis, then the x-coordinate of V is the average of the x-coordinates of P and Q, which is $\frac{1}{2}(2 + 8) = 5$.
Finally, the coordinates of V are $(5, -4)$.

Solution 2
Since the parabola intersects the x-axis at $P(2, 0)$ and $Q(8, 0)$, then the equation of the parabola will be of the form $y = a(x - 2)(x - 8)$ for some $a \neq 0$.
Completing the square, we obtain

$$y = a(x^2 - 10x + 16) = a((x - 5)^2 - 9) = a(x - 5)^2 - 9a$$

From this, we see that the vertex of this parabola has coordinates $(5, -9a)$.
Since the vertex of the parabola is below the x-axis, then $-9a < 0$ or $a > 0$.
Now $\triangle VPQ$ has base PQ along the x-axis (which has length $8 - 2 = 6$).
The corresponding height is the perpendicular distance from V to the x-axis. This equals $9a$, since $a > 0$.
Since the area of $\triangle VPQ$ is 12, then $\frac{1}{2}(6)(9a) = 12$ or $27a = 12$ which gives $a = \frac{4}{9}$.
Finally, substituting $a = \frac{4}{9}$ into $(5, -9a)$ gives the conclusion that the coordinates of V are $(5, -4)$.

4. (a) Rewriting $\sin^2\theta + 2\cos^2\theta = \frac{7}{4}$, we get $(\sin^2\theta + \cos^2\theta) + \cos^2\theta = \frac{7}{4}$.
Since $\sin^2\theta + \cos^2\theta = 1$ for any angle θ, then $1 + \cos^2\theta = \frac{7}{4}$ and so $\cos^2\theta = \frac{3}{4}$ or $\cos\theta = \pm\frac{\sqrt{3}}{2}$.
Since $0^\circ \leq \theta \leq 180^\circ$, then $\cos\theta = \frac{\sqrt{3}}{2}$ exactly when $\theta = 30^\circ$.
Since $0^\circ \leq \theta \leq 180^\circ$, then $\cos\theta = -\frac{\sqrt{3}}{2}$ exactly when $\theta = 150^\circ$.
Therefore, $\theta = 30^\circ$ or $\theta = 150^\circ$.
(We can check by substitution that each of these values satisfies the original equation.)

(b) Let the radius of the smaller circle be r cm and let the radius of the larger circle be R cm.
Thus, the circumference of the smaller circle is $2\pi r$ cm, the circumference of the larger circle is $2\pi R$ cm, the area of the smaller circle is πr^2 cm^2, and the area of the larger circle is πR^2 cm^2.
Since the sum of the radii of the two circles is 10 cm, then $r + R = 10$.
Since the circumference of the larger circle is 3 cm larger than the circumference of the smaller circle, then $2\pi R - 2\pi r = 3$, or $2\pi(R - r) = 3$.
Then the difference, in cm^2, between the area of the larger circle and the area of the smaller circle is

$$\pi R^2 - \pi r^2 = \pi(R-r)(R+r) = \tfrac{1}{2}[2\pi(R-r)](R+r) = \tfrac{1}{2}(3)(10) = 15$$

Therefore, the difference between the areas is 15 cm^2.

5. (a) When the price of \$$p$ is raised by $n\%$, the price is multiplied by $1 + \dfrac{n}{100}$.
When the new price is reduced by 20%, the new price is multiplied by $1 - \dfrac{20}{100} = \dfrac{80}{100}$.
Therefore, after these two price adjustments, the price is \$$p\left(1 + \dfrac{n}{100}\right)\left(\dfrac{80}{100}\right)$.
We are told that this final price is 20% higher than \$$p$, and so the final price equals \$$p\left(1 + \dfrac{20}{100}\right)$ or \$$p\left(\dfrac{120}{100}\right)$.
In other words,

$$\$p\left(1 + \frac{n}{100}\right)\left(\frac{80}{100}\right) = \$p\left(\frac{120}{100}\right)$$

Simplifying and using the fact that $p \neq 0$, we obtain $80\left(1 + \dfrac{n}{100}\right) = 120$.
Thus, $1 + \dfrac{n}{100} = \dfrac{120}{80} = \dfrac{3}{2} = \dfrac{150}{100}$ and so $\dfrac{n}{100} = \dfrac{50}{100}$ or $n = 50$.

(b) *Solution 1*
Let $m = f(n)$. The equation $f(f(n)) = 3$ becomes $f(m) = 3$.
Suppose that $f(m) = 3$ and m is odd.
By definition, we have $f(m) = m - 1 = 3$ and so $m = 4$, which is not odd, so this case cannot happen.
Suppose that $f(m) = 3$ and m is even.
By definition, we have $f(m) = m^2 - 1 = 3$ and so $m^2 = 4$ or $m = \pm 2$, each of which is even.
Therefore, if $f(f(n)) = 3$, then $f(n) = 2$ or $f(n) = -2$.
Suppose that $f(n) = 2$ or $f(n) = -2$ and n is odd.
By definition, we have $n - 1 = 2$ (giving $n = 3$) or $n - 1 = -2$ (giving $n = -1$). Each of these resulting values of n is odd.
Suppose that $f(n) = 2$ or $f(n) = -2$ and n is even.
Then $n^2 - 1 = 2$ or $n^2 - 1 = -2$ which give $n^2 = 3$ or $n^2 = -1$, neither of which is possible if n is an integer.
Thus, the integers n for which $f(f(n)) = 3$ are $n = 3$ and $n = -1$.
(We can check by substitution that each of these satisfies the original equation.)

Solution 2
We consider the cases of n even and n odd separately.

Suppose that n is even.
Then n^2 is even and so $f(n) = n^2 - 1$ must be odd.
Thus, $f(f(n)) = f(n^2 - 1) = (n^2 - 1) - 1 = n^2 - 2$, since $f(m) = m - 1$ when m is odd.
For n to be even and $f(f(n)) = 3$, we must have $n^2 - 2 = 3$ or $n^2 = 5$.
There are no integer solutions to this equation, and so there are no solutions in this case.

Suppose that n is odd.
Then $f(n) = n - 1$ must be even.
Thus, $f(f(n)) = f(n - 1) = (n - 1)^2 - 1 = n^2 - 2n + 1 - 1 = n^2 - 2n$.
For n to be odd and $f(f(n)) = 3$, we must have $n^2 - 2n = 3$ or $n^2 - 2n - 3 = 0$.
Factoring, we obtain $(n - 3)(n + 1) = 0$ and so $n = 3$ or $n = -1$, both of which are odd.

Thus, the integers n for which $f(f(n)) = 3$ are $n = 3$ and $n = -1$.
(We can check by substitution that each of these satisfies the original equation.)

6. (a) Since $10^y \neq 0$, the equation $\dfrac{1}{32} = \dfrac{x}{10^y}$ is equivalent to $10^y = 32x$.
So the given question is equivalent to asking for the smallest positive integer x for which $32x$ equals a positive integer power of 10.
Now $32 = 2^5$ and so $32x = 2^5x$.
For $32x$ to equal a power of 10, each factor of 2 must be matched with a factor of 5.
Therefore, x must be divisible by 5^5 (that is, x must include at least 5 powers of 5), and so $x \geq 5^5 = 3125$.
But $32(5^5) = 2^5 5^5 = 10^5$, and so if $x = 5^5 = 3125$, then $32x$ is indeed a power of 10, namely 10^5.
This tells us that the smallest positive integer x for which $\dfrac{1}{32} = \dfrac{x}{10^y}$ for some positive integer y is $x = 5^5 = 3125$.

(b) *Solution 1*
Since the three side lengths of a right-angled triangle form an arithemetic sequence and must include 60, then the three side lengths are $60, 60 + d, 60 + 2d$ or $60 - d, 60, 60 + d$ or $60 - 2d, 60 - d, 60$, for some $d \geq 0$.

For a triangle with sides of length $60, 60+d, 60+2d$ to be right-angled, by the Pythagorean Theorem, the following equivalent equations must be true:

$$\begin{aligned}
60^2 + (60 + d)^2 &= (60 + 2d)^2 \\
3600 + 3600 + 120d + d^2 &= 3600 + 240d + 4d^2 \\
0 &= 3d^2 + 120d - 3600 \\
0 &= d^2 + 40d - 1200 \\
0 &= (d + 60)(d - 20)
\end{aligned}$$

(Note that, since $d \geq 0$, then $60 + 2d$ must be the hypotenuse of the triangle.)
Since $d \geq 0$, then $d = 20$, which gives the triangle with side lengths $60, 80, 100$.
The longest side length is the hypotenuse and the shorter two sides meet at right angles, giving an area of $\frac{1}{2}(60)(80) = 2400$.

For a triangle with sides of length $60-d, 60, 60+d$ to be right-angled, by the Pythagorean Theorem, the following equivalent equations must be true:

$$\begin{aligned}(60-d)^2+60^2&=(60+d)^2\\3600-120d+d^2+3600&=3600+120d+d^2\\3600&=240d\\d&=15\end{aligned}$$

Since $d\geq 0$, then $d=15$ is admissible, which gives the triangle with side lengths $45, 60, 75$.
Using a similar analysis, the area of this triangle is $\frac{1}{2}(45)(60)=1350$.

For a triangle with sides of length $60-2d, 60-d, 60$ to be right-angled, by the Pythagorean Theorem, the following equivalent equations must be true:

$$\begin{aligned}(60-2d)^2+(60-d)^2&=60^2\\3600-240d+4d^2+3600-120d+d^2&=3600\\5d^2-360d+3600&=0\\d^2-72d+720&=0\\(d-60)(d-12)&=0\end{aligned}$$

Since $d\geq 0$, then $d=60$ or $d=12$, which give possible side lengths of $-60, 0, 60$ (which do not form a triangle) and $36, 48, 60$ (which do form a triangle).
Using a similar analysis, the area of this triangle is $\frac{1}{2}(36)(48)=864$.

Therefore, the possible values for the area of such a triangle are 2400, 1350, and 864.

Solution 2
Suppose that a triangle has side lengths in arithemetic sequence.
Then the side lengths can be written as $a-d, a, a+d$ for some $a>0$ and $d\geq 0$.
Note that $a-d\leq a\leq a+d$.
For such a triangle to be right-angled, by the Pythagorean Theorem, the following equivalent equations are true:

$$\begin{aligned}(a-d)^2+a^2&=(a+d)^2\\a^2-2ad+d^2+a^2&=a^2+2ad+d^2\\a^2&=4ad\end{aligned}$$

Since $a>0$, then $a=4d$, and so the side lengths of the triangle are $a-d=3d$, $a=4d$, and $a+d=5d$ for some $d\geq 0$.
(Note that such triangles are all similar to the 3-4-5 triangle.)
If such a triangle has 60 as a side length, then there are three possibilities:

(i) $3d=60$: This gives $d=20$ and side lengths $60, 80, 100$.
Since the triangle is right-angled and its hypotenuse has length 100, then its area will equal $\frac{1}{2}(60)(80)=2400$.

(ii) $4d=60$: This gives $d=15$ and side lengths $45, 60, 75$.
In a similar way to case (i), its area will equal $\frac{1}{2}(45)(60)=1350$.

(iii) $5d=60$: This gives $d=12$ and side lengths $36, 48, 60$.
In a similar way to case (i), its area will equal $\frac{1}{2}(36)(48)=864$.

Therefore, the possible values for the area of such a triangle are 2400, 1350, and 864.

7. (a) *Solution 1*

Suppose that Amrita paddles the kayak for p km and swims for s km.
Since Amrita leaves the kayak in the lake and it does not move, then Zhang swims p km and paddles the kayak for s km.
Note that each paddles at 7 km/h and each swims at 2 km/h and each takes exactly 90 minutes (or 1.5 hours) to complete the trip.
If $s < p$, then Amrita would paddle farther and swim less distance than Zhang and so would reach the other side in less time than Zhang.
If $s > p$, then Zhang would paddle farther and swim less distance than Amrita and so would reach the other side in less time than Amrita.
Since they each take 90 minutes, then we must have $s = p$.
Alternatively, since each paddles at 7 km/h and each swims at 2 km/h and each takes exactly 90 minutes (or 1.5 hours) to complete the trip, then we obtain the two equations

$$\frac{p}{7} + \frac{s}{2} = 1.5 \qquad \frac{p}{2} + \frac{s}{7} = 1.5$$

Using the fact that the right sides of these equations are equal, we obtain

$$\begin{aligned} \frac{p}{7} + \frac{s}{2} &= \frac{s}{7} + \frac{p}{2} \\ \frac{s}{2} - \frac{s}{7} &= \frac{p}{2} - \frac{p}{7} \\ s\left(\frac{1}{2} - \frac{1}{7}\right) &= p\left(\frac{1}{2} - \frac{1}{7}\right) \\ s &= p \end{aligned}$$

Therefore, $\frac{p}{7} + \frac{p}{2} = 1.5$ or $\frac{9}{14}p = 1.5 = \frac{3}{2}$ and so $p = \frac{7}{3}$.
For Amrita to paddle these $\frac{7}{3}$ km at 7 km/h, it takes $\frac{7/3}{7} = \frac{1}{3}$ hour, or 20 minutes.
For Zhang to swim these $\frac{7}{3}$ km at 2 km/h, it takes $\frac{7/3}{2} = \frac{7}{6}$ hour, or 70 minutes.
The kayak is not being paddled for the period of time from when Amrita stops paddling to the time when Zhang stops swimming, which is a period of $70 - 20 = 50$ minutes.

Solution 2

Let t_1 hours be the length of time during which Amrita paddles and Zhang swims.

Let t_2 hours be the length of time during which Amrita swims and Zhang swims; the kayak is not moving during this time.

Let t_3 hours be the length of time during which Amrita swims and Zhang paddles.

Let d km be the total distance across the lake.

Since Amrita paddles at 7 km/h and swims at 2 km/h, then $7t_1 + 2t_2 + 2t_3 = d$.

Since Zhang paddles at 7 km/h and swims at 2 km/h, then $2t_1 + 2t_2 + 7t_3 = d$.

Since the kayak travels at 7 km/h and does not move while both Amrita and Zhang are swimming, then $7t_1 + 0t_2 + 7t_3 = d$.

Since Amrita and Zhang each take 90 minutes ($\frac{3}{2}$ hours) to cross the lake, then the total time gives $t_1 + t_2 + t_3 = \frac{3}{2}$.

From $7t_1 + 2t_2 + 2t_3 = d$ and $2t_1 + 2t_2 + 7t_3 = d$, we obtain $7t_1 + 2t_2 + 2t_3 = 2t_1 + 2t_2 + 7t_3$ or $5t_1 = 5t_3$ and so $t_1 = t_3$.

Since $7t_1 + 2t_2 + 2t_3 = d$ and $7t_1 + 0t_2 + 7t_3 = d$ and $t_1 = t_3$, then $7t_1 + 2t_2 + 2t_1 = 7t_1 + 7t_1$ or $2t_2 = 5t_1$ or $t_2 = \frac{5}{2}t_1$.

Since $t_1 + t_2 + t_3 = \frac{3}{2}$, then $t_1 + \frac{5}{2}t_1 + t_1 = \frac{3}{2}$ or $\frac{9}{2}t_1 = \frac{3}{2}$ and so $t_1 = \frac{1}{3}$.

Thus, $t_2 = \frac{5}{2} \cdot \frac{1}{3} = \frac{5}{6}$ hours (or 50 minutes) is the period of time that the kayak is not moving.

(b) From the first equation, $x\left(\frac{1}{2} + y - 2x^2\right) = 0$, we obtain $x = 0$ or $\frac{1}{2} + y - 2x^2 = 0$.

From the second equation, $y\left(\frac{5}{2} + x - y\right) = 0$, we obtain $y = 0$ or $\frac{5}{2} + x - y = 0$.

If $x = 0$, the first equation is satisified.

For the second equation to be true in this case, we need $y = 0$ (giving the solution $(0, 0)$) or $\frac{5}{2} + 0 - y = 0$. The second equation gives $y = \frac{5}{2}$ (giving the solution $(0, \frac{5}{2})$).

If $y = 0$, the second equation is satisified.

For the first equation to be true in this case, we need $x = 0$ (giving the solution $(0, 0)$) or $\frac{1}{2} + 0 - 2x^2 = 0$. The second equation gives $x^2 = \frac{1}{4}$ or $x = \pm\frac{1}{2}$ (giving the solutions $(\frac{1}{2}, 0)$ and $(-\frac{1}{2}, 0)$).

So far, we have accounted for all solutions with $x = 0$ or $y = 0$.

If $x \neq 0$ and $y \neq 0$, then for both equations to be true, we need $\frac{1}{2} + y - 2x^2 = 0$ (or $1 + 2y - 4x^2 = 0$) and $\frac{5}{2} + x - y = 0$ (or $5 + 2x - 2y = 0$).

Adding these two equations, we obtain $6 + 2x - 4x^2 = 0$.

This is equivalent to $2x^2 - x - 3 = 0$ or $(2x - 3)(x + 1) = 0$, whose solutions are $x = \frac{3}{2}$ and $x = -1$.

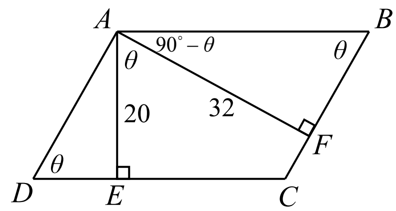
The equation $\frac{5}{2} + x - y = 0$ tells us that $y = x + \frac{5}{2}$.

If $x = \frac{3}{2}$, then $y = 4$; if $x = -1$, then $y = \frac{3}{2}$.

Therefore, the complete list of pairs that satisfy the given system of equations is

$$(x, y) = (0, 0), (0, \tfrac{5}{2}), (\tfrac{1}{2}, 0), (-\tfrac{1}{2}, 0), (\tfrac{3}{2}, 4), (-1, \tfrac{3}{2}) \ .$$

8. (a) Let $\angle EAF = \theta$.
Since $ABCD$ is a parallelogram, then AB and DC are parallel with $AB = DC$, and DA and CB are parallel with $DA = CB$.
Since AE is perpendicular to DC and AB and DC are parallel, then AE is perpendicular to AB.
In other words, $\angle EAB = 90^\circ$, and so $\angle FAB = 90^\circ - \theta$.
Since $\triangle AFB$ is right-angled at F and $\angle FAB = 90^\circ - \theta$, then $\angle ABF = \theta$.
Using similar arguments, we obtain that $\angle DAE = 90^\circ - \theta$ and $\angle ADE = \theta$.



Since $\cos(\angle EAF) = \cos\theta = \frac{1}{3}$ and $\cos^2\theta + \sin^2\theta = 1$, then

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\tfrac{1}{9}} = \sqrt{\tfrac{8}{9}} = \tfrac{2\sqrt{2}}{3}$$

(Note that $\sin\theta > 0$ since θ is an angle in a triangle.)
In $\triangle AFB$, $\sin\theta = \dfrac{AF}{AB}$ and $\cos\theta = \dfrac{FB}{AB}$.
Since $AF = 32$ and $\sin\theta = \frac{2\sqrt{2}}{3}$, then $AB = \dfrac{AF}{\sin\theta} = \dfrac{32}{2\sqrt{2}/3} = \dfrac{48}{\sqrt{2}} = 24\sqrt{2}$.
Since $AB = 24\sqrt{2}$ and $\cos\theta = \frac{1}{3}$, then $FB = AB\cos\theta = 24\sqrt{2}(\frac{1}{3}) = 8\sqrt{2}$.
In $\triangle AED$, $\sin\theta = \dfrac{AE}{AD}$ and $\cos\theta = \dfrac{DE}{AD}$.
Since $AE = 20$ and $\sin\theta = \frac{2\sqrt{2}}{3}$, then $AD = \dfrac{AE}{\sin\theta} = \dfrac{20}{2\sqrt{2}/3} = \dfrac{30}{\sqrt{2}} = 15\sqrt{2}$.
Since $AD = 15\sqrt{2}$ and $\cos\theta = \frac{1}{3}$, then $DE = AD\cos\theta = 15\sqrt{2}(\frac{1}{3}) = 5\sqrt{2}$.
(To calculate AD and DE, we could also have used the fact that $\triangle ADE$ is similar to $\triangle ABF$.)
Finally, the area of quadrilateral $AECF$ equals the area of parallelogram $ABCD$ minus the combined areas of $\triangle AFB$ and $\triangle ADE$.
The area of parallelogram $ABCD$ equals $AB \cdot AE = 24\sqrt{2} \cdot 20 = 480\sqrt{2}$.
The area of $\triangle AFB$ equals $\frac{1}{2}(AF)(FB) = \frac{1}{2}(32)(8\sqrt{2}) = 128\sqrt{2}$.
The area of $\triangle AED$ equals $\frac{1}{2}(AE)(DE) = \frac{1}{2}(20)(5\sqrt{2}) = 50\sqrt{2}$.
Thus, the area of quadrilateral $AECF$ is $480\sqrt{2} - 128\sqrt{2} - 50\sqrt{2} = 302\sqrt{2}$.

(b) Note that $x \neq 1$ since 1 cannot be the base of a logarithm. This tells us that $\log x \neq 0$.
Using the fact that $\log_a b = \dfrac{\log b}{\log a}$ and then using other logarithm laws, we obtain the following equivalent equations:

$$\begin{aligned}
\log_4 x - \log_x 16 &= \tfrac{7}{6} - \log_x 8 \\
\frac{\log x}{\log 4} - \frac{\log 16}{\log x} &= \frac{7}{6} - \frac{\log 8}{\log x} \qquad \text{(note that } x \neq 1\text{, so } \log x \neq 0) \\
\frac{\log x}{\log 4} &= \frac{7}{6} + \frac{\log 16 - \log 8}{\log x} \\
\frac{\log x}{\log(2^2)} &= \frac{7}{6} + \frac{\log(\frac{16}{8})}{\log x} \\
\frac{\log x}{2\log 2} &= \frac{7}{6} + \frac{\log 2}{\log x} \\
\frac{1}{2}\left(\frac{\log x}{\log 2}\right) &= \frac{7}{6} + \frac{\log 2}{\log x}
\end{aligned}$$

Letting $t = \dfrac{\log x}{\log 2} = \log_2 x$ and noting that $t \neq 0$ since $x \neq 1$, we obtain the following equations equivalent to the previous ones:

$$\begin{aligned}
\frac{t}{2} &= \frac{7}{6} + \frac{1}{t} \\
3t^2 &= 7t + 6 \qquad \text{(multiplying both sides by } 6t) \\
3t^2 - 7t - 6 &= 0 \\
(3t+2)(t-3) &= 0
\end{aligned}$$

Therefore, the original equation is equivalent to $t = -\frac{2}{3}$ or $t = 3$.
Converting back to the variable x, we obtain $\log_2 x = -\frac{2}{3}$ or $\log_2 x = 3$, which gives $x = 2^{-2/3}$ or $x = 2^3 = 8$.

9. (a) There are $2^{10} = 1024$ strings of ten letters, each of which is A or B, because there are 2 choices for each of the 10 positions in the string.

We determine the number of these strings that do not include the "substring" $ABBA$ (that is, that do not include consecutive letters $ABBA$) by counting the number of strings that do include the substring $ABBA$ and subtracting this total from 1024.

If a string includes the substring $ABBA$, there are 7 possible positions in which this substring could start ($ABBAxxxxxx$, $xABBAxxxxx$, ..., $xxxxxxABBA$).

There are 2 choices for each of the remaining 6 letters in such a string, so there are $7 \cdot 2^6 = 448$ occurrences of the substring $ABBA$ among the 1024 strings.

This does not mean that there are 448 strings that contain the substring $ABBA$. Since $ABBA$ can appear multiple times in a single string, this total will count some strings more than once. (For example, the string $ABBAAAABBA$ is included in both the first and seventh of these categories, so this string is counted twice.)

So we must "correct" this total of 448 by accounting for the strings in which $ABBA$ occurs more than once.

We note that, since two substrings of $ABBA$ can overlap in 0 letters (for example, $ABBAABBAxx$) or in 1 letter (for example, $ABBABBAxxx$), then the maximum number of times that the substring $ABBA$ can appear is 3, and there is only one such string: $ABBABBABBA$.

If a string contains two copies of $ABBA$ that overlap, then it must be of one of the following forms:

$$ABBABBAxxx \qquad xABBABBAxx \qquad xxABBABBAx \qquad xxxABBABBA$$

Since there are 4 choices for the starting position of $ABBABBA$ and 2 choices for each of the three unknown letters, then there are $4 \cdot 2^3 = 32$ occurrences of $ABBABBA$ among all of these strings.

But the string $ABBABBABBA$ is counted in each of the first and last categories, so we subtract 2 occurrences from this total to obtain 30, the total number of strings of ten letters that included exactly two overlapping copies of $ABBA$. (We'll count the string $ABBABBABBA$ later.)

If a string contains exactly two substrings of $ABBA$ and these do not overlap, then it must be of one of the following forms:

$$ABBAABBAxx \qquad ABBAxABBAx \qquad ABBAxxABBA$$
$$xABBAABBAx \qquad xABBAxABBA \qquad xxABBAABBA$$

Since there are 6 such forms and 2 choices for each of the 2 unknown letters in each case, then there appear to be $6 \cdot 2^2 = 24$ such strings.

This total includes the string $ABBABBABBA$ in the third category, so we subtract 1 from this total to obtain 23, the total number of strings of ten letters that include exactly two copies of $ABBA$ which do not overlap.

So there are 30 strings that contain exactly two overlapping substrings $ABBA$, 23 strings that contain exactly two non-overlapping substrings $ABBA$, and 1 string that contains exactly three substrings $ABBA$.

To get the total number of strings with one or more substrings $ABBA$ we take the total number of occurrences of $ABBA$ (448), subtract the number of strings with exactly two substrings $ABBA$ (since these were included twice in the original count), and subtract two times the number of strings with exactly three substrings $ABBA$ (since these were included three times in the original count).

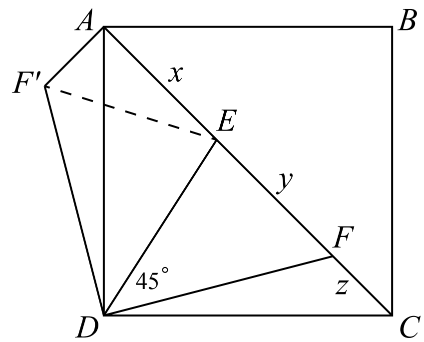
Therefore, there are $448 - 23 - 30 - 2 \cdot 1 = 393$ strings that include at least one substring

$ABBA$, and so there are $1024 - 393 = 631$ strings of ten letters that do not include the substring $ABBA$.

(b) *Solution 1*

Rotate $\triangle DFC$ through an angle of 90° counterclockwise about D, so that DC now lies along DA and F' is outside the square, as shown.

Join F' to E.



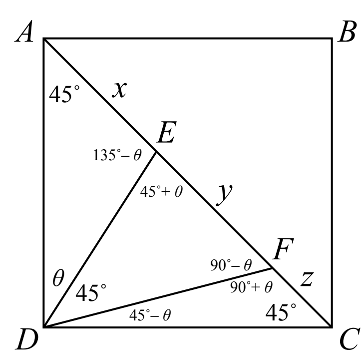
Since AC is a diagonal of square $ABCD$, then $\angle EAD = \angle FCD = 45^\circ$.
Since $\angle EAD = 45^\circ$ and $\angle F'AD = \angle FCD = 45^\circ$, then $\angle F'AE = 45^\circ + 45^\circ = 90^\circ$.
By the Pythagorean Theorem in $\triangle F'AE$, we have $F'E^2 = F'A^2 + AE^2$.
But $F'A = FC = z$ and $AE = x$, so $F'E^2 = z^2 + x^2$.
To show that $y^2 = x^2 + z^2$, it is sufficient to show that $F'E = y$.
Consider $\triangle F'DE$ and $\triangle FDE$.
Note that $F'D = FD$ and $\angle F'DA = \angle FDC$ by construction and $ED = ED$.
Also, $\angle F'DE = \angle F'DA + \angle EDA = \angle FDC + \angle EDA = 90^\circ - \angle EDF = 45^\circ$, which tells us that $\angle F'DE = \angle FDE = 45^\circ$.
Therefore, $\triangle F'DE$ is congruent to $\triangle FDE$ (side-angle-side), and so $F'E = FE = y$.
This gives us the desired conclusion that $y^2 = x^2 + z^2$.

Solution 2

Since AC is a diagonal of square $ABCD$, then $\angle EAD = \angle FCD = 45^\circ$.
Let $\angle ADE = \theta$.
Since the angles in a triangle have a sum of 180°, then

$$\angle AED = 180^\circ - \angle EAD - \angle ADE = 180^\circ - 45^\circ - \theta = 135^\circ - \theta$$

Since AEF is a straight angle, then $\angle DEF = 180^\circ - \angle AED = 180^\circ - (135^\circ - \theta) = 45^\circ + \theta$.
Continuing in this way, we find that $\angle EFD = 90^\circ - \theta$, $\angle DFC = 90^\circ + \theta$, and $\angle FDC = 45^\circ - \theta$.



Using the sine law in $\triangle AED$, we see that $\dfrac{AE}{\sin\angle ADE} = \dfrac{ED}{\sin\angle EAD}$ or $\dfrac{x}{\sin\theta} = \dfrac{ED}{\sin 45^\circ}$.

Using the sine law in $\triangle DEF$, we see that $\dfrac{EF}{\sin\angle EDF} = \dfrac{ED}{\sin\angle EFD}$ or $\dfrac{y}{\sin 45^\circ} = \dfrac{ED}{\sin(90^\circ - \theta)}$.

Using the sine law in $\triangle DEF$, we see that $\dfrac{EF}{\sin\angle EDF} = \dfrac{FD}{\sin\angle DEF}$ or $\dfrac{y}{\sin 45^\circ} = \dfrac{FD}{\sin(45^\circ+\theta)}$.

Using the sine law in $\triangle DFC$, we get $\dfrac{FC}{\sin\angle FDC} = \dfrac{FD}{\sin\angle DCF}$ or $\dfrac{z}{\sin(45^\circ-\theta)} = \dfrac{FD}{\sin 45^\circ}$.

Dividing the first of these equations by the second, we obtain $\dfrac{x\sin 45^\circ}{y\sin\theta} = \dfrac{\sin(90^\circ-\theta)}{\sin 45^\circ}$ or $\dfrac{x}{y} = \dfrac{\sin(90^\circ-\theta)\sin\theta}{\sin^2 45^\circ}$.

Dividing the fourth of these equations by the third, we obtain $\dfrac{z\sin 45^\circ}{y\sin(45^\circ-\theta)} = \dfrac{\sin(45^\circ+\theta)}{\sin 45^\circ}$ or $\dfrac{z}{y} = \dfrac{\sin(45^\circ+\theta)\sin(45^\circ-\theta)}{\sin^2 45^\circ}$.

Since $\sin(90^\circ-\alpha) = \cos\alpha$ for every angle α, then $\sin(90^\circ-\theta) = \cos\theta$.

Also, $\sin(45^\circ+\theta) = \sin(90^\circ-(45^\circ-\theta)) = \cos(45^\circ-\theta)$.

Using this and the fact that $\dfrac{1}{\sin^2 45^\circ} = \dfrac{1}{(1/\sqrt{2})^2} = 2$, the expressions for $\dfrac{x}{y}$ and $\dfrac{z}{y}$ become

$$\frac{x}{y} = 2\cos\theta\sin\theta = \sin 2\theta$$

and

$$\frac{z}{y} = 2\cos(45^\circ-\theta)\sin(45^\circ-\theta) = \sin(2(45^\circ-\theta)) = \sin(90^\circ-2\theta) = \cos 2\theta$$

Finally, this tells us that

$$\frac{x^2}{y^2} + \frac{z^2}{y^2} = \left(\frac{x}{y}\right)^2 + \left(\frac{z}{y}\right)^2 = \sin^2 2\theta + \cos^2 2\theta = 1$$

Since $\dfrac{x^2}{y^2} + \dfrac{z^2}{y^2} = 1$, then $x^2+z^2=y^2$, as required.

10. (a) Here, $k = 10$ and so there are 10 balls in each bag.
Since there are 10 balls in each bag, there are $10 \cdot 10 = 100$ pairs of balls that can be chosen.
Let a be the number on the first ball chosen and b be the number on the second ball chosen. To determine $P(10)$, we count the number of pairs (a, b) for which ab is divisible by 10.
If the number of pairs is m, then $P(10) = \dfrac{m}{100}$.
For ab to be divisible by 10, at least one of a and b must be a multiple of 5 and at least one of a and b must be even.
If $a = 10$ or $b = 10$, then the pair (a, b) gives a product ab divisible by 10.
In this case, we obtain the 19 pairs

$$(a, b) = (1, 10), (2, 10), \ldots, (9, 10), (10, 10), (10, 9), \ldots, (10, 2), (10, 1)$$

If neither a nor b equals 10, then either $a = 5$ or $b = 5$ in order for a or b to be divisible by 5.
In this case, the other of a and b must be even and not equal to 10. (We have already counted the pairs where $a = 10$ or $b = 10$.)
In this case, we obtain the 8 pairs

$$(a, b) = (5, 2), (5, 4), (5, 6), (5, 8), (2, 5), (4, 5), (6, 5), (8, 5)$$

From our work above, there are no additional pairs for which ab is divisible by 10.
Thus, there are $19 + 8 = 27$ pairs (a, b) for which ab is divisible by 10, which means that $P(10) = \frac{27}{100}$.
(We note that we could have made a 10 by 10 table that listed all possible combinations of a and b and their product, from which we could obtain $P(10)$.)

(b) Let n be a positive integer with $n \geq 2$.
Consider $f(n) = 2n - 1$. This is a polynomial in n.
We demonstrate that $P(n) \geq \dfrac{2n-1}{n^2}$ for all positive integers n with $n \geq 2$ and that $P(n) = \dfrac{2n-1}{n^2}$ for infinitely many positive integers n with $n \geq 2$.
Suppose that there are two bags, each containing n balls labelled from 1 to n.
Since there are n balls in each bag, there are n^2 pairs of balls that can be chosen.
Let a be the number on the first ball chosen and b be the number on the second ball chosen.
The pairs

$$(a, b) = (1, n), (2, n), \ldots, (n - 1, n), (n, n), (n, n - 1), \ldots, (n, 2), (n, 1)$$

each have the property that ab is divisible by n.
There are $(n - 1) + 1 + (n - 1) = 2n - 1$ of these pairs.
Therefore, at least $2n-1$ of the pairs of balls that can be chosen have labels whose product is divisible by n.
Since there are n^2 pairs that can be chosen and the number of pairs of balls with the desired property is at least $2n - 1$, then $P(n) \geq \dfrac{2n-1}{n^2}$.
This proves the first part of what we needed to prove.

Next, suppose that $n = p$, a prime number.
For ab to be divisible by p, then either a is divisible by p or b is divisible by p (or both). (This property is not true when p is not a prime number; for example, $2 \cdot 2$ is divisible by 4 even though neither factor is divisible by 4.)
Since $1 \leq a \leq p$ and $1 \leq b \leq p$, then if either a is divisible by p or b is divisible by p (or both), we must have either $a = p$ or $b = p$, or both.
In other words, ab is divisible by p exactly when (a, b) is in the list

$$(1, p), (2, p), \ldots, (p-1, p), (p, p), (p, p-1), \ldots, (p, 2), (p, 1)$$

There are $2p - 1$ pairs in this list and these are the only pairs for which ab is divisible by p.
Therefore, $P(n) = \dfrac{2n-1}{n^2}$ when n is a prime number.
Since there are infinitely many prime numbers, then $P(n) = \dfrac{2n-1}{n^2}$ for infinitely many positive integers n with $n \geq 2$.
Thus, $f(n) = 2n - 1$ is a polynomial with the desired properties.

(c) Let $N = 2^k$, where k is a positive integer with $k \geq 2$.
Suppose that there are two bags, each containing N balls labelled from 1 to N.
Since there are N balls in each bag, there are N^2 pairs of balls that can be chosen.
Let a be the number on the first ball chosen and b be the number on the second ball chosen.
Let j be a positive integer with $1 \leq j \leq k-1$.
Consider pairs of the form $(a, b) = (2^j x, 2^{k-j} y)$ where x and y are odd positive integers that keep a and b in the desired range.
Note that, in each case, $ab = (2^j x)(2^{k-j} y) = 2^k xy$ which is divisible by $N = 2^k$.
Since $1 \leq a \leq 2^k$, then $1 \leq 2^j x \leq 2^k$ and so $x \leq 2^{k-j}$.
Since half of the positive integers from 1 to 2^{k-j} are odd, then there are $\frac{1}{2}2^{k-j} = 2^{k-j-1}$ choices for x.
Similarly, there are $2^{k-(k-j)-1} = 2^{j-1}$ choices for y.
Note that each choice of x and y gives a unique pair (a, b).
For any fixed j, there are 2^{k-j-1} choices for x and 2^{j-1} choices for y.
Thus, there are $2^{k-j-1} \cdot 2^{j-1} = 2^{k-2}$ choices of this form for the pair (a, b).
So for a fixed j with $1 \leq j \leq k-1$, this method gives 2^{k-2} pairs (a, b) for which ab is divisible by N.
Since there are $k-1$ different values for j, then there are at least $(k-1)2^{k-2}$ pairs (a, b) for which ab is divisible by N. (Note that two pairs that come from different values of j will in fact be different since the number of factors of 2 in their values of a will be different.)
Since there are N^2 choices for (a, b), then

$$P(N) \geq \frac{(k-1)2^{k-2}}{N^2} = \frac{(k-1)2^k 2^{-2}}{N^2} = \frac{k-1}{4} \cdot \frac{1}{N}$$

When $\dfrac{k-1}{4} > 2016$, we will have $P(N) > 2016 \cdot \dfrac{1}{N}$.
The inequality $\dfrac{k-1}{4} > 2016$ is equivalent to $k - 1 > 8064$ or $k > 8065$.
We want to show that there exists a positive integer m with $P(m) > \dfrac{2016}{m}$.
Set $m = 2^{8066}$.
By the work above, $P(m) \geq \dfrac{8065}{4} \cdot \dfrac{1}{m} > \dfrac{2016}{m}$, as required.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015 Euclid Contest

Wednesday, April 15, 2015
(in North America and South America)

Thursday, April 16, 2015
(outside of North America and South America)

Solutions

1. (a) Evaluating, $\frac{10^2-9^2}{10+9}=\frac{100-81}{19}=\frac{19}{19}=1$.
Alternatively, we could factor 10^2-9^2 as a difference of squares to obtain

$$\frac{10^2-9^2}{10+9}=\frac{(10+9)(10-9)}{10+9}=10-9=1$$

noting that $10+9$, which we divided from the numerator and denominator, is not equal to 0.

(b) Since $\frac{x+1}{x+4}=4$, then $x+1=4(x+4)$ and so $x+1=4x+16$ or $3x=-15$.
Therefore, $3x+8=-15+8=-7$.
Alternatively, we could note that since $3x=-15$, then $x=-5$.
Thus, $3x+8=3(-5)+8=-15+8=-7$.

(c) Since $f(x)=2x-1$, then $f(3)=2(3)-1=5$.
Therefore, $(f(3))^2+2(f(3))+1=5^2+2(5)+1=25+10+1=36$.
Alternatively, we could note that since $f(x)=2x-1$, then

$$(f(x))^2+2(f(x))+1=(f(x)+1)^2=(2x-1+1)^2=4x^2$$

and so $(f(3))^2+2(f(3))+1=4(3^2)=36$.

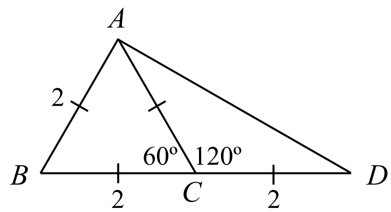
2. (a) Since $\sqrt{a}+\sqrt{a}=20$, then $2\sqrt{a}=20$ or $\sqrt{a}=10$, and so $a=10^2=100$.

(b) Let the radius of the larger circle be r.
Since the radius of the smaller circle is 1, then its area is $\pi\cdot 1^2=\pi$.
Since the area between the circles is equal to the area of the smaller circle, then the area of the larger circle is $\pi+\pi=2\pi$.
Thus, $\pi r^2=2\pi$ or $r^2=2$. Since $r>0$, then $r=\sqrt{2}$.

(c) Since 30 students had an average mark of 80, then the sum of the marks of these 30 students was $30\cdot 80=2400$.
After 2 students dropped the class, there were 28 students left. Their average mark was 82.
Thus, the sum of the marks of the remaining 28 students was $28\cdot 82=2296$.
Therefore, the sum of the marks of the 2 students who dropped the class was $2400-2296$ or 104, and so their average mark was $\frac{104}{2}=52$.

3. (a) *Solution 1*

Join AD.
Since $BC = CD$ and $BD = 4$, then $BC = CD = 2$. Also, $AB = BC = 2$.
Since $\triangle ABC$ is equilateral, then $\angle ABC = \angle ACB = 60^\circ$.
Since $\angle ACB = 60^\circ$, then $\angle ACD = 180^\circ - \angle ACB = 180^\circ - 60^\circ = 120^\circ$.



Since $AC = CD$, then $\triangle ACD$ is isosceles with $\angle CDA = \angle CAD$.
Each of these angles equals $\frac{1}{2}(180^\circ - \angle ACD) = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$.
Since $\angle ABD = 60^\circ$ and $\angle ADB = 30^\circ$, then $\angle BAD = 90^\circ$ and $\triangle DBA$ is a 30°-60°-90° triangle.
Therefore, $AD = \sqrt{3}AB = 2\sqrt{3}$.

Solution 2

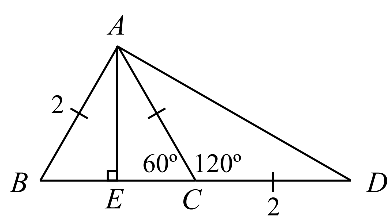
Join AD.
Since $BC = CD$ and $BD = 4$, then $BC = CD = 2$. Also, $AC = CD = 2$.
Since $\angle ACB = 60^\circ$, then $\angle ACD = 180^\circ - \angle ACB = 180^\circ - 60^\circ = 120^\circ$.
By the cosine law in $\triangle ACD$,

$$\begin{aligned} AD^2 &= AC^2 + CD^2 - 2(AC)(CD)\cos(\angle ACD) \\ &= 2^2 + 2^2 - 2(2)(2)\cos 120^\circ \\ &= 4 + 4 - 8(-\tfrac{1}{2}) \\ &= 12 \end{aligned}$$

Since $AD^2 = 12$ and $AD > 0$, then $AD = \sqrt{12} = 2\sqrt{3}$.

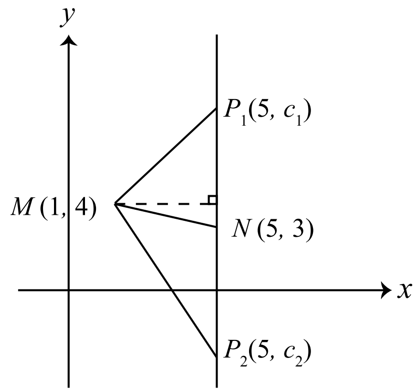
Solution 3

Join AD and drop a perpendicular from A to E on BC.
Since $BC = CD$ and $BD = 4$, then $BC = CD = 2$. Also, $AB = BC = 2$.
Since $\triangle ABC$ is equilateral, then $\angle ABC = \angle ACB = 60^\circ$.
Since $\angle ABC = 60^\circ$ and $\angle AEB = 90^\circ$, then $\triangle ABE$ is a 30°-60°-90° triangle.
Thus, $AE = \frac{\sqrt{3}}{2}AB = \sqrt{3}$.
Since $\angle ACB = 60^\circ$, then $\angle ACD = 180^\circ - \angle ACB = 180^\circ - 60^\circ = 120^\circ$.



Since $AC = CD$, then $\triangle ACD$ is isosceles with $\angle CDA = \angle CAD$.
Each of these angles equals $\frac{1}{2}(180^\circ - \angle ACD) = \frac{1}{2}(180^\circ - 120^\circ) = 30^\circ$.
But $\triangle DAE$ is then a 30°-60°-90° triangle, so $AD = 2AE = 2\sqrt{3}$.

(b) Points $N(5,3)$ and $P(5,c)$ lie on the same vertical line. We can consider NP as the base of $\triangle MNP$. Suppose that the length of this base is b.
The corresponding height of $\triangle MNP$ is the distance from $M(1,4)$ to the line through N and P. Since M lies on the vertical line $x=1$ and N and P lie on the vertical line $x=5$, then the height is $h=4$.



Since the area of $\triangle MNP$ is 14, then $\frac{1}{2}bh = 14$.
Since $h=4$, then $\frac{1}{2}b(4) = 14$ or $2b = 14$ and so $b=7$.
Therefore, $P(5,c)$ is a distance of 7 units away from $N(5,3)$.
Since NP is a vertical line segment, then $c = 3+7$ or $c = 3-7$, and so $c=10$ or $c=-4$.
The sum of these two values is $10+(-4) = 6$.
(We could also have noted that, since the two values of c will be symmetric about $y=3$, then the average of their values is 3 and so the sum of their values is $2\cdot 3 = 6$.)

4. (a) To find the y-intercept, we set $x=0$ and obtain

$$y = (-1)(-2)(-3) - (-2)(-3)(-4) = (-6) - (-24) = 18 \ .$$

To find the x-intercepts, we first simplify using common factors:

$$y = (x-1)(x-2)(x-3)-(x-2)(x-3)(x-4) = (x-2)(x-3)\left((x-1)-(x-4)\right) = 3(x-2)(x-3)$$

To find the x-intercepts, we set $y=0$ and obtain $3(x-2)(x-3) = 0$ which gives $x=2$ or $x=3$.
Therefore, the y-intercept is 18 and the x-intercepts are 2 and 3.

(b) To find the points of intersection of the graphs with equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$, we equate values of y and solve for x.
We want to find all values of a for which there are exactly two values of x which are solutions to $x^3 - x^2 + 3x - 4 = ax^2 - x - 4$.
Solving, we obtain

$$\begin{aligned} x^3 - x^2 + 3x - 4 &= ax^2 - x - 4 \\ x^3 - x^2 - ax^2 + 4x &= 0 \\ x^3 - (a+1)x^2 + 4x &= 0 \\ x(x^2 - (a+1)x + 4) &= 0 \end{aligned}$$

Therefore $x=0$ or $x^2 - (a+1)x + 4 = 0$.
Note that $x=0$ is not a solution to $x^2-(a+1)x+4=0$, since when $x=0$ is substituted

into the left side, we obtain 4 and not 0.
Therefore, for there to be exactly two points of intersection between the two graphs, the quadratic equation $x^2 - (a+1)x + 4 = 0$ must have exactly one solution.
Setting the discriminant equal to 0 (to obtain a single root), we obtain $(a+1)^2 - 4(1)(4) = 0$ or $(a+1)^2 = 16$, which gives $a + 1 = \pm 4$.
If $a + 1 = 4$, then $a = 3$; if $a + 1 = -4$, then $a = -5$.
Therefore, the values of a for which the graphs with equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$ intersect at exactly two points are $a = 3$ and $a = -5$.
(We can check that $y = x^3 - x^2 + 3x - 4$ and $y = 3x^2 - x - 4$ intersect exactly when $x = 0$ and $x = 2$, and that $y = x^3 - x^2 + 3x - 4$ and $y = -5x^2 - x - 4$ intersect exactly when $x = 0$ and $x = -2$.)

5. (a) Suppose that $AB = AC = DE = x$.
Since $DB = 9$, then $AD = x - 9$.
Since $EC = 8$, then $AE = x - 8$.
By the Pythagorean Theorem in $\triangle ADE$,

$$\begin{aligned} AD^2 + AE^2 &= DE^2 \\ (x-9)^2 + (x-8)^2 &= x^2 \\ x^2 - 18x + 81 + x^2 - 16x + 64 &= x^2 \\ x^2 - 34x + 145 &= 0 \\ (x-5)(x-29) &= 0 \end{aligned}$$

Therefore, $x = 5$ or $x = 29$.
Since $x \geq 9$ (because $AB \geq DB = 9$), then $DE = 29$.

(b) Since each list contains 6 consecutive positive integers and the smallest integers in the lists are a and b, then the positive integers in the first list are $a, a+1, a+2, a+3, a+4, a+5$ and the positive integers in the second list are $b, b+1, b+2, b+3, b+4, b+5$.
Note that $1 \leq a < b$.
We first determine the pairs (a, b) for which 49 will appear in the third list, then determine which of these pairs give a third list that contains no multiple of 64, and then finally keep only those pairs for which there is a number in the third list larger than 75.

The first bullet tells us that 49 is the product of an integer in the first list and an integer in the second list.
Since $49 = 7^2$ and 7 is prime, then these integers are either 1 and 49 or 7 and 7.
If 1 is in one of the lists, then either $a = 1$ or $b = 1$. Since $1 \leq a < b$, then it must be that $a = 1$.
If 49 is in the second list, then one of $b, b+1, b+2, b+3, b+4, b+5$ equals 49, and so $44 \leq b \leq 49$.
Therefore, for 1 and 49 to appear in the two lists, then (a, b) must be one of

$$(1,49), (1,48), (1,47), (1,46), (1,45), (1,44) .$$

If 7 appears in the first list, then one of $a, a+1, a+2, a+3, a+4, a+5$ equals 7, so $2 \leq a \leq 7$. Similarly, if 7 appears in the second list, then $2 \leq b \leq 7$.
Therefore, for 7 to appear in both lists, then, knowing that $a < b$, then (a, b) must be one of

$$(2,3), (2,4), (2,5), (2,6), (2,7), (3,4), (3,5), (3,6), (3,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7) .$$

The second bullet tells us that no pair of numbers in the first and second lists have a product that is a multiple of 64.
Given that the possible values of a and b are $1, 2, 3, 4, 5, 6, 7, 44, 45, 46, 47, 48, 49$, then the possible integers in the two lists are those integers from 1 to 12, inclusive, and from 44 to 54, inclusive. (For example, if the first number in one list is 7, then the remaining numbers in this list are 8, 9, 10, 11, 12.)
There is no multiple of 32 or 64 in these lists.
Thus, for a pair of integers from these lists to have a product that is a multiple of 64, one is a multiple of 4 and the other is a multiple of 16, or both are multiples of 8.
If $(a, b) = (1, 48), (1, 47), (1, 46), (1, 45), (1, 44)$, then 4 appears in the first list and 48 appears in the second list; these have a product of 192, which is $3 \cdot 64$.
If $(a, b) = (1, 49)$, there is a multiple of 4 but not of 8 in the first list, and a multiple of 4 but not of 8 in the second list, so there is no multiple of 64 in the third list.
If $(a, b) = (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7)$, then 8 appears in both lists, so 64 appears in the third list.
If $(a, b) = (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$, then there is no multiple of 8 or 16 in the first list and no multiple of 16 in the second list, so there is no multiple of 64 in the third list.
Therefore, after considering the first two bullets, the possible pairs (a, b) are $(1, 49), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$.

The third bullet tells us that there is at least one number in the third list that is larger than 75.
Given the possible pairs (a, b) are $(1, 49), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7)$, the corresponding pairs of largest integers in the lists are $(6, 54), (7, 8), (7, 9), (7, 10), (7, 11), (7, 12)$.
The corresponding largest integers in the third list are the products of the largest integers in the two lists; these products are $324, 56, 63, 70, 77, 84$, respectively.
Therefore, the remaining pairs (a, b) are $(1, 49), (2, 6), (2, 7)$

Having considered the three conditions, the possible pairs (a, b) are $(1, 49), (2, 6), (2, 7)$.

6. (a) We are told that when a, b and c are the numbers in consecutive sectors, then $b = ac$. This means that if a and b are the numbers in consecutive sectors, then the number in the next sector is $c = \dfrac{b}{a}$. (That is, each number is equal to the previous number divided by the one before that.)

Starting with the given 2 and 3 and proceeding clockwise, we obtain

$$2,\quad 3,\quad \frac{3}{2},\quad \frac{3/2}{3} = \frac{1}{2},\quad \frac{1/2}{3/2} = \frac{1}{3},\quad \frac{1/3}{1/2} = \frac{2}{3},\quad \frac{2/3}{1/3} = 2,\quad \frac{2}{2/3} = 3,\quad \frac{3}{2},\quad \ldots$$

After the first 6 terms, the first 2 terms (2 and 3) reappear, and so the first 6 terms will repeat again. (This is because each term comes from the previous two terms, so when two consecutive terms reappear, then the following terms are the same as when these two consecutive terms appeared earlier.)

Since there are 36 terms in total, then the 6 terms repeat exactly $\dfrac{36}{6} = 6$ times.

Therefore, the sum of the 36 numbers is $6\left(2 + 3 + \dfrac{3}{2} + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{2}{3}\right) = 6(2+3+2+1) = 48$.

(b) We consider two cases: $x > -1$ (that is, $x + 1 > 0$) and $x < -1$ (that is, $x + 1 < 0$). Note that $x \neq -1$.

Case 1: $x > -1$

We take the given inequality $0 < \dfrac{x^2 - 11}{x + 1} < 7$ and multiply through by $x + 1$, which is positive, to obtain $0 < x^2 - 11 < 7x + 7$.

Thus, $x^2 - 11 > 0$ and $x^2 - 11 < 7x + 7$.

From the first, we obtain $x^2 > 11$ and so $x > \sqrt{11}$ or $x < -\sqrt{11}$.

Since $x > -1$, then $x > \sqrt{11}$. (Note that $-\sqrt{11} < -1$.)

From the second, we obtain $x^2 - 7x - 18 < 0$ or $(x - 9)(x + 2) < 0$. Thus, $-2 < x < 9$. (Since $y = x^2 - 7x - 18$ represents a parabola opening upwards, its y-values are negative between its x-intercepts.)

Since $x > -1$ and $-2 < x < 9$, then $-1 < x < 9$.

Since $x > \sqrt{11}$ and $-1 < x < 9$, then the solution in this case is $\sqrt{11} < x < 9$.

Case 2: $x < -1$

We take the given inequality $0 < \dfrac{x^2 - 11}{x + 1} < 7$ and multiply through by $x + 1$, which is negative, to obtain $0 > x^2 - 11 > 7x + 7$.

Thus, $x^2 - 11 < 0$ and $x^2 - 11 > 7x + 7$.

From the first, we obtain $x^2 < 11$ and so $-\sqrt{11} < x < \sqrt{11}$.

Since $x < -1$ and $-\sqrt{11} < x < \sqrt{11}$, then $-\sqrt{11} < x < -1$.

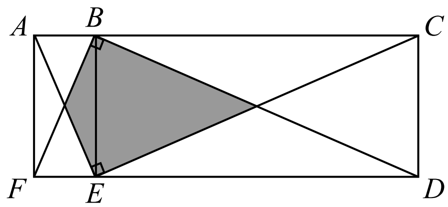
From the second, we obtain $x^2 - 7x - 18 > 0$ or $(x - 9)(x + 2) > 0$. Thus, $x < -2$ or $x > 9$. (Since $y = x^2 - 7x - 18$ represents a parabola opening upwards, its y-values are positive outside its x-intercepts.)

Since $x < -1$, we obtain $x < -2$.

Since $-\sqrt{11} < x < -1$ and $x < -2$, then the solution in this case is $-\sqrt{11} < x < -2$.

In summary, the values of x for which $0 < \dfrac{x^2 - 11}{x + 1} < 7$ those x with $-\sqrt{11} < x < -2$ and those x with $\sqrt{11} < x < 9$.

7. (a) Join BE.



Since $\triangle FBD$ is congruent to $\triangle AEC$, then $FB = AE$.
Since $\triangle FAB$ and $\triangle AFE$ are each right-angled, share a common side AF and have equal hypotenuses ($FB = AE$), then these triangles are congruent, and so $AB = FE$.
Now $BAFE$ has two right angles at A and F (so AB and FE are parallel) and has equal sides $AB = FE$ so must be a rectangle.
This means that $BCDE$ is also a rectangle.
Now the diagonals of a rectangle partition it into four triangles of equal area. (Diagonal AE of the rectangle splits the rectangle into two congruent triangles, which have equal area. The diagonals bisect each other, so the four smaller triangles all have equal area.)
Since $\frac{1}{4}$ of rectangle $ABEF$ is shaded and $\frac{1}{4}$ of rectangle $BCDE$ is shaded, then $\frac{1}{4}$ of the total area is shaded. (If the area of $ABEF$ is x and the area of $BCDE$ is y, then the total shaded area is $\frac{1}{4}x + \frac{1}{4}y$, which is $\frac{1}{4}$ of the total area $x + y$.)
Since $AC = 200$ and $CD = 50$, then the area of rectangle $ACDF$ is $200(50) = 10\,000$, so the total shaded area is $\frac{1}{4}(10\,000) = 2500$.

(b) Suppose that the arithmetic sequence $a_1, a_2, a_3, \ldots$ has first term a and common difference d.
Then, for each positive integer n, $a_n = a + (n-1)d$.
Since $a_1 = a$ and $a_2 = a + d$ and $a_1 \neq a_2$, then $d \neq 0$.
Since a_1, a_2, a_6 form a geometric sequence in that order, then $\dfrac{a_2}{a_1} = \dfrac{a_6}{a_2}$ or $(a_2)^2 = a_1 a_6$.
Substituting, we obtain

$$\begin{aligned} (a+d)^2 &= a(a+5d) \\ a^2 + 2ad + d^2 &= a^2 + 5ad \\ d^2 &= 3ad \\ d &= 3a \qquad (\text{since } d \neq 0) \end{aligned}$$

Therefore, $a_n = a + (n-1)d = a + (n-1)(3a) = (3n-2)a$ for each $n \geq 1$.
Thus, $a_4 = (3(4) - 2)a = 10a$, and $a_k = (3k-2)a$. (Note that $a_1 = (3(1) - 2)a = a$.)
For a_1, a_4, a_k to also form a geometric sequence then, as above, $(a_4)^2 = a_1 a_k$, and so

$$\begin{aligned} (10a)^2 &= (a)((3k-2)a) \\ 100a^2 &= (3k-2)a^2 \end{aligned}$$

Since $d \neq 0$ and $d = 3a$, then $a \neq 0$.
Since $100a^2 = (3k-2)a^2$ and $a \neq 0$, then $100 = 3k - 2$ and so $3k = 102$ or $k = 34$.
Checking, we note that $a_1 = a$, $a_4 = 10a$ and $a_{34} = 100a$ which form a geometric sequence with common ratio 10.
Therefore, the only possible value of k is $k = 34$.

8. (a) First, we note that since k is a positive integer, then $k \geq 1$.
Next, we note that the given parabola passes through the point $(0,-5)$ as does the given circle. (This is because if $x = 0$, then $y = \dfrac{0^2}{k} - 5 = -5$ and if $(x,y) = (0,-5)$, then $x^2 + y^2 = 0^2 + (-5)^2 = 25$, so $(0,-5)$ satisfies each of the equations.)
Therefore, for every positive integer k, the two graphs intersect in at least one point.
If $y = -5$, then $x^2 + (-5)^2 = 25$ and so $x^2 = 0$ or $x = 0$. In other words, there is one point on both parabola and circle with $y = -5$, namely $(0,-5)$.
Now, the given circle with equation $x^2 + y^2 = 25 = 5^2$ has centre $(0,0)$ and radius 5.
This means that the y-coordinates of points on this circle satisfy $-5 \leq y \leq 5$.
To find the other points of intersection, we re-write $y = \dfrac{x^2}{k} - 5$ as $ky = x^2 - 5k$ or $x^2 = ky + 5k$ and substitute into $x^2 + y^2 = 25$ to obtain

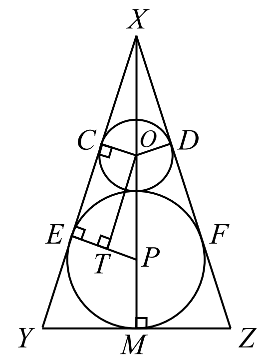
$$\begin{aligned} (ky + 5k) + y^2 &= 25 \\ y^2 + ky + (5k - 25) &= 0 \\ (y+5)(y + (k-5)) &= 0 \end{aligned}$$

and so $y = -5$ or $y = 5 - k$.
(We note that since the two graphs intersect at $y = -5$, then $(y+5)$ was going to be a factor of the quadratic equation $y^2 + ky + (5k - 25) = 0$. If we had not seen this, we could have used the quadratic formula.)
Therefore, for $y = 5 - k$ to give points on the circle, we need $-5 \leq 5 - k$ and $5 - k \leq 5$.
This gives $k \leq 10$ and $k \geq 0$.
Since k is a positive integer, the possible values of k to this point are $k = 1,2,3,4,5,6,7,8,9,10$.
If $k = 1$, then $y = 5 - 1 = 4$. In this case, $x^2 + 4^2 = 25$ or $x^2 = 9$ and so $x = \pm 3$.
This gives the two points $(3,4)$ and $(-3,4)$ which lie on the parabola and circle.
Consider the three points $A(3,4)$, $B(-3,4)$ and $C(0,-5)$.
Now AB is horizontal with $AB = 3 - (-3) = 6$. (This is the difference in x-coordinates.)
The vertical distance from AB to C is $4 - (-5) = 9$. (This is the difference in y-coordinates.)
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(6)(9) = 27$, which is a positive integer.
We now repeat these calculations for each of the other values of k by making a table:

k	y	$x = \pm\sqrt{25 - y^2}$	Base	Height	Area of triangle
1	4	± 3	$3 - (-3) = 6$	$4 - (-5) = 9$	27
2	3	± 4	$4 - (-4) = 8$	$3 - (-5) = 8$	32
3	2	$\pm\sqrt{21}$	$2\sqrt{21}$	7	$7\sqrt{21}$
4	1	$\pm\sqrt{24}$	$2\sqrt{24}$	6	$6\sqrt{24}$
5	0	± 5	10	5	25
6	-1	$\pm\sqrt{24}$	$2\sqrt{24}$	4	$4\sqrt{24}$
7	-2	$\pm\sqrt{21}$	$2\sqrt{21}$	3	$3\sqrt{21}$
8	-3	± 4	8	2	8
9	-4	± 3	6	1	3
10	-5	0			

When $k = 10$, we have $y = 5 - k = -5$ and $x = 0$ only, so there is only one point of intersection.
Finally, the values of k for which there are three points of intersection and for which the area of the resulting triangle is a positive integer are $k = 1,2,5,8,9$.

(b) Suppose that M is the midpoint of YZ.
Suppose that the centre of the smaller circle is O and the centre of the larger circle is P.
Suppose that the smaller circle touches XY at C and XZ at D, and that the larger circle touches XY at E and XZ at F.
Join OC, OD and PE.
Since OC and PE are radii that join the centres of circles to points of tangency, then OC and PE are perpendicular to XY.
Join XM. Since $\triangle XYZ$ is isosceles, then XM (which is a median by construction) is an altitude (that is, XM is perpendicular to YZ) and an angle bisector (that is, $\angle MXY = \angle MXZ$).
Now XM passes through O and P. (Since XC and XD are tangents from X to the same circle, then $XC = XD$. This means that $\triangle XCO$ is congruent to $\triangle XDO$ by side-side-side. This means that $\angle OXC = \angle OXD$ and so O lies on the angle bisector of $\angle CXD$, and so O lies on XM. Using a similar argument, P lies on XM.)



Draw a perpendicular from O to T on PE. Note that OT is parallel to XY (since each is perpendicular to PE) and that $OCET$ is a rectangle (since it has three right angles).
Consider $\triangle XMY$ and $\triangle OTP$.
Each triangle is right-angled (at M and at T).
Also, $\angle YXM = \angle POT$. (This is because OT is parallel to XY, since both are perpendicular to PE.)
Therefore, $\triangle XMY$ is similar to $\triangle OTP$.
Thus, $\dfrac{XY}{YM} = \dfrac{OP}{PT}$.
Now $XY = a$ and $YM = \frac{1}{2}b$.
Also, OP is the line segment joining the centres of two tangent circles, so $OP = r + R$.
Lastly, $PT = PE - ET = R - r$, since $PE = R$, $ET = OC = r$, and $OCET$ is a rectangle.
Therefore,

$$\begin{aligned}
\frac{a}{b/2} &= \frac{R+r}{R-r} \\
\frac{2a}{b} &= \frac{R+r}{R-r} \\
2a(R-r) &= b(R+r) \\
2aR - bR &= 2ar + br \\
R(2a-b) &= r(2a+b) \\
\frac{R}{r} &= \frac{2a+b}{2a-b} \qquad \text{(since } 2a > b \text{ so } 2a - b \neq 0\text{, and } r > 0\text{)}
\end{aligned}$$

Therefore, $\dfrac{R}{r} = \dfrac{2a+b}{2a-b}$.

9. Using logarithm rules $\log(uv) = \log u + \log v$ and $\log(s^t) = t \log s$ for all $u, v, s > 0$, the first equation becomes

$$\begin{aligned}
(\log x)(\log y) - 3\log 5 - 3\log y - \log 8 - \log x &= a \\
(\log x)(\log y) - \log x - 3\log y - \log 8 - \log 5^3 &= a \\
(\log x)(\log y) - \log x - 3\log y - \log(8 \cdot 125) &= a \\
(\log x)(\log y) - \log x - 3\log y - \log(1000) &= a \\
(\log x)(\log y) - \log x - 3\log y - 3 &= a
\end{aligned}$$

Similarly, the second equation becomes

$$\begin{aligned}
(\log y)(\log z) - 4\log 5 - 4\log y - \log 16 - \log z &= b \\
(\log y)(\log z) - 4\log y - \log z - 4\log 5 - \log 16 &= b \\
(\log y)(\log z) - 4\log y - \log z - \log(5^4 \cdot 16) &= b \\
(\log y)(\log z) - 4\log y - \log z - \log(10\,000) &= b \\
(\log y)(\log z) - 4\log y - \log z - 4 &= b
\end{aligned}$$

And the third equation becomes

$$\begin{aligned}
(\log z)(\log x) - 4\log 8 - 4\log x - 3\log 625 - 3\log z &= c \\
(\log z)(\log x) - 4\log x - 3\log z - 4\log 8 - 3\log 625 &= c \\
(\log z)(\log x) - 4\log x - 3\log z - \log(8^4 \cdot 625^3) &= c \\
(\log z)(\log x) - 4\log x - 3\log z - \log(2^{12} \cdot 5^{12}) &= c \\
(\log z)(\log x) - 4\log x - 3\log z - 12 &= c
\end{aligned}$$

Since each of the steps that we have made are reversible, the original system of equations is equivalent to the new system of equations

$$\begin{aligned}
(\log x)(\log y) - \log x - 3\log y - 3 &= a \\
(\log y)(\log z) - 4\log y - \log z - 4 &= b \\
(\log z)(\log x) - 4\log x - 3\log z - 12 &= c
\end{aligned}$$

Next, we make the substitution $X = \log x$, $Y = \log y$ and $Z = \log z$. (This is equivalent to saying $x = 10^X$, $y = 10^Y$ and $z = 10^Z$.)
This transforms the system of equations to the equivalent system

$$\begin{aligned}
XY - X - 3Y - 3 &= a \\
YZ - 4Y - Z - 4 &= b \\
XZ - 4X - 3Z - 12 &= c
\end{aligned}$$

We re-write the first of these three equations as $X(Y-1) - 3Y - 3 = a$ and then as $X(Y-1) - 3(Y-1) - 6 = a$ and then as $(X-3)(Y-1) = a+6$.
In a similar way, we re-write the second and third of these equations to obtain the equivalent system

$$\begin{aligned}
(X-3)(Y-1) &= a+6 \\
(Y-1)(Z-4) &= b+8 \\
(X-3)(Z-4) &= c+24
\end{aligned}$$

Next, we make the substitution $p = X - 3$, $q = Y - 1$ and $r = Z - 4$. (This is equivalent to saying $X = p + 3$, $Y = q + 1$ and $Z = r + 4$, or $x = 10^{p+3}$, $y = 10^{q+1}$ and $z = 10^{r+4}$.)
This transforms the original system of equations into the equivalent system

$$\begin{aligned} pq &= a + 6 \\ qr &= b + 8 \\ pr &= c + 24 \end{aligned}$$

We again note that this system of equations is equivalent to the initial system of equations, and each solution of this system corresponds with a solution of the initial system.

(a) Suppose that $a = -4$, $b = 4$ and $c = -18$.
Then the last version of the system is

$$\begin{aligned} pq &= 2 \\ qr &= 12 \\ pr &= 6 \end{aligned}$$

Multiplying the three equations together gives $p^2q^2r^2 = 2 \cdot 12 \cdot 6 = 144$.
Since $(pqr)^2 = 144$, then $pqr = \pm 12$.
Therefore, $r = \dfrac{pqr}{pq} = \dfrac{\pm 12}{2} = \pm 6$ and $p = \dfrac{pqr}{qr} = \dfrac{\pm 12}{12} = \pm 1$ and $q = \dfrac{pqr}{pr} = \dfrac{\pm 12}{6} = \pm 2$.
Therefore, the solutions to the last version of the system are $(p, q, r) = (1, 2, 6)$ and $(p, q, r) = (-1, -2, -6)$.
Converting back to the original variables, we see that the solutions to the original system when $(a, b, c) = (-4, 4, -18)$ are $(x, y, z) = (10^4, 10^3, 10^{10})$ and $(x, y, z) = (10^2, 10^{-1}, 10^{-2})$.

(b) We consider the various possibilities for the product, $(a + 6)(b + 8)(c + 24)$, of the right sides of the equations in the final form of the system above: whether it is positive, negative or equal to 0.

Case 1: $(a + 6)(b + 8)(c + 24) < 0$
As in (a), we multiply the three equations together to obtain $(pqr)^2 = (a+6)(b+8)(c+24)$.
Since the left side is at least 0 and the right side is negative, then there are no solutions to the system of equations in this case.

Case 2: $(a + 6)(b + 8)(c + 24) > 0$
As in (a), we multiply the three equations together to obtain $(pqr)^2 = (a+6)(b+8)(c+24)$.
Since $(pqr)^2 = (a + 6)(b + 8)(c + 24)$ and $(a + 6)(b + 8)(c + 24) > 0$, then $pqr = \pm\sqrt{(a + 6)(b + 8)(c + 24)}$.
Since $(a + 6)(b + 8)(c + 24) > 0$, then $\sqrt{(a + 6)(b + 8)(c + 24)}$ is well-defined.
Also, since $(a + 6)(b + 8)(c + 24) > 0$, then each of $a + 6$, $b + 8$, $c + 24$ is non-zero, so we can divide by each of these quantities.

As we did in (a), we can solve to obtain

$$\begin{aligned} p &= \frac{pqr}{qr} = \frac{\pm\sqrt{(a + 6)(b + 8)(c + 24)}}{b + 8} \\ q &= \frac{pqr}{pr} = \frac{\pm\sqrt{(a + 6)(b + 8)(c + 24)}}{c + 24} \\ r &= \frac{pqr}{pq} = \frac{\pm\sqrt{(a + 6)(b + 8)(c + 24)}}{a + 6} \end{aligned}$$

Since $(a+6)(b+8)(c+24) > 0$, these are all valid fractions and there are exactly two triples (p,q,r) that are solutions and so two triples (x,y,z) that are solutions to the original system.

Case 3: $(a+6)(b+8)(c+24) = 0$

Suppose that exactly one of $a+6$, $b+8$ and $c+24$ equals 0.
Without loss of generality, suppose that $a+6=0$, $b+8 \neq 0$ and $c+24 \neq 0$.
Since $pq = a+6 = 0$, then $p = 0$ or $q = 0$.
In this case, either $qr = b+8$ or $pr = c+24$ will equal 0, which contradicts our assumption that neither $b+8$ nor $c+24$ is 0.
Therefore, it cannot be the case that exactly one of $a+6$, $b+8$ and $c+24$ equals 0.

Suppose next that exactly two of $a+6$, $b+8$ and $c+24$ equal 0.
Without loss of generality, suppose that $a+6 = b+8 = 0$ and $c+24 \neq 0$.
Since $pr = c+24 \neq 0$, then $p \neq 0$ and $r \neq 0$.
Since $pq = a+6 = 0$ and $qr = b+8 = 0$ and $p \neq 0$ and $r \neq 0$, then $q = 0$.
In this case, any triple (p,q,r) with $q = 0$ and $pr = c+24 \neq 0$ is a solution to the system of equations.
Thus, when $a+6 = b+8 = 0$ and $c+24 \neq 0$ (that is, $(a,b,c) = (-6,-8,c)$ with $c \neq 24$), each triple $(p,q,r) = \left(p, 0, \dfrac{c+24}{p}\right)$ with $p \neq 0$ is a solution to the system of equations.
Each of these solutions corresponds to a solution to the original system of equations in (x,y,z), so if $(a,b,c) = (-6,-8,c)$ with $c \neq 0$, then there are infinite number of solutions to the system of equations.
Similarly, if $(a,b,c) = (-6,b,-24)$ with $b \neq -8$ (that is, if $p = a+6 = 0$ and $r = c+24 = 0$ but $q = b+8 \neq 0$) or $(a,b,c) = (a,-8,-24)$ with $a \neq -6$, then there are infinitely many solutions (x,y,z) to the original system of equations.
Finally, we must consider the case of $a+6 = b+8 = c+24 = 0$.
Here, we must solve the system of equations

$$\begin{aligned} pq &= 0 \\ qr &= 0 \\ pr &= 0 \end{aligned}$$

Each triple $(p,q,r) = (0,0,r)$ is a solution of this system and there are infinitely many such solutions. (This is not all of the solutions, but represents infinitely many solutions.)
Therefore, when $(a,b,c) = (-6,-8,-24)$, there are also infinitely many solutions to the original system of equations.

Therefore, the system of equations has an infinite number of solutions (x,y,z) precisely when $(a,b,c) = (-6,-8,c)$ for some real number c or $(a,b,c) = (-6,b,-24)$ for some real number b or $(a,b,c) = (a,-8,-24)$ for some real number a or $(a,b,c) = (-6,-8,-24)$. (This last triple is in fact included in each of the previous three families of triples.)

10. (a) The subsets of C_4 are:

$$\{\} \quad \{1\} \quad \{2\} \quad \{3\} \quad \{4\}$$
$$\{1,2\} \quad \{1,3\} \quad \{1,4\} \quad \{2,3\} \quad \{2,4\} \quad \{3,4\}$$
$$\{1,2,3\} \quad \{1,2,4\} \quad \{1,3,4\} \quad \{2,3,4\} \quad \{1,2,3,4\}$$

(There are 16 such subsets including the empty set $\{\}$ and the complete set $C_4 = \{1,2,3,4\}$.)
Consider the Furoni family $A = \{\{1,2\},\{1,3\},\{1,4\}\}$.
Each of the following subsets of C_4 is already an element of A: $\{1,2\},\{1,3\},\{1,4\}$.
Each of the following subsets of C_4 is a subset of one or more of the elements of A: $\{\},\{1\},\{2\},\{3\},\{4\}$.
Each of the following subsets of C_4 has the property that one or more of the elements of A is a subset of it: $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,2,3,4\}$.
Since a Furoni family of C_4 cannot contain two subsets of C_4 one of which is a subset of the other, none of the subsets in either of these two lists can be added to A to form a larger Furoni family.
This leaves the following subsets of C_4 to consider as possible elements to add to A: $\{2,3\},\{2,4\},\{3,4\},\{2,3,4\}$.
If $\{2,3,4\}$ is added to A to form $A' = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$, then A' is still a Furoni family of C_4 and none of $\{2,3\},\{2,4\},\{3,4\}$ can be added, since each is a subset of $\{2,3,4\}$. Therefore, A' is a Furoni family of C_4 to which no other subset can be added.
If any of $\{2,3\},\{2,4\},\{3,4\}$ is added to A, then $\{2,3,4\}$ cannot be added (since each of these three two elements sets is a subset of $\{2,3,4\}$) but each of the remaining two element sets can be still added without violating the conditions for being a Furoni family.
Thus, $A'' = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ is a Furoni family of C_4 to which no other subset can be added.
Therefore, the two Furoni families of C_4 that contain all of the elements of A and to which no further subsets of C_4 can be added are

$$A' = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\} \qquad A'' = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

(b) *Solution 1*

Suppose that n is a positive integer and F is a Furoni family of C_n that contains a_k elements that contain exactly k integers each, for each integer k from 0 to n, inclusive.
Consider each element E of F.
Each E is a subset of C_n. Suppose that a particular choice for E contains exactly k elements.
We use E to generate $k!(n-k)!$ permutations σ of the integers in $C_n = \{1,2,3,\ldots,n\}$ by starting with a permutation α of the elements of E and appending a permutation β of the elements in C_n not in E.
Since there are k elements in E, there are $k!$ possible permutations α.
Since there are $n-k$ elements in C_n that are not in E, there are $(n-k)!$ possible permutations β.
Each possible α can have each possible β appended to it, so there are $k!(n-k)!$ possible permutations $\sigma = \alpha|\beta$. (The notation "$\alpha|\beta$" means the permutation of C_n formed by writing out the permutation α (of the elements of E) followed by writing out the permutation β (of the elements of C_n not in E).)
Each of these $k!(n-k)!$ permutations generated by E is indeed different, since if two permutations $\sigma = \alpha|\beta$ and $\sigma' = \alpha'|\beta'$ are equal, then since α and α' are both permutations of the elements of E, then they have the same length and so $\alpha|\beta = \alpha'|\beta'$ means $\alpha = \alpha'$.

This then means that $\beta = \beta'$ and so the permutations started out the same.
We repeat this process for each of the elements E of F.
Since, for each k, there are a_k subsets of size k in F, then the total number of permutations that this generates is

$$a_0 0!(n-0)! + a_1 1!(n-1)! + \cdots + a_{n-1}(n-1)!(n-(n-1))1! + a_n n!(n-n)!$$

If each of these permutations is different, then this total is at most $n!$, since this is the total number of permutations of the elements of C_n.
Is it possible that two elements E and G of F generate identical permutations of the elements of C_n in this way?
Suppose that two permutations $\sigma = \alpha|\beta$ (generated by E) and $\sigma' = \alpha'|\beta'$ (generated by G) are identical.
Suppose that E contains k elements and G contains k' elements.
Either $k \leq k'$ or $k' \leq k$ (or both, if they are equal).
Without loss of generality, suppose that $k \leq k'$.
Then the length of α (which is k) is less than or equal to the length of α' (which is k').
But $\alpha|\beta = \alpha'|\beta'$, so this means that the first k entries in α' are equal to the first k entries in α.
But the entries in α are the elements of E and the entries of α' are the elements of G, so this means that E is a subset of G, which cannot be the case. This is a contradiction.
Therefore, each of the permutations generated by each of the subsets of C_n contained in F is unique.
Therefore,

$$a_0 0!(n-0)! + a_1 1!(n-1)! + \cdots + a_{n-1}(n-1)!(n-(n-1))1! + a_n n!(n-n)! \leq n!$$

Dividing both sides by $n!$, we obtain successively

$$\begin{aligned}
a_0 0!(n-0)! + a_1 1!(n-1)! + \cdots + a_{n-1}(n-1)!(n-(n-1))1! + a_n n!(n-n)! &\leq n! \\
a_0\frac{0!(n-0)!}{n!} + a_1\frac{1!(n-1)!}{n!} + \cdots + a_{n-1}\frac{(n-1)!(n-(n-1))1!}{n!} + a_n\frac{n!(n-n)!}{n!} &\leq 1 \\
a_0\frac{1}{\binom{n}{0}} + a_1\frac{1}{\binom{n}{1}} + \cdots + a_{n-1}\frac{1}{\binom{n}{n-1}} + a_n\frac{1}{\binom{n}{n}} &\leq 1 \\
\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} &\leq 1
\end{aligned}$$

as required.

Solution 2
Suppose that n is a positive integer and that F is a randomly chosen Furoni family of C_n.
Consider $L = \{\{\}, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,\ldots,n\}\}$.
The probability that the intersection of L and F is non-empty is at most 1.
Note that since each element of L is a subset of all of those to its right in the listing of L, then at most one of the elements of L can be in F.
If k is an integer with $k \geq 0$, the probability that $\{1,2,3,\ldots,k\}$ is an element of F is $\dfrac{a_k}{\binom{n}{k}}$, where a_k is the number of elements in F that contain exactly k integers:

There are $\binom{n}{k}$ subsets of C_n that contain exactly k integer.
The probability that any particular one of these subsets is $\{1,2,3,\ldots,k\}$ equals $\dfrac{1}{\binom{n}{k}}$.
Since a_k of these subsets are in F, then the probability that one of these a_k subsets is $\{1,2,3,\ldots,k\}$ equals $\dfrac{a_k}{\binom{n}{k}}$.
(Note that we use the convention that if $k = 0$, then $\{1,2,3,\ldots,k\} = \{\}$.)

The probability that any of the elements of L is in F is the sum of the probability of each element being in F, since at most one of the elements in L is in F.
Therefore,

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

as required.

(c) Set $M = \binom{n}{k}$ where $k = \frac{1}{2}n$ if n is even and $k = \frac{1}{2}(n-1)$ if n is odd.
Then $\binom{n}{r} \leq M$ for every integer r with $0 \leq r \leq n$. (Recall that the largest entries in Pascal's Triangle are the one or two entries in the middle of each row. We prove this algebraically at the end.)
From (b),

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

Multiplying through by M, we obtain

$$a_0\frac{M}{\binom{n}{0}} + a_1\frac{M}{\binom{n}{1}} + \cdots + a_{n-1}\frac{M}{\binom{n}{n-1}} + a_n\frac{M}{\binom{n}{n}} \leq M$$

Since M is at least as large as each binomial coefficient, then each of the fractions on the left side is larger than 1 and so

$$a_0 + a_1 + \cdots + a_{n-1} + a_n \leq a_0\frac{M}{\binom{n}{0}} + a_1\frac{M}{\binom{n}{1}} + \cdots + a_{n-1}\frac{M}{\binom{n}{n-1}} + a_n\frac{M}{\binom{n}{n}} \leq M$$

Therefore, the total number of elements in the Furoni family F, which is $a_0+a_1+\cdots+a_n$, is at most M.

Is it possible to find a Furoni family of size M?

Yes – the $M=\binom{n}{k}$ subsets of C_n of size k form a Furoni family, since no two sets of the same size can be subsets of each other without being equal. Therefore, the largest Furoni family of C_n has size $\binom{n}{k}$ when $n=2k$ or $n=2k+1$ for some non-negative integer k.

We now prove the algebraic result above.

First, we note that $\binom{n}{r}=\dfrac{n!}{r!(n-r)!}=\dfrac{n!}{(n-r)!(n-(n-r))!}=\binom{n}{n-r}$.

Therefore, if $\binom{n}{r}\leq\binom{n}{k}$ for all $r\leq k$, then $\binom{n}{r}\leq\binom{n}{k}$ for all k, since if $s>k$, then $s=n-r$ for some $r\leq k$ and so $\binom{n}{s}=\binom{n}{r}\leq\binom{n}{k}$.

Suppose first that $n=2k$ for some positive integer k.

We prove that $\binom{n}{r}\leq\binom{n}{k}$ for each integer r with $0\leq r\leq k$:

Since $n=2k$, then

$$\frac{\binom{n}{r}}{\binom{n}{k}}=\frac{\dfrac{(2k)!}{r!(2k-r)!}}{\dfrac{(2k)!}{k!k!}}=\frac{k!}{r!}\frac{k!}{(2k-r)!}$$

If $r=k-d$ for some non-negative integer d, then

$$\frac{k!}{r!}\frac{k!}{(2k-r)!}=\frac{k!k!}{(k-d)!(k+d)!}=\frac{k(k-1)\cdots(k-d+1)}{(k+1)(k+2)\cdots(k+d)}=\frac{k}{k+1}\frac{k-1}{k+2}\cdots\frac{k-d+1}{k+d}$$

Since the right side is the product of d non-negative fractions, each of which is smaller than 1, then their product is smaller than 1.

Thus, $\binom{n}{r}\leq\binom{n}{k}$ if $0\leq r\leq k$.

Suppose next that $n=2k+1$ for some non-negative integer k.

We prove that $\binom{n}{r}\leq\binom{n}{k}$ for each integer r with $0\leq r\leq k$:

Since $n=2k+1$, then

$$\frac{\binom{n}{r}}{\binom{n}{k}}=\frac{\dfrac{(2k+1)!}{r!(2k+1-r)!}}{\dfrac{(2k+1)!}{k!(k+1)!}}=\frac{k!}{r!}\frac{(k+1)!}{(2k+1-r)!}$$

If $r=k-d$ for some non-negative integer d, then

$$\begin{aligned}\frac{k!}{r!}\frac{(k+1)!}{(2k+1-r)!}&=\frac{k!(k+1)!}{(k-d)!(k+1+d)!}\\&=\frac{k(k-1)\cdots(k-d+1)}{(k+2)(k+3)\cdots(k+1+d)}\\&=\frac{k}{k+2}\frac{k-1}{k+3}\cdots\frac{k-d+1}{k+1+d}\end{aligned}$$

Since the right side is the product of d non-negative fractions, each of which is smaller than 1, then their product is smaller than 1.

Thus, $\binom{n}{r} \leq \binom{n}{k}$ if $0 \leq r \leq k$.

This completes our proof.

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Euclid Contest

Tuesday, April 15, 2014
(in North America and South America)

Wednesday, April 16, 2014
(outside of North America and South America)

Solutions

1. (a) Evaluating, $\dfrac{\sqrt{16}+\sqrt{9}}{\sqrt{16+9}} = \dfrac{4+3}{\sqrt{25}} = \dfrac{7}{5}$.

   (b) Since the sum of the angles in a triangle is 180°, then $(x-10)^\circ + (x+10)^\circ + x^\circ = 180^\circ$ or $(x-10)+(x+10)+x = 180$.
   Thus, $3x = 180$ and so $x = 60$.

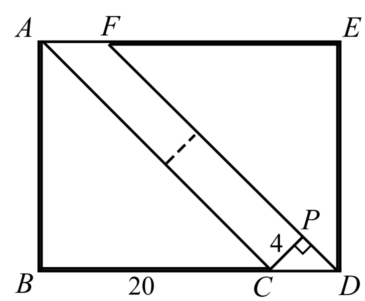
   (c) Suppose Bart earns $\$x$ per hour. In 4 hours, he earns $4 \times \$x = \$4x$.
   Then Lisa earns $\$2x$ per hour. In 6 hours, she earns $6 \times \$2x = \$12x$.
   Since they earn \$200 in total, then $4x + 12x = 200$ or $16x = 200$.
   Therefore, $x = 12.5$.
   Finally, since $2x = 25$, then Lisa earns \$25 per hour.

2. (a) The perimeter of the region includes the diameter and the semi-circle.
   Since the radius of the region is 10, then the length of its diameter is 20.
   Since the radius of the region is 10, then the circumference of an entire circle with this radius is $2\pi(10) = 20\pi$, so the arc length of the semi-circle is one-half of 20π, or 10π.
   Therefore, the perimeter of the region is $10\pi + 20$.

   (b) The x-intercepts of the parabola with equation $y = 10(x+2)(x-5)$ are -2 and 5.
   Since the line segment, PQ, joining these points is horizontal, then its length is the difference in the intercepts, or $5-(-2) = 7$.

   (c) The slope of the line joining the points $C(0,60)$ and $D(30,0)$ is $\dfrac{60-0}{0-30} = \dfrac{60}{-30} = -2$.
   Since this line passes through $C(0,60)$, then the y-intercept of the line is 60, and so an equation of the line is $y = -2x + 60$.
   We thus want to find the point of intersection, E, between the lines with equations $y = -2x + 60$ and $y = 2x$.
   Equating y-coordinates, we obtain $-2x + 60 = 2x$ or $4x = 60$, and so $x = 15$.
   Substituting $x = 15$ into the equation $y = 2x$, we obtain $y = 2(15) = 30$.
   Therefore, the coordinates of E are $(15,30)$.

3. (a) We note that $BD = BC + CD$ and that $BC = 20$ cm, so we need to determine CD.



   We draw a line from C to P on FD so that CP is perpendicular to DF.
   Since AC and DF are parallel, then CP is also perpendicular to AC.
   The distance between AC and DF is 4 cm, so $CP = 4$ cm.
   Since $\triangle ABC$ is isosceles and right-angled, then $\angle ACB = 45^\circ$.
   Thus, $\angle PCD = 180^\circ - \angle ACB - \angle PCA = 180^\circ - 45^\circ - 90^\circ = 45^\circ$.
   Since $\triangle CPD$ is right-angled at P and $\angle PCD = 45^\circ$, then $\triangle CPD$ is also an isosceles right-angled triangle.
   Therefore, $CD = \sqrt{2}CP = 4\sqrt{2}$ cm.
   Finally, $BD = BC + CD = (20 + 4\sqrt{2})$ cm.

(b) Manipulating the given equation and noting that $x \neq 0$ and $x \neq -\frac{1}{2}$ since neither denominator can equal 0, we obtain

$$\begin{aligned}
\frac{x^2+x+4}{2x+1} &= \frac{4}{x} \\
x(x^2+x+4) &= 4(2x+1) \\
x^3+x^2+4x &= 8x+4 \\
x^3+x^2-4x-4 &= 0 \\
x^2(x+1)-4(x+1) &= 0 \\
(x+1)(x^2-4) &= 0 \\
(x+1)(x-2)(x+2) &= 0
\end{aligned}$$

Therefore, $x=-1$ or $x=2$ or $x=-2$. We can check by substitution that each satisfies the original equation.

4. (a) *Solution 1*
Since $900 = 30^2$ and $30 = 2 \times 3 \times 5$, then $900 = 2^2 3^2 5^2$.
The positive divisors of 900 are those integers of the form $d = 2^a 3^b 5^c$, where each of a, b, c is 0, 1 or 2.
For d to be a perfect square, the exponent on each prime factor in the prime factorization of d must be even.
Thus, for d to be a perfect square, each of a, b, c must be 0 or 2.
There are two possibilities for each of a, b, c so $2 \times 2 \times 2 = 8$ possibilities for d.
These are $2^0 3^0 5^0 = 1$, $2^2 3^0 5^0 = 4$, $2^0 3^2 5^0 = 9$, $2^0 3^0 5^2 = 25$, $2^2 3^2 5^0 = 36$, $2^2 3^0 5^2 = 100$, $2^0 3^2 5^2 = 225$, and $2^2 3^2 5^2 = 900$.
Thus, 8 of the positive divisors of 900 are perfect squares.

Solution 2
The positive divisors of 900 are

$$1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 25, 30, 36, 45, 50, 60, 75, 90, 100, 150, 180, 225, 300, 450, 900$$

Of these, 1, 4, 9, 25, 36, 100, 225, and 900 are perfect squares ($1^2, 2^2, 3^2, 5^2, 6^2, 10^2, 15^2, 30^2$, respectively).
Thus, 8 of the positive divisors of 900 are perfect squares.

(b) In isosceles triangle ABC, $\angle ABC = \angle ACB$, so the sides opposite these angles (AC and AB, respectively) are equal in length.
Since the vertices of the triangle are $A(k,3)$, $B(3,1)$ and $C(6,k)$, then we obtain

$$\begin{aligned}
AC &= AB \\
\sqrt{(k-6)^2+(3-k)^2} &= \sqrt{(k-3)^2+(3-1)^2} \\
(k-6)^2+(3-k)^2 &= (k-3)^2+(3-1)^2 \\
(k-6)^2+(k-3)^2 &= (k-3)^2+2^2 \\
(k-6)^2 &= 4
\end{aligned}$$

Thus, $k-6=2$ or $k-6=-2$, and so $k=8$ or $k=4$.
We can check by substitution that each satisfies the original equation.

5. (a) Bottle A contains 40 g of which 10% is acid.
Thus, it contains $0.1 \times 40 = 4$ g of acid and $40 - 4 = 36$ g of water.
Bottle B contains 50 g of which 20% is acid.
Thus, it contains $0.2 \times 50 = 10$ g of acid and $50 - 10 = 40$ g of water.
Bottle C contains 50 g of which 30% is acid.
Thus, it contains $0.3 \times 50 = 15$ g of acid and $50 - 15 = 35$ g of water.
In total, the three bottles contain $40 + 50 + 50 = 140$ g, of which $4 + 10 + 15 = 29$ g is acid and $140 - 29 = 111$ g is water.
The new mixture has mass 60 g of which 25% is acid.
Thus, it contains $0.25 \times 60 = 15$ g of acid and $60 - 15 = 45$ g of water.
Since the total mass in the three bottles is initially 140 g and the new mixture has mass 60 g, then the remaining contents have mass $140 - 60 = 80$ g.
Since the total mass of acid in the three bottles is initially 29 g and the acid in the new mixture has mass 15 g, then the acid in the remaining contents has mass $29 - 15 = 14$ g.
This remaining mixture is thus $\dfrac{14\text{ g}}{80\text{ g}} \times 100\% = 17.5\%$ acid.

(b) Since $3x + 4y = 10$, then $4y = 10 - 3x$.
Therefore, when $3x + 4y = 10$,

$$\begin{aligned} x^2 + 16y^2 &= x^2 + (4y)^2 \\ &= x^2 + (10 - 3x)^2 \\ &= x^2 + (9x^2 - 60x + 100) \\ &= 10x^2 - 60x + 100 \\ &= 10(x^2 - 6x + 10) \\ &= 10(x^2 - 6x + 9 + 1) \\ &= 10((x - 3)^2 + 1) \\ &= 10(x - 3)^2 + 10 \end{aligned}$$

Since $(x - 3)^2 \geq 0$, then the minimum possible value of $10(x - 3)^2 + 10$ is $10(0) + 10 = 10$.
This occurs when $(x - 3)^2 = 0$ or $x = 3$.
Therefore, the minimum possible value of $x^2 + 16y^2$ when $3x + 4y = 10$ is 10.

6. (a) *Solution 1*
Suppose that the bag contains g gold balls.
We assume that Feridun reaches into the bag and removes the two balls one after the other.
There are 40 possible balls that he could remove first and then 39 balls that he could remove second. In total, there are 40(39) pairs of balls that he could choose in this way.
If he removes 2 gold balls, then there are g possible balls that he could remove first and then $g - 1$ balls that he could remove second. In total, there are $g(g - 1)$ pairs of gold balls that he could remove.
We are told that the probability of removing 2 gold balls is $\dfrac{5}{12}$.
Since there are 40(39) total pairs of balls that can be chosen and $g(g - 1)$ pairs of gold balls that can be chosen in this way, then $\dfrac{g(g-1)}{40(39)} = \dfrac{5}{12}$ which is equivalent to $g(g - 1) = \dfrac{5}{12}(40)(39) = 650$.

Therefore, $g^2 - g - 650 = 0$ or $(g-26)(g+25) = 0$, and so $g = 26$ or $g = -25$.
Since $g > 0$, then $g = 26$, so there are 26 gold balls in the bag.

Solution 2
Suppose that the bag contains g gold balls.
We assume that Feridun reaches into the bag and removes the two balls together.
Since there are 40 balls in the bag, there are $\binom{40}{2}$ pairs of balls that he could choose in this way.
Since there are g gold balls in the bag, then there are $\binom{g}{2}$ pairs of gold balls that he could choose in this way.
We are told that the probability of removing 2 gold balls is $\frac{5}{12}$.
Since there are $\binom{40}{2}$ pairs in total that can be chosen and $\binom{g}{2}$ pairs of gold balls that can be chosen in this way, then $\dfrac{\binom{g}{2}}{\binom{40}{2}} = \dfrac{5}{12}$ which is equivalent to $\binom{g}{2} = \frac{5}{12}\binom{40}{2}$.
Since $\binom{n}{2} = \dfrac{n(n-1)}{2}$, then this equation is equivalent to $\dfrac{g(g-1)}{2} = \dfrac{5}{12}\,\dfrac{40(39)}{2} = 325$.
Therefore, $g(g-1) = 650$ or $g^2 - g - 650 = 0$ or $(g-26)(g+25) = 0$, and so $g = 26$ or $g = -25$.
Since $g > 0$, then $g = 26$, so there are 26 gold balls in the bag.

(b) Suppose that the first term in the geometric sequence is $t_1 = a$ and the common ratio in the sequence is r.
Then the sequence, which has n terms, is $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$.
In general, the kth term is $t_k = ar^{k-1}$; in particular, the nth term is $t_n = ar^{n-1}$.
Since $t_1 t_n = 3$, then $a \cdot ar^{n-1} = 3$ or $a^2 r^{n-1} = 3$.
Since $t_1 t_2 \cdots t_{n-1} t_n = 59\,049$, then

$$\begin{aligned}
(a)(ar)\cdots(ar^{n-2})(ar^{n-1}) &= 59\,049 \\
a^n r r^2 \cdots r^{n-2} r^{n-1} &= 59\,049 \qquad \text{(since there are } n \text{ factors of } a \text{ on the left side)} \\
a^n r^{1+2+\cdots+(n-2)+(n-1)} &= 59\,049 \\
a^n r^{\frac{1}{2}(n-1)(n)} &= 59\,049
\end{aligned}$$

since $1 + 2 + \cdots + (n-2) + (n-1) = \frac{1}{2}(n-1)(n)$.
Since $a^2 r^{n-1} = 3$, then $(a^2 r^{n-1})^n = 3^n$ or $a^{2n} r^{(n-1)(n)} = 3^n$.
Since $a^n r^{\frac{1}{2}(n-1)(n)} = 59\,049$, then $\left(a^n r^{\frac{1}{2}(n-1)(n)}\right)^2 = 59\,049^2$ or $a^{2n} r^{(n-1)(n)} = 59\,049^2$.
Since the left sides of these equations are the same, then $3^n = 59\,049^2$.
Now

$$59\,049 = 3(19\,683) = 3^2(6561) = 3^3(2187) = 3^4(729) = 3^5(243) = 3^6(81) = 3^6 3^4 = 3^{10}$$

Since $59\,049 = 3^{10}$, then $59\,049^2 = 3^{20}$ and so $3^n = 3^{20}$, which gives $n = 20$.

7. (a) Let $a = x - 2013$ and let $b = y - 2014$.
The given equation becomes $\dfrac{ab}{a^2+b^2} = -\dfrac{1}{2}$, which is equivalent to $2ab = -a^2 - b^2$ and $a^2 + 2ab + b^2 = 0$.
This is equivalent to $(a+b)^2 = 0$ which is equivalent to $a + b = 0$.
Since $a = x - 2013$ and $b = y - 2014$, then $x - 2013 + y - 2014 = 0$ or $x + y = 4027$.

(b) Let $a = \log_{10} x$.
Then $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10\,000$ becomes $a^{\log_{10} a} = 10^4$.
Taking the base 10 logarithm of both sides and using the fact that $\log_{10}(a^b) = b \log_{10} a$, we obtain $(\log_{10} a)(\log_{10} a) = 4$ or $(\log_{10} a)^2 = 4$.
Therefore, $\log_{10} a = \pm 2$ and so $\log_{10}(\log_{10} x) = \pm 2$.

If $\log_{10}(\log_{10} x) = 2$, then $\log_{10} x = 10^2 = 100$ and so $x = 10^{100}$.
If $\log_{10}(\log_{10} x) = -2$, then $\log_{10} x = 10^{-2} = \frac{1}{100}$ and so $x = 10^{1/100}$.
Therefore, $x = 10^{100}$ or $x = 10^{1/100}$.

We check these answers in the original equation.
If $x = 10^{100}$, then $\log_{10} x = 100$.
Thus, $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 100^{\log_{10} 100} = 100^2 = 10\,000$.
If $x = 10^{1/100}$, then $\log_{10} x = 1/100 = 10^{-2}$.
Thus, $(\log_{10} x)^{\log_{10}(\log_{10} x)} = (10^{-2})^{\log_{10}(10^{-2})} = (10^{-2})^{-2} = 10^4 = 10\,000$.

8. (a) We use the cosine law in $\triangle ABD$ to determine the length of BD:

$$BD^2 = AB^2 + AD^2 - 2(AB)(AD)\cos(\angle BAD)$$

We are given that $AB = 75$ and $AD = 20$, so we need to determine $\cos(\angle BAD)$.
Now

$$\begin{aligned}\cos(\angle BAD) &= \cos(\angle BAC + \angle EAD) \\ &= \cos(\angle BAC)\cos(\angle EAD) - \sin(\angle BAC)\sin(\angle EAD) \\ &= \frac{AC}{AB}\frac{AD}{AE} - \frac{BC}{AB}\frac{ED}{AE}\end{aligned}$$

since $\triangle ABC$ and $\triangle ADE$ are right-angled.
Since $AB = 75$ and $BC = 21$, then by the Pythagorean Theorem,

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{75^2 - 21^2} = \sqrt{5625 - 441} = \sqrt{5184} = 72$$

since $AC > 0$.
Since $AC = 72$ and $CE = 47$, then $AE = AC - CE = 25$.
Since $AE = 25$ and $AD = 20$, then by the Pythagorean Theorem,

$$ED = \sqrt{AE^2 - AD^2} = \sqrt{25^2 - 20^2} = \sqrt{625 - 400} = \sqrt{225} = 15$$

since $ED > 0$.
Therefore,

$$\cos(\angle BAD) = \frac{AC}{AB}\frac{AD}{AE} - \frac{BC}{AB}\frac{ED}{AE} = \frac{72}{75}\frac{20}{25} - \frac{21}{75}\frac{15}{25} = \frac{1440 - 315}{75(25)} = \frac{1125}{75(25)} = \frac{45}{75} = \frac{3}{5}$$

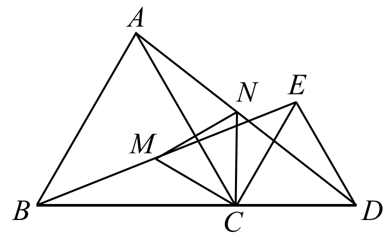
Finally,

$$\begin{aligned} BD^2 &= AB^2 + AD^2 - 2(AB)(AD)\cos(\angle BAD) \\ &= 75^2 + 20^2 - 2(75)(20)(\tfrac{3}{5}) \\ &= 5625 + 400 - 1800 \\ &= 4225 \end{aligned}$$

Since $BD > 0$, then $BD = \sqrt{4225} = 65$, as required.

(b) *Solution 1*

Consider $\triangle BCE$ and $\triangle ACD$.



Since $\triangle ABC$ is equilateral, then $BC = AC$.
Since $\triangle ECD$ is equilateral, then $CE = CD$.
Since BCD is a straight line and $\angle ECD = 60^\circ$, then $\angle BCE = 180^\circ - \angle ECD = 120^\circ$.
Since BCD is a straight line and $\angle BCA = 60^\circ$, then $\angle ACD = 180^\circ - \angle BCA = 120^\circ$.
Therefore, $\triangle BCE$ is congruent to $\triangle ACD$ ("side-angle-side").
Since $\triangle BCE$ and $\triangle ACD$ are congruent and CM and CN are line segments drawn from the corresponding vertex (C in both triangles) to the midpoint of the opposite side, then $CM = CN$.
Since $\angle ECD = 60^\circ$, then $\triangle ACD$ can be obtained by rotating $\triangle BCE$ through an angle of 60° clockwise about C.
This means that after this 60° rotation, CM coincides with CN.
In other words, $\angle MCN = 60^\circ$.
But since $CM = CN$ and $\angle MCN = 60^\circ$, then

$$\angle CMN = \angle CNM = \tfrac{1}{2}(180^\circ - \angle MCN) = 60^\circ$$

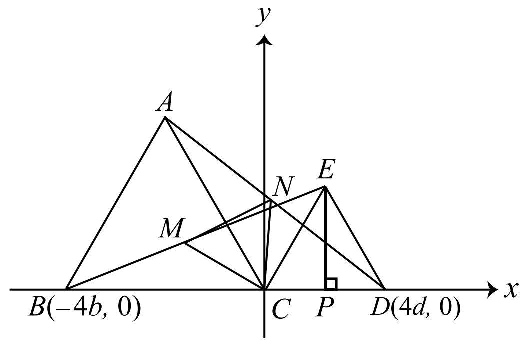
Therefore, $\triangle MNC$ is equilateral, as required.

Solution 2

We prove that $\triangle MNC$ is equilateral by introducing a coordinate system.

Suppose that C is at the origin $(0,0)$ with BCD along the x-axis, with B having coordinates $(-4b,0)$ and D having coordinates $(4d,0)$ for some real numbers $b,d>0$.

Drop a perpendicular from E to P on CD.



Since $\triangle ECD$ is equilateral, then P is the midpoint of CD.

Since C has coordinates $(0,0)$ and D has coordinates $(4d,0)$, then the coordinates of P are $(2d,0)$.

Since $\triangle ECD$ is equilateral, then $\angle ECD = 60^\circ$ and so $\triangle EPC$ is a 30°-60°-90° triangle and so $EP = \sqrt{3}CP = 2\sqrt{3}d$.

Therefore, the coordinates of E are $(2d, 2\sqrt{3}d)$.

In a similar way, we can show that the coordinates of A are $(-2b, 2\sqrt{3}b)$.

Now M is the midpoint of $B(-4b,0)$ and $E(2d,2\sqrt{3}d)$, so the coordinates of M are $\left(\frac{1}{2}(-4b+2d), \frac{1}{2}(0+2\sqrt{3}d)\right)$ or $(-2b+d, \sqrt{3}d)$.

Also, N is the midpoint of $A(-2b,2\sqrt{3}b)$ and $D(4d,0)$, so the coordinates of N are $\left(\frac{1}{2}(-2b+4d), \frac{1}{2}(2\sqrt{3}b+0)\right)$ or $(-b+2d, \sqrt{3}b)$.

To show that $\triangle MNC$ is equilateral, we show that $CM = CN = MN$ or equivalently that $CM^2 = CN^2 = MN^2$:

$$\begin{aligned}
CM^2 &= (-2b+d-0)^2 + (\sqrt{3}d-0)^2 \\
&= (-2b+d)^2 + (\sqrt{3}d)^2 \\
&= 4b^2 - 4bd + d^2 + 3d^2 \\
&= 4b^2 - 4bd + 4d^2 \\
CN^2 &= (-b+2d-0)^2 + (\sqrt{3}b-0)^2 \\
&= (-b+2d)^2 + (\sqrt{3}b)^2 \\
&= b^2 - 4bd + 4d^2 + 3b^2 \\
&= 4b^2 - 4bd + 4d^2 \\
MN^2 &= ((-2b+d)-(-b+2d))^2 + (\sqrt{3}d-\sqrt{3}b)^2 \\
&= (-b-d)^2 + 3(d-b)^2 \\
&= b^2 + 2bd + d^2 + 3d^2 - 6bd + 3b^2 \\
&= 4b^2 - 4bd + 4d^2
\end{aligned}$$

Therefore, $CM^2 = CN^2 = MN^2$ and so $\triangle MNC$ is equilateral, as required.

9. (a) Let $S = \sin^6 1^\circ + \sin^6 2^\circ + \sin^6 3^\circ + \cdots + \sin^6 87^\circ + \sin^6 88^\circ + \sin^6 89^\circ$.
Since $\sin\theta = \cos(90^\circ - \theta)$, then $\sin^6\theta = \cos^6(90^\circ - \theta)$, and so

$$\begin{aligned}
S &= \sin^6 1^\circ + \sin^6 2^\circ + \cdots + \sin^6 44^\circ + \sin^6 45^\circ \\
&\qquad + \cos^6(90^\circ - 46^\circ) + \cos^6(90^\circ - 47^\circ) + \cdots + \cos^6(90^\circ - 89^\circ) \\
&= \sin^6 1^\circ + \sin^6 2^\circ + \cdots + \sin^6 44^\circ + \sin^6 45^\circ + \cos^6 44^\circ + \cos^6 43^\circ + \cdots + \cos^6 1^\circ \\
&= (\sin^6 1^\circ + \cos^6 1^\circ) + (\sin^6 2^\circ + \cos^6 2^\circ) + \cdots + (\sin^6 44^\circ + \cos^6 44^\circ) + \sin^6 45^\circ
\end{aligned}$$

Since $\sin 45^\circ = \frac{1}{\sqrt{2}}$, then $\sin^6 45^\circ = \frac{1}{2^3} = \frac{1}{8}$.
Also, since

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) = (x+y)((x+y)^2 - 3xy)$$

then substituting $x = \sin^2\theta$ and $y = \cos^2\theta$, we obtain

$$\begin{aligned}
x^3 + y^3 &= (x+y)((x+y)^2 - 3xy) \\
\sin^6\theta + \cos^6\theta &= (\sin^2\theta + \cos^2\theta)((\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta) \\
\sin^6\theta + \cos^6\theta &= 1(1 - 3\sin^2\theta\cos^2\theta)
\end{aligned}$$

since $\sin^2\theta + \cos^2\theta = 1$.
Therefore,

$$\begin{aligned}
S &= (\sin^6 1^\circ + \cos^6 1^\circ) + (\sin^6 2^\circ + \cos^6 2^\circ) + \cdots + (\sin^6 44^\circ + \cos^6 44^\circ) + \sin^6 45^\circ \\
&= (1 - 3\sin^2 1^\circ \cos^2 1^\circ) + (1 - 3\sin^2 2^\circ \cos^2 2^\circ) + \cdots + (1 - 3\sin^2 44^\circ \cos^2 44^\circ) + \tfrac{1}{8} \\
&= 44 - (3\sin^2 1^\circ \cos^2 1^\circ + 3\sin^2 2^\circ \cos^2 2^\circ + \cdots + 3\sin^2 44^\circ \cos^2 44^\circ) + \tfrac{1}{8} \\
&= \tfrac{353}{8} - \tfrac{3}{4}(4\sin^2 1^\circ \cos^2 1^\circ + 4\sin^2 2^\circ \cos^2 2^\circ + \cdots + 4\sin^2 44^\circ \cos^2 44^\circ)
\end{aligned}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, then $4\sin^2\theta\cos^2\theta = \sin^2 2\theta$, which gives

$$\begin{aligned}
S &= \tfrac{353}{8} - \tfrac{3}{4}(4\sin^2 1^\circ \cos^2 1^\circ + 4\sin^2 2^\circ \cos^2 2^\circ + \cdots + 4\sin^2 44^\circ \cos^2 44^\circ) \\
&= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 88^\circ) \\
&= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 44^\circ + \sin^2 46^\circ + \cdots + \sin^2 86^\circ + \sin^2 88^\circ) \\
&= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 44^\circ + \\
&\qquad \cos^2(90^\circ - 46^\circ) + \cdots + \cos^2(90^\circ - 86^\circ) + \cos^2(90^\circ - 88^\circ)) \\
&= \tfrac{353}{8} - \tfrac{3}{4}(\sin^2 2^\circ + \sin^2 4^\circ + \cdots + \sin^2 44^\circ + \cos^2 44^\circ + \cdots + \cos^2 4^\circ + \cos^2 2^\circ) \\
&= \tfrac{353}{8} - \tfrac{3}{4}((\sin^2 2^\circ + \cos^2 2^\circ) + (\sin^2 4^\circ + \cos^2 4^\circ) + \cdots + (\sin^2 44^\circ + \cos^2 44^\circ)) \\
&= \tfrac{353}{8} - \tfrac{3}{4}(22) \qquad (\text{since } \sin^2\theta + \cos^2\theta = 1) \\
&= \tfrac{353}{8} - \tfrac{132}{8} \\
&= \tfrac{221}{8}
\end{aligned}$$

Therefore, since $S = \dfrac{m}{n}$, then $m = 221$ and $n = 8$ satisfy the required equation.

(b) First, we prove that $f(n) = \dfrac{n(n+1)(n+2)(n+3)}{24}$ in two different ways.

Method 1

If an n-digit integer has digits with a sum of 5, then there are several possibilities for the combination of non-zero digits used:

$$5 \quad 4,1 \quad 3,2 \quad 3,1,1 \quad 2,2,1 \quad 2,1,1,1 \quad 1,1,1,1,1$$

We count the number of possible integers in each case by determining the number of arrangements of the non-zero digits; we call the number of ways of doing this a. (For example, the digits 4 and 1 can be arranged as 4 1 or 1 4.) We then place the leftmost digit in such an arrangement as the leftmost digit of the n-digit integer (which must be non-zero) and choose the positions for the remaining non-zero digits among the remaining $n-1$ positions; we call the number of ways of doing this b. (For example, for the arrangement 1 4, the digit 1 is in the leftmost position and the digit 4 can be in any of the remaining $n-1$ positions.) We fill the rest of the positions with 0s. The number of possible integers in each case will be ab, since this method will create all such integers and for each of the a arrangements of the non-zero digits, there will be b ways of arranging the digits after the first one. We make a chart to summarize the cases, expanding each total and writing it as a fraction with denominator 24:

Case	a	b	ab (expanded)
5	1	1	$1 = \dfrac{24}{24}$
4, 1	2	$(n-1)$	$2(n-1) = \dfrac{48n-48}{24}$
3, 2	2	$(n-1)$	$2(n-1) = \dfrac{48n-48}{24}$
3, 1, 1	3	$\binom{n-1}{2}$	$3\binom{n-1}{2} = \dfrac{36n^2-108n+72}{24}$
2, 2, 1	3	$\binom{n-1}{2}$	$3\binom{n-1}{2} = \dfrac{36n^2-108n+72}{24}$
2, 1, 1, 1	4	$\binom{n-1}{3}$	$4\binom{n-1}{3} = \dfrac{16n^3-96n^2+176n-96}{24}$
1, 1, 1, 1, 1	1	$\binom{n-1}{4}$	$\binom{n-1}{4} = \dfrac{n^4-10n^3+35n^2-50n+24}{24}$

(Note that in the second and third cases we need $n \geq 2$, in the fourth and fifth cases we need $n \geq 3$, in the sixth case we need $n \geq 4$, and the seventh case we need $n \geq 5$. In each case, though, the given formula works for smaller positive values of n since it is equal to 0 in each case. Note also that we say $b = 1$ in the first case since there is exactly 1 way of placing 0s in all of the remaining $n-1$ positions.)

$f(n)$ is then the sum of the expressions in the last column of this table, and so

$$f(n) = \frac{n^4+6n^3+11n^2+6n}{24} = \frac{n(n+1)(n+2)(n+3)}{24}$$

as required.

Method 2

First, we create a correspondence between each integer with n digits and whose digits have

a sum of 5 and an arrangement of five 1s and $(n-1)$ Xs that begins with a 1.
We can then count these integers by counting the arrangements.
Starting with such an integer, we write down an arrangement of the above type using the following rule:

> The number of 1s to the left of the first X is the first digit of the number, the number of 1s between the first X and second X is the second digit of the number, and so on, with the number of 1s to the right of the $(n-1)$st X representing the nth digit of the number.

For example, the integer 1010020001 would correspond to 1XX1XXX11XXXX1.
In this way, each such integer gives an arrangement of the above type.
Similarly, each arrangement of this type can be associated back to a unique integer with the required properties by counting the number of 1s before the first X and writing this down as the leftmost digit, counting the number of 1s between the first and second Xs and writing this down as the second digit, and so on. Since a total of five 1s are used, then each arrangement corresponds with an integer with n digits whose digits have a sum of 5.
Therefore, there is a one-to-one correspondence between the integers and arrangements with the desired properties.
Thus, $f(n)$, which equals the number of such integers, also equals the number of such arrangements.
To count the number of such arrangements, we note that there are four 1s and $n-1$ Xs to arrange in the final $4+(n-1)=n+3$ positions, since the first position is occupied by a 1.
There are $\binom{n+3}{4}$ ways to choose the positions of the remaining four 1s, and so $\binom{n+3}{4}$ arrangements.
Thus, $f(n)=\binom{n+3}{4}=\dfrac{(n+3)!}{4!(n-1)!}=\dfrac{(n+3)(n+2)(n+1)(n)}{4!}=\dfrac{n(n+1)(n+2)(n+3)}{24}$.

Next, we need to determine the positive integers n between 1 and 2014, inclusive, for which the units digit of $f(n)$ is 1.
Now $f(n)=\dfrac{n(n+1)(n+2)(n+3)}{24}$ is an integer for all positive integers n, since it is counting the number of things with a certain property.
If the units digit of n is 0 or 5, then n is a multiple of 5.
If the units digit of n is 2 or 7, then $n+3$ is a multiple of 5.
If the units digit of n is 3 or 8, then $n+2$ is a multiple of 5.
If the units digit of n is 4 or 9, then $n+1$ is a multiple of 5.
Thus, if the units digit of n is 0, 2, 3, 4, 5, 7, 8, or 9, then $n(n+1)(n+2)(n+3)$ is a multiple of 5 and so $f(n)=\dfrac{n(n+1)(n+2)(n+3)}{24}$ is a multiple of 5, since the denominator contains no factors of 5 that can divide the factor from the numerator.
Therefore, if the units digit of n is 0, 2, 3, 4, 5, 7, 8, or 9, then $f(n)$ is divisible by 5, and so cannot have a units digit of 1.

So we consider the cases where n has a units digit of 1 or of 6; these are the only possible values of n for which $f(n)$ can have a units digit of 1.
We note that $3f(n)=\dfrac{n(n+1)(n+2)(n+3)}{8}$, which is a positive integer for all positive integers n.

Also, we note that if $f(n)$ has units digit 1, then $3f(n)$ has units digit 3, and if $3f(n)$ has units digit 3, then $f(n)$ must have units digit 1.
Therefore, determining the values of n for which $f(n)$ has units digit 1 is equivalent to determining the values of n for which $\dfrac{n(n+1)(n+2)(n+3)}{8}$ has units digit 3.

We consider the integers n in groups of 40. (Intuitively, we do this because the problem seems to involve multiples of 5 and multiples of 8, and $5 \times 8 = 40$.)
If n has units digit 1, then $n = 40k+1$ or $n = 40k+11$ or $n = 40k+21$ or $n = 40k+31$ for some integer $k \geq 0$.
If n has units digit 6, then $n = 40k+6$ or $n = 40k+16$ or $n = 40k+26$ or $n = 40k+36$ for some integer $k \geq 0$.

If $n = 40k+1$, then

$$\begin{aligned} 3f(n) &= \frac{n(n+1)(n+2)(n+3)}{8} \\ &= \frac{(40k+1)(40k+2)(40k+3)(40k+4)}{8} \\ &= (40k+1)(20k+1)(40k+3)(10k+1) \end{aligned}$$

The units digit of $40k+1$ is 1, the units digit of $20k+1$ is 1, the units digit of $40k+3$ is 3, and the units digit of $10k+1$ is 1, so the units digit of the product is the units digit of $(1)(1)(3)(1)$ or 3.
In a similar way, we treat the remaining seven cases and summarize all eight cases in a chart:

n	$3f(n)$ simplified	Units digit of $3f(n)$
$40k+1$	$(40k+1)(20k+1)(40k+3)(10k+1)$	3
$40k+11$	$(40k+11)(10k+3)(40k+13)(20k+7)$	3
$40k+21$	$(40k+21)(20k+11)(40k+23)(10k+6)$	8
$40k+31$	$(40k+31)(10k+8)(40k+33)(20k+17)$	8
$40k+6$	$(20k+3)(40k+7)(10k+2)(40k+9)$	8
$40k+16$	$(10k+4)(40k+17)(20k+9)(40k+19)$	8
$40k+26$	$(20k+13)(40k+27)(10k+7)(40k+29)$	3
$40k+36$	$(10k+9)(40k+37)(20k+19)(40k+39)$	3

(Note that, for example, when $n = 40k+16$, the simplified version of $3f(n)$ is $(10k+4)(40k+17)(20k+9)(40k+19)$, so the units digit of $3f(n)$ is the units digit of $(4)(7)(9)(9)$ which is the units digit of 2268, or 8.)

Therefore, $f(n)$ has units digit 1 whenever $n = 40k+1$ or $n = 40k+11$ or $n = 40k+26$ or $n = 40k+36$ for some integer $k \geq 0$.
There are 4 such integers n between each pair of consecutive multiples of 40.
Since $2000 = 50 \times 40$, then 2000 is the 50th multiple of 40, so there are $50 \times 4 = 200$ integers n less than 2000 for which the units digit of $f(n)$ is 1.
Between 2000 and 2014, inclusive, there are two additional integers: $n = 40(50)+1 = 2001$ and $n = 40(50)+11 = 2011$.
In total, 202 of the integers $f(1), f(2), \ldots, f(2014)$ have a units digit of 1.

10. Throughout this solution, we use "JB" to represent "jelly bean" or "jelly beans".
We use "T1" to represent "Type 1 move", "T2" to represent "Type 2 move", and so on.
We use "P0" to represent "position 0", "P1" to represent "position 1", and so on.
We represent the positions of the JB initially or after a move using an ordered tuple of non-negative integers representing the number of JB at P0, P1, P2, etc. For example, the tuple $(0,0,1,2,1)$ would represent 0 JB at P0, 0 JB at P1, 1 JB at P2, 2 JB at P3, and 1 JB at P4.

(a) To begin, we work backwards from the final state $(0,0,0,0,0,1)$.
The only move that could have put 1 JB at P5 is 1 T5.
Undoing this move removes 1 JB from P5 and adds 1 JB at P4 and 1 JB at P3, giving $(0,0,0,1,1,0)$.
The only move that could have put 1 JB at P4 is 1 T4.
Undoing this move removes 1 JB from P4 and adds 1 JB at P2 and 1 JB at P3, giving $(0,0,1,2,0,0)$.
The only moves that could put 2 JB at P3 are 2 T3s.
Undoing these moves removes 2 JB from P3, adds 2 JB at P1 and 2 JB at P2, giving $(0,2,3,0,0,0)$.
The only moves that could put 3 JB at P2 are 3 T2s.
Undoing these moves gives $(3,5,0,0,0,0)$.
The only moves that could put 5 JB at P1 are 5 T1s.
Undoing these moves removes 5 JB from P1 and adds 10 JB at P0, giving $(13,0,0,0,0,0)$.
Therefore, starting with $N = 13$ JB at P0 allows Fiona to win the game by making all of the moves as above in the reverse order.
In particular, from $(13,0,0,0,0,0)$, 5 T1s gives $(3,5,0,0,0,0)$, then 3 T2s give $(0,2,3,0,0,0)$, then 2 T3s give $(0,0,1,2,0,0)$, then 1 T4 gives $(0,0,0,1,1,0)$, then 1 T5 gives $(0,0,0,0,0,1)$, as required.

(b) Initial Set-up
First, we note that when Fiona starts with N JB (for some fixed positive integer N), then the game finishes in at most $N-1$ moves (since she eats exactly one JB on each move).
Second, we note that the positions of the JB in the final state as well as at any intermediate state (that is, after some number of moves) must be in the list P0, P1, ..., P$(N-1)$, since each JB can move at most 1 position to the right on any given move, so no JB can move more than $N-1$ positions to the right in at most $N-1$ moves.
This means that, starting with N JB, any state can be described using an N-tuple $(a_0, a_1, \ldots, a_{N-2}, a_{N-1})$, where a_i represents the number of JB at Pi in that state.

Introduction of Fibonacci Sequence and Important Fact #1 (IF1)
We define the Fibonacci sequence by $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.
The initial number of JB (N) and the number of JB at various positions are connected using the Fibonacci sequence in the following way.
At any state between the starting state (N JB at P0) and the final state, if there are a_i JB at Pi for each i from 0 to $N-1$, then

$$N = a_0F_2 + a_1F_3 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1} \qquad (*)$$

This is true because:

- It is true for the starting state, since here $(a_0, a_1, \ldots, a_{N-2}, a_{N-1}) = (N, 0, \ldots, 0, 0)$ and $F_2 = 1$, so the right side of $(*)$ equals $N(1) + 0$ or N
- A T1 does not change the value of the right side of $(*)$: Since a T1 changes the state $(a_0, a_1, a_2, \ldots, a_{N-2}, a_{N-1})$ to $(a_0 - 2, a_1 + 1, a_2, \ldots, a_{N-2}, a_{N-1})$, the right side of $(*)$

changes from

$$a_0F_2 + a_1F_3 + a_2F_4 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

to

$$(a_0 - 2)F_2 + (a_1 + 1)F_3 + a_2F_4 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

which is a difference of $-2F_2 + F_3 = -2(1) + 2 = 0$.

- A Ti for $i \geq 2$ does not change the value of the right side of ($*$): Since a Ti changes the state $(a_0, a_1, \ldots, a_{i-2}, a_{i-1}, a_i, \ldots, a_{N-2}, a_{N-1})$ to

$$(a_0, a_1, \ldots, a_{i-2} - 1, a_{i-1} - 1, a_i + 1, \ldots, a_{N-2}, a_{N-1})$$

the right side of ($*$) changes from

$$a_0F_2 + a_1F_3 + \cdots + a_{i-2}F_i + a_{i-1}F_{i+1} + a_iF_{i+2} + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1}$$

to

$$a_0F_2+a_1F_3+\cdots+(a_{i-2}-1)F_i+(a_{i-1}-1)F_{i+1}+(a_i+1)F_{i+2}+\cdots+a_{N-2}F_N+a_{N-1}F_{N+1}$$

which is a difference of $-F_i - F_{i+1} + F_{i+2} = 0$ since $F_{i+2} = F_{i+1} + F_i$.

This tells us that the value of the right side of ($*$) starts at N and does not change on any subsequent move.
Therefore, at any state $(a_0, a_1, \ldots, a_{N-2}, a_{N-1})$ after starting with N JB at P0, it is true that

$$N = a_0F_2 + a_1F_3 + \cdots + a_{N-2}F_N + a_{N-1}F_{N+1} \qquad (*)$$

To show that there is only one possible final state when Fiona wins the game, we assume that there are two possible winning final states starting from N JB and show that these in fact must be the same state.

Important Fact #2 (IF2)
To do this, we prove a property of Fibonacci numbers that will allow us to show that two sums of three or fewer non-consecutive Fibonacci numbers cannot be equal if the Fibonacci numbers used in each sum are not the same:

If x, y, z are positive integers with $2 \leq x < y < z$ and no pair of x, y, z are consecutive integers, then $F_z < F_y + F_z < F_x + F_y + F_z < F_{z+1}$.

Since each Fibonacci number is a positive integer, then $F_z < F_y + F_z < F_x + F_y + F_z$, so we must prove that $F_x + F_y + F_z < F_{z+1}$:

Since no two of x, y, z are consecutive and $x < y < z$, then $y < z - 1$.
Since y and z are positive integers, then $y \leq z - 2$.
Also, $x < y - 1 \leq z - 3$.
Since x and z are integers with $x < z - 3$, then $x \leq z - 4$.
Since the Fibonacci sequence is increasing from F_2 onwards, then

$$F_x + F_y + F_z \leq F_{z-4} + F_{z-2} + F_z < F_{z-3} + F_{z-2} + F_z = F_{z-1} + F_z = F_{z+1}$$

Since there is a "$<$" in this chain of inequalities and equalities, then we obtain that $F_x + F_y + F_z < F_{z+1}$, as required.

Completing the Proof

Recall from the statement of the problem that a winning state consists of three or fewer JB, each at a distinct position and no two at consecutive integer positions.

Suppose that, starting from N JB at P0, in a first winning final state with $a_d = 1$, each of a_b and a_c equal to 0 or 1 and all other $a_i = 0$, and in a second winning final state with $a_D = 1$, each of a_B and a_C equal to 0 or 1 and all other $a_i = 0$.

From IF1, this gives $N = a_bF_{b+2} + a_cF_{c+2} + F_{d+2}$ and $N = a_BF_{B+2} + a_CF_{C+2} + F_{D+2}$, and so $a_bF_{b+2} + a_cF_{c+2} + F_{d+2} = a_BF_{B+2} + a_CF_{C+2} + F_{D+2}$.

Starting from this last equation, we remove any common Fibonacci numbers from both sides. (Recall that each term on each side is either 0 or a Fibonacci number, and Fibonacci numbers on the same side are distinct.)

If there are no Fibonacci numbers remaining on each side, then the winning final states are the same, as required.

What happens if there are Fibonacci numbers remaining on either side? In this case, there must be Fibonacci numbers on each side, as otherwise we would have 0 equal to a non-zero number.

Suppose that the largest Fibonacci number remaining on the LS is F_k and the largest Fibonacci number remaining on the RS is F_m.

Since we have removed the common elements, then $k \neq m$, so we may assume that $k < m$; since k and m are integers, then $k \leq m - 1$.

Note that the RS must be greater than or equal to F_m, since it includes at least F_m.

Since the LS consists of at most three Fibonacci numbers, which are non-consecutive (since b, c, d are non-consecutive) and the largest of which is F_k, then IF2 tells us that the LS is less than F_{k+1}.

Since $k + 1 \leq m$, then the LS is less than F_m.

Since the LS is less than F_m and the RS is greater than or equal to F_m, we have a contradiction, since they are supposed to be equal.

Therefore, our assumption that Fibonacci numbers are left after removing the common numbers from each side is false.

In other words, the positions of the JB in each of the winning final states are the same, so there is indeed only one possible winning final state.

Therefore, if Fiona can win the game, then there is only one possible final state.

(c) From the statement of the problem and IF1, we know that Fiona can win the game starting with N JB at P0 only if N is equal to the sum of at most three distinct non-consecutive Fibonacci numbers.

To determine the closest positive integer N to 2014 for which Fiona can win the game, we can determine the closet positive integer to 2014 that can be written as the sum of at most three distinct Fibonacci numbers, no two of which are consecutive.

We write out terms in the Fibonacci sequence until we reach a term larger than 2014:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584$$

We note that $1597 + 377 + 34 = 2008$, which is 6 away from 2014. We will show that there we cannot achieve an answer closer to 2014. That is, we will show that we cannot achieve any of the integers from 2009 to 2019, inclusive.

Suppose that an integer from 2009 to 2019, inclusive, can be achieved.

The Fibonacci number 2584 cannot be included in our sum, as the sum would be too large.

If our sum includes no Fibonacci number larger than 987, then our sum is at most $987 + 377 + 144 = 1508$, which is not large enough.

Therefore, 1597 must be included in a sum equal to an integer in the range 2009 to 2019, inclusive.

The remaining 0, 1 or 2 Fibonacci numbers must have a sum in the range $2009-1597 = 412$ to $2019 - 1597 = 422$, inclusive.

No Fibonacci number larger than 377 can be used, otherwise the remaining sum would be too large.

If the remaining sum uses no Fibonacci number larger than 233, the sum is at most $233 + 89 = 322$, which is not in the desired range.

Therefore, 377 must be included in the remaining sum.

The remaining 0 or 1 Fibonacci numbers must have a sum in the range $412 - 377 = 35$ to $422 - 377 = 45$, inclusive.

There is no Fibonacci number in this range, so we cannot make a sum of at most three distinct, non-consecutive numbers that is closer to 2014 than 2008.

Note that $2008 = 1597 + 377 + 34$. Since $F_9 = 34$, $F_{14} = 377$ and $F_{17} = 1597$, the corresponding winning position would be 1 JB at each of P7, P12 and P15.

To complete our proof, we must show that we can actually achieve this final state:

> We start with the final state consisting of 1 JB at each of P7, P12 and P15 and play the game backwards as we did in (a).
>
> Since there is 1 JB at P15, it must have come from a T15.
>
> Undoing this move, we obtain a state consisting of 1 JB at each of P7, P12, P13 and P14. Note that the rightmost JB is now at P14.
>
> Since there is 1 JB at P14, it must have come from a T14.
>
> We undo this move and continue to undo moves that remove a JB from the rightmost position remaining at each step. This process will eventually move all of the JB back to P0.
>
> To win the game starting with $N = F_9 + F_{14} + F_{17}$, Fiona then uses all of these moves in the opposite order, in a similar way to the method in (a).
>
> Thus, Fiona can achieve the winning final state of 1 JB at each of P7, P12 and P15.

Therefore, if $N = F_9 + F_{14} + F_{17}$, then Fiona can win the game.

Thus, $N = 2008$ is the closest integer to 2014 for which Fiona can start with N JB at P0 and win the game.

The CENTRE for EDUCATION
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2013 Euclid Contest

Wednesday, April 17, 2013
(in North America and South America)

Thursday, April 18, 2013
(outside of North America and South America)

Solutions

1. (a) The expression $\sqrt{113+x}$ is an integer whenever $113+x$ is a perfect square.
To find the smallest positive integer x for which $113+x$ is a perfect square, we find the smallest perfect square larger than 113.
Since $10^2 = 100$ and $11^2 = 121$, then this perfect square must be 121.
Therefore, $113 + x = 121$ or $x = 8$.

(b) The average of 3 and 11 is $\frac{1}{2}(3+11) = 7$. Thus $a = 7$.
Using this with the given information, we see that the average of 7 and b is 11.
Therefore, $\frac{1}{2}(7+b) = 11$ or $7 + b = 22$ and so $b = 15$.
(Alternatively, we could note that since 7 is 4 less than 11 (the average), then b must be 4 more than 11, so $b = 11 + 4 = 15$.)

(c) Let c be Charlie's age in years and b be Bella's age in years.
From the first sentence, $c = 30 + b$.
From the second sentence, $c = 6b$.
Combining these, we obtain $6b = 30 + b$ or $5b = 30$, and so $b = 6$.
Since $c = 30 + b$, then $c = 36$, and so Charlie's age is 36.

2. (a) Since $\dfrac{21}{x} = \dfrac{7}{y}$, then $21 = \dfrac{7x}{y}$ or $\dfrac{x}{y} = \dfrac{21}{7} = 3$.

(b) *Solution 1*
Since

$$\frac{1}{3} \approx 0.3333 \qquad \frac{1}{4} = 0.25 \qquad \frac{1}{5} = 0.2 \qquad \frac{1}{6} \approx 0.1667$$

then $\dfrac{1}{5} < 0.2013$ and $0.2013 < \dfrac{1}{4}$, so n must equal 4.
(We should note as well that $\dfrac{1}{n}$ decreases as n increases, so this is the only integer value of n that works.)

Solution 2
Since $\dfrac{1}{n+1} < 0.2013$, then $n + 1 > \dfrac{1}{0.2013}$ or $n > \dfrac{1}{0.2013} - 1 \approx 3.9677$.
Since $\dfrac{1}{n} > 0.2013$, then $n < \dfrac{1}{0.2013} \approx 4.9677$.
Since n is a positive integer

* that is smaller than a number that is approximately 4.9677, and
* that is larger than a number that is approximately 3.9677,

then $n = 4$.

(c) Since AH is perpendicular to BC, then the area of $\triangle ABC$ equals $\frac{1}{2}(BC)(AH)$.
Since we are told that this area equals 84 and $AH = 8$, then $84 = \frac{1}{2}(BC)(8)$ or $4 \cdot BC = 84$ or $BC = 21$.
Also, since $\triangle AHB$ is right-angled at H, then by the Pythagorean Theorem,

$$BH = \sqrt{AB^2 - AH^2} = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$$

since $BH > 0$. (We could also have recognized two sides of a 6-8-10 right-angled triangle.)
Since $BC = 21$ and $BH = 6$, then $HC = BC - BH = 21 - 6 = 15$.

Since $\triangle AHC$ is right-angled at H, then by the Pythagorean Theorem,

$$AC = \sqrt{AH^2 + HC^2} = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$$

since $AC > 0$.
Finally, the perimeter of $\triangle ABC$ equals $AB + BC + AC$ or $10 + 21 + 17$, which equals 48.

3. (a) The *parity* of an integer is whether it is even or odd.
Since the Fibonacci sequence begins $1, 1, 2, 3, 5, 8, 13, 21, \ldots$, then the parities of the first eight terms are Odd, Odd, Even, Odd, Odd, Even, Odd, Odd.
In the sequence, if x and y are consecutive terms, then the next term is $x + y$.
In general, suppose that x and y are integers.
If x is even and y is even, then $x + y$ is even. If x is even and y is odd, then $x + y$ is odd. If x is odd and y is even, then $x + y$ is odd. If x is odd and y is odd, then $x + y$ is even.
Therefore, the parities of two consecutive terms x and y in the Fibonacci sequence determine the parity of the following term $x + y$.
Also, once there are two consecutive terms whose parities match the parities of two earlier consecutive terms in the sequence, then the parities will repeat in a cycle.
In particular, the parities of the fourth and fifth terms (Odd, Odd) are the same as the parities of the first and second terms (Odd, Odd).
Therefore, the parities in the sequence repeat the cycle Odd, Odd, Even.
This cycle has length 3.
Therefore, the 99th term in the Fibonacci sequence ends one of these cycles, since 99 is a multiple of 3.
In particular, the 99th term ends the 33rd cycle.
Each cycle contains two odd terms.
Therefore, the first 99 terms in the sequence include $2 \times 33 = 66$ odd terms.
Finally, the 100th term in the sequence begins a new cycle, so is odd.
Therefore, the first 100 terms include $66 + 1 = 67$ odd terms.

(b) Suppose that the first term in the given sequence is a and the common difference is d.
Then the first four terms are $a, a + d, a + 2d, a + 3d$.
From the given information, $a + (a + 2d) = 6$ and $(a + d) + (a + 3d) = 20$.
The first equation simplifies to $2a + 2d = 6$ or $a + d = 3$.
The second equation simplifies to $2a + 4d = 20$ or $a + 2d = 10$.
Therefore, $(a + 2d) - (a + d) = 10 - 3$ or $d = 7$.
Since $a + d = 3$ and $d = 7$, then $a = -4$.
Therefore, the tenth term in the sequence is $a + 9d = -4 + 9(7) = 59$.

4. (a) There are five odd digits: 1, 3, 5, 7, 9.
We consider the positive integers less than 1000 in three sets: those with one digit, those with two digits, and those with three digits.
There are 5 positive one-digit integers with one odd digit (namely 1, 3, 5, 7, 9).
Consider the two-digit positive integers with only odd digits.
Such an integer has the form XY where X and Y are digits.
There are five possibilities for each of X and Y (since each must be odd).
Therefore, there are $5 \times 5 = 25$ two-digit positive integers with only odd digits.
Consider the three-digit positive integers with only odd digits.
Such an integer has the form XYZ where X, Y and Z are digits.

There are five possibilities for each of X, Y and Z (since each must be odd).
Therefore, there are $5 \times 5 \times 5 = 125$ three-digit positive integers with only odd digits.
In total, there are $5 + 25 + 125 = 155$ positive integers less than 1000 with only odd digits.

(b) Combining the two terms on the right side of the second equation, we obtain $\frac{4}{7} = \frac{b+a}{ab}$.
Since $a + b = 16$, then $\frac{4}{7} = \frac{16}{ab}$ or $ab = \frac{16(7)}{4} = 28$.
Therefore, we have $a + b = 16$ and $ab = 28$.
From the first equation, $b = 16 - a$.
Substituting into the second equation, we obtain $a(16 - a) = 28$ or $16a - a^2 = 28$, which gives $a^2 - 16a + 28 = 0$.
Factoring, we obtain $(a - 14)(a - 2) = 0$.
Therefore, $a = 14$ or $a = 2$.
If $a = 14$, then $b = 16 - a = 2$.
If $a = 2$, then $b = 16 - a = 14$.
Therefore, the two solutions are $(a, b) = (14, 2), (2, 14)$.
(We note that since $\frac{1}{2} + \frac{1}{14} = \frac{7}{14} + \frac{1}{14} = \frac{8}{14} = \frac{4}{7}$, then both of these pairs are actually solutions to the original system of equations.)

5. (a) We make a table of the 36 possible combinations of rolls and the resulting sums:

	2	3	5	7	11	13
2	4	5	7	9	13	15
3	5	6	8	10	14	16
5	7	8	10	12	16	18
7	9	10	12	14	18	20
11	13	14	16	18	22	24
13	15	16	18	20	24	26

Of the 36 entries in the table, 6 are prime numbers (two entries each of 5, 7 and 13).
Therefore, the probability that the sum is a prime number is $\frac{6}{36}$ or $\frac{1}{6}$.
(Note that each sum is at least 4 and so must be odd to be prime. Since odd plus odd equals even, then the only possibilities that really need to be checked are even plus odd and odd plus even (that is, the first row and first column of the table).)

(b) First, we find the coordinates of V.
To do this, we use the given equation for the parabola and complete the square:

$$y = -x^2+4x+1 = -(x^2-4x-1) = -(x^2-4x+2^2-2^2-1) = -((x-2)^2-5) = -(x-2)^2+5$$

Therefore, the coordinates of the vertex V are $(2, 5)$.
Next, we find the coordinates of A and B.
Note that A and B are the points of intersection of the line with equation $y = -x + 1$ and the parabola with equation $y = -x^2 + 4x + 1$.
We equate y-values to obtain $-x + 1 = -x^2 + 4x + 1$ or $x^2 - 5x = 0$ or $x(x - 5) = 0$.
Therefore, $x = 0$ or $x = 5$.
If $x = 0$, then $y = -x + 1 = 1$, and so A (which is on the y-axis) has coordinates $(0, 1)$.
If $x = 5$, then $y = -x + 1 = -4$, and so B has coordinates $(5, -4)$.

We now have the points $V(2,5)$, $A(0,1)$, $B(5,-4)$.
This gives

$$\begin{aligned} AV^2 &= (0-2)^2 + (1-5)^2 = 20 \\ BV^2 &= (5-2)^2 + (-4-5)^2 = 90 \\ AB^2 &= (0-5)^2 + (1-(-4))^2 = 50 \end{aligned}$$

and so $AV^2 + BV^2 - AB^2 = 20 + 90 - 50 = 60$.

6. (a) Since ABC is a quarter of a circular pizza with centre A and radius 20 cm, then $AC = AB = 20$ cm.
We are also told that $\angle CAB = 90^\circ$ (one-quarter of 360°).
Since $\angle CAB = 90^\circ$ and A, B and C are all on the circumference of the circle, then CB is a diameter of the pan. (This is a property of circles: if X, Y and Z are three points on a circle with $\angle ZXY = 90^\circ$, then YZ must be a diameter of the circle.)
Since $\triangle CAB$ is right-angled and isosceles, then $CB = \sqrt{2}AC = 20\sqrt{2}$ cm.
Therefore, the radius of the circular plate is $\frac{1}{2}CB$ or $10\sqrt{2}$ cm.
Thus, the area of the circular pan is $\pi(10\sqrt{2}\text{ cm})^2 = 200\pi\text{ cm}^2$.
The area of the slice of pizza is one-quarter of the area of a circle with radius 20 cm, or $\frac{1}{4}\pi(20\text{ cm})^2 = 100\pi\text{ cm}^2$.
Finally, the fraction of the pan that is covered is the area of the slice of pizza divided by the area of the pan, or $\dfrac{100\pi\text{ cm}^2}{200\pi\text{ cm}^2} = \dfrac{1}{2}$.

(b) Suppose that the length of AF is x m.
Since the length of AB is 8 m, then the length of FB is $(8-x)$ m.
Since $\triangle MAF$ is right-angled and has an angle of 60°, then it is 30°-60°-90° triangle.
Therefore, $MF = \sqrt{3}AF$, since MF is opposite the 60° angle and AF is opposite the 30° angle.
Thus, $MF = \sqrt{3}x$ m.
Since $MP = 2$ m, then $PF = MF - MP = (\sqrt{3}x - 2)$ m.
We can now look at $\triangle BFP$ which is right-angled at F.
We have

$$\tan\theta = \frac{PF}{FB} = \frac{(\sqrt{3}x-2)\text{ m}}{(8-x)\text{ m}} = \frac{\sqrt{3}x-2}{8-x}$$

Therefore, $(8-x)\tan\theta = \sqrt{3}x - 2$ or $8\tan\theta + 2 = \sqrt{3}x + (\tan\theta)x$.
This gives $8\tan\theta + 2 = x(\sqrt{3} + \tan\theta)$ or $x = \dfrac{8\tan\theta + 2}{\tan\theta + \sqrt{3}}$.
Finally, $MF = \sqrt{3}x = \dfrac{8\sqrt{3}\tan\theta + 2\sqrt{3}}{\tan\theta + \sqrt{3}}$ m.

7. (a) Beginning with the given equation, we have

$$\begin{aligned} \frac{1}{\cos x} - \tan x &= 3 \\ \frac{1}{\cos x} - \frac{\sin x}{\cos x} &= 3 \\ 1 - \sin x &= 3\cos x \qquad \text{(since } \cos x \neq 0\text{)} \\ (1-\sin x)^2 &= 9\cos^2 x \qquad \text{(squaring both sides)} \\ 1 - 2\sin x + \sin^2 x &= 9(1-\sin^2 x) \\ 10\sin^2 x - 2\sin x - 8 &= 0 \\ 5\sin^2 x - \sin x - 4 &= 0 \\ (5\sin x + 4)(\sin x - 1) &= 0 \end{aligned}$$

Therefore, $\sin x = -\frac{4}{5}$ or $\sin x = 1$.
If $\sin x = 1$, then $\cos x = 0$ and $\tan x$ is undefined, which is inadmissible in the original equation.
Therefore, $\sin x = -\frac{4}{5}$.
(We can check that if $\sin x = -\frac{4}{5}$, then $\cos x = \pm\frac{3}{5}$ and the possibility that $\cos x = \frac{3}{5}$ satisfies the original equation, since in this case $\frac{1}{\cos x} = \frac{5}{3}$ and $\tan x = -\frac{4}{3}$ and the difference between these fractions is 3.)

(b) Since $f(x) = ax+b$, we can determine an expression for $g(x) = f^{-1}(x)$ by letting $y = f(x)$ to obtain $y = ax+b$. We then interchange x and y to obtain $x = ay + b$ which we solve for y to obtain $ay = x - b$ or $y = \frac{x}{a} - \frac{b}{a}$.
Therefore, $f^{-1}(x) = \frac{x}{a} - \frac{b}{a}$.
Note that $a \neq 0$. (This makes sense since the function $f(x) = b$ has a graph which is a horizontal line, and so cannot be invertible.)
Therefore, the equation $f(x) - g(x) = 44$ becomes $(ax+b) - \left(\frac{x}{a} - \frac{b}{a}\right) = 44$ or $\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 44 = 0x + 44$, and this equation is true for all x.
We can proceed in two ways.

Method #1: Comparing coefficients
Since the equation

$$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 0x + 44$$

is true for all x, then the coefficients of the linear expression on the left side must match the coefficients of the linear expression on the right side.
Therefore, $a - \frac{1}{a} = 0$ and $b + \frac{b}{a} = 44$.
From the first of these equations, we obtain $a = \frac{1}{a}$ or $a^2 = 1$, which gives $a = 1$ or $a = -1$.
If $a = 1$, the equation $b + \frac{b}{a} = 44$ becomes $b + b = 44$, which gives $b = 22$.

If $a=-1$, the equation $b+\frac{b}{a}=44$ becomes $b-b=44$, which is not possible.
Therefore, we must have $a=1$ and $b=22$, and so $f(x)=x+22$.

Method #2: Trying specific values for x
Since the equation

$$\left(a-\frac{1}{a}\right)x+\left(b+\frac{b}{a}\right)=0x+44$$

is true for all values of x, then it must be true for any specific values of x that we choose.
Choosing $x=0$, we obtain $0+\left(b+\frac{b}{a}\right)=44$ or $b+\frac{b}{a}=44$.
Choosing $x=b$, we obtain $\left(a-\frac{1}{a}\right)b+\left(b+\frac{b}{a}\right)=44$ or $ab+b=44$.
We can rearrange the first of these equations to get $\frac{ab+b}{a}=44$.
Using the second equation, we obtain $\frac{44}{a}=44$ or $a=1$.
Since $a=1$, then $ab+b=44$ gives $2b=44$ or $b=22$.
Thus, $f(x)=x+22$.

In summary, the only linear function f for which the given equation is true for all x is $f(x)=x+22$.

8. (a) First, we factor the left side of the given equation to obtain $a(a^2+2b)=2013$.
Next, we factor the integer 2013 as $2013=3\times 671=3\times 11\times 61$. Note that each of 3, 11 and 61 is prime, so we can factor 2013 no further. (We can find the factors of 3 and 11 using tests for divisibility by 3 and 11, or by systematic trial and error.)
Since $2013=3\times 11\times 61$, then the positive divisors of 2013 are

$$1,3,11,33,61,183,671,2013$$

Since a and b are positive integers, then a and a^2+2b are both positive integers.
Since a and b are positive integers, then $a^2\geq a$ and $2b>0$, so $a^2+2b>a$.
Since $a(a^2+2b)=2013$, then a and a^2+2b must be a divisor pair of 2013 (that is, a pair of positive integers whose product is 2013) with $a<a^2+2b$.
We make a table of the possibilities:

a	a^2+2b	$2b$	b
1	2013	2012	1006
3	671	662	331
11	183	62	31
33	61	-1028	N/A

Note that the last case is not possible, since b must be positive.
Therefore, the three pairs of positive integers that satisfy the equation are $(1,1006)$, $(3,331)$, $(11,31)$.
(We can verify by substitution that each is a solution of the original equation.)

(b) *Solution 1*

We successively manipulate the given equation to produce equivalent equations:

$$\begin{aligned}
\log_2(2^{x-1}+3^{x+1}) &= 2x-\log_2(3^x) \\
\log_2(2^{x-1}+3^{x+1})+\log_2(3^x) &= 2x \\
\log_2((2^{x-1}+3^{x+1})3^x) &= 2x \qquad (\text{using } \log_2 A+\log_2 B=\log_2 AB) \\
(2^{x-1}+3^{x+1})3^x &= 2^{2x} \qquad (\text{exponentiating both sides}) \\
2^{-1}2^x3^x+3^13^x3^x &= 2^{2x} \\
\tfrac{1}{2}\cdot 2^x3^x+3\cdot 3^{2x} &= 2^{2x} \\
2^x3^x+6\cdot 3^{2x} &= 2\cdot 2^{2x} \qquad (\text{multiplying by 2}) \\
2^x3^x+6\cdot(3^x)^2 &= 2\cdot(2^x)^2
\end{aligned}$$

Next, we make the substitution $a=2^x$ and $b=3^x$.
This gives $ab+6b^2=2a^2$ or $2a^2-ab-6b^2=0$.
Factoring, we obtain $(a-2b)(2a+3b)=0$.
Therefore, $a=2b$ or $2a=-3b$.
Since $a>0$ and $b>0$, then $a=2b$ which gives $2^x=2\cdot 3^x$.
Taking logs of both sides, we obtain $x\log 2=\log 2+x\log 3$ and so $x(\log 2-\log 3)=\log 2$ or $x=\dfrac{\log 2}{\log 2-\log 3}$.

Solution 2

We successively manipulate the given equation to produce equivalent equations:

$$\begin{aligned}
\log_2(2^{x-1}+3^{x+1}) &= 2x-\log_2(3^x) \\
\log_2(2^{x-1}+3^{x+1})+\log_2(3^x) &= 2x \\
\log_2((2^{x-1}+3^{x+1})3^x) &= 2x \qquad (\text{using } \log_2 A+\log_2 B=\log_2 AB) \\
(2^{x-1}+3^{x+1})3^x &= 2^{2x} \qquad (\text{exponentiating both sides}) \\
2^{-1}2^x3^x+3^13^x3^x &= 2^{2x} \\
\tfrac{1}{2}\cdot 2^x3^x+3\cdot 3^{2x} &= 2^{2x} \\
2^x3^x+6\cdot 3^{2x} &= 2\cdot 2^{2x} \qquad (\text{multiplying by 2}) \\
2^x3^x2^{-2x}+6\cdot 3^{2x}2^{-2x} &= 2 \qquad (\text{dividing both sides by } 2^{2x}\neq 0) \\
2^{-x}3^x+6\cdot 3^{2x}2^{-2x} &= 2 \\
\left(\tfrac{3}{2}\right)^x+6\left(\tfrac{3}{2}\right)^{2x} &= 2
\end{aligned}$$

Next, we make the substitution $t=\left(\frac{3}{2}\right)^x$, noting that $\left(\frac{3}{2}\right)^{2x}=\left(\left(\frac{3}{2}\right)^x\right)^2=t^2$.
Thus, we obtain the equivalent equations

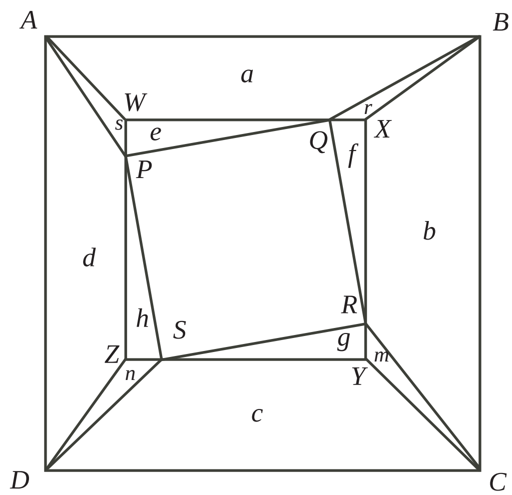
$$\begin{aligned}
t+6t^2 &= 2 \\
6t^2+t-2 &= 0 \\
(3t+2)(2t-1) &= 0
\end{aligned}$$

Therefore, $t=-\frac{2}{3}$ or $t=\frac{1}{2}$.
Since $t=\left(\frac{3}{2}\right)^x>0$, then we must have $t=\left(\frac{3}{2}\right)^x=\frac{1}{2}$.
Thus,

$$x=\log_{3/2}(1/2)=\frac{\log(1/2)}{\log(3/2)}=\frac{\log 1-\log 2}{\log 3-\log 2}=\frac{-\log 2}{\log 3-\log 2}=\frac{\log 2}{\log 2-\log 3}$$

9. (a) Suppose that the parallel line segments EF and WX are a distance of x apart.
This means that the height of trapezoid $EFXW$ is x.
Since the side length of square $EFGH$ is 10 and the side length of square $WXYZ$ is 6, then the distance between parallel line segments ZY and HG is $10 - 6 - x$ or $4 - x$.
Recall that the area of a trapezoid equals one-half times its height times the sum of the lengths of the parallel sides.
Thus, the area of trapezoid $EFXW$ is $\frac{1}{2}x(EF + WX) = \frac{1}{2}x(10 + 6) = 8x$.
Also, the area of trapezoid $GHZY$ is $\frac{1}{2}(4 - x)(HG + ZY) = \frac{1}{2}(4 - x)(10 + 6) = 32 - 8x$.
Therefore, the sum of the areas of trapezoids $EFXW$ and $GHZY$ is $8x + (32 - 8x) = 32$.
This sum is a constant and does not depend on the position of the inner square within the outer square, as required.

(b) We begin by "boxing in" square $PQRS$ by drawing horizontal and vertical lines through its vertices to form rectangle $WXYZ$, as shown. (Because the four quadrilaterals $ABQP$, $BCRQ$, $CDSR$, and $DAPS$ are convex, there will not be any configurations that look substantially different from this the diagram below.) We also label the various areas.



Since WX is parallel to AB, then quadrilateral $ABXW$ is a trapezoid. Similarly, quadrilaterals $BCYX$, $CDZY$, and $DAWZ$ are trapezoids.
We use the notation $|ABQP|$ to denote the area of quadrilateral $ABQP$, and similar notation for other areas.
Suppose that the side length of square $ABCD$ is x and the side length of square $PQRS$ is y.
Also, we let $\angle WPQ = \theta$.
Since each of $\triangle WPQ$, $\triangle XQR$, $\triangle YRS$, and $\triangle ZSP$ is right-angled and each of the four angles of square $PQRS$ is 90°, then $\angle WPQ = \angle XQR = \angle YRS = \angle ZSP = \theta$. This is because, for example,

$$\angle XQR = 180^\circ - \angle PQR - \angle WQP = 90^\circ - (180^\circ - \angle PWQ - \angle WPQ) = 90^\circ - (90^\circ - \theta) = \theta$$

This fact, together with the fact that $PQ = QR = RS = SP = y$, allows us to conclude that the four triangles $\triangle WPQ$, $\triangle XQR$, $\triangle YRS$, and $\triangle ZSP$ are congruent.
In particular, this tells us

* the four areas labelled e, f, g and h are equal (that is, $e = f = g = h$),
* $PZ = QW = RX = SY = y \sin\theta$, and
* $WP = XQ = YR = ZS = y \cos\theta$.

Combining these last two facts tells us that $WZ = XW = YX = ZY$, since, for example, $WZ = WP + PZ = ZS + SY = ZY$. In other words, $WXYZ$ is a square, with side length z, say.

Next, we show that $(a + r) + (c + n)$ is equal to $(b + m) + (d + s)$.
Note that the sum of these two quantities is the total area between square $ABCD$ and square $WXYZ$, so equals $x^2 - z^2$.
Thus, to show that the quantities are equal, it is enough to show that $(a + r) + (c + n)$ equals $\frac{1}{2}(x^2 - z^2)$.
Let the height of trapezoid $ABXW$ be k and the height of trapezoid $ZYCD$ be l.
Then $|ABXW| = a + r = \frac{1}{2}k(AB + WX) = \frac{1}{2}k(x + z)$.
Also, $|ZYCD| = c + n = \frac{1}{2}l(DC + ZY) = \frac{1}{2}l(x + z)$.
Since AB, WX, ZY, and DC are parallel, then the sum of the heights of trapezoid $ABXW$, square $WXYZ$, and trapezoid $ZYCD$ equals the height of square $ABCD$, so $k + z + l = x$, or $k + l = x - z$.
Therefore,

$$(a + r) + (c + n) = \tfrac{1}{2}k(x + z) + \tfrac{1}{2}l(x + z) = \tfrac{1}{2}(x + z)(k + l) = \tfrac{1}{2}(x + z)(x - z) = \tfrac{1}{2}(x^2 - z^2)$$

as required.
Therefore, $(a + r) + (c + n) = (b + m) + (d + s)$. We label this equation $(*)$.

Next, we show that $r + n = m + s$.
Note that $r = |\triangle QXB|$. This triangle can be viewed as having base QX and height equal to the height of trapezoid $ABXW$, or k.
Thus, $r = \frac{1}{2}(y\cos\theta)k$.
Note that $n = |\triangle SZD|$. This triangle can be viewed as having base SZ and height equal to the height of trapezoid $ZYCD$, or l.
Thus, $n = \frac{1}{2}(y\cos\theta)l$.
Combining these facts, we obtain

$$n + r = \tfrac{1}{2}(y\cos\theta)k + \tfrac{1}{2}(y\cos\theta)l = \tfrac{1}{2}y\cos\theta(k + l) = \tfrac{1}{2}y\cos\theta(x - z)$$

We note that this sum depends only on the side lengths of the squares and the angle of rotation of the inner square, so is independent of the position of the inner square within the outer square.
This means that we can repeat this analysis to obtain the same expression for $m + s$.
Therefore, $n + r = m + s$. We label this equation $(**)$.

We subtract $(*) - (**)$ to obtain $a + c = b + d$.
Finally, we can combine all of this information:

$$\begin{aligned}
&(|ABQP| + |CDSR|) - (|BCRQ| + |APSD|)\\
&\quad = (a + e + s + c + g + m) - (b + f + r + d + h + n)\\
&\quad = ((a + c) - (b + d)) + ((m + s) - (n + r)) + ((e + g) - (f + h))\\
&\quad = 0 + 0 + 0
\end{aligned}$$

since $a + c = b + d$ and $n + r = m + s$ and $e = f = g = h$.
Therefore, $|ABQP| + |CDSR| = |BCRQ| + |APSD|$, as required.

10. In each part, we use "partition" to mean "multiplicative partition". We also call the numbers being multiplied together in a given partition the "parts" of the partition.

(a) We determine the multiplicative partitions of 64 by considering the number of parts in the various partitions. Note that 64 is a power of 2 so any divisor of 64 is also a power of 2. In each partition, since the order of parts is not important, we list the parts in increasing order to make it easier to systematically find all of these.

- One part. There is one possibility: 64.
- Two parts. There are three possibilities: $64 = 2 \times 32 = 4 \times 16 = 8 \times 8$.
- Three parts. We start with the smallest possible first and second parts. We keep the first part fixed while adjusting the second and third parts. We then increase the first part and repeat.
  We get: $64 = 2 \times 2 \times 16 = 2 \times 4 \times 8 = 4 \times 4 \times 4$.
- Four parts. A partition of 64 with four parts must include at least two 2s, since if it didn't, it would include at least three parts that are at least 4, and so would be too large. With two 2s, the remaining two parts have a product of 16.
  We get: $64 = 2 \times 2 \times 2 \times 8 = 2 \times 2 \times 4 \times 4$.
- Five parts. A partition of 64 with five parts must include at least three 2s, since if it didn't, it would include at least three parts that are at least 4, and so would be too large. With three 2s, the remaining two parts have a product of 8.
  We get: $64 = 2 \times 2 \times 2 \times 2 \times 4$.
- Six parts. Since $64 = 2^6$, there is only one possibility: $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

Therefore, $P(64) = 1 + 3 + 3 + 2 + 1 + 1 = 11$.

(b) First, we note that $1000 = 10^3 = (2 \cdot 5)^3 = 2^3 5^3$.
We calculate the value of $P(p^3q^3)$ for two distinct prime numbers p and q. It will turn out that this value does not depend on p and q. This value will be the value of $P(1000)$, since 1000 has this form of prime factorization.

Let $n = p^3q^3$ for distinct prime numbers p and q.
The integer n has three prime factors equal to p.
In a given partition, these can be all together in one part (as p^3), can be split between two different parts (as p and p^2), or can be split between three different parts (as p, p and p). There are no other ways to divide up three divisors of p.
Similarly, n has three prime factors equal to q which can be divided in similar ways.
We determine $P(p^3q^3)$ by considering the possible combination of the number of parts divisible by p and the number of parts divisible by q and counting partitions in each case. In other words, we complete the following table:

		Number of parts divisible by p		
		1	2	3
Number of parts divisible by q	1			
	2			
	3			

We note that the table is symmetric, since the factors of p and q are interchangeable.
We proceed to consider cases, considering only those on the top left to bottom right diagonal and and those below this diagonal in the table.

Case 1: One part divisible by p, one part divisible by q

The partition must be p^3q^3 (n itself) or $p^3 \times q^3$.
There are two partitions in this case.

Case 2: One part divisible by p, two parts divisible by q

The three factors of p occur together as p^3. The three factors of q occur as q and q^2.
The p^3 can occur in one of the parts divisible by q or not.
This gives partitions $p^3 \times q \times q^2$ and $p^3q \times q^2$ and $q \times p^3q^2$.
There are three partitions in this case. Similarly, there are three partitions with one part divisible by q and two parts divisible by p.

Case 3: One part divisible by p, three parts divisible by q

The three factors of p occur together as p^3. The three factors of q occur as q, q and q.
The p^3 can occur in one of the parts divisible by q or not.
This gives partitions $p^3 \times q \times q \times q$ and $p^3q \times q \times q$.
(Note that the three divisors of q are interchangeable so p^3 only needs to be placed with one of them.)
There are two partitions in this case. Similarly, there are two partitions with one part divisible by q and three parts divisible by p.

Case 4: Two parts divisible by p, two parts divisible by q

The three factors of p occur as p and p^2. The three factors of q occur as q and q^2.
Each of p and p^2 can occur in one of the parts divisible by q or not.
If no part is a multiple of both p and q, we have one partition: $p \times p^2 \times q \times q^2$.
If one part is a multiple of both p and q, there are two choices for which power of p to include in this part and two choices for which power of q to include. (There is no choice for the remaining parts.) Thus, there are $2 \times 2 = 4$ such partitions:

$$p^2q^2 \times p \times q \qquad pq^2 \times p^2 \times q \qquad p^2q \times p \times q^2 \qquad pq \times p^2 \times q^2$$

If two parts are a multiple of both p and q, there are two ways to choose the power of p in the part containing just q, so there are two such partitions: $pq \times p^2q^2$ and $p^2q \times pq^2$.
There are seven partitions in this case.

Case 5: Two parts divisible by p, three parts divisible by q

The three factors of p occur as p and p^2. The three factors of q occur as q, q and q.
Each of p and p^2 can occur in one of the parts divisible by q or not.
If no part is a multiple of both p and q, we have one partition: $p \times p^2 \times q \times q \times q$.
If one part is a multiple of both p and q, there are two choices for which power of p to include in this part (since all powers of q are identical).
Thus, there are 2 such partitions: $p^2q \times p \times q \times q$ and $pq \times p^2 \times q \times q$.
If two parts are a multiple of both p and q, there is one partition, since all of the powers of q are identical: $pq \times p^2q \times q$.
There are four partitions in this case. Similarly, there are four partitions with two parts divisible by q and three parts divisible by p.

Case 6: Three parts divisible by p, three parts divisible by q

The three factors of p as p, p and p. The three factors of q appear as q, q and q.
Here, the number of parts in the partition that are multiples of both p and q can be 0, 1, 2 or 3. Since all of the powers of p and q are identical, the partitions are completely determined by this and are

$$p \times p \times p \times q \times q \times q \qquad p \times p \times pq \times q \times q \qquad p \times pq \times pq \times q \qquad pq \times pq \times pq$$

There are four partitions in this case.

Finally, we complete the table:

		Number of parts divisible by p		
		1	2	3
Number of parts divisible by q	1	2	3	2
	2	3	7	4
	3	2	4	4

Adding the entries in the table, we obtain $P(p^3q^3) = 31$.
Thus, $P(1000) = 31$.

(c) As in (b), the value of $P(n)$ depends only on the structure of the prime factorization of n, not on the actual primes in the factorization.
Therefore, $P(4 \times 5^m) = P(2^2 \times 5^m) = P(p^2q^m)$ for any distinct primes p and q.
Therefore, $P(4 \times 5^m) = P(p^2q^m) = P(5^2 \times 2^m) = P(25 \times 2^m)$.
We count the number of multiplicative partitions of $N = 5^2 \times 2^m$ by considering the placement of the 2s and 5s among the parts of the partitions.
Since N has only two factors of 5, these can occur in the same part, or in different parts.
Note that every factor of N is a product is of the form 5^j2^k for some integers j and k with $0 \leq j \leq 2$ and $0 \leq k \leq m$.

We first count the number of partitions where the two factors of 5 occur in the same part.
Consider one such partition.
In this partition, the part containing the two 5s will be of the form 5^22^k for some integer k with $0 \leq k \leq m$.
Thus, this partition will be of the form $5^22^k \times \mathcal{P}$, where $\mathcal{P}$ is a partition of 2^{m-k} (the remaining factors in N).
Since the order of parts does not matter, there are $P(2^{m-k})$ such partitions $\mathcal{P}$, and so this number of partitions of N of this form.
Since k ranges from 0 to m, then the number of partitions where the two 5s occur in the same part equals

$$P(2^m) + P(2^{m-1}) + \cdots + P(2^1) + P(2^0)$$

Next, we count the number of partitions where the two factors of 5 occur in different parts.
Consider one such partition.
In this partition, the parts containing the two 5s will be of the form 5×2^a and 5×2^b for some integers a and b with $0 \leq a, b \leq m$ and $a + b \leq m$.
Since the order of the parts within a partition does not matter, we can restrict a and b further by requiring that $0 \leq a \leq b \leq m$ and $a + b \leq m$ to avoid double-counting partitions.
Thus, this partition will be of the form $(5 \times 2^a) \times (5 \times 2^b) \times \mathcal{P}$, where $\mathcal{P}$ is a partition of 2^{m-a-b} (the remaining factors in N).
Since the order of parts does not matter, there are $P(2^{m-a-b})$ such partitions $\mathcal{P}$, and so this number of partitions of N of this form.

To determine the total number of partitions in this case, we need to add up $P(2^{m-a-b})$ over all possible pairs (a, b) satisfying $0 \leq a \leq b \leq m$ and $a + b \leq m$.
To do this, we focus on the possible values of $s = a + b$ and count the number of pairs (a, b) that give this sum.

If $s = a + b = 0$, there is one pair (a, b), namely $(a, b) = (0, 0)$.
If $s = a + b = 1$, there is one pair (a, b), namely $(a, b) = (0, 1)$.
If $s = a + b = 2$, there are two pairs (a, b), namely $(a, b) = (0, 2), (1, 1)$.
In general, if s is even, then $\frac{1}{2}s$ is an integer and so there are $(\frac{1}{2}s + 1)$ pairs (a, b), namely

$$(0, s), (1, s - 1), (2, s - 2), \ldots, (\tfrac{1}{2}s - 1, \tfrac{1}{2}s + 1), (\tfrac{1}{2}s, \tfrac{1}{2}s)$$

Any larger value of a would give a value of b smaller than a.
In general, if s is odd, then $\frac{1}{2}s - \frac{1}{2}$ is an integer and so there are $(\frac{1}{2}s - \frac{1}{2}) + 1 = (\frac{1}{2}s + \frac{1}{2})$ pairs (a, b), namely

$$(0, s), (1, s - 1), (2, s - 2), \ldots, (\tfrac{1}{2}s - \tfrac{3}{2}, \tfrac{1}{2}s + \tfrac{3}{2}), (\tfrac{1}{2}s - \tfrac{1}{2}, \tfrac{1}{2}s + \tfrac{1}{2})$$

Any larger value of a would give a value of b smaller than a.
To summarize, if $s = a + b$ is even, there are $(\frac{1}{2}s + 1)$ pairs (a, b) and if $s = a + b$ is odd, there are $(\frac{1}{2}s + \frac{1}{2})$ pairs (a, b).
Thus, as s increases from 0, the number of pairs (a, b) gives the sequence $1, 1, 2, 2, 3, 3, \ldots$.
The number in this sequence corresponding to the value of $a + b$ gives the number of times that $P(2^{m-a-b})$ should be included in the count of the total number of partitions in this case.
In other words, if $a + b = 0$, there are $1 \times P(2^m)$ partitions, if $a + b = 1$, there are $1 \times P(2^{m-1})$ partitions, if $a + b = 2$, there are $2 \times P(2^{m-2})$ partitions, etc.
We can rewrite this more compactly to say that for a given s, the number of pairs (a, b) is $\left\lfloor \frac{s+2}{2} \right\rfloor$ (where $\lfloor x \rfloor$ is the greatest integer less than or equal to x) and so the number of partitions is $\left\lfloor \frac{s+2}{2} \right\rfloor \times P(2^{m-s})$.
Therefore, the total number of partitions of N in this case is

$$1 \times P(2^m) + 1 \times P(2^{m-1}) + 2 \times P(2^{m-2}) + 2 \times P(2^{m-3}) + \cdots + \left\lfloor \frac{s+2}{2} \right\rfloor \times P(2^{m-s}) + \cdots$$
$$+ \left\lfloor \frac{m}{2} \right\rfloor \times P(2^1) + \left\lfloor \frac{m+2}{2} \right\rfloor \times P(2^0)$$

Combining the two cases and adding the corresponding expressions for the number of partitions, we obtain that the total number of partitions is

$$2 \times P(2^m) + 2 \times P(2^{m-1}) + 3 \times P(2^{m-2}) + 3 \times P(2^{m-3}) + \cdots + \left(1 + \left\lfloor \frac{s+2}{2} \right\rfloor\right) \times P(2^{m-s}) + \cdots$$
$$+ \left(1 + \left\lfloor \frac{m+1}{2} \right\rfloor\right) \times P(2^1) + \left(1 + \left\lfloor \frac{m+2}{2} \right\rfloor\right) \times P(2^0)$$

and so the desired sequence is

$$\begin{aligned} a_0 &= 2 \\ a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= 3 \\ &\vdots \\ a_s &= 1 + \left\lfloor \frac{s+2}{2} \right\rfloor \\ &\vdots \end{aligned}$$

The CENTRE for EDUCATION
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2012 Euclid Contest

Wednesday, April 11, 2012
(in North America and South America)

Thursday, April 12, 2012
(outside of North America and South America)

Solutions

1. (a) Since John buys 10 bags of apples, each of which contains 20 apples, then he buys a total of $10 \times 20 = 200$ apples.
Since he eats 8 apples a day, then it takes him $200 \div 8 = 25$ days to eat these apples.

(b) Evaluating,

$$\begin{aligned} & \sin(0^\circ) + \sin(60^\circ) + \sin(120^\circ) + \sin(180^\circ) + \sin(240^\circ) + \sin(300^\circ) + \sin(360^\circ) \\ = \ & 0 + \tfrac{\sqrt{3}}{2} + \tfrac{\sqrt{3}}{2} + 0 + (-\tfrac{\sqrt{3}}{2}) + (-\tfrac{\sqrt{3}}{2}) + 0 \\ = \ & 0 \end{aligned}$$

Alternatively, we could notice that $\sin(60^\circ) = -\sin(300^\circ)$ and $\sin(120^\circ) = -\sin(240^\circ)$ and $\sin(0^\circ) = \sin(180^\circ) = \sin(360^\circ) = 0$, so the sum is 0.

(c) Since the set of integers has a sum of 420 and an average of 60, then there are $420 \div 60 = 7$ integers in the set.
Since one integer is 120, then the remaining 6 integers have a sum of $420 - 120 = 300$ and so have an average of $300 \div 6 = 50$.

2. (a) Since $ax + ay = 4$, then $a(x + y) = 4$.
Since $x + y = 12$, then $12a = 4$ or $a = \frac{4}{12} = \frac{1}{3}$.

(b) Since the two lines are parallel, then their slopes are equal.
We re-write the given equations in the form "$y = mx + b$".
The first equation becomes $6y = -4x + 5$ or $y = -\frac{4}{6}x + \frac{5}{6}$ or $y = -\frac{2}{3}x + \frac{5}{6}$.
Since the first line is not vertical, then the second line is not vertical, and so $k \neq 0$.
The second equation becomes $ky = -6x + 3$ or $y = -\frac{6}{k}x + \frac{3}{k}$.
Therefore, $-\frac{2}{3} = -\frac{6}{k}$ and so $\frac{k}{6} = \frac{3}{2}$ or $k = 6 \times \frac{3}{2} = 9$.

(c) Adding the two equations, we obtain $x + x^2 = 2$ or $x^2 + x - 2 = 0$.
Factoring, we obtain $(x + 2)(x - 1) = 0$, and so $x = -2$ or $x = 1$.
From the first equation, $y = -x$. If $x = -2$, then $y = 2$ and if $x = 1$, then $y = -1$.
Therefore, the solutions are $(x, y) = (-2, 2)$ and $(x, y) = (1, -1)$.
(We can check that each of these solutions satisfies both equations.)

3. (a) Since the 200 g solution is 25% salt by mass, then $\frac{1}{4}$ of the mass (or 50 g) is salt and the rest (150 g) is water.
When water is added, the mass of salt does not change. Therefore, the 50 g of salt initially in the solution becomes 10% (or $\frac{1}{10}$) of the final solution by mass.
Therefore, the total mass of the final solution is $10 \times 50 = 500$ g.
Thus, the mass of water added is $500 - 200 = 300$ g.

(b) We are told that $F = \frac{9}{5}C + 32$.
From the given information $f = 2C + 30$.
We determine an expression for the error in terms of C by first determining when $f < F$.
The inequality $f < F$ is equivalent to $2C + 30 < \frac{9}{5}C + 32$ which is equivalent to $\frac{1}{5}C < 2$ which is equivalent to $C < 10$.
Therefore, $f < F$ precisely when $C < 10$.
Thus, for $-20 \leq C < 10$, the error equals $F - f = (\frac{9}{5}C + 32) - (2C + 30) = 2 - \frac{1}{5}C$.
Also, for $10 \leq C \leq 35$, the error equals $f - F = (2C + 30) - (\frac{9}{5}C + 32) = \frac{1}{5}C - 2$.
When $-20 \leq C < 10$, the error in terms of C is $2 - \frac{1}{5}C$ which is linear with negative slope, so is decreasing as C increases. Thus, the maximum value of error in this range for C occurs

when C is smallest, that is, when $C = -20$. This gives an error of $2 - \frac{1}{5}(-20) = 2 + 4 = 6$. When $10 \leq C \leq 35$, the error in terms of C is $\frac{1}{5}C - 2$ which is linear with positive slope, so is increasing as C increases. Thus, the maximum value of error in this range for C occurs when C is largest, that is, when $C = 35$. This gives an error of $\frac{1}{5}(35) - 2 = 7 - 2 = 5$.

Having considered the two possible ranges for C, the maximum possible error that Gordie would make is 6.

4. (a) *Solution 1*

Since the x-intercepts of the parabola with equation $y = 2(x-3)(x-5)$ are $x = 3$ and $x = 5$, then its axis of symmetry is at $x = \frac{1}{2}(3+5) = 4$.
If a horizontal line intersects the parabola at two points, then these points are symmetric across the axis of symmetry.
Since the line $y = k$ intersects the parabola at two points A and B with $AB = 6$, then each of A and B must be 3 units from the axis of symmetry.
Therefore, the x-coordinates of A and B are $4 - 3 = 1$ and $4 + 3 = 7$.
Thus, the coordinates of A and B, in some order, are $(1, k)$ and $(7, k)$.
Substituting $(1, k)$ into the equation of the parabola gives $k = 2(1-3)(1-5) = 16$.
(Substituting $(7, k)$ would give the same value of k.)

Solution 2

Let x_A be the x-coordinate of A and x_B be the x-coordinate of B. We may assume that A is to the left of B; that is, we assume that $x_A < x_B$. Since AB is horizontal and $AB = 6$, then $x_B - x_A = 6$.
Since A and B are the points of intersection between the line with equation $y = k$ and the parabola with equation $y = 2(x-3)(x-5)$, then we can solve for x_A and x_B by equating values of y to obtain the equation $k = 2(x-3)(x-5)$, which is equivalent to $k = 2(x^2 - 8x + 15)$ or $2x^2 - 16x + (30 - k) = 0$.
Using the quadratic formula, we obtain

$$x_A, x_B = \frac{16 \pm \sqrt{(-16)^2 - 4(2)(30-k)}}{2(2)}$$

Thus, $x_A = \dfrac{16 - \sqrt{16^2 - 4(2)(30-k)}}{2(2)}$ and $x_B = \dfrac{16 + \sqrt{16^2 - 4(2)(30-k)}}{2(2)}$.
Since $x_B - x_A = 6$, then

$$\begin{aligned}
\frac{16 + \sqrt{16^2 - 4(2)(30-k)}}{2(2)} - \frac{16 - \sqrt{16^2 - 4(2)(30-k)}}{2(2)} &= 6 \\
\frac{2\sqrt{16^2 - 4(2)(30-k)}}{2(2)} &= 6 \\
\sqrt{256 - (240 - 8k)} &= 12 \\
\sqrt{16 + 8k} &= 12 \\
16 + 8k &= 144 \\
8k &= 128 \\
k &= 16
\end{aligned}$$

Therefore, $k = 16$.
We can double check that the line with equation $y = 16$ intersects the parabola with equation $y = 2(x-3)(x-5)$ at the points $(1, 16)$ and $(7, 16)$, which are a distance 6 apart.

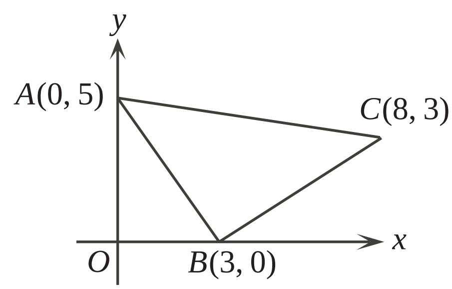
(b) Let $n = (3a + 6a + 9a + 12a + 15a) + (6b + 12b + 18b + 24b + 30b)$.
First, we simplify the given expression for n to obtain

$$n = (3a + 6a + 9a + 12a + 15a) + (6b + 12b + 18b + 24b + 30b) = 45a + 90b$$

We then factor the right side to obtain $n = 45(a + 2b) = 3^2 5^1 (a + 2b)$.
If $a + 2b = 5$, then $n = 3^2 5^2 = (3 \times 5)^2$, which is a perfect square.
Two pairs of positive integers (a, b) that satisfy $a + 2b = 5$ are $(a, b) = (3, 1)$ and $(a, b) = (1, 2)$.
Another value of $a + 2b$ for which n is a perfect square is $a + 2b = 20$, since here $n = 3^2 5^1 20 = 3^2 5^1 2^2 5^1 = 3^2 2^2 5^2 = (3 \times 2 \times 5)^2$.
A pair of positive integers (a, b) that satisfies $a + 2b = 20$ is $(18, 1)$.
Therefore, three pairs of positive integers (a, b) with the required property are $(3, 1)$, $(1, 2)$, $(18, 1)$.
(There are infinitely many other pairs with this property.)

5. (a) *Solution 1*
First, we calculate the side lengths of $\triangle ABC$:

$$\begin{aligned} AB &= \sqrt{(0-3)^2 + (5-0)^2} = \sqrt{34} \\ BC &= \sqrt{(3-8)^2 + (0-3)^2} = \sqrt{34} \\ AC &= \sqrt{(0-8)^2 + (5-3)^2} = \sqrt{68} \end{aligned}$$



Since $AB = BC$ and $AC = \sqrt{2}AB = \sqrt{2}BC$, then $\triangle ABC$ is an isosceles right-angled triangle, with the right angle at B.
Therefore, $\angle ACB = 45^\circ$.

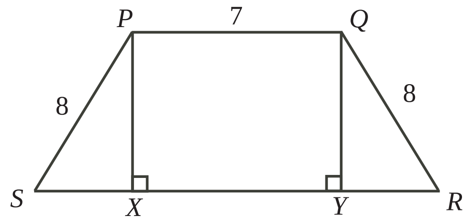
Solution 2
As in Solution 1, $AB = BC = \sqrt{34}$.
Line segment AB has slope $\frac{5-0}{0-3} = -\frac{5}{3}$.
Line segment BC has slope $\frac{0-3}{3-8} = \frac{3}{5}$.
Since the product of these two slopes is -1, then AB and BC are perpendicular.
Therefore, $\triangle ABC$ is right-angled at B.
Since $AB = BC$, then $\triangle ABC$ is an isosceles right-angled triangle, so $\angle ACB = 45^\circ$.

Solution 3
As in Solution 1, $AB = BC = \sqrt{34}$ and $AC = \sqrt{68}$.
Using the cosine law,

$$\begin{aligned} AB^2 &= AC^2 + BC^2 - 2(AC)(BC)\cos(\angle ACB) \\ 34 &= 68 + 34 - 2(\sqrt{68})(\sqrt{34})\cos(\angle ACB) \\ 0 &= 68 - 2(\sqrt{2}\sqrt{34})(\sqrt{34})\cos(\angle ACB) \\ 0 &= 68 - 68\sqrt{2}\cos(\angle ACB) \\ 68\sqrt{2}\cos(\angle ACB) &= 68 \\ \cos(\angle ACB) &= \frac{1}{\sqrt{2}} \end{aligned}$$

Since $\cos(\angle ACB) = \frac{1}{\sqrt{2}}$ and $0^\circ < \angle ACB < 180^\circ$, then $\angle ACB = 45^\circ$.

(b) Draw perpendiculars from P and Q to X and Y, respectively, on SR.



Since PQ is parallel to SR (because $PQRS$ is a trapezoid) and PX and QY are perpendicular to SR, then $PQYX$ is a rectangle.
Thus, $XY = PQ = 7$ and $PX = QY$.
Since $\triangle PXS$ and $\triangle QYR$ are right-angled with $PS = QR$ and $PX = QY$, then these triangles are congruent, and so $SX = YR$.
Since $XY = 7$ and $SR = 15$, then $SX + 7 + YR = 15$ or $2 \times SX = 8$ and so $SX = 4$.
By the Pythagorean Theorem in $\triangle PXS$,

$$PX^2 = PS^2 - SX^2 = 8^2 - 4^2 = 64 - 16 = 48$$

Now PR is the hypotenuse of right-angled $\triangle PXR$.
Since $PR > 0$, then by the Pythagorean Theorem,

$$PR = \sqrt{PX^2 + XR^2} = \sqrt{48 + (7+4)^2} = \sqrt{48 + 11^2} = \sqrt{48 + 121} = \sqrt{169} = 13$$

Therefore, $PR = 13$.

6. (a) *Solution 1*

There are two possibilities: either each player wins three games or one player wins more games than the other.
Since the probability that each player wins three games is $\frac{5}{16}$, then the probability that any one player wins more games than the other is $1 - \frac{5}{16} = \frac{11}{16}$.
Since each of Blaise and Pierre is equally likely to win any given game, then each must be equally likely to win more games than the other.
Therefore, the probability that Blaise wins more games than Pierre is $\frac{1}{2} \times \frac{11}{16} = \frac{11}{32}$.

Solution 2

We consider the results of the 6 games as a sequence of 6 Bs or Ps, with each letter a B if Blaise wins the corresponding game or P if Pierre wins.
Since the two players are equally skilled, then the probability that each wins a given game is $\frac{1}{2}$. This means that the probability of each letter being a B is $\frac{1}{2}$ and the probability of each letter being a P is also $\frac{1}{2}$.
Since each sequence consists of 6 letters, then the probability of a particular sequence occurring is $(\frac{1}{2})^6 = \frac{1}{64}$, because each of the letters is specified.
Since they play 6 games in total, then the probability that Blaise wins more games than Pierre is the sum of the probabilities that Blaise wins 4 games, that Blaise wins 5 games, and that Blaise wins 6 games.
If Blaise wins 6 games, then the sequence consists of 6 Bs. The probability of this is $\frac{1}{64}$, since there is only one way to arrange 6 Bs.
If Blaise wins 5 games, then the sequence consists of 5 Bs and 1 P. The probability of this is $6 \times \frac{1}{64} = \frac{6}{64}$, since there are 6 possible positions in the list for the 1 P (eg. PBBBBB,

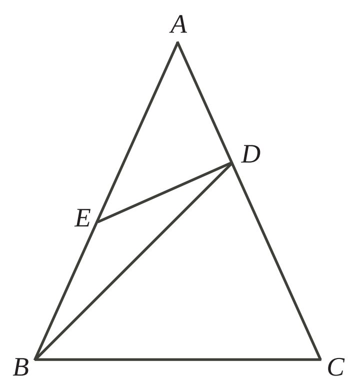
BPBBBB, BBPBBB, BBBPBB, BBBBPB, BBBBBP).
The probability that Blaise wins 4 games is $\binom{6}{2} \times \frac{1}{64} = \frac{15}{64}$, since there are $\binom{6}{2} = 15$ ways for 4 Bs and 2 Ps to be arranged.
Therefore, the probability that Blaise wins more games than Pierre is $\frac{1}{64} + \frac{6}{64} + \frac{15}{64} = \frac{22}{64} = \frac{11}{32}$.

(b) Using exponent rules and arithmetic, we manipulate the given equation:

$$\begin{aligned} 3^{x+2} + 2^{x+2} + 2^x &= 2^{x+5} + 3^x \\ 3^x 3^2 + 2^x 2^2 + 2^x &= 2^x 2^5 + 3^x \\ 9(3^x) + 4(2^x) + 2^x &= 32(2^x) + 3^x \\ 8(3^x) &= 27(2^x) \\ \frac{3^x}{2^x} &= \frac{27}{8} \\ \left(\frac{3}{2}\right)^x &= \left(\frac{3}{2}\right)^3 \end{aligned}$$

Since the two expressions are equal and the bases are equal, then the exponents must be equal, so $x = 3$.

7. (a) Since $AB = AC$, then $\triangle ABC$ is isosceles and $\angle ABC = \angle ACB$. Note that $\angle BAC = \theta$.

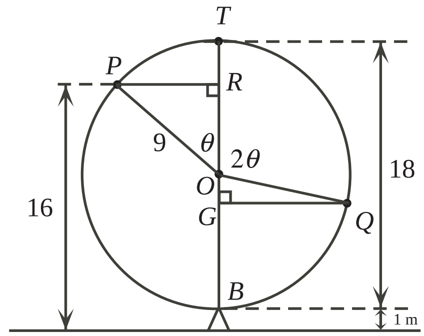


The angles in $\triangle ABC$ add to 180°, so $\angle ABC + \angle ACB + \angle BAC = 180^\circ$.
Thus, $2\angle ACB + \theta = 180^\circ$ or $\angle ABC = \angle ACB = \frac{1}{2}(180^\circ - \theta) = 90^\circ - \frac{1}{2}\theta$.
Now $\triangle BCD$ is isosceles as well with $BC = BD$ and so $\angle CDB = \angle DCB = 90^\circ - \frac{1}{2}\theta$.
Since the angles in $\triangle BCD$ add to 180°, then

$$\angle CBD = 180^\circ - \angle DCB - \angle CDB = 180^\circ - (90^\circ - \tfrac{1}{2}\theta) - (90^\circ - \tfrac{1}{2}\theta) = \theta$$

Now $\angle EBD = \angle ABC - \angle DBC = (90^\circ - \frac{1}{2}\theta) - \theta = 90^\circ - \frac{3}{2}\theta$.
Since $BE = ED$, then $\angle EDB = \angle EBD = 90^\circ - \frac{3}{2}\theta$.
Therefore, $\angle BED = 180^\circ - \angle EBD - \angle EDB = 180^\circ - (90^\circ - \frac{3}{2}\theta) - (90^\circ - \frac{3}{2}\theta) = 3\theta$.

(b) Let O be the centre of the ferris wheel and B the lowest point on the wheel.
Since the radius of the ferris wheel is 9 m (half of the diameter of 18 m) and B is 1 m above the ground, then O is $9 + 1 = 10$ m above the ground.
Let $\angle TOP = \theta$.



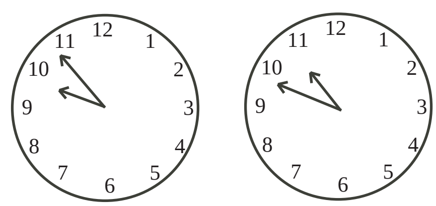
Since the ferris wheel rotates at a constant speed, then in 8 seconds, the angle through which the wheel rotates is twice the angle through which it rotates in 4 seconds. In other words, $\angle TOQ = 2\theta$.
Draw a perpendicular from P to R on TB and from Q to G on TB.
Since P is 16 m above the ground and O is 10 m above the ground, then $OR = 6$ m.
Since OP is a radius of the circle, then $OP = 9$ m.

Looking at right-angled $\triangle ORP$, we see that $\cos\theta = \dfrac{OR}{OP} = \dfrac{6}{9} = \dfrac{2}{3}$.
Since $\cos\theta = \frac{2}{3} < \frac{1}{\sqrt{2}} = \cos(45^\circ)$, then $\theta > 45^\circ$.
This means that $2\theta > 90^\circ$, which means that Q is below the horizontal diameter through O and so G is below O.
Since $\angle TOQ = 2\theta$, then $\angle QOG = 180^\circ - 2\theta$.
Kolapo's height above the ground at Q equals 1 m plus the length of BG.
Now $BG = OB - OG$. We know that $OB = 9$ m.
Also, considering right-angled $\triangle QOG$, we have

$$OG = OQ\cos(\angle QOG) = 9\cos(180^\circ - 2\theta) = -9\cos(2\theta) = -9(2\cos^2\theta - 1)$$

Since $\cos\theta = \frac{2}{3}$, then $OG = -9(2(\frac{2}{3})^2 - 1) = -9(\frac{8}{9} - 1) = 1$ m.
Therefore, $BG = 9 - 1 = 8$ m and so Q is $1 + 8 = 9$ m above the ground.

8. (a) *Solution 1*
The hour hand and minute hand both turn at constant rates. Since the hour hand moves $\frac{1}{12}$ of the way around the clock in 1 hour and the minute hand moves all of the way around the clock in 1 hour, then the minute hand turns 12 times as quickly as the hour hand.



Before *After*

Suppose also that the hour hand moves through an angle of x° between Before and After. Therefore, the minute hand moves through an angle of $(360^\circ - x^\circ)$ between Before and After, since these two angles add to 360°.

Since the minute hand moves 12 times as quickly as the hour hand, then $\dfrac{360^\circ - x^\circ}{x^\circ} = 12$ or $360 - x = 12x$ and so $13x = 360$, or $x = \frac{360}{13}$.
In one hour, the hour hand moves through $\frac{1}{12} \times 360^\circ = 30^\circ$.
Since the hour hand is moving for t hours, then we have $30^\circ t = (\frac{360}{13})^\circ$ and so $t = \frac{360}{30(13)} = \frac{12}{13}$.

Solution 2
Suppose that Jimmy starts painting x hours after 9:00 a.m. and finishes painting y hours after 10:00 a.m., where $0 < x < 1$ and $0 < y < 1$.
Since t is the amount of time in hours that he spends painting, then $t = (1-x)+y$, because he paints for $(1-x)$ hours until 10:00 a.m., and then for y hours until his finishing time.
The hour hand and minute hand both turn at constant rates.
The minute hand turns 360° in one hour and the hour hand turns $\frac{1}{12} \times 360^\circ = 30^\circ$ in one hour.
Thus, in x hours, where $0 < x < 1$, the minute hand turns $(360x)^\circ$ and the hour hand turns $(30x)^\circ$.
In the Before picture, the minute hand is $(360x)^\circ$ clockwise from the 12 o'clock position.
In the After picture, the minute hand is $(360y)^\circ$ clockwise from the 12 o'clock position.
The 9 is $9 \times 30^\circ = 270^\circ$ clockwise from the 12 o'clock position and the 10 is $10 \times 30^\circ = 300^\circ$ clockwise from the 12 o'clock position.
Therefore, in the Before picture, the hour hand is $270^\circ + (30x)^\circ$ clockwise from the 12 o'clock position, and in the After picture, the hour hand is $300^\circ + (30y)^\circ$ clockwise from the 12 o'clock position.
Because the hour and minute hands have switched places from the Before to the After positions, then we can equate the corresponding positions to obtain $(360x)^\circ = 300^\circ + (30y)^\circ$ (or $360x = 300 + 30y$) and $(360y)^\circ = 270^\circ + (30x)^\circ$ (or $360y = 270 + 30x$).
Dividing both equations by 30, we obtain $12x = 10 + y$ and $12y = 9 + x$.
Subtracting the second equation from the first, we obtain $12x - 12y = 10 + y - 9 - x$ or $-1 = 13y - 13x$.
Therefore, $y - x = -\frac{1}{13}$ and so $t = (1-x) + y = 1 + y - x = 1 - \frac{1}{13} = \frac{12}{13}$.

(b) We manipulate the given equation into a sequence of equivalent equations:

$$\begin{aligned}
\log_{5x+9}(x^2+6x+9) + \log_{x+3}(5x^2+24x+27) &= 4 \\
\frac{\log(x^2+6x+9)}{\log(5x+9)} + \frac{\log(5x^2+24x+27)}{\log(x+3)} &= 4 \qquad \text{(using the “change of base” formula)} \\
\frac{\log((x+3)^2)}{\log(5x+9)} + \frac{\log((5x+9)(x+3))}{\log(x+3)} &= 4 \qquad \text{(factoring)} \\
\frac{2\log(x+3)}{\log(5x+9)} + \frac{\log(5x+9)+\log(x+3)}{\log(x+3)} &= 4 \qquad \text{(using logarithm rules)} \\
2\left(\frac{\log(x+3)}{\log(5x+9)}\right) + \frac{\log(5x+9)}{\log(x+3)} + \frac{\log(x+3)}{\log(x+3)} &= 4 \qquad \text{(rearranging fractions)}
\end{aligned}$$

Making the substitution $t = \dfrac{\log(x+3)}{\log(5x+9)}$, we obtain successively

$$\begin{aligned} 2t + \frac{1}{t} + 1 &= 4 \\ 2t^2 + 1 + t &= 4t \\ 2t^2 - 3t + 1 &= 0 \\ (2t-1)(t-1) &= 0 \end{aligned}$$

Therefore, $t = 1$ or $t = \frac{1}{2}$.
If $\dfrac{\log(x+3)}{\log(5x+9)} = 1$, then $\log(x+3) = \log(5x+9)$ or $x + 3 = 5x + 9$, which gives $4x = -6$ or $x = -\frac{3}{2}$.
If $\dfrac{\log(x+3)}{\log(5x+9)} = \dfrac{1}{2}$, then $2\log(x+3) = \log(5x+9)$ or $\log((x+3)^2) = \log(5x+9)$ or $(x+3)^2 = 5x + 9$.
Here, $x^2 + 6x + 9 = 5x + 9$ or $x^2 + x = 0$ or $x(x+1) = 0$, and so $x = 0$ or $x = -1$.
Therefore, there are three possible values for x: $x = 0$, $x = -1$ and $x = -\frac{3}{2}$.
We should check each of these in the original equation.
If $x = 0$, the left side of the original equation is $\log_9 9 + \log_3 27 = 1 + 3 = 4$.
If $x = -1$, the left side of the original equation is $\log_4 4 + \log_2 8 = 1 + 3 = 4$.
If $x = -\frac{3}{2}$, the left side of the original equation is $\log_{3/2}(9/4) + \log_{3/2}(9/4) = 2 + 2 = 4$.
Therefore, the solutions are $x = 0, -1, -\frac{3}{2}$.

9. (a) Suppose that the auditorium with these properties has r rows and c columns of chairs.
Then there are rc chairs in total.
Each chair is empty, is occupied by a boy, or is occupied by a girl.
Since there are 14 boys in each row, then there are $14r$ chairs occupied by boys.
Since there are 10 girls in each column, then there are $10c$ chairs occupied by girls.
Since there are exactly 3 empty chairs, then the total number of chairs can also be written as $14r + 10c + 3$.
Therefore, $rc = 14r + 10c + 3$.
We proceed to find all pairs of positive integers r and c that satisfy this equation. We note that since there are 14 boys in each row, then there must be at least 14 columns (that is, $c \geq 14$) and since there are 10 girls in each column, then there must be at least 10 rows (that is, $r \geq 10$).
Manipulating the equation,

$$\begin{aligned} rc &= 14r + 10c + 3 \\ rc - 14r &= 10c + 3 \\ r(c-14) &= 10c + 3 \\ r &= \frac{10c+3}{c-14} \\ r &= \frac{10c - 140 + 143}{c-14} \\ r &= \frac{10c-140}{c-14} + \frac{143}{c-14} \\ r &= 10 + \frac{143}{c-14} \end{aligned}$$

Since r is an integer, then $10 + \dfrac{143}{c-14}$ is an integer, so $\dfrac{143}{c-14}$ must be an integer.
Therefore, $c - 14$ is a divisor of 143. Since $c \geq 14$, then $c - 14 \geq 0$, so $c - 14$ is a positive divisor of 143.
Since $143 = 11 \times 13$, then its positive divisors are $1, 11, 13, 143$.
We make a table of the possible values of $c - 14$ along with the resulting values of c, r (calculated using $r = 10 + \dfrac{143}{c-14}$) and rc:

$c - 14$	c	r	rc
1	15	153	2295
11	25	23	575
13	27	21	567
143	157	11	1727

Therefore, the four possible values for rc are $567, 575, 1727, 2295$. That is, the smallest possible number of chairs in the auditorium is 567.
(Can you create a grid with 27 columns and 21 rows that has the required properties?)

(b) *Solution 1*
We use the notation $|PMQN|$ to represent the area of quadrilateral $|PMQN|$, $|\triangle APD|$ to represent the area of $\triangle APD$, and so on.
We want to show that $|PMQN| = |\triangle APD| + |\triangle BQC|$.
This is equivalent to showing

$$|PMQN| + |\triangle DPN| + |\triangle CQN| = |\triangle APD| + |\triangle DPN| + |\triangle BQC| + |\triangle CQN|$$

which is equivalent to showing

$$|\triangle DMC| = |\triangle DAN| + |\triangle CBN|$$

since combining quadrilateral $PMQN$ with $\triangle DPN$ and $\triangle CQN$ gives $\triangle DMC$, combining $\triangle APD$ with $\triangle DPN$ gives $\triangle DAN$, and combining $\triangle BQC$ with $\triangle CQN$ gives $\triangle CBN$.
Suppose that DC has length x and DN has length tx for some t with $0 < t < 1$.
Then $NC = DC - DN = x - tx = (1 - t)x$.
Suppose also that the height of A above DC is a, the height of B above DC is b and the height of M above DC is m.

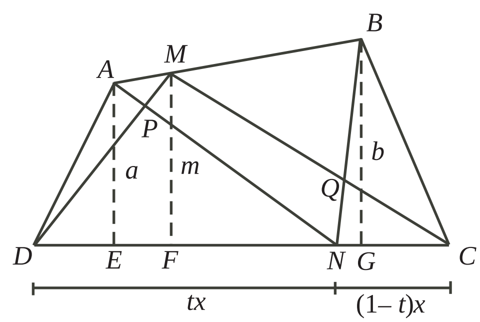


Figure 1

Then $|\triangle DAN| = \frac{1}{2}(tx)(a)$ and $|\triangle CBN| = \frac{1}{2}((1-t)x)b$ so

$$|\triangle DAN| + |\triangle CBN| = \tfrac{1}{2}(txa + (1-t)xb) = \tfrac{1}{2}x(ta + (1-t)b)$$

Also, $|\triangle DMC| = \frac{1}{2}xm$.
In order to prove that $|\triangle DMC| = |\triangle DAN| + |\triangle CBN|$, we need to show that $\frac{1}{2}xm$ equals

$\frac{1}{2}x(ta+(1-t)b)$ which is equivalent to showing that m is equal to $ta+(1-t)b$.

In Figure 2, we draw a horizontal line from A to BG, meeting MF at R and BG at S.

Since MF and BG are vertical and ARS is horizontal, then these line segments are perpendicular.

Since $AE=a$, $MF=m$ and $BG=b$, then $MR=m-a$ and $BS=b-a$.

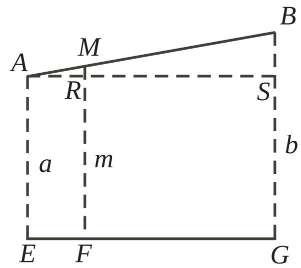


Figure 2

Now $\triangle ARM$ is similar to $\triangle ASB$, since each is right-angled and they share a common angle at A.

Therefore, $\dfrac{MR}{BS}=\dfrac{AM}{AB}=\dfrac{NC}{DC}$.

Since $MR=m-a$ and $BS=b-a$, then $\dfrac{MR}{BS}=\dfrac{m-a}{b-a}$.

Since $\dfrac{AM}{AB}=\dfrac{NC}{DC}$, then $\dfrac{MR}{BS}=\dfrac{(1-t)x}{x}=1-t$.

Comparing these two expressions, we obtain $\dfrac{m-a}{b-a}=(1-t)$ or $m-a=(b-a)(1-t)$ or $m=a+b(1-t)+(t-1)a=ta+(1-t)b$, as required.

This concludes the proof, and so $|PMQN|=|\triangle APD|+|\triangle BQC|$, as required.

Solution 2

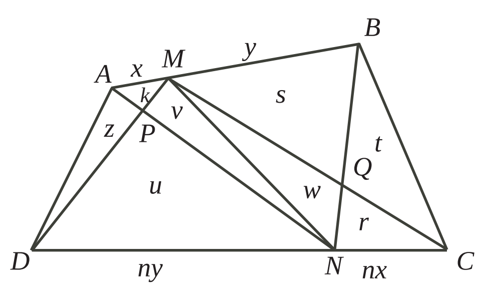
Let $AM=x$ and $MB=y$. Then $AB=x+y$ and so $\dfrac{AM}{AB}=\dfrac{x}{x+y}$.

Let $NC=nx$ for some real number n.

Since $\dfrac{NC}{DC}=\dfrac{AM}{AB}$, then $\dfrac{nx}{DC}=\dfrac{x}{x+y}$ and so $DC=n(x+y)$.

This tells us that $DN=DC-NC=n(x+y)-nx=ny$.

Join M to N and label the areas as shown in the diagram:

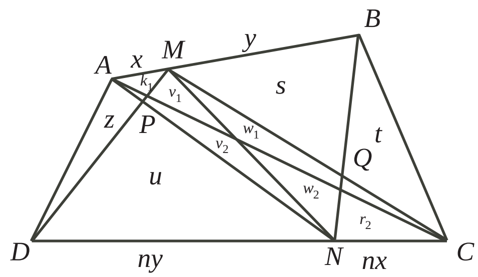


We repeatedly use the fact that triangles with a common height have areas in proportion to the lengths of their bases.

For example, $\triangle MDN$ and $\triangle MNC$ have a common height from line segment to DC to M and so the ratio of their areas equals the ratio of the lengths of their bases.

In other words, $\dfrac{w+r}{u+v}=\dfrac{nx}{ny}=\dfrac{x}{y}$. Thus, $w+r=\dfrac{x}{y}(u+v)$.

Also, the ratio of the area of $\triangle NAM$ to the area of $\triangle NMB$ equals the ratio of AM to MB.
This gives $\dfrac{k+v}{s+w} = \dfrac{x}{y}$ or $k+v = \dfrac{x}{y}(s+w)$.
Next, we join A to C and relabel the areas divided by this new line segment as shown:



(The unlabelled triangle adjacent to the one labelled k_1 has area k_2 and the unlabelled triangle adjacent to the one labelled r_2 has area r_1.)
Consider $\triangle ANC$ and $\triangle ADN$.
As above, the ratio of their areas equals the ratio of their bases.
Thus, $\dfrac{k_2+v_2+w_2+r_2}{z+u} = \dfrac{nx}{ny} = \dfrac{x}{y}$, and so $k_2+v_2+w_2+r_2 = \dfrac{x}{y}(z+u)$.
Consider $\triangle CAM$ and $\triangle CMB$.
As above, the ratio of their areas equals the ratio of their bases.
Thus, $\dfrac{k_1+v_1+w_1+r_1}{s+t} = \dfrac{x}{y}$, and so $k_1+v_1+w_1+r_1 = \dfrac{x}{y}(s+t)$.
Adding $k_2+v_2+w_2+r_2 = \dfrac{x}{y}(z+u)$ and $k_1+v_1+w_1+r_1 = \dfrac{x}{y}(s+t)$ gives

$$(k_1+k_2)+(v_1+v_2)+(w_1+w_2)+(r_1+r_2) = \frac{x}{y}(s+t+z+u)$$

or

$$k+v+w+r = \frac{x}{y}(s+t+z+u)$$

Since $k+v = \dfrac{x}{y}(s+w)$ and $w+r = \dfrac{x}{y}(u+v)$, then

$$\frac{x}{y}(s+w) + \frac{x}{y}(u+v) = \frac{x}{y}(s+t+z+u)$$

which gives

$$s+w+u+v = s+t+z+u$$

or

$$w+v = t+z$$

But $w+v$ is the area of quadrilateral $PMQN$, z is the area of $\triangle APD$ and t is the area of $\triangle BQC$. In other words, the area of quadrilateral $PMQN$ equals the sum of the areas of $\triangle APD$ and $\triangle PQC$, as required.

10. (a) The Eden sequences from $\{1,2,3,4,5\}$ are

$$1 \quad 3 \quad 5 \quad 1,2 \quad 1,4 \quad 3,4 \quad 1,2,3 \quad 1,2,5 \quad 1,4,5 \quad 3,4,5 \quad 1,2,3,4 \quad 1,2,3,4,5$$

There are 12 such sequences.

We present a brief justification of why these are all of the sequences.

* An Eden sequence of length 1 consists of a single odd integer. The possible choices are 1 and 3 and 5.
* An Eden sequence of length 2 consists of an odd integer followed by a larger even integer. Since the only possible even integers here are 2 and 4, then the possible sequences are 1, 2 and 1, 4 and 3, 4.
* An Eden sequence of length 3 starts with an Eden sequence of length 2 and appends (that is, adds to the end) a larger odd integer. Starting with 1,2, we form 1,2,3 and 1,2,5. Starting with 1,4, we form 1,4,5. Starting with 3,4, we form 3,4,5.
* An Eden sequence of length 4 starts with an Eden sequence of length 3 and appends a larger even integer. Since 2 and 4 are the only possible even integers, then the only possible sequence here is 1,2,3,4.
* An Eden sequence of length 5 from $\{1,2,3,4,5\}$ must include all 5 elements, so is 1,2,3,4,5.

(b) We will prove that, for all positive integers $n \geq 3$, we have $e(n) = e(n-1) + e(n-2) + 1$.
Thus, if $e(18) = m$, then $e(19) = e(18) + e(17) + 1 = m + 4181$ and

$$e(20) = e(19) + e(18) + 1 = (m + 4181) + m + 1$$

Since $e(20) = 17710$, then $17710 = 2m + 4182$ or $2m = 13528$ and so $m = 6764$.
Therefore, $e(18) = 6764$ and $e(19) = 6764 + 4181 = 10945$.

So we must prove that, for all positive integers $n \geq 3$, we have $e(n) = e(n-1)+e(n-2)+1$.

To simplify the reading, we use a number of abbreviations:

* ES means "Eden sequence"
* ES(m) means "Eden sequence from $\{1, 2, 3, \ldots, m\}$
* ESE and ESO mean "Eden sequence of even length" and "Eden sequence of odd length", respectively
* ESE(m) and ESO(m) mean "Eden sequence of even length from $\{1, 2, 3, \ldots, m\}$" and "Eden sequence of odd length from $\{1, 2, 3, \ldots, m\}$", respectively

Method 1

For each positive integer n, let $A(n)$ be the number of ESE(n), and let $B(n)$ be the number of ESO(n).
Then $e(n) = A(n) + B(n)$ for each positive integer n.
Note also that for each positive integer $n \geq 2$, we have $e(n) \geq e(n-1)$ and $A(n) \geq A(n-1)$ and $B(n) \geq B(n-1)$. This is because every ES($n-1$) is also an ES(n) because it satisfies the three required conditions. So there are at least as many ES(n) as there are ES($n-1$). (The same argument works to show that there are at least as many ESE(n) as there are ESE($n-1$), and at least as many ESO(n) as there are ESO($n-1$).)

Note that if k is a positive integer, then $2k + 1$ is odd and $2k$ is even.
The following four facts are true for every positive integer $k \geq 1$:

(i) $A(2k+1) = A(2k)$
(ii) $B(2k) = B(2k-1)$
(iii) $A(2k) = A(2k-1) + B(2k-1)$
(iv) $B(2k+1) = A(2k) + B(2k) + 1$

Here are justifications for these facts:

(i) An ESE must end with an even integer. Thus, an ESE$(2k+1)$ cannot include $2k+1$, since it would then have to include a larger even positive integer, which it cannot.
Therefore, an ESE$(2k+1)$ has largest term at most $2k$ and so is an ES$(2k)$.
Thus, $A(2k+1) \leq A(2k)$.
But from above, $A(2k+1) \geq A(2k)$, and so $A(2k+1) = A(2k)$.

(ii) An ESO must end with an odd integer. Thus, an ESO$(2k)$ cannot include $2k$, since it would then have to include a larger odd positive integer, which it cannot.
Therefore, an ESO$(2k)$ has largest term at most $2k-1$ and so is an ESO$(2k-1)$.
Thus, $B(2k) \leq B(2k-1)$.
But from above, $B(2k) \geq B(2k-1)$, and so $B(2k) = B(2k-1)$.

(iii) An ESE$(2k)$ either includes $2k$ or does not include $2k$.
If such a sequence includes $2k$, then removing the $2k$ produces an ESO$(2k-1)$. Also, every ESO$(2k-1)$ can be produced in this way.
Therefore, the number of sequences in this case is $B(2k-1)$.
If such a sequence does not include $2k$, then the sequence can be thought of as an ESE$(2k-1)$. Note that every ESE$(2k-1)$ is an ESE$(2k)$.
Therefore, the number of sequences in this case is $A(2k-1)$.
Thus, $A(2k) = A(2k-1) + B(2k-1)$.

(iv) An ESO$(2k+1)$ is either the one term sequence $2k+1$, or includes $2k+1$ and more terms, or does not include $2k+1$.
There is 1 sequence of the first kind.
As in (iii), there are $A(2k)$ sequences of the second kind and $B(2k)$ sequences of the third kind.
Thus, $B(2k+1) = 1 + A(2k) + B(2k)$.

Combining these facts, for each positive integer k, we obtain

$$\begin{aligned} e(2k+1) &= A(2k+1) + B(2k+1) \\ &= A(2k) + (A(2k) + B(2k) + 1) \\ &= (A(2k) + B(2k)) + A(2k) + 1 \\ &= e(2k) + (A(2k-1) + B(2k-1)) + 1 \\ &= e(2k) + e(2k-1) + 1 \end{aligned}$$

and

$$\begin{aligned} e(2k) &= A(2k) + B(2k) \\ &= (A(2k-1) + B(2k-1)) + B(2k-1) \\ &= e(2k-1) + (A(2k-2) + B(2k-2) + 1) \\ &= e(2k-1) + e(2k-2) + 1 \end{aligned}$$

Therefore, for all positive integers $n \geq 3$, we have $e(n) = e(n-1) + e(n-2) + 1$, as required.

Method 2

Let n be a positive integer with $n \geq 3$, and consider the ES(n).
We divide the sequences into three sets:

(i) The sequence 1 (there is 1 such sequence)

(ii) The sequences which begin with 1 and have more than 1 term

(iii) The sequences which do not begin with 1

We show that in case (ii) there are $e(n-1)$ sequences and in case (iii) there are $e(n-2)$ sequences. This will show that $e(n) = 1 + e(n-1) + e(n-2)$, as required.

(ii) Consider the set of ES(n) that begin with 1. We call this set of sequences P.
We remove the 1 from each of these and consider the set of resulting sequences. We call this set Q. Note that the number of sequences in P and in Q is the same.
Each of the sequences in Q includes numbers from the set $\{2, 3, \ldots, n\}$, is increasing, and has even terms in odd positions and odd terms in even positions (since each term has been shifted one position to the left).
The sequences in Q are in a one-to-one correspondence with the ES($n-1$) (we call this set of sequences R) and so there are exactly $e(n-1)$ of them (and so $e(n-1)$ sequences in P).
We can show that this one-to-one correspondence exists by subtracting 1 from each term of each sequence in Q, to form a set of sequences S. Each of the resulting sequences is distinct, includes numbers from the set $\{1, 2, 3, \ldots, n-1\}$, is increasing, and has odd terms in odd positions and even terms in even positions (since each term has been reduced by 1). Also, each sequence in R can be obtained in this way (since adding 1 to each term in one of these ES gives a distinct sequence in Q).
Therefore, the number of sequences in this case is $e(n-1)$.

(iii) Consider the set of ES(n) that do not begin with 1. We call this set of sequences T.
Since each sequence in T does not begin with 1, then the minimum number in each sequence is 3.
Thus, each of the sequences in T includes numbers from the set $\{3, 4, \ldots, n\}$, is increasing, and has odd terms in odd positions and even terms in even positions.
The sequences in T are in a one-to-one correspondence with the ES($n-2$) (we call this set of sequences U) and so there are exactly $e(n-2)$ of them.
We can show that this one-to-one correspondence exists by subtracting 2 from each term of each sequence in T, to form a set of sequences V. Each of the resulting sequences is distinct, includes numbers from the set $\{1, 2, 3, \ldots, n-2\}$, is increasing, and has odd terms in odd positions and even terms in even positions (since each term has been reduced by 2). Also, each sequence in U can be obtained in this way (since adding 2 to each term in one of these ES gives a distinct sequence in U).
Therefore, the number of sequences in this case is $e(n-2)$.

This concludes our proof and shows that $e(n) = 1 + e(n-1) + e(n-2)$, as required.

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

2011 Euclid Contest

Tuesday, April 12, 2011

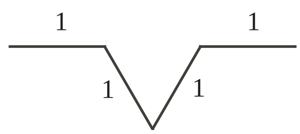
Solutions

1. (a) Since $(x+1)+(x+2)+(x+3)=8+9+10$, then $3x+6=27$ or $3x=21$ and so $x=7$.

   (b) Since $\sqrt{25+\sqrt{x}}=6$, then squaring both sides gives $25+\sqrt{x}=36$ or $\sqrt{x}=11$.
   Since $\sqrt{x}=11$, then squaring both sides again, we obtain $x=11^2=121$.
   Checking, $\sqrt{25+\sqrt{121}}=\sqrt{25+11}=\sqrt{36}=6$, as required.

   (c) Since $(a,2)$ is the point of intersection of the lines with equations $y=2x-4$ and $y=x+k$, then the coordinates of this point must satisfy both equations.
   Using the first equation, $2=2a-4$ or $2a=6$ or $a=3$.
   Since the coordinates of the point $(3,2)$ satisfy the equation $y=x+k$, then $2=3+k$ or $k=-1$.

2. (a) Since the side length of the original square is 3 and an equilateral triangle of side length 1 is removed from the middle of each side, then each of the two remaining pieces of each side of the square has length 1.
   Also, each of the two sides of each of the equilateral triangles that are shown has length 1.



   Therefore, each of the 16 line segments in the figure has length 1, and so the perimeter of the figure is 16.

   (b) Since $DC=DB$, then $\triangle CDB$ is isosceles and $\angle DBC=\angle DCB=15^\circ$.
   Thus, $\angle CDB=180^\circ-\angle DBC-\angle DCB=150^\circ$.
   Since the angles around a point add to 360°, then

$$\angle ADC=360^\circ-\angle ADB-\angle CDB=360^\circ-130^\circ-150^\circ=80^\circ\ .$$

   (c) By the Pythagorean Theorem in $\triangle EAD$, we have $EA^2+AD^2=ED^2$ or $12^2+AD^2=13^2$, and so $AD=\sqrt{169-144}=5$, since $AD>0$.
   By the Pythagorean Theorem in $\triangle ACD$, we have $AC^2+CD^2=AD^2$ or $AC^2+4^2=5^2$, and so $AC=\sqrt{25-16}=3$, since $AC>0$.
   (We could also have determined the lengths of AD and AC by recognizing 3-4-5 and 5-12-13 right-angled triangles.)
   By the Pythagorean Theorem in $\triangle ABC$, we have $AB^2+BC^2=AC^2$ or $AB^2+2^2=3^2$, and so $AB=\sqrt{9-4}=\sqrt{5}$, since $AB>0$.

3. (a) *Solution 1*
   Since we want to make $15-\dfrac{y}{x}$ as large as possible, then we want to subtract as little as possible from 15.
   In other words, we want to make $\dfrac{y}{x}$ as small as possible.
   To make a fraction with positive numerator and denominator as small as possible, we make the numerator as small as possible and the denominator as large as possible.
   Since $2\leq x\leq 5$ and $10\leq y\leq 20$, then we make $x=5$ and $y=10$.
   Therefore, the maximum value of $15-\dfrac{y}{x}$ is $15-\dfrac{10}{5}=13$.

Solution 2
Since y is positive and $2 \leq x \leq 5$, then $15 - \dfrac{y}{x} \leq 15 - \dfrac{y}{5}$ for any x with $2 \leq x \leq 5$ and positive y.
Since $10 \leq y \leq 20$, then $15 - \dfrac{y}{5} \leq 15 - \dfrac{10}{5}$ for any y with $10 \leq y \leq 20$.
Therefore, for any x and y in these ranges, $15 - \dfrac{y}{x} \leq 15 - \dfrac{10}{5} = 13$, and so the maximum possible value is 13 (which occurs when $x = 5$ and $y = 10$).

(b) *Solution 1*
First, we add the two given equations to obtain

$$(f(x) + g(x)) + (f(x) - g(x)) = (3x + 5) + (5x + 7)$$

or $2f(x) = 8x + 12$ which gives $f(x) = 4x + 6$.
Since $f(x) + g(x) = 3x + 5$, then $g(x) = 3x + 5 - f(x) = 3x + 5 - (4x + 6) = -x - 1$.
(We could also find $g(x)$ by subtracting the two given equations or by using the second of the given equations.)
Since $f(x) = 4x + 6$, then $f(2) = 14$.
Since $g(x) = -x - 1$, then $g(2) = -3$.
Therefore, $2f(2)g(2) = 2 \times 14 \times (-3) = -84$.

Solution 2
Since the two given equations are true for all values of x, then we can substitute $x = 2$ to obtain

$$\begin{aligned} f(2) + g(2) &= 11 \\ f(2) - g(2) &= 17 \end{aligned}$$

Next, we add these two equations to obtain $2f(2) = 28$ or $f(2) = 14$.
Since $f(2) + g(2) = 11$, then $g(2) = 11 - f(2) = 11 - 14 = -3$.
(We could also find $g(2)$ by subtracting the two equations above or by using the second of these equations.)
Therefore, $2f(2)g(2) = 2 \times 14 \times (-3) = -84$.

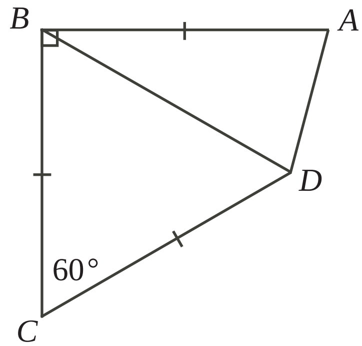
4. (a) We consider choosing the three numbers all at once.
We list the possible sets of three numbers that can be chosen:

$$\{1,2,3\}\ \{1,2,4\}\ \{1,2,5\}\ \{1,3,4\}\ \{1,3,5\}\ \{1,4,5\}\ \{2,3,4\}\ \{2,3,5\}\ \{2,4,5\}\ \{3,4,5\}$$

We have listed each in increasing order because once the numbers are chosen, we arrange them in increasing order.
There are 10 sets of three numbers that can be chosen.
Of these 10, the 4 sequences $1, 2, 3$ and $1, 3, 5$ and $2, 3, 4$ and $3, 4, 5$ are arithmetic sequences.
Therefore, the probability that the resulting sequence is an arithmetic sequence is $\frac{4}{10}$ or $\frac{2}{5}$.

(b) *Solution 1*

Join B to D.



Consider $\triangle CBD$.
Since $CB = CD$, then $\angle CBD = \angle CDB = \frac{1}{2}(180^\circ - \angle BCD) = \frac{1}{2}(180^\circ - 60^\circ) = 60^\circ$.
Therefore, $\triangle BCD$ is equilateral, and so $BD = BC = CD = 6$.
Consider $\triangle DBA$.
Note that $\angle DBA = 90^\circ - \angle CBD = 90^\circ - 60^\circ = 30^\circ$.
Since $BD = BA = 6$, then $\angle BDA = \angle BAD = \frac{1}{2}(180^\circ - \angle DBA) = \frac{1}{2}(180^\circ - 30^\circ) = 75^\circ$.
We calculate the length of AD.

<u>Method 1</u>

By the Sine Law in $\triangle DBA$, we have $\dfrac{AD}{\sin(\angle DBA)} = \dfrac{BA}{\sin(\angle BDA)}$.

Therefore, $AD = \dfrac{6\sin(30^\circ)}{\sin(75^\circ)} = \dfrac{6 \times \frac{1}{2}}{\sin(75^\circ)} = \dfrac{3}{\sin(75^\circ)}$.

<u>Method 2</u>

If we drop a perpendicular from B to P on AD, then P is the midpoint of AD since $\triangle BDA$ is isosceles. Thus, $AD = 2AP$.
Also, BP bisects $\angle DBA$, so $\angle ABP = 15^\circ$.
Now, $AP = BA\sin(\angle ABP) = 6\sin(15^\circ)$.
Therefore, $AD = 2AP = 12\sin(15^\circ)$.

<u>Method 3</u>

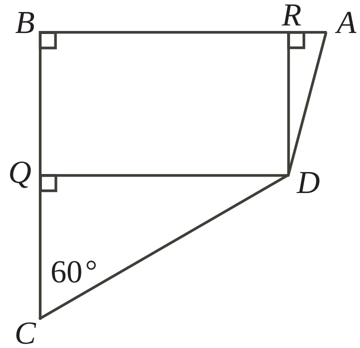
By the Cosine Law in $\triangle DBA$,

$$\begin{aligned} AD^2 &= AB^2 + BD^2 - 2(AB)(BD)\cos(\angle ABD) \\ &= 6^2 + 6^2 - 2(6)(6)\cos(30^\circ) \\ &= 72 - 72(\tfrac{\sqrt{3}}{2}) \\ &= 72 - 36\sqrt{3} \end{aligned}$$

Therefore, $AD = \sqrt{36(2-\sqrt{3})} = 6\sqrt{2-\sqrt{3}}$ since $AD > 0$.

Solution 2

Drop perpendiculars from D to Q on BC and from D to R on BA.



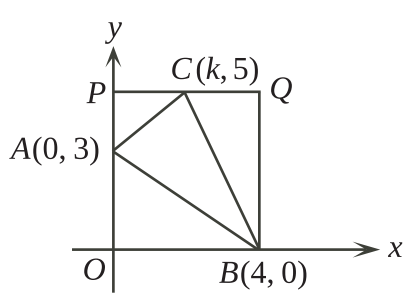
Then $CQ = CD\cos(\angle DCQ) = 6\cos(60^\circ) = 6 \times \frac{1}{2} = 3$.
Also, $DQ = CD\sin(\angle DCQ) = 6\sin(60^\circ) = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$.
Since $BC = 6$, then $BQ = BC - CQ = 6 - 3 = 3$.
Now quadrilateral $BQDR$ has three right angles, so it must have a fourth right angle and so must be a rectangle.
Thus, $RD = BQ = 3$ and $RB = DQ = 3\sqrt{3}$.
Since $AB = 6$, then $AR = AB - RB = 6 - 3\sqrt{3}$.
Since $\triangle ARD$ is right-angled at R, then using the Pythagorean Theorem and the fact that $AD > 0$, we obtain

$$AD = \sqrt{RD^2 + AR^2} = \sqrt{3^2 + (6 - 3\sqrt{3})^2} = \sqrt{9 + 36 - 36\sqrt{3} + 27} = \sqrt{72 - 36\sqrt{3}}$$

which we can rewrite as $AD = \sqrt{36(2 - \sqrt{3})} = 6\sqrt{2 - \sqrt{3}}$.

5. (a) Let n be the original number and N be the number when the digits are reversed. Since we are looking for the largest value of n, we assume that $n > 0$.
Since we want N to be 75% larger than n, then N should be 175% of n, or $N = \frac{7}{4}n$.
Suppose that the tens digit of n is a and the units digit of n is b. Then $n = 10a + b$.
Also, the tens digit of N is b and the units digit of N is a, so $N = 10b + a$.
We want $10b + a = \frac{7}{4}(10a + b)$ or $4(10b + a) = 7(10a + b)$ or $40b + 4a = 70a + 7b$ or $33b = 66a$, and so $b = 2a$.
This tells us that that any two-digit number $n = 10a + b$ with $b = 2a$ has the required property.
Since both a and b are digits then $b < 10$ and so $a < 5$, which means that the possible values of n are 12, 24, 36, and 48.
The largest of these numbers is 48.

(b) We "complete the rectangle" by drawing a horizontal line through C which meets the y-axis at P and the vertical line through B at Q.



Since C has y-coordinate 5, then P has y-coordinate 5; thus the coordinates of P are $(0,5)$.
Since B has x-coordinate 4, then Q has x-coordinate 4.
Since C has y-coordinate 5, then Q has y-coordinate 5.
Therefore, the coordinates of Q are $(4,5)$, and so rectangle $OPQB$ is 4 by 5 and so has area $4 \times 5 = 20$.
Now rectangle $OPQB$ is made up of four smaller triangles, and so the sum of the areas of these triangles must be 20.
Let us examine each of these triangles:

- $\triangle ABC$ has area 8 (given information)
- $\triangle AOB$ is right-angled at O, has height $AO = 3$ and base $OB = 4$, and so has area $\frac{1}{2} \times 4 \times 3 = 6$.
- $\triangle APC$ is right-angled at P, has height $AP = 5 - 3 = 2$ and base $PC = k - 0 = k$, and so has area $\frac{1}{2} \times k \times 2 = k$.
- $\triangle CQB$ is right-angled at Q, has height $QB = 5 - 0 = 5$ and base $CQ = 4 - k$, and so has area $\frac{1}{2} \times (4 - k) \times 5 = 10 - \frac{5}{2}k$.

Since the sum of the areas of these triangles is 20, then $8 + 6 + k + 10 - \frac{5}{2}k = 20$ or $4 = \frac{3}{2}k$ and so $k = \frac{8}{3}$.

6. (a) *Solution 1*

Suppose that the distance from point A to point B is d km.
Suppose also that r_c is the speed at which Serge travels while not paddling (i.e. being carried by just the current), that r_p is the speed at which Serge travels with no current (i.e. just from his paddling), and r_{p+c} his speed when being moved by both his paddling and the current.
It takes Serge 18 minutes to travel from A to B while paddling with the current.
Thus, $r_{p+c} = \dfrac{d}{18}$ km/min.
It takes Serge 30 minutes to travel from A to B with just the current.
Thus, $r_c = \dfrac{d}{30}$ km/min.
But $r_p = r_{p+c} - r_c = \dfrac{d}{18} - \dfrac{d}{30} = \dfrac{5d}{90} - \dfrac{3d}{90} = \dfrac{2d}{90} = \dfrac{d}{45}$ km/min.
Since Serge can paddle the d km from A to B at a speed of $\dfrac{d}{45}$ km/min, then it takes him 45 minutes to paddle from A to B with no current.

Solution 2

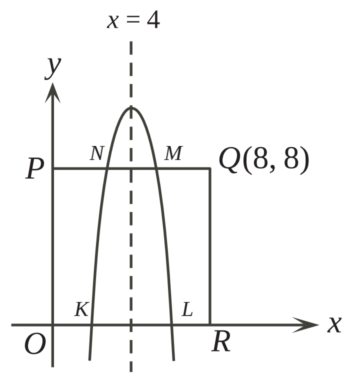
Suppose that the distance from point A to point B is d km, the speed of the current of the river is r km/h, and the speed that Serge can paddle is s km/h.
Since the current can carry Serge from A to B in 30 minutes (or $\dfrac{1}{2}$ h), then $\dfrac{d}{r} = \dfrac{1}{2}$.
When Serge paddles with the current, his speed equals his paddling speed plus the speed of the current, or $(s + r)$ km/h.
Since Serge can paddle with the current from A to B in 18 minutes (or $\dfrac{3}{10}$ h), then $\dfrac{d}{r+s} = \dfrac{3}{10}$.
The time to paddle from A to B with no current would be $\dfrac{d}{s}$ h.

Since $\dfrac{d}{r} = \dfrac{1}{2}$, then $\dfrac{r}{d} = 2$.
Since $\dfrac{d}{r+s} = \dfrac{3}{10}$, then $\dfrac{r+s}{d} = \dfrac{10}{3}$.
Therefore, $\dfrac{s}{d} = \dfrac{r+s}{d} - \dfrac{r}{d} = \dfrac{10}{3} - 2 = \dfrac{4}{3}$.
Thus, $\dfrac{d}{s} = \dfrac{3}{4}$, and so it would take Serge $\dfrac{3}{4}$ of an hour, or 45 minutes, to paddle from A to B with no current.

Solution 3
Suppose that the distance from point A to point B is d km, the speed of the current of the river is r km/h, and the speed that Serge can paddle is s km/h.
Since the current can carry Serge from A to B in 30 minutes (or $\dfrac{1}{2}$ h), then $\dfrac{d}{r} = \dfrac{1}{2}$ or $d = \frac{1}{2}r$.
When Serge paddles with the current, his speed equals his paddling speed plus the speed of the current, or $(s + r)$ km/h.
Since Serge can paddle with the current from A to B in 18 minutes (or $\dfrac{3}{10}$ h), then $\dfrac{d}{r+s} = \dfrac{3}{10}$ or $d = \frac{3}{10}(r + s)$.
Since $d = \frac{1}{2}r$ and $d = \frac{3}{10}(r + s)$, then $\frac{1}{2}r = \frac{3}{10}(r + s)$ or $5r = 3r + 3s$ and so $s = \frac{2}{3}r$.
To travel from A to B with no current, the time in hours that it takes is $\dfrac{d}{s} = \dfrac{\frac{1}{2}r}{\frac{2}{3}r} = \dfrac{3}{4}$, or 45 minutes.

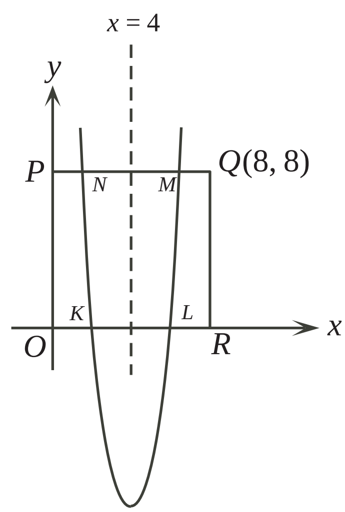
(b) First, we note that $a \neq 0$. (If $a = 0$, then the "parabola" $y = a(x - 2)(x - 6)$ is actually the horizontal line $y = 0$ which intersects the square all along OR.)
Second, we note that, regardless of the value of $a \neq 0$, the parabola has x-intercepts 2 and 6, and so intersects the x-axis at $(2, 0)$ and $(6, 0)$, which we call $K(2, 0)$ and $L(6, 0)$. This gives $KL = 4$.
Third, we note that since the x-intercepts of the parabola are 2 and 6, then the axis of symmetry of the parabola has equation $x = \frac{1}{2}(2 + 6) = 4$.
Since the axis of symmetry of the parabola is a vertical line of symmetry, then if the parabola intersects the two vertical sides of the square, it will intersect these at the same height, and if the parabola intersects the top side of the square, it will intersect it at two points that are symmetrical about the vertical line $x = 4$.
Fourth, we recall that a trapezoid with parallel sides of lengths a and b and height h has area $\frac{1}{2}h(a + b)$.
We now examine three cases.

Case 1: $a < 0$
Here, the parabola opens downwards.
Since the parabola intersects the square at four points, it must intersect PQ at points M and N. (The parabola cannot intersect the vertical sides of the square since it gets "narrower" towards the vertex.)



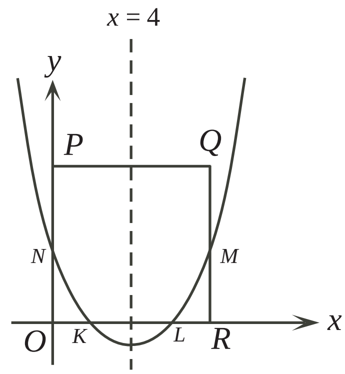
Since the parabola opens downwards, then $MN < KL = 4$.
Since the height of the trapezoid equals the height of the square (or 8), then the area of the trapezoid is $\frac{1}{2}h(KL + MN)$ which is less than $\frac{1}{2}(8)(4+4) = 32$.
But the area of the trapezoid must be 36, so this case is not possible.

Case 2: $a > 0$; M and N on PQ
We have the following configuration:



Here, the height of the trapezoid is 8, $KL = 4$, and M and N are symmetric about $x = 4$.
Since the area of the trapezoid is 36, then $\frac{1}{2}h(KL + MN) = 36$ or $\frac{1}{2}(8)(4 + MN) = 36$ or $4 + MN = 9$ or $MN = 5$.
Thus, M and N are each $\frac{5}{2}$ units from $x = 4$, and so N has coordinates $(\frac{3}{2}, 8)$.
Since this point lies on the parabola with equation $y = a(x-2)(x-6)$, then $8 = a(\frac{3}{2} - 2)(\frac{3}{2} - 6)$ or $8 = a(-\frac{1}{2})(-\frac{9}{2})$ or $8 = \frac{9}{4}a$ or $a = \frac{32}{9}$.

Case 3: $a > 0$; M and N on QR and PO
We have the following configuration:



Here, $KL = 4$, $MN = 8$, and M and N have the same y-coordinate.
Since the area of the trapezoid is 36, then $\frac{1}{2}h(KL + MN) = 36$ or $\frac{1}{2}h(4 + 8) = 36$ or $6h = 36$ or $h = 6$.
Thus, N has coordinates $(0, 6)$.
Since this point lies on the parabola with equation $y = a(x - 2)(x - 6)$, then $6 = a(0 - 2)(0 - 6)$ or $6 = 12a$ or $a = \frac{1}{2}$.

Therefore, the possible values of a are $\frac{32}{9}$ and $\frac{1}{2}$.

7. (a) *Solution 1*

Consider a population of 100 people, each of whom is 75 years old and who behave according to the probabilities given in the question.
Each of the original 100 people has a 50% chance of living at least another 10 years, so there will be $50\% \times 100 = 50$ of these people alive at age 85.
Each of the original 100 people has a 20% chance of living at least another 15 years, so there will be $20\% \times 100 = 20$ of these people alive at age 90.
Since there is a 25% (or $\frac{1}{4}$) chance that an 80 year old person will live at least another 10 years (that is, to age 90), then there should be 4 times as many of these people alive at age 80 than at age 90.
Since there are 20 people alive at age 90, then there are $4 \times 20 = 80$ of the original 100 people alive at age 80.
In summary, of the initial 100 people of age 75, there are 80 alive at age 80, 50 alive at age 85, and 20 people alive at age 90.
Because 50 of the 80 people alive at age 80 are still alive at age 85, then the probability that an 80 year old person will live at least 5 more years (that is, to age 85) is $\frac{50}{80} = \frac{5}{8}$, or 62.5%.

Solution 2

Suppose that the probability that a 75 year old person lives to 80 is p, the probability that an 80 year old person lives to 85 is q, and the probability that an 85 year old person lives to 90 is r.
We want to the determine the value of q.
For a 75 year old person to live at least another 10 years, they must live another 5 years (to age 80) and then another 5 years (to age 85). The probability of this is equal to pq. We are told in the question that this is equal to 50% or 0.5.
Therefore, $pq = 0.5$.

For a 75 year old person to live at least another 15 years, they must live another 5 years (to age 80), then another 5 years (to age 85), and then another 5 years (to age 90). The probability of this is equal to pqr. We are told in the question that this is equal to 20% or 0.2.
Therefore, $pqr = 0.2$
Similarly, since the probability that an 80 year old person will live another 10 years is 25%, then $qr = 0.25$.
Since $pqr = 0.2$ and $pq = 0.5$, then $r = \frac{pqr}{pq} = \frac{0.2}{0.5} = 0.4$.
Since $qr = 0.25$ and $r = 0.4$, then $q = \frac{qr}{r} = \frac{0.25}{0.4} = 0.625$.
Therefore, the probability that an 80 year old man will live at least another 5 years is 0.625, or 62.5%.

(b) Using logarithm rules, the given equation is equivalent to $2^{2\log_{10} x} = 3(2 \cdot 2^{\log_{10} x}) + 16$ or $(2^{\log_{10} x})^2 = 6 \cdot 2^{\log_{10} x} + 16$.
Set $u = 2^{\log_{10} x}$. Then the equation becomes $u^2 = 6u + 16$ or $u^2 - 6u - 16 = 0$.
Factoring, we obtain $(u - 8)(u + 2) = 0$ and so $u = 8$ or $u = -2$.
Since $2^a > 0$ for any real number a, then $u > 0$ and so we can reject the possibility that $u = -2$.
Thus, $u = 2^{\log_{10} x} = 8$ which means that $\log_{10} x = 3$.
Therefore, $x = 1000$.

8. (a) First, we determine the first entry in the 50th row.
Since the first column is an arithmetic sequence with common difference 3, then the 50th entry in the first column (the first entry in the 50th row) is $4 + 49(3) = 4 + 147 = 151$.
Second, we determine the common difference in the 50th row by determining the second entry in the 50th row.
Since the second column is an arithmetic sequence with common difference 5, then the 50th entry in the second column (that is, the second entry in the 50th row) is $7 + 49(5)$ or $7 + 245 = 252$.
Therefore, the common difference in the 50th row must be $252 - 151 = 101$.
Thus, the 40th entry in the 50th row (that is, the number in the 50th row and the 40th column) is $151 + 39(101) = 151 + 3939 = 4090$.

(b) We follow the same procedure as in (a).
First, we determine the first entry in the Rth row.
Since the first column is an arithmetic sequence with common difference 3, then the Rth entry in the first column (that is, the first entry in the Rth row) is $4 + (R - 1)(3)$ or $4 + 3R - 3 = 3R + 1$.
Second, we determine the common difference in the Rth row by determining the second entry in the Rth row.
Since the second column is an arithmetic sequence with common difference 5, then the Rth entry in the second column (that is, the second entry in the Rth row) is $7+(R-1)(5)$ or $7 + 5R - 5 = 5R + 2$.
Therefore, the common difference in the Rth row must be $(5R + 2) - (3R + 1) = 2R + 1$.
Thus, the Cth entry in the Rth row (that is, the number in the Rth row and the Cth column) is

$$3R + 1 + (C - 1)(2R + 1) = 3R + 1 + 2RC + C - 2R - 1 = 2RC + R + C$$

(c) Suppose that N is an entry in the table, say in the Rth row and Cth column.
From (b), then $N = 2RC + R + C$ and so $2N + 1 = 4RC + 2R + 2C + 1$.
Now $4RC + 2R + 2C + 1 = 2R(2C + 1) + 2C + 1 = (2R + 1)(2C + 1)$.
Since R and C are integers with $R \geq 1$ and $C \geq 1$, then $2R + 1$ and $2C + 1$ are each integers that are at least 3.
Therefore, $2N + 1 = (2R + 1)(2C + 1)$ must be composite, since it is the product of two integers that are each greater than 1.

9. (a) If $n = 2011$, then $8n - 7 = 16081$ and so $\sqrt{8n-7} \approx 126.81$.
Thus, $\dfrac{1+\sqrt{8n-7}}{2} \approx \dfrac{1+126.81}{2} \approx 63.9$.
Therefore, $g(2011) = 2(2011) + \left\lfloor \dfrac{1+\sqrt{8(2011)-7}}{2} \right\rfloor = 4022 + \lfloor 63.9 \rfloor = 4022 + 63 = 4085$.

(b) To determine a value of n for which $f(n) = 100$, we need to solve the equation

$$2n - \left\lfloor \frac{1+\sqrt{8n-7}}{2} \right\rfloor = 100 \qquad (*)$$

We first solve the equation

$$2x - \frac{1+\sqrt{8x-7}}{2} = 100 \qquad (**)$$

because the left sides of $(*)$ and $(**)$ do not differ by much and so the solutions are likely close together. We will try integers n in $(*)$ that are close to the solutions to $(**)$.
Manipulating $(**)$, we obtain

$$\begin{aligned} 4x - (1+\sqrt{8x-7}) &= 200 \\ 4x - 201 &= \sqrt{8x-7} \\ (4x-201)^2 &= 8x - 7 \\ 16x^2 - 1608x + 40401 &= 8x - 7 \\ 16x^2 - 1616x + 40408 &= 0 \\ 2x^2 - 202x + 5051 &= 0 \end{aligned}$$

By the quadratic formula,

$$x = \frac{202 \pm \sqrt{202^2 - 4(2)(5051)}}{2(2)} = \frac{202 \pm \sqrt{396}}{4} = \frac{101 \pm \sqrt{99}}{2}$$

and so $x \approx 55.47$ or $x \approx 45.53$.
We try $n = 55$, which is close to 55.47:

$$f(55) = 2(55) - \left\lfloor \frac{1+\sqrt{8(55)-7}}{2} \right\rfloor = 110 - \left\lfloor \frac{1+\sqrt{433}}{2} \right\rfloor$$

Since $\sqrt{433} \approx 20.8$, then $\dfrac{1+\sqrt{433}}{2} \approx 10.9$, which gives $\left\lfloor \dfrac{1+\sqrt{433}}{2} \right\rfloor = 10$.
Thus, $f(55) = 110 - 10 = 100$.
Therefore, a value of n for which $f(n) = 100$ is $n = 55$.

(c) We want to show that each positive integer m is in the range of f or the range of g, but not both.

To do this, we first try to better understand the "complicated" term of each of the functions – that is, the term involving the greatest integer function.

In particular, we start with a positive integer $k \geq 1$ and try to determine the positive integers n that give $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$.

By definition of the greatest integer function, the equation $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ is equivalent to the inequality $k \leq \dfrac{1+\sqrt{8n-7}}{2} < k+1$, from which we obtain the following set of equivalent inequalities

$$\begin{array}{ccccc} 2k & \leq & 1+\sqrt{8n-7} & < & 2k+2 \\ 2k-1 & \leq & \sqrt{8n-7} & < & 2k+1 \\ 4k^2-4k+1 & \leq & 8n-7 & < & 4k^2+4k+1 \\ 4k^2-4k+8 & \leq & 8n & < & 4k^2+4k+8 \\ \frac{1}{2}(k^2-k)+1 & \leq & n & < & \frac{1}{2}(k^2+k)+1 \end{array}$$

If we define $T_k = \frac{1}{2}k(k+1) = \frac{1}{2}(k^2+k)$ to be the kth triangular number for $k \geq 0$, then $T_{k-1} = \frac{1}{2}(k-1)(k) = \frac{1}{2}(k^2-k)$.

Therefore, $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ for $T_{k-1}+1 \leq n < T_k + 1$.

Since n is an integer, then $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ is true for $T_{k-1}+1 \leq n \leq T_k$.

When $k=1$, this interval is $T_0+1 \leq n \leq T_1$ (or $1 \leq n \leq 1$). When $k=2$, this interval is $T_1+1 \leq n \leq T_2$ (or $2 \leq n \leq 3$). When $k=3$, this interval is $T_2+1 \leq n \leq T_3$ (or $4 \leq n \leq 6$). As k ranges over all positive integers, these intervals include every positive integer n and do not overlap.

Therefore, we can determine the range of each of the functions f and g by examining the values $f(n)$ and $g(n)$ when n is in these intervals.

For each non-negative integer k, define $\mathcal{R}_k$ to be the set of integers greater than k^2 and less than or equal to $(k+1)^2$. Thus, $\mathcal{R}_k = \{k^2+1, k^2+2, \ldots, k^2+2k, k^2+2k+1\}$.

For example, $\mathcal{R}_0 = \{1\}$, $\mathcal{R}_1 = \{2,3,4\}$, $\mathcal{R}_2 = \{5,6,7,8,9\}$, and so on. Every positive integer occurs in exactly one of these sets.

Also, for each non-negative integer k define $\mathcal{S}_k = \{k^2+2, k^2+4, \ldots, k^2+2k\}$ and define $\mathcal{Q}_k = \{k^2+1, k^2+3, \ldots, k^2+2k+1\}$. For example, $\mathcal{S}_0 = \{\}$, $\mathcal{S}_1 = \{3\}$, $\mathcal{S}_2 = \{6,8\}$, $\mathcal{Q}_0 = \{1\}$, $\mathcal{Q}_1 = \{2,4\}$, $\mathcal{Q}_2 = \{5,7,9\}$, and so on. Note that $\mathcal{R}_k = \mathcal{Q}_k \cup \mathcal{S}_k$ so every positive integer occurs in exactly one $\mathcal{Q}_k$ or in exactly one $\mathcal{S}_k$, and that these sets do not overlap since no two $\mathcal{S}_k$'s overlap and no two $\mathcal{Q}_k$'s overlap and no $\mathcal{Q}_k$ overlaps with an $\mathcal{S}_k$.

We determine the range of the function g first.

For $T_{k-1}+1 \leq n \leq T_k$, we have $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k$ and so

$$\begin{array}{ccccc} 2T_{k-1}+2 & \leq & 2n & \leq & 2T_k \\ 2T_{k-1}+2+k & \leq & 2n+\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor & \leq & 2T_k+k \\ k^2-k+2+k & \leq & g(n) & \leq & k^2+k+k \\ k^2+2 & \leq & g(n) & \leq & k^2+2k \end{array}$$

Note that when n is in this interval and increases by 1, then the $2n$ term causes the value of $g(n)$ to increase by 2.
Therefore, for the values of n in this interval, $g(n)$ takes precisely the values k^2+2, $k^2+4, k^2+6, \ldots, k^2+2k$.
In other words, the range of g over this interval of its domain is precisely the set $\mathcal{S}_k$.
As k ranges over all positive integers (that is, as these intervals cover the domain of g), this tells us that the range of g is precisely the integers in the sets $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \ldots$.
(We could also include $\mathcal{S}_0$ in this list since it is the empty set.)

We note next that $f(1) = 2 - \left\lfloor \dfrac{1+\sqrt{8-7}}{2} \right\rfloor = 1$, the only element of $\mathcal{Q}_0$.

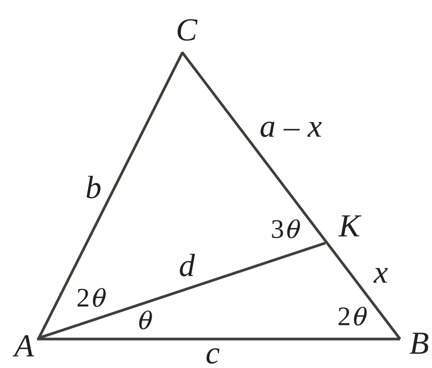
For $k \geq 1$ and $T_k + 1 \leq n \leq T_{k+1}$, we have $\left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor = k+1$ and so

$$\begin{array}{ccccc} 2T_k+2 & \leq & 2n & \leq & 2T_{k+1} \\ 2T_k+2-(k+1) & \leq & 2n - \left\lfloor \dfrac{1+\sqrt{8n-7}}{2} \right\rfloor & \leq & 2T_{k+1}-(k+1) \\ k^2+k+2-k-1 & \leq & f(n) & \leq & (k+1)(k+2)-k-1 \\ k^2+1 & \leq & f(n) & \leq & k^2+2k+1 \end{array}$$

Note that when n is in this interval and increases by 1, then the $2n$ term causes the value of $f(n)$ to increase by 2.
Therefore, for the values of n in this interval, $f(n)$ takes precisely the values k^2+1, $k^2+3, k^2+5, \ldots, k^2+2k+1$.
In other words, the range of f over this interval of its domain is precisely the set $\mathcal{Q}_k$.
As k ranges over all positive integers (that is, as these intervals cover the domain of f), this tells us that the range of f is precisely the integers in the sets $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$.

Therefore, the range of f is the set of elements in the sets $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \ldots$ and the range of g is the set of elements in the sets $\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2, \ldots$. These ranges include every positive integer and do not overlap.

10. (a) Suppose that $\angle KAB = \theta$.
Since $\angle KAC = 2\angle KAB$, then $\angle KAC = 2\theta$ and $\angle BAC = \angle KAC + \angle KAB = 3\theta$.
Since $3\angle ABC = 2\angle BAC$, then $\angle ABC = \frac{2}{3} \times 3\theta = 2\theta$.
Since $\angle AKC$ is exterior to $\triangle AKB$, then $\angle AKC = \angle KAB + \angle ABC = 3\theta$.
This gives the following configuration:



Now $\triangle CAK$ is similar to $\triangle CBA$ since the triangles have a common angle at C and $\angle CAK = \angle CBA$.

Therefore, $\dfrac{AK}{BA} = \dfrac{CA}{CB}$ or $\dfrac{d}{c} = \dfrac{b}{a}$ and so $d = \dfrac{bc}{a}$.

Also, $\dfrac{CK}{CA} = \dfrac{CA}{CB}$ or $\dfrac{a-x}{b} = \dfrac{b}{a}$ and so $a - x = \dfrac{b^2}{a}$ or $x = a - \dfrac{b^2}{a} = \dfrac{a^2-b^2}{a}$, as required.

(b) From (a), $bc = ad$ and $a^2 - b^2 = ax$ and so we obtain

$$\text{LS} = (a^2-b^2)(a^2-b^2+ac) = (ax)(ax+ac) = a^2x(x+c)$$

and

$$\text{RS} = b^2c^2 = (bc)^2 = (ad)^2 = a^2d^2$$

In order to show that LS = RS, we need to show that $x(x+c) = d^2$ (since $a > 0$).

Method 1: Use the Sine Law

First, we derive a formula for $\sin 3\theta$ which we will need in this solution:

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta+\theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta \\
&= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta \\
&= 2\sin\theta(1-\sin^2\theta) + (1-2\sin^2\theta)\sin\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

Since $\angle AKB = 180^\circ - \angle KAB - \angle KBA = 180^\circ - 3\theta$, then using the Sine Law in $\triangle AKB$ gives

$$\frac{x}{\sin\theta} = \frac{d}{\sin 2\theta} = \frac{c}{\sin(180^\circ - 3\theta)}$$

Since $\sin(180^\circ - X) = \sin X$, then $\sin(180^\circ - 3\theta) = \sin 3\theta$, and so $x = \dfrac{d\sin\theta}{\sin 2\theta}$ and $c = \dfrac{d\sin 3\theta}{\sin 2\theta}$. This gives

$$\begin{aligned}
x(x+c) &= \frac{d\sin\theta}{\sin 2\theta}\left(\frac{d\sin\theta}{\sin 2\theta} + \frac{d\sin 3\theta}{\sin 2\theta}\right) \\
&= \frac{d^2\sin\theta}{\sin^2 2\theta}(\sin\theta + \sin 3\theta) \\
&= \frac{d^2\sin\theta}{\sin^2 2\theta}(\sin\theta + 3\sin\theta - 4\sin^3\theta) \\
&= \frac{d^2\sin\theta}{\sin^2 2\theta}(4\sin\theta - 4\sin^3\theta) \\
&= \frac{4d^2\sin^2\theta}{\sin^2 2\theta}(1-\sin^2\theta) \\
&= \frac{4d^2\sin^2\theta\cos^2\theta}{\sin^2 2\theta} \\
&= \frac{4d^2\sin^2\theta\cos^2\theta}{(2\sin\theta\cos\theta)^2} \\
&= \frac{4d^2\sin^2\theta\cos^2\theta}{4\sin^2\theta\cos^2\theta} \\
&= d^2
\end{aligned}$$

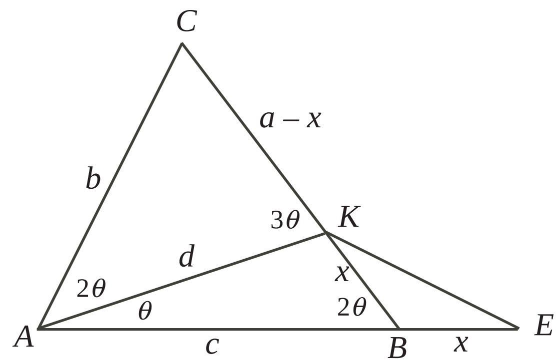
as required.

We could have instead used the formula $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ to show that $\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$, from which

$$\sin\theta(\sin 3\theta + \sin\theta) = \sin\theta(2\sin 2\theta\cos\theta) = 2\sin\theta\cos\theta\sin 2\theta = \sin^2 2\theta$$

Method 2: Extend AB

Extend AB to E so that $BE = BK = x$ and join KE.



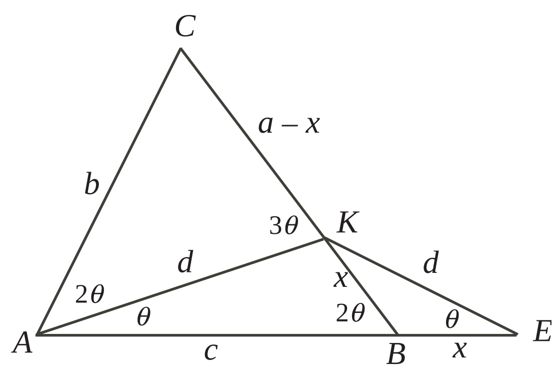
Now $\triangle KBE$ is isosceles with $\angle BKE = \angle KEB$.

Since $\angle KBA$ is the exterior angle of $\triangle KBE$, then $\angle KBA = 2\angle KEB = 2\theta$.

Thus, $\angle KEB = \angle BKE = \theta$.

But this also tells us that $\angle KAE = \angle KEA = \theta$.

Thus, $\triangle KAE$ is isosceles and so $KE = KA = d$.



So $\triangle KAE$ is similar to $\triangle BKE$, since each has two angles equal to θ.

Thus, $\frac{KA}{BK} = \frac{AE}{KE}$ or $\frac{d}{x} = \frac{c+x}{d}$ and so $d^2 = x(x+c)$, as required.

Method 3: Use the Cosine Law and the Sine Law

We apply the Cosine Law in $\triangle AKB$ to obtain

$$\begin{aligned} AK^2 &= BK^2 + BA^2 - 2(BA)(BK)\cos(\angle KBA) \\ d^2 &= x^2 + c^2 - 2cx\cos(2\theta) \\ d^2 &= x^2 + c^2 - 2cx(2\cos^2\theta - 1) \end{aligned}$$

Using the Sine Law in $\triangle AKB$, we get $\frac{x}{\sin\theta} = \frac{d}{\sin 2\theta}$ or $\frac{\sin 2\theta}{\sin\theta} = \frac{d}{x}$ or $\frac{2\sin\theta\cos\theta}{\sin\theta} = \frac{d}{x}$ and so $\cos\theta = \frac{d}{2x}$.

Combining these two equations,

$$\begin{aligned}
d^2 &= x^2 + c^2 - 2cx\left(\frac{2d^2}{4x^2} - 1\right) \\
d^2 &= x^2 + c^2 - \frac{cd^2}{x} + 2cx \\
d^2 + \frac{cd^2}{x} &= x^2 + 2cx + c^2 \\
d^2 + \frac{cd^2}{x} &= (x+c)^2 \\
xd^2 + cd^2 &= x(x+c)^2 \\
d^2(x+c) &= x(x+c)^2 \\
d^2 &= x(x+c)
\end{aligned}$$

as required (since $x + c \neq 0$).

(c) *Solution 1*

Our goal is to find a triple of positive integers that satisfy the equation in (b) and are the side lengths of a triangle.

First, we note that if (A, B, C) is a triple of real numbers that satisfies the equation in (b) and k is another real number, then the triple (kA, kB, kC) also satisfies the equation from (b), since

$$(k^2A^2 - k^2B^2)(k^2A^2 - k^2B^2 + kAkC) = k^4(A^2 - B^2)(A^2 - B^2 + AC) = k^4(B^2C^2) = (kB)^2(kC)^2$$

Therefore, we start by trying to find a triple (a, b, c) of rational numbers that satisfies the equation in (b) and forms a triangle, and then "scale up" this triple to form a triple (ka, kb, kc) of integers.

To do this, we rewrite the equation from (b) as a quadratic equation in c and solve for c using the quadratic formula.

Partially expanding the left side from (b), we obtain

$$(a^2 - b^2)(a^2 - b^2) + ac(a^2 - b^2) = b^2c^2$$

which we rearrange to obtain

$$b^2c^2 - c(a(a^2 - b^2)) - (a^2 - b^2)^2 = 0$$

By the quadratic formula,

$$c = \frac{a(a^2 - b^2) \pm \sqrt{a^2(a^2 - b^2)^2 + 4b^2(a^2 - b^2)^2}}{2b^2} = \frac{a(a^2 - b^2) \pm \sqrt{(a^2 - b^2)^2(a^2 + 4b^2)}}{2b^2}$$

Since $\angle BAC > \angle ABC$, then $a > b$ and so $a^2 - b^2 > 0$, which gives

$$c = \frac{a(a^2 - b^2) \pm (a^2 - b^2)\sqrt{a^2 + 4b^2}}{2b^2} = \frac{(a^2 - b^2)}{2b^2}(a \pm \sqrt{a^2 + 4b^2})$$

Since $a^2 + 4b^2 > 0$, then $\sqrt{a^2 + 4b^2} > a$, so the positive root is

$$c = \frac{(a^2 - b^2)}{2b^2}(a + \sqrt{a^2 + (2b)^2})$$

We try to find integers a and b that give a rational value for c. We will then check to see if this triple (a, b, c) forms the side lengths of a triangle, and then eventually scale these up to get integer values.
One way for the value of c to be rational (and in fact the only way) is for $\sqrt{a^2 + (2b)^2}$ to be an integer, or for a and $2b$ to be the legs of a Pythagorean triple.
Since $\sqrt{3^2 + 4^2}$ is an integer, then we try $a = 3$ and $b = 2$, which gives

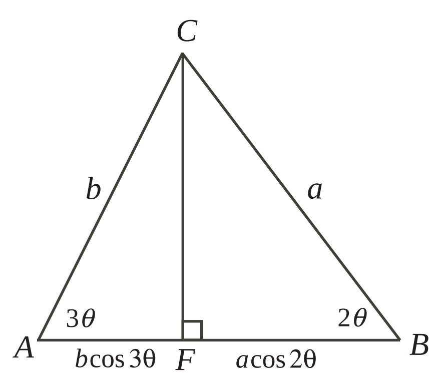
$$c = \frac{(3^2 - 2^2)}{2 \cdot 2^2}(3 + \sqrt{3^2 + 4^2}) = 5$$

and so $(a, b, c) = (3, 2, 5)$. Unfortunately, these lengths do not form a triangle, since $3 + 2 = 5$.
(The Triangle Inequality tells us that three positive real numbers a, b and c form a triangle if and only if $a + b > c$ and $a + c > b$ and $b + c > a$.)
We can continue to try small Pythagorean triples.
Now $15^2 + 8^2 = 17^2$, but $a = 15$ and $b = 4$ do not give a value of c that forms a triangle with a and b.
However, $16^2 + 30^2 = 34^2$, so we can try $a = 16$ and $b = 15$ which gives

$$c = \frac{(16^2 - 15^2)}{2 \cdot 15^2}(16 + \sqrt{16^2 + 30^2}) = \frac{31}{450}(16 + 34) = \frac{31}{9}$$

Now the lengths $(a, b, c) = (16, 15, \frac{31}{9})$ do form the sides of a triangle since $a + b > c$ and $a + c > b$ and $b + c > a$.
Since these values satisfy the equation from (b), then we can scale them up by a factor of $k = 9$ to obtain the triple $(144, 135, 31)$ which satisfies the equation from (b) and are the side lengths of a triangle.
(Using other Pythagorean triples, we could obtain other triples of integers that work.)

Solution 2
We note that the equation in (b) involves only a, b and c and so appears to depend only on the relationship between the angles $\angle CAB$ and $\angle CBA$ in $\triangle ABC$.
Using this premise, we use $\triangle ABC$, remove the line segment AK and draw the altitude CF.



Because we are only looking for one triple that works, we can make a number of assumptions that may or may not be true in general for such a triangle, but which will help us find an example.
We assume that 3θ and 2θ are both acute angles; that is, we assume that $\theta < 30^\circ$.
In $\triangle ABC$, we have $AF = b\cos 3\theta$, $BF = a\cos 2\theta$, and $CF = b\sin 3\theta = a\sin 2\theta$.
Note also that $c = b\cos 3\theta + a\cos 2\theta$.

One way to find the integers a, b, c that we require is to look for integers a and b and an angle θ with the properties that $b\cos 3\theta$ and $a\cos 2\theta$ are integers and $b\sin 3\theta = a\sin 2\theta$.
Using trigonometric formulae,

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= 2\cos^2\theta - 1 \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\
&\qquad \text{(from the calculation in (a), Solution 1, Method 1)} \\
\cos 3\theta &= \cos(2\theta+\theta) \\
&= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2\sin^2\theta\cos\theta \\
&= (2\cos^2\theta - 1)\cos\theta - 2(1-\cos^2\theta)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

So we can try to find an angle $\theta < 30^\circ$ with $\cos\theta$ a rational number and then integers a and b that make $b\sin 3\theta = a\sin 2\theta$ and ensure that $b\cos 3\theta$ and $a\cos 2\theta$ are integers.
Since we are assuming that $\theta < 30^\circ$, then $\cos\theta > \frac{\sqrt{3}}{2} \approx 0.866$.
The rational number with smallest denominator that is larger than $\frac{\sqrt{3}}{2}$ is $\frac{7}{8}$, so we try the acute angle θ with $\cos\theta = \frac{7}{8}$.
In this case, $\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{15}}{8}$, and so

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta = 2\times\tfrac{7}{8}\times\tfrac{\sqrt{15}}{8} = \tfrac{7\sqrt{15}}{32} \\
\cos 2\theta &= 2\cos^2\theta - 1 = 2\times\tfrac{49}{64} - 1 = \tfrac{17}{32} \\
\sin 3\theta &= 3\sin\theta - 4\sin^3\theta = 3\times\tfrac{\sqrt{15}}{8} - 4\times\tfrac{15\sqrt{15}}{512} = \tfrac{33\sqrt{15}}{128} \\
\cos 3\theta &= 4\cos^3\theta - 3\cos\theta = 4\times\tfrac{343}{512} - 3\times\tfrac{7}{8} = \tfrac{7}{128}
\end{aligned}$$

To have $b\sin 3\theta = a\sin 2\theta$, we need $\frac{33\sqrt{15}}{128}b = \frac{7\sqrt{15}}{32}a$ or $33b = 28a$.
To ensure that $b\cos 3\theta$ and $a\cos 2\theta$ are integers, we need $\frac{7}{128}b$ and $\frac{17}{32}a$ to be integers, and so a must be divisible by 32 and b must be divisible by 128.
The integers $a = 33$ and $b = 28$ satisfy the equation $33b = 28a$.
Multiplying each by 32 gives $a = 1056$ and $b = 896$ which satisfy the equation $33b = 28a$ and now have the property that b is divisible by 128 (with quotient 7) and a is divisible by 32 (with quotient 33).
With these values of a and b, we obtain $c = b\cos 3\theta + a\cos 2\theta = 896\times\frac{7}{128} + 1056\times\frac{17}{32} = 610$.
We can then check that the triple $(a, b, c) = (1056, 896, 610)$ satisfies the equation from (b), as required.
As in our discussion in Solution 1, each element of this triple can be divided by 2 to obtain the "smaller" triple $(a, b, c) = (528, 448, 305)$ that satisfies the equation too.
Using other values for $\cos\theta$ and integers a and b, we could obtain other triples (a, b, c) of integers that work.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2010 Euclid Contest

Wednesday, April 7, 2010

Solutions

1. (a) *Solution 1*
Since $3^x = 27$, then $3^{x+2} = 3^x 3^2 = 27 \cdot 9 = 243$.

Solution 2
Since $3^x = 27$ and $27 = 3^3$, then $x = 3$.
Therefore, $3^{x+2} = 3^5 = 243$.

(b) Since $2^5 3^{13} 5^9 x = 2^7 3^{14} 5^9$, then $x = \dfrac{2^7 3^{14} 5^9}{2^5 3^{13} 5^9} = 2^2 3^1 = 12$.

(c) The lines $y = x + 2$ and $y = -\frac{1}{2}x + 2$ both pass through the point B on the y-axis.
Since the y-intercept of the line $y = x + 2$ is 2, then B has coordinates $(0, 2)$.
Next, we find the x-intercepts of each of the two lines by setting $y = 0$.
If $y = x + 2$ and $y = 0$, then $x + 2 = 0$ or $x = -2$, so A has coordinates $(-2, 0)$.
If $y = -\frac{1}{2}x + 2$ and $y = 0$, then $0 = -\frac{1}{2}x + 2$ or $\frac{1}{2}x = 2$, and so $x = 4$.
Thus, C has coordinates $(4, 0)$.
Since BO and AC are perpendicular, then we can treat AC as the base of $\triangle ABC$ and BO as its height.
Note that $BO = 2$ and $AC = 4 - (-2) = 6$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2} \times AC \times BO = \frac{1}{2} \times 6 \times 2 = 6$.

2. (a) Let r, g and b be the masses of the red, green and blue packages, respectively.
We are told that $r + g + b = 60$, $r + g = 25$, and $g + b = 50$.
Subtracting the second equation from the first, we obtain $b = 60 - 25 = 35$.
Substituting into the third equation, we obtain $g = 50 - b = 50 - 35 = 15$.
Therefore, the mass of the green package is 15 kg.

(b) Suppose that a palindrome p is the sum of the three consecutive integers $a - 1$, a, $a + 1$.
In this case, $p = (a - 1) + a + (a + 1) = 3a$, so p is a multiple of 3.
The largest palindromes less than 200 are 191, 181, 171.
Note that 191 and 181 are not divisible by 3, but 171 is divisible by 3.
One way to check these without using a calculator is to use the test for divisibility by 3:

> A positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Therefore, 191 and 181 cannot be the sum of three consecutive integers.
The integer 171 can be written as $56 + 57 + 58$, so 171 is the largest palindrome less than 200 that is the sum of three consecutive integers.

(c) *Solution 1*
Since $(x + 1)(x - 1) = 8$, then $x^2 - 1 = 8$ or $x^2 = 9$.
Thus, $(x^2 + x)(x^2 - x) = x(x + 1)x(x - 1) = x^2(x + 1)(x - 1) = 9(8) = 72$.

Solution 2
Since $(x + 1)(x - 1) = 8$, then $x^2 - 1 = 8$ or $x^2 = 9$, so $x = \pm 3$.
If $x = 3$, then $(x^2 + x)(x^2 - x) = (3^2 + 3)(3^2 - 3) = (9 + 3)(9 - 3) = 12(6) = 72$.
If $x = -3$, then $(x^2 + x)(x^2 - x) = ((-3)^2 + (-3))((-3)^2 - (-3)) = (9 - 3)(9 + 3) = 72$.
In either case, $(x^2 + x)(x^2 - x) = 72$.

3. (a) *Solution 1*
Bea spends 60 minutes flying from H to F, 30 minutes at F, 45 minutes flying from F to G, 60 minutes at G, and then flies from G to H.

Thus, her total time is $60 + 30 + 45 + 60 = 195$ minutes plus the length of time that it takes her to fly from G to H.
Since Bea flies at a constant speed, then the ratio of the two distances equals the ratio of the corresponding times.
Therefore, $\dfrac{HF}{GF} = \dfrac{60 \text{ minutes}}{45 \text{ minutes}} = \dfrac{4}{3}$.
Since $\triangle FGH$ is right-angled at F, then $\triangle FGH$ must be similar to a 3-4-5 triangle, and so $\dfrac{HG}{GF} = \dfrac{5}{3}$.
In particular, this means that the ratio of the times flying H to G and F to G is also $\frac{5}{3}$.
Thus, it takes her $\frac{5}{3} \times 45 = 75$ minutes to fly from G to H.
In conclusion, Bea is away from her hive for $195 + 75 = 270$ minutes.

Solution 2
Bea spends 60 minutes flying from H to F, 30 minutes at F, 45 minutes flying from F to G, 60 minutes at G, and then flies from G to H.
Thus, her total time is $60 + 30 + 45 + 60 = 195$ minutes plus the length of time that it takes her to fly from G to H.
Since Bea flies at a constant speed, then the ratio of the two distances equals the ratio of the corresponding times.
Therefore, we can use the Pythagorean Theorem on the *times* to obtain

$$\text{Time } G \text{ to } H = \sqrt{(\text{Time } H \text{ to } F)^2 + (\text{Time } F \text{ to } G)^2} = \sqrt{60^2 + 45^2} = \sqrt{5625} = 75 \text{ min}$$

since the time is positive.
In conclusion, Bea is away from her hive for $195 + 75 = 270$ minutes.

(b) *Solution 1*
Since $\angle OPB = 90^\circ$, then OP and PB are perpendicular, so the product of their slopes is -1.
The slope of OP is $\dfrac{4-0}{p-0} = \dfrac{4}{p}$ and the slope of PB is $\dfrac{4-0}{p-10} = \dfrac{4}{p-10}$.
Therefore, we need

$$\begin{aligned} \frac{4}{p} \cdot \frac{4}{p-10} &= -1 \\ 16 &= -p(p-10) \\ p^2 - 10p + 16 &= 0 \\ (p-2)(p-8) &= 0 \end{aligned}$$

and so $p = 2$ or $p = 8$. Since each these steps is reversible, then $\triangle OPB$ is right-angled precisely when $p = 2$ and $p = 8$.

Solution 2
Since $\triangle OPB$ is right-angled at P, then $OP^2 + PB^2 = OB^2$ by the Pythagorean Theorem.
Note that $OB = 10$ since O has coordinates $(0, 0)$ and B has coordinates $(10, 0)$.
Also, $OP^2 = (p-0)^2 + (4-0)^2 = p^2 + 16$ and $PB^2 = (10-p)^2 + (4-0)^2 = p^2 - 20p + 116$.
Therefore,

$$\begin{aligned} (p^2 + 16) + (p^2 - 20p + 116) &= 10^2 \\ 2p^2 - 20p + 32 &= 0 \\ p^2 - 10p + 16 &= 0 \end{aligned}$$

and so $(p-2)(p-8)=0$, or $p=2$ or $p=8$. Since each these steps is reversible, then $\triangle OPB$ is right-angled precisely when $p=2$ and $p=8$.

4. (a) Suppose that Thurka bought x goats and y helicopters.
Then $19x+17y=201$.
Since x and y are non-negative integers, then $19x \leq 201$ so $x \leq 10$.
If $x=10$, then $17y=201-19x=11$, which does not have an integer solution because 11 is not divisible by 17.
If $x=9$, then $17y=201-19x=30$, which does not have an integer solution.
If $x=8$, then $17y=201-19x=49$, which does not have an integer solution.
If $x=7$, then $17y=201-19x=68$, so $y=4$.
Therefore, $19(7)+17(4)=201$, and so Thurka buys 7 goats and 4 helicopters.
(We can check that $x=0,1,2,3,4,5,6$ do not give values of y that work.)

(b) *Solution 1*
Manipulating algebraically,

$$\begin{aligned}
(x+8)^4 &= (2x+16)^2 \\
(x+8)^4-2^2(x+8)^2 &= 0 \\
(x+8)^2((x+8)^2-2^2) &= 0 \\
(x+8)^2((x+8)+2)((x+8)-2) &= 0 \\
(x+8)^2(x+10)(x+6) &= 0
\end{aligned}$$

Therefore, $x=-8$ or $x=-10$ or $x=-6$.

Solution 2
Manipulating algebraically,

$$\begin{aligned}
(x+8)^4 &= (2x+16)^2 \\
(x+8)^4-2^2(x+8)^2 &= 0 \\
(x+8)^2((x+8)^2-2^2) &= 0 \\
(x+8)^2(x^2+16x+64-4) &= 0 \\
(x+8)^2(x^2+16x+60) &= 0 \\
(x+8)^2(x+10)(x+6) &= 0
\end{aligned}$$

Therefore, $x=-8$ or $x=-10$ or $x=-6$.

Solution 3
Since $(x+8)^4=(2x+16)^2$, then $(x+8)^2=2x+16$ or $(x+8)^2=-(2x+16)$.
From the first equation, $x^2+16x+64=2x+16$ or $x^2+14x+48=0$ or $(x+6)(x+8)=0$.
From the second equation, $x^2+16x+64=-2x-16$ or $x^2+18x+80=0$ or $(x+10)(x+8)=0$.
Therefore, $x=-8$ or $x=-10$ or $x=-6$.

5. (a) *Solution 1*

We use the fact that $g(x) = g(f(f^{-1}(x)))$.
Since $f(x) = 2x + 1$, then to determine $f^{-1}(x)$ we solve $x = 2y + 1$ for y to get $2y = x - 1$ or $y = \frac{1}{2}(x - 1)$. Thus, $f^{-1}(x) = \frac{1}{2}(x - 1)$.
Since $g(f(x)) = 4x^2 + 1$, then

$$\begin{aligned} g(x) &= g(f(f^{-1}(x))) \\ &= g(f(\tfrac{1}{2}(x-1))) \\ &= 4(\tfrac{1}{2}(x-1))^2 + 1 \\ &= 4 \cdot \tfrac{1}{4}(x-1)^2 + 1 \\ &= (x-1)^2 + 1 \\ &= x^2 - 2x + 2 \end{aligned}$$

Solution 2

We use the expressions for $f(x)$ and $g(f(x))$ to construct $g(x)$.
Since $f(x)$ is linear and $g(f(x))$ is quadratic, then it is likely that $g(x)$ is also quadratic.
Since $f(x) = 2x + 1$, then $(f(x))^2 = 4x^2 + 4x + 1$.
Since $g(f(x))$ has no term involving x, then we subtract $2f(x)$ (to remove the $4x$ term) to get

$$(f(x))^2 - 2f(x) = (4x^2 + 4x + 1) - 2(2x + 1) = 4x^2 - 1$$

To get $g(f(x))$ from this, we add 2 to get $4x^2 + 1$.
Therefore, $g(f(x)) = (f(x))^2 - 2f(x) + 2$, and so an expression for $g(x)$ is $x^2 - 2x + 2$.

Solution 3

We use the expressions for $f(x)$ and $g(f(x))$ to construct $g(x)$.
Since $f(x)$ is linear and $g(f(x))$ is quadratic, then it is likely that $g(x)$ is also quadratic.
Suppose that $g(x) = ax^2 + bx + c$ for some real numbers a, b, c.
Then

$$\begin{aligned} g(f(x)) &= g(2x+1) \\ &= a(2x+1)^2 + b(2x+1) + c \\ &= a(4x^2 + 4x + 1) + b(2x+1) + c \\ &= 4ax^2 + (4a + 2b)x + (a + b + c) \end{aligned}$$

Since we are told that $g(f(x)) = 4x^2 + 1$, then we can compare coefficients to deduce that $4a = 4$ and $4a + 2b = 0$ and $a + b + c = 1$.
From the first equation, $a = 1$.
From the second equation, $b = -2a = -2$.
From the third equation, $c = 1 - a - b = 2$.
Therefore, an expression for $g(x)$ is $x^2 - 2x + 2$.

(b) *Solution 1*

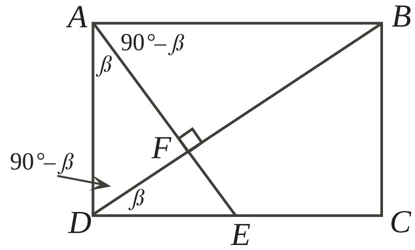
Since the sum of the first two terms is 40 and the sum of the first three terms is 76, then the third term is $76 - 40 = 36$.
Since the sum of the first three terms is 76 and the sum of the first four terms is 130, then the fourth term is $130 - 76 = 54$.
Since the third term is 36 and the fourth term is 54, then the common ratio in the geometric sequence is $\frac{54}{36} = \frac{3}{2}$.
Therefore, the fifth term is $54 \cdot \frac{3}{2} = 81$ and the sixth term is $81 \cdot \frac{3}{2} = \frac{243}{2}$.

Also, the second term is $36 \div \frac{3}{2} = 36 \cdot \frac{2}{3} = 24$ and the first term is $24 \div \frac{3}{2} = 24 \cdot \frac{2}{3} = 16$.
Thus, the first six terms of the sequence are 16, 24, 36, 54, 81, $\frac{243}{2}$.
Since the first term equals 2^4 and the common ratio is $\frac{3}{2}$, then the nth term in the sequence is $2^4\left(\dfrac{3}{2}\right)^{n-1} = \dfrac{3^{n-1}}{2^{n-5}}$.
When $n \geq 6$, this is a fraction whose numerator is odd and whose denominator is even, and so, when $n \geq 6$, the nth term is not an integer. (An odd integer is never divisible by an even integer.)
Therefore, there will be 5 integers in the sequence.

Solution 2
Suppose that a is the first term and r is the common ratio between consecutive terms (so that ar is the second term, ar^2 is the third term, and so on).
From the given information, $a+ar = 40$ and $a+ar+ar^2 = 76$ and $a+ar+ar^2+ar^3 = 130$.
Subtracting the first equation from the second, we obtain $ar^2 = 36$.
Subtracting the second equation from the third, we obtain $ar^3 = 54$.
Since $ar^3 = 54$ and $ar^2 = 36$, then $r = \dfrac{ar^3}{ar^2} = \dfrac{54}{36} = \dfrac{3}{2}$.
Since $ar^2 = 36$ and $r = \frac{3}{2}$, then $a(\frac{3}{2})^2 = 36$ or $\frac{9}{4}a = 36$ or $a = \frac{4}{9} \cdot 36 = 16$.
Since $a = 16$ and $r = \frac{3}{2}$, then the first six terms of the sequence are 16, 24, 36, 54, 81, $\frac{243}{2}$.
Since the first term equals 2^4 and the common ratio is $\frac{3}{2}$, then the nth term in the sequence is $2^4\left(\dfrac{3}{2}\right)^{n-1} = \dfrac{3^{n-1}}{2^{n-5}}$.
When $n \geq 6$, this is a fraction whose numerator is odd and whose denominator is even, and so, when $n \geq 6$, the nth term is not an integer. (An odd integer is never divisible by an even integer.)
Therefore, there will be 5 integers in the sequence.

6. (a) In a 30°-60°-90° triangle, the ratio of the side opposite the 90° to the side opposite the 60° angle is $2 : \sqrt{3}$.
Note that each of $\triangle ABC$, $\triangle ACD$, $\triangle ADE$, $\triangle AEF$, $\triangle AFG$, and $\triangle AGH$ is a 30°-60°-90° triangle.
Therefore, $\dfrac{AH}{AG} = \dfrac{AG}{AF} = \dfrac{AF}{AE} = \dfrac{AE}{AD} = \dfrac{AD}{AC} = \dfrac{AC}{AB} = \dfrac{2}{\sqrt{3}}$.
Thus, $AH = \frac{2}{\sqrt{3}}AG = \left(\frac{2}{\sqrt{3}}\right)^2 AF = \left(\frac{2}{\sqrt{3}}\right)^3 AE = \left(\frac{2}{\sqrt{3}}\right)^4 AD = \left(\frac{2}{\sqrt{3}}\right)^5 AC = \left(\frac{2}{\sqrt{3}}\right)^6 AB$.
(In other words, to get from $AB = 1$ to the length of AH, we multiply by the "scaling factor" $\frac{2}{\sqrt{3}}$ six times.)
Therefore, $AH = \left(\frac{2}{\sqrt{3}}\right)^6 = \frac{64}{27}$.

(b) *Solution 1*
Since $\triangle AFD$ is right-angled at F, then by the Pythagorean Theorem,
$$AD = \sqrt{AF^2 + FD^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$
since $AD > 0$.
Let $\angle FAD = \beta$.
Since $ABCD$ is a rectangle, then $\angle BAF = 90° - \beta$.
Since $\triangle AFD$ is right-angled at F, then $\angle ADF = 90° - \beta$.
Since $ABCD$ is a rectangle, then $\angle BDC = 90° - (90° - \beta) = \beta$.



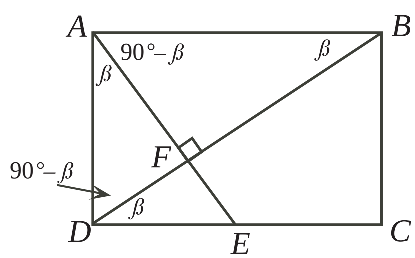
Therefore, $\triangle BFA$, $\triangle AFD$, and $\triangle DFE$ are all similar as each is right-angled and has either an angle of β or an angle of $90^\circ - \beta$ (and hence both of these angles).
Therefore, $\dfrac{AB}{AF} = \dfrac{DA}{DF}$ and so $AB = \frac{4(2\sqrt{5})}{2} = 4\sqrt{5}$.
Also, $\dfrac{FE}{FD} = \dfrac{FD}{FA}$ and so $FE = \frac{2(2)}{4} = 1$.
Since $ABCD$ is a rectangle, then $BC = AD = 2\sqrt{5}$, and $DC = AB = 4\sqrt{5}$.
Finally, the area of quadrilateral $BCEF$ equals the area of $\triangle DCB$ minus the area $\triangle DFE$.
Thus, the required area is

$$\tfrac{1}{2}(DC)(CB) - \tfrac{1}{2}(DF)(FE) = \tfrac{1}{2}(4\sqrt{5})(2\sqrt{5}) - \tfrac{1}{2}(2)(1) = 20 - 1 = 19$$

Solution 2
Since $\triangle AFD$ is right-angled at F, then by the Pythagorean Theorem,

$$AD = \sqrt{AF^2 + FD^2} = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$$

since $AD > 0$.
Let $\angle FAD = \beta$.
Since $ABCD$ is a rectangle, then $\angle BAF = 90^\circ - \beta$. Since $\triangle BAF$ is right-angled at F, then $\angle ABF = \beta$.
Since $\triangle AFD$ is right-angled at F, then $\angle ADF = 90^\circ - \beta$.
Since $ABCD$ is a rectangle, then $\angle BDC = 90^\circ - (90^\circ - \beta) = \beta$.



Looking at $\triangle AFD$, we see that $\sin\beta = \dfrac{FD}{AD} = \dfrac{2}{2\sqrt{5}} = \dfrac{1}{\sqrt{5}}$, $\cos\beta = \dfrac{AF}{AD} = \dfrac{4}{2\sqrt{5}} = \dfrac{2}{\sqrt{5}}$, and $\tan\beta = \dfrac{FD}{AF} = \dfrac{2}{4} = \dfrac{1}{2}$.
Since $AF = 4$ and $\angle ABF = \beta$, then $AB = \dfrac{AF}{\sin\beta} = \dfrac{4}{\frac{1}{\sqrt{5}}} = 4\sqrt{5}$.
Since $FD = 2$ and $\angle FDE = \beta$, then $FE = FD\tan\beta = 2 \cdot \frac{1}{2} = 1$.
Since $ABCD$ is a rectangle, then $BC = AD = 2\sqrt{5}$, and $DC = AB = 4\sqrt{5}$.
Finally, the area of quadrilateral $EFBC$ equals the area of $\triangle DCB$ minus the area $\triangle DFE$.
Thus, the required area is

$$\tfrac{1}{2}(DC)(CB) - \tfrac{1}{2}(DF)(FE) = \tfrac{1}{2}(4\sqrt{5})(2\sqrt{5}) - \tfrac{1}{2}(2)(1) = 20 - 1 = 19$$

7. (a) Using the facts that $9 = 3^2$ and $27 = 3^3$, and the laws for manipulating exponents, we have

$$\begin{aligned} 3^{x-1}9^{\frac{3}{2x^2}} &= 27 \\ 3^{x-1}(3^2)^{\frac{3}{2x^2}} &= 3^3 \\ 3^{x-1}3^{\frac{3}{x^2}} &= 3^3 \\ 3^{x-1+\frac{3}{x^2}} &= 3^3 \end{aligned}$$

When two powers of 3 are equal, their exponents must be equal so

$$\begin{aligned} x - 1 + \frac{3}{x^2} &= 3 \\ x^3 - x^2 + 3 &= 3x^2 \qquad \text{(multiplying by } x^2\text{)} \\ x^3 - 4x^2 + 3 &= 0 \end{aligned}$$

Since $x = 1$ satisfies the equation, then $x - 1$ is a factor of the left side. Using long division or synthetic division, we can factor this out to get $(x-1)(x^2 - 3x - 3) = 0$.
Using the quadratic formula, the quadratic equation $x^2 - 3x - 3 = 0$ has roots

$$x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-3)}}{2} = \frac{3 \pm \sqrt{21}}{2}$$

Therefore, the solutions to the original equation are $x = 1$ and $x = \dfrac{3 \pm \sqrt{21}}{2}$.

(b) To determine the points of intersection, we equate y values of the two curves and obtain $\log_{10}(x^4) = (\log_{10} x)^3$.
Since $\log_{10}(a^b) = b\log_{10} a$, the equation becomes $4\log_{10} x = (\log_{10} x)^3$.
We set $u = \log_{10} x$ and so the equation becomes $4u = u^3$, or $u^3 - 4u = 0$.
We can factor the left side as $u^3 - 4u = u(u^2 - 4) = u(u+2)(u-2)$.
Therefore, $u(u+2)(u-2) = 0$, and so $u = 0$ or $u = -2$ or $u = 2$.
Therefore, $\log_{10} x = 0$ or $\log_{10} x = -2$ or $\log_{10} x = 2$.
Therefore, $x = 1$ or $x = \frac{1}{100}$ or $x = 100$.
Finally, we must calculate the y-coordinates of the points of intersection. Since one of the original curves is $y = (\log_{10} x)^3$, we can calculate the corresponding values of y by using the fact that $y = u^3$.
The corresponding values of y are $y = 0^3 = 0$ and $y = (-2)^3 = -8$ and $y = 2^3 = 8$.
Therefore, the points of intersection are $(1, 0)$, $(\frac{1}{100}, -8)$ and $(100, 8)$.

8. (a) If Oi-Lam tosses 3 heads, then George has no coins to toss, so cannot toss exactly 1 head.
If Oi-Lam tosses 2, 1 or 0 heads, then George has at least one coin to toss, so can toss exactly 1 head.
Therefore, the following possibilities exist:

* Oi-Lam tosses 2 heads out of 3 coins and George tosses 1 head out of 1 coin
* Oi-Lam tosses 1 head out of 3 coins and George tosses 1 head out of 2 coins
* Oi-Lam tosses 0 heads out of 3 coins and George tosses 1 head out of 3 coins

We calculate the various probabilities.
If 3 coins are tossed, there are 8 equally likely possibilities: HHH, HHT, HTH, THH, TTH, THT, HTT, TTT. Each of these possibilities has probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Therefore,

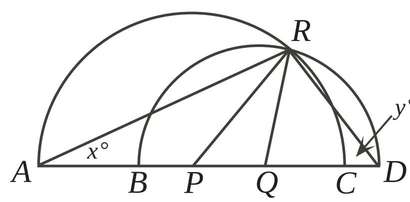
* the probability of tossing 0 heads out of 3 coins is $\frac{1}{8}$
* the probability of tossing 1 head out of 3 coins is $\frac{3}{8}$
* the probability of tossing 2 heads out of 3 coins is $\frac{3}{8}$
* the probability of tossing 3 heads out of 3 coins is $\frac{1}{8}$

If 2 coins are tossed, there are 4 equally likely possibilities: HH, HT, TH, TT. Each of these possibilities has probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Therefore, the probability of tossing 1 head out of 2 coins is $\frac{2}{4} = \frac{1}{2}$.
If 1 coin is tossed, the probability of tossing 1 head is $\frac{1}{2}$.
To summarize, the possibilities are

* Oi-Lam tosses 2 heads out of 3 coins (with probability $\frac{3}{8}$) and George tosses 1 head out of 1 coin (with probability $\frac{1}{2}$)
* Oi-Lam tosses 1 head out of 3 coins (with probability $\frac{3}{8}$) and George tosses 1 head out of 2 coins (with probability $\frac{1}{2}$)
* Oi-Lam tosses 0 heads out of 3 coins (with probability $\frac{1}{8}$) and George tosses 1 head out of 3 coins (with probability $\frac{3}{8}$)

Therefore, the overall probability is $\frac{3}{8} \cdot \frac{1}{2} + \frac{3}{8} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{3}{8} = \frac{27}{64}$.

(b) Suppose $\angle PAR = x^\circ$ and $\angle QDR = y^\circ$.



Since PR and PA are radii of the larger circle, then $\triangle PAR$ is isosceles.
Thus, $\angle PRA = \angle PAR = x^\circ$.
Since QD and QR are radii of the smaller circle, then $\triangle QRD$ is isosceles.
Thus, $\angle QRD = \angle QDR = y^\circ$.
In $\triangle ARD$, the sum of the angles is 180°, so $x^\circ + (x^\circ + 40^\circ + y^\circ) + y^\circ = 180^\circ$ or $2x + 2y = 140$ or $x + y = 70$.
Therefore, $\angle CPD = x^\circ + 40^\circ + y^\circ = (x + y + 40)^\circ = 110^\circ$.

9. (a) (i) *Solution 1*

$$\begin{aligned}
\text{LS} &= \cot\theta - \cot 2\theta \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta} \\
&= \frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\sin\theta\sin 2\theta} \\
&= \frac{\sin(2\theta - \theta)}{\sin\theta\sin 2\theta} \\
&= \frac{\sin\theta}{\sin\theta\sin 2\theta} \\
&= \frac{1}{\sin 2\theta} \\
&= \text{RS}
\end{aligned}$$

as required.

Solution 2

$$\begin{aligned}
\text{LS} &= \cot\theta - \cot 2\theta \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{\sin 2\theta} \\
&= \frac{\cos\theta}{\sin\theta} - \frac{\cos 2\theta}{2\sin\theta\cos\theta} \\
&= \frac{2\cos^2\theta - \cos 2\theta}{2\sin\theta\cos\theta} \\
&= \frac{2\cos^2\theta - (2\cos^2\theta - 1)}{\sin 2\theta} \\
&= \frac{1}{\sin 2\theta} \\
&= \text{RS}
\end{aligned}$$

as required.

(ii) We use (i) to note that $\dfrac{1}{\sin 8^\circ} = \cot 4^\circ - \cot 8^\circ$ and $\dfrac{1}{\sin 16^\circ} = \cot 8^\circ - \cot 16^\circ$ and so on. Thus,

$$\begin{aligned}
S &= \frac{1}{\sin 8^\circ} + \frac{1}{\sin 16^\circ} + \frac{1}{\sin 32^\circ} + \cdots + \frac{1}{\sin 4096^\circ} + \frac{1}{\sin 8192^\circ} \\
&= (\cot 4^\circ - \cot 8^\circ) + (\cot 8^\circ - \cot 16^\circ) + (\cot 16^\circ - \cot 32^\circ) + \\
&\qquad \cdots + (\cot 2048^\circ - \cot 4096^\circ) + (\cot 4096^\circ - \cot 8192^\circ) \\
&= \cot 4^\circ - \cot 8192^\circ
\end{aligned}$$

since the sum "telescopes".
Since the cotangent function has a period of 180°, and 8100° is a multiple of 180°, then $\cot 8192^\circ = \cot 92^\circ$.
Therefore,

$$\begin{aligned}
S &= \cot 4^\circ - \cot 92^\circ \\
&= \frac{\cos 4^\circ}{\sin 4^\circ} - \frac{\cos 92^\circ}{\sin 92^\circ} \\
&= \frac{\cos 4^\circ}{\sin 4^\circ} - \frac{-\sin 2^\circ}{\cos 2^\circ} \\
&= \frac{\cos 4^\circ}{2\sin 2^\circ\cos 2^\circ} + \frac{\sin 2^\circ}{\cos 2^\circ} \\
&= \frac{\cos 4^\circ + 2\sin^2 2^\circ}{2\sin 2^\circ\cos 2^\circ} \\
&= \frac{(1 - 2\sin^2 2^\circ) + 2\sin^2 2^\circ}{\sin 4^\circ} \\
&= \frac{1}{\sin 4^\circ}
\end{aligned}$$

Therefore, $\alpha = 4^\circ$.

(b) *Solution 1*

We use the notation $A = \angle BAC$, $B = \angle ABC$ and $C = \angle ACB$.

We need to show that $A < \frac{1}{2}(B + C)$. Since the sum of the angles in $\triangle ABC$ is 180°, then $B + C = 180^\circ - A$, and so this inequality is equivalent to $A < \frac{1}{2}(180^\circ - A)$ which is equivalent to $\frac{3}{2}A < 90^\circ$ or $A < 60^\circ$.

So we need to show that $A < 60^\circ$.

We know that $a < \frac{1}{2}(b + c)$. Thus, $2a < b + c$ and so $4a^2 < b^2 + c^2 + 2bc$ because all quantities are positive.

Using the cosine law in $\triangle ABC$, we obtain $a^2 = b^2 + c^2 - 2bc\cos A$.

Therefore,

$$\begin{aligned}
4a^2 &< b^2 + c^2 + 2bc \\
4(b^2 + c^2 - 2bc\cos A) &< b^2 + c^2 + 2bc \\
4b^2 + 4c^2 - 8bc\cos A &< b^2 + c^2 + 2bc \\
4b^2 + 4c^2 - 8bc\cos A &< b^2 + c^2 + 2bc + 3(b - c)^2 \qquad (\text{since } (b - c)^2 \geq 0) \\
4b^2 + 4c^2 - 8bc\cos A &< b^2 + c^2 + 2bc + 3b^2 - 6bc + 3c^2 \\
4b^2 + 4c^2 - 8bc\cos A &< 4b^2 + 4c^2 - 4bc \\
-8bc\cos A &< -4bc \\
\cos A &> \tfrac{1}{2} \qquad (\text{since } 8bc > 0)
\end{aligned}$$

Since $2a < b + c$, then a cannot be the longest side of $\triangle ABC$ (that is, we cannot have $a \geq b$ and $a \geq c$), so A must be an acute angle.

Therefore, $\cos A > \frac{1}{2}$ implies $A < 60^\circ$, as required.

Solution 2

We use the notation $A = \angle BAC$, $B = \angle ABC$ and $C = \angle ACB$.

We need to show that $A < \frac{1}{2}(B + C)$. Since the sum of the angles in $\triangle ABC$ is 180°, then $B + C = 180^\circ - A$, and so this inequality is equivalent to $A < \frac{1}{2}(180^\circ - A)$ which is equivalent to $\frac{3}{2}A < 90^\circ$ or $A < 60^\circ$.

So we need to show that $A < 60^\circ$.

We know that $a < \frac{1}{2}(b + c)$ which implies $2a < b + c$.

Using the sine law in $\triangle ABC$, we obtain $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, which gives $b = \dfrac{a \sin B}{\sin A}$ and $c = \dfrac{a \sin C}{\sin A}$.

Therefore, we obtain equivalent inequalities

$$\begin{aligned}
2a &< b + c \\
2a &< \frac{a \sin B}{\sin A} + \frac{a \sin C}{\sin A} \\
2a \sin A &< a \sin B + a \sin C \qquad (\text{since } \sin A > 0 \text{ for } 0^\circ < A < 180^\circ) \\
2 \sin A &< \sin B + \sin C
\end{aligned}$$

since $a > 0$. Next, we use the trigonometric formula $\sin B + \sin C = 2\sin\left(\dfrac{B + C}{2}\right)\cos\left(\dfrac{B - C}{2}\right)$.

Since $\cos\theta \leq 1$ for any θ, then $\sin B + \sin C \leq 2\sin\left(\dfrac{B + C}{2}\right) \cdot 1 = 2\sin\left(\dfrac{B + C}{2}\right)$.

Therefore,

$$\begin{aligned}
2\sin A &< \sin B + \sin C \le 2\sin\left(\frac{B+C}{2}\right) \\
2\sin A &< 2\sin\left(\frac{B+C}{2}\right) \\
2\sin A &< 2\sin\left(\frac{180^\circ - A}{2}\right) \\
4\sin(\tfrac{1}{2}A)\cos(\tfrac{1}{2}A) &< 2\sin(90^\circ - \tfrac{1}{2}A) \\
2\sin(\tfrac{1}{2}A)\cos(\tfrac{1}{2}A) &< \cos(\tfrac{1}{2}A)
\end{aligned}$$

Since $0^\circ < A < 180^\circ$, then $\cos(\frac{1}{2}A) > 0$, so $\sin(\frac{1}{2}A) < \frac{1}{2}$.
Since $2a < b + c$, then a cannot be the longest side of $\triangle ABC$, so A must be an acute angle.
Therefore, $\frac{1}{2}A < 30^\circ$ or $A < 60^\circ$, as required.

10. Denote the side lengths of a triangle by a, b and c, with $0 < a \le b \le c$.
In order for these lengths to form a triangle, we need $c < a + b$ and $b < a + c$ and $a < b + c$.
Since $0 < a \le b \le c$, then $b < a + c$ and $a < b + c$ follow automatically, so only $c < a + b$ ever needs to be checked.
Instead of directly considering triangles and sets of triangle, we can consider triples (a, b, c) and sets of triples (a, b, c) with the appropriate conditions.
For each positive integer $k \ge 3$, we use the notation S_k to denote the set of triples of positive integers (a, b, c) with $0 < a \le b \le c$ and $c < a + b$ and $a + b + c = k$.
In this case, $c < a + b$ and $a + b + c = k$, so $c + c < a + b + c = k$, so $2c < k$ or $c < \frac{1}{2}k$.
Also, if $0 < a \le b \le c$ and $a + b + c = k$, then $k = a + b + c \le c + c + c$, so $3c \ge k$ or $c \ge \frac{1}{3}k$.

(a) Consider $T(10)$, which is the number of elements in S_{10}.
We want to find all possible triples (a, b, c) of integers with $0 < a \le b \le c$ and $c < a + b$ and $a + b + c = 10$.
We need $c < \frac{10}{2} = 5$ and $c \ge \frac{10}{3}$. Thus, $c = 4$.
Therefore, we need $0 < a \le b \le 4$ and $a + b = 6$.
There are two possibilities: $(a, b, c) = (2, 4, 4)$ or $(a, b, c) = (3, 3, 4)$.
Therefore, $T(10) = 2$.

Consider $T(11)$. We want to find all possible triples (a, b, c) of integers with $0 < a \le b \le c$ and $c < a + b$ and $a + b + c = 11$.
We need $c < \frac{11}{2}$ and $c \ge \frac{11}{3}$. Thus, $c = 4$ or $c = 5$.
If $c = 4$, we need $0 < a \le b \le 4$ and $a + b = 7$.
There is only one possibility: $(a, b, c) = (3, 4, 4)$.
If $c = 5$, we need $0 < a \le b \le 5$ and $a + b = 6$.
There are three possibilities: $(a, b, c) = (1, 5, 5)$ or $(a, b, c) = (2, 4, 5)$ or $(a, b, c) = (3, 3, 5)$.
Therefore, $T(11) = 4$.

Consider $T(12)$. We want to find all possible triples (a, b, c) of integers with $0 < a \le b \le c$ and $c < a + b$ and $a + b + c = 12$.
We need $c < \frac{12}{2}$ and $c \ge \frac{12}{3}$. Thus, $c = 4$ or $c = 5$.
If $c = 4$, we need $0 < a \le b \le 4$ and $a + b = 8$.
There is only one possibility: $(a, b, c) = (4, 4, 4)$.

If $c = 5$, we need $0 < a \leq b \leq 5$ and $a + b = 7$.
There are two possibilities: $(a, b, c) = (2, 5, 5)$ or $(a, b, c) = (3, 4, 5)$.
Therefore, $T(12) = 3$.

(b) We show that $T(2m) = T(2m - 3)$ by creating a one-to-one correspondence between the triples in S_{2m} and the triples S_{2m-3}.
Note that S_{2m} is the set of triples (a, b, c) of positive integers with $0 < a \leq b \leq c$, with $c < a + b$, and with $a + b + c = 2m$.
Also, S_{2m-3} is the set of triples (A, B, C) of positive integers with $0 < A \leq B \leq C$, with $C < A + B$, and with $A + B + C = 2m - 3$.

Consider a triple (a, b, c) in S_{2m} and a corresponding triple $(a - 1, b - 1, c - 1)$.
We show that $(a - 1, b - 1, c - 1)$ is in S_{2m-3}:

* Since (a, b, c) is in S_{2m}, then $c < \frac{1}{2}(2m) = m$. This means that $b \leq c \leq m - 1$, so $a = 2m - b - c \geq 2$. Therefore, $a - 1$, $b - 1$ and $c - 1$ are positive integers since a, b and c are positive integers with $2 \leq a \leq b \leq c$.
* Since $2 \leq a \leq b \leq c$, then $1 \leq a - 1 \leq b - 1 \leq c - 1$, so $0 < a - 1 \leq b - 1 \leq c - 1$.
* Since $a + b + c = 2m$, then $c = 2m - (a + b)$ so $a + b$ and c have the same parity.
Since $c < a + b$, then $c \leq a + b - 2$. (In other words, it cannot be the case that $c = a + b - 1$.) Therefore, $c - 1 \leq (a - 1) + (b - 1) - 1$; that is, $c - 1 < (a - 1) + (b - 1)$.
* Since $a + b + c = 2m$, then $(a - 1) + (b - 1) + (c - 1) = 2m - 3$.

Therefore, $(a - 1, b - 1, c - 1)$ is in S_{2m-3}, since it satisfies all of the conditions of S_{2m-3}.
Note as well that two different triples in S_{2m} correspond to two different triples in S_{2m-3}.
Thus, every triple in S_{2m} corresponds to a different triple in S_{2m-3}.
Thus, $T(2m) \leq T(2m - 3)$.

Consider a triple (A, B, C) in S_{2m-3} and a corresponding triple $(A + 1, B + 1, C + 1)$.
We show that $(A + 1, B + 1, C + 1)$ is in S_{2m}:

* Since (A, B, C) is in S_{2m-3}, then A, B and C are positive integers, so $A + 1$, $B + 1$ and $C + 1$ are positive integers.
* Since $0 < A \leq B \leq C$, then $1 < A+1 \leq B+1 \leq C+1$, so $0 < A+1 \leq B+1 \leq C+1$.
* Since $C < A + B$, then $C + 1 < (A + 1) + (B + 1) - 1$ so $C + 1 < (A + 1) + (B + 1)$.
* Since $A + B + C = 2m - 3$, then $(A + 1) + (B + 1) + (C + 1) = 2m$.

Therefore, $(A + 1, B + 1, C + 1)$ is in S_{2m}.
Note again that two different triples in S_{2m-3} correspond to two different triples in S_{2m}.
Thus, every triple in S_{2m-3} corresponds to a different triple in S_{2m}.
Therefore, $T(2m - 3) \leq T(2m)$.

Since $T(2m) \leq T(2m - 3)$ and $T(2m - 3) \leq T(2m)$, then $T(2m) = T(2m - 3)$.

(c) We will use two important facts:

(F1) $T(2m) = T(2m - 3)$ for every positive integer $m \geq 3$, and

(F2) $T(k) \leq T(k + 2)$ for every positive integer $k \geq 3$

We proved (F1) in (b).
Next, we prove (F2):

Consider a triple (a, b, c) in S_k and a corresponding triple $(a, b + 1, c + 1)$. We show that the triple $(a, b + 1, c + 1)$ is in S_{k+2}:

* Since a, b and c are positive integers, then a, $b + 1$ and $c + 1$ are positive integers.

* Since $0 < a \leq b \leq c$, then $0 < a \leq b+1 \leq c+1$.
* Since $c < a+b$, then $c+1 < a+(b+1)$.
* Since $a+b+c=k$, then $a+(b+1)+(c+1)=k+2$.

Therefore, $(a, b+1, c+1)$ is in S_{k+2}. Note that, using this correspondence, different triples in S_k correspond different triples in S_{k+2}. Thus, every triple in S_k corresponds to a different triple in S_{k+2}. This proves that $T(k) \leq T(k+2)$.

Suppose that $n = N$ is the smallest positive integer for which $T(n) > 2010$.
Then N must be odd:

If N was even, then by (F1), $T(N-3) = T(N) > 2010$ and so $n = N-3$ would be an integer smaller than N with $T(n) > 2010$. This contradicts the fact that $n = N$ is the smallest such integer.

Therefore, we want to find the smallest odd positive integer N for which $T(N) > 2010$.
Next, we note that if we can find an odd positive integer n such that $T(n) > 2010 \geq T(n-2)$, then we will have found the desired value of n:

This is because n and $n-2$ are both odd, and by property (F2), any smaller odd positive integer k will give $T(k) \leq T(n-2) \leq 2010$ and any larger odd positive integer m will give $T(m) \geq T(n) > 2010$.

We show that $N = 309$ is the desired value of N by showing that $T(309) > 2010$ and $T(307) \leq 2010$.

Calculation of $T(309)$

We know that $\frac{309}{3} \leq c < \frac{309}{2}$, so $103 \leq c \leq 154$.
For each admissible value of c, we need to count the number of pairs of positive integers (a, b) with $a \leq b \leq c$ and $a+b = 309-c$.
For example, if $c = 154$, then we need $a \leq b \leq 154$ and $a+b = 155$.
This gives pairs $(1, 154), (2, 153), \ldots, (76, 79), (77, 78)$, of which there are 77.
Also, if $c = 153$, then we need $a \leq b \leq 153$ and $a+b = 156$.
This gives pairs $(3, 153), \ldots, (77, 79), (78, 78)$, of which there are 76.
In general, if c is even, then the minimum possible value of a occurs when b is as large as possible – that is, when $b = c$, so $a \geq 309-2c$.
Also, the largest possible value of a occurs when a and b are as close to equal as possible.
Since c is even, then $309-c$ is odd, so a and b cannot be equal, but they can differ by 1.
In this case, $a = 154 - \frac{1}{2}c$ and $b = 155 - \frac{1}{2}c$.
Therefore, if c is even, there are $(154 - \frac{1}{2}c) - (309-2c) + 1 = \frac{3}{2}c - 154$ possible pairs (a, b) and so $\frac{3}{2}c - 154$ possible triples.
In general, if c is odd, then the minimum possible value of a occurs when b is as large as possible – that is, when $b = c$, so $a \geq 309-2c$.
Also, the largest possible value of a occurs when a and b are as close to equal as possible.
Since c is odd, then $309-c$ is even, so a and b can be equal. In this case, $a = \frac{1}{2}(309-c)$.
Therefore, if c is odd, there are $\frac{1}{2}(309-c) - (309-2c) + 1 = \frac{3}{2}c - \frac{307}{2}$ possible pairs (a, b) and so $\frac{3}{2}c - \frac{307}{2}$ possible triples.
The possible even values of c are $104, 106, \ldots, 152, 154$ (there are 26 such values) and the possible odd values of c are $103, 105, \ldots, 151, 153$ (there are 26 such values).

Therefore,

$$\begin{aligned}
T(309) &= \left(\tfrac{3}{2}(104) - 154\right) + \left(\tfrac{3}{2}(106) - 154\right) + \cdots + \left(\tfrac{3}{2}(154) - 154\right) + \\
&\qquad \left(\tfrac{3}{2}(103) - \tfrac{307}{2}\right) + \left(\tfrac{3}{2}(105) - \tfrac{307}{2}\right) + \cdots + \left(\tfrac{3}{2}(153) - \tfrac{307}{2}\right) \\
&= \tfrac{3}{2}(104 + 106 + \cdots + 154) - 26 \cdot 154 + \tfrac{3}{2}(103 + 105 + \cdots + 153) - 26 \cdot \tfrac{307}{2} \\
&= \tfrac{3}{2}(103 + 104 + 105 + 106 + \cdots + 153 + 154) - 26 \cdot 154 - 26 \cdot \tfrac{307}{2} \\
&= \tfrac{3}{2} \cdot \tfrac{1}{2}(103 + 154)(52) - 26 \cdot 154 - 26 \cdot \tfrac{307}{2} \\
&= \tfrac{3}{2}(26)(257) - 26 \cdot 154 - 26 \cdot \tfrac{307}{2} \\
&= 2028
\end{aligned}$$

Therefore, $T(309) > 2010$, as required.

Calculation of $T(307)$

We know that $\frac{307}{3} \leq c < \frac{307}{2}$, so $103 \leq c \leq 153$.
For each admissible value of c, we need to count the number of pairs of positive integers (a, b) with $a \leq b \leq c$ and $a + b = 307 - c$.
This can be done in a similar way to the calculation of $T(309)$ above.
If n is even, there are $\frac{3}{2}c - 153$ possible triples.
If n is odd, there are $\frac{3}{2}c - \frac{305}{2}$ possible triples.
The possible even values of c are $104, 106, \ldots, 150, 152$ (there are 25 such values) and the possible odd values of c are $103, 105, \ldots, 151, 153$ (there are 26 such values).
Therefore,

$$\begin{aligned}
T(307) &= \left(\tfrac{3}{2}(104) - 153\right) + \left(\tfrac{3}{2}(106) - 153\right) + \cdots + \left(\tfrac{3}{2}(152) - 153\right) + \\
&\qquad \left(\tfrac{3}{2}(103) - \tfrac{305}{2}\right) + \left(\tfrac{3}{2}(105) - \tfrac{305}{2}\right) + \cdots + \left(\tfrac{3}{2}(153) - \tfrac{305}{2}\right) \\
&= \tfrac{3}{2}(104 + 106 + \cdots + 152) - 25 \cdot 153 + \tfrac{3}{2}(103 + 105 + \cdots + 153) - 26 \cdot \tfrac{305}{2} \\
&= \tfrac{3}{2}(103 + 104 + 105 + 106 + \cdots + 152 + 153) - 25 \cdot 153 - 26 \cdot \tfrac{305}{2} \\
&= \tfrac{3}{2} \cdot \tfrac{1}{2}(103 + 153)(51) - 25 \cdot 153 - 26 \cdot \tfrac{305}{2} \\
&= \tfrac{3}{2}(51)(128) - 25 \cdot 153 - 26 \cdot \tfrac{305}{2} \\
&= 2002
\end{aligned}$$

Therefore, $T(307) < 2010$, as required.

Therefore, the smallest positive integer n such that $T(n) > 2010$ is $n = 309$.

As a final note, we discuss briefly how one could guess that the answer was near $N = 309$.

Consider the values of $T(n)$ for small odd positive integers n.
In (a), by considering the possible values of c from smallest (roughly $\frac{1}{3}n$) to largest (roughly $\frac{1}{2}n$), we saw that $T(11) = 1 + 3 = 4$.
If we continue to calculate $T(n)$ for a few more small odd values of n we will see that:

$$\begin{aligned}
T(13) &= 2 + 3 = 5 \\
T(15) &= 1 + 2 + 4 = 7 \\
T(17) &= 1 + 3 + 4 = 8 \\
T(19) &= 2 + 3 + 5 = 10 \\
T(21) &= 1 + 2 + 4 + 5 = 12 \\
T(23) &= 1 + 3 + 4 + 6 = 14
\end{aligned}$$

The pattern that seems to emerge is that for n odd, $T(n)$ is roughly equal to the sum of the integers from 1 to $\frac{1}{4}n$, with one out of every three integers removed.
Thus, $T(n)$ is roughly equal to $\frac{2}{3}$ of the sum of the integers from 1 to $\frac{1}{4}n$.
Therefore, $T(n) \approx \frac{2}{3} \cdot \frac{1}{2}(\frac{1}{4}n)(\frac{1}{4}n+1) \approx \frac{2}{3} \cdot \frac{1}{2}(\frac{1}{4}n)^2 \approx \frac{1}{48}n^2$.

It makes sense to look for an odd positive integer n with $T(n) \approx 2010$.
Thus, we are looking for a value of n that roughly satisfies $\frac{1}{48}n^2 \approx 2010$ or $n^2 \approx 96480$ or $n \approx 310$.
Since n is odd, then it makes sense to consider $n = 309$, as in the solution above.

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2009 Euclid Contest

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Solutions

1. (a) We rewrite $6x + 3y = 21$ as $3y = -6x + 21$ or $y = -2x + 7$.
Thus, the slope is -2.

(b) *Solution 1*
Since the slope of the line segment is 3, then $\dfrac{c-0}{5-1} = 3$, and so $\dfrac{c}{4} = 3$ or $c = 12$.

Solution 2
Since the slope of the line segment is 3, then for every unit that we move to the right, we move 3 units up.
Since $(5, c)$ is 4 units to the right of $(1, 0)$, then it is $3(4) = 12$ units up from $(1, 0)$, so $c = 0 + 12 = 12$.

(c) *Solution 1*
The given line segment joins $(0, 4)$ to $(8, -4)$, so has slope $\dfrac{4-(-4)}{0-8} = \dfrac{8}{-8} = -1$.
Since the y-intercept of the line segment is 4, then the equation of the line passing through A and B is $y = -x + 4$.
Since the point (k, k) lies on the line, then $k = -k + 4$ or $2k = 4$ and so $k = 2$.

Solution 2
We label the point (k, k) as K.
Since K lies on the line segment AB, then the slope of AK equals the slope of AB.
Line segment AB joins $(0, 4)$ to $(8, -4)$, so has slope $\dfrac{4-(-4)}{0-8} = \dfrac{8}{-8} = -1$.
Line segment AK joins $(0, 4)$ to (k, k), so has slope $\dfrac{k-4}{k-0}$.
Therefore, $\dfrac{k-4}{k} = -1$ or $k - 4 = -k$ or $2k = 4$ and so $k = 2$.

2. (a) *Solution 1*
If a quadratic equation has the form $ax^2 + bx + c = 0$, then the sum of its roots is $-\dfrac{b}{a}$.
Here, the sum of the roots must be $-\left(\frac{(-6)}{1}\right) = 6$.

Solution 2
Since $x^2 - 6x - 7 = 0$, then $(x - 7)(x + 1) = 0$.
Thus, the roots are $x = 7$ and $x = -1$.
The sum of these roots is $7 + (-1) = 6$.

(b) *Solution 1*
If a quadratic equation has the form $ax^2 + bx + c = 0$, then the product of its roots is $\dfrac{c}{a}$.
Here, the product of the roots must be $\frac{-20}{5} = -4$.

Solution 2
Since $5x^2 - 20 = 0$, then $x^2 - 4 = 0$ or $(x - 2)(x + 2) = 0$.
Thus, the roots are $x = 2$ and $x = -2$.
The product of these roots is $2(-2) = -4$.

(c) *Solution 1*
If a cubic equation has the form $a^3 + bx^2 + cx + d = 0$, then the sum of its roots is $-\dfrac{b}{a}$.
Here, the sum of the three roots is $-\left(\frac{-6}{1}\right) = 6$.
The average of three numbers is their sum divided by 3, so the average of the three roots is $\frac{6}{3} = 2$.

Solution 2
Since $x^3 - 6x^2 + 5x = 0$, then $x(x^2 - 6x + 5) = 0$ or $x(x-5)(x-1) = 0$.
The three roots of this equation are $x = 0$, $x = 1$ and $x = 5$.
The average of these numbers is $\frac{1}{3}(0+1+5) = \frac{1}{3}(6) = 2$.

3. (a) Since $AB = AD = BD$, then $\triangle BDA$ is equilateral.
Thus, $\angle ABD = \angle ADB = \angle DAB = 60^\circ$.
Also, $\angle DAE = 180^\circ - \angle ADE - \angle AED = 180^\circ - 60^\circ - 90^\circ = 30^\circ$.
Since CAE is a straight line, then $\angle CAD = 180^\circ - \angle DAE = 180^\circ - 30^\circ = 150^\circ$.
Now $AC = AD$ so $\triangle CAD$ is isosceles, which gives $\angle CDA = \angle DCA$.
Since the sum of the angles in $\triangle CAD$ is 180° and $\angle CDA = \angle DCA$, then

$$\angle CDA = \tfrac{1}{2}(180^\circ - \angle CAD) = \tfrac{1}{2}(180^\circ - 150^\circ) = 15^\circ$$

Thus, $\angle CDB = \angle CDA + \angle ADB = 15^\circ + 60^\circ = 75^\circ$.

(b) *Solution 1*
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

We calculate the area of $\triangle ADB$ is two different ways.
First, using AB as base and AD as height, we obtain an area of $\frac{1}{2}(40)(30) = 600$.
Next, using DB as base and AF as height, we obtain an area of $\frac{1}{2}(50)x = 25x$.
We must have $25x = 600$ and so $x = \frac{600}{25} = 24$.

Solution 2
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

Since $\triangle DAB$ is right-angled at A, then $\sin(\angle ADB) = \dfrac{AB}{BD} = \dfrac{40}{50} = \dfrac{4}{5}$.
But $\triangle ADF$ is right-angled at F and $\angle ADF = \angle ADB$.
Therefore, $\sin(\angle ADF) = \dfrac{AF}{AD} = \dfrac{x}{30}$.
Thus, $\dfrac{x}{30} = \dfrac{4}{5}$ and so $x = \frac{4}{5}(30) = 24$.

Solution 3
Since $ABCD$ is a rectangle, then $AB = CD = 40$ and $AD = BC = 30$.
By the Pythagorean Theorem, $BD^2 = AD^2 + AB^2$ and since $BD > 0$, then

$$BD = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$$

Note that $\triangle BFA$ is similar to $\triangle BAD$, since each is right-angled and they share a common angle at B.
Thus, $\dfrac{AF}{AB} = \dfrac{AD}{BD}$ and so $\dfrac{x}{30} = \dfrac{40}{50}$ which gives $x = \dfrac{30(40)}{50} = 24$.

4. (a) *Solution 1*

The sum of the terms in an arithmetic sequence is equal to the average of the first and last terms times the number of terms.

If n is the number of terms in the sequence, then $\frac{1}{2}(1+19)n = 70$ or $10n = 70$ and so $n = 7$.

Solution 2

Let n be the number of terms in the sequence and d the common difference.

Since the first term is 1 and the nth term equals 19, then $1+(n-1)d = 19$ and so $(n-1)d = 18$.

Since the sum of the terms in the sequence is 70, then $\frac{1}{2}n(1+1+(n-1)d) = 70$.

Thus, $\frac{1}{2}n(2+18) = 70$ or $10n = 70$ and so $n = 7$.

(b) *Solution 1*

Since the given equation is true for all values of x, then it is true for any particular value of x that we try.

If $x = -3$, the equation becomes $a(-3+b(0)) = 2(3)$ or $-3a = 6$ and so $a = -2$.

If $x = 0$, the equation becomes $-2(0+b(3)) = 2(6)$ or $-6b = 12$ and so $b = -2$.

Therefore, $a = -2$ and $b = -2$.

Solution 2

We expand both sides of the equation:

$$\begin{aligned} a(x+b(x+3)) &= 2(x+6) \\ a(x+bx+3b) &= 2x+12 \\ ax+abx+3ab &= 2x+12 \\ (a+ab)x+3ab &= 2x+12 \end{aligned}$$

Since this equation is true for all values of x, then the coefficients on the left side and right side must be equal, so $a+ab = 2$ and $3ab = 12$.

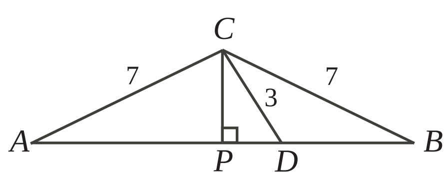
From the second equation, $ab = 4$ so the first equation becomes $a+4 = 2$ or $a = -2$.

Since $ab = 4$, then $-2b = 4$ and so $b = -2$.

Thus, $a = b = -2$.

5. (a) *Solution 1*

Drop a perpendicular from C to P on AD.



Since $\triangle ACB$ is isosceles, then $AP = PB$.

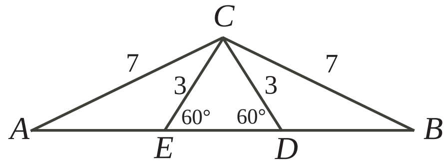
Since $\triangle CDP$ is a 30°-60°-90° triangle, then $PD = \frac{1}{2}(CD) = \frac{3}{2}$.

Thus, $AP = AD - PD = 8 - \frac{3}{2} = \frac{13}{2}$.

This tells us that $DB = PB - PD = AP - PD = \frac{13}{2} - \frac{3}{2} = 5$.

Solution 2

Since $\triangle ACB$ is symmetric about the vertical line through C, we can reflect CD in this vertical line, finding point E on AD with $CE = 3$ and $\angle CED = 60°$.



Then $\triangle CDE$ has two 60° angles, so must have a third, and so is equilateral.
Therefore, $ED = CD = CE = 3$ and so $DB = AE = AD - ED = 8 - 3 = 5$.

Solution 3
Since $\angle CDB = 180^\circ - \angle CDA = 180^\circ - 60^\circ = 120^\circ$, then using the cosine law in $\triangle CDB$, we obtain

$$\begin{aligned} CB^2 &= CD^2 + DB^2 - 2(CD)(DB)\cos(\angle CDB) \\ 7^2 &= 3^2 + DB^2 - 2(3)(DB)\cos(120^\circ) \\ 49 &= 9 + DB^2 - 6(DB)\left(-\tfrac{1}{2}\right) \\ 0 &= DB^2 + 3DB - 40 \\ 0 &= (DB-5)(DB+8) \end{aligned}$$

Since $DB > 0$, then $DB = 5$.

(b) *Solution 1*
Since $\triangle ABC$ is right-angled at C, then $\sin B = \cos A$.
Therefore, $2\cos A = 3\tan A = \dfrac{3\sin A}{\cos A}$ or $2\cos^2 A = 3\sin A$.
Using the fact that $\cos^2 A = 1 - \sin^2 A$, this becomes $2 - 2\sin^2 A = 3\sin A$ or $2\sin^2 A + 3\sin A - 2 = 0$ or $(2\sin A - 1)(\sin A + 2) = 0$.
Since $\sin A$ is between -1 and 1, then $\sin A = \frac{1}{2}$.
Since A is an acute angle, then $A = 30^\circ$.

Solution 2
Since $\triangle ABC$ is right-angled at C, then $\sin B = \dfrac{b}{c}$ and $\tan A = \dfrac{a}{b}$.
Thus, the given equation is $\dfrac{2b}{c} = \dfrac{3a}{b}$ or $2b^2 = 3ac$.
Using the Pythagorean Theorem, $b^2 = c^2 - a^2$ and so we obtain $2c^2 - 2a^2 = 3ac$ or $2c^2 - 3ac - 2a^2 = 0$.
Factoring, we obtain $(c - 2a)(2c + a) = 0$.
Since a and c must both be positive, then $c = 2a$.
Since $\triangle ABC$ is right-angled, the relation $c = 2a$ means that $\triangle ABC$ is a 30°-60°-90° triangle, with $A = 30^\circ$.

6. (a) The number of integers between 100 and 999 inclusive is $999 - 100 + 1 = 900$.
An integer n in this range has three digits, say a, b and c, with the hundreds digit equal to a.
Note that $0 \leq b \leq 9$ and $0 \leq c \leq 9$ and $1 \leq a \leq 9$.
To have $a + b + c = 24$, then the possible triples for a, b, c in some order are 9,9,6; 9,8,7; 8,8,8. (There cannot be three 9's. If there are two 9's, the the other digit equals 6. If there is one 9, the second and third digits add to 15 but are both less than 9, so must equal 8 and 7. If there are zero 9's, the maximum for each digit is 8, and so each digt must be 8 in order for the sum of all three to equal 24.)
If the digits are 9, 9 and 6, there are 3 arrangements: 996, 969, 699.

If the digits are 9, 8 and 7, there are 6 arrangements: 987, 978, 897, 879, 798, 789.
If the digits are 8, 8 and 8, there is only 1 arrangement: 888.
Therefore, there are $3+6+1=10$ integers n in the range 100 to 999 with the sum of the digits of n equal to 24.
The required probability equals the number of possible values of n with the sum of digits equal to 24 divided by the total number of integers in the range, or $\frac{10}{900}=\frac{1}{90}$.

(b) Since Alice drives at 60 km/h, then she drives 1 km every minute.
Since Alice drove from G to F in 45 minutes, then the distance from G to F is 45 km.
Let the distance from E to G be d km and let Bob's speed be B km/h.
Since Bob drove from G to E in 20 minutes (or $\frac{1}{3}$ of an hour), then $\dfrac{d}{B}=\dfrac{1}{3}$. Thus, $d=\frac{1}{3}B$.
The time that it took Bob to drive from F to G was $\frac{45}{B}$ hours.
The time that it took Alice to drive from E to G was $\frac{d}{60}$ hours.
Since the time that it took each of Alice and Bob to reach G was the same, then $\dfrac{d}{60}=\dfrac{45}{B}$ and so $Bd=45(60)=2700$.
Thus, $B\left(\frac{1}{3}B\right)=2700$ so $B^2=8100$ or $B=90$ since $B>0$.
Therefore, Bob's speed was 90 km/h.

7. (a) Completing the square on the original parabola, we obtain

$$y=x^2-2x+4=x^2-2x+1-1+4=(x-1)^2+3$$

Therefore, the vertex of the original parabola is $(1,3)$.
Since the new parabola is a translation of the original parabola and has x-intercepts 3 and 5, then its equation is $y=1(x-3)(x-5)=x^2-8x+15$.
Completing the square here, we obtain

$$y=x^2-8x+15=x^2-8x+16-16+15=(x-4)^2-1$$

Therefore, the vertex of the new parabola is $(4,-1)$.
Thus, the point $(1,3)$ is translated p units to the right and q units down to reach $(4,-1)$, so $p=3$ and $q=4$.

(b) First, we determine the coordinates of A.
The area of $\triangle ABC$ is 4. We can think of AC as its base, and its height being the distance from B to the x-axis.
If the coordinates of A are $(a,0)$, then the base has length $4-a$ and the height is 4.
Thus, $\frac{1}{2}(4-a)(4)=4$, so $4-a=2$ and so $a=2$.
Therefore, the coordinates of A are $(2,0)$.

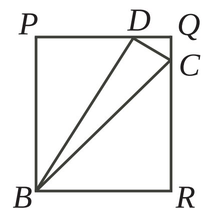
Next, we determine the equation of the parabola.
The parabola has x-intercepts 2 and 4, so has equation $y=k(x-2)(x-4)$.
Since the parabola passes through $(0,-4)$ as well, then $-4=k(-2)(-4)$ so $k=-\frac{1}{2}$.
Therefore, the parabola has equation $y=-\frac{1}{2}(x-2)(x-4)$.

Next, we determine the coordinates of D, the vertex of the parabola.
Since the x-intercepts are 2 and 4, then the x-coordinate of the vertex is the average of these, or 3.

The y-coordinate of D can be obtained from the equation of the parabola; we obtain $y = -\frac{1}{2}(3-2)(3-4) = -\frac{1}{2}(1)(-1) = \frac{1}{2}$.
Thus, the coordinates of D are $(3, \frac{1}{2})$.

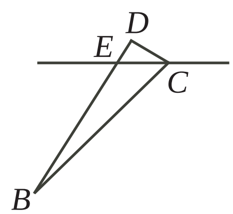
Lastly, we determine the area of $\triangle BDC$, whose vertices have coordinates $B(0,-4)$, $D(3, \frac{1}{2})$, and $C(4,0)$.

Method 1
We proceed be "completing the rectangle". That is, we draw the rectangle with horizontal sides along the lines $y = \frac{1}{2}$ and $y = -4$ and vertical sides along the lines $x = 0$ and $x = 4$. We label this rectangle as $BPQR$.



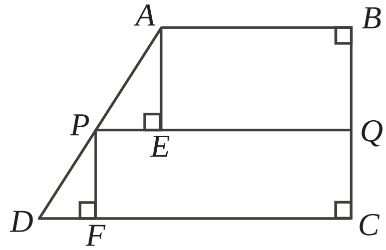
The area of $\triangle BDC$ equals the area of the rectangle minus the areas of $\triangle BPD$, $\triangle DQC$ and $\triangle CRB$.
Rectangle $BPQR$ has height $4 + \frac{1}{2} = \frac{9}{2}$ and width 4.
$\triangle BPD$ has height $\frac{9}{2}$ and base 3.
$\triangle DQC$ has height $\frac{1}{2}$ and base 1.
$\triangle CRB$ has height 4 and base 4.
Therefore, the area of $\triangle BDC$ is $4(\frac{9}{2}) - \frac{1}{2}(\frac{9}{2})(3) - \frac{1}{2}(\frac{1}{2})(1) - \frac{1}{2}(4)(4) = 18 - \frac{27}{4} - \frac{1}{4} - 8 = 3$.

Method 2
We determine the coordinates of E, the point where BD crosses the x-axis.



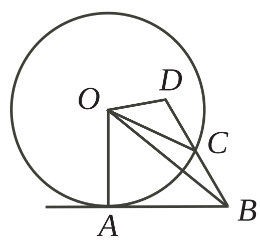
Once we have done this, then the area of $\triangle BDC$ equals the sum of the areas of $\triangle ECB$ and $\triangle ECD$.
Since B has coordinates $(0,-4)$ and D has coordinates $(3, \frac{1}{2})$, then the slope of BD is $\dfrac{\frac{1}{2} - (-4)}{3 - 0} = \dfrac{\frac{9}{2}}{3} = \dfrac{3}{2}$.
Since B is on the y-axis, then the equation of the line through B and D is $y = \frac{3}{2}x - 4$.
To find the x-coordinate of E, we set $y = 0$ to obtain $0 = \frac{3}{2}x - 4$ or $\frac{3}{2}x = 4$ or $x = \frac{8}{3}$.
We think of EC as the base of each of the two smaller triangles. Note that $EC = 4 - \frac{8}{3} = \frac{4}{3}$.
Thus, the area of $\triangle ECD$ is $\frac{1}{2}(\frac{4}{3})(\frac{1}{2}) = \frac{1}{3}$.
Also, the area of $\triangle ECB$ is $\frac{1}{2}(\frac{4}{3})(4) = \frac{8}{3}$.
Therefore, the area of $\triangle BDC$ is $\frac{1}{3} + \frac{8}{3} = 3$.

8. (a) Since PQ is parallel to AB, then it is parallel to DC and is perpendicular to BC.
Drop perpendiculars from A to E on PQ and from P to F on DC.



Then $ABQE$ and $PQCF$ are rectangles. Thus, $EQ = x$, which means that $PE = r - x$ and $FC = r$, which means that $DF = y - r$.
Let $BQ = b$ and $QC = c$. Thus, $AE = b$ and $PF = c$.
The area of trapezoid $ABQP$ is $\frac{1}{2}(x + r)b$.
The area of trapezoid $PQCD$ is $\frac{1}{2}(r + y)c$.
Since these areas are equal, then $\frac{1}{2}(x + r)b = \frac{1}{2}(r + y)c$, which gives $\dfrac{x + r}{r + y} = \dfrac{c}{b}$.
Since AE is parallel to PF, then $\angle PAE = \angle DPF$ and $\triangle AEP$ is similar to $\triangle PFD$.
Thus, $\dfrac{AE}{PE} = \dfrac{PF}{DF}$ which gives $\dfrac{b}{r - x} = \dfrac{c}{y - r}$ or $\dfrac{c}{b} = \dfrac{y - r}{r - x}$.
Combining $\dfrac{x + r}{r + y} = \dfrac{c}{b}$ and $\dfrac{c}{b} = \dfrac{y - r}{r - x}$ gives $\dfrac{x + r}{r + y} = \dfrac{y - r}{r - x}$ or $(x+r)(r-x) = (r+y)(y-r)$.
From this, we get $r^2 - x^2 = y^2 - r^2$ or $2r^2 = x^2 + y^2$, as required.

(b) Join O to A, B and C.



Since AB is tangent to the circle at A, then $\angle OAB = 90^\circ$.
By the Pythagorean Theorem in $\triangle OAB$, we get $OA^2 + AB^2 = OB^2$ or $r^2 + p^2 = OB^2$.
In $\triangle ODC$, we have $OD = DC = q$ and $OC = r$.
By the cosine law,

$$\begin{aligned} OC^2 &= OD^2 + DC^2 - 2(OD)(DC)\cos(\angle ODC) \\ r^2 &= q^2 + q^2 - 2q^2\cos(\angle ODC) \\ \cos(\angle ODC) &= \frac{2q^2 - r^2}{2q^2} \end{aligned}$$

In $\triangle ODB$, we have $\angle ODB = \angle ODC$.
Thus, using the cosine law again,

$$\begin{aligned} OB^2 &= OD^2 + DB^2 - 2(OD)(DB)\cos(\angle ODB) \\ &= q^2 + (2q)^2 - 2(q)(2q)\left(\frac{2q^2 - r^2}{2q^2}\right) \\ &= q^2 + 4q^2 - 2(2q^2 - r^2) \\ &= q^2 + 2r^2 \end{aligned}$$

So $OB^2 = r^2 + p^2 = q^2 + 2r^2$, which gives $p^2 = q^2 + r^2$, as required.

9. (a) First, we convert each of the logarithms to a logarithm with base 2:

$$\begin{aligned} 1+\log_4 x &= 1+\frac{\log_2 x}{\log_2 4} = 1+\frac{\log_2 x}{2} = 1+\tfrac{1}{2}\log_2 x \\ \log_8 4x &= \frac{\log_2 4x}{\log_2 8} = \frac{\log_2 4+\log_2 x}{3} = \tfrac{2}{3}+\tfrac{1}{3}\log_2 x \end{aligned}$$

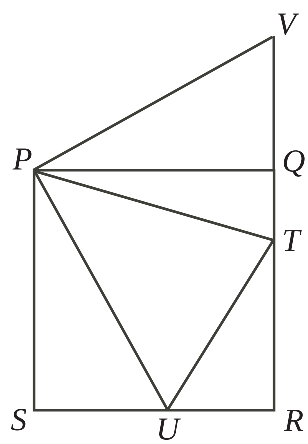
Let $y=\log_2 x$. Then the three terms are y, $1+\frac{1}{2}y$, and $\frac{2}{3}+\frac{1}{3}y$. Since these three are in geometric sequence, then

$$\begin{aligned} \frac{y}{1+\frac{1}{2}y} &= \frac{1+\frac{1}{2}y}{\frac{2}{3}+\frac{1}{3}y} \\ y(\tfrac{2}{3}+\tfrac{1}{3}y) &= (1+\tfrac{1}{2}y)^2 \\ \tfrac{2}{3}y+\tfrac{1}{3}y^2 &= 1+y+\tfrac{1}{4}y^2 \\ 8y+4y^2 &= 12+12y+3y^2 \\ y^2-4y-12 &= 0 \\ (y-6)(y+2) &= 0 \end{aligned}$$

Therefore, $y=\log_2 x=6$ or $y=\log_2 x=-2$, which gives $x=2^6=64$ or $x=2^{-2}=\frac{1}{4}$.

(b) *Solution 1*

Rotate a copy of $\triangle PSU$ by 90° counterclockwise around P, forming a new triangle PQV. Note that V lies on the extension of RQ.



Then $PV=PU$ by rotation.

Also, $\angle VPT=\angle VPQ+\angle QPT=\angle UPS+\angle QPT=90^\circ-\angle UPT=90^\circ-45^\circ$.

This tells us that $\triangle PTU$ is congruent to $\triangle PTV$, by "side-angle-side".

Thus, the perimeter of $\triangle RUT$ equals

$$\begin{aligned} UR+RT+UT &= UR+RT+TV \\ &= UR+RT+TQ+QV \\ &= UR+RQ+SU \\ &= SU+UR+RQ \\ &= SR+RQ \\ &= 8 \end{aligned}$$

That is, the perimeter of $\triangle RUT$ always equals 8, so the maximum possible perimeter is 8.

Solution 2

Let $\angle SPU = \theta$. Note that $0^\circ \le \theta \le 45^\circ$.

Then $\tan\theta = \dfrac{SU}{PS}$, so $SU = 4\tan\theta$.

Since $SR = 4$, then $UR = SR - SU = 4 - 4\tan\theta$.

Since $\angle UPT = 45^\circ$, then $\angle QPT = 90^\circ - 45^\circ - \theta = 45^\circ - \theta$.

Thus, $\tan(45^\circ - \theta) = \dfrac{QT}{PQ}$ and so $QT = 4\tan(45^\circ - \theta)$.

Since $QR = 4$, then $RT = 4 - 4\tan(45^\circ - \theta)$.

But $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, so $\tan(45^\circ - \theta) = \dfrac{\tan(45^\circ) - \tan\theta}{1 + \tan(45^\circ)\tan\theta} = \dfrac{1 - \tan\theta}{1 + \tan\theta}$, since $\tan(45^\circ) = 1$.

This gives $RT = 4 - 4\left(\dfrac{1 - \tan\theta}{1 + \tan\theta}\right) = \dfrac{4 + 4\tan\theta}{1 + \tan\theta} - \dfrac{4 - 4\tan\theta}{1 + \tan\theta} = \dfrac{8\tan\theta}{1 + \tan\theta}$.

By the Pythagorean Theorem in $\triangle URT$, we obtain

$$\begin{aligned}
UT &= \sqrt{UR^2 + RT^2} \\
&= \sqrt{(4 - 4\tan\theta)^2 + \left(\frac{8\tan\theta}{1 + \tan\theta}\right)^2} \\
&= 4\sqrt{(1 - \tan\theta)^2 + \left(\frac{2\tan\theta}{1 + \tan\theta}\right)^2} \\
&= 4\sqrt{\left(\frac{1 - \tan^2\theta}{1 + \tan\theta}\right)^2 + \left(\frac{2\tan\theta}{1 + \tan\theta}\right)^2} \\
&= 4\sqrt{\frac{1 - 2\tan^2\theta + \tan^4\theta + 4\tan^2\theta}{(1 + \tan\theta)^2}} \\
&= 4\sqrt{\frac{1 + 2\tan^2\theta + \tan^4\theta}{(1 + \tan\theta)^2}} \\
&= 4\sqrt{\frac{(1 + \tan^2\theta)^2}{(1 + \tan\theta)^2}} \\
&= 4\left(\frac{1 + \tan^2\theta}{1 + \tan\theta}\right)
\end{aligned}$$

Therefore, the perimeter of $\triangle URT$ is

$$\begin{aligned}
UR + RT + UT &= 4 - 4\tan\theta + \frac{8\tan\theta}{1 + \tan\theta} + 4\left(\frac{1 + \tan^2\theta}{1 + \tan\theta}\right) \\
&= 4\left(\frac{1 - \tan^2\theta}{1 + \tan\theta} + \frac{2\tan\theta}{1 + \tan\theta} + \frac{1 + \tan^2\theta}{1 + \tan\theta}\right) \\
&= 4\left(\frac{2 + 2\tan\theta}{1 + \tan\theta}\right) \\
&= 8
\end{aligned}$$

Thus, the perimeter is always 8, regardless of the value of θ, so the maximum possible perimeter is 8.

10. Throughout this problem, we represent the states of the n plates as a string of 0's and 1's (called a *binary string*) of length n of the form $p_1p_2\cdots p_n$, with the rth digit from the left (namely p_r) equal to 1 if plate r contains a gift and equal to 0 if plate r does not. We call a binary string of length n *allowable* if it satisfies the requirements – that is, if no two adjacent digits both equal 1. Note that digit p_n is also "adjacent" to digit p_1, so we cannot have $p_1 = p_n = 1$.

(a) Suppose that $p_1 = 1$.
Then $p_2 = p_7 = 0$, so the string is of the form $10p_3p_4p_5p_60$.
Since $k = 3$, then 2 of p_3, p_4, p_5, p_6 equal 1, but in such a way that no two adjacent digits are both 1.
The possible strings in this case are 1010100, 1010010 and 1001010.

Suppose that $p_1 = 0$. Then p_2 can equal 1 or 0.
If $p_2 = 1$, then $p_3 = 0$ as well. This means that the string is of the form $010p_4p_5p_6p_7$, which is the same as the general string in the first case, but shifted by 1 position around the circle, so there are again 3 possibilities.
If $p_2 = 0$, then the string is of the form $00p_3p_4p_5p_6p_7$ and 3 of the digits p_3, p_4, p_5, p_6, p_7 equal 1 in such a way that no 2 adjacent digits equal 1.
There is only 1 way in which this can happen: 0010101.
Overall, this gives 7 possible configurations, so $f(7, 3) = 7$.

(b) *Solution 1*
An allowable string $p_1p_2\cdots p_{n-1}p_n$ has $(p_1, p_n) = (1, 0)$, $(0, 1)$, or $(0, 0)$.
Define $g(n, k, 1, 0)$ to be the number of allowable strings of length n, containing k 1's, and with $(p_1, p_n) = (1, 0)$.
We define $g(n, k, 0, 1)$ and $g(n, k, 0, 0)$ in a similar manner.
Note that $f(n, k) = g(n, k, 1, 0) + g(n, k, 0, 1) + g(n, k, 0, 0)$.

Consider the strings counted by $g(n, k, 0, 1)$.
Since $p_n = 1$, then $p_{n-1} = 0$. Since $p_1 = 0$, then p_2 can equal 0 or 1.
We remove the first and last digits of these strings.
We obtain strings $p_2p_3\cdots p_{n-2}p_{n-1}$ that is strings of length $n-2$ containing $k-1$ 1's.
Since $p_{n-1} = 0$, then the first and last digits of these strings are not both 1. Also, since the original strings did not contain two consecutive 1's, then these new strings does not either.
Therefore, $p_2p_3\cdots p_{n-2}p_{n-1}$ are allowable strings of length $n-2$ containing $k-1$ 1's, with $p_{n-1} = 0$ and $p_2 = 1$ or $p_2 = 0$.
The number of such strings with $p_2 = 1$ and $p_{n-1} = 0$ is $g(n-2, k-1, 1, 0)$ and the number of such strings with $p_2 = 0$ and $p_{n-1} = 0$ is $g(n-2, k-1, 0, 0)$.
Thus, $g(n, k, 0, 1) = g(n-2, k-1, 1, 0) + g(n-2, k-1, 0, 0)$.

Consider the strings counted by $g(n, k, 0, 0)$.
Since $p_1 = 0$ and $p_n = 0$, then we can remove p_n to obtain strings $p_1p_2\cdots p_{n-1}$ of length $n-1$ containing k 1's. These strings are allowable since $p_1 = 0$ and the original strings were allowable.
Note that we have $p_1 = 0$ and p_{n-1} is either 0 or 1.
So the strings $p_1p_2\cdots p_{n-1}$ are allowable strings of length $n-1$ containing k 1's, starting with 0, and ending with 0 or 1.
The number of such strings with $p_1 = 0$ and $p_{n-1} = 0$ is $g(n-1, k, 0, 0)$ and the number of such strings with $p_1 = 0$ and $p_{n-1} = 1$ is $g(n-1, k, 0, 1)$.
Thus, $g(n, k, 0, 0) = g(n-1, k, 0, 0) + g(n-1, k, 0, 1)$.

Consider the strings counted by $g(n,k,1,0)$.
Here, $p_1 = 1$ and $p_n = 0$. Thus, p_{n-1} can equal 0 or 1. We consider these two sets separately.
If $p_{n-1} = 0$, then the string $p_1p_2 \cdots p_{n-1}$ is an allowable string of length $n-1$, containing k 1's, beginning with 1 and ending with 0.
Therefore, the number of strings counted by $g(n,k,1,0)$ with $p_{n-1} = 0$ is equal to $g(n-1,k,1,0)$.
If $p_{n-1} = 1$, then the string $p_2p_3 \cdots p_{n-1}$ is of length $n-2$, begins with 0 and ends with 1. Also, it contains $k-1$ 1's (having removed the original leading 1) and is allowable since the original string was.
Therefore, the number of strings counted by $g(n,k,1,0)$ with $p_{n-1} = 1$ is equal to $g(n-2,k-1,0,1)$.

Therefore,

$$\begin{aligned} f(n,k) &= g(n,k,1,0) + g(n,k,0,1) + g(n,k,0,0) \\ &= (g(n-1,k,1,0) + g(n-2,k-1,0,1)) \\ &\quad +(g(n-2,k-1,1,0) + g(n-2,k-1,0,0)) \\ &\quad +(g(n-1,k,0,0) + g(n-1,k,0,1)) \\ &= (g(n-1,k,1,0) + g(n-1,k,0,1) + g(n-1,k,0,0)) \\ &\quad +(g(n-2,k-1,0,1) + g(n-2,k-1,1,0) + g(n-2,k-1,0,0)) \\ &= f(n-1,k) + f(n-2,k-1) \end{aligned}$$

as required.

Solution 2
We develop an explicit formula for $f(n,k)$ by building these strings.
Consider the allowable strings of length n that include k 1's. Either $p_n = 0$ or $p_n = 1$.

Consider first the case when $p_n = 0$. (Here, p_1 can equal 0 or 1.)
These strings are all of the form $p_1p_2p_3 \cdots p_{n-1}0$.
In this case, since a 1 is always followed by a 0 and the strings end with 0, we can build these strings using blocks of the form 10 and 0. Any combination of these blocks will be an allowable string, as each 1 will always be both preceded and followed by a 0.
Thus, these strings can all be built using k 10 blocks and $n-2k$ 0 blocks. This gives k 1's and $k + (n-2k) = n-k$ 0's. Note that any string built with these blocks will be allowable and will end with a 0, and any such allowable string can be built in this way.
The number of ways of arranging k blocks of one kind and $n-2k$ blocks of another kind is $\binom{k+(n-2k)}{k}$, which simplifies to $\binom{n-k}{k}$.
Consider next the case when $p_n = 1$.
Here, we must have $p_{n-1} = p_1 = 0$, since these are the two digits adjacent to p_n.
Thus, these strings are all of the form $0p_2p_3 \cdots 01$.
Consider the strings formed by removing the first and last digits.
These strings are allowable, are of length $n-2$, include $k-1$ 1's, end with 0, and can begin with 0 or 1.
Again, since a 1 is always followed by a 0 and the strings end with 0, we can build these strings using blocks of the form 10 and 0. Any combination of these blocks will be an allowable string, as each 1 will always be both preceded and followed by a 0.
Translating our method of counting from the first case, there are $\binom{(n-2)-(k-1)}{k-1}$ or

$\binom{n-k-1}{k-1}$ such strings.

Thus, $f(n,k) = \binom{n-k}{k} + \binom{n-k-1}{k-1}$ such strings.

To prove the desired fact, we will use the fact that $\binom{m}{r} = \binom{m-1}{r} + \binom{m-1}{r-1}$, which we prove at the end.

Now

$$\begin{aligned}
&f(n-1,k) + f(n-2,k-1)\\
&= \binom{(n-1)-k}{k} + \binom{(n-1)-k-1}{k-1} + \binom{(n-2)-(k-1)}{k-1} + \binom{(n-2)-(k-1)-1}{(k-1)-1}\\
&= \binom{n-k-1}{k} + \binom{n-k-2}{k-1} + \binom{n-k-1}{k-1} + \binom{n-k-2}{k-2}\\
&= \binom{n-k-1}{k} + \binom{n-k-1}{k-1} + \binom{n-k-2}{k-1} + \binom{n-k-2}{k-2}\\
&= \binom{n-k}{k} + \binom{n-k-1}{k-1} \qquad \text{(using the identity above)}\\
&= f(n,k)
\end{aligned}$$

as required.

To prove the identity, we expand the terms on the right side:

$$\begin{aligned}
\binom{m-1}{r} + \binom{m-1}{r-1} &= \frac{(m-1)!}{r!(m-r-1)!} + \frac{(m-1)!}{(r-1)!(m-r)!}\\
&= \frac{(m-1)!(m-r)}{r!(m-r-1)!(m-r)} + \frac{r(m-1)!}{r(r-1)!(m-r)!}\\
&= \frac{(m-1)!(m-r)}{r!(m-r)!} + \frac{r(m-1)!}{r!(m-r)!}\\
&= \frac{(m-1)!(m-r+r)}{r!(m-r)!}\\
&= \frac{(m-1)!m}{r!(m-r)!}\\
&= \frac{m!}{r!(m-r)!}\\
&= \binom{m}{r}
\end{aligned}$$

as required.

(c) We use the formula for $f(n,k)$ developed in Solution 2 to (b). In order to look at divisibility, we need to first simplify the formula:

$$\begin{aligned}
f(n,k) &= \binom{n-k}{k} + \binom{n-k-1}{k-1} \\
&= \frac{(n-k)!}{k!(n-k-k)!} + \frac{(n-k-1)!}{(k-1)!((n-k-1)-(k-1))!} \\
&= \frac{(n-k)!}{k!(n-2k)!} + \frac{(n-k-1)!}{(k-1)!(n-2k)!} \\
&= \frac{(n-k-1)!(n-k)}{k!(n-2k)!} + \frac{(n-k-1)!k}{k!(n-2k)!} \\
&= \frac{(n-k-1)!(n-k+k)}{k!(n-2k)!} \\
&= \frac{n(n-k-1)!}{k!(n-2k)!} \\
&= \frac{n(n-k-1)(n-k-2)\cdots(n-2k+2)(n-2k+1)}{k!}
\end{aligned}$$

Now that we have written $f(n,k)$ as a product, it is significantly easier to look at divisibility.

Note that $2009 = 41 \times 49 = 7^2 \times 41$, so we need $f(n,k)$ to be divisible by 41 and by 7 twice. For this to be the case, the numerator of $f(n,k)$ must have at least one more factor of 41 and at least two more factors of 7 than the denominator.

Also, we want to minimize $n+k$, so we work to keep n and k as small as possible.

If $n = 49$ and $k = 5$, then

$$f(49,5) = \frac{49(43)(42)(41)(40)}{5!} = \frac{49(43)(42)(41)(40)}{5(4)(3)(2)(1)} = 49(43)(14)(41)$$

which is divisible by 2009.

We show that this pair minimizes the value of $n+k$ with a value of 54.

We consider the possible cases by looking separately at the factors of 41 and 7 that must occur. We focus on the factor of 41 first.

For the numerator to contain a factor of 41, either n is divisible by 41 or one of the terms in the product $(n-k-1)(n-k-2)\cdots(n-2k+1)$ is divisible by 41.

<u>Case 1: n is divisible by 41</u>

We already know that $n = 82$ is too large, so we consider $n = 41$. From the original interpretation of $f(n,k)$, we see that $k \leq 20$, as there can be no more than 20 gifts placed on 41 plates.

Here, the numerator becomes 41 times the product of $k-1$ consecutive integers, the largest of which is $40-k$.

Now the numerator must also contain at least two factors of 7 more than the denominator. But the denominator is the product of k consecutive integers. Since the numerator contains the product of $k-1$ consecutive integers and the denominator contains the product of k consecutive integers, then the denominator will always include at least as many multiples of 7 as the numerator (since there are more consecutive integers in the product in the denominator). Thus, it is impossible for the numerator to contain even one more

additional factor of 7 than the denominator.
Therefore, if $n = 41$, then $f(n, k)$ cannot be divisible by 2009.

Case 2: n is not divisible by 41
This means that the factor of 41 in the numerator must occur in the product

$$(n-k-1)(n-k-2)\cdots(n-2k+1)$$

In this case, the integer 41 must occur in this product, since an occurrence of 82 would make n greater than 82, which does not minimize $n+k$.
So we try to find values of n and k that include the integer 41 in this list.
Note that $n-k-1$ is the largest factor in the product and $n-2k+1$ is the smallest.
Since 41 is contained somewhere in the product, then $n-2k+1 \leq 41$ (giving $n \leq 40+2k$) and $41 \leq n-k-1$ (giving $n \geq 42+k$).
Combining these restrictions, we get $42+k \leq n \leq 40+2k$.

Now, we focus on the factors of 7.
Either n is not divisible by 7 or n is divisible by 7.

* If n is not divisible by 7, then at least two factors of 7 must be included in the product

  $$(n-k-1)(n-k-2)\cdots(n-2k+1)$$

  which means that either $k \geq 8$ (to give two multiples of 7 in this list of $k-1$ consecutive integers) or one of the factors is divisible by 49.
  - If $k \geq 8$, then $n \geq 42+k \geq 50$ so $n+k \geq 58$, which is not minimal.
  - If one of the factors is a multiple of 49, then 49 must be included in the list so $n-2k+1 \leq 49$ (giving $n \leq 48+2k$) and $49 \leq n-k-1$ (giving $n \geq 50+k$).
    In this case, we already know that $42+k \leq n \leq 40+2k$ and now we also have $50+k \leq n \leq 48+2k$.
    For these ranges to overlap, we need $50+k \leq 40+2k$ and so $k \geq 10$, which means that $n \geq 50+k \geq 60$, and so $n+k \geq 70$, which is not minimal.
* Next, we consider the case where n is a multiple of 7.
  Here, $42+k \leq n \leq 40+2k$ (to include 41 in the product) and n is a multiple of 7.
  Since k is at least 2 by definition, then $n \geq 42+k \geq 44$, so n is at least 49.
  If n was 56 or more, we do not get a minimal value for $n+k$.
  Thus, we need to have $n = 49$. In this case, we do not need to look for another factor of 7 in the list.
  To complete this case, we need to find the smallest value of k for which 49 is in the range from $42+k$ to $40+2k$ because we need to have $42+k \leq n \leq 40+2k$.
  This value of k is $k = 5$, which gives $n+k = 49+5 = 54$.

Since $f(49, 5)$ is divisible by 2009, as determined above, then this is the case that minimizes $n+k$, giving a value of 54.

Canadian
Mathematics
Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2008 Euclid Contest

Tuesday, April 15, 2008

Solutions

1. (a) *Solution 1*

By the Pythagorean Theorem in $\triangle ADB$,

$$AB^2 = BD^2 + DA^2 = 9^2 + 12^2 = 81 + 144 = 225$$

so $AB = \sqrt{225} = 15$ since $AB > 0$.

By the Pythagorean Theorem in $\triangle ADC$,

$$DC^2 = CA^2 - AD^2 = 20^2 - 12^2 = 400 - 144 = 256$$

so $DC = \sqrt{256} = 16$ since $AD > 0$.

The perimeter of $\triangle ABC$ is

$$AB + BC + CA = AB + (BD + DC) + CA = 15 + (9 + 16) + 20 = 60$$

Solution 2

Since $BD : DA = 9 : 12 = 3 : 4$ and $\triangle BDA$ is right-angled at B, then $\triangle ADB$ is similar to a 3-4-5 triangle. Thus, $AB = \frac{5}{3}BD = 15$.

Since $CA : DA = 20 : 12 = 5 : 3$ and $\triangle ADC$ is right-angled at D, then $\triangle ADC$ is similar to a 3-4-5 triangle. Thus, $DC = \frac{4}{5}CA = 16$.

Therefore, the perimeter of $\triangle ABC$ is

$$AB + BC + CA = AB + (BD + DC) + CA = 15 + (9 + 16) + 20 = 60$$

(b) *Solution 1*

Since $P(5, 4)$ is the midpoint of $A(a, 0)$ and $B(8, b)$, then 5 is the average of the x-coordinates of A and B and 4 is the average of the y-coordinates of A and B.

Therefore, $5 = \frac{1}{2}(a + 8)$ so $10 = a + 8$ or $a = 2$.

Also, $4 = \frac{1}{2}(0 + b)$ so $8 = 0 + b$ or $b = 8$.

Thus, $a = 2$ and $b = 8$.

Solution 2

$P(5, 4)$ is the midpoint of $A(a, 0)$ and $B(8, b)$.

To get from A to P, we move 4 units up, so to get from P to B we move 4 units up. Therefore, the y-coordinate of B is $4 + 4 = 8$, so $b = 8$.

To get from P to B, we move 3 units to the right, so to get from P to A, we move 3 units to the left. Therefore, the x-coordinate of A is $5 - 3 = 2$, so $a = 2$.

Thus, $a = 2$ and $b = 8$.

(c) The line $ax + y = 30$ passes through the point $(6, 12)$, so $6a + 12 = 30$ or $6a = 18$ or $a = 3$.

This tells us that the line $x + 3y = k$ also passes through the point $(6, 12)$, so $6 + 3(12) = k$ or $k = 42$. Therefore, $k = 42$.

2. (a) *Solution 1*

Since $(c, 7)$ lies on the parabola, then $7 = (c-2)(c-8)+7$ so $(c-2)(c-8) = 0$.
Thus, $c = 2$ or $c = 8$. Since $c \neq 2$, then $c = 8$.

Solution 2

The parabola has equation $y = (x-2)(x-8)+7 = x^2 - 10x + 16 + 7 = x^2 - 10x + 23$.
Completing the square,

$$y = x^2 - 10x + 25 - 25 + 23 = (x-5)^2 - 2$$

so the axis of symmetry of the parabola is $x = 5$.
Since $(2, 7)$ lies on the parabola and is 3 units to the left of the axis of symmetry, then $(5+3, 7) = (8, 7)$ is also on the parabola.
Thus, $c = 8$.

(b) *Solution 1*

Since $(2, 7)$ and $(8, 7)$ lie on the parabola, the axis of symmetry lies halfway between these two points, so has equation $x = 5$.
The vertex lies on the axis of symmetry, so has x-coordinate equal to 5.
The y-coordinate of the vertex is thus $y = (5-2)(5-8)+7 = -9+7 = -2$.
Therefore, the vertex has coordinates $(5, -2)$.

Solution 2

The parabola has equation $y = (x-2)(x-8)+7 = x^2 - 10x + 16 + 7 = x^2 - 10x + 23$.
Completing the square,

$$y = x^2 - 10x + 25 - 25 + 23 = (x-5)^2 - 2$$

so the vertex has coordinates $(5, -2)$.

(c) Since the line passes through $A(5, 0)$ and $B(4, -1)$, it has slope $m = \dfrac{0-(-1)}{5-4} = \dfrac{1}{1} = 1$.
Thus, the equation of the line is $y = x + b$ for some number b.
Since $A(5, 0)$ lies on the line, then $0 = 5 + b$ so $b = -5$, so the line has equation $y = x - 5$.
To find the intersection of the line with parabola, we use the two equations and equate:

$$\begin{aligned}(x-2)(x-8)+7 &= x-5 \\ x^2 - 10x + 16 + 7 &= x - 5 \\ x^2 - 11x + 28 &= 0 \\ (x-4)(x-7) &= 0\end{aligned}$$

Therefore, $x = 4$ or $x = 7$.
However, we already have the point where $x = 4$, so we consider $x = 7$.
We need to find the y-coordinate of this point. To do so, it is easier to use the equation

of the line than the equation of the parabola. Thus, $y = 7 - 5 = 2$.
Therefore, the other point is $(7, 2)$.

3. (a) *Solution 1*

Suppose that the middle number inside the square frame is x.
Then the other numbers in the middle row inside the frame are $x - 1$ and $x + 1$.
Also, the other numbers in the left column are $(x-1)-7 = x-8$ and $(x-1)+7 = x+6$, since there are 7 numbers in each row of the large grid. This tells us that the other numbers in the first and third rows are $x - 7$, $x - 6$, $x + 7$, and $x + 8$.
Therefore, the sum of the numbers in the frame is
$x + x - 1 + x + 1 + x - 8 + x - 7 + x - 6 + x + 6 + x + 7 + x + 8 = 9x$
Thus, the sum of numbers in the frame is 9 times the middle number.
(We can check this using the given example. We might also have been able to see initially that the average of the numbers inside the square frame is always the middle number.)
For the sum of the numbers inside the frame to be 279, the middle number in the frame must be $\frac{1}{9}(279) = 31$.
(We can check that this is correct by placing the frame in this position and adding.)

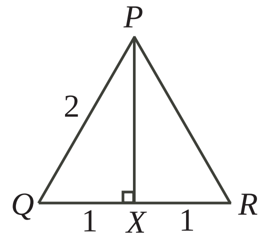
Solution 2

If we start the frame in a given position and slide it one column to the left, the sum of the numbers inside the frame decreases by 9 because the original first and second columns stay inside the frame and the original third column is replaced by a column 3 to the left. In this case, the three numbers entering the frame are each 3 less than the three numbers leaving the frame, so the overall sum decreases by $3 \times 3 = 9$.
Similarly, if the frame slides one column to the right, the sum inside the frame increases by 9.
Also, if the frame slides one row down, the sum inside the frame increases by 63, as the three numbers entering the frame are each 21 larger than the three numbers leaving the frame.
Similarly, if the frame slides one row up, the sum inside the frame decreases by 63.
The original sum inside the frame is 108. We want the new sum to be 279, so we want the sum to increase by $279 - 108 = 171$.
Now $171 = 3(63) - 2(9)$, so if we slide the frame 3 rows down and then 2 columns to the left, the sum will increase by 171 and become 279.
In this case, the new middle number is $12 + 3(7) - 2 = 31$.

(b) In Figure A, the circle has diameter of length 2, so has radius 1, and thus area πr^2 which is $\pi 1^2 = \pi \approx 3.14$.

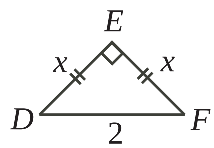
In Figure B, the figure has area equal to twice the area of an equilateral triangle of side length 2.

Consider a single equilateral triangle, $\triangle PQR$, with side length 2.
Drop a perpendicular from P to X on QR.
Since $\triangle PQR$ is equilateral, then $QX = XR = \frac{1}{2}QR = 1$.



Since $\angle PQR = 60^\circ$, then $\triangle PQX$ is 30°-60°-90° triangle, so $PX = \sqrt{3}QX = \sqrt{3}(1) = \sqrt{3}$.
Therefore, the area of $\triangle PQR$ is $\frac{1}{2}(2)(\sqrt{3}) = \sqrt{3}$.
Thus, the area of Figure B is $2\sqrt{3} \approx 3.46$.

In Figure C, the figure has area equal to twice the area of an isosceles right-angled triangle with hypotenuse of length 2.



Suppose that $DE = EF = x$.
Since the triangle is isosceles right-angled, then $x = \frac{1}{\sqrt{2}}(2) = \sqrt{2}$.
Thus, the area of this triangle is $\frac{1}{2}(\sqrt{2})(\sqrt{2}) = \frac{1}{2}(2) = 1$.
Thus, the area of Figure C is 2.

Since $2 < \pi < 2\sqrt{3}$ (because $2 < 3.14 < 3.46$), then the figure with the smallest area is Figure C and the figure with the largest area is Figure B.

4. (a) Since $PF = 20$ m and $\angle PAF = 40^\circ$, then $\dfrac{PF}{AF} = \tan(40^\circ)$ so $AF = \dfrac{20 \text{ m}}{\tan(40^\circ)}$.
Since B is halfway from A to F, then

$$BF = \tfrac{1}{2}AF = \frac{10 \text{ m}}{\tan(40^\circ)}$$

and so

$$\tan(\angle FBP) = \frac{PF}{BF} = \frac{20 \text{ m}}{\left(\frac{10 \text{ m}}{\tan(40^\circ)}\right)} = 2\tan(40^\circ) \approx 1.678$$

Thus, $\angle FBP \approx 59.21^\circ$, and so $\angle FBP$ is 59°, to the nearest degree.

(b) By the cosine law in $\triangle CBA$,

$$\begin{aligned} CA^2 &= CB^2 + BA^2 - 2(CB)(BA)\cos(\angle CBA) \\ CA^2 &= 16^2 + 21^2 - 2(16)(21)\cos(60^\circ) \\ CA^2 &= 256 + 441 - 2(16)(21)(\tfrac{1}{2}) \\ CA^2 &= 256 + 441 - (16)(21) \\ CA^2 &= 361 \\ CA &= \sqrt{361} = 19 \qquad (\text{since } CA > 0) \end{aligned}$$

In $\triangle CAD$, $\angle CDA = 180^\circ - \angle DCA - \angle DAC = 180^\circ - 45^\circ - 30^\circ = 105^\circ$.
By the sine law in $\triangle CDA$,

$$\begin{aligned} \frac{CD}{\sin(\angle DAC)} &= \frac{CA}{\sin(\angle CDA)} \\ CD &= \frac{19\sin(30^\circ)}{\sin(105^\circ)} \\ CD &= \frac{19(\frac{1}{2})}{\sin(105^\circ)} \\ CD &= \frac{19}{2\sin(105^\circ)} \\ CD &\approx 9.835 \end{aligned}$$

so, to the nearest tenth, CD equals 9.8.
(Note that we could have used

$$\begin{aligned} \sin(105^\circ) &= \sin(60^\circ + 45^\circ) = \sin(60^\circ)\cos(45^\circ) + \cos(60^\circ)\sin(45^\circ) \\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} \end{aligned}$$

to say that $CD = \dfrac{19}{2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)} = \dfrac{19\sqrt{2}}{\sqrt{3}+1}$ exactly, and then evaluated this expression.)

5. (a) Consider P on AB with CP perpendicular to AB. Note that $CP = 12$.
Since the small circle with centre A is tangent to the large circle with centre C, then AC equals the sum of the radii of these circles, or $AC = 4 + 9 = 13$. Similarly, $BC = 13$.
This tells us that $\triangle APC$ is congruent to $\triangle BPC$ (they have equal hypotenuses and each is right-angled and has a common side), so $BP = AP$.
By the Pythagorean Theorem in $\triangle APC$,

$$AP^2 = AC^2 - PC^2 = 13^2 - 12^2 = 169 - 144 = 25$$

so $AP = 5$ (since $AP > 0$).
Therefore, $BP = AP = 5$ and so $AB = 10$.
Since it takes the bug 5 seconds to walk this distance, then in 1 second, the bug walks a distance of $\frac{10}{5} = 2$.

(b) For the parabola to have its vertex on the x-axis, the equation

$$y = kx^2 + (5k+3)x + (6k+5) = 0$$

must have two equal real roots.
That is, its discriminant must equal 0, and so

$$\begin{aligned}(5k+3)^2 - 4k(6k+5) &= 0\\ 25k^2 + 30k + 9 - 24k^2 - 20k &= 0\\ k^2 + 10k + 9 &= 0\\ (k+1)(k+9) &= 0\end{aligned}$$

Therefore, $k = -1$ or $k = -9$.

6. (a) Since $f(x) = f(x-1) + f(x+1)$, then $f(x+1) = f(x) - f(x-1)$, and so

$$\begin{aligned}f(1) &= 1\\ f(2) &= 3\\ f(3) &= f(2) - f(1) = 3 - 1 = 2\\ f(4) &= f(3) - f(2) = 2 - 3 = -1\\ f(5) &= f(4) - f(3) = -1 - 2 = -3\\ f(6) &= f(5) - f(4) = -3 - (-1) = -2\\ f(7) &= f(6) - f(5) = -2 - (-3) = 1 = f(1)\\ f(8) &= f(7) - f(6) = 1 - (-2) = 3 = f(2)\end{aligned}$$

Since the value of f at an integer depends only on the values of f at the two previous integers, then the fact that the first several values form a cycle with $f(7) = f(1)$ and $f(8) = f(2)$ tells us that the values of f will always repeat in sets of 6.
Since 2008 is 4 more than a multiple of 6 (as $2008 = 4 + 2004 = 4 + 6(334)$), then $f(2008) = f(2008 - 6(334)) = f(4) = -1$.

(b) *Solution 1*
Since a, b, c form an arithmetic sequence, then we can write $a = b - d$ and $c = b + d$ for some real number d.
Since $a + b + c = 60$, then $(b-d) + b + (b+d) = 60$ or $3b = 60$ or $b = 20$.
Therefore, we can write a, b, c as $20 - d, 20, 20 + d$.
(We could have written a, b, c instead as $a, a+d, a+2d$ and arrived at the same result.)
Thus, $a - 2 = 20 - d - 2 = 18 - d$ and $c + 3 = 20 + d + 3 = 23 + d$, so we can write $a - 2, b, c + 3$ as $18 - d, 20, 23 + d$.

Since these three numbers form a geometric sequence, then

$$\begin{aligned} \frac{20}{18-d} &= \frac{23+d}{20} \\ 20^2 &= (23+d)(18-d) \\ 400 &= -d^2-5d+414 \\ d^2+5d-14 &= 0 \\ (d+7)(d-2) &= 0 \end{aligned}$$

Therefore, $d=-7$ or $d=2$.

If $d=-7$, then $a=27$, $b=20$ and $c=13$.

If $d=2$, then $a=18$, $b=20$ and $c=22$.

(We can check that, in each case, $a-2, b, c+3$ is a geometric sequence.)

Solution 2

Since a, b, c form an arithmetic sequence, then $c-b=b-a$ or $a+c=2b$.

Since $a+b+c=60$, then $2b+b=60$ or $3b=60$ or $b=20$.

Thus, $a+c=40$, so $a=40-c$.

Therefore, we can write a, b, c as $40-c, 20, c$.

Also, $a-2=40-c-2=38-c$, so we can write $a-2, b, c+3$ as $38-c, 20, c+3$.

Since these three numbers form a geometric sequence, then

$$\begin{aligned} \frac{20}{38-c} &= \frac{c+3}{20} \\ 20^2 &= (38-c)(c+3) \\ 400 &= -c^2+35c+114 \\ c^2-35d+286 &= 0 \\ (c-13)(c-22) &= 0 \end{aligned}$$

Therefore, $c=13$ or $c=22$.

If $c=13$, then $a=27$, so $a=27$, $b=20$ and $c=13$.

If $c=22$, then $a=18$, so $a=18$, $b=20$ and $c=22$.

(We can check that, in each case, $a-2, b, c+3$ is a geometric sequence.)

7. (a) Since the average of three consecutive multiples of 3 is a, then a is the middle of these three integers, so the integers are $a-3, a, a+3$.

Since the average of four consecutive multiples of 4 is $a+27$, then $a+27$ is halfway in between the second and third of these multiples (which differ by 4), so the second and third of the multiples are $(a+27)-2=a+25$ and $(a+27)+2=a+29$, so the four integers are $a+21, a+25, a+29, a+33$.

(We have used in these two statements the fact that if a list contains an odd number of integers, then there is a middle integer in the list, and if the list contains an even number of integers, then the "middle" integer is between two integers from the list.)

The smallest of these seven integers is $a-3$ and the largest is $a+33$.

The average of these two integers is $\frac{1}{2}(a-3+a+33)=\frac{1}{2}(2a+30)=a+15$.

Since $a+15=42$, then $a=27$.

(b) Suppose that Billy removes the ball numbered x from his bag and that Crystal removes the ball numbered y from her bag.

Then $b=1+2+3+4+5+6+7+8+9-x=45-x$.

Also, $c=1+2+3+4+5+6+7+8+9-y=45-y$.

Hence, $b-c=(45-x)-(45-y)=y-x$.

Since $1 \leq x \leq 9$ and $1 \leq y \leq 9$, then $-8 \leq y-x \leq 8$.

(This is because $y-x$ is maximized when y is largest (that is, $y=9$) and x is smallest (that is, $x=1$), so $y-x \leq 9-1=8$. Similarly, $y-x \geq -8$.)

Since $b-c=y-x$ is between -8 and 8, then for it to be a multiple of 4, $b-c=y-x$ can be -8, -4, 0, 4, or 8.

Since each of Billy and Crystal chooses 1 ball from 9 balls and each ball is equally likely to be chosen, then the probability of any specific ball being chosen from one of their bags is $\frac{1}{9}$. Thus, the probability of any specific pair of balls being chosen (one from each bag) is $\frac{1}{9} \times \frac{1}{9}=\frac{1}{81}$.

Therefore, to compute the desired probability, we must count the number of pairs (x,y) where $y-x$ is -8, -4, 0, 4, 8, and multiply this result by $\frac{1}{81}$.

<u>Method 1</u>

If $y-x=-8$, then (x,y) must be $(9,1)$.

If $y-x=8$, then (x,y) must be $(1,9)$.

If $y-x=-4$, then (x,y) can be $(5,1),(6,2),(7,3),(8,4),(9,5)$.

If $y-x=4$, then (x,y) can be $(1,5),(2,6),(3,7),(4,8),(5,9)$.

If $y-x=0$, then (x,y) can be $(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)$.

There are thus 21 pairs (x,y) that work, so the desired probability is $\frac{21}{81}=\frac{7}{27}$.

<u>Method 2</u>

If $x=9$, then for $y-x$ to be a multiple of 4, y could be 9, 5 or 1.

If $x=8$, then for $y-x$ to be a multiple of 4, y could be 8 or 4.

If $x=7$, then for $y-x$ to be a multiple of 4, y could be 7 or 3.

If $x=6$, then for $y-x$ to be a multiple of 4, y could be 6 or 2.

If $x=5$, then for $y-x$ to be a multiple of 4, y could be 9, 5 or 1.

If $x=4$, then for $y-x$ to be a multiple of 4, y could be 8 or 4.

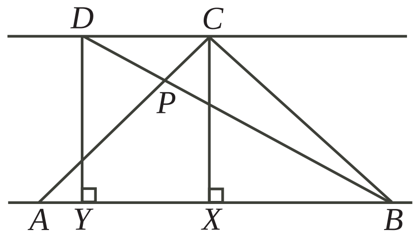
If $x=3$, then for $y-x$ to be a multiple of 4, y could be 7 or 3.

If $x=2$, then for $y-x$ to be a multiple of 4, y could be 6 or 2.

If $x=1$, then for $y-x$ to be a multiple of 4, y could be 9, 5 or 1.

There are thus 21 pairs (x, y) that work, so the desired probability is $\frac{21}{81} = \frac{7}{27}$.

8. (a) Since $AC = CB$, then $\triangle ACB$ is isosceles and right-angled, so $\angle CAB = \angle CBA = 45^\circ$.
Drop perpendiculars from C and D to X and Y on AB.



Since $\triangle ACB$ is isosceles, then $AX = XB = \frac{1}{2}AB = 1$.
Since $\angle CAX = 45^\circ$, then $\triangle AXC$ is isosceles and right-angled, so $CX = AX = 1$.
Since AB is parallel to DC, then $DY = CX = 1$.
Consider $\triangle BDY$. We know that $\angle DYB = 90^\circ$, $DB = 2$ and $DY = 1$.
This tells us that $\triangle BDY$ is a 30°-60°-90° triangle, as one of its legs and its hypotenuse are in the ratio $1 : 2$.
Therefore, $\angle DBY = 30^\circ$, and so $\angle DBC = \angle CBA - \angle DBY = 45^\circ - 30^\circ = 15^\circ$.

(b) *Solution 1*
Let $AP = x$ and $QP = h$.
Since QP is parallel to CB, then QP is perpendicular to BA.
Consider trapezoid $CBPQ$. We can think of this as having parallel bases of lengths 4 and h and height 5. Thus, its area is $\frac{1}{2}(4 + h)(5)$.
However, we can also compute its area by adding the areas of $\triangle CBR$ (which is $\frac{1}{2}(4)(3)$), $\triangle CRQ$ (which is given as 5), and $\triangle RPQ$ (which is $\frac{1}{2}(2)(h)$).
Thus,

$$\begin{aligned}
\tfrac{1}{2}(4 + h)(5) &= \tfrac{1}{2}(4)(3) + 5 + \tfrac{1}{2}(2)(h) \\
20 + 5h &= 12 + 10 + 2h \\
3h &= 2 \\
h &= \tfrac{2}{3}
\end{aligned}$$

Now, $\triangle APQ$ is similar to $\triangle ABC$, as each has a right angle and they share a common angle at A. Thus,

$$\begin{aligned}
\frac{AP}{PQ} &= \frac{AB}{BC} \\
(AP)(BC) &= (PQ)(AB) \\
4x &= \tfrac{2}{3}(x + 5) \\
4x &= \tfrac{2}{3}x + \tfrac{10}{3} \\
\tfrac{10}{3}x &= \tfrac{10}{3} \\
x &= 1
\end{aligned}$$

Therefore, $AP = x = 1$.

Solution 2

Let $AP = x$ and $QP = h$.
Since QP is parallel to CB, then QP is perpendicular to BA.
Since $\triangle ABC$ is right-angled at B, its area is $\frac{1}{2}(4)(5+x) = 10 + 2x$.
However, we can look at the area of the $\triangle ABC$ in terms of its four triangular pieces: $\triangle CBR$ (which has area $\frac{1}{2}(4)(3)$), $\triangle CRQ$ (which has area 5), $\triangle QPR$ (which has area $\frac{1}{2}h(2)$), and $\triangle QPA$ (which has area $\frac{1}{2}xh$).
Therefore, $10 + 2x = 6 + 5 + h + \frac{1}{2}xh$ so $xh - 4x + 2h + 2 = 0$.

Now, $\triangle APQ$ is similar to $\triangle ABC$, as each has a right angle and they share a common angle at A. Thus,

$$\begin{aligned}
\frac{AP}{PQ} &= \frac{AB}{BC} \\
(AP)(BC) &= (PQ)(AB) \\
x(4) &= h(x+5) \\
4x &= hx + 5h \\
-5h &= hx - 4x
\end{aligned}$$

Substituting this into the equation above, $xh + 2h - 4x + 2 = 0$ becomes $-5h + 2h + 2 = 0$ or $3h = 2$ or $h = \frac{2}{3}$.
Lastly, we solve for x by subsituting our value for h: $-5(\frac{2}{3}) = \frac{2}{3}x - 4x$ or $-\frac{10}{3} = -\frac{10}{3}x$ and so $x = 1$.
Therefore, $AP = x = 1$.

9. (a) *Solution 1*

Rewriting the equation, we obtain

$$\begin{aligned}
2^{x+2}5^{6-x} &= 2^{x^2}5^{x^2} \\
1 &= 2^{x^2}2^{-2-x}5^{x^2}5^{x-6} \\
1 &= 2^{x^2-x-2}5^{x^2+x-6} \\
0 &= (x^2 - x - 2)\log_{10} 2 + (x^2 + x - 6)\log_{10} 5 \\
0 &= (x-2)(x+1)\log_{10} 2 + (x-2)(x+3)\log_{10} 5 \\
0 &= (x-2)[(x+1)\log_{10} 2 + (x+3)\log_{10} 5] \\
0 &= (x-2)[(\log_{10} 2 + \log_{10} 5)x + (\log_{10} 2 + 3\log 105)] \\
0 &= (x-2)[(\log_{10} 10)x + \log_{10}(2 \cdot 5^3)] \\
0 &= (x-2)(x + \log_{10} 250)
\end{aligned}$$

Therefore, $x = 2$ or $x = -\log_{10} 250$.

Solution 2

We take base 10 logarithms of both sides:

$$\begin{aligned} \log_{10}(2^{x+2}5^{6-x}) &= \log_{10}(10^{x^2}) \\ \log_{10}(2^{x+2}) + \log_{10}(5^{6-x}) &= x^2 \\ (x+2)\log_{10} 2 + (6-x)\log_{10} 5 &= x^2 \\ x(\log_{10} 2 - \log_{10} 5) + (2\log_{10} 2 + 6\log_{10} 5) &= x^2 \\ x^2 - x(\log_{10} 2 - \log_{10} 5) - (2\log_{10} 2 + 6\log_{10} 5) &= 0 \end{aligned}$$

Now, $\log_{10} 2 + \log_{10} 5 = \log_{10} 10 = 1$ so $\log_{10} 5 = 1 - \log_{10} 2$, so we can simplify the equation to

$$x^2 - x(2\log_{10} 2 - 1) - (6 - 4\log_{10} 2) = 0$$

This is a quadratic equation in x, so should have at most 2 real solutions.

By the quadratic formula,

$$\begin{aligned} x &= \frac{(2\log_{10} 2 - 1) \pm \sqrt{(2\log_{10} 2 - 1)^2 - 4(1)(-(6 - 4\log_{10} 2))}}{2(1)} \\ &= \frac{(2\log_{10} 2 - 1) \pm \sqrt{4(\log_{10} 2)^2 - 4(\log_{10} 2) + 1 + 24 - 16\log_{10} 2}}{2} \\ &= \frac{(2\log_{10} 2 - 1) \pm \sqrt{4(\log_{10} 2)^2 - 20(\log_{10} 2) + 25}}{2} \\ &= \frac{(2\log_{10} 2 - 1) \pm \sqrt{(2\log_{10} 2 - 5)^2}}{2} \\ &= \frac{(2\log_{10} 2 - 1) \pm (5 - 2\log_{10} 2)}{2} \end{aligned}$$

since $5 - 2\log_{10} 2 > 0$.

Therefore,

$$x = \frac{(2\log_{10} 2 - 1) + (5 - 2\log_{10} 2)}{2} = \frac{4}{2} = 2$$

or

$$x = \frac{(2\log_{10} 2 - 1) - (5 - 2\log_{10} 2)}{2} = \frac{4\log_{10} 2 - 6}{2} = 2\log_{10} 2 - 3$$

(Note that at any point, we could have used a calculator to convert to decimal approximations and solve.)

(b) First, we rewrite the system as

$$\begin{array}{ccccc} x & + & \log_{10} x & = & y - 1 \\ (y-1) & + & \log_{10}(y-1) & = & z - 2 \\ (z-2) & + & \log_{10}(z-2) & = & x \end{array}$$

Second, we make the substitution $a = x$, $b = y - 1$ and $c = z - 2$, allowing us to rewrite the system as

$$\begin{aligned} a+\log_{10} a &= b \qquad (1) \\ b+\log_{10} b &= c \qquad (2) \\ c+\log_{10} c &= a \qquad (3) \end{aligned}$$

Third, we observe that $(a,b,c)=(1,1,1)$ is a solution, since $1+\log_{10} 1 = 1+0 = 1$.
Next, if $a>1$, then $\log_{10} a > 0$, so from (1),

$$b = a + \log_{10} a > a + 0 = a > 1$$

so $\log_{10} b > 0$, so from (2),

$$c = b + \log_{10} b > b + 0 = b > a > 1$$

so $\log_{10} c > 0$, so from (3),

$$a = c + \log_{10} c > c + 0 = c > b > a > 1$$

But this says that $a > c > b > a$, which is a contradiction.
Therefore, a cannot be larger than 1.
Lastly, if $0 < a < 1$ (a cannot be negative), then $\log_{10} a < 0$, so from (1),

$$b = a + \log_{10} a < a + 0 = a < 1$$

so $\log_{10} b < 0$, so from (2),

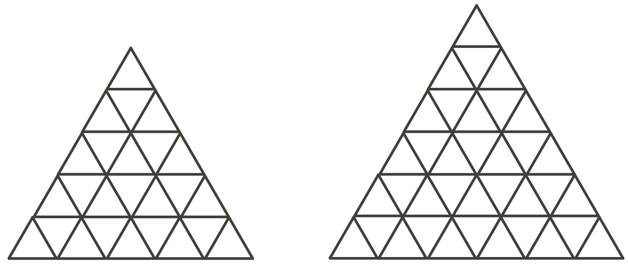
$$c = b + \log_{10} b < b + 0 = b < a < 1$$

so $\log_{10} c < 0$, so from (3),

$$a = c + \log_{10} c > c + 0 = c < b < a < 1$$

But this says that $a < c < b < a$, which is a contradiction.
Therefore, a cannot be smaller than 1 either.
Thus, a must equal 1.
If $a = 1$, then $b = a + \log_{10} a = 1 + \log_{10} 1 = 1 + 0 = 1$ from (1), which will similarly give $c = 1$ from (2).
Thus, the only solution to the system is $(a,b,c) = (1,1,1) = (x, y-1, z-2)$ since a cannot be either larger than or smaller than 1, so $(x,y,z) = (1,2,3)$.

10. (a) If $n = 5$, there are 10 downward-pointing triangles with side length 1 and 3 downward-pointing triangles with side length 2, so $f(5) = 10 + 3 = 13$.



If $n = 6$, there are 15 downward-pointing triangles with side length 1, 6 downward-pointing triangles with side length 2, and 1 downward-pointing triangle with side length 3, so $f(6) = 15 + 6 + 1 = 22$.

(b) *Solution 1*

We determine explicit formulas for $f(2k)$ and $f(2k-1)$ in terms of k and use these formulas to show that $f(2k) - f(2k-1) = k^2$.

Start with a large triangle of side length n. (We will split into cases with n even and odd later.)

We call the ith horizontal line from the top "row i". Note that row i has length i. We label the unit points along the row, starting with 0 at the left-hand end and ending with i at the right-hand end. (These are the points where the diagonal lines intersect row i. We will refer to a generic such point with the variable j.)

Consider first downward-pointing triangles with side length $m = 1$. We count these by counting the possible locations for their bottom vertex.

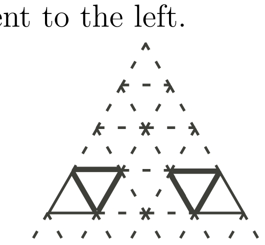
There are no such bottom vertices in row 1.

There is one such bottom vertex in row 2, at $j = 1$.

There are two such bottom vertices in row 3, at $j = 1$ and $j = 2$.

This continues, with $n - 1$ such bottom vertices in row n, at $j = 1$ to $j = n - 1$.

In general, there are $i - 1$ such bottom vertices in row i:

> To see this, we show that the leftmost such vertex is at $j = 1$ and the rightmost is at $j = i - 1$. We obtain the other bottom vertices by translating the downward-pointing triangle.
>
> The leftmost such vertex is at $j = 1$ because the downward-pointing triangle can be completed to form a parallelogram starting at this leftmost vertex and drawing a 1 unit horizontal line segment to the left.
>
> 
>
> Since the downward-pointing triangle is inside the large triangle, this parallelogram must also be, so $j = 1$. (This tells us that we cannot go any further to the left.)

Similarly, the rightmost such vertex is at $j = i - 1$, which we can see by drawing a parallelogram to the right.

Therefore, there are $1 + 2 + \cdots + (n-2) + (n-1) = \frac{1}{2}(n-1)(n)$ downward-pointing triangles of side length $m = 1$, as there are $i-1$ such triangles for each row from $i = 2$ to $i = n$.

For a general side length m of a downward-pointing triangle, we can argue as above that in row i, the leftmost bottom vertex will occur at $j = m$ and the rightmost at $j = i - m$.
For there to be any such triangles, we need $m \leq i - m$ (or $2m \leq i$) so the leftmost vertex is not to the right of the rightmost vertex.
At row i, there are thus $(i - m) - m + 1 = i + 1 - 2m$ possible locations for the bottom vertex.
(When $i = 2m$ (the smallest possible value of i), there is $2m + 1 - 2m = 1$ location for the vertex.
When $i = n$ (the largest possible value of i), there are $n+1-2m$ locations for the vertex.)

For a fixed positive integer n, what are the permissible values of m? Certainly, $m \geq 1$.
If $n = 2k$ for some positive integer k, then $2m \leq n = 2k$ since the largest possible downward-triangle that can be fit has its bottom vertex on the bottom row, so $m \leq k$.
If $n = 2k - 1$ for some positive integer k, then $2m \leq n = 2k - 1$ so $m \leq k - 1$.

Thus, for a fixed permissible value of m, the total number of downward-pointing triangles of side length m is

$$1 + 2 + \cdots + (n + 1 - 2m) = \tfrac{1}{2}(n+1-2m)(n+2-2m) \qquad (*)$$

which is the sum of $i + 1 - 2m$ from $i = 2m$ to $i = n$, because we look at all possible locations for the bottom vertex.

For $n = 2k$, the permissible values of m are $m = 1$ to $m = k$, so we add up the formula in $(*)$ for $m = 1$ to $m = k$:

$$\begin{aligned}
f(2k) &= \sum_{m=1}^{k} \tfrac{1}{2}(2k+1-2m)(2k+2-2m) \\
&= \sum_{l=1}^{k} \tfrac{1}{2}(2l-1)(2l) \qquad \text{(letting } l = k+1-m\text{)} \\
&= \sum_{l=1}^{k} (2l-1)(l) \\
&= \sum_{l=1}^{k} (2l^2 - l) \\
&= 2\sum_{l=1}^{k} l^2 - \sum_{l=1}^{k} l \\
&= 2\left(\tfrac{1}{6}k(k+1)(2k+1)\right) - \tfrac{1}{2}k(k+1) \\
&= k(k+1)\left(\tfrac{1}{3}(2k+1) - \tfrac{1}{2}\right) \\
&= k(k+1)\left(\tfrac{2}{3}k - \tfrac{1}{6}\right) \\
&= \frac{k(k+1)(4k-1)}{6}
\end{aligned}$$

For $n = 2k-1$, the permissible values of m are $m = 1$ to $m = k-1$, so we add up the formula in $(*)$ for $m = 1$ to $m = k-1$:

$$\begin{aligned}
f(2k-1) &= \sum_{m=1}^{k-1} \tfrac{1}{2}(2k-2m)(2k+1-2m) \\
&= \sum_{l=1}^{k-1} \tfrac{1}{2}(2l)(2l+1) \qquad \text{(letting } l = k-m\text{)} \\
&= \sum_{l=1}^{k-1} l(2l+1) \\
&= \sum_{l=1}^{k-1} (2l^2 + l) \\
&= 2\sum_{l=1}^{k-1} l^2 + \sum_{l=1}^{k-1} l \\
&= 2\left(\tfrac{1}{6}(k-1)(k)(2k-1)\right) + \tfrac{1}{2}(k-1)(k) \\
&= k(k-1)\left(\tfrac{1}{3}(2k-1) + \tfrac{1}{2}\right) \\
&= k(k-1)\left(\tfrac{2}{3}k + \tfrac{1}{6}\right) \\
&= \frac{k(k-1)(4k+1)}{6}
\end{aligned}$$

Therefore,

$$\begin{aligned} f(2k) - f(2k-1) &= \frac{k(k+1)(4k-1)}{6} - \frac{k(k-1)(4k+1)}{6} \\ &= \tfrac{1}{6}k((k+1)(4k-1) - (k-1)(4k+1)) \\ &= \tfrac{1}{6}k((4k^2+3k-1) - (4k^2-3k-1)) \\ &= \tfrac{1}{6}k(6k) \\ &= k^2 \end{aligned}$$

as required.

Solution 2

As in Solution 1, we can show that if $n = 2k$, we can fit in downward-pointing triangles of sizes $m = 1$ to $m = k$ and if $n = 2k - 1$, we can fit in downward-pointing triangles of sizes $m = 1$ to $m = k - 1$.

Consider $f(2k) - f(2k - 1)$. To calculate this quantity, we must determine how many *additional* downward-pointing triangles can be put into the large triangle of size $n = 2k$ instead of that of size $n = 2k - 1$.

Since the triangle of size $2k - 1$ can be put inside the triangle of size $2k$ with their top vertices coinciding, then any new downward-pointing triangles inside the triangle of size $n = 2k$ all have their bottom vertices in row $n = 2k$.

We count the number of such triangles by considering the possible values of m.

If $m = 1$, Solution 1 tells us that there are $2k + 1 - 2(1) = 2k - 1$ such triangles.

If $m = 2$, there are $2k + 1 - 2(2) = 2k - 3$ such triangles.

For a general m, there are $2k + 1 - 2m$ such triangles.

The value of $f(2k) - f(2k - 1)$ is equal to the sum of $2k + 1 - 2m$ over all possible values of m.

Therefore,

$$\begin{aligned} f(2k) - f(2k-1) &= \sum_{m=1}^{k}(2k+1-2m) \\ &= \sum_{m=1}^{k}(2k+1) - 2\sum_{m=1}^{k} m \\ &= k(2k+1) - 2\left(\tfrac{1}{2}k(k+1)\right) \\ &= 2k^2 + k - (k^2 + k) \\ &= k^2 \end{aligned}$$

as required.

(c) From Solution 1 to (b), we know that

$$f(2k) = \frac{k(k+1)(4k-1)}{6} \qquad \text{and} \qquad f(2k-1) = \frac{k(k-1)(4k+1)}{6}$$

We rewrite each of these in terms of n.

In the case of n even, since $n = 2k$, then $k = \frac{1}{2}n$, so

$$f(n) = f(2k) = \frac{\frac{1}{2}n(\frac{1}{2}n+1)(4(\frac{1}{2}n)-1)}{6} = \frac{\frac{1}{2}n(\frac{1}{2}n+1)(2n-1)}{6} = \frac{n(n+2)(2n-1)}{24}$$

In the case of n odd, since $n = 2k-1$, then $k = \frac{1}{2}(n+1)$, so

$$\begin{aligned} f(n) &= f(2k-1) = \frac{\frac{1}{2}(n+1)(\frac{1}{2}(n+1)-1)(4(\frac{1}{2}(n+1))+1)}{6} \\ &= \frac{\frac{1}{2}(n+1)(\frac{1}{2}(n+1)-1)(2n+3)}{6} = \frac{(n+1)(n-1)(2n+3)}{24} \end{aligned}$$

<u>Case 1: n is even</u>

If $f(n)$ is divisible by n, then $f(n) = nq$ for some integer q.

Thus,

$$\begin{aligned} nq &= \frac{n(n+2)(2n-1)}{24} \\ 24nq &= n(n+2)(2n-1) \\ 24q &= (n+2)(2n-1) \qquad (\text{since } n \neq 0) \end{aligned}$$

Thus, we need $(n+2)(2n-1)$ to be a multiple of 24.

Since $2n-1$ is odd, then $n+2$ must be a multiple of 8, so $n+2 = 8a$ for some integer a, so $n = 8a-2$.

Therefore, $24q = 8a(2(8a-2)-1)$ or $3q = a(16a-5)$.

Therefore, we still need $a(16a-5)$ to be a multiple of 3.

Since 3 is a prime number, then either a is divisible by 3 or $16a-5 = 3(5a-2)+(a+1)$ is divisible by 3.

If a is divisible by 3, then $a = 3b$ for some integer b.

If $16a-5$ is divisible by 3, then since $16a-5 = 3(5a-2)+(a+1)$, we have that $a+1 = (16a-5)-3(5a-2)$ is divisible by 3, as it is the difference of two multiples of 3. Therefore, $a+1 = 3b$ for some integer b. Therefore, $n = 8(3b)-2 = 24b-2$ or $n = 8(3b-1)-2 = 24b-10$ for some integer b.

We have proven that if $f(n)$ is divisible by n, then $n = 24b-2$ or $n = 24b-10$. We need to verify that each of these forms for n works for all b.

If $n = 24b-2$, then

$$f(n) = f(24b-2) = \frac{(24b-2)(24b)(48b-5)}{24} = b(24b-2)(48b-5)$$

which is divisible by $24b-2$, and so $f(n)$ is always divisible by n in this case.

If $n = 24b-10$, then

$$f(n) = f(24b-10) = \frac{(24b-10)(24b-8)(48b-21)}{24} = (24b-10)(3b-1)(16b-7)$$

which is divisible by $24b - 10$, and so $f(n)$ is always divisible by n in this case.

Thus, each of these forms for n works for all positive integers b.

Case 2: n is odd

If $f(n)$ is divisible by n, then $f(n) = nq$ for some integer q.

Thus,

$$\begin{aligned} nq &= \frac{(n+1)(n-1)(2n+3)}{24} \\ 24nq &= (n+1)(n-1)(2n+3) \\ 24nq &= (n^2-1)(2n+3) \\ 24nq &= 2n^3+3n^2-2n-3 \\ 3 &= 2n^3+3n^2-2n-24nq \\ 3 &= n(2n^2+3n-2-24q) \end{aligned}$$

Since the right side is divisible by n, then the left side must be as well, so n divides into 3 which gives us that $n = 1$ or $n = 3$.

(Note that $f(1) = 0$ which is divisible by 1 and $f(3) = 3$ which is divisible by 3, so both of these cases do work.)

Therefore, $f(n)$ is divisible by n when $n = 1$, $n = 3$, $n = 24b - 10$ for all positive integers b or $n = 24b - 2$ for all positive integers b.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2007 Euclid Contest

Tuesday, April 17, 2007

Solutions

1. (a) Since $(a-1, a+1)$ lies on the line $y = 2x - 3$, then $a + 1 = 2(a-1) - 3$ or $a + 1 = 2a - 5$ or $a = 6$.

   (b) *Solution 1*

   To get from P to Q, we move 3 units right and 4 units up.

   Since $PQ = QR$ and R lies on the line through Q, then we must use the same motion to get from Q to R.

   Therefore, to get from $Q(0,4)$ to R, we move 3 units right and 4 units up, so the coordinates of R are $(3,8)$.

   Solution 2

   The line through $P(-3,0)$ and $Q(0,4)$ has slope $\dfrac{4-0}{0-(-3)} = \dfrac{4}{3}$ and y-intercept 4, so has equation $y = \frac{4}{3}x + 4$.

   Thus, R has coordinates $(a, \frac{4}{3}a + 4)$ for some $a > 0$.

   Since $PQ = QR$, then $PQ^2 = QR^2$, so

$$\begin{aligned}(-3)^2 + 4^2 &= a^2 + \left(\tfrac{4}{3}a + 4 - 4\right)^2 \\ 25 &= a^2 + \tfrac{16}{9}a^2 \\ \tfrac{25}{9}a^2 &= 25 \\ a^2 &= 9\end{aligned}$$

   so $a = 3$ since $a > 0$.

   Thus, R has coordinates $(3, \frac{4}{3}(3) + 4) = (3, 8)$.

   (c) Since $OP = 9$, then the coordinates of P are $(9,0)$.

   Since $OP = 9$ and $OA = 15$, then by the Pythagorean Theorem,

$$AP^2 = OA^2 - OP^2 = 15^2 - 9^2 = 144$$

   so $AP = 12$.

   Since P has coordinates $(9,0)$ and A is 12 units directly above P, then A has coordinates $(9,12)$.

   Since $PB = 4$, then B has coordinates $(13,0)$.

   The line through $A(9,12)$ and $B(13,0)$ has slope $\dfrac{12-0}{9-13} = -3$ so, using the point-slope form, has equation $y - 0 = -3(x - 13)$ or $y = -3x + 39$.

2. (a) Since $\cos(\angle BAC) = \dfrac{AB}{AC}$ and $\cos(\angle BAC) = \frac{5}{13}$ and $AB = 10$, then $AC = \frac{13}{5}AB = 26$.

   Since $\triangle ABC$ is right-angled at B, then by the Pythagorean Theorem,

   $BC^2 = AC^2 - AB^2 = 26^2 - 10^2 = 576$ so $BC = 24$ since $BC > 0$.

   Therefore, $\tan(\angle ACB) = \dfrac{AB}{BC} = \dfrac{10}{24} = \dfrac{5}{12}$.

(b) Since $2\sin^2 x + \cos^2 x = \frac{25}{16}$ and $\sin^2 x + \cos^2 x = 1$ (so $\cos^2 x = 1 - \sin^2 x$), then we get

$$\begin{aligned} 2\sin^2 x + (1 - \sin^2 x) &= \tfrac{25}{16} \\ \sin^2 x &= \tfrac{25}{16} - 1 \\ \sin^2 x &= \tfrac{9}{16} \\ \sin x &= \pm\tfrac{3}{4} \end{aligned}$$

so $\sin x = \frac{3}{4}$ since $\sin x > 0$ because $0^\circ < x < 90^\circ$.

(c) Since $\triangle ABC$ is isosceles and right-angled, then $\angle BAC = 45^\circ$.
Also, $AC = \sqrt{2}AB = \sqrt{2}(2\sqrt{2}) = 4$.
Since $\angle EAB = 75^\circ$ and $\angle BAC = 45^\circ$, then $\angle CAE = \angle EAB - \angle BAC = 30^\circ$.
Since $\triangle AEC$ is right-angled and has a 30° angle, then $\triangle AEC$ is a 30°-60°-90° triangle.
Thus, $EC = \frac{1}{2}AC = 2$ (since EC is opposite the 30° angle) and $AE = \frac{\sqrt{3}}{2}AC = 2\sqrt{3}$ (since AE is opposite the 60° angle).
In $\triangle CDE$, $ED = DC$ and $\angle EDC = 60^\circ$, so $\triangle CDE$ is equilateral.
Therefore, $ED = CD = EC = 2$.
Overall, the perimeter of $ABCDE$ is

$$AB + BC + CD + DE + EA = 2\sqrt{2} + 2\sqrt{2} + 2 + 2 + 2\sqrt{3} = 4 + 4\sqrt{2} + 2\sqrt{3}$$

3. (a) From the given information, the first term in the sequence is 2007 and each term starting with the second can be determined from the previous term.
The second term is $2^3 + 0^3 + 0^3 + 7^3 = 8 + 0 + 0 + 343 = 351$.
The third term is $3^3 + 5^3 + 1^3 = 27 + 125 + 1 = 153$.
The fourth term is $1^3 + 5^3 + 3^3 = 27 + 125 + 1 = 153$.
Since two consecutive terms are equal, then every term thereafter will be equal, because each term depends only on the previous term and a term of 153 always makes the next term 153.
Thus, the 2007th term will be 153.

(b) The nth term of sequence A is $n^2 - 10n + 70$.
Since sequence B is arithmetic with first term 5 and common difference 10, then the nth term of sequence B is equal to $5 + 10(n-1) = 10n - 5$. (Note that this formula agrees with the first few terms.)
For the nth term of sequence A to be equal to the nth term of sequence B, we must have

$$\begin{aligned} n^2 - 10n + 70 &= 10n - 5 \\ n^2 - 20n + 75 &= 0 \\ (n-5)(n-15) &= 0 \end{aligned}$$

Therefore, $n = 5$ or $n = 15$. That is, 5th and 15th terms of sequence A and sequence B are equal to each other.

4. (a) *Solution 1*

Rearranging and then squaring both sides,

$$\begin{aligned} 2+\sqrt{x-2} &= x-2 \\ \sqrt{x-2} &= x-4 \\ x-2 &= (x-4)^2 \\ x-2 &= x^2-8x+16 \\ 0 &= x^2-9x+18 \\ 0 &= (x-3)(x-6) \end{aligned}$$

so $x=3$ or $x=6$.

We should check both solutions, because we may have introduced extraneous solutions by squaring.

If $x=3$, the left side equals $2+\sqrt{1}=3$ and the right side equals 1, so $x=3$ must be rejected.

If $x=6$, the left side equals $2+\sqrt{4}=4$ and the right side equals 4, so $x=6$ is the only solution.

Solution 2

Suppose $u=\sqrt{x-2}$.

The equation becomes $2+u=u^2$ or $u^2-u-2=0$ or $(u-2)(u+1)=0$.

Therefore, $u=2$ or $u=-1$.

But we cannot have $\sqrt{x-2}=-1$ (as square roots are always non-negative).

Therefore, $\sqrt{x-2}=2$ or $x-2=4$ or $x=6$.

We can check as in Solution 1 that $x=6$ is indeed a solution.

(b) *Solution 1*

From the diagram, the parabola has x-intercepts $x=3$ and $x=-3$.

Therefore, the equation of the parabola is of the form $y=a(x-3)(x+3)$ for some real number a.

Triangle ABC can be considered as having base AB (of length $3-(-3)=6$) and height OC (where O is the origin).

Suppose C has coordinates $(0,-c)$. Then $OC=c$.

Thus, the area of $\triangle ABC$ is $\frac{1}{2}(AB)(OC)=3c$. But we know that the area of $\triangle ABC$ is 54, so $3c=54$ or $c=18$.

Since the parabola passes through $C(0,-18)$, then this point must satisfy the equation of the parabola.

Therefore, $-18=a(0-3)(0+3)$ or $-18=-9a$ or $a=2$.

Thus, the equation of the parabola is $y=2(x-3)(x+3)=2x^2-18$.

Solution 2

Triangle ABC can be considered as having base AB (of length $3-(-3)=6$) and height OC (where O is the origin).

Suppose C has coordinates $(0,-c)$. Then $OC = c$.

Thus, the area of $\triangle ABC$ is $\frac{1}{2}(AB)(OC) = 3c$. But we know that the area of $\triangle ABC$ is 54, so $3c = 54$ or $c = 18$.

Therefore, the parabola has vertex $C(0,-18)$, so has equation $y = a(x-0)^2 - 18$.

(The vertex of the parabola must lie on the y-axis since its roots are equally distant from the y-axis, so C must be the vertex.)

Since the parabola passes through $B(3,0)$, then these coordinates satisfy the equation, so $0 = 3^2a - 18$ or $9a = 18$ or $a = 2$.

Therefore, the equation of the parabola is $y = 2x^2 - 18$.

5. (a) The perimeter of the sector is made up of two line segments (of total length $5+5=10$) and one arc of a circle.

Since $\dfrac{72^\circ}{360^\circ} = \dfrac{1}{5}$, then the length of the arc is $\dfrac{1}{5}$ of the total circumference of a circle of radius 5.

Thus, the length of the arc is $\frac{1}{5}(2\pi(5)) = 2\pi$.

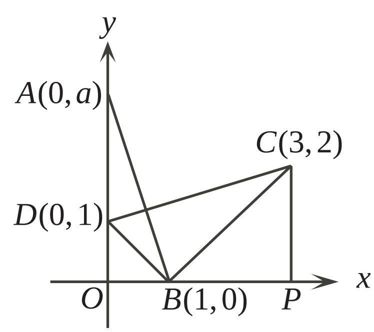
Therefore, the perimeter of the sector is $10 + 2\pi$.

(b) $\triangle AOB$ is right-angled at O, so has area $\frac{1}{2}(AO)(OB) = \frac{1}{2}a(1) = \frac{1}{2}a$.

We next need to calculate the area of $\triangle BCD$.

Method 1: Completing the trapezoid

Drop a perpendicular from C to $P(3,0)$ on the x-axis.



Then $DOPC$ is a trapezoid with parallel sides DO of length 1 and PC of length 2 and height OP (which is indeed perpendicular to the parallel sides) of length 3.

The area of the trapezoid is thus $\frac{1}{2}(DO+PC)(OP) = \frac{1}{2}(1+2)(3) = \frac{9}{2}$.

But the area of $\triangle BCD$ equals the area of trapezoid $DOPC$ minus the areas of $\triangle DOB$ and $\triangle BPC$.

$\triangle DOB$ is right-angled at O, so has area $\frac{1}{2}(DO)(OB) = \frac{1}{2}(1)(1) = \frac{1}{2}$.

$\triangle BPC$ is right-angled at P, so has area $\frac{1}{2}(BP)(PC) = \frac{1}{2}(2)(2) = 2$.

Thus, the area of $\triangle DBC$ is $\frac{9}{2} - \frac{1}{2} - 2 = 2$.

(A similar method for calculating the area of $\triangle DBC$ would be to drop a perpendicular to Q on the y-axis, creating a rectangle $QOPC$.)

Method 2: $\triangle DBC$ is right-angled

The slope of line segment DB is $\dfrac{1-0}{0-1} = -1$.

The slope of line segment BC is $\dfrac{2-0}{3-1} = 1$.

Since the product of these slopes is -1 (that is, their slopes are negative reciprocals), then DB and BC are perpendicular.

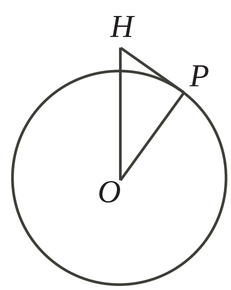
Therefore, the area of $\triangle DBC$ is $\frac{1}{2}(DB)(BC)$.

Now $DB = \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}$ and $BC = \sqrt{(3-1)^2 + (2-0)^2} = \sqrt{8}$.

Thus, the area of $\triangle DBC$ is $\frac{1}{2}\sqrt{2}\sqrt{8} = 2$.

Since the area of $\triangle AOB$ equals the area of $\triangle DBC$, then $\frac{1}{2}a = 2$ or $a = 4$.

6. (a) Suppose that O is the centre of the planet, H is the place where His Highness hovers in the helicopter, and P is the furthest point on the surface of the planet that he can see.



Then HP must be a tangent to the surface of the planet (otherwise he could see further), so OP (a radius) is perpendicular to HP (a tangent).

We are told that $OP = 24$ km.

Since the helicopter hovers at a height of 2 km, then $OH = 24 + 2 = 26$ km.

Therefore, $HP^2 = OH^2 - OP^2 = 26^2 - 24^2 = 100$, so $HP = 10$ km.

Therefore, the distance to the furthest point that he can see is 10 km.

(b) Since we know the measure of $\angle ADB$, then to find the distance AB, it is enough to find the distances AD and BD and then apply the cosine law.

In $\triangle DBE$, we have $\angle DBE = 180^\circ - 20^\circ - 70^\circ = 90^\circ$, so $\triangle DBE$ is right-angled, giving $BD = 100\cos(20^\circ) \approx 93.969$.

In $\triangle DAC$, we have $\angle DAC = 180^\circ - 50^\circ - 45^\circ = 85^\circ$.

Using the sine law, $\dfrac{AD}{\sin(50^\circ)} = \dfrac{CD}{\sin(85^\circ)}$, so $AD = \dfrac{150\sin(50^\circ)}{\sin(85^\circ)} \approx 115.346$.

Finally, using the cosine law in $\triangle ABD$, we get

$$\begin{aligned}
AB^2 &= AD^2 + BD^2 - 2(AD)(BD)\cos(\angle ADB) \\
AB^2 &\approx (115.346)^2 + (93.969)^2 - 2(115.346)(93.969)\cos(35^\circ) \\
AB^2 &\approx 4377.379 \\
AB &\approx 66.16
\end{aligned}$$

Therefore, the distance from A to B is approximately 66 m.

7. (a) Using rules for manipulating logarithms,

$$\begin{aligned}
(\sqrt{x})^{\log_{10} x} &= 100 \\
\log_{10}\left((\sqrt{x})^{\log_{10} x}\right) &= \log_{10} 100 \\
(\log_{10} x)(\log_{10} \sqrt{x}) &= 2 \\
(\log_{10} x)(\log_{10} x^{\frac{1}{2}}) &= 2 \\
(\log_{10} x)(\tfrac{1}{2}\log_{10} x) &= 2 \\
(\log_{10} x)^2 &= 4 \\
\log_{10} x &= \pm 2 \\
x &= 10^{\pm 2}
\end{aligned}$$

Therefore, $x = 100$ or $x = \frac{1}{100}$.
(We can check by substitution that each is indeed a solution.)

(b) *Solution 1*

Without loss of generality, suppose that square $ABCD$ has side length 1.
Suppose next that $BF = a$ and $\angle CFB = \theta$.
Since $\triangle CBF$ is right-angled at B, then $\angle BCF = 90^\circ - \theta$.
Since GCF is a straight line, then $\angle GCD = 180^\circ - 90^\circ - (90^\circ - \theta) = \theta$.
Therefore, $\triangle GDC$ is similar to $\triangle CBF$, since $\triangle GDC$ is right-angled at D.
Thus, $\dfrac{GD}{DC} = \dfrac{BC}{BF}$ or $\dfrac{GD}{1} = \dfrac{1}{a}$ or $GD = \dfrac{1}{a}$.
So $AF = AB + BF = 1 + a$ and $AG = AD + DG = 1 + \dfrac{1}{a} = \dfrac{a+1}{a}$.
Thus, $\dfrac{1}{AF} + \dfrac{1}{AG} = \dfrac{1}{1+a} + \dfrac{a}{a+1} = \dfrac{a+1}{a+1} = 1 = \dfrac{1}{AB}$, as required.

Solution 2

We attach a set of coordinate axes to the diagram, with A at the origin, AG lying along the positive y-axis and AF lying along the positive x-axis.
Without loss of generality, suppose that square $ABCD$ has side length 1, so that C has coordinates $(1, 1)$. (We can make this assumption without loss of generality, because if the square had a different side length, then each of the lengths in the problem would be scaled by the same factor.)

Suppose that the line through G and F has slope m.
Since this line passes through $(1,1)$, its equation is $y-1=m(x-1)$ or $y=mx+(1-m)$.
The y-intercept of this line is $1-m$, so G has coordinates $(0,1-m)$.
The x-intercept of this line is $\dfrac{m-1}{m}$, so F has coordinates $\left(\dfrac{m-1}{m},0\right)$. (Note that $m\neq 0$ as the line cannot be horizontal.)
Therefore,

$$\frac{1}{AF}+\frac{1}{AG}=\frac{m}{m-1}+\frac{1}{1-m}=\frac{m}{m-1}+\frac{-1}{m-1}=\frac{m-1}{m-1}=1=\frac{1}{AB}$$

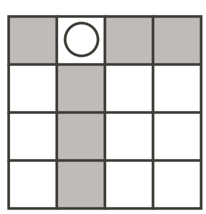
as required.

Solution 3
Join A to C.
We know that the sum of the areas of $\triangle GCA$ and $\triangle FCA$ equals the area of $\triangle GAF$.
The area of $\triangle GCA$ (thinking of AG as the base) is $\frac{1}{2}(AG)(DC)$, since DC is perpendicular to AG.
Similarly, the area of $\triangle FCA$ is $\frac{1}{2}(AF)(CB)$.
Also, the area of $\triangle GAF$ is $\frac{1}{2}(AG)(AF)$.
Therefore,

$$\begin{aligned}\tfrac{1}{2}(AG)(DC)+\tfrac{1}{2}(AF)(CB) &= \tfrac{1}{2}(AG)(AF)\\ \frac{(AG)(DC)}{(AG)(AF)(AB)}+\frac{(AF)(CB)}{(AG)(AF)(AB)} &= \frac{(AG)(AF)}{(AG)(AF)(AB)}\\ \frac{1}{AF}+\frac{1}{AG} &= \frac{1}{AB}\end{aligned}$$

as required, since $AB=DC=CB$.

8. (a) We consider placing the three coins individually.
Place one coin randomly on the grid.
When the second coin is placed (in any one of 15 squares), 6 of the 15 squares will leave two coins in the same row or column and 9 of the 15 squares will leave the two coins in different rows and different columns.



Therefore, the probability that the two coins are in different rows and different columns is $\frac{9}{15}=\frac{3}{5}$.
There are 14 possible squares in which the third coin can be placed.

Of these 14 squares, 6 lie in the same row or column as the first coin and an additional 4 lie the same row or column as the second coin. Therefore, the probability that the third coin is placed in a different row and a different column than each of the first two coins is $\frac{4}{14} = \frac{2}{7}$.

Therefore, the probability that all three coins are placed in different rows and different columns is $\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$.

(b) Suppose that $AB = c$, $AC = b$ and $BC = a$.

Since DG is parallel to AC, $\angle BDG = \angle BAC$ and $\angle DGB = \angle ACB$, so $\triangle DGB$ is similar to $\triangle ACB$.

(Similarly, $\triangle AED$ and $\triangle ECF$ are also both similar to $\triangle ABC$.)

Suppose next that $DB = kc$, with $0 < k < 1$.

Then the ratio of the side lengths of $\triangle DGB$ to those of $\triangle ACB$ will be $k : 1$, so $BG = ka$ and $DG = kb$.

Since the ratio of the side lengths of $\triangle DGB$ to $\triangle ACB$ is $k : 1$, then the ratio of their areas will be $k^2 : 1$, so the area of $\triangle DGB$ is k^2 (since the area of $\triangle ACB$ is 1).

Since $AB = c$ and $DB = kc$, then $AD = (1-k)c$, so using similar triangles as before, $DE = (1-k)a$ and $AE = (1-k)b$. Also, the area of $\triangle ADE$ is $(1-k)^2$.

Since $AC = b$ and $AE = (1-k)b$, then $EC = kb$, so again using similar triangles, $EF = kc$, $FC = ka$ and the area of $\triangle ECF$ is k^2.

Now the area of trapezoid $DEFG$ is the area of the large triangle minus the combined areas of the small triangles, or $1 - k^2 - k^2 - (1-k)^2 = 2k - 3k^2$.

We know that $k \geq 0$ by its definition. Also, since G is to the left of F, then $BG + FC \leq BC$ or $ka + ka \leq a$ or $2ka \leq a$ or $k \leq \frac{1}{2}$.

Let $f(k) = 2k - 3k^2$.

Since $f(k) = -3k^2 + 2k + 0$ is a parabola opening downwards, its maximum occurs at its vertex, whose k-coordinate is $k = -\frac{2}{2(-3)} = \frac{1}{3}$ (which lies in the admissible range for k).

Note that $f(\frac{1}{3}) = \frac{2}{3} - 3(\frac{1}{9}) = \frac{1}{3}$.

Therefore, the maximum area of the trapezoid is $\frac{1}{3}$.

9. (a) The vertex of the first parabola has x-coordinate $x = -\frac{1}{2}b$.

Since each parabola passes through P, then

$$\begin{aligned} f\left(-\tfrac{1}{2}b\right) &= g\left(-\tfrac{1}{2}b\right) \\ \tfrac{1}{4}b^2 + b(-\tfrac{1}{2}b) + c &= -\tfrac{1}{4}b^2 + d(-\tfrac{1}{2}b) + e \\ \tfrac{1}{4}b^2 - \tfrac{1}{2}b^2 + c &= -\tfrac{1}{4}b^2 - \tfrac{1}{2}bd + e \\ \tfrac{1}{2}bd &= e - c \\ bd &= 2(e-c) \end{aligned}$$

as required. (The same result can be obtained by using the vertex of the second parabola.)

(b) *Solution 1*

The vertex, P, of the first parabola has x-coordinate $x = -\frac{1}{2}b$ so has y-coordinate $f(-\frac{1}{2}b) = \frac{1}{4}b^2 - \frac{1}{2}b^2 + c = -\frac{1}{4}b^2 + c$.

The vertex, Q, of the second parabola has x-coordinate $x = \frac{1}{2}d$ so has y-coordinate $g(\frac{1}{2}d) = -\frac{1}{4}d^2 + \frac{1}{2}d^2 + c = \frac{1}{4}d^2 + e$.

Therefore, the slope of the line through P and Q is

$$\begin{aligned} \frac{(-\frac{1}{4}b^2 + c) - (\frac{1}{4}d^2 + e)}{-\frac{1}{2}b - \frac{1}{2}d} &= \frac{-\frac{1}{4}(b^2 + d^2) - (e - c)}{-\frac{1}{2}b - \frac{1}{2}d} \\ &= \frac{-\frac{1}{4}(b^2 + d^2) - \frac{1}{2}bd}{-\frac{1}{2}b - \frac{1}{2}d} \\ &= \frac{-\frac{1}{4}(b^2 + 2bd + d^2)}{-\frac{1}{2}(b + d)} \\ &= \frac{1}{2}(b + d) \end{aligned}$$

Using the point-slope form of the line, the line thus has equation

$$\begin{aligned} y &= \tfrac{1}{2}(b + d)(x - (-\tfrac{1}{2}b)) + (-\tfrac{1}{4}b^2 + c) \\ &= \tfrac{1}{2}(b + d)x + \tfrac{1}{4}b^2 + \tfrac{1}{4}bd - \tfrac{1}{4}b^2 + c \\ &= \tfrac{1}{2}(b + d)x + \tfrac{1}{4}bd + c \\ &= \tfrac{1}{2}(b + d)x + \tfrac{1}{2}(e - c) + c \\ &= \tfrac{1}{2}(b + d)x + \tfrac{1}{2}(e + c) \end{aligned}$$

so the y-intercept of the line is $\frac{1}{2}(e + c)$.

Solution 2

The equations of the two parabolas are $y = x^2 + bx + c$ and $y = -x^2 + dx + e$.

Adding the two equations, we obtain $2y = (b + d)x + (c + e)$ or $y = \frac{1}{2}(b + d)x + \frac{1}{2}(c + e)$.

This last equation is the equation of a line.

Points P and Q, whose coordinates satisfy the equation of each parabola, must satisfy the equation of the line, and so lie on the line.

But the line through P and Q is unique, so this is the equation of the line through P and Q.

Therefore, the line through P and Q has slope $\frac{1}{2}(b + d)$ and y-intercept $\frac{1}{2}(c + e)$.

10. (a) First, we note that since the circle and lines XY and XZ are fixed, then the quantity $XY + XZ$ is fixed.

Since VT and VY are tangents from the same point V to the circle, then $VT = VY$.

Since WT and WZ are tangents from the same point W to the circle, then $WT = WZ$.

Therefore, the perimeter of $\triangle VXW$ is

$$\begin{aligned} XV + XW + VW &= XV + XW + VT + WT \\ &= XV + XW + VY + WZ \\ &= XV + VY + XW + WZ \\ &= XY + XZ \end{aligned}$$

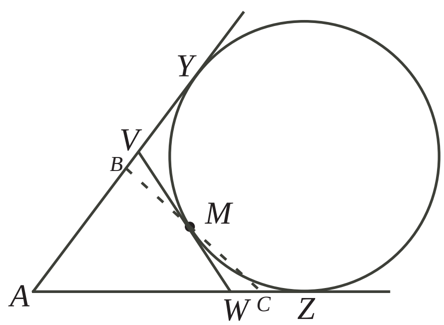
which is constant, by our earlier comment.

Therefore, the perimeter of $\triangle VXW$ always equals $XY + XZ$, which does not depend on the position of T.

(b) *Solution 1*

A circle can be drawn that is tangent to the lines AB extended and AC extended, that passes through M, and that has M on the left side of the circle. (The fact that such a circle can be drawn and that this circle is unique can be seen by starting with a small circle tangent to the two lines and expanding the circle, keeping it tangent to the two lines, until it has M on the left side of its circumference.) Suppose that this circle is tangent to AB and AC extended at Y and Z, respectively.

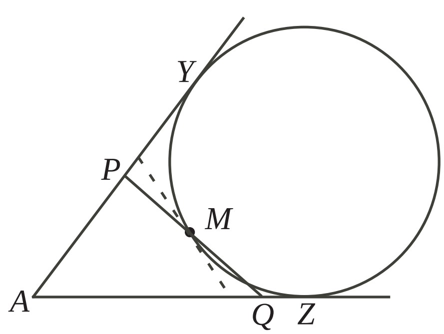
Draw a line tangent to the circle at M that cuts AB (extended) at V and AC (extended) at W.



We prove that $\triangle AVW$ has the minimum perimeter of all triangles that can be drawn with their third side passing through M.

From (a), we know that the perimeter of $\triangle AVW$ equals $AY + AZ$.

Consider a different triangle APQ formed by drawing another line through M. Note that this line PMQ cannot be tangent to the circle, so must cut the circle in two places (at M and at another point).



This line, however, will be tangent to a new circle that is tangent to AB and AC at Y' and Z'. But PMQ cuts the original circle at two points, then this new circle must be formed by shifting the original circle to the right. In other words, Y' and Z' will be further along AB and AC than Y and Z.
But the perimeter of $\triangle APQ$ will equal $AY' + AZ'$ by (a) and $AY' + AZ' > AY + AZ$, so the perimeter of $\triangle APQ$ is greater than that of $\triangle AVW$.
Therefore, the perimeter is minimized when the line through M is tangent to the circle.

We now must determine the perimeter of $\triangle AVW$. Note that it is sufficient to determine the length of AZ, since the perimeter of $\triangle AVW$ equals $AY + AZ$ and $AY = AZ$, so the perimeter of $\triangle AVW$ is twice the length of AZ.
First, we calculate $\angle VAW = \angle BAC$ using the cosine law:

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC) \\ 14^2 &= 10^2 + 16^2 - 2(10)(16)\cos(\angle BAC) \\ 196 &= 356 - 320\cos(\angle BAC) \\ 320\cos(\angle BAC) &= 160 \\ \cos(\angle BAC) &= \tfrac{1}{2} \\ \angle BAC &= 60^\circ \end{aligned}$$

Next, we add coordinates to the diagram by placing A at the origin $(0, 0)$ and AC along the positive x-axis. Thus, C has coordinates $(16, 0)$.
Since $\angle BAC = 60^\circ$ and $AB = 10$, then B has coordinates $(10\cos(60^\circ), 10\sin(60^\circ))$ or $(5, 5\sqrt{3})$.
Since M is the midpoint of BC, then M has coordinates $\left(\frac{1}{2}(5 + 16), \frac{1}{2}(5\sqrt{3} + 0)\right)$ or $\left(\frac{21}{2}, \frac{5}{2}\sqrt{3}\right)$.
Suppose the centre of the circle is O and the circle has radius r.
Since the circle is tangent to the two lines AY and AZ, then the centre of the circle lies on the angle bisector of $\angle BAC$, so lies on the line through the origin that makes an angle of 30° with the positive x-axis. The slope of this line is thus $\tan(30^\circ) = \frac{1}{\sqrt{3}}$.
The centre O will have y-coordinate r, since a radius from the centre to AZ is perpendicular to the x-axis. Thus, O has coordinates $(\sqrt{3}r, r)$ and Z has coordinates $(\sqrt{3}r, 0)$.
Thus, the perimeter of the desired triangle is $2AZ = 2\sqrt{3}r$.
Since the circle has centre $(\sqrt{3}r, r)$ and radius r, then its equation is
$(x - \sqrt{3}r)^2 + (y - r)^2 = r^2$.
Since M lies on the circle, then when we substitute the coordinates of M, we obtain an

equation for r:

$$\begin{aligned}
\left(\tfrac{21}{2}-\sqrt{3}r\right)^2+\left(\tfrac{5}{2}\sqrt{3}-r\right)^2 &= r^2\\
\tfrac{441}{4}-21\sqrt{3}r+3r^2+\tfrac{75}{4}-5\sqrt{3}r+r^2 &= r^2\\
3r^2-26\sqrt{3}r+129 &= 0\\
(\sqrt{3}r)^2-2(13)(\sqrt{3}r)+169-40 &= 0\\
(\sqrt{3}r-13)^2 &= 40\\
\sqrt{3}r-13 &= \pm 2\sqrt{10}\\
r &= \frac{13\pm 2\sqrt{10}}{\sqrt{3}}\\
r &= \frac{13\sqrt{3}\pm 2\sqrt{30}}{3}
\end{aligned}$$

(Alternatively, we could have used the quadratic formula instead of completing the square.)

Therefore, $r=\dfrac{13\sqrt{3}+2\sqrt{30}}{3}$ since we want the circle with the larger radius that passes through M and is tangent to the two lines. (Note that there is a smaller circle "inside" M and a larger circle "outside" M.)

Therefore, the minimum perimeter is $2\sqrt{3}r=\dfrac{26(3)+4\sqrt{90}}{3}=26+4\sqrt{10}$.

Solution 2

As in Solution 1, we prove that the triangle with minimum perimeter has perimeter equal to $AY+AZ$.

Next, we must determine the length of AY.

As in Solution 1, we can show that $\angle YAZ=60^\circ$.

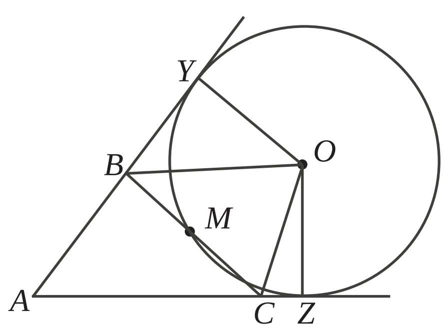
Suppose the centre of the circle is O and the circle has radius r.

Since the circle is tangent to AY and to AZ at Y and Z, respectively, then OY and OZ are perpendicular to AY and AZ.

Also, joining O to A bisects $\angle YAZ$ (since the circle is tangent to AY and AZ), so $\angle YAO=30^\circ$.

Thus, $AY=\sqrt{3}YO=\sqrt{3}r$. Also, $AZ=AY=\sqrt{3}r$.

Next, join O to B and to C.



Since $AB=10$, then $BY=AY-AB=\sqrt{3}r-10$.

Since $AC=10$, then $CZ=AZ-AC=\sqrt{3}r-16$.

Since $\triangle OBY$ is right-angled at Y, then

$$OB^2 = BY^2 + OY^2 = (\sqrt{3}r - 10)^2 + r^2$$

Since $\triangle OCZ$ is right-angled at Z, then

$$OC^2 = CZ^2 + OZ^2 = (\sqrt{3}r - 16)^2 + r^2$$

In $\triangle OBC$, since $BM = MC$, then $OB^2 + OC^2 = 2BM^2 + 2OM^2$. (See the end for a proof of this.)
Therefore,

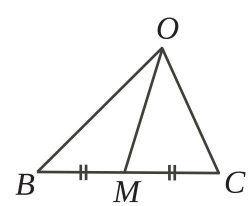
$$\begin{aligned}(\sqrt{3}r - 10)^2 + r^2 + (\sqrt{3}r - 16)^2 + r^2 &= 2(7^2) + 2r^2 \\ 3r^2 - 20\sqrt{3}r + 100 + r^2 + 3r^2 - 32\sqrt{3}r + 256 + r^2 &= 98 + 2r^2 \\ 6r^2 - 52\sqrt{3}r + 258 &= 0 \\ 3r^2 - 26\sqrt{3}r + 129 &= 0\end{aligned}$$

As in Solution 1, $r = \dfrac{13\sqrt{3} + 2\sqrt{30}}{3}$, and so the minimum perimeter is

$$2\sqrt{3}r = \frac{26(3) + 4\sqrt{90}}{3} = 26 + 4\sqrt{10}$$

We could have noted, though, that since we want to find $2\sqrt{3}r$, then setting $z = \sqrt{3}r$, the equation $3r^2 - 26\sqrt{3}r + 129 = 0$ becomes $z^2 - 26z + 129 = 0$. Completing the square, we get $(z - 13)^2 = 40$, so $z = 13 \pm 2\sqrt{10}$, whence the perimeter is $26 + 4\sqrt{10}$ in similar way.

We must still justify that, in $\triangle OBC$, we have $OB^2 + OC^2 = 2BM^2 + 2OM^2$.



By the cosine law in $\triangle OBM$,

$$OB^2 = OM^2 + BM^2 - 2(OM)(BM)\cos(\angle OMB)$$

By the cosine law in $\triangle OCM$,

$$OC^2 = OM^2 + CM^2 - 2(OM)(CM)\cos(\angle OMC)$$

But $BM = CM$ and $\angle OMC = 180^\circ - \angle OMB$, so $\cos(\angle OMC) = -\cos(\angle OMB)$.
Therefore, our two equations become

$$\begin{aligned}OB^2 &= OM^2 + BM^2 - 2(OM)(BM)\cos(\angle OMB) \\ OC^2 &= OM^2 + BM^2 + 2(OM)(BM)\cos(\angle OMB)\end{aligned}$$

Adding, we obtain $OB^2 + OC^2 = 2OM^2 + 2BM^2$, as required.
(Notice that this result holds in any triangle with a median drawn in.)

Canadian
Mathematics
Competition
An activity of the Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

2006 Euclid Contest

Wednesday, April 19, 2006

Solutions

1. (a) Answer: 0

Solution 1

Since $3x - 3y = 24$, then $x - y = 8$.
To determine the x-intercept we set $y = 0$ and obtain $x = 8$.
To determine the y-intercept we set $x = 0$ and obtain $y = -8$.
Thus, the sum of the intercepts is $8 + (-8) = 0$.

Solution 2

To determine the x-intercept we set $y = 0$ and obtain $3x = 24$ or $x = 8$.
To determine the y-intercept we set $x = 0$ and obtain $-3y = 24$ or $y = -8$.
Thus, the sum of the intercepts is $8 + (-8) = 0$.

Solution 3

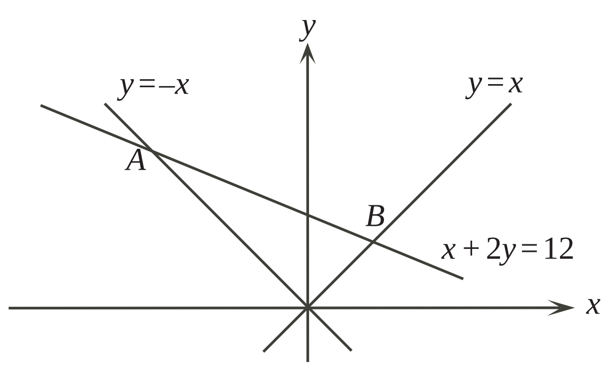
Since $3x - 3y = 24$, then $x - y = 8$ or $y = x - 8$.
This tells us immediately that the y-intercept of the line is $y = -8$ and that the x-intercept (obtained by setting $y = 0$) is $x = 8$.
Thus, the sum of the intercepts is $8 + (-8) = 0$.

(b) Answer: 20

Since $(1, 1)$ is the point of intersection of the two lines, then it must satisfy the equation of each line.
From the first line, $p(1) = 12$ or $p = 12$.
From the second line, $2(1) + q(1) = 10$ or $q = 8$.
Therefore, $p + q = 20$.

(c) *Solution 1*

To determine B, the point of intersection of the lines $y = x$ and $x + 2y = 12$, we set $y = x$ in the second equation to obtain $x + 2x = 12$ or $3x = 12$ or $x = 4$.



Since $y = x$, B has coordinates $(4, 4)$.
To determine A, the point of intersection of the lines $y = -x$ and $x + 2y = 12$, we set $y = -x$ in the second equation to obtain $x - 2x = 12$ or $-x = 12$ or $x = -12$.
Since $y = -x$, A has coordinates $(-12, 12)$.
The length of AB equals the distance between A and B, or

$$\sqrt{(4-(-12))^2 + (4-12)^2} = \sqrt{16^2 + (-8)^2} = \sqrt{320} = 8\sqrt{5}$$

Solution 2

We determine the coordinates of A and B as in Solution 1.

Since the slopes of $y = x$ (slope of 1) and $y = -x$ (slope of -1) are negative reciprocals, then these lines are perpendicular, so $\angle AOB = 90^\circ$.
Since B has coordinates $(4, 4)$, then OB has length $\sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$.
Since A has coordinates $(-12, 12)$, then OA has length $\sqrt{(-12)^2 + 12^2} = \sqrt{288} = 12\sqrt{2}$.
Using the Pythagorean Theorem on triangle AOB,

$$AB = \sqrt{OB^2 + OA^2} = \sqrt{32 + 288} = \sqrt{320} = 8\sqrt{5}$$

2. (a) Answer: 9
For the average of two digits to be 5, their sum must be 10.
The two-digit positive integers whose digits sum to 10 are 19, 28, 37, 46, 55, 64, 73, 82, 91, of which there are 9.

(b) Answer: $n = 45$
Solution 1
Suppose that n has digits AB. Then $n = 10A + B$.
The average of the digits of n is $\frac{A+B}{2}$.
Putting a decimal point between the digits of n is equivalent to dividing n by 10, so the resulting number is $\frac{10A+B}{10}$.
So we want to determine A and B so that

$$\begin{aligned} \frac{10A+B}{10} &= \frac{A+B}{2} \\ 10A + B &= 5(A+B) \\ 5A &= 4B \end{aligned}$$

Since A and B are digits such that $5A = 4B$, then $A = 4$ and $B = 5$ is the only possibility.
Therefore, $n = 45$.
(We can quickly check that the average of the digits of n is 4.5, the number obtained by putting a decimal point between the digits of n.)

Solution 2
When we compute the average of two digits, the result is either an integer or a half-integer (ie. a decimal number of the form $a.5$).
Therefore, the possible averages are 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5, 8.0, 8.5, 9.0. (0.0 is not possible as an two-digit integer cannot start with 0.)
From this list, the only one equal to the average of the two digits forming it is 4.5.
Therefore, $n = 45$ (formed by removing the decimal point from 4.5).

(c) *Solution 1*
When the average of three integers is 28, their sum is $3(28) = 84$.
When the average of five integers is 34, their sum is $5(34) = 170$.
In this case, the difference between the sum of the five integers and the sum of the three integers is $s + t$ which must equal $170 - 84 = 86$.
Therefore, $s + t = 86$ and so the average of s and t is $\frac{s+t}{2} = 43$.

Solution 2
Suppose the first three integers are a, b and c.

Then $\dfrac{a+b+c}{3} = 28$ or $a+b+c = 84$.
Also, $\dfrac{a+b+c+s+t}{5} = 34$ or $a+b+c+s+t = 170$.
Thus, $s+t = (a+b+c+s+t) - (a+b+c) = 170 - 84 = 86$ and so the average of s and t is $\dfrac{s+t}{2} = 43$.

Solution 3
Suppose that the average of s and t is A.
Since the average of the initial three numbers is 28, the average of s and t is A, and the average of all five numbers is 34, then 34 must be $\frac{2}{5}$ of the way from 28 to A. The difference between 34 and 28 is 6, so the total difference between A and 28 must be $\frac{5}{2}(6) = 15$.
Thus, A, the average of s and t, is $28 + 15 = 43$.

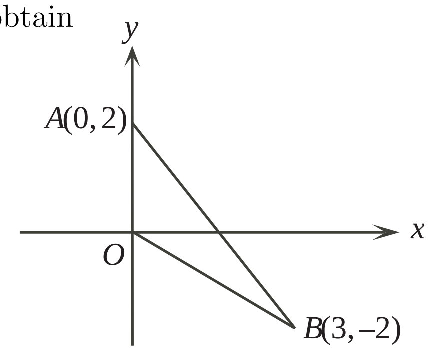
3. (a) Answer: $(21, -1)$
Solution 1
The x-intercepts of the given parabola are $x = 20$ and $x = 22$.
The x-coordinate of the vertex of the parabola is the average of the x-intercepts, or $\frac{1}{2}(20+22) = 21$.
When $x = 21$, $y = (21-20)(21-22) = -1$.
Thus, the coordinates of the vertex are $(21, -1)$.

Solution 2
We expand the right side of the equation of the parabola to obtain $y = x^2 - 42x + 440$.
Next we complete the square to obtain

$$y = x^2 - 2(21)x + 21^2 - 21^2 + 440 = (x-21)^2 - 21^2 + 440 = (x-21)^2 - 441 + 440 = (x-21)^2 - 1$$

From this form, we immediately see that the coordinates of the vertex are $(21, -1)$.

(b) Consider the parabola $y = x^2 + 2 = (x-0)^2 + 2$.
The coordinates of its vertex are $A(0, 2)$.
Consider the parabola $y = x^2 - 6x + 7$.
Completing the square, we obtain $y = (x-3)^2 - 9 + 7 = (x-3)^2 - 2$.
The coordinates of its vertex are $B(3, -2)$.
Therefore, the vertices of $\triangle OAB$ are $O(0, 0)$, $A(0, 2)$, $B(3, -2)$.
Sketching this triangle, we obtain

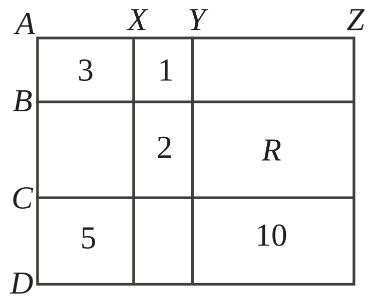


We can consider this triangle as having base OA (of length 2) and height, equal to the distance from B to the y-axis, of 3.
Thus, $\triangle OAB$ has area $\frac{1}{2}(2)(3) = 3$.

4. (a) Answer: $R = 12$

Solution 1

We label some of the points in the diagram.



Looking at the middle column of rectangles, each has the same width, so the ratio of their areas equals the ratio of their heights. Thus, $AB : BC = 1 : 2$.

Looking at the rectangles in the first column, the area of the middle rectangle must be twice the area of the top rectangle, or 6.

Thus, $BC : CD = 6 : 5$ by the reasoning above.

So, looking at the third column, $R : 10 = 6 : 5$ or $R = 12$.

Solution 2

Let the width of the first column be x.

Since the area of the top left rectangle is 3, the height of the first row is $\frac{3}{x}$.

Since the area of the bottom left rectangle is 5, the height of the third row is $\frac{5}{x}$.

Since the height of the first row is $\frac{3}{x}$ and the area of the top middle rectangle is 1, the width of the middle column is $\frac{x}{3}$.

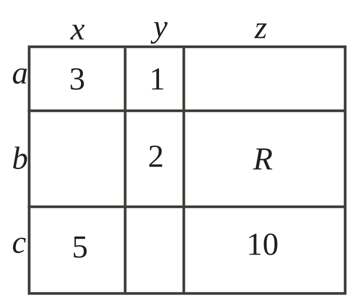
Thus, the height of the middle row is $\frac{6}{x}$, since the area of the middle rectangle is 2.

Since the height of the third row is $\frac{5}{x}$ and the area of the bottom right rectangle is 10, then the width of the third column is $2x$.

Since the rectangle labelled R has height $\frac{6}{x}$ and width $2x$, then it has area 12.

Solution 3

We label some of the lengths in the diagram.

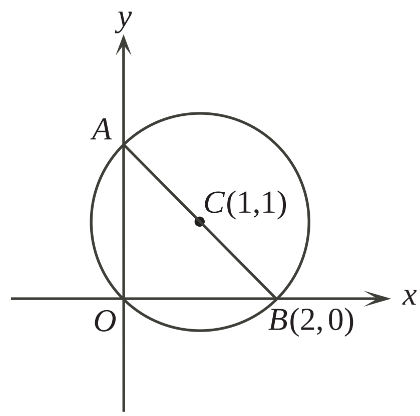


From the given information, $ax = 3$, $ay = 1$, $by = 2$, $bz = R$, $cx = 5$ and $cz = 10$.

We want to determine bz.

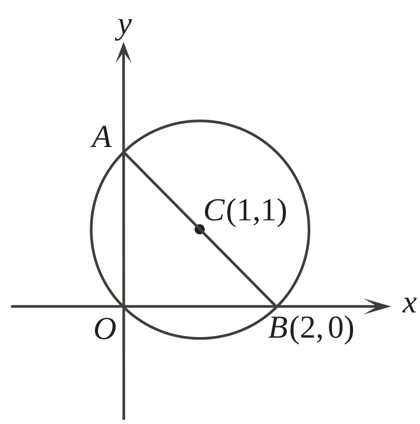
But $bz = \dfrac{(ax)(by)(cz)}{(ay)(cx)} = \dfrac{(3)(2)(10)}{(1)(5)} = 12$, so $R = 12$.

(b) *Solution 1*
Since $\angle AOB = 90^\circ$, AB is a diameter of the circle.
Join AB.

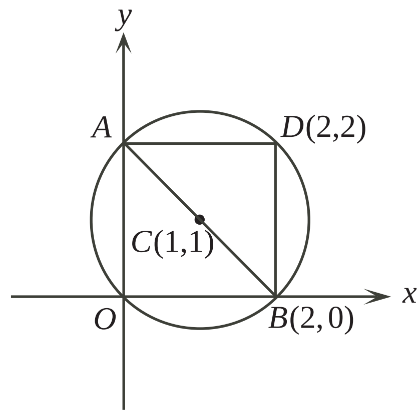


Since C is the centre of the circle and AB is a diameter, then C is the midpoint of AB, so A has coordinates $(0,2)$.
Therefore, the area of the part of the circle inside the first quadrant is equal to the area of $\triangle AOB$ plus the area of the semi-circle above AB.
The radius of the circle is equal to the distance from C to B, or $\sqrt{(1-2)^2+(1-0)^2} = \sqrt{2}$, so the area of the semi-circle is $\frac{1}{2}\pi(\sqrt{2})^2 = \pi$.
The area of $\triangle AOB$ is $\frac{1}{2}(OB)(AO) = \frac{1}{2}(2)(2) = 2$.
Thus, the area of the part of the circle inside the first quadrant is $\pi + 2$.

Solution 2
Since $\angle AOB = 90^\circ$, AB is a diameter of the circle.
Join AB.



Since C is the centre of the circle and AB is a diameter, then C is the midpoint of AB, so A has coordinates $(0,2)$.
Thus, $AO = BO$.
We "complete the square" by adding point $D(2,2)$, which is on the circle, by symmetry.



The area of the square is 4.
The radius of the circle is equal to the distance from C to B, or $\sqrt{(1-2)^2+(1-0)^2}=\sqrt{2}$, so the area of the circle is $\pi(\sqrt{2})^2 = 2\pi$.
The area of the portion of the circle outside the square is thus $2\pi - 4$. This area is divided into four equal sections (each of area $\frac{1}{4}(2\pi-4) = \frac{1}{2}\pi - 1$), two of which are the only portions of the circle outside the first quadrant.
Therefore, the area of the part of the circle inside the the first quadrant is $2\pi - 2(\frac{1}{2}\pi - 1) = \pi + 2$.

Two additional ways to find the coordinates of A:

* The length of OC is $\sqrt{1^2+1^2}=\sqrt{2}$.
  Since C is the centre of the circle and O lies on the circle, then the circle has radius $\sqrt{2}$.
  Since the circle has centre $(1,1)$ and radius $\sqrt{2}$, its equation is $(x-1)^2+(y-1)^2=2$.
  To find the coordinates of A, we substitute $x=0$ to obtain $(0-1)^2+(y-1)^2=2$ or $(y-1)^2=1$, and so $y=0$ or $y=2$.
  Since $y=0$ gives us the point O, then $y=2$ gives us A, ie. A has coordinates $(0,2)$.
* Since O and A are both on the circle and each has a horizontal distance of 1 from C, then their vertical distances from C must be same, ie. must each be 1.
  Thus, A has coordinates $(0,2)$.

5. (a) Answer: $\frac{2}{5}$
Since there are 5 choices for a and 3 choices for b, there are fifteen possible ways of choosing a and b.
If a is even, a^b is even; if a is odd, a^b is odd.
So the choices of a and b which give an even value for a^b are those where a is even, or 6 of the choices (since there are two even choices for a and three ways of choosing b for each of these). (Notice that in fact the value of b does not affect whether a^b is even or odd, so the probability depends only on the choice of a.)
Thus, the probability is $\frac{6}{15}=\frac{2}{5}$.

(b) Starting with 4 blue hats and 2 green hats, the probability that Julia removes a blue hat is $\frac{4}{6}=\frac{2}{3}$. The result would be 3 blue hats and 3 green hats, since a blue hat is replaced with a green hat.
In order to return to 4 blue hats and 2 green hats from 3 blue and 3 green, Julia would need remove a green hat (which would be replaced by a blue hat). The probability of her

removing a green hat from 3 blue and 3 green is $\frac{3}{6} = \frac{1}{2}$.
Summarizing, the probability of choosing a blue hat and then a green hat is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

Starting with 4 blue hats and 2 green hats, the probability that Julia removes a green hat is $\frac{2}{6} = \frac{1}{3}$. The result would be 5 blue hats and 1 green hat, since a green hat is replaced with a blue hat.
In order to return to 4 blue hats and 2 green hats from 5 blue and 1 green, Julia would need remove a blue hat (which would be replaced by a green hat). The probability of her removing a green hat from 5 blue and 1 green is $\frac{5}{6}$.
Summarizing, the probability of choosing a green hat and then a blue hat is $\frac{1}{3} \times \frac{5}{6} = \frac{5}{18}$.

These are the only two ways to return to 4 blue hats and 2 green hats after two turns – removing a blue hat then a green, or removing a green then a blue.
Therefore, the total probability of returning to 4 blue hats and 2 green hats after two turns is $\frac{1}{3} + \frac{5}{18} = \frac{11}{18}$.

6. (a) Answer: $a = 1$
Adding the two equations, we obtain

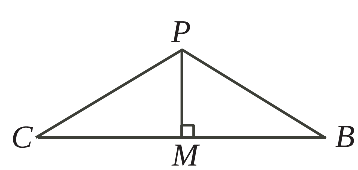
$$\begin{aligned}
\sin^2 x + \cos^2 x + \sin^2 y + \cos^2 y &= \frac{3}{2}a + \frac{1}{2}a^2 \\
2 &= \frac{3}{2}a + \frac{1}{2}a^2 \\
4 &= 3a + a^2 \\
0 &= a^2 + 3a - 4 \\
0 &= (a+4)(a-1)
\end{aligned}$$

and so $a = -4$ or $a = 1$.
However, $a = -4$ is impossible, since this would give $\sin^2 x + \cos^2 y = -6$, whose left side is non-negative and whose right side is negative.
Therefore, the only possible value for a is $a = 1$.

(We can check that angles $x = 90^\circ$ and $y = 45^\circ$ give $\sin^2 x + \cos^2 y = \frac{3}{2}$ and $\cos^2 x + \sin^2 y = \frac{1}{2}$, so $a = 1$ is indeed possible.)

(b) From the given information, $PC = PB$.
If we can calculate the length of PC, we can calculate the value of h, since we already know the length of AC.

Now $\triangle CPB$ is isosceles with $PC = PB$, $BC = 2$ and $\angle BPC = 120^\circ$.
Since $\triangle CPB$ is isosceles, $\angle PCB = \angle PBC = 30^\circ$.



Join P to the midpoint, M, of BC.
Then PM is perpendicular to BC, since $\triangle PCB$ is isosceles.

Therefore, $\triangle PMC$ is right-angled, has $\angle PCM = 30^\circ$ and has $CM = 1$.
Thus, $PC = \frac{2}{\sqrt{3}}$.
(There are many other techniques that we can use to calculate the length of PC.)

Returning to $\triangle APC$, we see $AP^2 = AC^2 - PC^2$ or $h^2 = 2^2 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3}$, and so $h = \sqrt{\frac{8}{3}} = 2\sqrt{\frac{2}{3}} = \frac{2\sqrt{6}}{3} \approx 1.630$.
Therefore, the height is approximately 1.63 m or 163 cm.

7. (a) Answer: $k = 233$

Solution 1

We calculate the first 15 terms, writing each as an integer times a power of 10:

$$2, 5, 10, 5 \times 10, 5 \times 10^2, 5^2 \times 10^3, 5^3 \times 10^5, 5^5 \times 10^8, 5^8 \times 10^{13}, 5^{13} \times 10^{21}, 5^{21} \times 10^{34},$$

$$5^{34} \times 10^{55}, 5^{55} \times 10^{89}, 5^{89} \times 10^{144}, 5^{144} \times 10^{233}$$

Since the 15th term equals an odd integer times 10^{233}, then the 15th term ends with 233 zeroes.

Solution 2

To obtain the 6th term, we calculate $50 \times 500 = 25 \times 1000$.
Each of the 4th and 5th terms equals an odd integer followed by a number of zeroes, so the 6th term also equals an odd integer followed by a number of zeroes, where the number of zeroes is the sum of the numbers of zeroes at the ends of the 4th and 5th terms.
This pattern will continue. Thus, starting with the 6th term, the number of zeroes at the end of the term will be the sum of the number of zeroes at the ends of the two previous terms.
This tells us that, starting with the 4th term, the number of zeroes at the ends of the terms is

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233$$

Therefore, the 15th term ends with 233 zeroes.

(b) *Solution 1*

Since a, b and c are consecutive terms in an arithmetic sequence, then $b = a + d$ and $c = a + 2d$ for some number d.
Therefore,

$$\begin{aligned} a^2 - bc &= a^2 - (a+d)(a+2d) = a^2 - a^2 - 3ad - 2d^2 = -3ad - 2d^2 \\ b^2 - ac &= (a+d)^2 - a(a+2d) = a^2 + 2ad + d^2 - a^2 - 2ad = d^2 \\ c^2 - ab &= (a+2d)^2 - a(a+d) = a^2 + 4ad + 4d^2 - a^2 - ad = 3ad + 4d^2 \end{aligned}$$

Thus,

$$(b^2 - ac) - (a^2 - bc) = d^2 - (-3ad - 2d^2) = 3d^2 + 3ad$$

and

$$(c^2 - ab) - (b^2 - ac) = (3ad + 4d^2) - d^2 = 3d^2 + 3ad$$

Therefore, $(b^2 - ac) - (a^2 - bc) = (c^2 - ab) - (b^2 - ac)$, so the sequence $a^2 - bc$, $b^2 - ac$ and $c^2 - ab$ is arithmetic.

Solution 2

Since a, b and c are consecutive terms in an arithmetic sequence, then $a = b - d$ and $c = b + d$ for some number d.

Therefore,

$$\begin{aligned} a^2 - bc &= (b-d)^2 - b(b+d) = b^2 - 2bd + d^2 - b^2 - bd = -3bd + d^2 \\ b^2 - ac &= b^2 - (b-d)(b+d) = b^2 - b^2 + d^2 = d^2 \\ c^2 - ab &= (b+d)^2 - (b-d)b = b^2 + 2bd + d^2 - b^2 + bd = 3bd + d^2 \end{aligned}$$

Thus,

$$(b^2 - ac) - (a^2 - bc) = d^2 - (-3bd + d^2) = 3bd$$

and

$$(c^2 - ab) - (b^2 - ac) = (3bd + d^2) - d^2 = 3bd$$

Therefore, $(b^2 - ac) - (a^2 - bc) = (c^2 - ab) - (b^2 - ac)$, so the sequence $a^2 - bc$, $b^2 - ac$ and $c^2 - ab$ is arithmetic.

Solution 3

To show that $a^2 - bc$, $b^2 - ac$ and $c^2 - ab$ form an arithmetic sequence, we can show that $(c^2 - ab) + (a^2 - bc) = 2(b^2 - ac)$.

Since a, b and c form an arithmetic sequence, then $a + c = 2b$.

Now

$$\begin{aligned} (c^2 - ab) + (a^2 - bc) &= c^2 + a^2 - b(a+c) \\ &= c^2 + a^2 + 2ac - b(a+c) - 2ac \\ &= (c+a)^2 - b(a+c) - 2ac \\ &= (c+a)(a+c-b) - 2ac \\ &= 2b(2b-b) - 2ac \\ &= 2b^2 - 2ac \\ &= 2(b^2 - ac) \end{aligned}$$

as required.

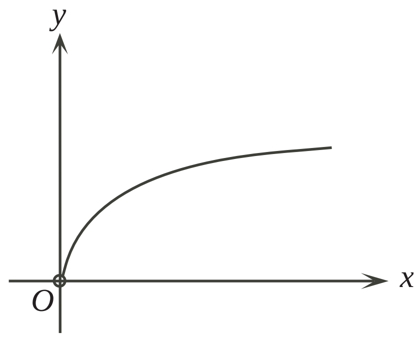
8. (a) We use logarithm rules to rearrange the equation to solve for y:

$$\begin{aligned} \log_2 x - 2\log_2 y &= 2 \\ \log_2 x - \log_2(y^2) &= 2 \\ \log_2\left(\frac{x}{y^2}\right) &= 2 \\ \frac{x}{y^2} &= 2^2 \\ \tfrac{1}{4}x &= y^2 \\ y &= \pm\tfrac{1}{2}\sqrt{x} \end{aligned}$$

But since the domain of the $\log_2$ function is all positive real numbers, we must have $x > 0$ and $y > 0$, so we can reject the negative square root to obtain

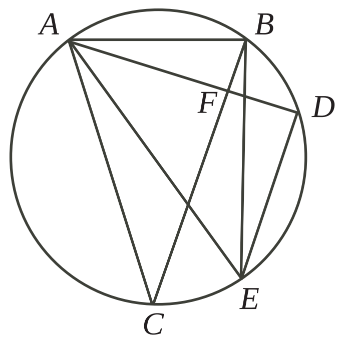
$$y = \tfrac{1}{2}\sqrt{x}, \quad x > 0$$

The graph of this function is:



(b) *Solution 1*

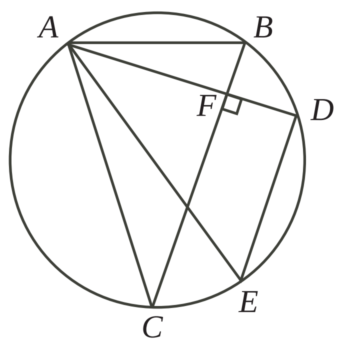
Join A to E and C, and B to E.



Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$.
Therefore, AE is a diameter.
Now $\angle EAC = \angle EBC$ since both are subtended by EC.
Therefore, $\angle EAC + \angle ABC = \angle EBC + \angle ABC = \angle EBA$ which is indeed equal to 90° as required, since AE is a diameter.

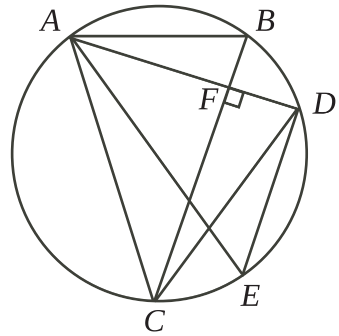
Solution 2

Join A to E and C.



Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$.
Therefore, AE is a diameter.
Thus, $\angle ECA = 90^\circ$.
Now $\angle ABC = \angle AEC$ since both are subtended by AC.
Now $\angle EAC + \angle ABC = \angle EAC + \angle AEC = 180^\circ - \angle ECA$ using the sum of the angles in $\triangle AEC$.
But $\angle ECA = 90^\circ$, so $\angle EAC + \angle AEC = 90^\circ$.

Solution 3
Join A to E and C, and C to D.



Since DE is parallel to BC and AD is perpendicular to BC, then AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$.
Therefore, AE is a diameter.
Now $\angle ABC = \angle ADC$ since both are subtended by AC.
Also $\angle EAC = \angle EDC$ since both are subtended by EC.
So $\angle EAC + \angle ABC = \angle EDC + \angle ADC = \angle ADE = 90^\circ$.

9. (a) *Solution 1*
Since $\sin^2 x + \cos^2 x = 1$, then $\cos^2 x = 1 - \sin^2 x$, so

$$\begin{aligned} f(x) &= \sin^6 x + (1-\sin^2 x)^3 + k(\sin^4 x + (1-\sin^2 x)^2) \\ &= \sin^6 x + 1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x + k(\sin^4 x + 1 - 2\sin^2 x + \sin^4 x) \\ &= (1+k) - (3+2k)\sin^2 x + (3+2k)\sin^4 x \end{aligned}$$

Therefore, if $3 + 2k = 0$ or $k = -\frac{3}{2}$, then $f(x) = 1 + k = -\frac{1}{2}$ for all x and so is constant.
(If $k \neq -\frac{3}{2}$, then we get

$$\begin{aligned} f(0) &= 1+k \\ f\left(\tfrac{1}{4}\pi\right) &= (1+k) - (3+2k)\left(\tfrac{1}{2}\right) + (3+2k)\left(\tfrac{1}{4}\right) = \tfrac{1}{4} + \tfrac{1}{2}k \\ f\left(\tfrac{1}{6}\pi\right) &= (1+k) - (3+2k)\left(\tfrac{1}{4}\right) + (3+2k)\left(\tfrac{1}{16}\right) = \tfrac{7}{16} + \tfrac{5}{8}k \end{aligned}$$

which cannot be all equal for any single value of k, so $f(x)$ is not constant if $k \neq -\frac{3}{2}$.)

Solution 2
Since $\sin^2 x + \cos^2 x = 1$, then

$$\begin{aligned} f(x) &= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x) + k(\sin^4 x + \cos^4 x) \\ &= (\sin^4 + 2\sin^2 x\cos^2 x + \cos^4 x - 3\sin^2 x\cos^2 x) \\ &\qquad k(\sin^4 x + 2\sin^2 x\cos^2 x + \cos^4 x - 2\sin^2 x\cos^2 x) \\ &= ((\sin^2 x + \cos^2 x)^2 - 3\sin^2 x\cos^2 x) + k((\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x) \\ &= 1 - 3\sin^2 x\cos^2 x + k(1 - 2\sin^2 x\cos^2 x) \\ &= (1+k) - (3+2k)\sin^2 x\cos^2 x \end{aligned}$$

Therefore, if $3 + 2k = 0$ or $k = -\frac{3}{2}$, then $f(x) = 1 + k = -\frac{1}{2}$ for all x and so is constant.
(We can check as in Solution 1 that if $k \neq -\frac{3}{2}$, then $f(x)$ is not constant.)

Solution 3
For $f(x)$ to be constant, we need $f'(x) = 0$ for all values of x.
Calculating using the Chain Rule,

$$\begin{aligned} f'(x) &= 6\sin^5 x\cos x - 6\cos^5 x\sin x + k(4\sin^3 x\cos x - 4\cos^3 x\sin x) \\ &= 2\sin x\cos x(3(\sin^4 x - \cos^4 x) + 2k(\sin^2 x - \cos^2 x)) \\ &= 2\sin x\cos x(\sin^2 x - \cos^2 x)(3(\sin^2 x + \cos^2 x) + 2k) \\ &= 2\sin x\cos x(\sin^2 x - \cos^2 x)(3 + 2k) \end{aligned}$$

If $3 + 2k = 0$ or $k = -\frac{3}{2}$, then $f'(x) = 0$ for all x, so $f(x)$ is constant.
(If $3 + 2k \neq 0$, then choosing $x = \frac{1}{6}\pi$ for example gives $f'(x) \neq 0$ so $f(x)$ is not constant.)

(b) *Solution 1*
Using the simplified version of $f(x)$ from Solution 1 of (a), we have

$$f(x) = (1 + k) - (3 + 2k)\sin^2 x + (3 + 2k)\sin^4 x$$

and so we want to solve

$$\begin{aligned} 0.3 - (1.6)\sin^2 x + (1.6)\sin^4 x &= 0 \\ 16\sin^4 x - 16\sin^2 x + 3 &= 0 \\ (4\sin^2 x - 3)(4\sin^2 x - 1) &= 0 \end{aligned}$$

Therefore, $\sin^2 x = \frac{1}{4}, \frac{3}{4}$, and so $\sin x = \pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}$.
Therefore,

$$x = \tfrac{1}{6}\pi + 2\pi k, \tfrac{5}{6}\pi + 2\pi k, \tfrac{7}{6}\pi + 2\pi k, \tfrac{11}{6}\pi + 2\pi k, \tfrac{1}{3}\pi + 2\pi k, \tfrac{2}{3}\pi + 2\pi k, \tfrac{4}{3}\pi + 2\pi k, \tfrac{5}{3}\pi + 2\pi k$$

for $k \in \mathbb{Z}$.

Solution 2
Using the simplified version of $f(x)$ from Solution 2 of (a), we have

$$f(x) = (1 + k) - (3 + 2k)\sin^2 x\cos^2 x$$

and so we want to solve

$$\begin{aligned} 0.3 - (1.6)\sin^2 x\cos^2 x &= 0 \\ 0.3 - (1.6)\sin^2 x(1 - \sin^2 x) &= 0 \\ 1.6\sin^4 x - 1.6\sin^2 x + 0.3 &= 0 \end{aligned}$$

and the solution concludes as in Solution 1.

Solution 3
Using the simplified version of $f(x)$ from Solution 2 of (a), we have

$$f(x) = (1 + k) - (3 + 2k)\sin^2 x\cos^2 x$$

Using the fact that $\sin 2x = 2\sin x\cos x$, we can further simplify $f(x)$ to

$$f(x) = (1 + k) - \tfrac{1}{4}(3 + 2k)\sin^2 2x$$

and so we want to solve

$$\begin{aligned}
0.3 - \tfrac{1}{4}(1.6)\sin^2 2x &= 0 \\
4\sin^2 2x &= 3 \\
\sin^2 2x &= \tfrac{3}{4}
\end{aligned}$$

and so $\sin 2x = \pm\frac{\sqrt{3}}{2}$.
Therefore,

$$2x = \tfrac{1}{3}\pi + 2\pi k, \tfrac{2}{3}\pi + 2\pi k, \tfrac{4}{3}\pi + 2\pi k, \tfrac{5}{3}\pi + 2\pi k$$

for $k \in \mathbb{Z}$, and so

$$x = \tfrac{1}{6}\pi + \pi k, \tfrac{1}{3}\pi + \pi k, \tfrac{2}{3}\pi + \pi k, \tfrac{5}{6}\pi + \pi k$$

for $k \in \mathbb{Z}$.
(Note that this solution, while appearing different, does agree with that from Solution 1, since here each of the four families of solutions has "$+\pi k$" and in Solution 1 each of the eight families has "$+2\pi k$".)

(c) *Solution 1*
Using the simplified version of $f(x)$ from Solution 1 of (a), we have

$$f(x) = (1+k) - (3+2k)\sin^2 x + (3+2k)\sin^4 x$$

We want to determine the values of k for which there is an a such that $f(a) = 0$.
From (a), if $k = -\frac{3}{2}$, $f(x)$ is constant and equal to $-\frac{1}{2}$, so has no roots.
Let $u = \sin^2 x$.
Then u takes all values between 0 and 1 as $\sin x$ takes all values between -1 and 1.
Then we want to determine for which k the equation

$$(3+2k)u^2 - (3+2k)u + (1+k) = 0 \qquad (*)$$

has a solution for u with $0 \leq u \leq 1$.
First, we must ensure that the equation (*) has real solutions, ie.

$$\begin{aligned}
(3+2k)^2 - 4(3+2k)(1+k) &\geq 0 \\
(3+2k)(3+2k-4(1+k)) &\geq 0 \\
(3+2k)(-1-2k) &\geq 0 \\
(3+2k)(1+2k) &\leq 0
\end{aligned}$$

This is true if and only if $-\frac{3}{2} < k \leq -\frac{1}{2}$. (We omit $k = -\frac{3}{2}$ because of the earlier comment.)
Next, we have to check for which values of k the equation (*) has a solution u with $0 \leq u \leq 1$. We may assume that $-\frac{3}{2} < k \leq -\frac{1}{2}$.
To do this, we solve the equation (*) using the quadratic formula to obtain

$$u = \frac{(3+2k) \pm \sqrt{(3+2k)^2 - 4(3+2k)(1+k)}}{2(3+2k)}$$

or

$$u = \frac{(3+2k) \pm \sqrt{-(3+2k)(1+2k)}}{2(3+2k)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{-\frac{1+2k}{3+2k}}$$

Since $k > -\frac{3}{2}$ then $3 + 2k > 0$.
For u to be between 0 and 1, we need to have

$$0 \leq \sqrt{-\frac{1+2k}{3+2k}} \leq 1$$

Thus

$$0 \leq -\frac{1+2k}{3+2k} \leq 1$$

Since $-\frac{3}{2} < k \leq -\frac{1}{2}$ then $3 + 2k > 0$ and $1 + 2k \leq 0$, so the left inequality is true.
Therefore, we need $-\dfrac{1+2k}{3+2k} \leq 1$ or $-(1+2k) \leq (3+2k)$ (we can multiply by $(3+2k)$ since it is positive), and so $-4 \leq 4k$ or $k \geq -1$.
Combining with $-\frac{3}{2} < k \leq -\frac{1}{2}$ gives $-1 \leq k \leq -\frac{1}{2}$.

Solution 2
Using the simplified version of $f(x)$ from Solution 3 of (b), we have

$$f(x) = (1+k) - \tfrac{1}{4}(3+2k)\sin^2 2x$$

If we tried to solve $f(x) = 0$, we would obtain

$$(1+k) - \tfrac{1}{4}(3+2k)\sin^2 2x = 0$$

or

$$\sin^2 2x = \frac{4(1+k)}{3+2k}$$

(From (a), if $k = -\frac{3}{2}$, $f(x)$ is constant and equal to $-\frac{1}{2}$, so has no roots.)
In order to be able to solve this (first for $\sin 2x$, then for $2x$ then for x), we therefore need

$$0 \leq \frac{4(1+k)}{3+2k} \leq 1$$

If $3 + 2k > 0$, we can multiply the inequality by $3 + 2k$ to obtain

$$0 \leq 4(1+k) \leq 3+2k$$

and so we get $k \geq -1$ from the left inequality and $k \leq -\frac{1}{2}$ from the right inequality.
Combining these with $-\frac{3}{2} < k$, we obtain $-1 \leq k \leq -\frac{1}{2}$.

If $3 + 2k < 0$, we would obtain $0 \geq 4(1+k) \geq 3+2k$ which would give $k \leq -1$ and $k \geq -\frac{1}{2}$, which are inconsistent.

Therefore, $-1 \leq k \leq -\frac{1}{2}$.

There were many other clever approaches to be taken in this problem:

- Deriving either of the expressions

$$f(x) = \left(k + \frac{3}{2}\right)(\sin^4 x + \cos^4 x) - \frac{1}{2}$$

or

$$f(x) = (1+k) - \frac{1}{4}(3+2k)\sin^2 2x$$

led to some simpler algebra.

– In (c), deriving the equation

$$(1-\sin^2 x)(\sin^2 x) = \frac{k+1}{2k+3}$$

and rewriting the left side as $-(\sin^2 x - \frac{1}{2})^2 + \frac{1}{4}$ allowed one student to conclude that the right side lies between 0 and $\frac{1}{4}$, thus quickly obtaining the range of values for k.

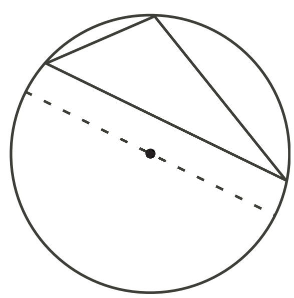
10. Through this solution, we will use the following facts:

When an acute triangle is inscribed in a circle:

- each of the three angles of the triangle is the angle inscribed in the major arc defined by the side of the triangle by which it is subtended,
- each of the three arcs into which the circle is divided by the vertices of the triangles is less than half of the circumference of the circle, and
- it contains the centre of the circle.

Why are these facts true?

- Consider a chord of a circle which is not a diameter.
  Then the angle subtended in the major arc of this circle is an acute angle and the angle subtended in the minor arc is an obtuse angle.
  Now consider an acute triangle inscribed in a circle.
  Since each angle of the triangle is acute, then each of the three angles is inscribed in the major arc defined by the side of the triangle by which it is subtended.
- It follows that each arc of the circle that is outside the triangle must be a minor arc, thus less than the circumference of the circle.
- Lastly, if the centre was outside the triangle, then we would be able to draw a diameter of the circle with the triangle entirely on one side of the diameter.

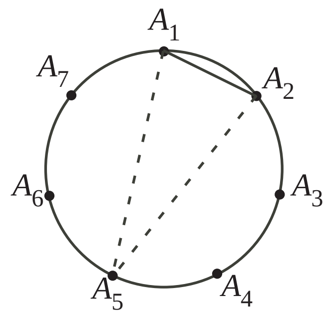


In this case, one of the arcs of the circle cut off by one of the sides of the triangle would have to be a major arc, which cannot happen, because of the above.
Therefore, the centre is contained inside the triangle.

(a) Since there are $N = 7$ points from which the triangle's vertices can be chosen, there are $\binom{7}{3} = 35$ triangles in total.
We compute the number of acute triangles.

Fix one of the vertices of such a triangle at A_1.
We construct the triangle by choosing the other two vertices in ascending subscript order.
We choose the vertices by considering the arc length from the previous vertex – each of

these arc lengths must be smaller than half the total circumference of the circle.
Since there are 7 equally spaced points on the circle, we assume the circumference is 7, so the arc length formed by each side must be at most 3.
Since the first arc length is at most 3, the second point can be only A_2, A_3 or A_4.
If the second point is A_2, then since the second and third arc lengths are each at most 3, then the third point must be A_5. (Since the second arc length is at most 3, then the third point cannot be any further along than A_5. However, the arc length from A_5 around to A_1 is 3, so it cannot be any closer than A_5.)



If the second point is A_3, the third point must be A_5 or A_6.
If the second point is A_4, the third point must be A_5 or A_6 or A_7.
Therefore, there are 6 acute triangles which include A_1 as one of its vertices.

How many acute triangles are there in total?
We can repeat the above process for each of the 6 other points, giving $7 \times 6 = 42$ acute triangles.
But each triangle is counted three times here, as it has been counted once for each of its vertices.
Thus, there are $\dfrac{7 \times 6}{3} = 14$ acute triangles.
Therefore, the probability that a randomly chosen triangle is acute if $\frac{14}{35} = \frac{2}{5}$.

(b) *Solution 1*
Since there are $N = 2k$ points from which the triangles vertices can be chosen, there are $\dbinom{2k}{3} = \dfrac{2k(2k-1)(2k-2)}{6}$ triangles in total.
We compute the number of acute triangles as in (a) by counting the number of acute triangles with one vertex fixed at A_1 and then multiply by $2k$ and divide by 3 as in (a).

Fix one vertex at A_1 and suppose that the circumference of the circle is $2k$.
The diametrically opposite point from A_1 is A_{k+1}.
Since the triangle cannot be entirely on one side of a diameter, then the second vertex must be between A_2 and A_k inclusive and the other between A_{k+2} and A_{2k} inclusive. This will ensure that the first and third arcs are shorter than half of the circumference of the circle.
If the second vertex is at A_2, then the third vertex must be no further than A_{k+1} for the second arc length to be shorter than k. Thus, there are no possibilities.
If the second vertex is at A_3, then the third vertex must be no further than A_{k+2}. (In this case, the arc between the second and third vertices has length $k-1$, as does the arc between the third and first vertices.) Thus, there is one possibility.
If the second vertex is at A_4, then the third vertex must be no further than A_{k+3}. In this

case, if the third vertex is A_{k+3}, the arc between the second and third vertices has length $k-1$ and the arc between the third and first vertices has length $k-2$. If the third vertex is A_{k+2}, these lengths are reversed. If the third vertex is neither of these points, then one of the two arcs will have length greater than $k-1$. Thus, there are two possibilities (A_{k+2} and A_{k+3}).

In general, if the second vertex is at A_j (with $2 \le j \le k$), then the third vertex must be no further than A_{j+k-1}, so there are $j+k-1-(k+1) = j-2$ possibilities. (This is because if the third vertex is A_i, then $i-j < k$ and $(2k+1)-i < k$ for the two arc lengths to be less than k. Therefore, $i < j+k$ and $i > k+1$ so i runs from $k+2$ to $j+k-1$, which is $j-2$ possibilities in total.)

As j runs from 2 to k, then $j-2$ runs from 0 to $k-2$, for a total of $0+1+2+\cdots+(k-2) = \frac{1}{2}(k-2)(k-1)$ acute triangles with one vertex fixed at A_1.

So in total there are $\frac{1}{3}(2k) \times \frac{1}{2}(k-2)(k-1) = \frac{1}{6}(2k)(k-2)(k-1)$ acute triangles.

Therefore, the probability is $\dfrac{\frac{1}{6}(2k)(k-2)(k-1)}{\frac{1}{6}(2k)(2k-1)(2k-2)} = \dfrac{k-2}{4k-2}$.

Solution 2

Again, we note that the total number of triangles is $\binom{2k}{3}$ if $N = 2k$.

We calculate the number of acute triangles with one vertex fixed at A_1.

Let the circumference of the circle be $2k$.

The vertices of any triangle partition this circumference into lengths a, b and c, reading clockwise from A_1.

Our strategy is to count the number of acute triangles in this configuration, and then multiply by $\dfrac{2k}{3}$ as in Solution 1 to obtain the total number of acute triangles.

We look for solutions to the equation $a+b+c = 2k$ with $a, b, c \ge 1$. (These are the partitions of the circumference of the circle.) Since we are looking for acute triangles, we must also have $a, b, c < k$ as in the preamble. Each different solution to the equation subject to these restrictions gives us different triangle and vice-versa.

So we count the number of integer solutions to $a+b+c = 2k$ with $1 \le a, b, c < k$.

Consider the transformation $a' = k-a$, $b' = k-b$ and $c' = k-c$.

Since $a, b, c < k$, then $a', b', c' > 0$.

Also, $a+b+c = 2k$ if and only if $3k-(a+b+c) = k$ if and only if $(k-a)+(k-b)+(k-c) = k$ if and only if $a'+b'+c' = k$.

So this transformation gives us a one-to-one correspondence between the acute triangles between the acute triangles on $2k$ vertices with one vertex fixed at A_1, and *all* triangles on k vertices with one vertex fixed at A_1. (Since we can "undo" this tranformation (ie. find its inverse), then it is a one-to-one correspondence (ie. a bijection).)

The total number of triangles on k points with one vertex fixed is $\binom{k-1}{2}$ since there are 2 vertices to choose from the remaining $k-1$ points.

Therefore, the total number of acute triangles with one vertex fixed at A_1 is also $\binom{k-1}{2}$, so the total number of acute triangles is

$$\frac{2k}{3}\binom{k-1}{2} = \frac{2k}{3}\frac{(k-1)(k-2)}{2} = \frac{k(k-1)(k-2)}{3}$$

To obtain the probability, we divide by $\binom{2k}{3}$ to obtain

$$\frac{6}{(2k)(2k-1)(2k-2)}\frac{k(k-1)(k-2)}{3} = \frac{k-2}{4k-2}$$

(c) From (b), the probability is $\frac{k-2}{4k-2}$.

We want to determine the values of k for which $\frac{k-2}{4k-2} = \frac{a}{2007}$ for some positive integer a.
Cross-multiplying, $2007(k-2) = a(4k-2)$.
Since the right is even, the left side must be even, so $k-2$ is even, so k is even, say $k = 2m$ for some positive integer $m \geq 1$.
Then $2007(2m-2) = a(8m-2)$ or $2007(m-1) = a(4m-1)$.
Since

$$(4m-1) - 4(m-1) = 3 \qquad (*)$$

then the possible positive common divisors of $4m-1$ and $m-1$ are 1 and 3 (since any common divisor of $4m-1$ and $m-1$ must also divide into 3 by (*)).
In other words, $\gcd(4m-1, m-1) = 1$ or $\gcd(4m-1, m-1) = 3$.

If $\gcd(4m-1, m-1) = 1$ then $4m-1$ and $m-1$ have no common factors larger than 1.
Since $2007(m-1) = a(4m-1)$, then $4m-1$ divides into $2007(m-1)$ and so $4m-1$ divides into 2007, since $4m-1$ and $m-1$ have no common factors.
Now $2007 = 9 \times 223 = 3^2 \times 223$, so the positive divisors of 2007 are 1, 3, 9, 223, 669, 2007.
The divisors having the form $4m-1$ for some positive integer m are 3, 233 and 2007, giving:

$4m-1$	3	233	2007
m	1	56	502
a	0	495	2004
k		112	1004

($m = 1$ gives $a = 0$ which is inadmissible, so there is no value for k.)

If $\gcd(4m-1, m-1) = 3$, then $m-1$ is divisible by 3, so we write $m-1 = 3p$ for some non-negative integer p.
Thus, $4m-1 = 12p+3$ and so $2007(m-1) = a(4m-1)$ becomes $2007(3p) = a(12p+3)$ or $2007p = a(4p+1)$.
Note that $\gcd(4p+1, p) = 1$, since $\gcd(12p+3, 3p) = 3$.
Since $4p+1$ divides into $2007p$ and has no common factors with p, then $4p+1$ divides into 2007.
The divisors of 2007 having the form $4p+1$ are 1, 9 and 669, giving:

$4p+1$	1	9	669
p	0	2	167
m	1	7	502
a	0	446	501
k		14	1004

($m = 1$ gives $a = 0$ which is inadmissible, so there is no value for k.)

Thus, the possible values of k are 14, 112 and 1004.

Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

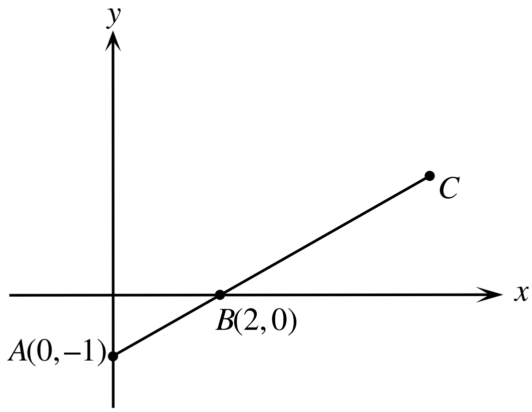
2005 Euclid Contest

Tuesday, April 19, 2005

Solutions

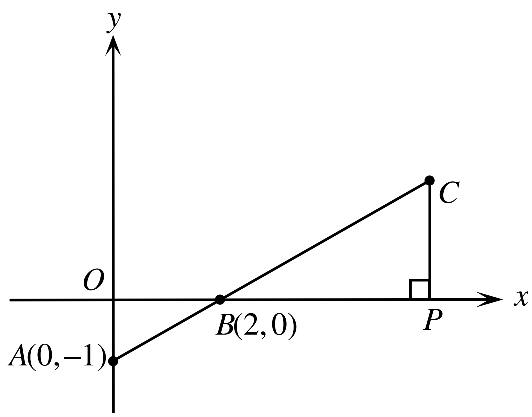
1. (a) ANSWER: $a = 5$
Since (a, a) lies on the line $3x - y = 10$, then $3a - a = 10$ or $2a = 10$ or $a = 5$.

(b) ANSWER: $(6, 2)$
Solution 1
To get from A to B, we move 2 units to the right and 1 unit up.



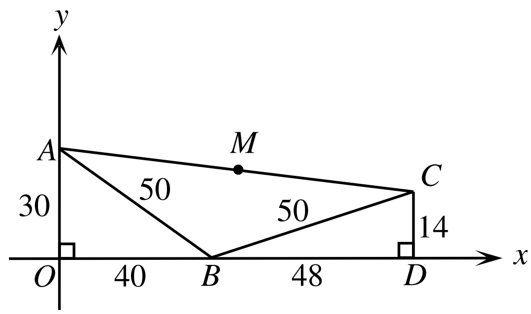
Since C lies on the same straight line as A and B, then to get from B to C we move 2 units to the right and 1 unit up twice, or 4 units to the right and 2 units up.
Thus, the coordinates of C are $(6, 2)$.

Solution 2
Label the origin as O and drop a perpendicular from C to P on the x-axis.



Then $\triangle AOB$ is similar to $\triangle CPB$ since both are right-angled and they have equal angles at B.
Since $BC = 2AB$, then $CP = 2AO = 2(1) = 2$ and $BP = 2BO = 2(2) = 4$.
Therefore, the coordinates of C are $(2 + 4, 0 + 2) = (6, 2)$.

(c) By the Pythagorean Theorem, $AO^2 = AB^2 - OB^2 = 50^2 - 40^2 = 900$, so $AO = 30$.
Therefore, the coordinates of A are $(0, 30)$.
By the Pythagorean Theorem, $CD^2 = CB^2 - BD^2 = 50^2 - 48^2 = 196$, so $CD = 14$.



Therefore, the coordinates of C are $(40+48,14)=(88,14)$.
Since M is the midpoint of AC, then the coordinates of M are

$$\left(\frac{1}{2}(0+88),\frac{1}{2}(30+14)\right)=(44,22)$$

2. (a) Answer: $x=-2$

Solution 1
Since $y=2x+3$, then $4y=4(2x+3)=8x+12$.
Since $4y=8x+12$ and $4y=5x+6$, then $8x+12=5x+6$ or $3x=-6$ or $x=-2$.

Solution 2
Since $4y=5x+6$, then $y=\frac{5}{4}x+\frac{6}{4}=\frac{5}{4}x+\frac{3}{2}$.
Since $y=2x+3$ and $y=\frac{5}{4}x+\frac{3}{2}$, then $2x+3=\frac{5}{4}x+\frac{3}{2}$ or $\frac{3}{4}x=-\frac{3}{2}$ or $x=-2$.

Solution 3
Since the second equation contains a "$5x$", we multiply the first equation by $\frac{5}{2}$ to obtain a $5x$ term, and obtain $\frac{5}{2}y=5x+\frac{15}{2}$.
Subtracting this from $4y=5x+6$, we obtain $\frac{3}{2}y=-\frac{3}{2}$ or $y=-1$.
Since $y=-1$, then $-1=2x+3$ or $2x=-4$ or $x=-2$.

(b) Answer: $a=6$

Solution 1
Adding the three equations together, we obtain $a-3b+b+2b+7c-2c-5c=-10+3+13$ or $a=6$.

Solution 2
Multiplying the second equation by 3, we obtain $3b-6c=9$.
Adding this new equation to the first equation, we obtain $c=-1$.
Substituting this back into the original second equation, we obtain $b=3+2c=1$.
Substituting into the third equation, $a=-2b+5c+13=-2-5+13=6$.

(c) *Solution 1*
Let J be John's score and M be Mary's score.
Since two times John's score was 60 more than Mary's score, then $2J=M+60$.
Since two times Mary's score was 90 more than John's score, then $2M=J+90$.
Adding these two equations, we obtain $2J+2M=M+J+150$ or $J+M=150$ or $\frac{J+M}{2}=75$.
Therefore, the average of their two scores was 75.
(Note that we didn't have to solve for their individual scores.)

Solution 2
Let J be John's score and M be Mary's score.
Since two times John's score was 60 more than Mary's score, then $2J = M + 60$, so $M = 2J - 60$.
Since two times Mary's score was 90 more than John's score, then $2M = J + 90$.
Substituting the first equation into the second, we obtain

$$\begin{aligned} 2(2J-60) &= J+90 \\ 4J-120 &= J+90 \\ 3J &= 210 \\ J &= 70 \end{aligned}$$

Substituting into $M = 2J - 60$ gives $M = 80$.
Therefore, the average of their scores (ie. the average of 70 and 80) is 75.

3. (a) Answer: $x = 50$
Simplifying using exponent rules,

$$2(16^{12}) + 2(8^{16}) = 2((2^4)^{12}) + 2((2^3)^{16}) = 2(2^{48}) + 2(2^{48}) = 4(2^{48}) = 2^2(2^{48}) = 2^{50}$$

Therefore, since $2^x = 2(16^{12}) + 2(8^{16}) = 2^{50}$, then $x = 50$.

(b) *Solution 1*
We factor the given equation $(f(x))^2 - 3f(x) + 2 = 0$ as $(f(x) - 1)(f(x) - 2) = 0$.
Therefore, $f(x) = 1$ or $f(x) = 2$.
If $f(x) = 1$, then $2x - 1 = 1$ or $2x = 2$ or $x = 1$.
If $f(x) = 2$, then $2x - 1 = 2$ or $2x = 3$ or $x = \frac{3}{2}$.
Therefore, the values of x are $x = 1$ or $x = \frac{3}{2}$.

Solution 2
Since $f(x) = 2x - 1$ and $(f(x))^2 - 3f(x) + 2 = 0$, then

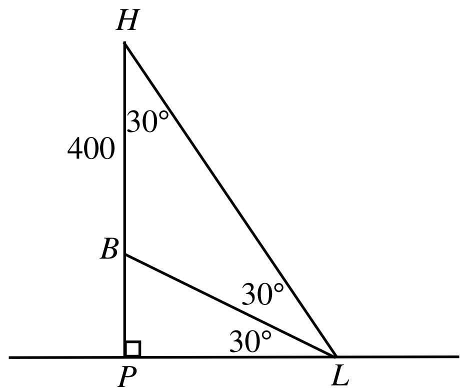
$$\begin{aligned} (2x-1)^2 - 3(2x-1) + 2 &= 0 \\ 4x^2 - 4x + 1 - 6x + 3 + 2 &= 0 \\ 4x^2 - 10x + 6 &= 0 \\ 2x^2 - 5x + 3 &= 0 \\ (x-1)(2x-3) &= 0 \end{aligned}$$

Therfore, $x = 1$ or $x = \frac{3}{2}$.

4. (a) Answer: $\frac{14}{15}$
Solution 1
The possible pairs of numbers on the tickets are (listed as ordered pairs): $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, $(4,6)$, and $(5,6)$.
There are fifteen such pairs. (We treat the pair of tickets numbered 2 and 4 as being the same as the pair numbered 4 and 2.)
The pairs for which the smaller of the two numbers is less than or equal to 4 are $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, and $(4,6)$.
There are fourteen such pairs.
Therefore, the probability of selecting such a pair of tickets is $\frac{14}{15}$.

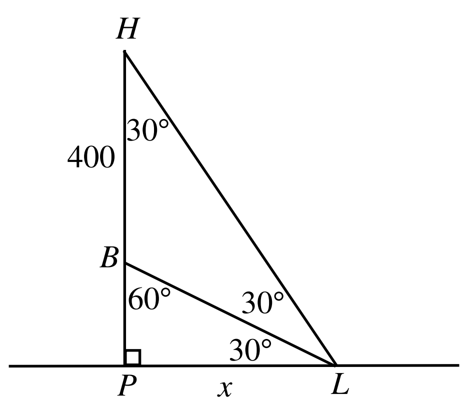
Solution 2
We find the probability that the smaller number on the two tickets is NOT less than or equal to 4.
Therefore, the smaller number on the two tickets is at least 5.
Thus, the pair of numbers must be 5 and 6, since two distinct numbers less than or equal to 6 are being chosen.
As in Solution 1, we can determine that there are fifteen possible pairs that we can selected.
Therefore, the probability that the smaller number on the two tickets is NOT less than or equal to 4 is $\frac{1}{15}$, so the probability that the smaller number on the two tickets IS less than or equal to 4 is $1 - \frac{1}{15} = \frac{14}{15}$.

(b) *Solution 1*
Since $\angle HLP = 60°$ and $\angle BLP = 30°$, then $\angle HLB = \angle HLP - \angle BLP = 30°$.
Also, since $\angle HLP = 60°$ and $\angle HPL = 90°$, then $\angle LHP = 180° - 90° - 60° = 30°$.



Therefore, $\triangle HBL$ is isosceles and $BL = HB = 400$ m.
In $\triangle BLP$, $BL = 400$ m and $\angle BLP = 30°$, so $LP = BL\cos(30°) = 400\left(\frac{\sqrt{3}}{2}\right) = 200\sqrt{3}$ m.
Therefore, the distance between L and P is $200\sqrt{3}$ m.

Solution 2
Since $\angle HLP = 60°$ and $\angle BLP = 30°$, then $\angle HLB = \angle HLP - \angle BLP = 30°$.
Also, since $\angle HLP = 60°$ and $\angle HPL = 90°$, then $\angle LHP = 180° - 90° - 60° = 30°$.
Also, $\angle LBP = 60°$.
Let $LP = x$.



Since $\triangle BLP$ is 30°-60°-90°, then $BP : LP = 1 : \sqrt{3}$, so $BP = \frac{1}{\sqrt{3}}LP = \frac{1}{\sqrt{3}}x$.

Since $\triangle HLP$ is 30°-60°-90°, then $HP : LP = \sqrt{3} : 1$, so $HP = \sqrt{3}LP = \sqrt{3}x$.
But $HP = HB + BP$ so

$$\begin{aligned}
\sqrt{3}x &= 400 + \frac{1}{\sqrt{3}}x \\
3x &= 400\sqrt{3} + x \\
2x &= 400\sqrt{3} \\
x &= 200\sqrt{3}
\end{aligned}$$

Therefore, the distance from L to P is $200\sqrt{3}$ m.

5. (a) ANSWER: (6,5)
After 2 moves, the goat has travelled $1 + 2 = 3$ units.
After 3 moves, the goat has travelled $1 + 2 + 3 = 6$ units.
Similarly, after n moves, the goat has travelled a total of $1 + 2 + 3 + \cdots + n$ units.
For what value of n is $1 + 2 + 3 + \cdots + n$ equal to 55?
The fastest way to determine the value of n is by adding the first few integers until we obtain a sum of 55. This will be $n = 10$.
(We could also do this by remembering that $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$ and solving for n this way.)
So we must determine the coordinates of the goat after 10 moves.
We consider first the x-coordinate.
Since starting at $(0, 0)$ the goat has moved 2 units in the positive x direction, 4 units in the negative x direction, 6 units in the positive x direction, 8 units in the negative x direction and 10 units in the positive x direction, so its x coordinate should be $2-4+6-8+10 = 6$.
Similarly, its y-coordinate should be $1 - 3 + 5 - 7 + 9 = 5$.
Therefore, after having travelled a distance of 55 units, the goat is at the point $(6, 5)$.

(b) *Solution 1*
Since the sequence 4, $4r$, $4r^2$ is also arithmetic, then the difference between $4r^2$ and $4r$ equals the difference between $4r$ and 4, or

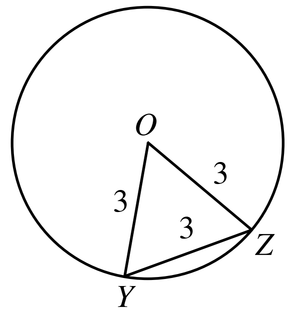
$$\begin{aligned}
4r^2 - 4r &= 4r - 4 \\
4r^2 - 8r + 4 &= 0 \\
r^2 - 2r + 1 &= 0 \\
(r - 1)^2 &= 0
\end{aligned}$$

Therefore, the only value of r is $r = 1$.

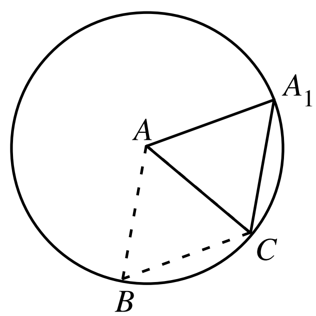
Solution 2
Since the sequence 4, $4r$, $4r^2$ is also arithmetic, then we can write $4r = 4 + d$ and $4r^2 = 4 + 2d$ for some real number d. (Here, d is the common difference in this arithmetic sequence.)
Then $d = 4r - 4$ and $2d = 4r^2 - 4$ or $d = 2r^2 - 2$.
Therefore, equating the two expressions for d, we obtain $2r^2-2 = 4r-4$ or $2r^2-4r+2 = 0$ or $r^2 - 2r + 1 = 0$ or $(r - 1)^2 = 0$.
Therefore, the only value of r is $r = 1$.

6. (a) ANSWER: 4π
First, we notice that whenever an equilateral triangle of side length 3 is placed inside a

circle of radius 3 with two of its vertices on the circle, then the third vertex will be at the centre of the circle.
This is because if we place $\triangle XYZ$ with Y and Z on the circle and connect Y and Z to the centre O, then $OY = OZ = 3$, so $\triangle OYZ$ is equilateral (since all three sides have length 3). Thus $\triangle XYZ$ and $\triangle OYZ$ must be the same, so X is at the same point as O.

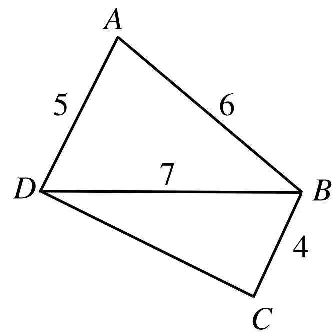


Thus, in the starting position, A is at the centre of the circle.
As the triangle is rotated about C, the point B traces out an arc of a circle of radius 3.
What fraction of the circle is traced out?
When point A reaches point A_1 on the circle, we have $AC = 3$ and $CA_1 = 3$. Since A is at the centre of the circle, then $AA_1 = 3$ as well, so $\triangle AA_1C$ is equilateral, and $\angle A_1CA = 60^\circ$, so the triangle has rotated through 60°.



Therefore, B has traced out $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a circle of radius 3.
Notice that A has also traced out an arc of the same length. When A reaches the circle, we have A and C on the circle, so B must be at the centre of the circle.
Thus, on the next rotation, B again rotates through $\frac{1}{6}$ of a circle of radius 3 as it moves to the circle.
On the third rotation, the triangle rotates about B, so B does not move. After three rotations, the triangle will have A at the centre and B and C on the circle, with the net result that the triangle has rotated 180° about the centre of the circle.
Thus, to return to its original position, the triangle must undergo three more of these rotations, and B will behave in the same way as it did for the first three rotations.
Thus, in total, B moves four times along an arc equal to $\frac{1}{6}$ of a circle of radius 3.
Therefore, the distance travelled by B is $4(\frac{1}{6})(2\pi(3)) = 4\pi$.

(b) In order to determine CD, we must determine one of the angles (or at least some information about one of the angles) in $\triangle BCD$.
To do this, we look at $\angle A$ use the fact that $\angle A + \angle C = 180^\circ$.



Using the cosine law in $\triangle ABD$, we obtain

$$\begin{aligned} 7^2 &= 5^2 + 6^2 - 2(5)(6)\cos(\angle A) \\ 49 &= 61 - 60\cos(\angle A) \\ \cos(\angle A) &= \frac{1}{5} \end{aligned}$$

Since $\cos(\angle A) = \frac{1}{5}$ and $\angle A + \angle C = 180^\circ$, then $\cos(\angle C) = -\cos(180^\circ - \angle A) = -\frac{1}{5}$.
(We could have calculated the actual size of $\angle A$ using $\cos(\angle A) = \frac{1}{5}$ and then used this to calculate the size of $\angle C$, but we would introduce the possibility of rounding error by doing this.)
Then, using the cosine law in $\triangle BCD$, we obtain

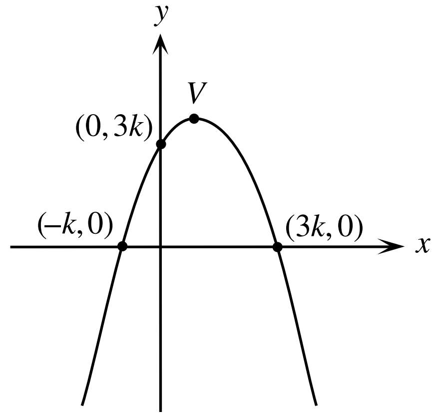
$$\begin{aligned} 7^2 &= 4^2 + CD^2 - 2(4)(CD)\cos(\angle C) \\ 49 &= 16 + CD^2 - 8(CD)\left(-\frac{1}{5}\right) \\ 0 &= 5CD^2 + 8CD - 165 \\ 0 &= (5CD + 33)(CD - 5) \end{aligned}$$

So $CD = -\frac{33}{5}$ or $CD = 5$. (We could have also determined these roots using the quadratic formula.)
Since CD is a length, it must be positive, so $CD = 5$.

(We could have also proceeded by using the sine law in $\triangle BCD$ to determine $\angle BDC$ and then found the size of $\angle DBC$, which would have allowed us to calculate CD using the sine law. However, this would again introduce the potential of rounding error.)

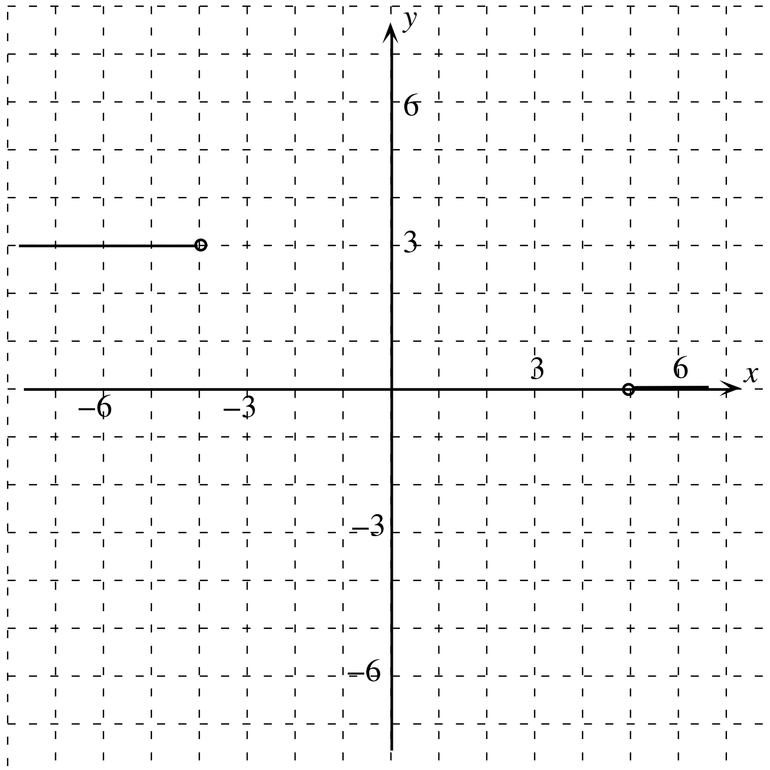
7. (a) Answer: Maximum $= 5$, Minimum $= 1$
We rewrite by completing the square as $f(x) = \sin^2 x - 2\sin x + 2 = (\sin x - 1)^2 + 1$.
Therefore, since $(\sin x - 1)^2 \geq 0$, then $f(x) \geq 1$, and in fact $f(x) = 1$ when $\sin x = 1$ (which occurs for instance when $x = 90^\circ$).
Thus, the minimum value of $f(x)$ is 1.
To maximize $f(x)$, we must maximize $(\sin x - 1)^2$.
Since $-1 \leq \sin x \leq 1$, then $(\sin x - 1)^2$ is maximized when $\sin x = -1$ (for instance, when $x = 270^\circ$). In this case, $(\sin x - 1)^2 = 4$, so $f(x) = 5$.
Thus, the maximum value of $f(x)$ is 5.

(b) From the diagram, the x-intercepts of the parabola are $x = -k$ and $x = 3k$.

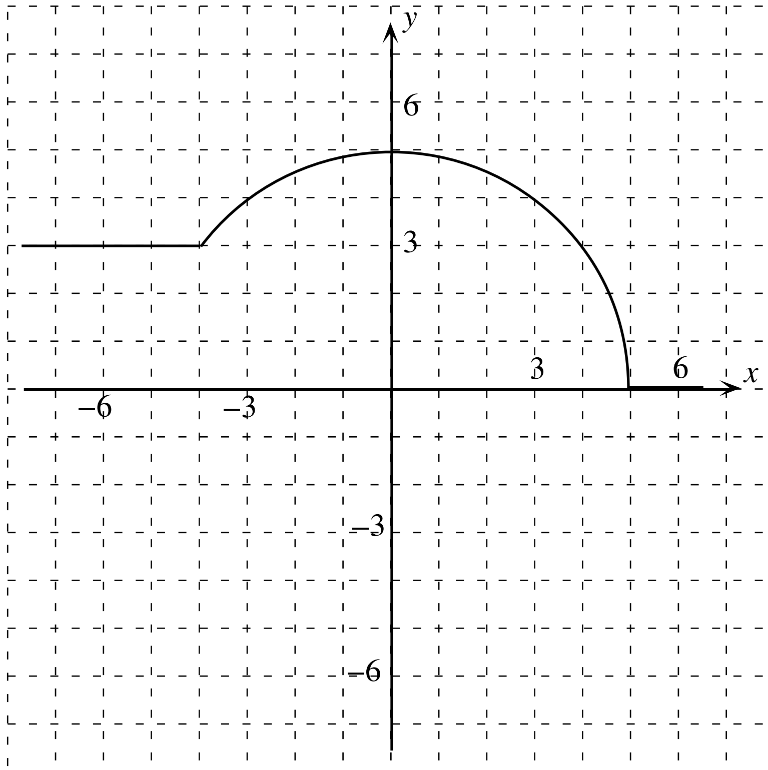


Since we are given that $y = -\frac{1}{4}(x-r)(x-s)$, then the x-intercepts are r and s, so r and s equal $-k$ and $3k$ in some order.
Therefore, we can rewrite the parabola as $y = -\frac{1}{4}(x-(-k))(x-3k)$.
Since the point $(0,3k)$ lies on the parabola, then $3k = -\frac{1}{4}(0+k)(0-3k)$ or $12k = 3k^2$ or $k^2 - 4k = 0$ or $k(k-4) = 0$.
Thus, $k = 0$ or $k = 4$.
Since the two roots are distinct, then we cannot have $k = 0$ (otherwise both x-intercepts would be 0).
Thus, $k = 4$.
This tells us that the equation of the parabola is $y = -\frac{1}{4}(x+4)(x-12)$ or $y = -\frac{1}{4}x^2 + 2x + 12$.
We still have to determine the coordinates of the vertex, V.
Since the x-intercepts of the parabola are -4 and 12, then the x-coordinate of the vertex is the average of these intercepts, or 4.
(We could have also used the fact that the x-coordinate is $-\dfrac{b}{2a} = -\dfrac{2}{2(-\frac{1}{4})}$.)
Therefore, the y-coordinate of the vertex is $y = -\frac{1}{4}(4^2) + 2(4) + 12 = 16$.
Thus, the coordinates of the vertex are $(4, 16)$.

8. (a) We look at the three pieces separately.
If $x < -4$, $f(x) = 4$ so $g(x) = \sqrt{25 - [f(x)]^2} = \sqrt{25 - 4^2} = \sqrt{9} = 3$.
So $g(x)$ is the horizontal line $y = 3$ when $x < -4$.
If $x > 5$, $f(x) = -5$ so $g(x) = \sqrt{25 - [f(x)]^2} = \sqrt{25 - (-5)^2} = \sqrt{0} = 0$.
So $g(x)$ is the horizontal line $y = 0$ when $x > 5$.
So far, our graph looks like this:

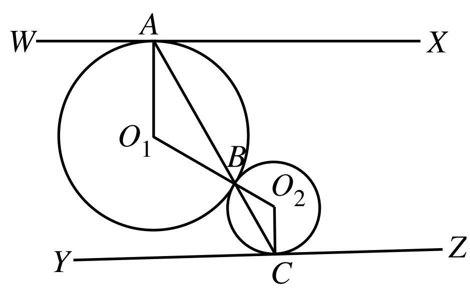


If $-4 \le x \le 5$, $f(x) = -x$ so $g(x) = \sqrt{25 - [f(x)]^2} = \sqrt{25 - (-x)^2} = \sqrt{25 - x^2}$.
What is this shape?
If $y = g(x)$, then we have $y = \sqrt{25 - x^2}$ or $y^2 = 25 - x^2$ or $x^2 + y^2 = 25$.
Therefore, this shape is a section of the upper half (since y is a positive square-root) of the circle $x^2 + y^2 = 25$, ie. the circle with centre $(0, 0)$ and radius 5.
We must check the endpoints.
When $x = -4$, we have $g(-4) = \sqrt{25 - (-4))^2} = 3$.
When $x = 5$, we have $g(5) = \sqrt{25 - 5^2} = 0$.
Therefore, the section of the circle connects up with the other two sections of our graph already in place.
Thus, our final graph is:



(b) *Solution 1*

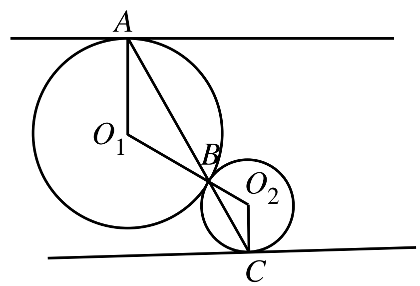
Let the centres of the two circles be O_1 and O_2.
Join A and B to O_1 and B and C to O_2.
Designate two points W and X on either side of A on one tangent line, and two points Y and Z on either side of C on the other tangent line.



Let $\angle XAB = \theta$.
Since WX is tangent to the circle with centre O_1 at A, then O_1A is perpendicular to WX, so $\angle O_1AB = 90^\circ - \theta$.
Since $O_1A = O_1B$ because both are radii, then $\triangle AO_1B$ is isosceles, so $\angle O_1BA = \angle O_1AB = 90^\circ - \theta$.
Since the two circles are tangent at B, then the line segment joining O_1 and O_2 passes through B, ie. O_1BO_2 is a straight line segment.
Thus, $\angle O_2BC = \angle O_1BA = 90^\circ - \theta$, by opposite angles.
Since $O_2B = O_2C$, then similarly to above, $\angle O_2CB = \angle O_2BC = 90^\circ - \theta$.
Since YZ is tangent to the circle with centre O_2 at C, then O_2C is perpendicular to YZ.
Thus, $\angle YCB = 90^\circ - \angle O_2CB = \theta$.
Since $\angle XAB = \angle YCB$, then WX is parallel to YZ, by alternate angles, as required.

Solution 2

Let the centres of the two circles be O_1 and O_2.
Join A and B to O_1 and B and C to O_2.
Since AO_1 and BO_1 are radii of the same circle, $AO_1 = BO_1$ so $\triangle AO_1B$ is isosceles, so $\angle O_1AB = \angle O_1BA$.



Since BO_2 and CO_2 are radii of the same circle, $BO_2 = CO_2$ so $\triangle BO_2C$ is isosceles, so $\angle O_2BC = \angle O_2CB$.
Since the two circles are tangent at B, then O_1BO_2 is a line segment (ie. the line segment joining O_1 and O_2 passes through the point of tangency of the two circles).
Since O_1BO_2 is straight, then $\angle O_1BA = \angle O_2BC$, by opposite angles.
Thus, $\angle O_1AB = \angle O_1BA = \angle O_2BC = \angle O_2CB$.
This tells us that $\triangle AO_1B$ is similar to $\triangle BO_2C$, so $\angle AO_1B = \angle BO_2C$ or $\angle AO_1O_2 = \angle CO_2O_1$.
Therefore, AO_1 is parallel to CO_2, by alternate angles.
But A and C are points of tangency, AO_1 is perpendicular to the tangent line at A and CO_2 is perpendicular to the tangent line at C.
Since AO_1 and CO_2 are parallel, then the two tangent lines must be parallel.

9. (a) *Solution 1*

We have $(x-p)^2 + y^2 = r^2$ and $x^2 + (y-p)^2 = r^2$, so at the points of intersection,

$$\begin{aligned} (x-p)^2 + y^2 &= x^2 + (y-p)^2 \\ x^2 - 2px + p^2 + y^2 &= x^2 + y^2 - 2py + p^2 \\ -2px &= -2py \end{aligned}$$

and so $x = y$ (since we may assume that $p \neq 0$ otherwise the two circles would coincide).
Therefore, a and b are the two solutions of the equation $(x-p)^2 + x^2 = r^2$ or $2x^2 - 2px + (p^2 - r^2) = 0$ or $x^2 - px + \frac{1}{2}(p^2 - r^2)$=0.
Using the relationship between the sum and product of roots of a quadratic equation and its coefficients, we obtain that $a + b = p$ and $ab = \frac{1}{2}(p^2 - r^2)$.
(We could have solved for a and b using the quadratic formula and calculated these directly.)
So we know that $a + b = p$.
Lastly, $a^2 + b^2 = (a+b)^2 - 2ab = p^2 - 2\left(\frac{1}{2}(p^2 - r^2)\right) = r^2$, as required.

Solution 2

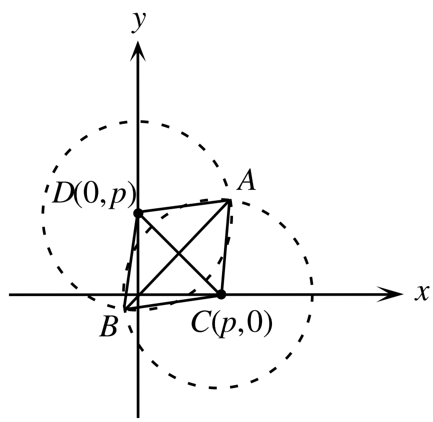
Since the circles are reflections of one another in the line $y = x$, then the two points of intersection must both lie on the line $y = x$, ie. A has coordinates (a, a) and B has coordinates (b, b).
Therefore, $(a-p)^2 + a^2 = r^2$ and $(b-p)^2 + b^2 = r^2$, since these points lie on both circles.

Subtracting the two equations, we get

$$\begin{aligned}
(b-p)^2-(a-p)^2+b^2-a^2 &= 0\\
((b-p)-(a-p))((b-p)+(a-p))+(b-a)(b+a) &= 0\\
(b-a)(a+b-2p)+(b-a)(b+a) &= 0\\
(b-a)(a+b-2p+b+a) &= 0\\
2(b-a)(a+b-p) &= 0
\end{aligned}$$

Since $a \neq b$, then we must have $a+b=p$, as required.
Since $a+b=p$, then $a-p=-b$, so substituting back into $(a-p)^2+a^2=r^2$ gives $(-b)^2+a^2=r^2$, or $a^2+b^2=r^2$, as required.

(b) We first draw a diagram.



We know that C has coordinates $(p,0)$ and D has coordinates $(0,p)$.
Thus, the slope of line segment CD is -1.
Since the points A and B both lie on the line $y=x$, then the slope of line segment AB is 1.
Therefore, AB is perpendicular to CD, so $CADB$ is a kite, and so its area is equal to $\frac{1}{2}(AB)(CD)$.
(We could derive this by breaking quadrilateral $CADB$ into $\triangle CAB$ and $\triangle DAB$.)
Since C has coordinates $(p,0)$ and D has coordinates $(0,p)$, then $CD=\sqrt{p^2+(-p)^2}=\sqrt{2p^2}$.
(We do not know if p is positive, so this is not necessarily equal to $\sqrt{2}p$.)
We know that A has coordinates (a,a) and B has coordinates (b,b), so

$$\begin{aligned}
AB &= \sqrt{(a-b)^2+(a-b)^2}\\
&= \sqrt{2a^2-4ab+2b^2}\\
&= \sqrt{2(a^2+b^2)-4ab}\\
&= \sqrt{2r^2-4\left(\frac{1}{2}(p^2-r^2)\right)}\\
&= \sqrt{4r^2-2p^2}
\end{aligned}$$

Therefore, the area of quadrilateral $CADB$ is

$$\frac{1}{2}(AB)(CD)=\frac{1}{2}\sqrt{4r^2-2p^2}\sqrt{2p^2}=\sqrt{2r^2p^2-p^4}$$

To maximize this area, we must maximize $2r^2p^2 - p^4 = 2r^2(p^2) - (p^2)^2$.
Since r is fixed, we can consider this as a quadratic polynomial in p^2. Since the coefficient of $(p^2)^2$ is negative, then this is a parabola opening downwards, so we find its maximum value by finding its vertex.
The vertex of $2r^2(p^2) - (p^2)^2$ is at $p^2 = -\dfrac{2r^2}{2(-1)} = r^2$.
So the maximum area of the quadrilateral occurs when p is chosen so that $p^2 = r^2$.
Since $p^2 = r^2$, then $(a+b)^2 = p^2 = r^2$ so $a^2 + 2ab + b^2 = r^2$.
Since $a^2 + b^2 = r^2$, then $2ab = 0$ so either $a = 0$ or $b = 0$, and so either A has coordinates $(0,0)$ or B has coordinates $(0,0)$, ie. either A is the origin or B is the origin.

(c) In (b), we calculated that $AB = \sqrt{4r^2 - 2p^2} = \sqrt{2}\sqrt{2r^2 - p^2}$.
Since r and p are integers (and we assume that neither r nor p is 0), then $2r^2 - p^2 \neq 0$, so the minimum possible non-negative value for $2r^2 - p^2$ is 1, since $2r^2 - p^2$ must be an integer.
Therefore, the minimum possible distance between A and B should be $\sqrt{2}\sqrt{1} = \sqrt{2}$.
Can we find positive integers p and r that give us this value?
Yes – if $r = 5$ and $p = 7$, then $2r^2 - p^2 = 1$, so $AB = \sqrt{2}$.
(There are in fact an infinite number of positive integer solutions to the equation $2r^2 - p^2 = 1$ or equivalently $p^2 - 2r^2 = -1$. This type of equation is called Pell's Equation.)

10. (a) We proceed directly.
On the first pass from left to right, Josephine closes all of the even numbered lockers, leaving the odd ones open.
The second pass proceeds from right to left. Before the pass, the lockers which are open are 1, 3, …, 47, 49.
On the second pass, she shuts lockers 47, 43, 39, …, 3.
The third pass proceeds from left to right. Before the pass, the lockers which are open are 1, 5, …, 45, 49.
On the third pass, she shuts lockers 5, 13, …, 45.
This leaves lockers 1, 9, 17, 25, 33, 41, 49 open.
On the fourth pass, from right to left, lockers 41, 25 and 9 are shut, leaving 1, 17, 33, 49.
On the fifth pass, from left to right, lockers 17 and 49 are shut, leaving 1 and 33 open.
On the sixth pass, from right to left, locker 1 is shut, leaving 33 open.
Thus, $f(50) = 33$.

(b)&(c) *Solution 1*
First, we note that if $n = 2k$ is even, then $f(n) = f(2k) = f(2k-1) = f(n-1)$. See Solution 2 for this justification.
Therefore, we only need to look for odd values of n in parts (b) and (c).

Suppose that there was an n so that $f(n) = 2005$, ie. 2005 is the last locker left open.
On the first pass, Josephine closes every other locker starting at the beginning, so she closes all lockers numbered m with $m \equiv 0 \pmod{2}$.
This leaves only odd-numbered lockers open, ie. only lockers m with $m \equiv 1$ or $3 \pmod{4}$.
On her second pass, she closes every other open locker, starting from the right-hand end. Thus, she will close every fourth locker from the original row.
Since we want 2005 to be left open and $2005 \equiv 1 \pmod{4}$, then she must close all lockers numbered m with $m \equiv 3 \pmod{4}$.
This leaves open only the lockers m with $m \equiv 1 \pmod{4}$, or equivalently lockers with $m \equiv 1$ or $5 \pmod{8}$.

On her third pass, she closes every other open locker, starting from the left-hand end. Thus, she will close every eighth locker from the original row.
Since locker 1 is still open, then she starts by closing locker 5, and so closes all lockers m with $m \equiv 5 \pmod 8$.
But since $2005 \equiv 5 \pmod 8$, then she closes locker 2005 on this pass, a contradiction.
Therefore, there can be no integer n with $f(n) = 2005$.

Next, we show that there are infinitely many positive integers n such that $f(n) = f(2005)$. To do this, we first make a table of what happens when there are 2005 lockers in the row. We record the pass #, the direction of the pass, the leftmost locker that is open, the rightmost locker that is open, all open lockers before the pass, which lockers will be closed on the pass, and which lockers will be left open after the pass:

Pass #	Dir.	L Open	R Open	Open	To close	Leaves Open
1	L to R	1	2005	All	$\equiv 0 \pmod 2$	$\equiv 1 \pmod 2$
2	R to L	1	2005	$\equiv 1, 3 \pmod 4$	$\equiv 3 \pmod 4$	$\equiv 1 \pmod 4$
3	L to R	1	2005	$\equiv 1, 5 \pmod 8$	$\equiv 5 \pmod 8$	$\equiv 1 \pmod 8$
4	R to L	1	2001	$\equiv 1, 9 \pmod{16}$	$\equiv 9 \pmod{16}$	$\equiv 1 \pmod{16}$
5	L to R	1	2001	$\equiv 1, 17 \pmod{32}$	$\equiv 17 \pmod{32}$	$\equiv 1 \pmod{32}$
6	R to L	1	1985	$\equiv 1, 33 \pmod{64}$	$\equiv 33 \pmod{64}$	$\equiv 1 \pmod{64}$
7	L to R	1	1985	$\equiv 1, 65 \pmod{128}$	$\equiv 65 \pmod{128}$	$\equiv 1 \pmod{128}$
8	R to L	1	1921	$\equiv 1, 129 \pmod{256}$	$\equiv 1 \pmod{256}$	$\equiv 129 \pmod{256}$
9	L to R	129	1921	$\equiv 129, 385 \pmod{512}$	$\equiv 385 \pmod{512}$	$\equiv 129 \pmod{512}$
10	R to L	129	1665	$\equiv 129, 641 \pmod{1024}$	$\equiv 129 \pmod{1024}$	$\equiv 641 \pmod{1024}$
11	L to R	641	1665	$\equiv 641, 1665 \pmod{2048}$	$\equiv 1665 \pmod{2048}$	$\equiv 641 \pmod{2048}$

Since there is only one integer between 1 and 2005 congruent to $641 \pmod{2048}$, then there is only one locker left open: locker 641.
Notice also that on any pass s, the "class" of lockers which are closed depends on what the number of the leftmost (on an odd-numbered pass) or rightmost (on an even-numbered pass) open locker number is congruent to mod 2^s.

Consider $n = 2005 + 2^{2a}$, where $2^{2a} > 2005$, ie. $a \geq 6$.
We show that $f(n) = f(2005) = 641$. (See Solution 2 for a justification of why we might try these values of n.)
Suppose we were to try to make a table as above to calculate $f(n)$.
Then the first 11 passes in the table would be identical to the table above, except for the rightmost open number; this number in the new table would be the number above plus 2^{2a}.
What will happen after pass 11?
After pass 11, the lockers which are open are lockers with numbers $\equiv 641 \pmod{2048}$. Thus, the leftmost open locker is 641 and the rightmost is $2^{2a} + 641$.
As the 12th pass starts, the lockers which are still open are those with numbers $\equiv 641$ or $2689 \pmod{2^{12}}$.
Since the rightmost open locker number $(2^{2a} + 641)$ is congruent to $641 \pmod{2^{12}}$, then all lockers with numbers $\equiv 2689 \pmod{2^{12}}$ are closed, leaving open only those lockers with numbers $\equiv 641 \pmod{2^{12}}$.
So after this 12th pass, the lockers which are open are 641, $641 + 2^{12}$, $641 + 2(2^{12})$, $641 + 3(2^{12})$, …, $641 + 2^{2a-12}(2^{12}) = 641 + 2^{2a}$.
The number of open lockers is $2^{2a-12} + 1$.

If we can now show that whenever we start with a number of lockers of the form $2^{2c}+1$, the last locker remaining open is the leftmost locker, then we will be done, since of the lockers left open above ($2^{2a-12}+1$ of them, ie. 2 to an even power plus 1), then the last locker remaining open will be the leftmost one, that is locker 641, so $f(2^{2a}+2005) = 641 = f(2005)$.

So consider a row of $2^{2c}+1$ lockers.
Notice that on any pass, if the number of lockers is odd, then the number of lockers which will be closed is one-half of one less than the total number of lockers, and the first and last lockers will be left open.
So on the first pass, there are 2^{2c-1} lockers closed, leaving $2^{2c}+1-2^{2c-1} = 2^{2c-1}+1$ lockers open, ie. an odd number of lockers open.
On the next pass, there are 2^{2c-2} lockers closed (since there are an odd number of lockers open to begin), leaving $2^{2c-2}+1$ lockers open.
This continues, until there are $2^1+1=3$ lockers open just before an even-numbered (ie. right to left) pass. Thus, the middle of these three lockers will be closed, leaving only the original leftmost and rightmost lockers open.
On the last pass (an odd-numbered pass from left to right), the rightmost locker will be closed, leaving only the leftmost locker open.
Therefore, starting with a row of $2^{2c}+1$ open lockers, the leftmost locker will be the last remaining open.

Translating this to the above, we see that the leftmost locker of the $2^{2a-12}+1$ still open is the last left open, ie. $f(2^{2a}+2005) = 641 = f(2005)$ if $a \geq 6$.

Therefore, there are infinitely many positive integers n for which $f(n) = f(2005)$.

Solution 2
First, we calculate $f(n)$ for n from 1 to 32, to get a feeling for what happens. We obtain 1, 1, 3, 3, 1, 1, 3, 3, 9, 9, 11, 11, 9, 9, 11, 11, 1, 1, 3, 3, 1, 1, 3, 3, 9, 9, 11, 11, 9, 9, 11, 11. This will help us to establish some patterns.

Next, we establish two recursive formulas for $f(n)$.

First, from our pattern, it looks like $f(2m) = f(2m-1)$.
Why is this true in general?
Consider a row of $2m$ lockers.
On the first pass through , Josephine shuts all of the even numbered lockers, leaving open lockers 1, 3, $\ldots$, $2m-1$.
These are exactly the same open lockers as if she had started with $2m-1$ lockers in total.
Thus, as she starts her second pass from right to left, the process will be the same now whether she started with $2m$ lockers or $2m-1$ lockers.
Therefore, $f(2m) = f(2m-1)$.
This tells us that we need only focus on the values of $f(n)$ where n is odd.

Secondly, we show that $f(2m-1) = 2m+1-2f(m)$.
(It is helpful to connect $n = 2m-1$ to a smaller case.)
Why is this formula true?
Starting with $2m-1$ lockers, the lockers left open after the first pass are 1, 3, $\ldots$, $2m-1$, ie. m lockers in total.
Suppose $f(m) = p$. As Josephine begins her second pass, which is from right to left, we can think of this as being like the first pass through a row of m lockers.
Thus, the last open locker will be the pth locker, counting from the right hand end, from the list 1, 3, $\ldots$, $2m-1$.
The first locker from the right is $2m-1 = 2m+1-2(1)$, the second is $2m-3 = 2m+1-2(2)$, and so on, so the pth locker is $2m+1-2p$.
Therefore, the final open locker is $2m+1-2p$, ie. $f(2m-1) = 2m+1-2p = 2m+1-2f(m)$.

Using these two formulae repeatedly,

$$\begin{aligned} f(4k+1) &= f(2(2k+1)-1) \\ &= 2(2k+1)+1-2f(2k+1) \\ &= 4k+3-2f(2(k+1)-1) \\ &= 4k+3-2(2(k+1)+1-2f(k+1)) \\ &= 4k+3-2(2k+3-2f(k+1)) \\ &= 4f(k+1)-3 \end{aligned}$$

and

$$\begin{aligned}
f(4k+3) &= f(2(2k+2)-1) \\
&= 2(2k+2)+1-2f(2k+2) \\
&= 4k+5-2f(2k+1) \\
&= 4k+5-2f(2(k+1)-1) \\
&= 4k+5-2(2(k+1)+1-2f(k+1)) \\
&= 4k+5-2(2k+3-2f(k+1)) \\
&= 4f(k+1)-1
\end{aligned}$$

From our initial list of values of $f(n)$, it appears as if $f(n)$ cannot leave a remainder of 5 or 7 when divided by 8. So we use these recursive relations once more to try to establish this:

$$\begin{aligned}
f(8l+1) &= 4f(2l+1)-3 \qquad (\text{since } 8l+1 = 4(2l)+1) \\
&= 4(2l+3-2f(l+1))-3 \\
&= 8l+9-8f(l+1) \\
&= 8(l-f(l+1))+9
\end{aligned}$$

$$\begin{aligned}
f(8l+3) &= 4f(2l+1)-1 \qquad (\text{since } 8l+3 = 4(2l)+3) \\
&= 4(2l+3-2f(l+1))-1 \\
&= 8l+11-8f(l+1) \\
&= 8(l-f(l+1))+11
\end{aligned}$$

Similarly, $f(8l+5) = 8l+9-8f(l+1)$ and $f(8l+7) = 8l+11-8f(l+1)$.
Therefore, since any odd positive integer n can be written as $8l+1$, $8l+3$, $8l+5$ or $8l+7$, then for any odd positive integer n, $f(n)$ is either 9 more or 11 more than a multiple of 8.
Therefore, for any odd positive integer n, $f(n)$ cannot be 2005, since 2005 is not 9 more or 11 more than a multiple of 8.
Thus, for every positive integer n, $f(n) \neq 2005$, since we only need to consider odd values of n.

Next, we show that there are infinitely many positive integers n such that $f(n) = f(2005)$. We do this by looking at the pattern we initially created and conjecturing that

$$f(2005) = f(2005 + 2^{2a})$$

if $2^{2a} > 2005$. (We might guess this by looking at the connection between $f(1)$ and $f(3)$ with $f(5)$ and $f(7)$ and then $f(1)$ through $f(15)$ with $f(17)$ through $f(31)$. In fact, it appears to be true that $f(m + 2^{2a}) = f(m)$ if $2^{2a} > m$.)

Using our formulae from above,

$$\begin{aligned}
f(2005+2^{2a}) &= 4f(502+2^{2a-2})-3 & (2005+2^{2a}=4(501+2^{2a-2})+1)\\
&= 4f(501+2^{2a-2})-3 &\\
&= 4(4f(126+2^{2a-4})-3)-3 & (501+2^{2a-2}=4(125+2^{2a-4})+1)\\
&= 16f(126+2^{2a-4})-15 &\\
&= 16f(125+2^{2a-4})-15 &\\
&= 16(4f(32+2^{2a-6})-3)-15 & (125+2^{2a-4}=4(31+2^{2a-6})+1)\\
&= 64f(32+2^{2a-6})-63 &\\
&= 64f(31+2^{2a-6})-63 &\\
&= 64(4f(8+2^{2a-8})-1)-63 & (31+2^{2a-6}=4(7+2^{2a-8})+3)\\
&= 256f(8+2^{2a-8})-127 &\\
&= 256f(7+2^{2a-8})-127 &\\
&= 256(4f(2+2^{2a-10})-1)-127 & (7+2^{2a-8}=4(1+2^{2a-10})+3)\\
&= 1024f(2+2^{2a-10})-383 &\\
&= 1024f(1+2^{2a-10})-383 &
\end{aligned}$$

(Notice that we could have removed the powers of 2 from inside the functions and used this same approach to show that $f(2005) = 1024f(1) - 383 = 641$.)

But, $f(2^{2b}+1) = 1$ for every positive integer b.
Why is this true? We can prove this quickly by induction.
For $b = 1$, we know $f(5) = 1$.
Assume that the result is true for $b = B - 1$, for some positive integer $B \geq 2$.
Then $f(2^{2B}+1) = f(4(2^{2B-2})+1) = 4f(2^{2B-2}+1) - 3 = 4(1) - 3 = 1$ by our induction hypothesis.

Therefore, if $a \geq 6$, then $f(1+2^{2a-10}) = f(1+2^{2(a-5)}) = 1$ so

$$f(2005+2^{2a}) = 1024(1) - 383 = 641 = f(2005)$$

so there are infinitely many integers n for which $f(n) = f(2005)$.

Solution 3

We conjecture a formula for $f(n)$ and prove this formula by induction, using the formulae that we proved in Solution 2.

We start by writing the positive integer n in its binary representation, ie. we write

$$n = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \cdots + b_{2p-1} \cdot 2^{2p-1} + b_{2p} \cdot 2^{2p}$$

where each of $b_0, b_1, \cdots, b_{2p}$ is 0 or 1 with either $b_{2p} = 1$, or $b_{2p} = 0$ and $b_{2p-1} = 1$.

Thus, in binary, n is equal to either $(b_{2p}b_{2p-1}\cdots b_1b_0)_2$ or $(b_{2p-1}\cdots b_1b_0)_2$.

We then conjecture that if n is odd (which tells us that $b_0 = 1$ for sure), then

$$f(n) = b_0 + b_1 \cdot 2 + b_3 \cdot 2^3 + \cdots + b_{2p-1} \cdot 2^{2p-1}$$

In other words, we omit the even-numbered powers of 2 from n. Looking at a few examples: $7 = 4 + 2 + 1$, so $f(7) = 2 + 1 = 3$, $13 = 8 + 4 + 1$, so $f(13) = 8 + 1 = 9$, and $27 = 16 + 8 + 2 + 1$, so $f(27) = 8 + 2 + 1 = 11$.

We already know that if n is even, then $f(n) = f(n - 1)$ (we proved this in Solution 2).

Let's assume that we've proved this formula. (We'll prove it at the end.)

We can now solve parts (b) and (c) very quickly using our formula.

Are then any values of n such that $f(n) = 2005$?

Writing 2005 as a sum of powers of 2 (ie. in binary), we get

$$2005 = 1024 + 512 + 256 + 128 + 64 + 16 + 4 + 1$$

Since the representation of 2005 does not use only odd-numbered powers of 2, then there is no n for which $f(n) = 2005$.

Lastly, we need to prove that there are infinitely many positive integers n for which $f(n) = f(2005)$.

To do this, we note that if $n = 2005 + 2^{2a}$ for some $a \geq 6$, then the last 11 binary digits of n agree with those of 2005 and the only 1s in the representation of $n = 2005 + 2^{2a}$ in positions corresponding to odd-numbered powers of 2 come from the 2005 portion (since the extra "1" from 2^{2a} corresponds to an even-numbered power of 2).

Therefore, since we calculate $f(2005 + 2^{2a})$ by looking at only the odd-numbered powers of 2, then $f(2005 + 2^{2a}) = f(2005)$ for all integers $a \geq 6$.

Therefore, there are infinitely many positive integers n for which $f(n) = f(2005)$.

We now must prove that this formula is true. We use strong induction.

Looking at the list in Solution 2, we can quickly see that the result holds for all odd values of n with $n \leq 31$. (We only need to establish this for a couple of small values of n to serve as base cases.)

Assume that the result holds for all odd positive integers n up to $n = N - 2$ for some odd positive integer N.

Consider $n = N$.

Case 1: $N = 4q + 1$

Here we can write

$$N = 1 + b_2 \cdot 2^2 + \cdots + b_{2p-1} \cdot 2^{2p-1} + b_{2p} \cdot 2^{2p}$$

and so

$$q = b_2 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$$

Note that $q < N - 2$ since $4q + 1 = N$, so $q = \frac{1}{4}N - \frac{1}{4}$.
From our formulae in Solution 2, $f(N) = f(4q + 1) = 4f(q + 1) - 3$.
If q is even, then $b_2 = 0$ and so $q+1$ is odd and $q+1 = 1+b_3 \cdot 2+\cdots+b_{2p-1} \cdot 2^{2p-3}+b_{2p} \cdot 2^{2p-2}$.
If q is odd, then $b_2 = 1$, so $q = 1 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$ and $q + 1$ is even, so $f(q + 1) = f(q)$.
In either case,

$$f(q+1) = f(1+b_3 \cdot 2+\cdots+b_{2p-1} \cdot 2^{2p-3}+b_{2p} \cdot 2^{2p-2}) = 1+b_3 \cdot 2+b_5 \cdot 2^3+\cdots b_{2p-1} \cdot 2^{2p-3}$$

by our Induction Hypothesis.
Therefore,

$$f(N) = 4(1 + b_3 \cdot 2 + b_5 \cdot 2^3 + \cdots b_{2p-1} \cdot 2^{2p-3}) - 3 = 1 + b_3 \cdot 2^3 + b_5 \cdot 2^5 + \cdots + b_{2p-1} \cdot 2^{2p-1}$$

as we would like, since $b_1 = 0$.

Case 2: $N = 4q + 3$
Here we can write

$$N = 1 + 2 + b_2 \cdot 2^2 + \cdots + b_{2p-1} \cdot 2^{2p-1} + b_{2p} \cdot 2^{2p}$$

and so

$$q = b_2 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$$

Note that $q < N - 2$ since $4q + 3 = N$.
From our formulae in Solution 2, $f(N) = f(4q + 3) = 4f(q + 1) - 1$.
If q is even, then $b_2 = 0$ and so $q+1$ is odd and $q+1 = 1+b_3 \cdot 2+\cdots+b_{2p-1} \cdot 2^{2p-3}+b_{2p} \cdot 2^{2p-2}$.
If q is odd, then $b_2 = 1$, so $q = 1 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$ and $q + 1$ is even, so $f(q + 1) = f(q)$.
In either case,

$$f(q+1) = f(1+b_3 \cdot 2+\cdots+b_{2p-1} \cdot 2^{2p-3}+b_{2p} \cdot 2^{2p-2}) = 1+b_3 \cdot 2+b_5 \cdot 2^3+\cdots b_{2p-1} \cdot 2^{2p-3}$$

by our Induction Hypothesis.
Therefore,

$$f(N) = 4(1+b_3 \cdot 2+b_5 \cdot 2^3+\cdots b_{2p-1} \cdot 2^{2p-3})-1 = 1+2+\cdot b_3 \cdot 2^3+b_5 \cdot 2^5+\cdots+b_{2p-1} \cdot 2^{2p-1}$$

as we would like.

Therefore, by strong induction, our formula holds. This complete our proof.

Solution 4
First, we note that if $n = 2k$ is even, then $f(n) = f(2k) = f(2k-1) = f(n-1)$. See Solution 2 for this justification.
Therefore, we only need to look for odd values of n in parts (b) and (c).

Write the number n in binary as $n = (b_{2p}b_{2p-1}\cdots b_2b_1 1)_2$, where each digit is either 0 or 1. Here, we allow the possibility of $b_{2p} = 0$ if $b_{2p-1} = 1$. Since n is odd, then the last digit must be 1, as shown in the representation of n.
We conjecture that if $n = (b_{2p}b_{2p-1}\cdots b_2b_1 1)_2$, then $f(n) = (b_{2p-1}0b_{2p-3}0\cdots b_3 0 b_1 1)_2$, ie. we take the binary representation of n and make every digit corresponding to an even power of 2 into a 0.

Assume that we have proven this formula. (We will prove it below.) We can now quickly solve (b) and (c).
Is there an integer n such that $f(n) = 2005$?
Since $2005 = 1024 + 512 + 256 + 128 + 64 + 16 + 4 + 1$, then $2005 = (11111010101)_2$.
Thus, the binary representation of 2005 does not have only 0's in the digits corresponding to even powers of 2, so 2005 cannot be $f(n)$ for any n.

Why are there infinitely many positive integers n for which $f(n) = 2005$?
Consider $n = 2005 + 2^{2a}$ for some positive integer n, where $2^{2a} > 2005$, ie. $n \geq 6$.
Then the binary representation of n is $n = (10\cdots 011111010101)_2$, where the leading 1 is in a digit corresponding to an even power of 2, and so is zeroed when f is applied.
Therefore, $f(n) = (00\cdots 001010000001)_2 = (1010000001)_2 = f(2005)$.
Thus, there are infinitely many positive integers n for which $f(n) = f(2005)$.

Lastly, we must prove that our formula is true.
Write the numbers from 1 to n in binary in a list from top to bottom:

$$\begin{array}{cccccc}
\ldots & 0 & 0 & 0 & 0 & 1 \\
\ldots & 0 & 0 & 0 & 1 & 0 \\
\ldots & 0 & 0 & 0 & 1 & 1 \\
\ldots & 0 & 0 & 1 & 0 & 0 \\
\ldots & 0 & 0 & 1 & 0 & 1 \\
\ldots & 0 & 0 & 1 & 1 & 0 \\
\ldots & 0 & 0 & 1 & 1 & 1 \\
\ldots & 0 & 1 & 0 & 0 & 0 \\
 & & \vdots & & & \\
\ldots & b_4 & b_3 & b_2 & b_1 & 1
\end{array}$$

On odd-numbered passes through the lockers, Josephine moves from left to right, corresponding to downwards in this list. On even-numbered passes through the lockers, Josephine moves from right to left, corresponding to upwards in this list.

On the first pass, we remove every other number from this list, moving downwards. Thus, we remove every even number, or all of those $\equiv 0 \pmod 2$, or all of those with 0th binary digit of 0.
Therefore, after the first pass, only those with a 0th binary digit of 1 remain, and the 1st binary digit (ie. corresponding to 2^1) alternates between 0 and 1, since the numbers in the

list alternate between 1 (mod 4) and 3 (mod 4).

On the second pass through the list, which is upwards, we remove every other remaining number. Since the numbers remaining the list alternate between ending in 01 and 11, and we do not remove the last number, then we leave all those numbers ending in $b_1 1$.
(Since we are removing every fourth number from the original list, the final two binary digits of the remaining numbers should all be the same.)
What remains in our list after two passes? The numbers which remain are all congruent to the same thing (an odd number) modulo 4.

Consider the third pass.
Since one out of every four of the original numbers remains and all of the remaining numbers are odd, then the first number still in the list is less than 4.
Since every number remaining in the list is congruent to the same thing modulo 4, then the last three digits alternate $0b_1 1$ and $1b_1 1$ (ie. the last two binary digits are the same).
Since the first number is less than 4, then it ends in $0b_1 1$.
Since we now remove every other number remaining, then we remove all those numbers with last three binary digits $1b_1 1$, leaving only those with last three digits $0b_1 1$. Thus, all remaining numbers are congruent to the same number modulo 8.
What is the last number remaining in the list?
If the original last number in the list was $...b_3 0 b_1 1$ (ie. $b_2 = 0$), then this number still remains.
If the last number before the third pass was $...b_3 1 b_1 1$ (ie. $b_2 = 1$), then the second last remaining number would be $(...b_3 1 b_1 1)_2 - 4 = (...b_3 0 b_1 1)_2$, and it is this second last number which remains. In either case, the last remaining number is $...b_3 0 b_1 1$.

Consider now a general even-numbered pass (say, pass number $2m$ moving through the list from bottom to top).
The last number in the list (ie. the first encountered) will be $...b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$ and the numbers in the list will alternate between ending $...10 b_{2m-3} 0 \cdots b_3 0 b_1 1$ and ending $...00 b_{2m-3} 0 \cdots b_3 0 b_1 1$ (since every 2^{2m-1}th number from the original list remains).
The last number in the list will not be removed, so we will remove all numbers not agreeing with the last number in the $(2m-1)$th digit, ie. we are left with all numbers ending in $...b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$. This leaves us with every 2^{2m}th number from our original list.
Since all remaining numbers are odd, then the smallest number remaining in the list is smaller than 2^{2m}, so ends in $...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$.

On the next (odd-numbered pass), the list begins with all numbers ending in either $...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$ or $...1 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$.
Since the first number encountered ends in $...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, then we remove all numbers ending in $...1 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, leaving only those ending in $...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, ie. every 2^{2m+1}th number from the original list.
Just before this pass, the largest number remaining ended in $...b_{2m+1} b_{2m} b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$.
After this pass, the largest number remaining will end in $...b_{2m+1} 0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, by the same argument we used in the third pass.

Thus, the process continues as expected, and the final number remaining in the list will be $b_{2p-1} 0 b_{2p-3} 0 \cdots b_3 0 b_1 1$, so $f(n) = (b_{2p-1} 0 b_{2p-3} 0 \cdots b_3 0 b_1 1)_2$

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2004 Solutions
Euclid Contest

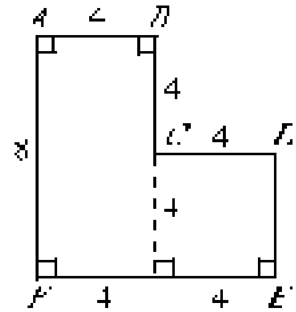
for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. (a) *Solution*

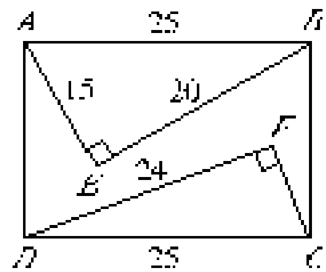
Because all of the angles in the figure are right angles, then $BC = DE = 4$.
Thus, we can break up the figure into a 4 by 8 rectangle and a 4 by 4 square, by extending BC to hit FE. Therefore, the area of the figure is $(8)(4)+(4)(4)=48$.



Answer: 48

(b) *Solution*

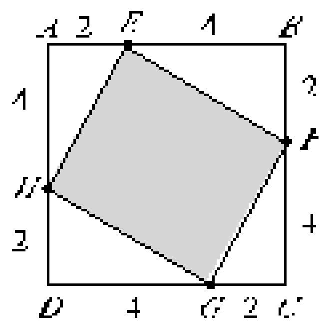
By the Pythagorean Theorem in triangle ABE, $AB^2 = 15^2 + 20^2 = 625$, so $AB = 25$.
Since $ABCD$ is a rectangle, $CD = AB = 25$, so by the Pythagorean Theorem in triangle CFD, we have $625 = 25^2 = 24^2 + CF^2$, so $CF^2 = 625 - 576 = 49$, or $CF = 7$.



Answer: 7

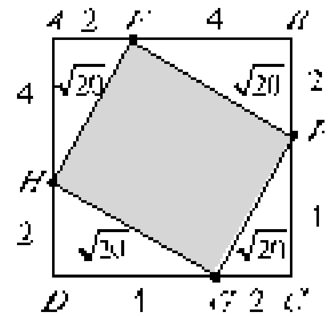
(c) *Solution 1*

Since $ABCD$ is a square of side length 6 and each of $AE : EB$, $BF : FC$, $CG : GD$, and $DH : HA$ is equal to 1 : 2, then $AE = BF = CG = DH = 2$ and $EB = FC = GD = HA = 4$.
Thus, each of the triangles HAE, EBF, FCG, and GDH is right-angled, with one leg of length 2 and the other of length 4.
Then the area of $EFGH$ is equal to the area of square $ABCD$ minus the combined area of the four triangles, or $6^2 - 4\left[\frac{1}{2}(2)(4)\right] = 36 - 16 = 20$ square units.



Solution 2

Since $ABCD$ is a square of side length 6 and each of $AE:EB$, $BF:FC$, $CG:GD$, and $DH:HA$ is equal to 1 : 2, then $AE = BF = CG = DH = 2$ and $EB = FC = GD = HA = 4$.



Thus, each of the triangles HAE, EBF, FCG, and GDH is right-angled, with one leg of length 2 and the other of length 4.

By the Pythagorean Theorem,

$EF = FG = GH = HE = \sqrt{2^2 + 4^2} = \sqrt{20}$.

Since the two triangles HAE and EBF are congruent (we know the lengths of all three sides of each), then $\angle AHE = \angle BEF$. But $\angle AHE + \angle AEH = 90^\circ$, so $\angle BEF + \angle AEH = 90^\circ$, so $\angle HEF = 90^\circ$.

In a similar way, we can show that each of the four angles of $EFGH$ is a right-angle, and so $EFGH$ is a square of side length $\sqrt{20}$.

Therefore, the area of $EFGH$ is $\left(\sqrt{20}\right)^2 = 20$ square units.

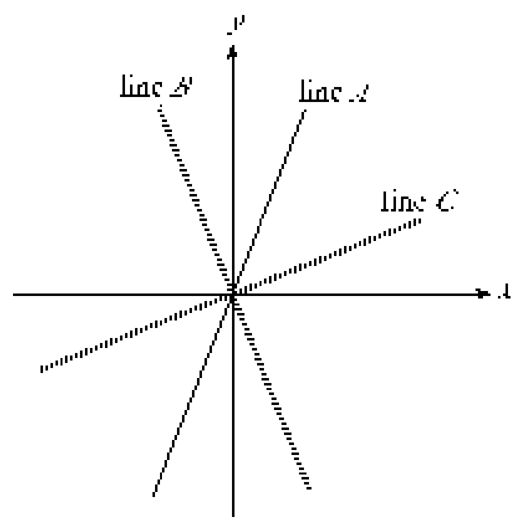
2. (a) Rearranging the equation of the given line $3x - y = 6$, we get $y = 3x - 6$, so the given line has y-intercept –6.

Since horizontal lines have the general form $y = a$ for some constant a, then the horizontal line with y-intercept –6 is the line $y = -6$.

Answer: $y = -6$

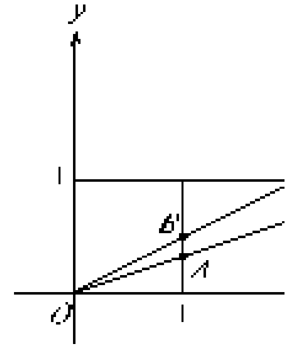
(b) When line A with equation $y = 2x$ is reflected in the y-axis, the resulting line (line B) has equation $y = -2x$. (Reflecting a line in the y-axis changes the sign of the slope.)

Since the slope of line B is –2 and line C is perpendicular to line B, then the slope of line C is $\frac{1}{2}$ (the slopes of perpendicular lines are negative reciprocals).



Answer: $\frac{1}{2}$

(c) *Solution 1*



Consider the line through O and P. To get from O to P, we go right 2 and up 1. Since B lies on this line and to get from O to B we go over 1, then we must go up $\frac{1}{2}$, to keep the ratio constant.

Consider the line through O and Q. To get from O to Q, we go right 3 and up 1. Since A lies on this line and to get from O to A we go over 1, then we must go up $\frac{1}{3}$, to keep the ratio constant.

Therefore, since A and B lie on the same vertical line, then $AB = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

Solution 2

Since the line through P passes through the origin, then its equation is of the form $y = mx$. Since it passes through the point $(2,1)$, then $1 = 2m$, so the line has equation $y = \frac{1}{2}x$. Since B has x-coordinate 1, then $y = \frac{1}{2}(1) = \frac{1}{2}$, so B has coordinates $\left(1,\frac{1}{2}\right)$.

Similarly, we can determine that the equation of the line through Q is $y = \frac{1}{3}x$, and so A has coordinates $\left(1,\frac{1}{3}\right)$.

Therefore, since A and B lie on the same vertical line, then $AB = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

3. (a) *Solution 1*

If the sequence has common difference d, then we can write the sequence as $a - 2d$, $a - d$, a, $a + d$, $a + 2d$.

From the given information, $(a - 2d) + (a - d) = 2$, or $2a - 3d = 2$.

Also, $(a + d) + (a + 2d) = -18$, or $2a + 3d = -18$.

We want to determine the third term, which is a.

We can get an equation involving a only by adding the two equations to get $4a = (2a - 3d) + (2a + 3d) = 2 + (-18) = -16$, and so a, the third term, is –4.

Solution 2

If the sequence has first term a and common difference d, then we can write the sequence as a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$.

From the given information, $a + (a + d) = 2$, or $2a + d = 2$.

Also, $(a + 3d) + (a + 4d) = -18$, or $2a + 7d = -18$.

Subtracting these two equations, we obtain $6d = -20$ or $d = -\frac{10}{3}$.

Substituting back into the first equation, $2a = 2 - \left(-\frac{10}{3}\right) = \frac{16}{3}$ and so $a = \frac{8}{3}$.

Therefore, the third term is $a + 2d = \frac{8}{3} + 2\left(-\frac{10}{3}\right) = -\frac{12}{3} = -4$.

Answer: -4

(b) *Solution 1*

Since $(x-y)^2 = x^2 - 2xy + y^2$ and $(x+y)^2 = x^2 + 2xy + y^2$, then $(x+y)^2 - (x-y)^2 = 4xy$.

Thus, we have $(x+y)^2 = \left(4\sqrt{2}\right)^2 + 4(56) = 32 + 224 = 256$, and so $x+y=16$ or $x+y=-16$.

Since we are told that there are two values, then these two values must be 16 and –16.

Solution 2

From the first equation, $x = y + 4\sqrt{2}$, so substituting into the second equation,

$$\left(y + 4\sqrt{2}\right)y = 56$$

$$y^2 + 4\sqrt{2}y - 56 = 0$$

$$y = \frac{-4\sqrt{2} \pm \sqrt{\left(4\sqrt{2}\right)^2 - 4(1)(-56)}}{2}$$

$$y = \frac{-4\sqrt{2} \pm \sqrt{256}}{2}$$

$$y = -2\sqrt{2} \pm 8$$

If $y = -2\sqrt{2} + 8$, then $x = y + 4\sqrt{2} = 2\sqrt{2} + 8$, so $x + y = 16$.

If $y = -2\sqrt{2} - 8$, then $x = y + 4\sqrt{2} = 2\sqrt{2} - 8$, so $x + y = -16$.

Therefore, the two values are 16 and –16.

4. (a) *Solution 1*

There are 36 possibilities for the pair of numbers on the faces when the dice are thrown.

For the product of the two numbers, each of which is between 1 and 6, to be divisible by 5, one of the two numbers must be equal to 5.

Therefore, the possible pairs for the faces are

$$(1,5),\ (2,5),\ (3,5),\ (4,5),\ (5,5),\ (6,5),\ (5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,6)$$

ie. there are 11 possibilities.

Thus, the probability is $\frac{11}{36}$.

Solution 2

For the product of the two numbers, each of which is between 1 and 6, to be divisible by 5, one of the two numbers must be equal to 5.

When the two dice are thrown, the probability that the first die has a 5 on the top face and any number appears on the second die has any number on the top face is $\frac{1}{6} \times 1 = \frac{1}{6}$.

Also, the probability that any number appears on the first die and a 5 appears on the second die is $1 \times \frac{1}{6} = \frac{1}{6}$.

If we consider the sum of these probabilities, we have double-counted the possibility that a 5 occurs on both dice, which happens with probability $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.
Therefore, the required probability is $\frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$.

Answer: $\frac{11}{36}$

(b) First, we compute an expression for the composition of the two given functions:

$$\begin{aligned} f(g(x)) &= f(ax+b) \\ &= (ax+b)^2 - (ax+b) + 2 \\ &= a^2x^2 + 2abx + b^2 - ax - b + 2 \\ &= a^2x^2 + (2ab-a)x + \left(b^2 - b + 2\right) \end{aligned}$$

But we already know that $f(g(x)) = 9x^2 - 3x + 2$, so comparing coefficients, we see that

$$a^2 = 9 \qquad (1)$$
$$2ab - a = -3 \qquad (2)$$
$$b^2 - b + 2 = 2 \qquad (3)$$

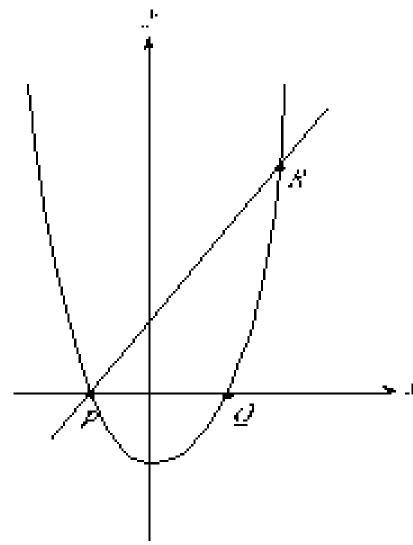
From the first equation, $a = 3$ or $a = -3$.
From the third equation, $b^2 - b = b(b-1) = 0$ so $b = 0$ or $b = 1$.
There are thus 4 possible pairs (a, b) which could solve the problem. We will check which pairs work by looking at the second equation.
From the second equation, $a(2b-1) = -3$, so if $a = 3$ then $b = 0$, and if $a = -3$ then $b = 1$.
Therefore, the possible ordered pairs (a, b) are $(3,0)$ and $(-3,1)$.

5. (a)

$$\begin{aligned} 16^x &= 2^{x+5} - 2^{x+4} \\ \left(2^4\right)^x &= 2^{x+4}\left(2^1 - 1\right) \\ 2^{4x} &= 2^{x+4}(1) \\ 2^{4x} &= 2^{x+4} \\ 4x &= x + 4 \qquad \text{(equating exponents)} \\ 3x &= 4 \\ x &= \tfrac{4}{3} \end{aligned}$$

Answer: $x = \frac{4}{3}$

(b) Point P is the point where the line $y = 3x + 3$ crosses the $x$axis, and so has coordinates $(-1,0)$.



Therefore, one of the roots of the parabola $y = x^2 + tx - 2$ is $x = -1$, so

$$0 = (-1)^2 + t(-1) - 2$$
$$0 = 1 - t - 2$$
$$t = -1$$

The parabola now has equation $y = x^2 - x - 2 = (x+1)(x-2)$ (we already knew one of the roots so this helped with the factoring) and so its two x-intercepts are –1 and 2, ie. P has coordinates $(-1,0)$ and Q has coordinates $(2,0)$.

We now have to find the coordinates of the point R. We know that R is one of the two points of intersection of the line and the parabola, so we equate their equations:

$$3x + 3 = x^2 - x - 2$$
$$0 = x^2 - 4x - 5$$
$$0 = (x+1)(x-5)$$

(Again, we already knew one of the solutions to this equation ($x = -1$) so this made factoring easier.) Since R does not have x-coordinate –1, then R has x-coordinate $x = 5$. Since R lies on the line, then $y = 3(5) + 3 = 18$, so R has coordinates $(5,18)$.

We can now calculate the area of triangle PQR. This triangle has base of length 3 (from P to Q) and height of length 18 (from the x-axis to R), and so has area $\frac{1}{2}(3)(18) = 27$.

Thus, $t = -1$ and the area of triangle PQR is 27.

6. (a) In order to use as many coins as possible, Lori should use coins with smaller values wherever possible.

Can Lori make \$1.34 without using the loonie? The total value of all of the other coins is $3(\$0.25) + 3(\$0.10) + 3(\$0.05) + 5(\$0.01) = \$1.25$, so Lori needs to use the loonie to pay for the toy helicopter. Thus, she needs to try to make \$0.34 with as many coins as possible.

Next, Lori must use 4 pennies in order to make \$0.34 (since each other coin's value is a multiple of 5 cents). So she has used 5 coins thus far, and still needs to try to make \$0.30 with as many coins as possible.

In order to make \$0.30 using as many of the quarters, dimes and nickels as possible, she should use 2 nickels and 2 dimes, or 4 coins. (If a quarter is used, a single nickel completes the \$0.30 with 2 coins only. At least 2 dimes must be used.)

Therefore, the maximum number of coins she can use is 9.

Answer: 9

(b) When the dimensions were increased by n % from 10 by 15, the new dimensions were $10\left(1+\frac{n}{100}\right)$ by $15\left(1+\frac{n}{100}\right)$.

When the resolution was decreased by n percent, the new resolution was $75\left(1-\frac{n}{100}\right)$.

(Note that n cannot be larger than 100, since the resolution cannot be decreased by more than 100%.)

Therefore, the number of pixels in the new image is

$$\left[10\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right]\times\left[15\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right]$$

Since we know that the number of pixels in the new image is 345 600, then

$$\begin{aligned}\left[10\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right]\times\left[15\left(1+\frac{n}{100}\right)\times 75\left(1-\frac{n}{100}\right)\right] &= 345600\\ [10\times 75]\times[15\times 75]\times\left(1+\frac{n}{100}\right)^2\times\left(1-\frac{n}{100}\right)^2 &= 345600\\ 843750\left(1+\frac{n}{100}\right)^2\left(1-\frac{n}{100}\right)^2 &= 345600\\ \left(1-\frac{n^2}{100^2}\right)^2 &= 0.4096\\ 1-\frac{n^2}{100^2} &= \pm 0.64\\ 1-\frac{n^2}{100^2} &= 0.64 \qquad (n \text{ cannot be larger than } 100)\\ \frac{n^2}{100^2} &= 0.36\\ \frac{n}{100} &= 0.6 \qquad (\text{since } n \text{ must be positive})\\ n &= 60\end{aligned}$$

Thus, $n = 60$.

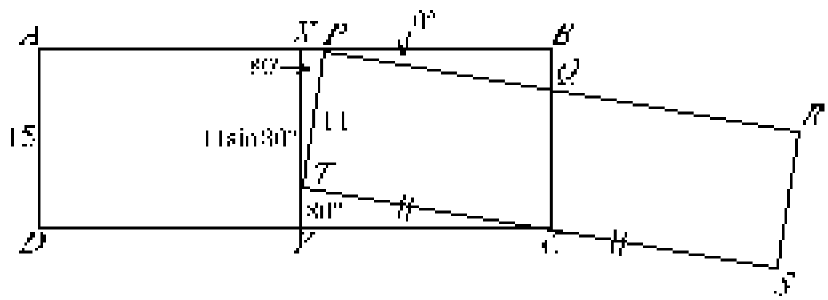
7. (a) We first calculate the length of AC using the cosine law:

$$\begin{aligned}AC^2 &= 7^2+8^2-2(7)(8)\cos(120^\circ)\\ AC^2 &= 49+64-112\left(-\tfrac{1}{2}\right)\\ AC^2 &= 169\\ AC &= 13\end{aligned}$$

Since triangle ABC is right-angled and isosceles, then $x = AB = \sqrt{2}(AC) = 13\sqrt{2}$.

Answer: $x = 13\sqrt{2}$

(b) First, we draw a line through T which is perpendicular to AB. This line cuts AB at X and CD at Y.
Since $\angle TPR$ is a right angle, then $\angle XPT = 80^\circ$. Thus, $XT = 11\sin(80^\circ)$.
Since $XY = 15$, then $TY = 15 - 11\sin(80^\circ)$.



But triangle XPT is right-angled at X, so since $\angle XPT = 80^\circ$, then $\angle XTP = 10^\circ$, and so $\angle YTC = 80^\circ$, since $\angle TYC = 90^\circ$.

Thus, $TC = \dfrac{TY}{\cos(80^\circ)} = \dfrac{15 - 11\sin(80^\circ)}{\cos(80^\circ)}$, and so the length of the drawer is

$$TS = 2TC = \frac{30 - 22\sin(80^\circ)}{\cos(80^\circ)} \approx 47.9949.$$

Thus, to the nearest tenth of a centimetre, the length of the drawer is 48.0 cm.
[Note that there are many different ways to do this problem.]

8. (a) Consider the right side of the given equation:

$$\begin{aligned} T^3 + bT + c &= \left(x^2 + \frac{1}{x^2}\right)^3 + b\left(x^2 + \frac{1}{x^2}\right) + c \\ &= \left(x^4 + 2 + \frac{1}{x^4}\right)\left(x^2 + \frac{1}{x^2}\right) + b\left(x^2 + \frac{1}{x^2}\right) + c \\ &= x^6 + 3x^2 + \frac{3}{x^2} + \frac{1}{x^6} + b\left(x^2 + \frac{1}{x^2}\right) + c \\ &= x^6 + \frac{1}{x^6} + (b+3)\left(x^2 + \frac{1}{x^2}\right) + c \end{aligned}$$

For this expression to be equal to $x^6 + \dfrac{1}{x^6}$ for all values of x, we want $b + 3 = 0$ or $b = -3$ and $c = 0$.

(b) *Solution 1*

We start with $x^3 + \frac{1}{x^3} = 2\sqrt{5}$ and square both sides:

$$\left(x^3 + \frac{1}{x^3}\right)^2 = \left(2\sqrt{5}\right)^2$$

$$x^6 + 2 + \frac{1}{x^6} = 20$$

$$x^6 + \frac{1}{x^6} = 18$$

Using the result from part (a) and letting $T = x^2 + \frac{1}{x^2}$, we see that $T^3 - 3T = 18$.

So we would like to factor the equation $T^3 - 3T - 18 = 0$.

After some trial and error, we can see that $T = 3$ is a solution, so by the Factor Theorem, $(T-3)\left(T^2 + 3T + 6\right) = 0$, and $T^2 + 3T + 6$ has no real roots.

Therefore, $T = 3$, ie. $x^2 + \frac{1}{x^2} = 3$.

Solution 2

Let $t = x + \frac{1}{x}$. Since we saw in (a) that $x^6 + \frac{1}{x^6} = \left(x^2 + \frac{1}{x^2}\right)^3 - 3\left(x^2 + \frac{1}{x^2}\right)$, then it makes sense that $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = t^3 - 3t$.

Therefore, we have $t^3 - 3t = 2\sqrt{5}$ or $t^3 - 3t - 2\sqrt{5} = 0$.

Since $\left(\sqrt{5}\right)^3 = 5\sqrt{5}$, then $t = \sqrt{5}$ is a solution to this equation, so factoring we obtain $\left(t - \sqrt{5}\right)\left(t^2 + \sqrt{5}t + 2\right) = 0$. The quadratic factor has discriminant $\left(\sqrt{5}\right)^2 - 4(1)(2) = -3 < 0$ and so has no real roots.

Therefore, $t = x + \frac{1}{x} = \sqrt{5}$.

Squaring, we obtain

$$\left(x + \frac{1}{x}\right)^2 = \left(\sqrt{5}\right)^2$$

$$x^2 + 2 + \frac{1}{x^2} = 5$$

$$x^2 + \frac{1}{x^2} = 3$$

Solution 3

We start with $x^3 + \dfrac{1}{x^3} = 2\sqrt{5}$ and square both sides:

$$\left(x^3 + \frac{1}{x^3}\right)^2 = \left(2\sqrt{5}\right)^2$$

$$x^6 + 2 + \frac{1}{x^6} = 20$$

$$x^6 + \frac{1}{x^6} = 18$$

From part (a), if $T = x^2 + \dfrac{1}{x^2}$, then $T^3 = \left(x^2 + \dfrac{1}{x^2}\right)^3 = x^6 + 3\left(x^2 + \dfrac{1}{x^2}\right) + \dfrac{1}{x^6} = 18 + 3T$.

So we would like to factor the equation $T^3 - 3T - 18 = 0$.

After some trial and error, we can see that $T = 3$ is a solution, so by the Factor Theorem, $(T-3)\left(T^2 + 3T + 6\right) = 0$, and $T^2 + 3T + 6$ has no real roots.

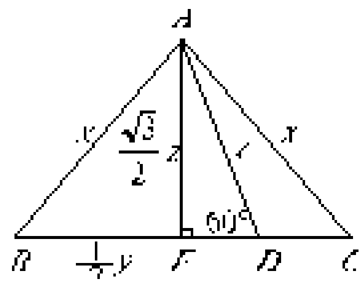
Therefore, $T = 3$, ie. $x^2 + \dfrac{1}{x^2} = 3$.

9. (a) *Solution 1*

From A, drop a perpendicular to BC.

From triangle ABE, $AE^2 = x^2 - \frac{1}{4}y^2$.

From triangle ADE, $AE = \frac{\sqrt{3}}{2}z$ or $AE^2 = \frac{3}{4}z^2$.



Equating these two expressions, we get $x^2 - \frac{1}{4}y^2 = \frac{3}{4}z^2$ or $4x^2 - y^2 = 3z^2$.

If $x = 7$ and $z = 5$, then

$$196 - y^2 = 75$$

$$y^2 = 121$$

$$y = \pm 11$$

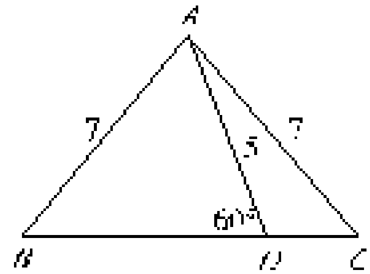
Therefore, the Kirk triplet with $x = 7$ and $z = 5$ is $(7,11,5)$.

Solution 2

By the cosine law in triangle ADB,

$$7^2 = 5^2 + BD^2 - 2(5)(BD)\cos(60^\circ)$$
$$0 = BD^2 - 5BD - 24$$
$$0 = (BD - 8)(BD + 3)$$

Since BD is a length, $BD = 8$.



Since $\angle ADC = 120^\circ$, then by the cosine law in triangle ADC,

$$7^2 = 5^2 + DC^2 - 2(5)(DC)\cos(120^\circ)$$
$$0 = DC^2 + 5DC - 24$$
$$0 = (DC + 8)(DC - 3)$$

Since DC is a length, then $DC = 3$. Thus, $y = BD + DC = 11$.

Therefore, the Kirk triplet with $x = 7$ and $z = 5$ is $(7,11,5)$.

(b) *Solution 1*

If $4x^2 - y^2 = 3z^2$ (from (a) Solution 1) and $z = 5$, then

$$4x^2 - y^2 = 75$$
$$(2x + y)(2x - y) = 75$$

Each of $2x + y$ and $2x - y$ is a positive integer and their product is 75. Note that $2x + y$ is bigger than $2x - y$. The factors of 75 are 1, 3, 5, 15, 25, 75.

Looking at each of the possibilities,

$2x + y$	$2x - y$	x	y
75	1	19	37
25	3	7	11
15	5	5	5

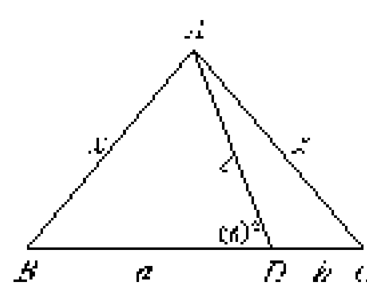
so the two possible Kirk triplets with $z = 5$ are $(19,37,5)$ and $(7,11,5)$.

Solution 2

Let $BD = a$ and $DC = b$.

By the cosine law in triangles ABD and ADC,

$$x^2 = 5^2 + a^2 - 2(5)(a)\cos(60^\circ) \quad \text{and}$$
$$x^2 = a^2 - 5a + 25$$
$$x^2 = 5^2 + b^2 - 2(5)(b)\cos(120^\circ)$$



$x^2 = 5^2 + b^2 - 2(5)(b)\cos(120^\circ)$

$x^2 = b^2 + 5b + 25$

Subtracting the second equation from the first, we obtain

$0 = a^2 - b^2 - 5a - 5b$

$0 = (a+b)(a-b-5)$

$0 = y(a-b-5)$

Since y is not 0, then $a = b + 5$.

We know $y = a + b = 2b + 5$, so $2b = y - 5$.

But $x^2 = b^2 + 5b + 25$, so

$$4x^2 = 4b^2 + 20b + 100$$

$$4x^2 = (y-5)^2 + 10(y-5) + 100$$

$$4x^2 = y^2 + 75$$

$$4x^2 - y^2 = 75$$

and thus we continue as in Solution 1 to determine that the only Kirk triplets with $z = 5$ are $(19,37,5)$ and $(7,11,5)$.

(c) In order determine the appropriate Kirk triplet, we need to find a way to determine Kirk triplets. We model our approach after that in part (b), Solution 1.

Drop a perpendicular from A to F on BC.

Since triangle ABC is isosceles, then F is the midpoint of BC.

Also, triangle ADF is a 30-60-90 triangle, with $AD = z$.

Thus, $FD = \frac{1}{2}z$ and $AF = \frac{\sqrt{3}}{2}z$.

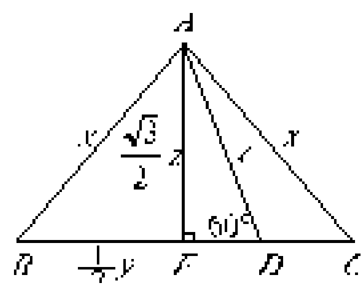
Then triangle ABF is right-angled at F with $AF = \frac{\sqrt{3}}{2}z$, $AB = x$ and $BF = \frac{1}{2}y$.

Thus, by the Pythagorean Theorem,

$$x^2 = \left(\tfrac{1}{2}y\right)^2 + \left(\tfrac{\sqrt{3}}{2}z\right)^2$$

$$4x^2 - y^2 = 3z^2$$

$$(2x+y)(2x-y) = 3z^2$$



Each of $2x + y$ and $2x - y$ is a positive integer and their product is $3z^2$. Note that $2x + y$ is bigger than $2x - y$. Since z is a prime number, the factors of $3z^2$ are 1, 3, z, $3z$, z^2, $3z^2$. (Notice that if z is equal to 2, these are not in ascending order, and if z equals 3, then there is duplication in this list.)

Looking at each of the possibilities,

$2x+y$	$2x-y$	x	y

$3z^2$	1	$\dfrac{3z^2+1}{4}$	$\dfrac{3z^2-1}{2}$
z^2	3	$\dfrac{z^2+3}{4}$	$\dfrac{z^2-3}{2}$
$3z$	z	z	z

so the only two possible Kirk triplets with a fixed value for z are $\left(\dfrac{3z^2+1}{4}, \dfrac{3z^2-1}{2}, z\right)$ and $\left(\dfrac{z^2+3}{4}, \dfrac{z^2-3}{2}, z\right)$.

We would like to determine the Kirk triplets with $\cos(\angle ABC)$ is as close to 0.99 as possible.

Looking back at triangle ABF, we see that $\cos(\angle ABC) = \dfrac{\frac{1}{2}y}{x}$, so in our two cases

$$\cos(\angle ABC) = \frac{3z^2-1}{3z^2+1} = 1 - \frac{2}{3z^2+1} \text{ or } \cos(\angle ABC) = \frac{z^2-3}{z^2+3} = 1 - \frac{6}{z^2+3}.$$

In order to make $\cos(\angle ABC)$ close to 0.99, we thus make either $\dfrac{2}{3z^2+1}$ or $\dfrac{6}{z^2+3}$ close to 0.01.

In other words we make $3z^2+1$ close to 200, or z^2+3 close to 600.

In the first case, since $3(8)^2+1 = 193$ and $3(9)^2+1 = 244$, and since z must be a prime number, then we try $z = 7$ and $z = 11$, and obtain $\cos(\angle ABC) = \dfrac{3z^2-1}{3z^2+1}$ to be 0.986487 and 0.994506.

In the second case, since $24^2+3 = 579$ and $25^2+3 = 628$, and since z must be a prime number, then we try $z = 23$ and $z = 29$, and obtain $\cos(\angle ABC) = \dfrac{z^2-3}{z^2+3}$ to be 0.988722 and 0.993644.

So $\cos(\angle ABC)$ appears to be as close to 0.99 as possible when $z = 23$.

We should double-check to confirm that we actually get a triplet of integers in this case!

In the second case, when $z = 23$, we get the triplet $(133, 263, 23)$.

Thus, the Kirk triplet $(133, 263, 23)$ gives $\cos(\angle ABC)$ is as close to 0.99 as possible.

(Note: We could have proceeded analogously to the second approach to 9(b) to obtain $4x^2 - y^2 = 3z^2$ in this way, and then proceeded as above in this solution.)

10. (a) We start by placing the two 4's. We systematically try each pair of possible positions – from positions 1 and 5 to positions 4 and 8. For each of these positions, we try placing

the two 3's in each pair of possible positions, and then see if the two 2's and two 1's will fit.

(We can reduce our work by noticing that if a Skolem sequence has the two 4's in positions 1 and 5, then reversing the sequence will give a Skolem sequence with the two 4's in positions 4 and 8. So we only need to consider putting the two 4's in positions 1 and 5, and in positions 2 and 6. The remaining possibilities can be dealt with by reversing.)

Thus, the six possible Skolem sequences of order 4 are:

(4, 2, 3, 2, 4, 3, 1, 1) and its reverse, (1, 1, 3, 4, 2, 3, 2, 4)
(4, 1, 1, 3, 4, 2, 3, 2) and its reverse, (2, 3, 2, 4, 3, 1, 1, 4)
(3, 4, 2, 3, 2, 4, 1, 1) and its reverse, (1, 1, 4, 2, 3, 2, 4, 3)

(b) Since we are trying to create a Skolem sequence of order 9, then there are 18 positions to fill with 10 odd numbers and 8 even numbers.

We are told that $s_{18} = 8$, so we must have $s_{10} = 8$, since the 8's must be 8 positions apart.

By condition III, between the two 8's, there can be only one odd integer. But there are 7 positions between the two 8's and only 6 remaining even numbers to place. Thus, all 6 remaining even numbers are placed between the two 8's. The only way in which this is possible is with the two 6's next to the two 8's, then the two 4's, then the two 2's. (The two 8's are 8 positions apart, and the two 6's must be 6 positions apart.)

Thus, the sequence so far is:

(__, __, 1, __, __, __, __, __, __, 8, 6, 4, 2, __, 2, 4, 6, 8)

The numbers that we have left to place are 1, 3, 3, 5, 5, 7, 7, 9, 9, and empty positions are 1, 2, 4, 5, 6, 7, 8, 9, 14.

Since the 9's must be 9 positions apart, they must be placed in positions 5 and 14.

Thus, we have

(__, __, 1, __, 9, __, __, __, __, 8, 6, 4, 2, 9, 2, 4, 6, 8)

The remaining 1 must be placed in position 2 or 4. If it is placed in position 2, then the 7's can only go in positions 1 and 8, giving

(7, 1, 1, __, 9, __, __, 7, __, 8, 6, 4, 2, 9, 2, 4, 6, 8)

But we now cannot place both the two 3's and the two 5's. (The placing of one of these pairs means that the other pair cannot be placed.)

We conclude that the only possibility is that the remaining 1 must be placed in position 4.

This gives

(__, __, 1, 1, 9, __, __, __, __, 8, 6, 4, 2, 9, 2, 4, 6, 8)

with 3, 3, 5, 5, 7, 7 left to be placed in positions 1, 2, 6, 7, 8, 9.

Now the two 3's must be placed in positions 6 and 9, so the 7's must be placed in positions 1 and 8, and finally the 5's must be placed in positions 2 and 7.

Therefore, the only Skolem sequence satisfying the given conditions is

(7, 5, 1, 1, 9, 3, 5, 7, 3, 8, 6, 4, 2, 9, 2, 4, 6, 8)

(c) *Solution 1*

Assume that there is a Skolem sequence of order n, where n is of the form $4k+2$ or $4k+3$. (We will deal with both cases together.)

Let P_1 be the position number of the left-most 1 in the Skolem sequence. Then P_1+1 is the position number of the other 1.

Similarly, let $P_2,\ P_3,\ \ldots,\ P_n$ be the position numbers of the left-most 2, 3, …, n, respectively, in the Skolem sequence. Then $P_2+2,\ P_3+3,\ \ldots,\ P_n+n$ are the position numbers of the other occurrences of 2, 3, …, n, respectively.

Since the Skolem sequence exists, then the numbers $P_1,\ P_2,\ P_3,\ \ldots,\ P_n,\ P_1+1,\ P_2+2,\ P_3+3,\ \ldots,\ P_n+n$ are a rearrangement of all of the position numbers, ie. the numbers 1, 2, …, $2n$.

Thus,

$$\begin{aligned} P_1+P_2+\cdots+P_n+(P_1+1)+(P_2+2)+\cdots+(P_n+n) &= 1+2+\cdots+2n \\ 2(P_1+P_2+\cdots+P_n)+(1+2+\cdots+n) &= 1+2+\cdots+2n \\ 2(P_1+P_2+\cdots+P_n)+\frac{n(n+1)}{2} &= \frac{2n(2n+1)}{2} \\ 2(P_1+P_2+\cdots+P_n)+\frac{n(n+1)}{2} &= n(2n+1) \qquad (**) \end{aligned}$$

using the fact that $1+2+\cdots+k=\dfrac{k(k+1)}{2}$.

We now look at the parities of the terms in equation (**) in each of our two cases.

If $n=4k+2$, then $\dfrac{n(n+1)}{2}=\dfrac{(4k+2)(4k+3)}{2}=(2k+1)(4k+3)$ which is the product of two odd numbers, so is odd, and $n(2n+1)=(4k+2)(8k+5)$, which is the product of an even number and an odd number, so is even.

Thus, (**) is Even + Odd = Even, a contradiction.

If $n=4k+3$, then $\dfrac{n(n+1)}{2}=\dfrac{(4k+3)(4k+4)}{2}=(4k+3)(2k+2)$ which is the product of an odd number and an even number, so is even, and $n(2n+1)=(4k+3)(8k+7)$, which is the product of two odd numbers, so is odd.

Thus, (**) is Even + Even = Odd, a contradiction.

Therefore, in either case, a contradiction is reached, so there cannot exist a Skolem sequence of order n, if n is of the form $4k+2$ or $4k+3$, where k is a non-negative integer.

Solution 2

Assume that there is a Skolem sequence of order n, where n is of the form $4k+2$ or $4k+3$.

Let l be an integer between 1 and n.

If l is even, then the two position numbers in which l is placed differ by an even number (namely l), and so are either both odd or both even.

If l is odd, then the two position numbers in which l is placed differ by an odd number (namely l), and so one is odd and the other is even.

<u>Case 1:</u> $n = 4k+2$

Between 1 and n, there are $2k+1$ even numbers $(2, 4, \ldots, 4k+2)$ and $2k+1$ odd numbers $(1, 3, \ldots, 4k+1)$.

The position numbers in a Skolem sequence of order $n = 4k+2$ are the integers 1, 2, $\ldots$, $8k+4$, so there are $4k+2$ even position numbers and $4k+2$ odd position numbers.

Considering the $2k+1$ odd numbers in the sequence gives us $2k+1$ odd position numbers and $2k+1$ even position numbers, ie. an odd number of odd position numbers and an odd number of even position numbers.

Each even number in the sequence will contribute either two odd position numbers or two even position numbers. In other words, all of the even numbers in the sequence contribute an even number of odd position numbers.

Thus, the total number of odd position numbers is odd plus even, which is odd. This is a contradiction, because we know there must be $4k+2$ odd position numbers.

<u>Case 2:</u> $n = 4k+3$

Between 1 and n, there are $2k+1$ even numbers $(2, 4, \ldots, 4k+2)$ and $2k+2$ odd numbers $(1, 3, \ldots, 4k+3)$.

The position numbers in a Skolem sequence of order $n = 4k+3$ are the integers $1, 2, \ldots, 8k+6$, so there are $4k+3$ even position numbers and $4k+3$ odd position numbers.

Considering the $2k+2$ odd numbers in the sequence gives us $2k+2$ odd position numbers and $2k+2$ even position numbers, ie. an even number of odd position numbers and an even number of even position numbers.

Each even number in the sequence will contribute either two odd position numbers or two even position numbers. In other words, all of the even numbers in the sequence contribute an even number of odd position numbers.

Thus, the total number of odd position numbers is even plus even, which is even. This is a contradiction, because we know there must be $4k+3$ odd position numbers.

Therefore, in either case, a contradiction is reached, so there cannot exist a Skolem sequence of order n, if n is of the form $4k+2$ or $4k+3$, where k is a non-negative integer.

Canadian Mathematics Competition

An activity of The Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2003 Solutions
Euclid Contest

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. (a) *Solution 1*

Since the x-intercepts of the parabola are 2 and 4, then the axis of symmetry of the parabola is $x = 3$.
Since the point $(0,8)$ is on the parabola, its image after a reflection across the vertical line $x = 3$ is the point $(6,8)$. Thus, $a = 6$.

Solution 2

Since the x-intercepts of the parabola are 2 and 4, then the equation of the parabola is of the form $y = A(x-2)(x-4)$.
Since $(0,8)$ lies on the parabola, then $8 = A(-2)(-4)$ or $A = 1$.
Therefore, the parabola has equation $y = (x-2)(x-4) = x^2 - 6x + 8$.
But the point $(a,8)$ lies on the parabola, so

$$8 = a^2 - 6a + 8$$
$$0 = a^2 - 6a$$
$$0 = a(a-6)$$

Since $a \neq 0$, then $a = 6$. Answer: $a = 6$

(b) *Solution 1*

Since the quadratic equation has two equal roots, then the expression on the left must be a perfect square. Since the leading coefficient is 1 and the coefficient of the x-term is 6, then the expression must be $(x+3)^2 = x^2 + 6x + 9$. By comparing expressions, $k = 9$.

Solution 2

Since the quadratic equation has two equal roots, the discriminant is 0, ie.
$6^2 - 4(1)(k) = 0$ or $4k = 36$ or $k = 9$. Answer: $k = 9$

(c) From the given information, the point $(1,4)$ lies on the parabola, so $4 = 1^2 - 3(1) + c$ or $c = 6$.
We now find the points of intersection of the parabola and the line by equating:

$$2x + 2 = x^2 - 3x + 6$$
$$0 = x^2 - 5x + 4$$
$$0 = (x-1)(x-4)$$

Thus the points of intersection have x-coordinates $x = 1$ and $x = 4$.
Substituting $x = 4$ into the line $y = 2x + 2$, we get the point $(4,10)$.
Therefore, the second point of intersection is $(4,10)$.

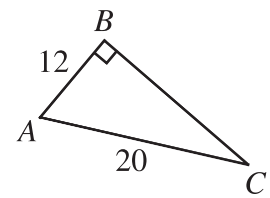
2. (a) Rearranging the equation,

$$3\sin(x) = \cos(15^\circ)$$
$$\sin(x) = \tfrac{1}{3}\cos(15^\circ)$$
$$\sin(x) \approx 0.3220$$

Using a calculator, $x \approx 18.78^\circ$. To the nearest tenth of a degree, $x = 18.8^\circ$.

Answer: $x = 18.8^\circ$

(b) *Solution 1*



Since $\sin C = \dfrac{AB}{AC}$, then $AB = AC\sin C = 20\left(\frac{3}{5}\right) = 12$.

By Pythagoras, $BC^2 = AC^2 - AB^2 = 20^2 - 12^2 = 256$ or $BC = 16$.

Solution 2

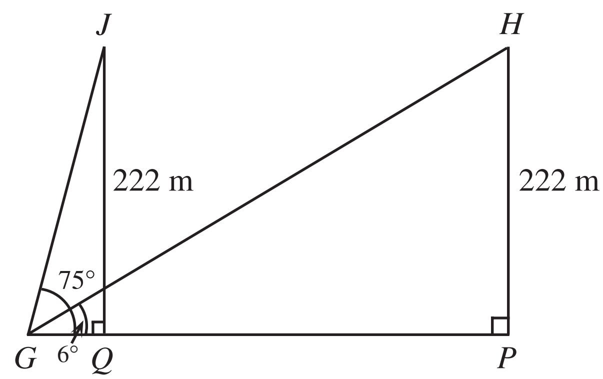
Using the standard trigonometric ratios, $BC = AC\cos C$.

Since $\sin C = \frac{3}{5}$, then $\cos^2 C = 1 - \sin^2 C = 1 - \frac{9}{25} = \frac{16}{25}$ or $\cos C = \frac{4}{5}$. (Notice that $\cos C$ is positive since angle C is acute in triangle ABC.)

Therefore, $BC = 20\left(\frac{4}{5}\right) = 16$.

Answer: $BC = 16$

(c) Let G be the point where the goat is standing, H the position of the helicopter when the goat first measures the angle, P the point directly below the helicopter at this time, J the position of the helicopter one minute later, and Q the point directly below the helicopter at this time.



Using the initial position of the helicopter, $\tan(6^\circ) = \dfrac{HP}{PG}$ or $PG = \dfrac{222}{\tan(6^\circ)} \approx 2112.19$ m.

Using the second position of the helicopter, $\tan(75^\circ) = \dfrac{JQ}{QG}$ or $QG = \dfrac{222}{\tan(75^\circ)} \approx 59.48$ m.

So in the one minute that has elapsed, the helicopter has travelled
$2112.19 \text{ m} - 59.48 \text{ m} = 2052.71 \text{ m}$ or 2.0527 km.

Therefore, in one hour, the helicopter will travel $60(2.0527) = 123.162$ km.

Thus, the helicopter is travelling 123 km/h.

3. (a) Since we are looking for the value of $f(9)$, then it makes sense to use the given equation and to set $x = 3$in order to obtain $f(9) = 2f(3) + 3$.
So we need to determine the value of $f(3)$. We use the equation again and set $x = 0$ since we will then get $f(3)$ on the left side and $f(0)$ (whose value we already know) on the right side, ie.
$$f(3) = 2f(0) + 3 = 2(6) + 3 = 15$$
Thus, $f(9) = 2(15) + 3 = 33$. Answer: $f(9) = 33$

(b) *Solution 1*
We solve the system of equations for $f(x)$ and $g(x)$.
Dividing out the common factor of 2 from the second equation, we get
$f(x) + 2g(x) = x^2 + 2$.
Subtracting from the first equation, we get $g(x) = x + 4$.
Thus, $f(x) = x^2 + 2 - 2g(x) = x^2 + 2 - 2(x + 4) = x^2 - 2x - 6$.
Equating $f(x)$ and $g(x)$, we obtain
$$x^2 - 2x - 6 = x + 4$$
$$x^2 - 3x - 10 = 0$$
$$(x - 5)(x + 2) = 0$$
Therefore, $x = 5$ or $x = -2$.

Solution 2
Instead of considering the equation $f(x) = g(x)$, we consider the equation $f(x) - g(x) = 0$, and we try to obtain an expression for $f(x) - g(x)$ by manipulating the two given equations.
In fact, after some experimentation, we can see that
$$\begin{aligned} f(x) - g(x) &= 2\left(2f(x) + 4g(x)\right) - 3\left(f(x) + 3g(x)\right) \\ &= 2\left(2x^2 + 4\right) - 3\left(x^2 + x + 6\right) \\ &= x^2 - 3x - 10 \end{aligned}$$
So to solve $f(x) - g(x) = 0$, we solve $x^2 - 3x - 10 = 0$ or $(x - 5)(x + 2) = 0$. Therefore, $x = 5$ or $x = -2$.

4. (a) *Solution 1*
We label the 5 skaters A, B, C, D, and E, where D and E are the two Canadians.
There are then $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways of arranging these skaters in their order of finish (for example, ADBCE indicates that A finished first, D second, etc.), because there are 5 choices for the winner, 4 choices for the second place finisher, 3 choices for the third place finisher, etc.

If the two Canadians finish without winning medals, then they must finish fourth and fifth. So the D and E are in the final two positions, and A, B and C in the first three. There are $3! = 6$ ways of arranging the A, B and C, and $2! = 2$ ways to arrange the D and E. Thus, there are $6 \times 2 = 12$ ways or arranging the skaters so that neither Canadian wins a medal.
Therefore, the probability that neither Canadian wins a medal is

$$\frac{\text{\# of ways where Canadians don't win medals}}{\text{Total \# of arrangements}} = \frac{12}{120} = \frac{1}{10}$$

Solution 2
We label the 5 skaters as A, B, C, D, and E, where D and E are the two Canadians. In any race, two of the skaters finish fourth and fifth. Also, any pair of skaters are equally as likely to finish fourth and fifth, since the probability of every skater is equally likely to finish in a given position.
How many pairs of 2 skaters can we form from the 5 skaters? There are ten such pairs:
{A,B}, {A,C}, {A,D}, {A,E}, {B,C}, {B,D}, {B,E}, {C,D}, {C,E}, {D,E}
Only one of these ten pairs is made up of the two Canadians. Therefore, the probability is $\frac{1}{10}$, since one out of ten choices gives the desired result.

Answer: $\frac{1}{10}$

(b) *Solution 1*
Since the least common multiple of 3, 5, 10 and 15 is 30, then we can count the number of positive integers less than or equal to 30 satisfying these conditions, and multiply the total by 10 to obtain the number less than 300. (This is because each group of 30 consecutive integers starting with 1 more than a multiple of 30 will have the same number of integers having these properties, because we can subtract 30 from each one and not change these properties.)
So from 1 to 30, we have:
3, 5, 6, 9, 12, 18, 21, 24, 25, 27
Thus there are 10 less than or equal to 30, and so 100 such positive integers less than or equal to 300.

Solution 2
We proceed by doing a (careful!) count.
The number of positive multiples of 3 less than or equal to 300 is 100.
The number of positive multiples of 5 less than or equal to 300 is 60.
Thus, we have 160 candidates, but have included multiples of 15 twice (since 15 is a multiple of each of 3 and 5), and have also included multiples of 10.
The number of multiples of 15 less than or equal to 300 is 20, so to remove the multiples of 15, we must remove 40 from 160 to get 120 positive integers less than or equal to 300 which are multiples of 3 or 5 but not of 15.

This total still included some multiples of 10 that are less or equal to 300 (but not all, since we have already removed 30, for instance).
In fact, there are 30 multiples of 10 less than or equal 300, 10 of which are multiples of 15 as well (that is, the multiples of 30). So we must remove 20 from the total of 120.
We then obtain that there are 100 positive integers less than or equal to 300 which are multiples of 3 or 5, but not of 10 or 15.

5. (a) Since the signs alternate every three terms, it makes sense to look at the terms in groups of 6.
The sum of the first 6 terms is $1+3+5-7-9-11=-18$.
The sum of the next 6 terms is $13+15+17-19-21-23=-18$.
In fact, the sum of each group of 6 terms will be the same, since in each group, 12 has been added to the numerical value of each term when compared to the previous group of 6, so overall 12 has been added three times and subtracted three times.
Since we are looking for the sum of the first 300 terms, then we are looking at 50 groups of 6 terms, so the sum must be $50(-18)=-900$.

Answer: -900

(b) Let the two digit integer have tens digit a and units digit b. Then the given information tells us

$$a^2+10b=b^2+10a$$
$$a^2-b^2-10a+10b=0$$
$$(a+b)(a-b)-10(a-b)=0$$
$$(a-b)(a+b-10)=0$$

and so $a=b$ or $a+b=10$.
So the possibilities for the integer are 11, 22, 33, 44, 55, 66, 77, 88, 99, 19, 28, 37, 46, 55, 64, 73, 82, 91. We now must determine which integers in this list are prime.
We can quickly reject all multiples of 11 bigger than 11 and all of the even integers, to reduce the list to 11, 19, 37, 73, 91.
All of these are prime, except for $91=13\times 7$.
Therefore, the required integers are 11, 19, 37, and 73.

6. (a) *Solution 1*
In 24 minutes, the number of atoms of isotope A has halved 4 times, so the initial number of atoms is $2^4=16$ times the number of atoms of isotope A at time 24 minutes.
But there were initially half as many atoms of isotope B as of isotope B, so there was 8 times the final number of atoms. Therefore, the number of atoms of isotope B halves 3 times in the 24 minutes, so it takes 8 minutes for the number of atoms of isotope B to halve.

Solution 2

Initially, there is twice as many atoms of isotope A as of isotope B, so let the original numbers of atoms of each be $2x$ and x, respectively.

Considering isotope A, after 24 minutes, if it loses half of its atoms every 6 minutes, there will be $2x\left(\frac{1}{2}\right)^{\frac{24}{6}}$ atoms remaining.

Similarly for isotope B, after 24 minutes, there will be $x\left(\frac{1}{2}\right)^{\frac{24}{T}}$ atoms remaining, where T is the length of time (in minutes) that it takes for the number of atoms to halve.

From the given information,

$$
\begin{aligned}
2x\left(\tfrac{1}{2}\right)^{\frac{24}{6}} &= x\left(\tfrac{1}{2}\right)^{\frac{24}{T}} \\
2\left(\tfrac{1}{2}\right)^{4} &= \left(\tfrac{1}{2}\right)^{\frac{24}{T}} \\
\left(\tfrac{1}{2}\right)^{3} &= \left(\tfrac{1}{2}\right)^{\frac{24}{T}} \\
\frac{24}{T} &= 3 \\
T &= 8
\end{aligned}
$$

Therefore, it takes 8 minutes for the number of atoms of isotope B to halve.

Answer: 8 minutes

(b) *Solution 1*

Using the facts that $\log_{10} A + \log_{10} B = \log_{10} AB$ and that $\log_{10} A - \log_{10} B = \log_{10} \frac{A}{B}$, then we can convert the two equations to

$$
\begin{aligned}
\log_{10}\left(x^3y^2\right) &= 11 \\
\log_{10}\left(\frac{x^2}{y^3}\right) &= 3
\end{aligned}
$$

Raising both sides to the power of 10, we obtain

$$
\begin{aligned}
x^3y^2 &= 10^{11} \\
\frac{x^2}{y^3} &= 10^3
\end{aligned}
$$

To eliminate the y's, we raise the first equation to the power 3 and the second to the power 2 to obtain

$$
\begin{aligned}
x^9y^6 &= 10^{33} \\
\frac{x^4}{y^6} &= 10^6
\end{aligned}
$$

and multiply to obtain $x^9x^4 = x^{13} = 10^{39} = 10^{33}10^6$.

Therefore, since $x^{13} = 10^{39}$, then $x = 10^3$.

Substituting back into $x^3y^2 = 10^{11}$, we get $y^2 = 10^2$, and so $y = \pm 10$. However, substituting into $\frac{x^2}{y^3} = 10^3$ we see that y must be positive, so $y = 10$.

Therefore, the solution to the system of equation is $x = 10^3$ and $y = 10$.

Solution 2

Since the domain of the logarithm is the positive real numbers, then the quantities $\log_{10}(x^3)$ and $\log_{10}(y^3)$ tell us that x and y are positive.

Using the fact that $\log_{10}(a^b) = b\log_{10}(a)$, we rewrite the equations as

$$3\log_{10} x + 2\log_{10} y = 11$$
$$2\log_{10} x - 3\log_{10} y = 3$$

We solve the system of equations for $\log_{10} x$ and $\log_{10} y$ by multiplying the first equation by 3 and adding two times the second equation in order to eliminate $\log_{10} y$.

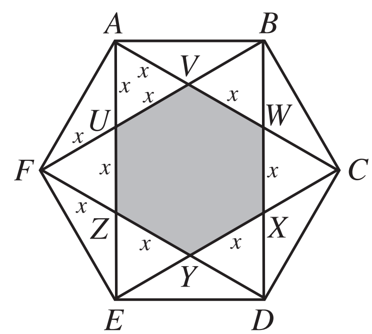
Thus we obtain $13\log_{10} x = 39$ or $\log_{10} x = 3$.

Substituting back into the first equation, we obtain $\log_{10} y = 1$.

Therefore, $x = 10^3$ and $y = 10$.

7. (a) *Solution 1*

We label the vertices of the shaded hexagon U, V, W, X, Y, and Z.



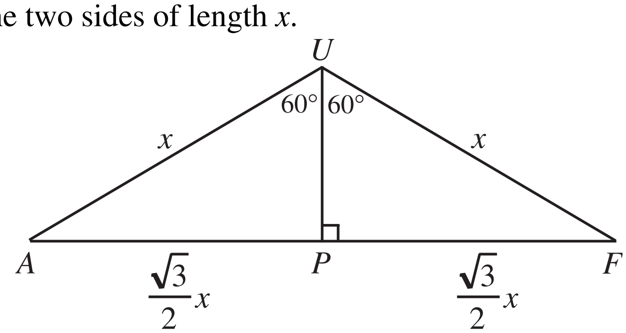
By symmetry, all of the six triangles with two vertices on the inner hexagon and one on the outer hexagon (eg. triangle UVA) are congruent equilateral triangles.

In order to determine the area of the inner hexagon, we determine the ratio of the side lengths of the two hexagons.

Let the side length of the inner hexagon be x. Then $AU = UF = x$.

Then triangle AUF has a 120° between the two sides of length x.

If we draw a perpendicular from U to point P on side AF, then UP divides ΔAUF into two $30^\circ - 60^\circ - 90^\circ$ triangles. Thus, $FP = PA = \frac{\sqrt{3}}{2}x$ and so $AF = \sqrt{3}x$.

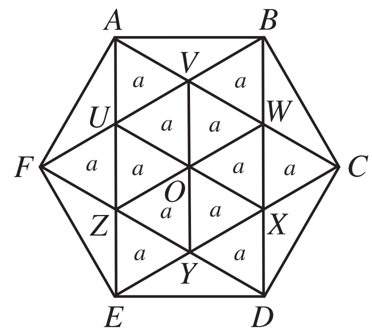


So the ratio of the side lengths of the hexagons is $\sqrt{3} : 1$, and so the ratio of their areas is $\left(\sqrt{3}\right)^2 : 1 = 3 : 1$.

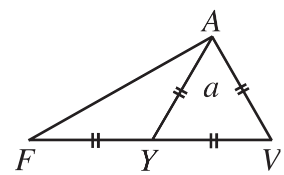
Since the area of the larger hexagon is 36, then the area of the inner hexagon is 12.

Solution 2

We label the vertices of the hexagon U, V, W, X, Y, and Z.
By symmetry, all of the six triangles with two vertices on the inner hexagon and one on the outer hexagon (eg. triangle UVA) are congruent equilateral triangles.
We also join the opposite vertices of the inner hexagon, ie. we join U to X, V to Y, and W to Z. (These 3 line segments all meet at a single point, say O.) This divides the inner hexagon into 6 small equilateral triangles identical to the six earlier mentioned equilateral triangles.



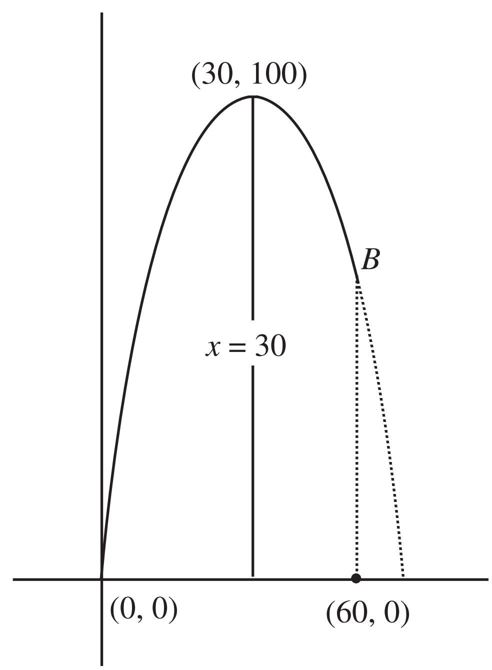
Let the area of one of these triangles be a. Then we can label the 12 small equilateral triangles as all having area a.
But triangle AUF also has area a, because if we consider triangle AFV, then AU is a median (since $FU = AU = UV$ by symmetry) and so divides triangle AFV into two triangles of equal area. Since the area of triangle AUV is a, then the area of triangle AUF is also a.



Therefore, hexagon $ABCDEF$ is divided into 18 equal areas. Thus, $a = 2$ since the area of the large hexagon is 36.
Since the area of $UVWXYZ$ is $6a$, then its area is 12.

Answer: 12

(b) We assign coordinates to the diagram, with the mouth of the cannon at the point $(0,0)$, with the positive x-axis in the horizontal direction towards the safety net from the cannon, and the positive y-axis upwards from $(0,0)$.
Since Herc reaches his maximum height when his horizontal distance is 30 m, then the axis of symmetry of the parabola is the line $x = 30$.
Since the parabola has a root at $x = 0$, then the other root must be at $x = 60$.
Therefore, the parabola has the form $y = ax(x-60)$.
In order to determine the value of a, we note that Herc passes through the point $(30,100)$, and so



$$100 = 30a(-30)$$
$$a = -\tfrac{1}{9}$$

Thus, the equation of the parabola is $y = -\frac{1}{9}x(x-60)$.

(Alternatively, we could say that since the parabola has its maximum point at $(30,100)$, then it must be of the form $y = a(x-30)^2 + 100$.
Since the parabola passes through $(0,0)$, then we have

$$0 = a(0-30)^2 + 100$$
$$0 = 900a + 100$$
$$a = -\tfrac{1}{9}$$

Thus, the parabola has the equation $y = -\frac{1}{9}(x-30)^2 + 100$.)

We would like to find the points on the parabola which have y-coordinate 64, so we solve

$$64 = -\tfrac{1}{9}x(x-60)$$
$$0 = x^2 - 60x + 576$$
$$0 = (x-12)(x-48)$$

Since we want a point after Herc has passed his highest point, then $x = 48$, ie. the horizontal distance from the cannon to the safety net is 48 m.

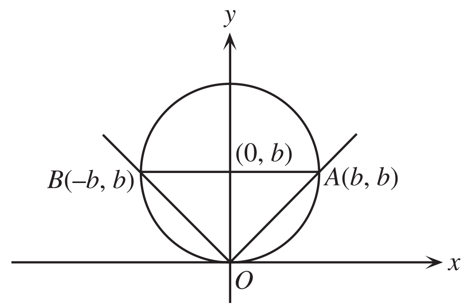
8. (a) Since both the circle with its centre on the y-axis and the graph of $y = |x|$ are symmetric about the y-axis, then for each point of intersection between these two graphs, there should be a corresponding point of intersection symmetrically located across the y-axis.
Thus, since there are exactly three points of intersection, then one of these points must be on the y-axis, ie. has x-coordinate 0. Since this point is on the graph of $y = |x|$, then this point must be $(0,0)$.
Since the circle has centre on the y-axis (say, has coordinates $(0,b)$), then its radius is equal to b (and b must be positive for there to be three points of intersection).
So the circle has equation $x^2 + (y-b)^2 = b^2$. Where are the other two points of intersection? We consider the points with x positive and use symmetry to get the other point of intersection.

When $x \geq 0$, then $y = |x|$ has equation $y = x$. Substituting into the equation of the circle,

$$x^2 + (x-b)^2 = b^2$$
$$2x^2 - 2bx = 0$$
$$2x(x-b) = 0$$

Therefore, the points of intersection are $(0,0)$ and (b,b) on the positive side of the y-axis, and so at the point $(-b,b)$ on the negative side of the y-axis.

Thus the points O, A and B are the points $(0,0)$, (b,b) and $(-b,b)$.



Since the radius of the circle is b, then the area of the circle is πb^2.

Triangle OAB has a base from $(-b,b)$ to (b,b) of length $2b$, and a height from the line $y = b$ to the point $(0,0)$ of length b, and so an area of $\frac{1}{2}b(2b) = b^2$.

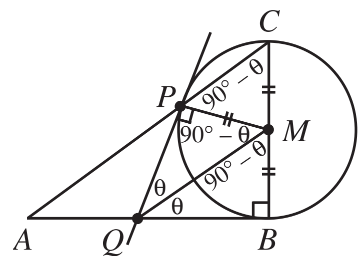
Therefore, the ratio of the area of the triangle to the area of the circle is $b^2 : \pi b^2 = 1 : \pi$.

(b) *Solution 1*

Since M is the midpoint of a diameter of the circle, M is the centre of the circle.

Join P to M. Since QP is tangent to the circle, PM is perpendicular to QP.

Since PM and BM are both radii of the circle, then $PM = MB$.



Therefore, ΔQPM and ΔQBM are congruent (Hypotenuse - Side).

Thus, let $\angle MQB = \angle MQP = \theta$. So $\angle QMB = \angle QMP = 90^\circ - \theta$

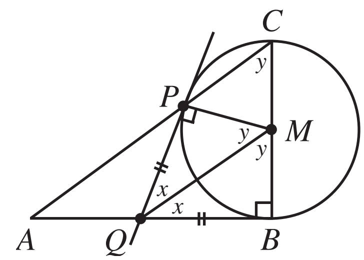
Then $\angle PMC = 180^\circ - \angle PMQ - \angle BMQ = 180^\circ - \left(90^\circ - \theta\right) - \left(90^\circ - \theta\right) = 2\theta$.

But ΔPMC is isosceles with $PM = MC$ since PM and MC are both radii.

Therefore, $\angle CPM = \frac{1}{2}\left(180^\circ - \angle PMC\right) = 90^\circ - \theta$.

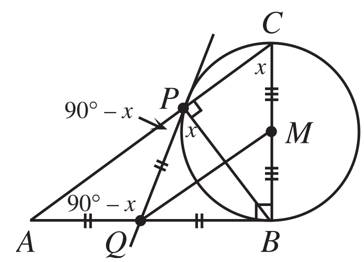
But then $\angle CPM = \angle PMQ$, and since PM is a transversal between AC and QM, then QM is parallel to AC because of equal alternating angles.

Solution 2



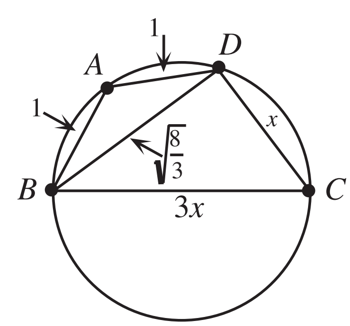
Join M to P and B to P.
Since QP and QB are tangents to the circle coming from the same point, they have the same length.
Since QM joins the point of intersection of the tangents to the centre of the circle, then by symmetry, $\angle PQM = \angle BQM$ and $\angle PMQ = \angle BMQ$. So let $\angle PQM = \angle BQM = x$ and $\angle PMQ = \angle BMQ = y$.
Looking at ΔQMB, we see that $x + y = 90^\circ$, since ΔQMB is right-angled.
Now if we consider the chord PB, we see that its central angle is $2y$, so any angle that it subtends on the circle (eg. $\angle PCB$) is equal to y.
Thus, $\angle ACB = \angle QMB$, so QM is parallel to AC.

Solution 3



Join PB.
Since QP is tangent to the circle, then by the Tangent-Chord Theorem, $\angle QPB = \angle PCB = x$ (ie. the inscribed angle of a chord is equal to the angle between the tangent and chord.
Since BC is a diameter of the circle, then $\angle CPB = 90^\circ$ and so $\angle APB = 90^\circ$, whence $\angle APQ = 90^\circ - \angle QPB = 90^\circ - x$.
Looking at ΔABC, we see that $\angle PAQ = 90^\circ - x$, so $\angle PAQ = \angle APQ$, and so $AQ = QP$.
But QP and QB are both tangents to the circle (QB is tangent since it is perpendicular to a radius), so $QP = QB$.
But then $AQ = QB$ and $BM = MC$, so Q is the midpoint of AB and M is the midpoint of BC. Thus we can conclude that QM is parallel to AC.
(To justify this last statement, we can show very easily that ΔQBM is similar to ΔABC, and so show that $\angle CAB = \angle MQB$.)

9. *Solution 1*



Consider ΔBAD. Since we know the lengths of sides BA and AD and the cosine of the angle between them, we can calculate the length of BD using the cosine law:

$$\begin{aligned} BD &= \sqrt{BA^2 + AD^2 - 2(BA)(AD)\cos\angle BAD} \\ &= \sqrt{2 - 2\left(-\tfrac{1}{3}\right)} \\ &= \sqrt{\tfrac{8}{3}} \end{aligned}$$

Next, let $x = \cos\angle ABC$. Note that $DC = x$.

Since $ABCD$ is a cyclic quadrilateral, then $\angle ADC = 180^\circ - \angle ABC$, and so $\cos\angle ADC = -\cos\angle ABC = -x$.
Similarly, $\cos\angle BCD = -\cos\angle BAD = \frac{1}{3}$ (since $ABCD$ is a cyclic quadrilateral).
So we can now use the cosine law simultaneously in ΔADC and ΔABC (since side AC is common) in order to try to solve for BC:

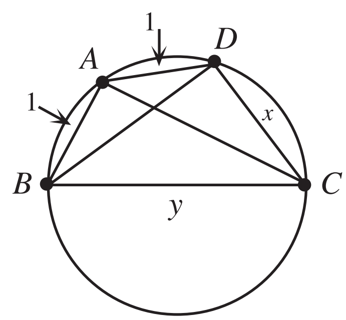
$$\begin{aligned} 1^2 + x^2 - 2(1)(x)\cos\angle ADC &= 1^2 + BC^2 - 2(1)(BC)\cos\angle ABC \\ 1^2 + x^2 - 2(1)(x)(-x) &= 1^2 + BC^2 - 2(1)(BC)(x) \\ 0 &= BC^2 - 2(BC)x - 3x^2 \\ 0 &= (BC - 3x)(BC + x) \end{aligned}$$

Since x is already a side length, then x must be positive (ie. $\angle ABC$ is acute), so $BC = 3x$.

Since $\cos\angle BCD = \frac{1}{3}$ and sides DC and BC are in the ratio 1 : 3, then ΔBCD must indeed be right-angled at D. (We could prove this by using the cosine law to calculate $BD^2 = 8x^2$ and then noticing that $DC^2 + BD^2 = BC^2$.)
Since ΔBCD is right-angled at D, then BC is a diameter of the circle.

Solution 2

Let $x = \cos\angle ABC = CD$, and let $BC = y$.
Since the opposite angles in a cyclic quadrilateral are supplementary, their cosines are negatives of each other. Thus, $\cos\angle ADC = -x$ and $\cos\angle BCD = \frac{1}{3}$.



Next, we use the cosine law four times: twice to calculate AC^2 in the two triangles ABC and ADC, and then twice to calculate BD^2 in the triangles ADB and CDB to obtain:

$$\begin{aligned} 1^2 + x^2 - 2(1)(x)\cos\angle ADC &= 1^2 + y^2 - 2(1)(y)\cos\angle ABC \\ 1 + x^2 - 2x(-x) &= 1 + y^2 - 2y(x) \\ 0 &= y^2 - 2xy - 3x^2 \\ 0 &= (y - 3x)(y + x) \end{aligned}$$

and

$$\begin{aligned} 1^2 + 1^2 - 2(1)(1)\cos\angle BAD &= x^2 + y^2 - 2xy\cos\angle BCD \\ 2 - 2\left(-\tfrac{1}{3}\right) &= x^2 + y^2 - 2xy\left(\tfrac{1}{3}\right) \\ \tfrac{8}{3} &= x^2 + y^2 - \tfrac{2}{3}xy \end{aligned}$$

From the first equation, since x is already a side length and so is positive, we must have that $y = 3x$.
Substituting into the second equation, we obtain

$$\tfrac{8}{3} = x^2 + (3x)^2 - \tfrac{2}{3}x(3x)$$
$$\tfrac{8}{3} = 8x^2$$
$$x = \tfrac{1}{\sqrt{3}}$$

since x must be positive. Thus, since $y = 3x$, then $y = \sqrt{3}$.
Looking then at ΔBDC, we have side lengths $BC = \sqrt{3}$, $CD = \frac{1}{\sqrt{3}}$ and $BD = \sqrt{\frac{8}{3}}$. (The last is from the left side of the second cosine law equation.) Thus, $BC^2 = CD^2 + BD^2$, and so ΔBDC is right-angled at D, whence BC is a diameter of the circle.

10. (a) To show that 8 is a savage integer, we must partition the set $\{1,2,3,4,5,6,7,8\}$ according to the given criteria.
Since the sum of the integers from 1 to 8 is 36, then the sum of the elements in each of the sets A, B, and C must be 12.
C must contain both 3 and 6.
A can contain only the numbers 1, 5, 7, and may not contain all of these.
B can contain only the numbers 2, 4, 8, and may not contain all of these.
So if we let $C = \{1,2,3,6\}$, $A = \{5,7\}$ and $B = \{4,8\}$, then these sets have the desired properties.
Therefore, 8 is a savage integer.

(b) We use the strategy of putting all of the multiples of 3 between 1 and n in the set C, all of the remaining even numbers in the set B, and all of the remaining numbers in the set A. The sums of these sets will not likely all be equal, but we then try to adjust the sums to by moving elements out of A and B into C, as we did in part (a), to try to make these sums equal. (Notice that we can't move elements either into A or B, or out of C.) We will use the notation $|C|$ to denote the sum of the elements of C.

Since we are considering the case of n even and we want to examine multiples of 3 less than or equal to n, it makes sense to consider n as having one of the three forms $6k$, $6k+2$ or $6k+4$. (These forms allow us to quickly tell what the greatest multiple of 3 less than n is.)

Case 1: $n = 6k$

In this case, C contains at least the integers 3, 6, 9, …, $6k$, and so the sum of C is greater than one-third of the sum of the integers from 1 to n, since if we divide the integers from 1 to $n = 6k$ into groups of 3 consecutive integers starting with 1, 2, 3, then the set C will always contain the largest of the 3.

Case 2: $n = 6k + 4$

Here, the sum of the integers from 1 to $n = 6k + 4$ is
$\frac{1}{2}(6k+4)(6k+5) = 18k^2 + 27k + 10 = 3\left(6k^2 + 9k + 3\right) + 1$, which is never divisible by 3. Therefore, n cannot be savage in this case because the integers from 1 to n cannot be partitioned into 3 sets with equal sums.

Case 3: $n = 6k + 2$

Here, the sum of the integers from 1 to $n = 6k + 2$ is
$\frac{1}{2}(6k+2)(6k+3) = 18k^2 + 15k + 3$, so the sum of the elements of each of the sets A, B and C should be $6k^2 + 5k + 1$, so that the sums are equal.
In this case C, contains at least the integers 3, 6, 9, …, $6k$, and so
$|C| \geq 3 + 6 + 9 + \cdots 6k = 3(1 + 2 + 3 + \cdots + 2k) = 3\left(\frac{1}{2}(2k)(2k+1)\right) = 6k^2 + 3k$

The set A contains at most the integers 1, 3, 5, 7, …, $6k + 1$, but does not contain the odd multiples of 3 less than n, ie. the integers 3, 9, 15, …, $6k - 3$. Therefore,
$$
\begin{aligned}
|A| &\leq (1 + 3 + 5 + \cdots + 6k + 1) - (3 + 9 + \cdots + 6k - 3) \\
&= \tfrac{1}{2}(3k+1)[1 + 6k + 1] - \tfrac{1}{2}(k)[3 + 6k - 3] \\
&= (3k+1)(3k+1) - k(3k) \\
&= 6k^2 + 6k + 1
\end{aligned}
$$
(To compute the sum of each of these arithmetic sequences, we use the fact that the sum of an arithmetic sequence is equal to half of the number of terms times the sum of the first and last terms.)

The set B contains at most the integers 2, 4, 6, 8, …, $6k + 2$, but does not contain the even multiples of 3 less than n, ie. the integers 6, 12, …, $6k$. Therefore,
$$
\begin{aligned}
|B| &\leq (2 + 4 + 6 + \cdots + 6k + 2) - (6 + 12 + \cdots + 6k) \\
&= \tfrac{1}{2}(3k+1)[2 + 6k + 2] - \tfrac{1}{2}(k)[6 + 6k] \\
&= (3k+1)(3k+2) - k(3k+3) \\
&= 6k^2 + 6k + 2
\end{aligned}
$$

Thus, the set C is $2k + 1$ short of the desired sum, while the set A has a sum that is k too big and the set B has a sum that is $k + 1$ too big.

So in order to correct this, we would like to move elements from A adding to k, and elements from B which add to $k + 1$ all to set C.

Since we are assuming that n is savage, then this is possible, which means that $k+1$ must be even since every element in B is even, so the sum of any number of elements of B is even.
Therefore, k is odd, and so $k = 2l+1$ for some integer l, and so $n = 6(2l+1)+2 = 12l+8$, ie. $\frac{n+4}{12}$ is an integer.

Having examined all cases, we see that if n is an even savage integer, then $\frac{n+4}{12}$ is an integer.

(c) From (b), the only possible even savage integers less than 100 are those satisfying the condition that $\frac{n+4}{12}$ is an integer, ie. 8, 20, 32, 44, 56, 68, 80, 92. We already know that 8 is savage, so we examine the remaining 7 possibilities.
We make a table of the possibilities, using the notation from (b):

n	k	Sum of elements to remove from A	Sum of elements to remove from B	Possible?
20	3	3	4	No – cannot remove a sum of 3 from A.
32	5	5	6	Yes – remove 5 from A, 2 and 4 from B
44	7	7	8	Yes – remove 7 from A, 8 from B
56	9	9	10	No – cannot remove a sum of 9 from A.
68	11	11	12	Yes – remove 11 from A, 4 and 8 from B
80	13	13	14	Yes – remove 13 from A, 14 from B
92	15	15	16	No – cannot remove a sum of 15 from A (since could only use 1, 5, 7, 11, 13)

Therefore, the only even savage integers less than 100 are 8, 32, 44, 68 and 80.

Canadian
Mathematics
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2002 Solutions
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1. (a) *Solution 1 (Midpoint Formula)*
Since M is the midpoint of the line segment joining R and S, then looking at the x-coordinate of M,

$$\begin{aligned} 7 &= \frac{1+a}{2} \\ 14 &= 1+a \\ a &= 13 \end{aligned}$$

Solution 2 (Slopes)
Since the slope of RM is equal to the slope of MS, then

$$\begin{aligned} \frac{3}{6} &= \frac{3}{a-7} \\ a-7 &= 6 \\ a &= 13 \end{aligned}$$

Solution 3 (Distances)
Since $RM = MS$ or $RM^2 = MS^2$, then

$$\begin{aligned} 36 &= 3^2 + (a-7)^2 \\ 0 &= a^2 - 14a + 13 \\ 0 &= (a-13)(a-1) \end{aligned}$$

Therefore, $a = 13$ or $a = 1$, but we reject $a = 1$, since $(1,10)$ does not lie on the line.
Thus, $a = 13$.

Answer: $a = 13$

(b) The base of ΔPQR has length 8, and the height has length $k-2$ (since $k > 0$).
Since the area of ΔPQR is 24, then

$$\begin{aligned} \tfrac{1}{2}(8)(k-2) &= 24 \\ 4k - 8 &= 24 \\ 4k &= 32 \\ k &= 8 \end{aligned}$$

Answer: $k = 8$

(c) We first determine the point of intersection of lines $y = 2x + 3$ and $y = 8x + 15$, and then substitute this point into the line $y = 5x + b$, since it lies on all three lines.
So we set the first two equations equal to each other:

$$\begin{aligned} 2x+3 &= 8x+15 \\ -12 &= 6x \\ x &= -2 \end{aligned}$$

Substituting $x=-2$ into the first equation, we obtain $y=2(-2)+3=-1$, so the point of intersection is $(-2,-1)$, which must lie on the third line.

Thus,

$$\begin{aligned} -1 &= 5(-2)+b \\ b &= 9 \end{aligned}$$

Therefore, the value of b is 9.

2. (a) *Solution 1*

Since $x=4$ is a root, then $4^2-3(4)+c=0$ or $c=-4$.

Therefore, the quadratic equation is $x^2-3x-4=0$, which we can factor as $(x-4)(x+1)=0$. (This factorization is made easier since we already know one of the roots.) Therefore, the second root is $x=-1$.

Solution 2

The sum of the roots of $x^2-3x+c=0$ is $-\left(\frac{-3}{1}\right)=3$, so since one root is 4, the second root must be $x=-1$.

Answer: $x=-1$

(b) *Solution 1*

Since the two expressions are the same, then they must have the same value when we substitute any value for *x*. In particular, substitute $x=2$, and so we get

$$\begin{aligned} \frac{2\left(2^2\right)+1}{2^2-3} &= 2+\frac{A}{2^2-3} \\ 9 &= 2+A \\ A &= 7 \end{aligned}$$

Solution 2

We compare the two expressions

$$\begin{aligned} \frac{2x^2+1}{x^2-3} &= 2+\frac{A}{x^2-3} \\ &= \frac{2\left(x^2-3\right)}{x^2-3}+\frac{A}{x^2-3} \\ &= \frac{2x^2-6+A}{x^2-3} \end{aligned}$$

Since the expressions are the same, the numerators must be the same, and so $-6+A=1$ or $A=7$.

Solution 3

$$\frac{2x^2+1}{x^2-3}=\frac{2x^2-6+7}{x^2-3}$$
$$=\frac{2\left(x^2-3\right)+7}{x^2-3}$$
$$=2+\frac{7}{x^2-3}$$

Therefore, $A=7$.

Answer: $A=7$

(c) *Solution 1*

The original parabola can be written as

$$y=(x-3)(x-1)$$

which means its roots are $x=3$ and $x=1$.

When this parabola is shifted 5 units to the right, the parabola obtained must thus have roots $x=3+5=8$ and $x=1+5=6$.

Therefore, the new parabola is

$$y=(x-8)(x-6)$$
$$=x^2-14x+48$$

and so $d=48$.

Solution 2

The original parabola $y=x^2-4x+3$ can be written as $y=(x-2)^2-1$, and so its vertex has coordinates $(2,-1)$. To get the vertex of the new parabola, we shift the vertex of the original parabola 5 units to the right to the point $(7,-1)$. Substituting this point into the new parabola, we obtain

$$-1=7^2-14(7)+d$$
$$-1=49-98+d$$
$$d=48$$

[An easier version of this solution is to recognize that if the original parabola passes through $(1,0)$, then $(6,0)$ must be on the translated parabola. Thus, $0=36-14(6)+d$ or $d=48$, as above.]

Solution 3

To carry out a translation of 5 units to the right, we can define new coordinates X and Y, with $(x, y) = (X - 5, Y)$. So in these new coordinates the parabola will have equation

$$\begin{aligned} Y &= (X-5)^2 - 4(X-5) + 3 \\ &= X^2 - 10X + 25 - 4X + 20 + 3 \\ &= X^2 - 14X + 48 \end{aligned}$$

Comparing this with the given equation, we see that $d = 48$.

3. (a) We make a table of the possible selections of balls a, b, c that give $a = b + c$:

a	b	c
2	1	1
3	1	2
3	2	1
4	1	3
4	2	2
4	3	1

Therefore, since there are 6 ways to get the required sum, then the probability that he wins the prize is $\frac{6}{64} = \frac{3}{32}$.

Answer: $\frac{3}{32}$

(b) Since the product of the three integers is 216, then

$$\begin{aligned} a(ar)(ar^2) &= 216 \\ a^3r^3 &= 216 \\ (ar)^3 &= 6^3 \\ ar &= 6 \end{aligned}$$

Now we are given that a is a positive integer, but r is not necessarily an integer. However, we do know that the sequence is *increasing*, so $r > 1$, and thus $a < 6$. We check the possibilities for a between 1 and 5, and determine whether these possibilities for a yield a value for a that makes the third term (ar^2) an integer (we already know that $ar = 6$, so is an integer):

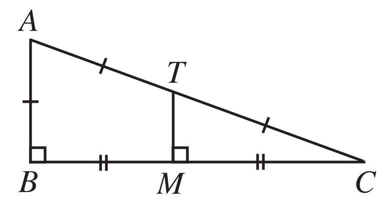
a	r	ar	ar^2
1	6	6	36
2	3	6	18
3	2	6	12

4	$\frac{3}{2}$	6	9
5	$\frac{6}{5}$	6	$\frac{36}{5}$

So the four sequences that satisfy the required conditions are:

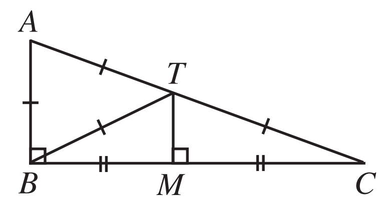
1, 6, 36;
2, 6, 18;
3, 6, 12;
4, 6, 9.

4. (a) *Solution 1*



Since MT is the perpendicular bisector of BC, then $BM = MC$, and TM is perpendicular to BC.
Therefore, ΔCMT is similar to ΔCBA, since they share a common angle and each have a right angle.
But $\frac{CM}{CB} = \frac{1}{2}$ so $\frac{CT}{CA} = \frac{CM}{CB} = \frac{1}{2}$, and thus $CT = AT = AB$, ie. $\frac{AB}{AC} = \frac{1}{2}$ or $\sin(\angle ACB) = \frac{1}{2}$.
Therefore, $\angle ACB = 30^\circ$.

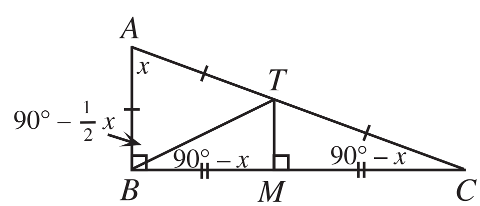
Solution 2



Since $TM \| AB$, and $CM = MB$, then $CT = TA = AB$.
Join T to B.
Since $\angle ABC = 90^\circ$, then AC is the diameter of a circle passing through A, C and B, with T as its centre.

Thus, $TA = AB = BT$ (all radii), and so ΔABT is equilateral. Therefore, $\angle BAC = 60^\circ$, and so $\angle ACB = 30^\circ$.

Solution 3



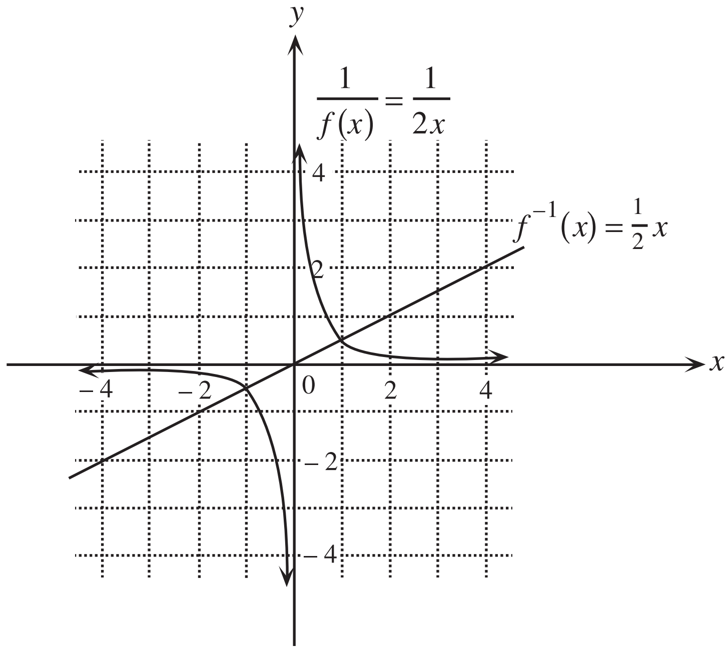
Join T to B, and let $\angle BAC = x$. Thus, $\angle ACB = 90^\circ - x$
As in Solution 1 or Solution 2, ΔATB is isosceles, so $\angle ABT = 90^\circ - \frac{1}{2}x$.
Since ΔTBM is congruent to ΔTCM (common side; right angle; equal side), then $\angle TBC = \angle ACB = 90^\circ - x$
Looking at $\angle ABC$, we see that

$$\left(90^\circ - \tfrac{1}{2}x\right) + (90^\circ - x) = 90^\circ$$
$$90^\circ = \tfrac{3}{2}x$$
$$x = 60^\circ$$

Therefore, $\angle ACB = 30^\circ$.

Answer: $\angle ACB = 30^\circ$

(b) (i)



(ii) *Solution 1*

From the graph in (i), the points where $f^{-1}(x) = \dfrac{1}{f(x)}$ are $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$.

Solution 2

We determine the functions $f^{-1}(x)$ and $\dfrac{1}{f(x)}$ explicitly.

To get $f^{-1}(x)$, we start with $y = 2x$, interchange x and y to get $x = 2y$, and solve for y to get $y = \frac{1}{2}x$ or $f^{-1}(x) = \frac{1}{2}x$.

Also, $\dfrac{1}{f(x)} = \dfrac{1}{2x}$.

Setting these functions equal,

$$\frac{1}{2}x = \frac{1}{2x}$$
$$x^2 = 1$$
$$x = \pm 1$$

Substituting into $f^{-1}(x)$, we obtain the points $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$.

(iii) Using $f(x) = 2x$, we see that $f\left(\frac{1}{2}\right) = 1$, and so

$$f^{-1}\left(\frac{1}{f\left(\frac{1}{2}\right)}\right)=f^{-1}\left(\frac{1}{1}\right)$$
$$=f^{-1}(1)$$
$$=\tfrac{1}{2}(1)$$
$$=\tfrac{1}{2}$$

where $f^{-1}(1)$ is determined either from the explicit form of $f^{-1}(x)$ or from the graph.

5. (a) Combining the logarithms,

$$\log_5(x+3)+\log_5(x-1)=1$$
$$\log_5((x+3)(x-1))=1$$
$$\log_5\left(x^2+2x-3\right)=1$$
$$x^2+2x-3=5$$
$$x^2+2x-8=0$$
$$(x+4)(x-2)=0$$

Therefore, $x=-4$ or $x=2$. Substituting the two values for x back into the original equation, we see that $x=2$ works, but that $x=-4$ does not, since we cannot take the logarithm of a negative number.

Answer: $x=2$

(b) (i) From the table we have two pieces of information, so we substitute both of these into the given formula.

$$2.75=a(3.00)^b$$
$$3.75=a(6.00)^b$$

We can now proceed in either of two ways to solve for b.

Method 1 to find b

Dividing the second equation by the first, we obtain

$$\frac{3.75}{2.75}=\frac{a(6.00)^b}{a(3.00)^b}=\frac{(6.00)^b}{(3.00)^b}=\left(\frac{6.00}{3.00}\right)^b=2^b$$

or

$$2^b\approx1.363636$$

Taking logarithms of both sides,

$$\log\left(2^b\right) \approx \log(1.363636)$$
$$b\log(2) \approx \log(1.363636)$$
$$b \approx \frac{\log(1.363636)}{\log(2)}$$
$$b \approx 0.4475$$

Method 2 to find b

Taking logarithms of both sides of the above equations, we obtain

$$\log(2.75) = \log\left(a(3.00)^b\right)$$
$$= \log(a) + \log\left((3.00)^b\right)$$
$$= \log(a) + b\log(3.00)$$

Similarly,

$$\log(3.75) = \log(a) + b\log(6.00)$$

Subtracting the first equation from the second, we obtain

$$\log(3.75) - \log(2.75) = b\big(\log(6.00) - \log(3.00)\big)$$
$$b = \frac{\log(3.75) - \log(2.75)}{\log(6.00) - \log(3.00)}$$
$$b \approx 0.4475$$

We now continue in the same way for both methods.

Substituting this value for b back into the first equation above,

$$2.75 \approx a(3.00)^{0.4475}$$
$$a \approx \frac{2.75}{(3.00)^{0.4475}}$$
$$a \approx 1.6820$$

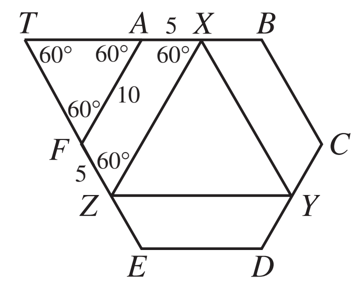
Therefore, to two decimal places, $a = 1.68$ and $b = 0.45$.

(ii) To determine the time to cook a goose of mass 8.00 kg, we substitute $m = 8.00$ into the given formula:

$$t = am^b$$
$$\approx 1.68(8.00)^{0.45}$$
$$\approx 4.2825$$

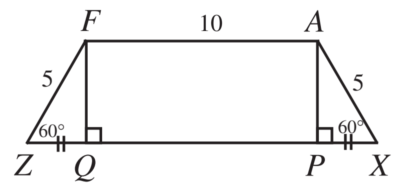
Thus, it will take about 4.28 h until his goose is cooked.

6. (a) *Solution 1*



Extend *XA* and *ZF* to meet at point *T*.
By symmetry, $\angle AXZ = \angle FZX = 60^\circ$ and $\angle TAF = \angle TFA = 60^\circ$, and so ΔTAF and ΔTXZ are both equilateral triangles.
Since $AF = 10$, then $TA = 10$, which means $TX = 10 + 5 = 15$, and so $XZ = TX = 15$.

Solution 2



We look at the quadrilateral *AXZF*.
Since *ABCDEF* is a regular hexagon, then $\angle FAX = \angle AFZ = 120^\circ$.
Note that $AF = 10$, and also $AX = FZ = 5$ since *X* and *Z* are midpoints of their respective sides.
By symmetry, $\angle AXZ = \angle FZX = 60^\circ$, and so *AXZF* is a trapezoid.
Drop perpendiculars from *A* and *F* to *P* and *Q*, respectively, on *XZ*.
By symmetry again, $PX = QZ$. Now, $PX = AX\cos 60^\circ = 5\left(\frac{1}{2}\right) = \frac{5}{2}$.
Since *APQF* is a rectangle, then $PQ = 10$.
Therefore, $XZ = XP + PQ + QZ = \frac{5}{2} + 10 + \frac{5}{2} = 15$.

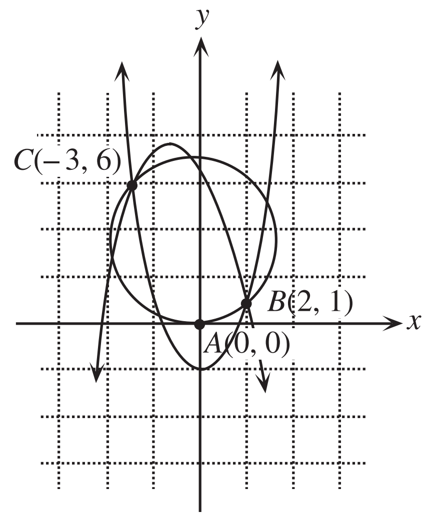
Answer: $XZ = 15$

(b) We first determine the three points through which the circle passes.
The first point is the origin $(0,0)$.
The second and third points are found by determining the points of intersection of the two parabolas $y = x^2 - 3$ and $y = -x^2 - 2x + 9$. We do this by setting the *y* values equal.

$$\begin{aligned} x^2 - 3 &= -x^2 - 2x + 9 \\ 2x^2 + 2x - 12 &= 0 \\ x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \end{aligned}$$

so $x = -3$ or $x = 2$.

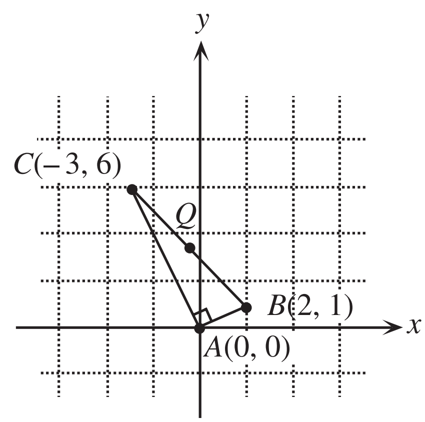
We determine the points of intersection by substituting into the first parabola.
If $x = 2$, $y = 2^2 - 3 = 1$, so the point of intersection is $(2,1)$.
If $x = -3$, $y = (-3)^2 - 3 = 6$, so the point of intersection is $(-3,6)$.
Therefore, the circle passes through the three points $A(0,0)$, $B(2,1)$ and $C(-3,6)$.
Let the centre of the circle be the point $Q(a,b)$.



Finding the centre of the circle can be done in a variety of ways.

Method 1 $\left(\angle CAB = 90^\circ\right)$

We notice that the line segment joining $A(0,0)$ to $B(2,1)$ has slope $\frac{1}{2}$, and the line segment joining $A(0,0)$ to $C(-3,6)$ has slope -2, and so the two lines are perpendicular (since $\frac{1}{2}(-2) = -1$). Therefore, $\angle BAC = 90^\circ$.
Since *BC* is a chord of the circle which subtends an angle of 90° at point *A* on the circle, then *BC* is a diameter of the circle. Therefore, the centre of the circle is the midpoint of *BC*, which is the point $\left(-\frac{1}{2}, \frac{7}{2}\right)$.



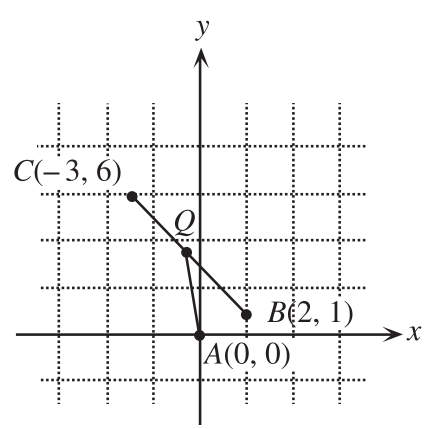
Method 2 (Equal radii)

We use the fact *Q* is of equal distance from each of the points *A*, *B* and *C*. In particular $QA^2 = QB^2 = QC^2$ or

$$x^2 + y^2 = (x-2)^2 + (y-1)^2 = (x+3)^2 + (y-6)^2$$

From the first equality,

$$x^2 + y^2 = (x-2)^2 + (y-1)^2$$
$$4x + 2y = 5$$



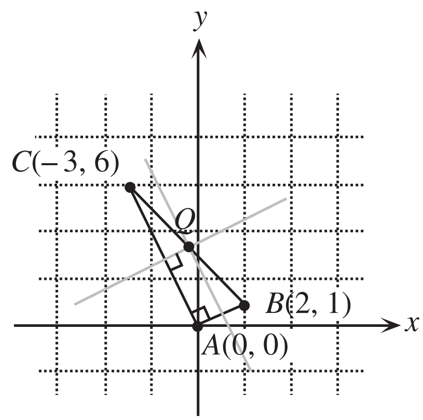
From the second equality,

$$\begin{aligned}(x-2)^2+(y-1)^2 &= (x+3)^2+(y-6)^2\\ -10x+10y &= 40\\ y &= x+4\end{aligned}$$

Substituting the equation above into into $4x+2y=5$, we obtain $4x+2(x+4)=5$ or $6x=-3$ or $x=-\frac{1}{2}$. Thus, $y=-\frac{1}{2}+4=\frac{7}{2}$, and so the centre of the circle is $\left(-\frac{1}{2},\frac{7}{2}\right)$.

Method 3 (Perpendicular bisectors)

We determine the equations for the perpendicular bisectors of *AB* and *AC*. The centre is the point of intersection of these two lines.



Since *AB* has slope $\frac{1}{2}$, then the slope of its perpendicular bisector is -2. Since the midpoint of *AB* is $\left(1,\frac{1}{2}\right)$, then the perpendicular bisector is $y-\frac{1}{2}=-2(x-1)$ or $y=-2x+\frac{5}{2}$.

Since *AC* has slope -2, then the slope of its perpendicular bisector is $\frac{1}{2}$. Since the midpoint of *AB* is $\left(-\frac{3}{2},3\right)$, then the perpendicular bisector is $y-3=\frac{1}{2}\left(x+\frac{3}{2}\right)$ or $y=\frac{1}{2}x+\frac{15}{4}$.

To find the point of intersection of these two lines, we set them equal:

$$\begin{aligned}-2x+\tfrac{5}{2} &= \tfrac{1}{2}x+\tfrac{15}{4}\\ -\tfrac{5}{4} &= \tfrac{5}{2}x\\ x &= -\tfrac{1}{2}\end{aligned}$$

From this, $y=-2\left(-\frac{1}{2}\right)+\frac{5}{2}=\frac{7}{2}$, and so the centre of the circle is $\left(-\frac{1}{2},\frac{7}{2}\right)$.

7. (a) *Solution 1*

Using a known formula for the area of a triangle, $A=\frac{1}{2}ab\sin C$,

$$\begin{aligned}18 &= \tfrac{1}{2}(2x+1)(2x)\sin 30^\circ\\ 36 &= (2x+1)(2x)\left(\tfrac{1}{2}\right)\\ 0 &= 2x^2+x-36\\ 0 &= (2x+9)(x-4)\end{aligned}$$

and so $x=4$ or $x=-\frac{9}{2}$. Since *x* is positive, then $x=4$.

Solution 2

Draw a perpendicular from A to P on BC.

Using ΔAPC, $AP = AC\sin 30^\circ = 2x\left(\frac{1}{2}\right) = x$.

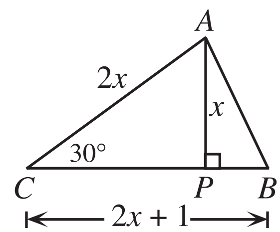
Now AP is the height of ΔABC, so Area $= \frac{1}{2}(BC)(AP)$.

Then

$$18 = \tfrac{1}{2}(2x+1)(x)$$
$$0 = 2x^2 + x - 36$$
$$0 = (2x+9)(x-4)$$

and so $x = 4$ or $x = -\frac{9}{2}$.

Since x is positive, then $x = 4$.



Answer: $x = 4$

(b) Let the length of the ladder be L.

Then $AC = L\cos 70^\circ$ and $BC = L\sin 70^\circ$. Also,
$A'C = L\cos 55^\circ$ and $B'C = L\sin 55^\circ$.

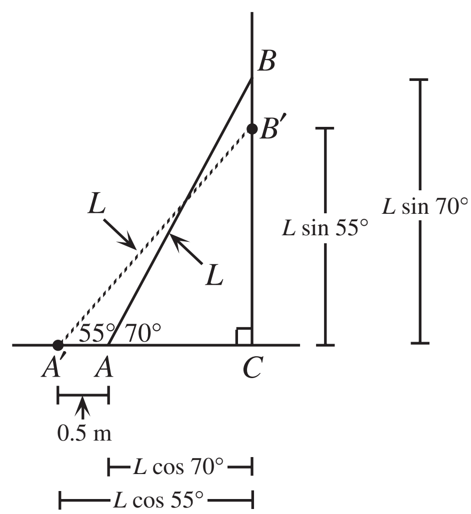
Since $A'A = 0.5$, then

$$0.5 = L\cos 55^\circ - L\cos 70^\circ$$

$$L = \frac{0.5}{\cos 55^\circ - \cos 70^\circ} \qquad (*)$$

Therefore,

$$\begin{aligned} BB' &= BC - B'C \\ &= L\sin 70^\circ - L\sin 55^\circ \\ &= L\left(\sin 70^\circ - \sin 55^\circ\right) \\ &= \frac{(0.5)\left(\sin 70^\circ - \sin 55^\circ\right)}{\left(\cos 55^\circ - \cos 70^\circ\right)} \qquad \text{(from (*))} \\ &\approx 0.2603 \text{ m} \end{aligned}$$



Therefore, to the nearest centimetre, the distance that the ladder slides down the wall is 26 cm.

8. (a) *Solution 1*

In total, there are $\frac{1}{2} \times 5 \times 20 = 50$ games played, since each of 5 teams plays 20 games (we divide by 2 since each game is double-counted).

In each game, there is either a loss or a tie.

The number of games with a loss is $44 + y$ from the second column, and the number of games with a tie is $\frac{1}{2}(11 + z)$ (since any game ending in a tie has 2 ties).

So

$$50 = 44 + y + \tfrac{1}{2}(11 + z)$$
$$100 = 88 + 2y + 11 + z$$
$$1 = 2y + z$$

Since y and z are non-negative integers, $z = 1$ and $y = 0$. So $x = 19$ since Team E plays 20 games.

Solution 2

In any game played, the final result is either both teams earning a tie, or one team earning a win, and the other getting a loss. Therefore, the total number of wins among all teams equals the total number of losses, ie.

$$25 + x = 44 + y$$
$$x - y = 19 \qquad (1)$$

Also, since team E plays 20 games, then

$$x + y + z = 20 \qquad (2)$$

So from (1), x must be at least 19, and from (2), x can be at most 20.
Lastly, we know that the total of all of the teams numbers of ties must be even, ie. $11 + z$ is even, ie. z is odd.
Since x is at least 19, then z can be at most 1 by (2).
Therefore, $z = 1$. Thus, $x = 19$ and $y = 0$.

Solution 3

In any game played, the final result is either both teams earning a tie, or one team earning a win, and the other getting a loss. Therefore, the total number of wins among all teams equals the total number of losses, ie.

$$25 + x = 44 + y$$
$$x - y = 19 \qquad (1)$$

Also, since team E plays 20 games, then

$$x + y + z = 20 \qquad (2)$$

So from (1), x must be at least 19, and from (2), x can be at most 20.
Consider the possibility that $x = 20$. From (2), then $y = z = 0$, which does not agree with (1).
Thus, the only possibility is $x = 19$. From (1), $y = 0$, and so $z = 1$ from (2). (These three values agree with both equations (1) and (2).)

(b) *Solution 1*

Assume such a sequence a, b, c, d exists. (We proceed by contradiction.)
Since the sum of any two consecutive terms is positive, $a+b>0$, $b+c>0$, and $c+d>0$. Adding these three inequalities, $(a+b)+(b+c)+(c+d)>0$ or $a+2b+2c+d>0$.
We are going to show that this statement contradicts the facts that are known about the sequence. We are told that the sum of any three consecutive terms is negative, ie. $a+b+c<0$ and $b+c+d<0$. Adding these two inequalities, $(a+b+c)+(b+c+d)<0$ or $a+2b+2c+d<0$.
This is a contradiction, since the two conditions $a+2b+2c+d>0$ and $a+2b+2c+d<0$ cannot occur simultaneously.
Therefore, our original assumption is false, and so no such sequence exists.

Solution 2

Assume such a sequence a, b, c, d exists. (We proceed by contradiction.)
We consider two cases.

Case 1: $a \le 0$

In this case, $b>0$ since $a+b>0$.
Then, since $a+b+c<0$, we must have that $c<0$.
But $c+d>0$, so $d>0$.
This means that we have $b>0$ and $c+d>0$, ie. $b+c+d>0$.
But from the conditions on the sequence, $b+c+d<0$, a contradiction.
Therefore, no such sequence exists with $a \le 0$.

Case 2: $a>0$

In this case, it is not immediately clear whether b has to be positive or negative.
However, we do know that $a+b>0$ and $a+b+c<0$, so it must be true that $c<0$.
Then since $b+c>0$ and $c+d>0$, we must have both $b>0$ and $d>0$. But then $b+c+d=b+(c+d)>0$ since $c+d>0$ and $b>0$.
This is again a contradiction.
Therefore, no such sequence exists with $a>0$.

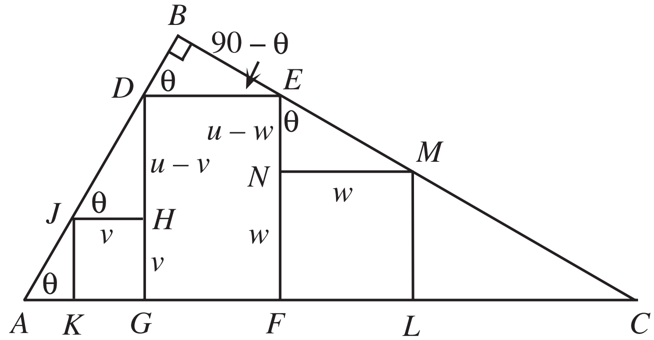
9. (a) Let $\angle BAC = \theta$. Then by parallel lines,

$\angle DJH = \angle BDE = \theta$.

Thus, $\angle BED = 90^\circ - \theta$ and so $\angle NEM = \theta$ since $\angle DEF = 90^\circ$.

Since $DG = u$ and $HG = v$, then $DH = u - v$.

Similarly, $EN = u - w$.



Looking at ΔDHJ and ΔMNE, we see that $\tan\theta = \dfrac{u-v}{v}$ and $\tan\theta = \dfrac{w}{u-w}$.

Therefore,

$$\frac{u-v}{v} = \frac{w}{u-w}$$

$$(u-v)(u-w) = vw$$

$$u^2 - uv - uw + vw = vw$$

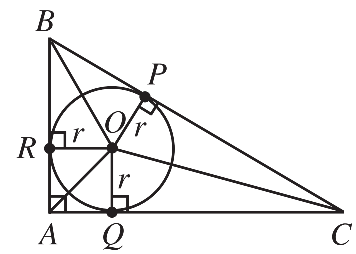
$$u(u - v - w) = 0$$

and since $u \neq 0$, we must have $u - v - w = 0$ or $u = v + w$.

[Note: If $u = 0$, then the height of rectangle $DEFG$ is 0, ie. D coincides with point A and E coincides with point C, which says that we must also have $v = w = 0$, ie. the squares have no place to go!]

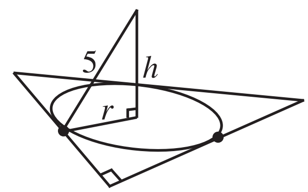
(b) Consider the cross-section of the sphere in the plane defined by the triangle. This cross-section will be a circle, since any cross-section of a sphere is a circle. This circle will be tangent to the three sides of the triangle, ie. will be the inscribed circle (or incircle) of the triangle. Let the centre of this circle be O, and its radius be r. We calculate the value of r.

Join O to the three points of tangency, P, Q, R, and to the three vertices A, B, C. Then OP, OQ and OR (radii) will form right angles with the three sides of the triangle. Consider the three triangles ΔAOB, ΔBOC and ΔCOA. Each of these triangles has a height of r and they have bases 15, 9 and 12, respectively. Since the area of ΔABC is equal to the sum of the areas of ΔAOB, ΔBOC, and ΔCOA,



So comparing areas,

$$\tfrac{1}{2}(9)(12) = \tfrac{1}{2}(9)(r) + \tfrac{1}{2}(12)(r) + \tfrac{1}{2}(15)(r)$$
$$54 = \tfrac{1}{2}r(9+12+15)$$
$$r = 3$$

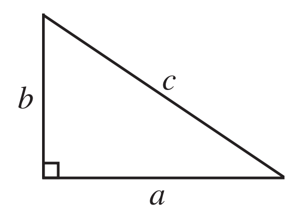


Now join the centre of the cross-sectional circle to the centre of the sphere and let this distance be h. Now, the line joining the centre of the circle to the centre of the sphere will be perpendicular to the plane of the triangle, so we can form a right-angled triangle by joining the centre of the sphere to any point on the circumference of the cross-sectional circle. By Pythagoras,

$$h^2 + r^2 = 25$$
$$h = 4$$

This tells us that the top of the sphere is 9 units above the plane of the triangle, since the top of the sphere is 5 units above the centre of the sphere.

10. (a) Consider a Pythagorean triangle with integer side lengths a, b, c satisfying $a^2 + b^2 = c^2$. To show that this triangle is Heronian, we must show that it has an integer area. Now we know that the area is equal to $\tfrac{1}{2}ab$, so we must show that either a or b is an even integer.



Suppose that both a and b are odd. (We proceed by contradiction.)
In this case, let $a = 2k+1$ and $b = 2l+1$. Then both a^2 and b^2 are odd, and so c^2 is even since $a^2 + b^2 = c^2$. Therefore, c itself must be even, so let $c = 2m$.
Therefore,

$$(2k+1)^2 + (2l+1)^2 = (2m)^2$$
$$4k^2 + 4k + 1 + 4l^2 + 4l + 1 = 4m^2$$
$$4\left(k^2 + k + l^2 + l\right) + 2 = 4\left(m^2\right)$$

But the right side is a multiple of 4, and the left side is not a multiple of 4. This is a contradiction.
Therefore, one of a or b must be even, and so the area of the triangle is an integer.
Thus, any Pythagorean triangle is Heronian.

(b) We examine the first few smallest Pythagorean triples:

3 4 5 $\left(3^2 = 4+5\right)$

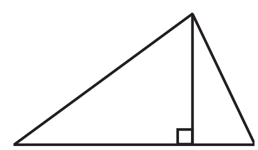
5	12	13	$\left(5^2 = 12 + 13\right)$
6	8	10	(Does not fit pattern)
7	24	25	$\left(7^2 = 24 + 25\right)$

It appears from the first few examples that perhaps we can form a Pythagorean triple by using any odd number greater than 1 as its shortest leg.
Next, we notice from the pattern that the sum of the second leg and the hypotenuse is the square of the shortest leg, and that these two side lengths differ by 1.
Will this pattern always hold?
Let $a = 2k + 1$ with $k \geq 1$. (This formula will generate all odd integers greater than or equal to 3.) Can we always find b so that $c = b + 1$ and $a^2 + b^2 = c^2$?
Consider the equation

$$(2k+1)^2 + b^2 = (b+1)^2$$
$$4k^2 + 4k + 1 + b^2 = b^2 + 2b + 1$$
$$4k^2 + 4k = 2b$$
$$b = 2k^2 + 2k$$

So we can always find a b to make the equation true. Therefore, since a can be any odd integer greater than or equal to 3, then we can make any odd number the shortest leg of a Pythagorean triangle, namely the Pythagorean triangle $a = 2k + 1$, $b = 2k^2 + 2k$, $c = 2k^2 + 2k + 1$. (Check that $a^2 + b^2 = c^2$ does indeed hold here!)

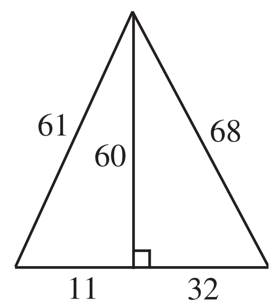
(c) We consider forming a triangle by joining two Pythagorean triangles along a common side. Since any Pythagorean triangle is Heronian, then the triangle that is formed by joining two Pythagorean triangles in the manner shown will have integer side lengths and will have integer area, thus making it Heronian.



So again, we make a list of Pythagorean triples

3	4	5
5	12	13
6	8	10
7	24	25
8	15	17
9	40	41
10	24	26
11	60	61

We notice that we can scale any Pythagorean triangle by an integer factor and obtain another Pythagorean triangle. This will enable us to create two Pythagorean triangles with a common side length.
Also, we note that when joining two Pythagorean triangles, the hypotenuse of each triangle becomes a side length in the new triangle. Since we cannot have a side length divisible by 3, 5, 7 or 11, this eliminates the 3-4-5, 6-8-10, and 7-24-25 triangles from the list above.
Suppose we scale the 8-15-17 triangle by a factor of 4 to obtain 32-60-68 and join to the 11-60-61 triangle in the manner shown.
Thus we obtain a 43-61-68 triangle, which has integer area because its height is an even integer.
Therefore, a 43-61-68 triangle is Heronian.



[It is worth noting that this is not the *only* such triangle, but it is the one with the shortest sides.]

Canadian
Mathematics
Competition

*An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario*

2001 Solutions
Euclid Contest (Grade 12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING
Awards

1. (a) What are the values of x such that $(2x-3)^2 = 9$?

Solution 1

$$(2x-3)^2 = 9$$
$$(2x-3)^2 - 9 = 0$$
$$(2x-3-3)(2x-3+3) = 0 \qquad \text{[by difference of squares]}$$
$$(2x-6)(2x) = 0$$
so $x = 3$ or $x = 0$.

Solution 2

$(2x-3)^2 = 9$
$2x-3 = 3$ or $2x-3 = -3$
Therefore $x = 3$ or $x = 0$.

Solution 3

$4x^2 - 12x + 9 = 9$
$4x^2 - 12x = 0$
$4x(x-3) = 0$
Therefore $x = 0$ or $x = 3$.

(b) If $f(x) = x^2 - 3x - 5$, what are the values of k such that $f(k) = k$?

Solution

If $f(k) = k$, then $k^2 - 3k - 5 = k$
$$k^2 - 4k - 5 = 0$$
$$(k-5)(k+1) = 0$$
so $k = 5$ or $k = -1$.

(c) Determine all (x, y) such that $x^2 + y^2 = 25$ and $x - y = 1$.

Solution 1 (Algebraic)

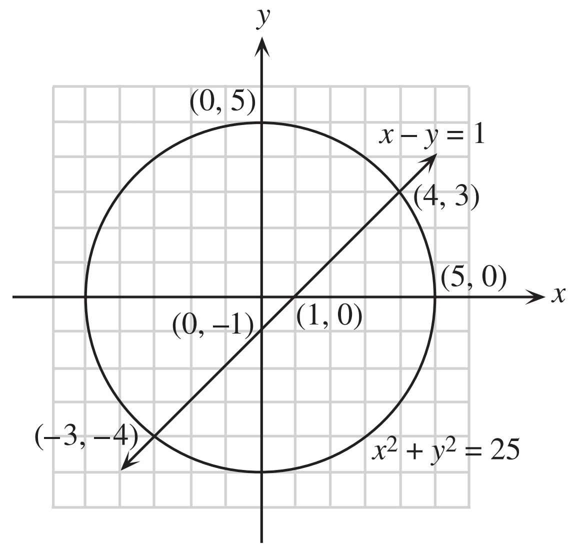
Since $x - y = 1$, then $x = y + 1$ (or $y = x - 1$).
So since $x^2 + y^2 = 25$, then
$$(y+1)^2 + y^2 = 25 \qquad \text{or} \qquad x^2 + (x-1)^2 = 25$$

$$y^2+2y+1+y^2=25 \qquad x^2+x^2-2x+1=25$$
$$2y^2+2y-24=0 \qquad 2x^2-2x-24=0$$
$$y^2+y-12=0 \qquad x^2-x-12=0$$
$$(y+4)(y-3)=0 \qquad (x-4)(x+3)=0$$
$$y=-4,3 \qquad x=4,-3$$

and using $x=y+1$, we get $x=-3,4$. | and using $y=x-1$, we get $y=3,-4$.

So the solutions are $(x,y)=(-3,-4),(4,3)$.

Solution 2 (Graphical)
Placing each of $x^2+y^2=25$ and $x-y=1$ on a grid we have the diagram at the right.

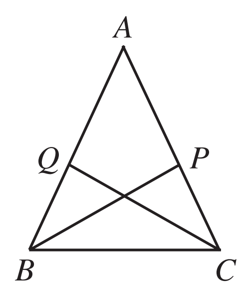


Therefore, the solutions are $(x,y)=(-3,-4)$, $(4,3)$.

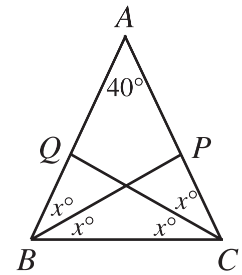
2. (a) The vertex of the parabola $y=(x-b)^2+b+h$ has coordinates $(2,5)$. What is the value of h?

Solution
Since the x-coordinate of the vertex is 2, then $b=2$.
Since the y-coordinate of the vertex is 5, then $b+h=5$. Since $b=2$, then $h=3$.

(b) In the isosceles triangle ABC, $AB=AC$ and $\angle BAC=40°$. Point P is on AC such that BP is the bisector of $\angle ABC$. Similarly, Q is on AB such that CQ bisects $\angle ACB$. What is the size of $\angle APB$, in degrees?



Solution



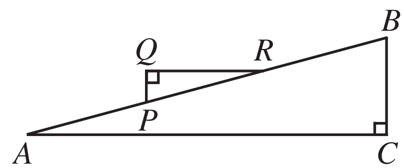
Let $\angle ABC = 2x°$. Since ΔABC is isosceles, then $\angle ACB = 2x°$.
Since BP bisects $\angle ABC$, $\angle ABP = \angle CBP = x°$.
Similarly, $\angle ACQ = \angle BCQ = x°$.
The angles in ΔABC add to $180°$, so

$$40° + 2x° + 2x° = 180°$$
$$x = 35.$$

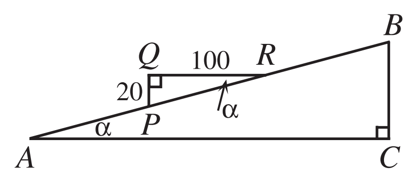
In ΔAPB, the angles add to $180°$, so

$$40° + 35° + \angle APB = 180°$$
$$\angle APB = 105°.$$

(c) In the diagram, $AB = 300$, $PQ = 20$, and $QR = 100$. Also, QR is parallel to AC. Determine the length of BC, to the nearest integer.

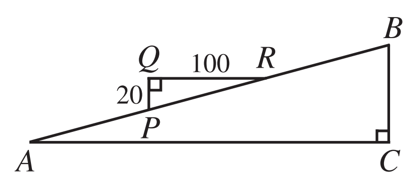


Solution 1



Since $QR \| AC$, $\angle QRP = \angle BAC = \alpha$ (alternating angles).
From ΔRPQ, $\tan \alpha = \frac{1}{5}$.
In ΔACB, since $\tan \alpha = \dfrac{1}{5} = \dfrac{BC}{AC}$, let $BC = x$ and $AC = 5x$. (This argument could also be made by just using the fact that ΔRQP and ΔACB are similar.)
By Pythagoras, $x^2 + 25x^2 = 300^2$, $x = \sqrt{\frac{90\,000}{25}} \doteq 58.83$.
Therefore $BC = 59$ m to the nearest metre.

Solution 2



Since $QR \| AC$, $\angle QRP = \angle BAC$ (alternating angles).
This means $\Delta ABC \sim \Delta RPQ$ (two equal angles).
By Pythagoras,

$$PR^2 = QP^2 + QR^2$$
$$PR = \sqrt{100^2 + 20^2} = \sqrt{10\,400}.$$

Since $\Delta ABC \sim \Delta RPQ$,

$$\frac{BC}{AB} = \frac{PQ}{RP}$$
$$BC = \frac{AB \cdot PQ}{RP}$$
$$= \frac{300 \cdot 20}{\sqrt{10\,400}}$$
$$\doteq 58.83$$

BC is 59 m (to the nearest metre).

3. (a) In an increasing sequence of numbers with an odd number of terms, the difference between any two consecutive terms is a constant d, and the middle term is 302. When the last 4 terms are removed from the sequence, the middle term of the resulting sequence is 296. What is the value of d?

Solution 1

Let the number of terms in the sequence be $2k+1$.
We label the terms $a_1, a_2, ..., a_{2k+1}$.
The middle term here is $a_{k+1} = 302$.
Since the difference between any two consecutive terms in this increasing sequence is d, $a_{m+1} - a_m = d$ for $m = 1, 2, ..., 2k$.
When the last 4 terms are removed, the last term is now a_{2k-3} so the middle term is then $a_{k-1} = 296$. (When four terms are removed from the end, the middle term shifts two terms to the left.)
Now $6 = a_{k+1} - a_{k-1} = (a_{k+1} - a_k) + (a_k - a_{k-1}) = d + d = 2d$.
Therefore $d = 3$.

Solution 2

If the last four terms are removed from the sequence this results in 302 shifting 2 terms to the left in the new sequence meaning that $302 - 296 = 2d$, $d = 3$.

(b) There are two increasing sequences of five consecutive integers, each of which have the property that the sum of the squares of the first three integers in the sequence equals the sum of the squares of the last two. Determine these two sequences.

Solution

Let n be the smallest integer in one of these sequences.
So we want to solve the equation $n^2 + (n+1)^2 + (n+2)^2 = (n+3)^2 + (n+4)^2$ (translating the given problem into an equation).
Thus $n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 = n^2 + 6n + 9 + n^2 + 8n + 16$

$$n^2 - 8n - 20 = 0$$
$$(n-10)(n+2) = 0.$$

So $n = 10$ or $n = -2$.
Therefore, the sequences are 10, 11, 12, 13, 14 and $-2, -1, 0, 1, 2$.

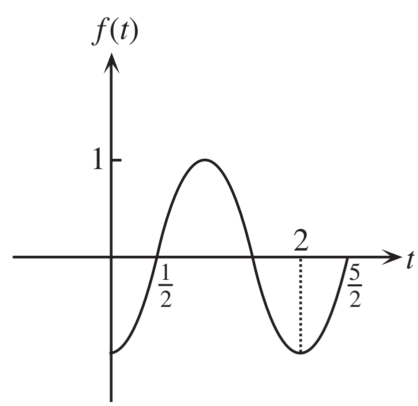
Verification
$(-2)^2 + (-1)^2 + 0^2 = 1^2 + 2^2 = 5$ and $10^2 + 11^2 + 12^2 = 13^2 + 14^2 = 365$

4. (a) If $f(t) = \sin\left(\pi t - \frac{\pi}{2}\right)$, what is the smallest positive value of t at which $f(t)$ attains its minimum value?

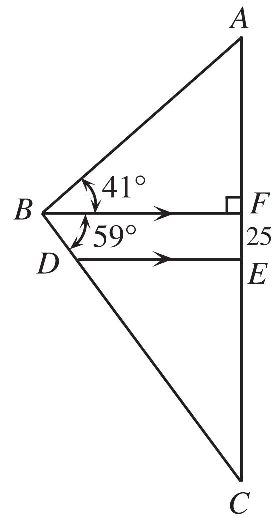
Solution 1
Since $t > 0$, $\pi t - \frac{\pi}{2} > -\frac{\pi}{2}$. So $\sin\left(\pi t - \frac{\pi}{2}\right)$ first attains its minimum value when

$$\pi t - \frac{\pi}{2} = \frac{3\pi}{2}$$
$$t = 2.$$

Solution 2
Rewriting $f(t)$ as, $f(t) = \sin\left[\pi\left(t - \frac{1}{2}\right)\right]$.
Thus $f(t)$ has a period $\frac{2\pi}{\pi} = 2$ and appears in the diagram at the right.
Thus $f(t)$ attains its minimum at $t = 2$. Note that $f(t)$ attains a minimum value at $t = 0$ but since $t > 0$, the required answer is $t = 2$.



(b) In the diagram, $\angle ABF = 41°$, $\angle CBF = 59°$, DE is parallel to BF, and $EF = 25$. If $AE = EC$, determine the length of AE, to 2 decimal places.



Solution

Let the length of $AE = EC$ be x.
Then $AF = x - 25$.

In, ΔBCF, $\dfrac{x+25}{BF} = \tan(59°)$.

In ΔABF, $\dfrac{x-25}{BF} = \tan(41°)$.

Solving for BF in these two equations and equating,

$$BF = \frac{x+25}{\tan 59°} = \frac{x-25}{\tan 41°}$$

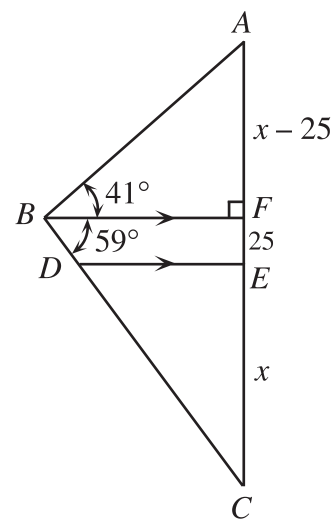
so $(\tan 41°)(x+25) = (\tan 59°)(x-25)$

$$25(\tan 59° + \tan 41°) = x(\tan 59° - \tan 41°)$$

$$x = \frac{25(\tan 59° + \tan 41°)}{\tan 59° - \tan 41°}$$

$$x \doteq 79.67.$$

Therefore the length of AE is 79.67.



5. (a) Determine all integer values of x such that $(x^2-3)(x^2+5) < 0$.

Solution

Since $x^2 \ge 0$ for all x, $x^2+5>0$. Since $(x^2-3)(x^2+5)<0$, $x^2-3<0$, so $x^2<3$ or $-\sqrt{3} < x < \sqrt{3}$. Thus $x = -1, 0, 1$.

(b) At present, the sum of the ages of a husband and wife, P, is six times the sum of the ages of their children, C. Two years ago, the sum of the ages of the husband and wife was ten times the sum of the ages of the same children. Six years from now, it will be three times the sum of the ages of the same children. Determine the number of children.

Solution

Let n be the number of children.

At the present, $P = 6C$, where P and C are as given. (1)

Two years ago, the sum of the ages of the husband and wife was $P - 4$, since they were each two years younger.

Similarly, the sum of the ages of the children was $C - n(2)$ (n is the number of children).

So two years ago, $P - 4 = 10(C - 2n)$ (2), from the given condition.

Similarly, six years from now, $P + 12 = 3(C + 6n)$ (3), from the given condition.

We want to solve for n.

Substituting (1) into each of (2) and (3),

$$6C - 4 = 10(C - 2n) \quad \text{or} \quad 20n - 4C = 4 \quad \text{or} \quad 5n - C = 1$$
$$6C + 12 = 3(C + 6n) \quad \text{or} \quad -18n + 3C = -12 \quad \text{or} \quad -6n + C = -4$$

Adding these two equations, $-n = -3$, so $n = 3$.

Therefore, there were three children.

6. (a) Four teams, A, B, C, and D, competed in a field hockey tournament. Three coaches predicted who would win the Gold, Silver and Bronze medals:

Medal	Gold	Silver	Bronze
Team			

- Coach 1 predicted Gold for A, Silver for B, and Bronze for C,
- Coach 2 predicted Gold for B, Silver for C, and Bronze for D,
- Coach 3 predicted Gold for C, Silver for A, and Bronze for D.

Each coach predicted exactly one medal winner correctly. Complete the table **in the answer booklet** to show which team won which medal.

Solution

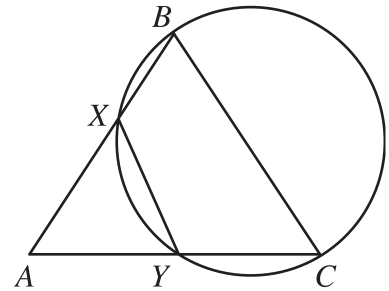
If A wins gold, then Coach 1 has one right. For Coach 3 to get one right, D must win bronze, since A cannot win silver. Since D wins bronze, Coach 2 gets one right. So C can't win silver, so B does which means Coach 1 has two right, which can't happen. So A doesn't win gold.

If B wins gold, then Coach 2 has one right. For Coach 1 to get one right, C wins bronze, as B can't win silver.

For Coach 3 to get one right, A wins silver.

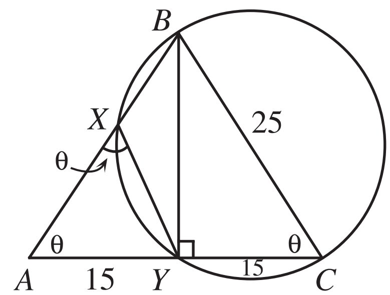
So Gold to B, Silver to A and Bronze to C satisfies the conditions.

(b) In triangle ABC, $AB = BC = 25$ and $AC = 30$. The circle with diameter BC intersects AB at X and AC at Y. Determine the length of XY.



Solution 1

Join BY. Since BC is a diameter, then $\angle BYC = 90°$.
Since $AB = BC$, ΔABC is isosceles and BY is an altitude in ΔABC, then $AY = YC = 15$.
Let $\angle BAC = \theta$.
Since ΔABC is isosceles, $\angle BCA = \theta$.
Since $BCYX$ is cyclic, $\angle BXY = 180 - \theta$ and so $\angle AXY = \theta$.
Thus ΔAXY is isosceles and so $XY = AY = 15$.
Therefore $XY = 15$.



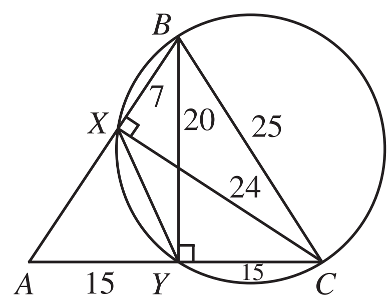
Solution 2

Join BY. $\angle BYC = 90°$ since it is inscribed in a semicircle.
ΔBAC is isosceles, so altitude BY bisects the base.
Therefore $BY = \sqrt{25^2 - 15^2} = 20$.
Join CX. $\angle CXB = 90°$ since it is also inscribed in a semicircle.
The area of ΔABC is

$$\tfrac{1}{2}(AC)(BY) = \tfrac{1}{2}(AB)(CX)$$

$$\cancel{\tfrac{1}{2}}(30)(20) = \cancel{\tfrac{1}{2}}(25)(CX)$$

$$CX = \tfrac{600}{25} = 24.$$



From ΔABY we conclude that $\cos \angle ABY = \dfrac{BY}{AB} = \dfrac{20}{25} = \dfrac{4}{5}$.
In ΔBXY, applying the Law of Cosines we get $(XY)^2 = (BX)^2 + (BY)^2 - 2(BX)(BY)\cos \angle XBY$.
Now (by Pythagoras ΔBXC),

$$\begin{aligned} BX^2 &= BC^2 - CX^2 \\ &= 25^2 - 24^2 \\ &= 49 \\ BX &= 7. \end{aligned}$$

Therefore $XY^2 = 7^2 + 20^2 - 2(7)(20)\frac{4}{5}$

$$\begin{aligned} &= 49 + 400 - 224 \\ &= 225. \end{aligned}$$

Therefore $XY = 15$.

7\. (a) What is the value of x such that $\log_2\left(\log_2(2x-2)\right) = 2$?

Solution

$$\begin{aligned} \log_2\left(\log_2(2x-2)\right) &= 2 \\ \log_2(2x-2) &= 2^2 \\ 2x-2 &= 2^{\left(2^2\right)} \\ 2x-2 &= 2^4 \\ 2x-2 &= 16 \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

(b) Let $f(x) = 2^{kx} + 9$, where k is a real number. If $f(3) : f(6) = 1 : 3$, determine the value of $f(9) - f(3)$.

Solution

From the given condition,

$$\begin{aligned} \frac{f(3)}{f(6)} = \frac{2^{3k}+9}{2^{6k}+9} &= \frac{1}{3} \\ 3\left(2^{3k}+9\right) &= 2^{6k}+9 \\ 0 &= 2^{6k} - 3\left(2^{3k}\right) - 18. \end{aligned}$$

We treat this as a quadratic equation in the variable $x = 2^{3k}$, so

$$\begin{aligned} 0 &= x^2 - 3x - 18 \\ 0 &= (x-6)(x+3). \end{aligned}$$

Therefore, $2^{3k} = 6$ or $2^{3k} = -3$. Since $2^a > 0$ for any a, then $2^{3k} \neq -3$.

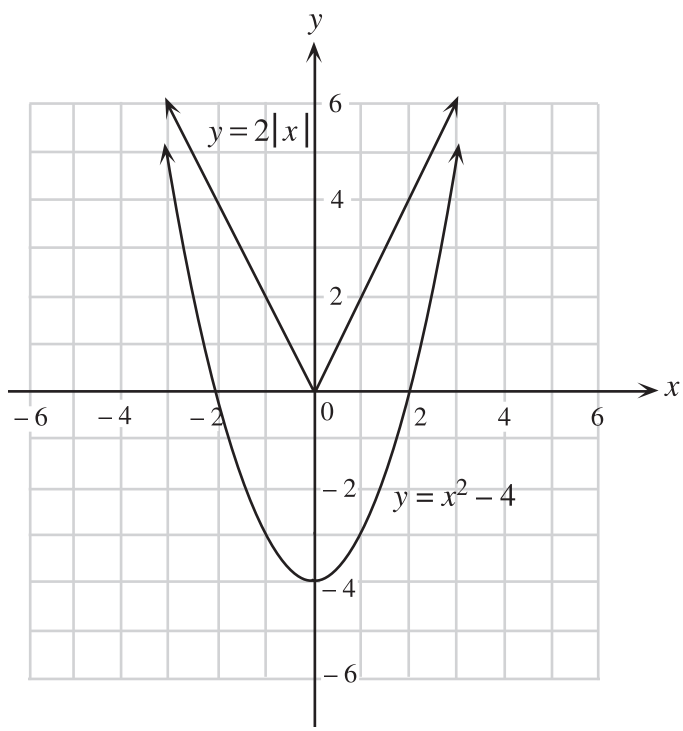
So $2^{3k} = 6$. We could solve for k here, but this is unnecessary.

$$\begin{aligned} \text{We calculate } f(9)-f(3) &= \left(2^{9k}+9\right)-\left(2^{3k}+9\right) \\ &= 2^{9k}-2^{3k} \\ &= \left(2^{3k}\right)^3-2^{3k} \\ &= 6^3-6 \\ &= 210. \end{aligned}$$

Therefore $f(9)-f(3)=210$.

8. (a) On the grid provided in the answer booklet, sketch $y=x^2-4$ and $y=2|x|$.

Solution



(b) Determine, with justification, all values of k for which $y=x^2-4$ and $y=2|x|+k$ do **not** intersect.

Solution

Since each of these two graphs is symmetric about the y-axis (i.e. both are even functions), then we only need to find k so that there are no points of intersection with $x \geq 0$.
So let $x \geq 0$ and consider the intersection between $y=2x+k$ and $y=x^2-4$.
Equating, we have, $2x+k=x^2-4$.
Rearranging, we want $x^2-2x-(k+4)=0$ to have no solutions.

For no solutions, the discriminant is negative, i.e.

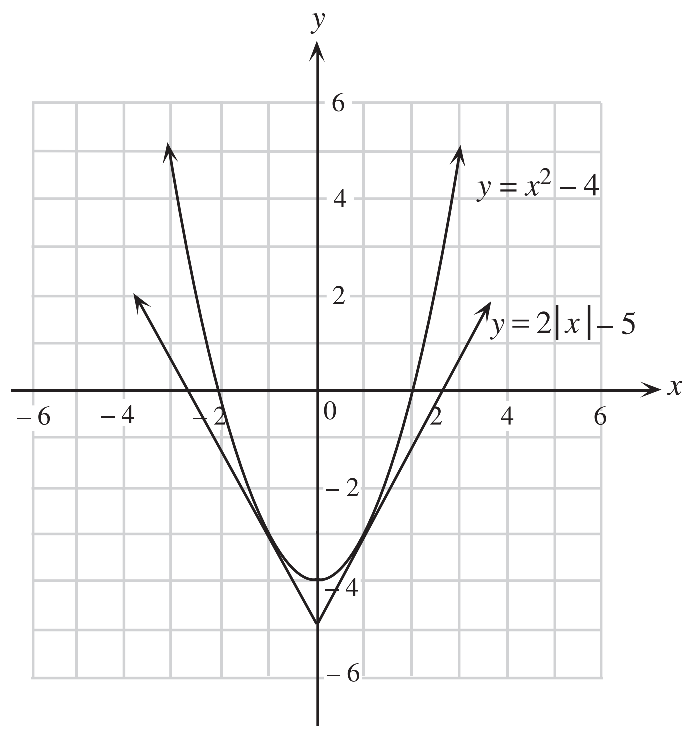
$$20 + 4k < 0$$
$$4k < -20$$
$$k < -5.$$

So $y = x^2 - 4$ and $y = 2|x| + k$ have no intersection points when $k < -5$.

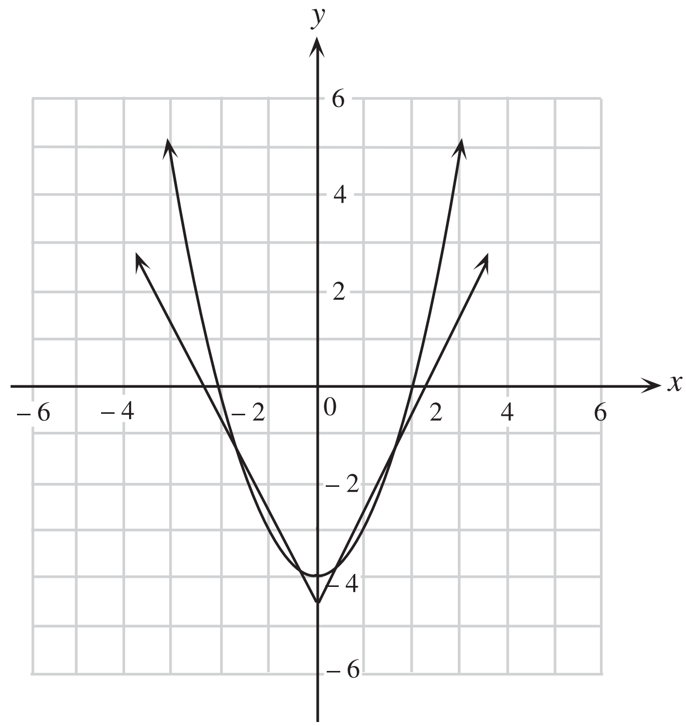
(c) State the values of k for which $y = x^2 - 4$ and $y = 2|x| + k$ intersect in exactly two points. (Justification is not required.)

Solution Analysing Graphs

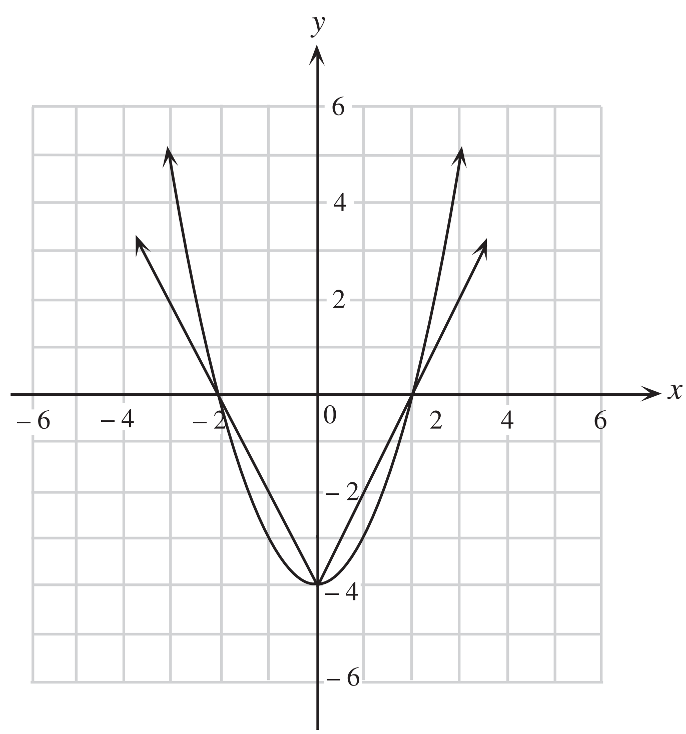
For $k < -5$, there are no points of intersection. When $k = -5$, the graph with equation $y = 2|x| + k$ is tangent to the graph with equation $y = x^2 - 4$ for both $x \geq 0$ and $x \leq 0$. So $k = -5$ is one possibility for two intersection points.



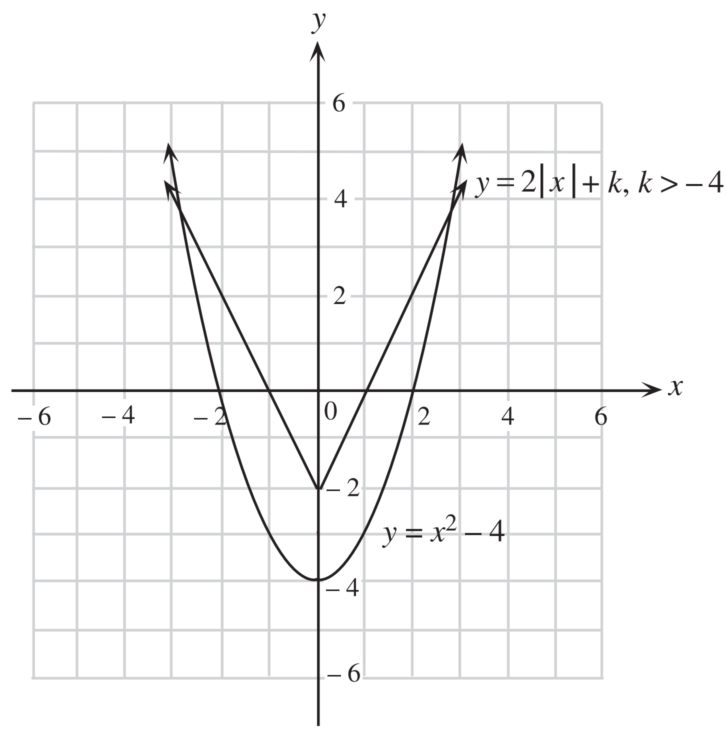
For $-5<k<-4$ a typical graph appears on the right.
i.e. for $-5<k<-4$, there will be 4 points of intersection.



When $k=-4$, a typical graph appears on the right.

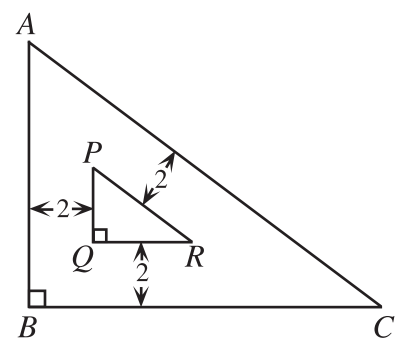


So when $k > -4$, there will only be two points of intersection, as the contact point at the cusp of $y = 2|x| - 4$ will be eliminated. An example where $k = -2$ is shown.



So the possibility for exactly two distinct points of intersection are $k = -5$, $k > -4$.

9. Triangle ABC is right-angled at B and has side lengths which are integers. A second triangle, PQR, is located inside ΔABC as shown, such that its sides are parallel to the sides of ΔABC and the distance between parallel lines is 2. Determine the side lengths of all possible triangles ABC, such that the area of ΔABC is 9 times that of ΔPQR.



Solution 1

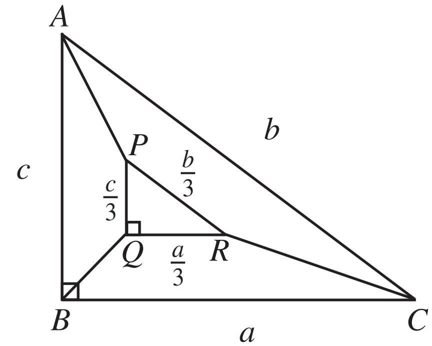
Let the sides of ΔABC be $AB = c$, $BC = a$, $AC = b$, a, b, c are all integers.

Since the sides of ΔPQR are all parallel to the sides of ΔABC, then ΔABC is similar to ΔPQR.

Now the ratio of areas of ΔABC to ΔPQR is $9 = 3^2$ to 1, so the ratio of side lengths will be 3 to 1.

So the sides of ΔPQR are $PQ = \frac{c}{3}$, $QR = \frac{a}{3}$, $PR = \frac{b}{3}$.

So we can label the diagram as indicated.
We join the corresponding vertices of the two triangles as



Area of trapezoid $BQRC$
Area of trapezoid $CRPA$
Area of trapezoid $APQB$
+ Area of ΔPQR

Area of ΔABC.

Doing so gives,

$$2\left(\frac{2}{3}a\right)+2\left(\frac{2}{3}b\right)+2\left(\frac{2}{3}c\right)+\frac{ac}{18}=\frac{ac}{72}$$

Or upon simplifying $ac=3a+3b+3c$ (Note that this relationship can be derived in a variety of ways.)

$$\begin{aligned} ac &= 3c+3b+3a \\ ac-3c-3a &= 3b \\ ac-3c-3a &= 3\sqrt{a^2+c^2} && \text{(since } b=\sqrt{a^2+c^2}\text{)} \\ a^2c^2+9c^2+9a^2-6ac^2-6a^2c+18ac &= 9\left(a^2+c^2\right) && \text{(squaring both sides)} \\ ac(ac-6c-6a+18) &= 0 \\ ac-6c-6a+18 &= 0 && \text{(as } ac\neq 0\text{)} \\ c(a-6) &= 6a-18 \\ c &= \frac{6a-18}{a-6} \\ c &= 6+\frac{18}{a-6}. \end{aligned}$$

Since a is a side of a triangle, $a>0$. We are now looking for positive integer values such that $\frac{18}{a-6}$ is also an integer.

The only possible values for a are 3, 7, 8, 9, 12, 15 and 24.

Tabulating the possibilities and calculating values for b and c gives,

a	3	7	8	9	12	15	24
c	0	24	15	12	9	8	7
b	–	25	17	15	15	17	25

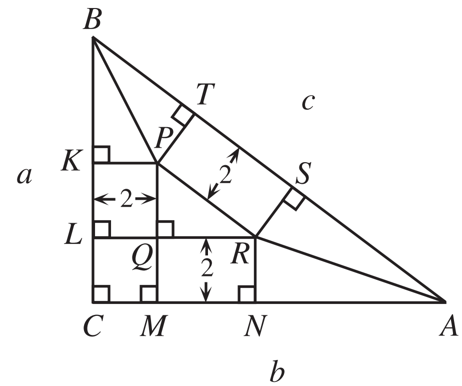
Thus the only possibilities for the triangle are $(7,24,25)$, $(8,15,7)$ and $(9,12,15)$.

Solution 2

The two triangles are similar with areas in the ratio 1:9.

Therefore the sides are in the ratio 1:3.

Let $a=BC$, $b=CA$, $c=BA$.



Then $\frac{a}{3} = PQ$, $\frac{b}{3} = QR$, $\frac{c}{3} = PR$.

Locate points K, L on BC; M, N on CA; and T, S on AB as shown.

$$BC = BK + KL + LC$$

$$a = BK + \tfrac{a}{3} + 2$$

Therefore $BK = \frac{2}{3}a - 2$.

In a similar way, $AN = \frac{2}{3}b - 2$.

Now $\Delta BKP \cong \Delta BTP$ and $\Delta ANR \cong \Delta ASR$, both by HL.

Therefore $BT = BK = \frac{2}{3}a - 2$ and $AS = AN = \frac{2}{3}b - 2$.

Now, $AB = AS + ST + BT$

$$c = \tfrac{2}{3}b - 2 + \tfrac{c}{3} + \tfrac{2}{3}a - 2$$

$$\tfrac{2}{3}c = \tfrac{2}{3}b + \tfrac{2}{3}a - 4$$

$$c = b + a - 6$$

$$b = c + (6 - a).$$

By Pythagoras, $a^2 + b^2 = c^2$

$$a^2 + [c + (6 - a)]2 = c^2$$

$$a^2 + \cancel{c^2} + 2c(6 - a) + (6 - a)2 = \cancel{c^2}$$

$$a^2 + (6 - a)^2 = -2c(6 - a)$$

$$2a^2 - 12a + 36 = 2c(a - 6)$$

$$a^2 - 6a + 18 = c(a - 6)$$

$$c = \frac{a^2 - 6a + 18}{a - 6}$$

$$c = \frac{a(a - 6) + 18}{a - 6}$$

$$c = a + \frac{18}{a - 6}.$$

Since a and c are integers, $a - 6$ is a divisor of 18.

Also since $b < c$ and $b = c + (6 - a)$, we conclude that $6 - a < 0$ so $a - 6 > 0$.

Thus $a - 6$ can be 1, 2, 3, 6, 9, 18.

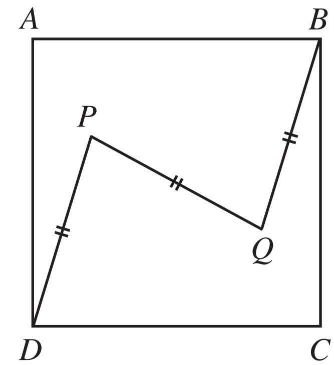
The values of a are: 7, 8, 9, 12, 15, 24.

Matching values for c: 25, 17, 15, 15, 17, 25

Matching values for b: 24, 15, 12, 9, 8, 7

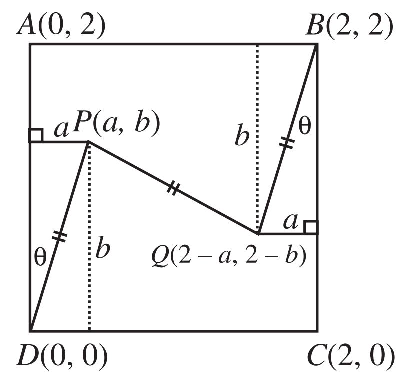
The distinct triangles are (7, 24, 25), (8, 15, 17) and (9, 12, 15).

10. Points P and Q are located inside the square $ABCD$ such that DP is parallel to QB and $DP = QB = PQ$. Determine the minimum possible value of $\angle ADP$.



Solution 1

Placing the information on the coordinate axes, the diagram is indicated to the right.



We note that P has coordinates (a, b).

By symmetry (or congruency) we can label lengths a and b as shown. Thus Q has coordinates $(2-a, 2-b)$.

Since $PD = PQ$, $a^2 + b^2 = (2-2a)^2 + (2-2b)^2$

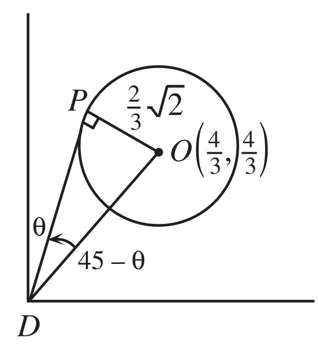
or, $\quad 3a^2 + 3b^2 - 8a - 8b + 8 = 0$

$$\left(a - \tfrac{4}{3}\right)^2 + \left(b - \tfrac{4}{3}\right)^2 = \tfrac{8}{9}$$

P is on a circle with centre $O\left(\frac{4}{3}, \frac{4}{3}\right)$ with $r = \frac{2}{3}\sqrt{2}$.

The minimum angle for θ occurs when DP is tangent to the circle.

So we have the diagram noted to the right.



Since OD makes an angle of $45°$ with the x-axis then $\angle PDO = 45 - \theta$ and $OD = \frac{4}{3}\sqrt{2}$.

Therefore $\sin(45 - \theta) = \dfrac{\frac{2}{3}\sqrt{2}}{\frac{4}{3}\sqrt{2}} = \dfrac{1}{2}$ which means $45° - \theta = 30°$ or $\theta = 15°$.

Thus the minimum value for θ is $15°$.

Solution 2

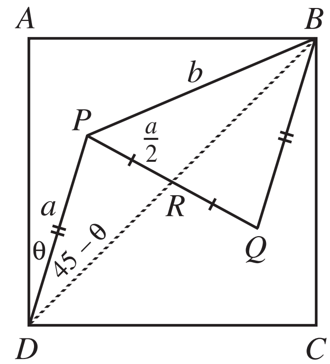
Let $AB = BC = CD = DA = 1$.

Join D to B. Let $\angle ADP = \theta$. Therefore, $\angle PDB = 45 - \theta$.

Let $PD = a$ and $PB = b$ and $PQ = \dfrac{a}{2}$.

We now establish a relationship between a and b.
In ΔPDB, $b^2 = a^2 + 2 - 2(a)\left(\sqrt{2}\right)\cos(45-\theta)$

or, $\cos(45-\theta) = \dfrac{a^2 - b^2 + 2}{2\sqrt{2}a}$ (1)



In ΔPDR, $\left(\dfrac{a}{2}\right)^2 = a^2 + \left(\dfrac{\sqrt{2}}{2}\right)^2 - 2a\dfrac{\sqrt{2}}{2}\cos(45-\theta)$

or, $\cos(45-\theta) = \dfrac{\frac{3}{4}a^2 + \frac{1}{2}}{a\sqrt{2}}$ (2)

Comparing (1) and (2) gives, $\dfrac{a^2 - b^2 + 2}{2\sqrt{2}a} = \dfrac{\frac{3}{4}a^2 + \frac{1}{2}}{a\sqrt{2}}$.

Simplifying this, $a^2 + 2b^2 = 2$

or, $b^2 = \dfrac{2-a^2}{2}$.

Now $\cos(45-\theta) = \dfrac{a^2 + 2 - \left(\dfrac{2-a^2}{2}\right)}{2a\sqrt{2}} = \dfrac{1}{4\sqrt{2}}\left(3a + \dfrac{2}{a}\right)$.

Now considering $3a + \dfrac{2}{a}$, we know $\left(\sqrt{3a} - \sqrt{\dfrac{2}{a}}\right)^2 \geq 0$

or, $3a + \dfrac{2}{a} \geq 2\sqrt{6}$.

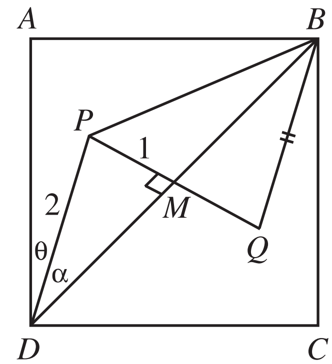
Thus, $\cos(45-\theta) \geq \dfrac{1}{4\sqrt{2}}\left(2\sqrt{6}\right) = \dfrac{\sqrt{3}}{2}$

$\cos(45-\theta) \geq \dfrac{\sqrt{3}}{2}$.

$\cos(45-\theta)$ has a minimum value for $45° - \theta = 30°$ or $\theta = 15°$.

Solution 3

Join BD. Let BD meet PQ at M. Let $\angle ADP = \theta$.
By interior alternate angles, $\angle P = \angle Q$ and $\angle PDM = \angle QBM$.
Thus $\Delta PDM \cong \Delta QBM$ by A.S.A., so $PM = QM$ and $DM = BM$.
So M is the midpoint of BD and the centre of the square.



Without loss of generality, let $PM = 1$. Then $PD = 2$.
Since $\theta + \alpha = 45°$ (see diagram), θ will be minimized when α is maximized.

Consider ΔPMD.

Using the sine law, $\dfrac{\sin\alpha}{1} = \dfrac{\sin(\angle PMD)}{2}$.

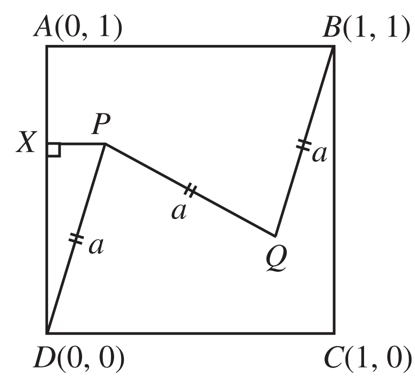
To maximize α, we maximize $\sin\alpha$.

But $\sin\alpha = \dfrac{\sin(\angle PMD)}{2}$, so it is maximized when $\sin(\angle PMD) = 1$.

In this case, $\sin\alpha = \frac{1}{2}$, so $\alpha = 30°$.

Therefore, $\theta = 45° - 30° = 15°$, and so the minimum value of θ is 15°.

Solution 4



We place the diagram on a coordinate grid, with $D(0,0)$, $C(1,0)$, $B(0,1)$, $A(1,1)$.
Let $PD = PQ = QB = a$, and $\angle ADP = \theta$.
Drop a perpendicular from P to AD, meeting AD at X.
Then $PX = a\sin\theta$, $DX = a\cos\theta$.
Therefore the coordinates of P are $(a\sin\theta, a\cos\theta)$.
Since $PD \| BQ$, then $\angle QBC = \theta$.
So by a similar argument (or by using the fact that PQ are symmetric through the centre of the square), the coordinates of Q are $(1 - a\sin\theta, 1 + a\cos\theta)$.

Now $(PQ)^2 = a^2$, so $(1 - 2a\sin\theta)^2 + (1 - 2a\cos\theta)^2 = a^2$

$$2 + 4a^2\sin^2\theta + 4a^2\cos^2\theta - 4a(\sin\theta + \cos\theta) = a^2$$

$$2 + 4a^2 - a^2 = 4a(\sin\theta + \cos\theta)$$

$$\frac{2 + 3a^2}{4a} = \sin\theta + \cos\theta$$

$$\frac{2 + 3a^2}{4\sqrt{2}a} = \frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta = \cos(45°)\sin\theta + \sin(45°)\cos\theta$$

$$\frac{2 + 3a^2}{4\sqrt{2}a} = \sin(\theta + 45°)$$

Now $\left(a - \sqrt{\tfrac{2}{3}}\right)^2 \geq 0$

$$a^2 - 2a\sqrt{\tfrac{2}{3}} + \tfrac{2}{3} \geq 0$$

$$3a^2 - 2a\sqrt{6} + 2 \geq 0$$

$$3a^2 + 2 \geq 2a\sqrt{6}$$

$$\frac{3a^2 + 2}{4\sqrt{2}a} \geq \frac{\sqrt{3}}{2}$$

and equality occurs when $a = \sqrt{\tfrac{2}{3}}$.

So $\sin(\theta + 45°) \geq \frac{\sqrt{3}}{2}$ and thus since $0° \leq \theta \leq 90°$, then $\theta + 45° \geq 60°$ or $\theta \geq 15°$.

Therefore the minimum possible value of $\angle ADP$ is $15°$.

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
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2000 Solutions

Euclid Contest

(Grade12)

for

The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

Awards

1. (a) If $x+27^{\frac{1}{3}}=125^{\frac{1}{3}}$, what is the value of x?

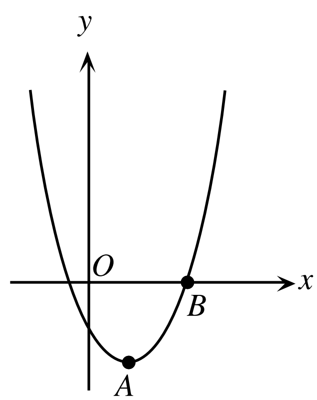
Solution
$125^{\frac{1}{3}}=5$, $27^{\frac{1}{3}}=3$
Therefore, $x=5-3=2$.

(b) The line $y=ax+c$ is parallel to the line $y=2x$ and passes through the point $(1,5)$. What is the value of c?

Solution
Since the two given lines are parallel, the line $y=ax+c$ has slope 2 and is of the form, $y=2x+c$. Since $(1,5)$ is on the line, $5=2(1)+c$

$$c=3.$$

(c) The parabola with equation $y=(x-2)^2-16$ has its vertex at A and intersects the x-axis at B, as shown. Determine the equation for the line passing through A and B.

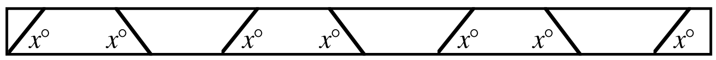


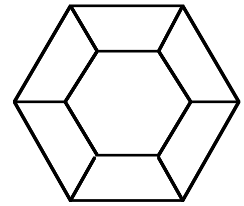
Solution
For $y=0$, $(x-2)^2-16=0$

$$[(x-2)-4][(x-2)+4]=0$$

Therefore $x=6$ or $x=-2$.
Thus, the x-intercepts of the parabola are -2 and 6, and B has coordinates $(6,0)$.
The vertex of the parabola is at $A(2,-16)$.
Equation of line containing $(6,0)$ and $(2,-16)$ has slope $\dfrac{-16}{2-6}=4$.
Thus the line has equation, $\dfrac{y-0}{x-6}=4 \Leftrightarrow y=4x-24$.

2. (a) Six identical pieces are cut from a board, as shown in the diagram. The angle of each cut is $x°$. The pieces are assembled to form a hexagonal picture frame as shown. What is the value of x?

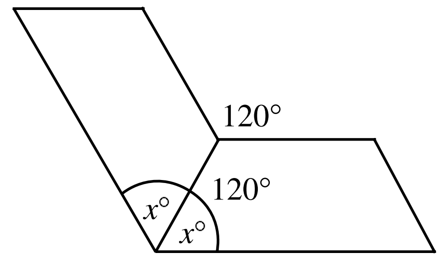




Solution

Each interior angle of a regular hexagon is $120°$.
Putting the frame together we would have the following

$$2x = 120 \text{ (in degrees)}$$
$$x = 60°$$



(b) If $\log_{10} x = 3 + \log_{10} y$, what is the value of $\dfrac{x}{y}$?

Solution

$\log_{10} x - \log_{10} y = 3$

$\Leftrightarrow \log_{10}\left(\dfrac{x}{y}\right) = 3$

$\Leftrightarrow \dfrac{x}{y} = 10^3 = 1000$

(c) If $x + \dfrac{1}{x} = \dfrac{13}{6}$, determine all values of $x^2 + \dfrac{1}{x^2}$.

Solution 1 'Squaring both sides'

$\left(x + \dfrac{1}{x}\right)^2 = \left(\dfrac{13}{6}\right)^2$; squaring

$x^2 + 2 + \dfrac{1}{x^2} = \dfrac{169}{36}$

$x^2 + \dfrac{1}{x^2} = \dfrac{169}{32} - 2$

$x^2 + \dfrac{1}{x^2} = \dfrac{169}{36} - \dfrac{72}{36} = \dfrac{97}{36}$

Solution 2 'Creating a quadratic equation and solving'

$6x\left(x + \dfrac{1}{x}\right) = 6x\left(\dfrac{13}{6}\right)$

$6x^2 + 6 = 13x$

$6x^2 - 13x + 6 = 0$

$(3x-2)(2x-3) = 0$

$x = \frac{2}{3}$ ***or*** $x = \frac{3}{2}$

For $x = \frac{2}{3}$, $x^2 + \frac{1}{x^2}$

$= \left(\frac{2}{3}\right)^2 + \frac{1}{\left(\frac{2}{3}\right)^2}$

$= \frac{4}{9} + \frac{9}{4}$

$= \frac{81+16}{36}$

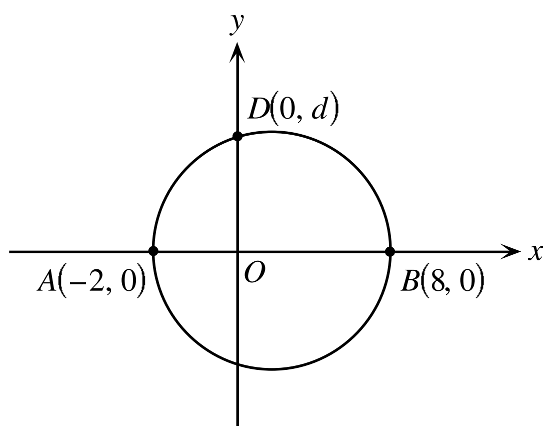
$= \frac{97}{36}$

For $x = \frac{3}{2}$, $\left(\frac{3}{2}\right)^2 + \frac{1}{\left(\frac{3}{2}\right)^2}$

$= \frac{9}{4} + \frac{4}{9}$

$= \frac{97}{36}$

3. (a) A circle, with diameter AB as shown, intersects the positive y-axis at point $D(0, d)$. Determine d.



Solution 1

The centre of the circle is $(3, 0)$ and the circle has a radius of 5.

Thus $\sqrt{d^2 + 3^2} = 5$

$d^2 = 5^2 - 3^2$

$d^2 = 16$

Therefore $d = 4$, since $d > 0$.

Solution 2

Since AB is a diameter of the circle, $\angle ADB = 90°$ and $\angle AOD = 90°$.

$\Delta ADO \sim \Delta DBO$

Therefore, $\frac{OD}{AO} = \frac{BO}{OD}$

and $d^2 = 2(8)$

$d^2 = 16$

$d = 4$, since $d > 0$.

Solution 3

$\angle ADB = \angle AOD = \angle BOD = 90°$

In ΔAOD, $AD^2 = 4 + d^2$.

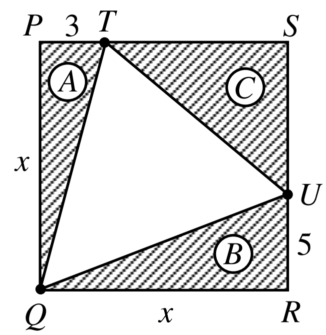
In ΔBOD, $DB^2 = 64 + d^2$.

In ΔADB, $\left(4+d^2\right)+\left(64+d^2\right)=100$

$$2d^2 = 32$$

$$d = 4,\ d > 0.$$

(b) A square $PQRS$ with side of length x is subdivided into four triangular regions as shown so that area Ⓐ + area Ⓑ = area Ⓒ. If $PT = 3$ and $RU = 5$, determine the value of x.



Solution

Since the side length of the square is x, $TS = x - 3$ and $VS = x - 5$

Area of triangle $A = \frac{1}{2}(3)(x)$.

Area of triangle $B = \frac{1}{2}(5)(x)$

Area of triangle $C = \frac{1}{2}(x-5)(x-3)$.

From the given information, $\frac{1}{2}(3x) + \frac{1}{2}(5x) = \frac{1}{2}(x-5)(x-3)$.

$3x + 5x = x^2 - 8x + 15$

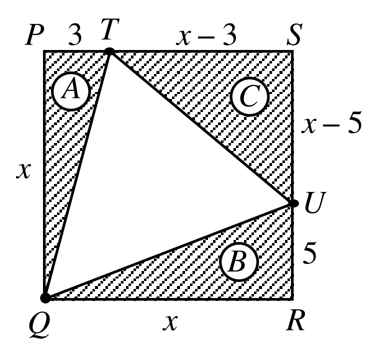
$x^2 - 16x + 15 = 0$

$(x-15)(x-1) = 0$

Thus $x = 15$ ***or*** $x = 1$.

Therefore $x = 15$ since $x = 1$ is inadmissible.

Labelled diagram



4. (a) A die, with the numbers 1, 2, 3, 4, 6, and 8 on its six faces, is rolled. After this roll, if an odd number appears on the top face, all odd numbers on the die are doubled. If an even number appears on the top face, all the even numbers are halved. If the given die changes in this way, what is the probability that a 2 will appear on the second roll of the die?

Solution

There are only two possibilities on the first roll - it can either be even or odd.

Possibility 1 'The first roll is odd'

The probability of an odd outcome on the first roll is $\frac{1}{3}$.

After doubling all the numbers, the possible outcomes on the second roll would now be 2, 2, 6, 4, 6, 8 with the probability of a 2 being $\frac{1}{3}$.

Thus the probability of a 2 on the second roll would be $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

Possibility 2 'The first is even'

The probability of an even outcome on the first roll is $\frac{2}{3}$.

After halving all the numbers, the possible outcomes on the second roll would be 1, 1, 3, 2, 3, 8.

The probability of a 2 on the second die would now be $\frac{1}{6}$.

Thus the probability of a 2 on the second roll is $\frac{2}{3} \times \frac{1}{6} = \frac{1}{9}$.

The probability of a 2 appear on the top face is $\frac{1}{9} + \frac{1}{9} = \frac{2}{9}$.

b) The table below gives the final standings for seven of the teams in the English Cricket League in 1998. At the end of the year, each team had played 17 matches and had obtained the total number of points shown in the last column. Each win W, each draw D, each bonus bowling point A, and each bonus batting point B received w, d, a and b points respectively, where w, d, a and b are positive integers. *No points* are given for a loss. Determine the values of w, d, a and b if total points awarded are given by the formula: Points $= w \times W + d \times D + a \times A + b \times B$.

Final Standings

	W	*Losses*	*D*	*A*	*B*	*Points*
Sussex	6	7	4	30	63	201
Warks	6	8	3	35	60	200
Som	6	7	4	30	54	192
Derbys	6	7	4	28	55	191
Kent	5	5	7	18	59	178
Worcs	4	6	7	32	59	176
Glam	4	6	7	36	55	176

Solution

There are a variety of ways to find the unknowns.

The most efficient way is to choose equations that have like coefficients. Here is one way to solve the problem using this method.

For Sussex: $6w+4d+30a+63b=201$

For Som: $6w+4d+30a+54b=192$

Subtracting, $9b=9 \quad b=1$

If b = 1

For Derbys: $6w+4d+28a+55=191$

$$6w+4d+28a=136 \qquad (1)$$

For Sussex: $6w+4d+30a+63=201$

$$6w+4d+30a=138 \qquad (2)$$

Subtracting, (2) – (1) $\quad 2a=2$

$$a=1.$$

We can now calculate d and w by substituting $a=1$, $b=1$ into a pair of equations.

An efficient way of doing this is by substituting $a=1$, $b=1$ into Som and Worcs.

For Som: $6w+4d+84=192$

$$6w+4d=108 \qquad (3)$$

For Worcs: $6w+3d+85=200$

$$6w+3d=105 \qquad (4)$$

Subtracting, (3) – (4) $\quad d=3$.

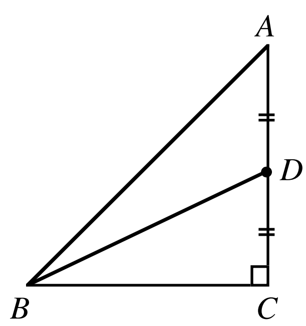
Substituting $d=3$ in either (3) or (4), $6w+4(3)=108$ (substituting in (3))

$$6w=96$$

$$w=16.$$

Therefore $w=16$, $d=3$, $a=b=1$.

5. (a) In the diagram, $AD=DC$, $\sin\angle DBC=0.6$ and $\angle ACB=90°$. What is the value of $\tan\angle ABC$?



Solution

Let $DB=10$.

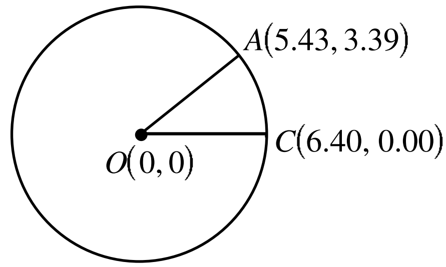
Therefore, $DC=AD=6$.

By the theorem of Pythagoras, $BC^2=10^2-6^2=64$.

Therefore, $BC=8$.

Thus, $\tan \angle ABC = \frac{12}{8} = \frac{3}{2}$.

(b) On a cross-sectional diagram of the Earth, the x and y-axes are placed so that $O(0, 0)$ is the centre of the Earth and $C(6.40, 0.00)$ is the location of Cape Canaveral. A space shuttle is forced to land on an island at $A(5.43, 3.39)$, as shown. Each unit represents 1 000 km. Determine the distance from Cape Canaveral to the island, measured on the surface of the earth, to the nearest 10 km.



Solution

Calculating $\angle AOC$

Calculating arc length

Distance

Solution

$\tan \angle AOC = \frac{3.39}{5.43}$

$\angle AOC = \tan^{-1}\left(\frac{3.39}{5.43}\right) = 31.97°$

The arc length $\widehat{AC} = \frac{31.97}{360°}[(2\pi)(6.40)] = 3.57$ units

The distance is approximately 3570 km.

6. (a) Let $\lfloor x \rfloor$ represent the greatest integer which is less than or equal to x. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 2.6 \rfloor = 2$. If x is positive and $x\lfloor x \rfloor = 17$, what is the value of x?

Solution

We deduce that $4 < x < 5$.

Otherwise, if $x \le 4$, $x\lfloor x \rfloor \le 16$, and if $x \ge 5$, $x\lfloor x \rfloor \ge 25$.

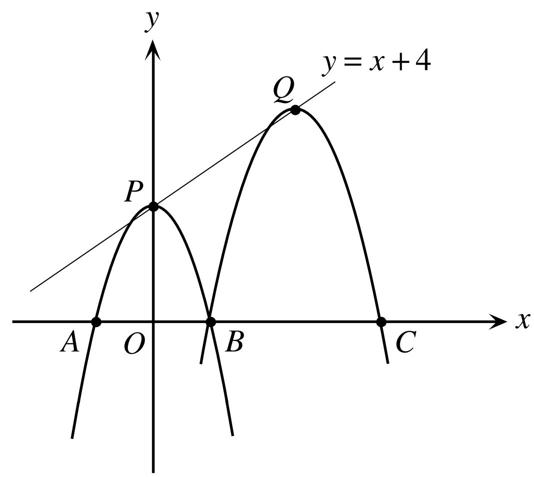
Therefore $\lfloor x \rfloor = 4$

Since $x\lfloor x \rfloor = 17$

$$4x = 17$$

$$x = 4.25$$

(b) The parabola $y = -x^2 + 4$ has vertex P and intersects the x-axis at A and B. The parabola is translated from its original position so that its vertex moves along the line $y = x + 4$ to the point Q. In this position, the parabola intersects the x-axis at B and C. Determine the coordinates of C.



Solution 1

The parabola $y = -x^2 + 4$ has vertex $P(0, 4)$ and intersects the x-axis at $A(-2, 0)$ and $B(2, 0)$. The intercept $B(2, 0)$ has its pre-image, B' on the parabola $y = -x^2 + 4$. To find B', we find the point of intersection of the line passing through $B(2, 0)$, with slope 1, and the parabola $y = -x^2 + 4$.

The equation of the line is $y = x - 2$.

Intersection points, $x - 2 = -x^2 + 4$

$$x^2 + x - 6 = 0$$
$$(x + 3)(x - 2) = 0.$$

Therefore, $x = -3$ or $x = 2$.

For $x = -3$, $y = -3 - 2 = -5$. Thus B' has coordinates $(-3, -5)$.

If $(-3, -5) \to (2, 0)$ then the required general translation mapping $y = -x^2 + 4$ onto the parabola with vertex Q is $(x, y) \to (x + 5, y + 5)$.

Possibility 1

Using the general translation, we find the coordinates of Q to be,
$P(0, 4) \to Q(0 + 5, 4 + 5) = Q(5, 9)$.

If C is the reflection of B in the axis of symmetry of the parabola, i.e. $x = 5$, C has coordinates $(8, 0)$.

Possibility 2

If B' has coordinates $(-3, -5)$ then C' is the reflection of B' in the y-axis. Thus C' has coordinates $(3, -5)$.

If we apply the general translation then C has coordinates $(3 + 5, -5 + 5)$ or $(8, 0)$.

Thus C has coordinates $(8, 0)$.

Possibility 3

Using the general translation, we find the coordinates of Q to be,
$P(0, 4) \to Q(0 + 5, 4 + 5) = Q(5, 9)$.

The equation of the image parabola is $y = -(x - 5)^2 + 9$.

To find its intercepts, $-(x-5)^2+9=0$

$$(x-5)^2=9$$

$$x-5=\pm 3.$$

Therefore $x=8$ or $x=2$.
Thus C has coordinates $(8,0)$.

Solution 2

The translation moving the parabola with equation $y=-x^2+4$ onto the parabola with vertex Q is $T(t,t)$ because the slope of the line $y=x+4$ is 1.
The pre-image of B' is $(2-t,-t)$.
Since B' is on the parabola with vertex P, we have

$$-t=-(2-t)^2+4$$

$$-t=-4+4t-t^2+4$$

$$t^2-5t=0$$

$$t(t-5)=0.$$

Therefore, $t=0$ or $t=5$.
Thus B' is $(-3,-5)$.
Let C have coordinates $(c,0)$.
The pre-image of C is $(c-5,-5)$.
Therefore, $-5=-(c-5)^2+4$.
Or, $(c-5)^2=9$.
Therefore $c-5=3$ or $c-5=-3$.

$$c=8 \text{ or } c=2$$

Thus C has coordinates $(8,0)$.

Solution 3

The translation moving the parabola with equation $y=-x^2+4$ onto the parabola with vertex Q is $T(p,p)$ because the slope of the line $y=x+4$ is 1.
Q will have coordinates $(p,p+4)$.
Thus the equation of the image parabola is $y=-(x-p)^2+p+4$.
Since $(2,0)$ is on the parabola,

$$0=-(2-p)^2+p+4$$

$$p^2-5p=0$$

$$p(p-5)=0.$$

Therefore $p=0$ or $p=5$.
The coordinates of Q are $(5,9)$.

As in solution 1, we can use either reflection properties or the equation of the parabola to find that C has coordinates $(8,0)$.

7. (a) A cube has edges of length n, where n is an integer. Three faces, meeting at a corner, are painted red. The cube is then cut into n^3 smaller cubes of unit length. If exactly 125 of these cubes have no faces painted red, determine the value of n.

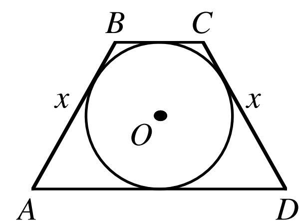
Solution

If we remove the cubes which have red paint, we are left with a smaller cube with measurements, $(n-1)\times(n-1)\times(n-1)$

Thus, $(n-1)^3 = 125$

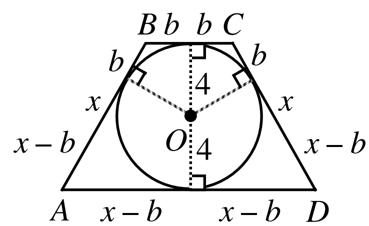
$$n = 6.$$

(b) In the isosceles trapezoid $ABCD$, $AB = CD = x$. The area of the trapezoid is 80 and the circle with centre O and radius 4 is tangent to the four sides of the trapezoid. Determine the value of x.



Solution

Using the tangent properties of a circle, the lengths of line segments are as shown on the diagram.



Area of trapezoid $ABCD = \frac{1}{2}(8)(BC + AD)$

$$= 4(2b + 2x - 2b)$$

$$= 8x.$$

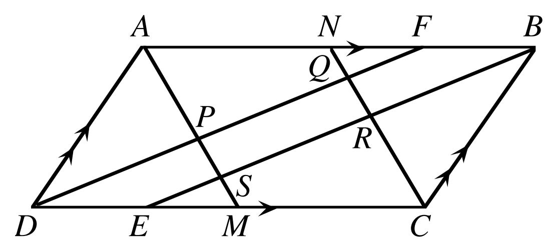
Thus, $8x = 80$.

Therefore, $x = 10$.

8. In parallelogram $ABCD$, $AB = a$ and $BC = b$, where $a > b$. The points of intersection of the angle bisectors are the vertices of quadrilateral $PQRS$.

(a) Prove that $PQRS$ is a rectangle.

(b) Prove that $PR = a - b$.



Solution

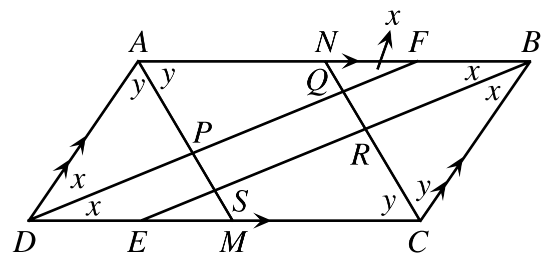
(a) In a parallelogram opposite angles are equal.

Since DF and BE bisect the two angles, let $\angle ADF = \angle CDF = \angle ABE = \angle CBE$
$= x$ (in degrees)

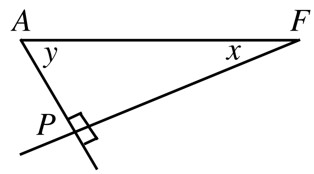
Also $\angle CDF = \angle AFD = x$ (alternate angles)

Let $\angle DAM = \angle BAM = \angle DCN = \angle BCN = y$ (in degrees)

For any parallelogram, any two consecutive angles add to $180°$, $\therefore 2x+2y=180$
or, $x+y=90$.



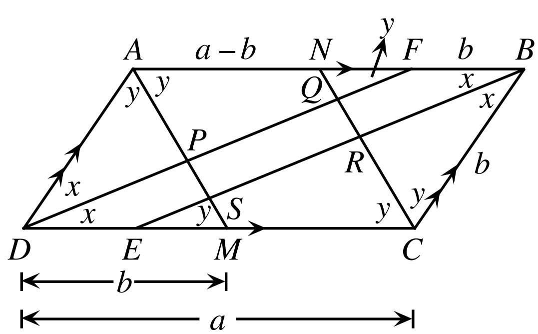
Therefore in ΔPAF, $\angle APF = 90°$.



Using similar reasoning and properties of parallel lines we get right angles at Q, R and S.
Thus $PQRS$ is a rectangle.

Solution

(b) Since AM is a bisector of $\angle DAB$, let $\angle DAM = \angle BAM = y$.
Also, $\angle DMA = y$ (alternate angles)
This implies that ΔADM is isosceles.
Using the same reasoning in ΔCBN, we see that it is also isosceles and so the diagram may now be labelled as:



$AN = a - b$

Thus ΔADM and ΔCBN are identical isosceles triangles.
Also, $AM \parallel NC$ (corresponding angles)
or, $AP \parallel NR$.
By using properties of isosceles triangles (or congruency), $AP = NR$ implying that $APRN$ is a parallelogram.
Thus $AN = PR$ and since $AN = a - b$, $PR = a - b$ (as required)

9. A permutation of the integers 1, 2, ..., n is a listing of these integers in some order. For example, $(3, 1, 2)$ and $(2, 1, 3)$ are two different permutations of the integers 1, 2, 3. A permutation $(a_1, a_2, ..., a_n)$ of the integers 1, 2, ..., n is said to be "fantastic" if $a_1 + a_2 + ... + a_k$ is divisible by k, for ***each*** k from 1 to n. For example, $(3, 1, 2)$ is a fantastic permutation of 1, 2, 3 because 3 is divisible by 1, $3+1$ is divisible by 2, and $3+1+2$ is divisible by 3. However, $(2, 1, 3)$ is not fantastic because $2+1$ is not divisible by 2.

(a) Show that no fantastic permutation exists for $n = 2000$.

(b) Does a fantastic permutation exist for $n = 2001$? Explain.

Solution

(a) In our consideration of whether there is a fantastic permutation for $n = 2000$, we start by looking at the 2000th position.
Using our definition of fantastic permutation, it is necessary that $2000 \mid (1+2+3+\cdots+2000)$.
Since $1+2+3+\cdots+2000 = \frac{(2000)(2001)}{2} = (1000)(2001)$, it is required that
$2000 \mid 1000(2001)$.
This is not possible and so no fantastic permutation exists for $n = 2000$.

Solution

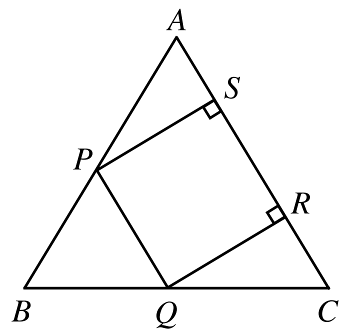
(b) The sum of the integers from 1 to 2001 is $\frac{(2001)(2002)}{2} = (2001)(1001)$ which is divisible by 2001. If t_1, t_2, ..., t_{2001} is a fantastic permutation, when we remove t_{2001} from the above sum, and what remains must be divisible by 2000.
We now consider $t_1 + t_2 + \cdots + t_{2000}$ and determine what integer is not included in the permutation.

$$\begin{aligned} t_1 + t_2 + \cdots + t_{2000} &= \frac{(2001)(2002)}{2} - t_{2001} \\ &= (1001)(2001) - t_{2001} \end{aligned}$$

Since $(1001)(2001) = 2\,003\,001$, t_{2001} must be a number of the form $k001$ where k is odd.
The only integer less than or equal to 2001 with this property is 1001. Therefore $t_{2001} = 1001$.

So the sum up to t_{2000} is $2\,003\,001 - 1001 = 2\,002\,000$.
When we remove t_{2000} we must get a multiple of 1999.
The largest multiple of 1999 less than $2\,002\,000$ is $(1999)(1001) = 2\,000\,999$. This would make $t_{2000} = 2\,002\,000 - 2\,000\,999 = 1001$ which is impossible since $t_{2000} \neq t_{2001}$. If we choose lesser multiples of 1999 to subtract from $2\,002\,000$ we will get values of t_{2000} which are greater than 2001, which is also not possible.
Thus, a fantastic permutation is not possible for $n = 2001$.

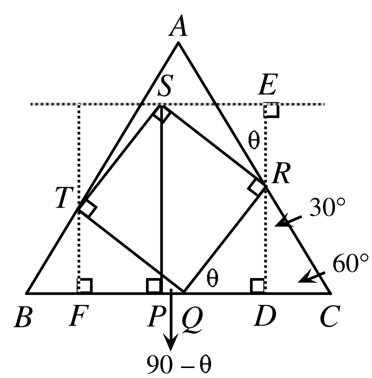
10. An equilateral triangle ABC has side length 2. A square, $PQRS$, is such that P lies on AB, Q lies on BC, and R and S lie on AC as shown. The points P, Q, R, and S move so that P, Q and R always remain on the sides of the triangle and S moves from AC to AB through the interior of the triangle. If the points P, Q, R and S always form the vertices of a square, show that the path traced out by S is a straight line parallel to BC.



In essence, this solution establishes that the perpendicular distance from S to BC is $s(\sin\theta+\cos\theta)$ and then showing that this is a constant by finding $s(\sin\theta+\cos\theta)$ as part of the base which is itself a constant length.

Solution

Let $\angle RQC=\theta$ and from S draw a line perpendicular to the base at P.
Then $\angle TQB=180-(90+\theta)=90-\theta$.
Let s be the length of the side of the square.
From R draw a line perpendicular to BC at D and then through S draw a line parallel to BC. From R draw a line perpendicular to this line at E.



From ΔRQD, $RD=s\sin\theta$.
Since $\angle QRD=90-\theta$ then $\angle SRE=\theta$.
From ΔSER, $ER=s\cos\theta$.
The perpendicular distance from S to BC is $RD+ER=s\sin\theta+s\cos\theta$ which we must now show is a constant.

We can now take each of the lengths DC, DQ, PF, FB and express them in terms of s.

From ΔRDC which is a $30°-60°-90°$ triangle, $\dfrac{DC}{RD}=\dfrac{1}{\sqrt{3}}$.

Since $RD=s\sin\theta$ (from above)

$$DC=\frac{1}{\sqrt{3}}(s\sin\theta)=\frac{\sqrt{3}}{3}s\sin\theta.$$

From ΔRDQ, $\dfrac{QD}{RQ} = \cos\theta$, $QD = s\cos\theta$.

From ΔTFQ, $\sin\theta = \dfrac{FQ}{s}$ and $\cos\theta = \dfrac{TF}{s}$.
or, $FQ = s\sin\theta$ and $TF = s\sin\theta$.

From ΔTFB, $\dfrac{BF}{TF} = \dfrac{1}{\sqrt{3}}$, $BF = \dfrac{1}{\sqrt{3}}TF = \dfrac{1}{\sqrt{3}}s\cos\theta = \dfrac{\sqrt{3}}{3}s\cos\theta$.

Since $DC + QD + FQ + BF = 2$, $\dfrac{\sqrt{3}}{3}s\sin\theta + s\cos\theta + s\sin\theta + \dfrac{\sqrt{3}}{3}s\cos\theta = 2$.

$$\frac{\sqrt{3}}{3}(s\cos\theta + s\sin\theta) + (s\cos\theta + s\sin\theta) = 2$$

$$s\cos\theta + s\sin\theta = \frac{2}{\left(\frac{\sqrt{3}}{3}+1\right)}$$

Thus $s\cos\theta + s\sin\theta$ is a constant and the path traced out by S is a straight line parallel to BC.

Note: A number of enquiries have been made about this question. Several individuals have made the comment that it is not possible to do this under the given conditions. What is not mentioned, and what is not realized, is that the size of the square *changes*. This makes it possible for the square to exist under the given conditions.

Canadian
Mathematics
Competition

An activity of The Centre for Education
in Mathematics and Computing,
University of Waterloo, Waterloo, Ontario

1999 Solutions

Euclid Contest

(Grade12)

for the

NATIONAL BANK OF CANADA

Awards

1. (a) If $x^{-1} = 3^{-1} + 4^{-1}$, what is the value of x?

Solution

$$\frac{1}{x} = \frac{1}{3} + \frac{1}{4}$$

$$\frac{1}{x} = \frac{7}{12}$$

$$\therefore x = \frac{12}{7}$$

(b) If the point $P(-3, 2)$ is on the line $3x + 7ky = 5$, what is the value of k?

Solution

Since P is on the line, its coordinates must satisfy the equation of the line.

Thus, $3(-3) + 7k(2) = 5$

$$14k = 14$$

$$k = 1$$

(c) If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

Solution 1

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

Substituting

For $x = 2$, $1 - \frac{1}{2} - \frac{3}{2}$

$= -1$

For $x = -1$, $1 + 1 - 6$

$= -4$

Solution 2

$$1 - \frac{1}{x} - \frac{6}{x^2} = \frac{x^2 - x - 6}{x^2}$$

$$= \frac{\left(x^2 - x - 2\right) - 4}{x^2}$$

$$= \frac{-4}{x^2} \quad (\text{since } x^2 - x - 2 = 0)$$

But $x^2 - x - 2 = (x-2)(x+1) = 0$ $\therefore x = 2$ or $x = -1$.

Substituting $x = 2$, <u>or</u> $x = -1$,

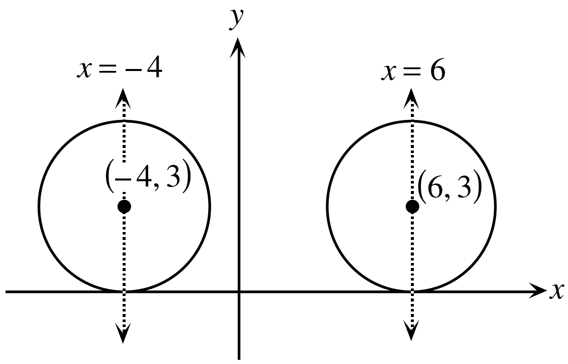
$\frac{-4}{4}$ $\qquad$ $= \frac{-4}{1}$

$= -1.$ $\qquad$ $= -4$

2. (a) The circle defined by the equation $(x+4)^2+(y-3)^2=9$ is moved horizontally until its centre is on the line $x=6$. How far does the centre of the circle move?

Solution
The identifying centre is $(-4, 3)$.

Therefore the centre moves 10 units.



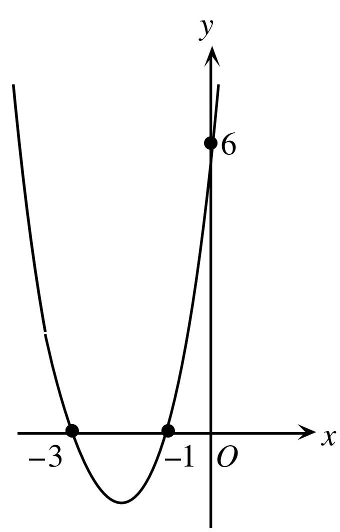
(b) The parabola defined by the equation $y=(x-1)^2-4$ intersects the x-axis at the points P and Q. If (a, b) is the mid-point of the line segment PQ, what is the value of a?

Solution 1
Intercepts of parabola are 3 and –1.
Midpoint is 1, $\therefore a=1$.

Solution 2
Axis of symmetry is $x=1$.
(or vertex is at $(1,-4)$)

By symmetry, $a=1$.

(c) Determine an equation of the quadratic function shown in the diagram.



Solution 1

Let the equation of the parabola be $y = ax^2 + bx + c$.
The parabola passes through the points $(-3, 0)$, $(-1, 0)$ and $(0, 6)$.
Substituting $(0, 6)$ gives, $6 = 0 + 0 + c$, $c = 6$.
Substituting $(-3, 0)$ and $(-1, 0)$ gives,

$$0 = 9a - 3b + 6$$

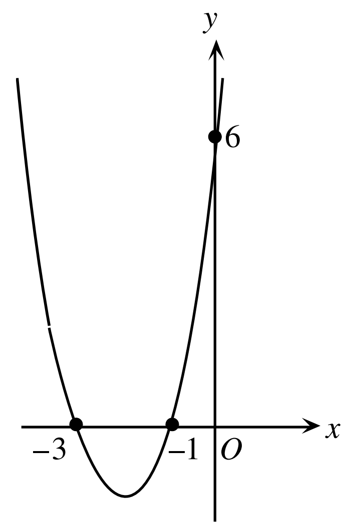
and $\quad 0 = a - b + 6$.
Solving gives, $a = 2$, $b = 8$.
The equation is $y = 2x^2 + 8x + 6$.

Solution 2

The general form of the quadratic function is, $y = k(x+3)(x+1)$.
Since $(0, 6)$ is on the parabola,

$$6 = k(0+3)(0+1)$$
$$\therefore k = 2.$$

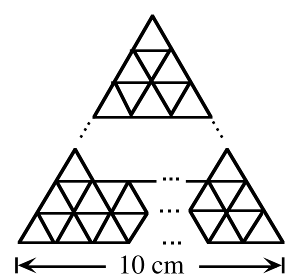
The equation is $y = 2(x+3)(x+1)$.



Solution 3

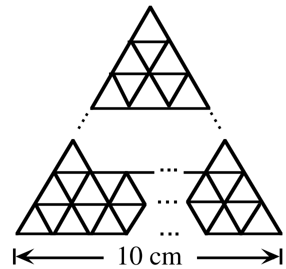
Let the equation of the parabola be $y = a(x+2)^2 + c$.
Since $(0, 6)$ is on parabola, $6 = 4a + c$,
and $(-1, 0)$ is on parabola, $\quad 0 = a + c$.
Solving, $a = 2$, $c = -2$.
$\therefore$ Equation is $y = 2(x+2)^2 - 2$.

3. (a) How many equilateral triangles of side 1 cm, placed as shown in the diagram, are needed to completely cover the interior of an equilateral triangle of side 10 cm?



Solution1

If we proceed by pattern recognition, we find after row 1 we have a total of 1 triangle, after two rows we have 2^2 or 4 triangles. After ten rows we have 10^2 or 100 triangles.



Solution 2

This solution is based on the fact that the ratio of areas for similar triangles is the square of the ratio of corresponding sides. Thus the big triangle with side length ten times that of the smaller triangle has 100 times the area.

(b) The populations of Alphaville and Betaville were equal at the end of 1995. The population of Alphaville decreased by 2.9% during 1996, then increased by 8.9% during 1997, and then increased by 6.9% during 1998. The population of Betaville increased by $r\,\%$ in each of the three years. If the populations of the towns are equal at the end of 1998, determine the value of r correct to one decimal place.

Solution

If P is the original population of Alphaville and Betaville,

$$P(.971)(1.089)(1.069) = P\left(1+\frac{r}{100}\right)^3 \qquad (1)$$

$$1.1303 = \left(1+\frac{r}{100}\right)^3 \qquad (2)$$

From here,

Possibility 1

$$1+\frac{r}{100} = (1.1303)^{\frac{1}{3}}$$

$$1+\frac{r}{100} = 1.0416$$

$$r \doteq 4.2\%$$

Or, ***Possibility 2***

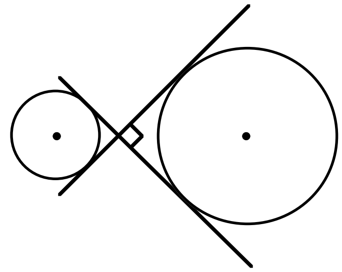
$$3\log\left(1+\frac{r}{100}\right) = \log 1.1303$$

$$\log\left(1+\frac{r}{100}\right) = .01773$$

$$1+\frac{r}{100} = 1.0416$$

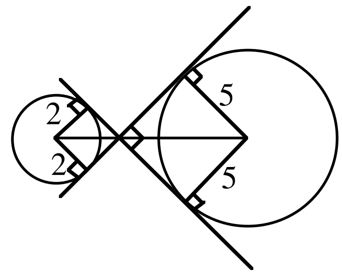
$$r \doteq 4.2\%$$

4. (a) In the diagram, the tangents to the two circles intersect at $90°$ as shown. If the radius of the smaller circle is 2, and the radius of the larger circle is 5, what is the distance between the centres of the two circles?



Solution

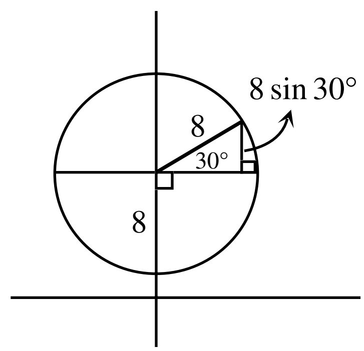
The distance from the centre of smaller circle to the point of intersection is $2\sqrt{2}$.
The distance from the centre of larger circle to the point of intersection is $5\sqrt{2}$.
Therefore the total distance is $7\sqrt{2}$.



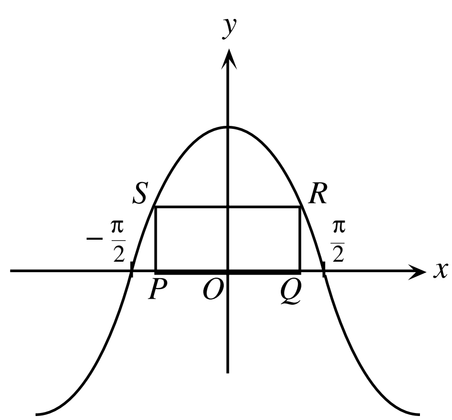
(b) A circular ferris wheel has a radius of 8 m and rotates at a rate of $12°$ per second. At $t = 0$, a seat is at its lowest point which is 2 m above the ground. Determine how high the seat is above the ground at $t = 40$ seconds.

Solution

At $t = 40$, the seat would have rotated $480°$ or $120°$ from its starting position.
We draw the triangle as shown.
The height above the ground is
$2 + 8 + 8\sin 30°$
$= 14$ m.

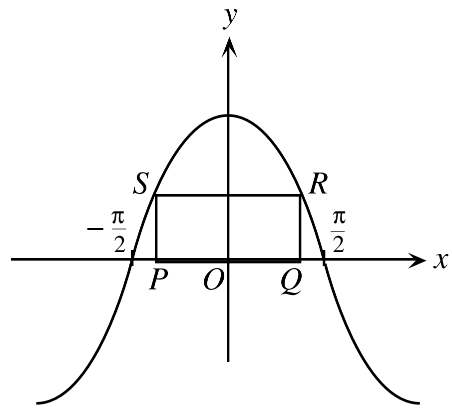


5. (a) A rectangle $PQRS$ has side PQ on the x-axis and touches the graph of $y = k\cos x$ at the points S and R as shown. If the length of PQ is $\frac{\pi}{3}$ and the area of the rectangle is $\frac{5\pi}{3}$, what is the value of k?



Solution

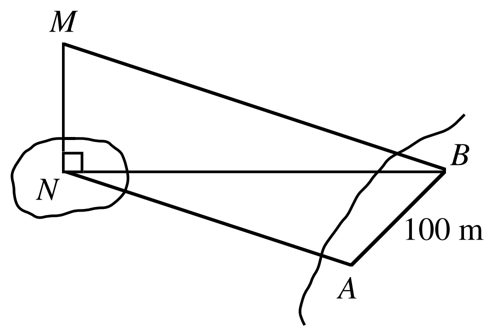
If $PQ = \frac{\pi}{3}$, then by symmetry the coordinates of R are $\left(\frac{\pi}{6}, k\cos\frac{\pi}{6}\right)$.



Area of rectangle $PQRS = \frac{\pi}{3}\left(k\cos\frac{\pi}{6}\right) = \frac{\pi}{3}(k)\left(\frac{\sqrt{3}}{2}\right)$

But $\frac{\sqrt{3}k\pi}{6} = \frac{5\pi}{3}$ $\quad \therefore k = \frac{10}{\sqrt{3}}$ or $\frac{10}{3}\sqrt{3}$.

(b) In determining the height, MN, of a tower on an island, two points A and B, 100 m apart, are chosen on the same horizontal plane as N. If $\angle NAB = 108°$, $\angle ABN = 47°$ and $\angle MBN = 32°$, determine the height of the tower to the nearest metre.



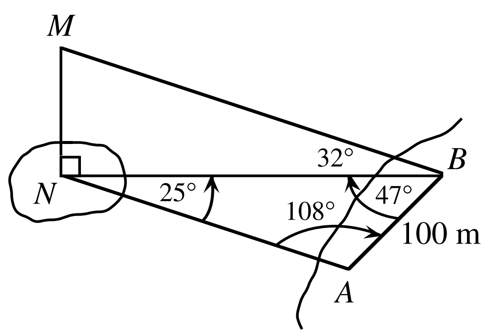
Solution

In ΔBAN, $\angle BNA = 25°$

Using the Sine Law in ΔBAN,

$$\frac{NB}{\sin 108°} = \frac{100}{\sin 25°}$$

Therefore $NB = \frac{100\sin 108°}{\sin 25°} \approx 225.04$,

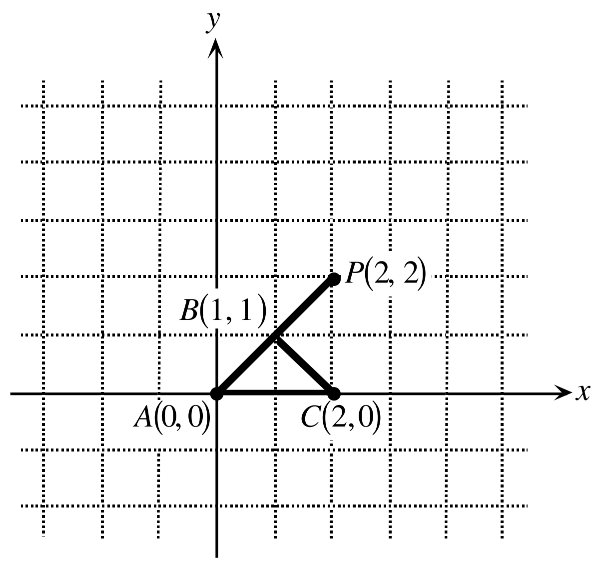


Now in ΔMNB, $\frac{MN}{NB} = \tan 32°$

$$MN = \frac{100\sin 108°}{\sin 25°} \times \tan 32° \doteq 140.6$$

The tower is approximately 141 m high.

6. (a) The points A, P and a third point Q (not shown) are the vertices of a triangle which is similar to triangle ABC. What are the coordinates of all possible positions for Q?

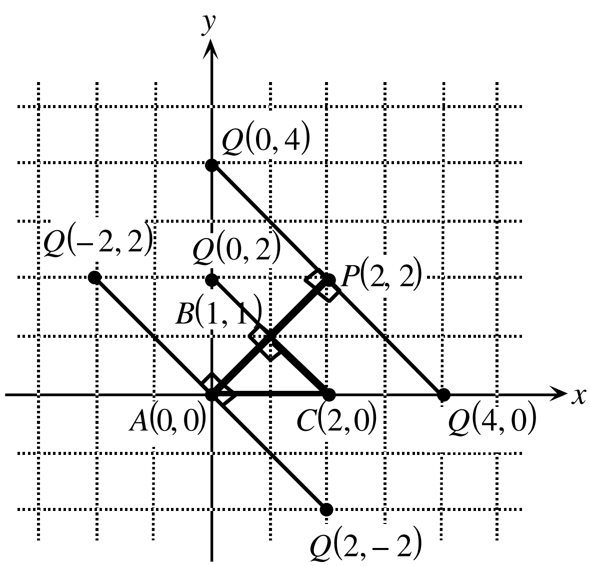


Solution

$Q(4, 0), Q(0, 4)$

$Q(2, 0), Q(0, 2)$

$Q(-2, 2), Q(2, -2)$



(b) Determine the coordinates of the points of intersection of the graphs of $y = \log_{10}(x-2)$ and $y = 1 - \log_{10}(x+1)$.

Solution

The intersection takes place where,

$$\log_{10}(x-2) = 1 - \log_{10}(x+1)$$

$$\log_{10}(x-2) + \log_{10}(x+1) = 1$$

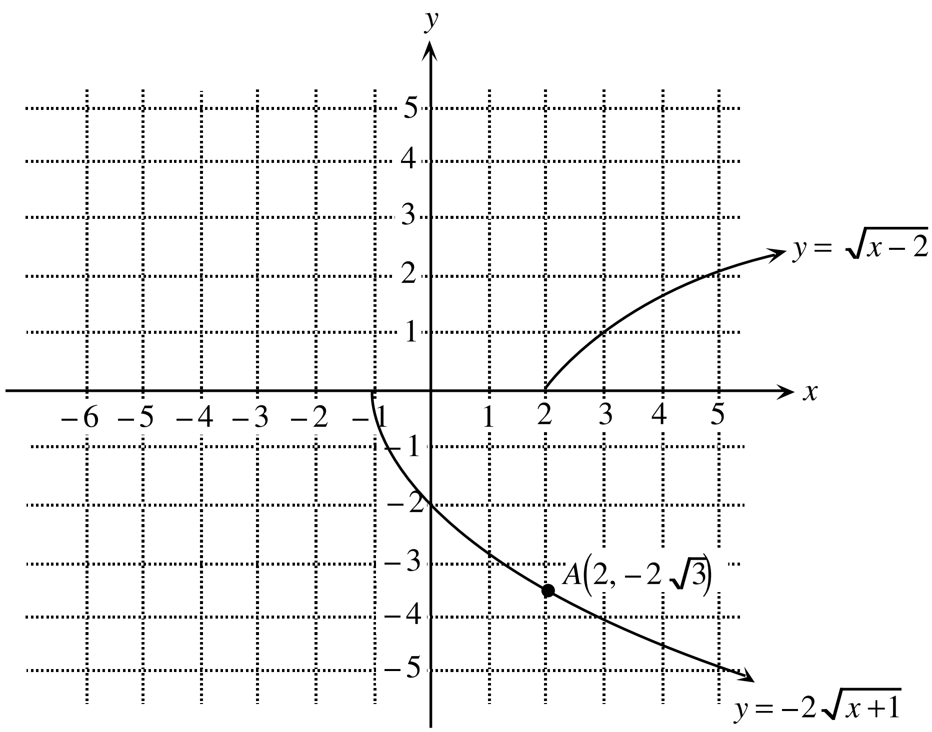
$$\log_{10}\left(x^2 - x - 2\right) = 1$$

$$x^2 - x - 2 = 10$$
$$x^2 - x - 12 = 0$$
$$(x-4)(x+3) = 0$$
$$x = 4 \ \ or \ -3$$

For $x = -3$, y is not defined.
For $x = 4$, $y = \log_{10} 2 \doteq 0.3$.
The graphs therefore intersect at $\left(4, \log_{10} 2\right)$.

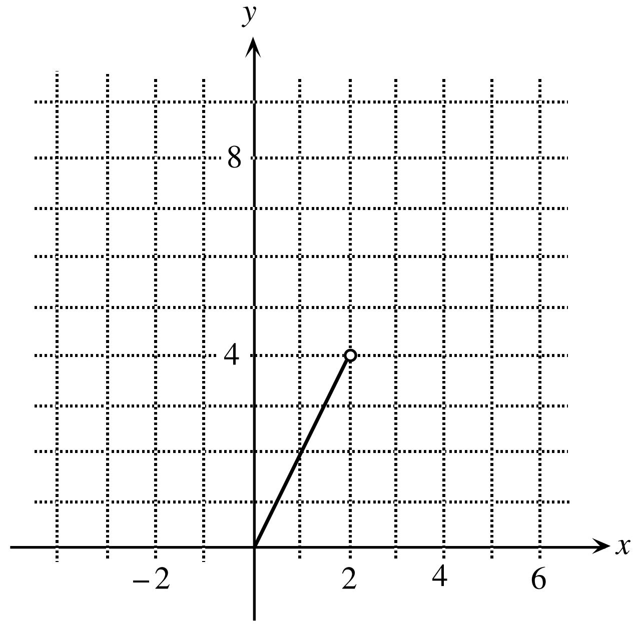
7. (a) On the grid provided in the answer booklet, draw the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2}$. For what value(s) of k will the graphs of the functions $y = -2\sqrt{x+1}$ and $y = \sqrt{x-2} + k$ intersect? (Assume x and k are real numbers.)

Solution

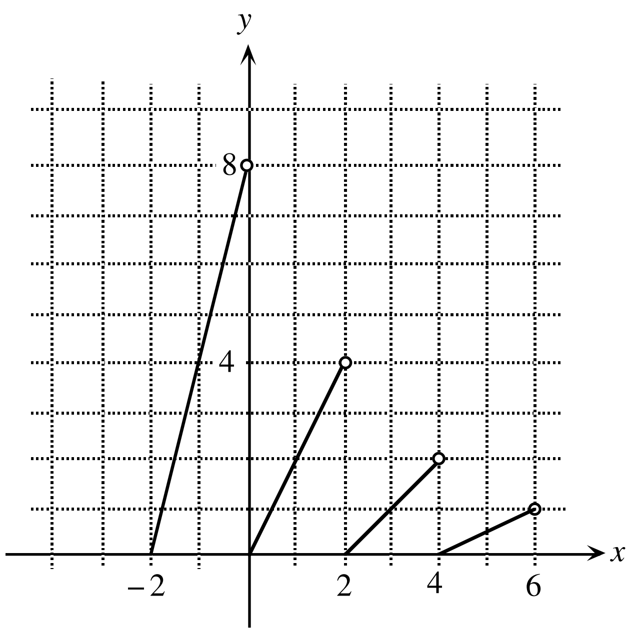


The graph of $y = \sqrt{x-2} + k$ is identical in size and shape compared to that of $y = \sqrt{x-2}$. The parameter, k, just means that the graph can take any position on the line $x = 2$ for all real values of k. If we allow $y = \sqrt{x-2} + k$ to move and it slides down to point A, this implies that for $k = -2\sqrt{3}$ the graphs will intersect. The graphs will also intersect for $k < -2\sqrt{3}$ which is easily seen graphically. Thus the required values for k are $k \le -2\sqrt{3}$, .

(b) Part of the graph for $y = f(x)$ is shown, $0 \le x < 2$.
If $f(x+2) = \frac{1}{2} f(x)$ for all real values of x, draw the graph for the intervals, $-2 \le x < 0$ and $2 \le x < 6$.



Solution



In (b) students did not know how to use the functional equation, $f(x+2) = \frac{1}{2} f(x)$. We will give an example to indicate how we might use this notation to get the required graph.

Let $x = 1$ which gives us $f(3) = \frac{1}{2} f(1)$ when we substitute into the given equation. Since $f(1) = 2$ (which we read from the given graph) we can say that $f(3) = \frac{1}{2}(2) = 1$. This means that (3, 1) is a point on the new graph. We can proceed from here by considering different values for x.

8. (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a. Prove that each of these parabolas pass through a common point and determine the coordinates of this point.

Solution 1
Since $y = x^2 + 2ax + a$ for all a, $a \in R$, it must be true for $a = 0$ and $a = 1$.
For $a = 0$, $y = x^2$; for $a = 1$, $y = x^2 + 2x + 1$.

By comparison, (or substitution)
$$x^2 = x^2 + 2x + 1$$
$$\therefore x = \frac{-1}{2}$$
$$\Rightarrow y = \frac{1}{4}$$
We must verify that $x = \frac{-1}{2}$, $y = \frac{1}{4}$ satisfies the original.

Verification: $y = x^2 + 2ax + a = \left(\frac{-1}{2}\right)^2 + 2a\left(\frac{-1}{2}\right) + a = \frac{1}{4} - a + a = \frac{1}{4}$

$\therefore \left(\frac{-1}{2}, \frac{1}{4}\right)$ is a point on $y = x^2 + 2ax + a$, $a \in R$.

Note:
Students can choose values other than $a = 0$, $a = 1$ to achieve the same result.

Solution 2
If $y = x^2 + 2ax + a$ represents a parabola for all real values of a then it is true for all a and b where $a \neq b$.
So, $y = x^2 + 2ax + a$ and $y = x^2 + 2bx + b$ (by substitution of a and b into $y = x^2 + 2ax + a$)
Since we are looking for common point, $x^2 + 2ax + a = x^2 + 2bx + b$
$$2ax - 2bx + a - b = 0$$
$$a(2x+1) - b(2x+1) = 0$$
$$(a-b)(2x+1) = 0$$
Since $a \neq b$, $2x + 1 = 0 \Rightarrow x = \frac{-1}{2}$ and $y = \frac{1}{4}$.

Solution 3
(1) The parabola can be written as, $y = x^2 + a(2x+1)$.
(2) If $2x + 1 = 0$, then $x = \frac{-1}{2}$ and $y = \frac{1}{4}$ by substitution

Line (2) is true for all values of a and hence $\left(\frac{-1}{2}, \frac{1}{4}\right)$ is a point that is always on the given parabola.

Solution 4
Let the common point be (p, q) for all a.

$\therefore \quad p = q^2 + 2ap + a$

For $a = 0$, $q = p^2$

For $a = 1$, $q = p^2 + 2p + 1$

$\therefore \quad 2p + 1 = 0$, $p = \frac{-1}{2}$ and $q = \frac{1}{4}$

Hence the point is $\left(\frac{-1}{2}, \frac{1}{4}\right)$.

Verification as in Solution 1.

(b) The vertices of the parabolas in part (a) lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

Solution

Calculating the coordinates of the vertex of $y = x^2 + 2ax + a$,

$$y = x^2 + 2ax + a^2 - a^2 + a$$

$$y = (x + a)^2 - a^2 + a$$

$\therefore$ Vertex is $\left(-a, -a^2 + a\right)$.

We can determine the required by letting $x = -a$ and $y = -a^2 + a$.

Substitute $a = -x$ into $y = -a^2 + a$ to obtain $y = -x^2 - x$.

Completing the square of $y = -x^2 - x$ gives $y = -\left(x + \frac{1}{2}\right)^2 + \frac{1}{4}$.

$\therefore \left(\frac{-1}{2}, \frac{1}{4}\right)$ is the vertex of $y = -x^2 - x$.

9. A 'millennium' series is any series of consecutive integers with a sum of 2000. Let *m* represent the first term of a 'millennium' series.

(a) Determine the minimum value of *m*.

(b) Determine the smallest possible positive value of *m*.

Solution 1 - Parts (a) and (b)

Series is, $m + (m+1) + (m+2) + \ldots + \left(m + (k-1)\right) = 2000$

Therefore, $\frac{(m+k-1)(m+k)}{2} - \frac{(m-1)m}{2} = 2000$

$$k(2m + k - 1) = 4000$$

Parity Argument

If *k* is odd then $2m + k - 1$ is even and vice-versa.

(Note: This is true because if *k* is odd then $k - 1$ is even as is $2m$ so $2m + k - 1$, is itself even. A similar argument can be made for *k* an even integer to show that $2m + k - 1$ is odd.)

One of the factors of 4000 must be 1, 5, 25 or 125 which gives the eight cases listed below.

Note:

If we had used $m+(m+1)+...+(m+k)=2000$, we would have arrived at $(k+1)(2m+k)=4000$ and then the parity argument is virtually identical to that presented just above.

Listing of Possibilities

k	$2m+k-1$	m
1	4000	2000
5	800	398
25	160	68
125	32	–46
4000	1	–1999
800	5	–397
160	25	–67
32	125	47

(a) minimum value of m is -1999

(b) smallest possible positive value of m is 47

Solution 2 - Parts (a) and (b)

Note that this argument is very similar to the previous but initially it looks different.

With n integers in the series: $m+m+1+...+\left[m+(n-1)\right]=2000$

$$\frac{1}{2}n(n+2m-1)=2000$$

$$n^2+(2m-1)n-4000=0$$

Since n is a positive integer this expression factors.

Since the sum of the roots is $-(2m-1)$, an odd integer, the roots must be one odd and one even.

The product of the roots is 4000.

The odd divisors of 4000 are 1, 5, 25 *or* 125.

Factorization	$(2m-1)$	m
$(n-1)(n+4000)$	3999	2000
$(n-5)(n+800)$	795	398
$(n-25)(n+160)$	135	68
$(n-125)(n+32)$	-93	-46
$(n+1)(n-4000)$	-3999	–1999
$(n+5)(n-800)$	-795	-397
$(n+25)(n-160)$	-135	-67
$(n+125)(n-32)$	93	47

(a) -1999

(b) 47

Possible Solution - Part (a)

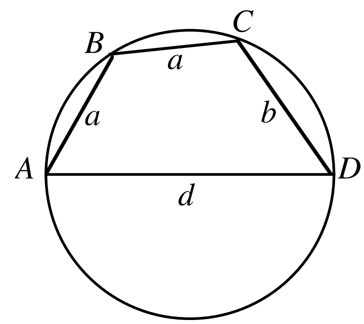
If we start with a negative number m and add consecutive integers the sum will remain negative until we have added the integers from m to $|m|$ at which time the sum will be 0. To reach a positive sum we add one more term, giving us a sum of $|m|+1$. Thus if we add the numbers -1999, ..., 1999, 2000 we get a sum of 2000. However, if we start with an integer m less than -1999 and add until we get to $|m|+1$ the sum will be greater than 2000, and will get even larger if we add further integers. Thus the minimum value for m is -1999.

This argument recognizes that -1999 is an answer but also justifies that $m=-1999$ is the minimum answer.

10. $ABCD$ is a cyclic quadrilateral, as shown, with side $AD=d$, where d is the diameter of the circle. $AB=a$, $BC=a$ and $CD=b$. If a, b and d are integers $a \neq b$,

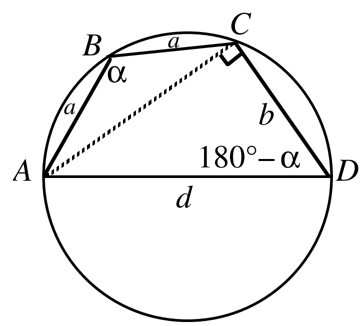
(a) prove that d cannot be a prime number.

(b) determine the *minimum* value of d.



Solution

(a) Join A to C and since $\angle ACD$ is in a semicircle, $\angle ACD = 90°$.



Let $\angle ABC = \alpha$, $\therefore \angle CDA = 180° - \alpha$ (cyclic quad.)

From ΔABC, $AC^2 = a^2 + a^2 - 2a^2 \cos\alpha$.

From ΔACD, $AC^2 = d^2 - b^2$ and $\cos(180° - \alpha) = \dfrac{b}{d}$

or $\cos\alpha = \dfrac{-b}{d}$.

By substitution, $d^2 - b^2 = 2a^2 - 2a^2\left(\dfrac{-b}{d}\right)$

$$d^3 - db^2 = 2a^2d + 2a^2b$$

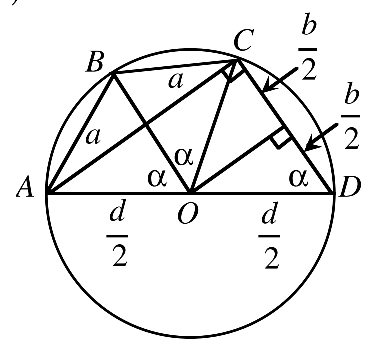
$$d\left(d^2 - b^2\right) = 2a^2(d+b)$$

$$2a^2 = d(d-b),\ d \neq b$$

Note that this relationship could also be reached in the following way:

A Possible Second Method for deriving, $2a^2 = d(d-b)$

Using ΔOBC for example,



From ΔOBC, $a^2 = \frac{d^2}{4} + \frac{d^2}{4} - 2\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)\cos\alpha$

$$\therefore a^2 = \frac{d^2}{2}(1-\cos\alpha)$$

But $\cos\alpha = \dfrac{\frac{b}{2}}{\frac{d}{2}} = \dfrac{b}{d}$.

By substitution, $2a^2 = d(d-b)$ as before.

From here we can use the above relationships to arrive at a contradiction.
From $2a^2 = d(d-b)$, if we start by assuming d is a prime then $d = 2$ or $d \geq 3$.

Case 1 $d = 2$

If we make the substitution $d = 2$ in $2a^2 = d(d-b)$ then we have,

$$2a^2 = 2(2-b)$$
$$a^2 = 2-b$$
$$b + a^2 = 2.$$

Since a and b are integers then this implies $a = b = 1$ which is not possible since we are told that a and b must be different.

Case 2 $d \geq 3$, d a prime

Students should start by looking at the relationship, $2a^2 = d(d-b)$. If $d \geq 3$ and we look at the left and right side of this relationship then d must divide into the left hand side. Since $d > 2$, it is not possible that $d | 2$. So clearly, then, d must divide a^2. However, we made the assumption that d is prime and greater than 2 so d not only divides a^2 but d also divides a. This is not possible, however, since d is the diameter of the circle and is larger than a. Our original assumption that d was prime must be incorrect and so d must be a composite number.

Note:

Students should observe that there was nothing in our proof that prevents d from being a composite number. In part (b) an example is given to show this possibility.

Solution

(b) Note that d is not prime so $d \neq 2, 3, 5, 7$, etc.

Try $d = 4$, $2a^2 = 4(4-b)$

$$a^2 = 2(4-b).$$

If $b = 1$ or 3 then $a^2 = 6$ or 2 so a is not an integer.

If $b = 2$ then $a = 2$ but $a \neq b$ so this is not possible.

Try $d = 6$, $2a^2 = 6(6-b)$, $a^2 = 3(6-b)$.
If $b = 1$, 2, 4 or 5 then a is not an integer.
If $b = 3$ then $a = 3$ but $a \neq b$ as before.

Try $d = 8$, $2a^2 = 8(8-b)$, $a^2 = 4(8-b)$.
$\therefore b = 7$ gives $a = 2$, an acceptable solution.
So the minimum value of d is 8.

Canadian Mathematics Competition

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35th Anniversary 1963 – 1998

1998 Solutions

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1. (a) If one root of $x^2 + 2x - c = 0$ is $x = 1$, what is the value of c?

Solution 1
If $x = 1$, by substituting, $c = 3$.

Solution 2
By division,

$$\begin{array}{r|l} & \quad\; x+3 \\ \hline x-1 & x^2+2x-c \\ & \underline{x^2-x} \\ & \qquad 3x-c \\ & \qquad \underline{3x-3} \\ & \qquad -c+3 \end{array}$$

If the remainder is zero, $-c + 3 = 0$
$c = 3$.

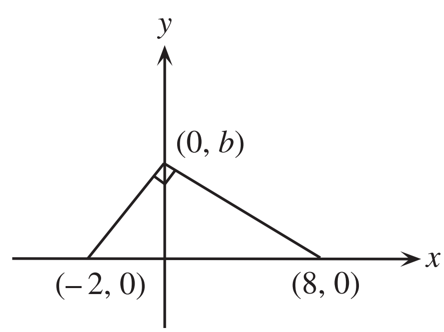
(b) If $2^{2x-4} = 8$, what is the value of x?

Solution
$2^{2x-4} = 2^3$
Therefore, $2x - 4 = 3$
$x = \frac{7}{2}$.

(c) Two perpendicular lines with x-intercepts -2 and 8 intersect at $(0, b)$. Determine all values of b.

Solution 1
If the lines are perpendicular their slopes are negative reciprocals.
Thus, $\frac{b}{-8} \times \frac{b}{2} = -1$

$b^2 = 16$, $b = \pm 4$.



Solution 2
Using Pythagoras, $\left[(b-0)^2 + (0-8)^2\right] + \left[(b-0)^2 + (0+2)^2\right] = 10^2$
$2b^2 = 32$, $b = \pm 4$.

Solution 3
The vertices of the triangle represents three points on a circle with $(-2, 0)$ and $(8, 0)$ being the

coordinates of the end points of the diameter. This circle has centre $C(3, 0)$ and $r = 5$. The equation for this circle is $(x-3)^2 + y^2 = 25$ and if we want to find the y-intercepts we let $x = 0$ which gives $b = \pm 4$.

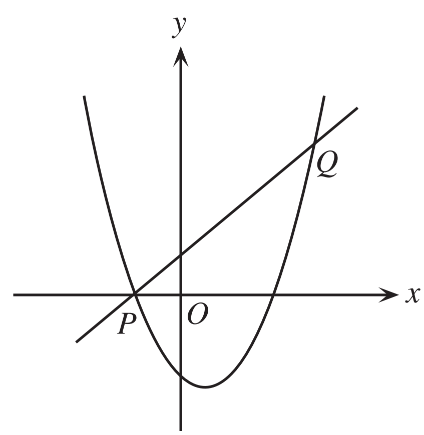
2. (a) The vertex of $y = (x-1)^2 + b$ has coordinates $(1, 3)$. What is the y-intercept of this parabola?

Solution
The vertex of parabola is $(1, b)$.
Therefore, $b = 3$.
The required equation is now $y = (x-1)^2 + 3$.
For the y-intercept, let $x = 0$.
Thus, $y_{\text{int}} = (0-1)^2 + 3 = 4$.

(b) What is the area of ΔABC with vertices $A(-3, 1)$, $B(5, 1)$ and $C(8, 7)$?

Solution
Drawing the diagram gives a triangle with a height of 6 and a base of 8 units.
The triangle has an area of 24 square units.

(c) In the diagram, the line $y = x + 1$ intersects the parabola $y = x^2 - 3x - 4$ at the points P and Q. Determine the coordinates of P and Q.



Solution
Consider the system of equations $y = x + 1$, $y = x^2 - 3x - 4$.
Comparison gives $x + 1 = x^2 - 3x - 4$

$$x^2 - 4x - 5 = 0$$
$$(x-5)(x+1) = 0.$$

Therefore $x = 5$ or $x = -1$.
If $x = 5$, $y = 6$ and if $x = -1$, $y = 0$.
The required coordinates are $P(-1, 0)$ and $Q(5, 6)$.

3. (a) The graph of $y = m^x$ passes through the points $(2, 5)$ and $(5, n)$. What is the value of mn?

Solution

Since $(2, 5)$ is on $y = m^x$, $5 = m^2$.

Since $(5, n)$ is on $y = m^x$, $n = m^5$.

So $mn = m(m^5) = m^6 = (m^2)^3 = 5^3 = 125$.

(b) Jane bought 100 shares of stock at \$10.00 per share. When the shares increased to a value of \$*N* each, she made a charitable donation of all the shares to the Euclid Foundation. She received a tax refund of 60% on the total value of her donation. However, she had to pay a tax of 20% on the increase in the value of the stock. Determine the value of *N* if the difference between her tax refund and the tax paid was \$1000.

Solution

Jane's charitable donation to the Euclid Foundation was $100N$ dollars.

Her tax refund was 60% of $100N$ or $60N$ dollars.

The increase in the value of her stock was $100(N-10)$ or $(100N-1000)$ dollars.

Jane's tax payment was 20% of $100N-1000$ or $20N-200$.

From the given, $60N-(20N-200)=1000$

Upon simplification, $40N = 800$

$N = 20$.

Therefore the value of *N* was 20.

4. (a) Consider the sequence $t_1 = 1$, $t_2 = -1$ and $t_n = \left(\frac{n-3}{n-1}\right)t_{n-2}$ where $n \geq 3$. What is the value of t_{1998}?

Solution 1

Calculating some terms, $t_1 = 1$, $t_2 = -1$, $t_3 = 0$, $t_4 = \frac{-1}{3}$, $t_5 = 0$, $t_6 = \frac{-1}{5}$ etc.

By pattern recognition, $t_{1998} = \frac{-1}{1997}$.

Solution 2

$$\begin{aligned} t_{1998} &= \frac{1995}{1997}t_{1996} = \frac{1995}{1997} \times \frac{1993}{1995}t_{1994} \\ &= \frac{1995}{1997} \cdot \frac{1993}{1995} \cdot \frac{1991}{1993} \cdots \frac{3}{5} \cdot \frac{1}{3}t_2 \\ &= \frac{-1}{1997} \end{aligned}$$

(b) The *n*th term of an arithmetic sequence is given by $t_n = 555 - 7n$.

If $S_n = t_1 + t_2 + ... + t_n$, determine the smallest value of *n* for which $S_n < 0$.

Solution 1

This is an arithmetic sequence in which $a = 548$ and $d = -7$.

Therefore, $S_n = \frac{n}{2}\left[2(548) + (n-1)(-7)\right] = \frac{n}{2}\left[-7n + 1103\right]$.

We now want $\frac{n}{2}(-7n + 1103) < 0$.

Since $n > 0$, $-7n + 1103 < 0$

$$n > 157\tfrac{4}{7}.$$

Therefore the smallest value of *n* is 158.

Solution 2

For this series we want, $\sum_{k=1}^{n} t_k < 0$, or $\sum_{k=1}^{n} (555 - 7k) < 0$.

Rewriting, $555n - 7\dfrac{(n)(n+1)}{2} < 0$

$$1110n - 7n^2 - 7n < 0$$

$$7n^2 - 1103n > 0$$

or, $n > \dfrac{1103}{7}$.

The smallest value of *n* is 158.

Solution 3

We generate the series as 548, 541, 534, ..., 2, -5, ..., -544, -551.

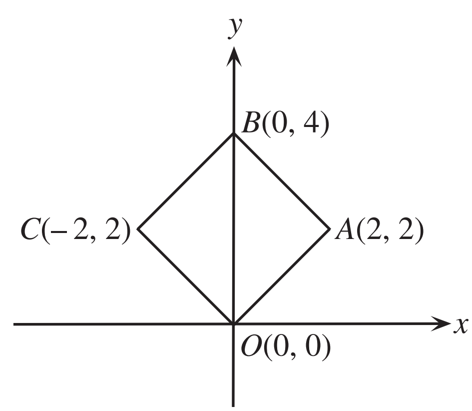
If we pair the series from front to back the sum of each pair is -3.

Including all the pairs $548 - 551$, $541 - 544$ and so on there would be 79 pairs which give a sum of -237.

If the last term, -551, were omitted we would have a positive sum.

Therefore we need all 79 pairs or 158 terms.

5. (a) A square $OABC$ is drawn with vertices as shown. Find the equation of the circle with largest area that can be drawn inside the square.



Solution

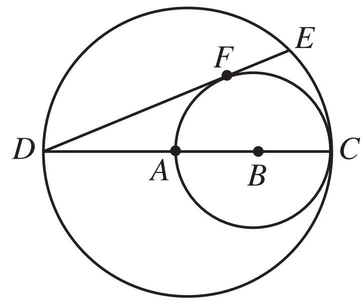
The square has a side length of $2\sqrt{2}$.

The diameter of the inscribed circle is $2\sqrt{2}$, so its radius is $\sqrt{2}$.

The centre of the circle is $(0, 2)$.

The required equation is $x^2 + (y-2)^2 = 2$ or $x^2 + y^2 - 4y + 2 = 0$.

(b) In the diagram, DC is a diameter of the larger circle centred at A, and AC is a diameter of the smaller circle centred at B. If DE is tangent to the smaller circle at F, and $DC = 12$, determine the length of DE.



Solution

Join B to F and C to E.

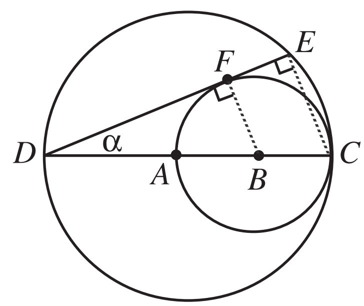
$FB \perp DE$ and DFE is a tangent.

Since DC is a diameter, $\angle DEC = 90°$.

Thus $FB \parallel EC$.

By Pythagoras, $DF = \sqrt{9^2 - 3^2} = \sqrt{72}$.

Using similar triangles (or the side splitting theorem) we have,



$$\frac{DE}{DF} = \frac{DC}{DB}$$

$$\frac{DE}{6\sqrt{2}} = \frac{4}{3}$$

$$DE = 8\sqrt{2} \text{ or } \sqrt{128}$$

OR

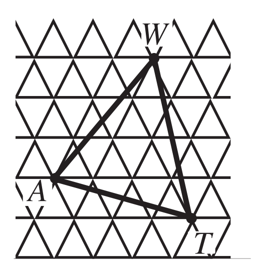
$$\frac{EC}{FB} = \frac{12}{9}$$

$$EC = \tfrac{4}{3} FB$$

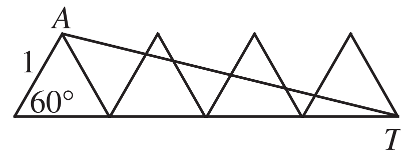
$$EC = 4$$

By Pythagoras, $DE = 8\sqrt{2}$ or $\sqrt{128}$.

6. (a) In the grid, each small equilateral triangle has side length 1. If the vertices of ΔWAT are themselves vertices of small equilateral triangles, what is the area of ΔWAT?



Solution 1

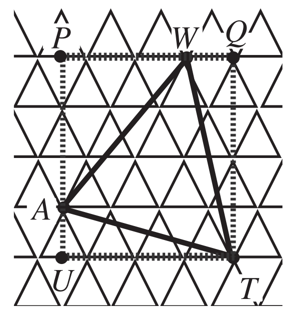


$AT^2 = 1^2 + 4^2 - 2(1)(4)\cos 60° = 13$

Since ΔWAT is an equilateral triangle with a side of length $\sqrt{13}$, its height will be $\frac{\sqrt{3}}{2}\left(\sqrt{13}\right)$. The area of ΔWAT is thus, $\frac{1}{2}\left[\left(\frac{\sqrt{3}}{2}\right)\left(\sqrt{13}\right)\right]\sqrt{13} = \frac{13}{4}\sqrt{3}$. It is also possible to use the formula for the area of a triangle,

Area $= \frac{1}{2}ab \sin c$. Since the triangle is equilateral, area of $\Delta WAT = \dfrac{\sqrt{3}AT^2}{4} = \dfrac{13\sqrt{3}}{4}$.

Solution 2



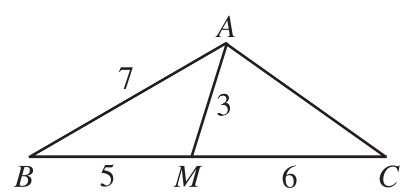
Since the small triangles have sides 1, they have a height of $\frac{\sqrt{3}}{2}$.

Consider rectangle $PQTU$.

Then

$$
\begin{aligned}
|\Delta WAT| &= |PQTU| - |\Delta APW| - |\Delta WQT| - |\Delta TUA| \\
&= (PQ)(QT) - \tfrac{1}{2}(AP)(PW) - \tfrac{1}{2}(WQ)(QT) - \tfrac{1}{2}(TU)(UA) \\
&= (3.5)\left(2\sqrt{3}\right) - \tfrac{1}{2}\left(\tfrac{3\sqrt{3}}{2}\right)(2.5) - \tfrac{1}{2}(1)\left(2\sqrt{3}\right) - \tfrac{1}{2}(3.5)\left(\tfrac{\sqrt{3}}{2}\right) \\
&= 7\sqrt{3} - \frac{15\sqrt{3}}{4} \\
&= \frac{13\sqrt{3}}{4}
\end{aligned}
$$

(b) In ΔABC, M is a point on BC such that $BM = 5$ and $MC = 6$. If $AM = 3$ and $AB = 7$, determine the exact value of AC.



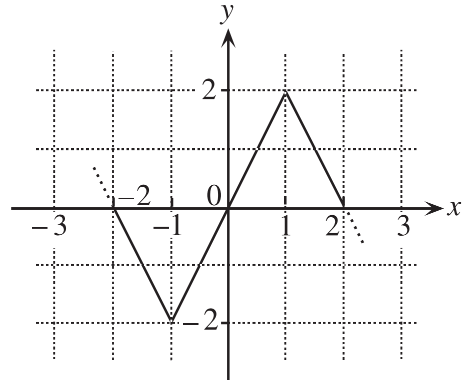
Solution

From ΔABM, $\cos \angle B = \dfrac{3^2 - 7^2 - 5^2}{-2(7)(5)} = \dfrac{13}{14}$.

From ΔABC, $AC^2 = 7^2 + 11^2 - 2(7)(11)\left(\frac{13}{14}\right) = 27$.

Therefore, $AC = \sqrt{27}$.

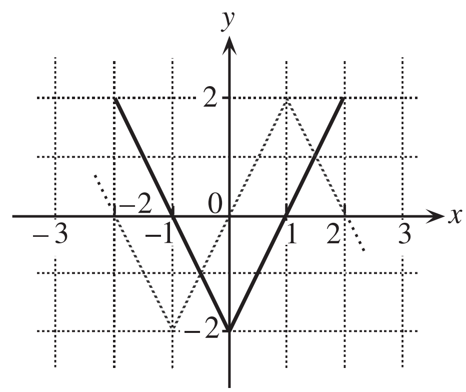
7. (a) The function $f(x)$ has period 4. The graph of one period of $y=f(x)$ is shown in the diagram. Sketch the graph of $y=\frac{1}{2}[f(x-1)+f(x+3)]$, for $-2\le x\le 2$.



Solution 1

(a)

x	$f(x)$	$f(x-1)$	$f(x+3)$	$\frac{1}{2}[f(x-1)+f(x+3)]$
-2	0	2	2	2
-1	-2	0	0	0
0	0	-2	-2	-2
1	2	0	0	0
2	0	2	2	2



Now plot the points and join them with straight line segments.

Solution 2

Since $f(x)$ has period 4, $f(x+3)=f(x-1)$.

Therefore, $y=\frac{1}{2}[f(x-1)+f(x+3)]=\frac{1}{2}[f(x-1)+f(x-1)]=f(x-1)$.

The required graph is that of $y=f(x-1)$ which is formed by shifting the given graph ***1 unit*** to the right.

(b) If x and y are real numbers, determine all solutions (x, y) of the system of equations

$$x^2-xy+8=0$$
$$x^2-8x+y=0.$$

Solution 1

Subtracting,

$$\begin{array}{r} x^2-xy+8=0 \\ x^2-8x+y=0 \\ \hline -xy+8x+8-y=0 \\ 8(1+x)-y(1+x)=0 \\ (8-y)(1+x)=0 \end{array}$$

$y=8 \quad \text{or} \quad x=-1$

If $y=8$, both equations become $x^2-8x+8=0$, $x=4\pm2\sqrt{2}$.
If $x=-1$ both equations become $y+9=0$, $y=-9$.
The solutions are $(-1,-9), \left(4+2\sqrt{2},8\right)$ and $\left(4-2\sqrt{2},8\right)$.

Solution 2

If $x^2-xy+8=0$, $y=\dfrac{x^2+8}{x}$.

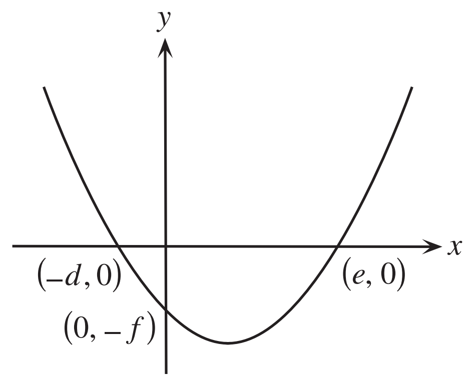
And $x^2-8x+y=0$ implies $y=8x-x^2$.

Equating, $\dfrac{x^2+8}{x}=8x-x^2$

or, $x^3-7x^2+8=0$.

By inspection, $x=-1$ is a root.
By division, $x^3-7x^2+8=(x+1)\left(x^2-8x+8\right)$.
As before, the solutions are $(-1,-9)$, $\left(4\pm2\sqrt{2},8\right)$.

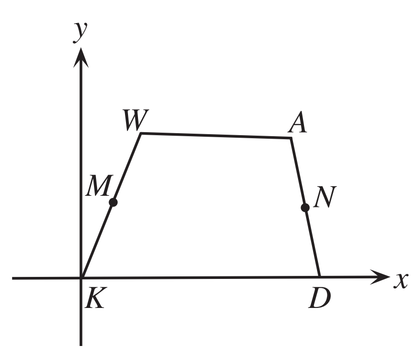
8. (a) In the graph, the parabola $y=x^2$ has been translated to the position shown. Prove that $de=f$.



Solution

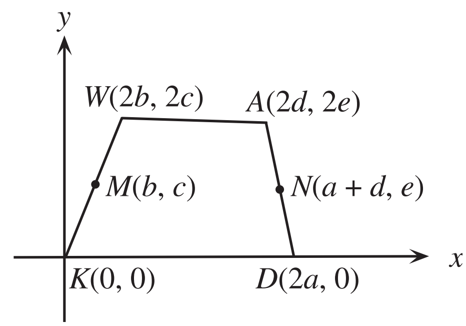
Since the given graph is congruent to $y=x^2$ and has x-intercepts $-d$ and e, its general form is $y=(x+d)(x-e)$.
To find the y-intercept, let $x=0$. Therefore y-intercept = $-de$.
We are given that the y-intercept is $-f$.
Therefore $-f=-de$ or $f=de$.

(b) In quadrilateral $KWAD$, the midpoints of KW and AD are M and N respectively. If $MN=\frac{1}{2}(AW+DK)$, prove that WA is parallel to KD.



Solution 1

Establish a coordinate system with $K(0, 0)$, $D(2a, 0)$ on the x-axes. Let W be $(2b, 2c)$ and A be $(2d, 2e)$.
Thus M is (b, c) and N is $(a+d, e)$.



KD has slope 0 and slope $WA = \dfrac{e-c}{d-b}$.

Since $MN = \frac{1}{2}(AW + DK)$

$$\sqrt{(a+d-b)^2 + (e-c)^2}$$
$$= \tfrac{1}{2}\left(2a + \sqrt{(2d-2b)^2 + (2e-2c)^2}\right)$$
$$= \tfrac{1}{2}\left(2a + 2\sqrt{(d-b)^2 + (e-c)^2}\right)$$

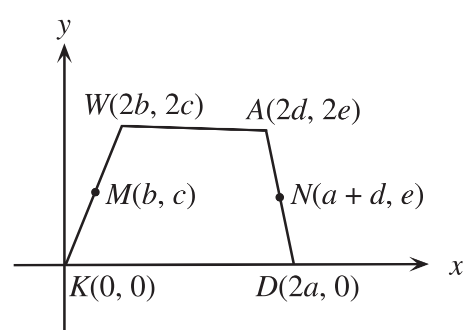
Squaring both sides gives,

$$(a+d-b)^2 + (e-c)^2 = a^2 + 2a\sqrt{(d-b)^2 + (e-c)^2} + (d-b)^2 + (e-c)^2$$
$$a^2 + 2a(d-b) + (d-b)^2 = a^2 + 2a\sqrt{(d-b)^2 + (e-c)^2} + (d-b)^2$$

Simplifying and dividing by $2a$ we have, $d-b = \sqrt{(d-b)^2 + (e-c)^2}$.
Squaring, $(d-b)^2 = (d-b)^2 + (e-c)^2$.
Therefore $(e-c)^2 = 0$ or $e = c$.
Since $e = c$ then slope of WA is 0 and $KD \| AW$.

Solution 2

Join A to K and call P the mid-point of AK.
Join M to P, N to P and M to N.
In ΔKAW, P and M are the mid-points of KA and KW.



Therefore, $MP = \frac{1}{2}WA$.
Similarly in ΔKAD, $PN = \frac{1}{2}KD$.
Therefore $MP + PN = MN$.
As a result M, P and N cannot form the vertices of a triangle but must form a straight line.
So if MPN is a straight line with $MP \| WA$ and $PN \| KD$ then $WA \| KD$ as required.

Solution 3

We are given that $\overrightarrow{AN} = \overrightarrow{ND}$ and $\overrightarrow{WM} = \overrightarrow{MK}$.
Using vectors,
(1) $\overrightarrow{MN} = \overrightarrow{MW} + \overrightarrow{WA} + \overrightarrow{AN}$ (from quad. $MWAN$)
(2) $\overrightarrow{MN} = \overrightarrow{MK} + \overrightarrow{KD} + \overrightarrow{DN}$ (from quad. $KMND$)
It is also possible to write, $\overrightarrow{MN} = -\overrightarrow{MW} + \overrightarrow{KD} - \overrightarrow{AN}$,
(3) (This comes from taking statement (2) and making appropriate substitutions.)

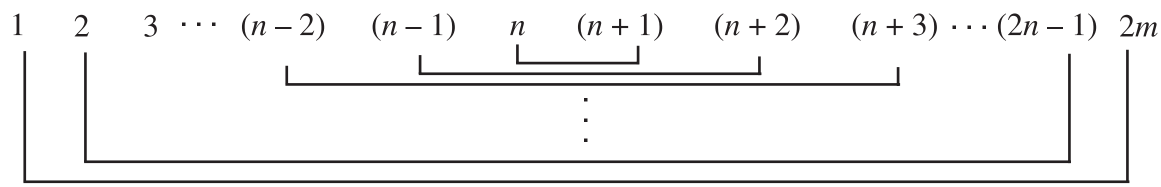
If we add (1) and (3) we find, $2\overrightarrow{MN} = \overrightarrow{WA} + \overrightarrow{KD}$.

But it is given that $2\left|\overrightarrow{MN}\right| = \left|\overrightarrow{AW}\right| + \left|\overrightarrow{DK}\right|$.

From these two previous statements, $\overrightarrow{MN}$ must be parallel to $\overrightarrow{WA}$ and $\overrightarrow{KD}$ otherwise $2\left|\overrightarrow{MN}\right| < \left|\overrightarrow{AW}\right| + \left|\overrightarrow{DK}\right|$.

Therefore, $WA \parallel KD$.

9. Consider the first $2n$ natural numbers. Pair off the numbers, as shown, and multiply the two members of each pair. Prove that there is no value of n for which two of the n products are equal.



Solution 1

The sequence is $1(2n), 2(2n-1), 3(2n-2), ..., k(2n-k+1), ..., p(2n-p+1), ..., n(n+1)$.

In essence we are asking the question, 'is it possible that $k(2n-k+1) = p(2n-p+1)$ where p and k are both less than or equal to n?'

$k(2n-k+1) = p(2n-p+1)$ (supposing them to be equal)

$$2nk - k^2 + k = 2np - p^2 + p$$
$$p^2 - k^2 + 2nk - 2np + k - p = 0$$
$$(p-k)(p+k) + 2n(k-p) + (k-p) = 0$$
$$(p-k)[(p+k) - 2n - 1] = 0$$
$$(p-k)(p+k-2n-1) = 0$$

Since p and k are both less than or equal to n, it follows $p + k - 2n - 1 \neq 0$. Therefore $p = k$ and they represent the same pair. Thus the required is proven.

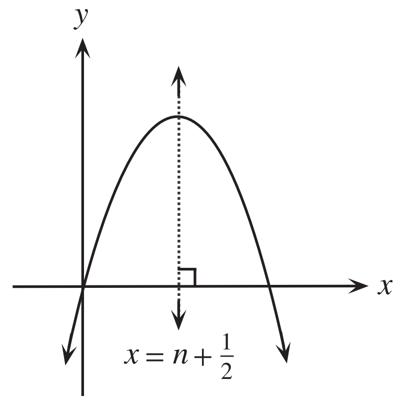
Solution 2

The products are $1(2n+1-1)$, $2(2n+1-2)$, $3(2n+1-3), ..., n(2n+1-n)$.

Consider the function, $y = x(2n+1-x) = -x^2 + (2n+1)x = f(x)$.

The graph of this function is a parabola, opening down, with its vertex at $x = n + \frac{1}{2}$.

The products are the y-coordinates of the points on the parabola corresponding to $x = 1, 2, 3, ..., n$. Since all the points are to the left of the vertex, no two have the same y-coordinate.
Thus the products are distinct.



Solution 3

The sum of these numbers is $\dfrac{2n(2n+1)}{2}$ or $n(2n+1)$.

Their average is $\dfrac{n(2n+1)}{2n} = n + \dfrac{1}{2}$.

The $2n$ numbers can be rewritten as,

$$n+\frac{1}{2}-\left(\frac{2n-1}{2}\right), \cdots, n+\frac{1}{2}-\frac{3}{2}, n+\frac{1}{2}-\frac{1}{2}, n+\frac{1}{2}+\frac{1}{2}, n+\frac{1}{2}+\frac{3}{2}, \cdots, n+\frac{1}{2}+\left(\frac{2n-1}{2}\right).$$

The product pairs, starting from the middle and working outward are

$$P_1 = \left(n+\tfrac{1}{2}\right)^2 - \tfrac{1}{4}$$

$$P_2 = \left(n+\tfrac{1}{2}\right)^2 - \tfrac{9}{4}$$

$$\vdots$$

$$P_n = \left(n+\frac{1}{2}\right)^2 - \left(\frac{2n-1}{2}\right)^2$$

Each of the numbers $\left(\dfrac{2k-1}{2}\right)^2$ is distinct for $k = 1, 2, 3, ..., n$ and hence no terms of P_k are equal.

Solution 4

The sequence is $1(2n), 2(2n-1), 3(2n-2), ..., n[2n-(n-1)]$.

This sequence has exactly n terms.

When the kth term is subtracted from the $(k+1)$th term the difference is

$(k+1)[2n-k] - k[2n-(k-1)] = 2(n-k)$. Since $n > k$, this is a positive difference.

Therefore each term is greater than the term before, so no two terms are equal.

10. The equations $x^2+5x+6=0$ and $x^2+5x-6=0$ ***each*** have integer solutions whereas only one of the equations in the pair $x^2+4x+5=0$ and $x^2+4x-5=0$ has integer solutions.
    (a) Show that if $x^2+px+q=0$ and $x^2+px-q=0$ ***both*** have integer solutions, then it is possible to find integers a and b such that $p^2=a^2+b^2$. (i.e. (a, b, p) is a Pythagorean triple).

(b) Determine q in terms of a and b.

Solution

(a) We have that $x^2+px+q=0$ and $x^2+px-q=0$ both have integer solutions.

For $x^2+px+q=0$, its roots are $\frac{-p\pm\sqrt{p^2-4q}}{2}$.

In order that these roots be integers, p^2-4q must be a perfect square.

Therefore, $p^2-4q=m^2$ for some positive integer m.

Similarly for $x^2+px-q=0$, it has roots $\frac{-p\pm\sqrt{p^2+4q}}{2}$ and in order that these roots be integers p^2+4q must be a perfect square.

Thus $p^2+4q=n^2$ for some positive integer n.

Adding gives $2p^2=m^2+n^2$ (with $n\ge m$ since $n^2=p^2+4q$
$\ge p^2-4q=m^2$)

And so $p^2=\frac{1}{2}m^2+\frac{1}{2}n^2=\left(\frac{n+m}{2}\right)^2+\left(\frac{n-m}{2}\right)^2$.

We note that m and n have the same parity since $m^2=p^2-4q\equiv p^2 \pmod 2$ and $n^2\equiv p^2+4q\equiv p^2 \pmod 2$.

Since $\frac{n+m}{2}$ and $\frac{n-m}{2}$ are positive integers then $p^2=a^2+b^2$ where $a=\frac{n+m}{2}$ and $b=\frac{n-m}{2}$.

(b) From (a), $a=\frac{n+m}{2}$ and $b=\frac{n-m}{2}$ or $n=a+b$ and $m=a-b$.

From before, $p^2+4q=n^2$

$$\begin{aligned}4q^2&=n^2-p^2\\&=(a+b)^2-\left(a^2+b^2\right)\\4q&=2ab.\end{aligned}$$

Therefore, $q=\frac{ab}{2}$.