The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2024 Fryer Contest

Thursday, April 4, 2024
(in North America and South America)

Friday, April 5, 2024
(outside of North America and South America)

Solutions

1. (a) The 5th term is obtained by adding 6 to the 4th term. Thus, the 5th term is $21 + 6 = 27$.

   (b) *Solution 1*
   The 6th term is obtained by adding 6 to the 5th term. Thus, the 6th term is $27 + 6 = 33$, and so the average of the 4th, 5th and 6th terms is $\frac{21+27+33}{3} = \frac{81}{3} = 27$.

   Solution 2
   The 4th term is 6 less than the 5th term, and the 6th term is 6 more than the 5th term, and so the average of the 4th, 5th and 6th terms is the 5th term, which is 27.

   (c) The nth term ($n \geq 2$) is obtained by adding $n - 1$ 6s to the first term, 3. For example, the 2nd term is $3 + 1 \times 6$, the 3rd term is $3 + 2 \times 6$, the 4th term is $3 + 3 \times 6$, and so on.
   In general, the nth term is given by $3 + (n - 1) \times 6$.
   Therefore, the 20th term is $3 + 19 \times 6 = 3 + 114 = 117$.

   (d) *Solution 1*
   Since each new term is obtained by adding 6 to the previous term and $\frac{1000}{6} \approx 166.7$, then it makes sense to begin by determining the 166th term. The 166th term is $3+165\times 6 = 993$, the next term is $993 + 6 = 999$ (still less than 1000), and so the smallest term that is greater than 1000 is $999 + 6 = 1005$.
   (We note that 1005 is the 168th term and is equal to $3 + 167 \times 6$.)

   Solution 2
   From part (c), the nth term is given by the expression $3 + (n - 1) \times 6$.
   We want the smallest term that is greater than 1000. To begin, we find the smallest possible value of n for which $3 + (n - 1) \times 6 > 1000$.
   Solving this inequality, we get

$$\begin{aligned} 3 + (n-1) \times 6 &> 1000 \\ 3 + 6n - 6 &> 1000 \\ 6n - 3 &> 1000 \\ 6n &> 1003 \\ n &> \tfrac{1003}{6} \approx 167.2 \end{aligned}$$

   Since n must be an integer, the first term number to exceed 1000 is the 168th term, and its value is $3 + 167 \times 6 = 1005$.

2. (a) At Store 2, 50% of the shirts dropped off were red, and so the other 50% were blue.
   That is, Ella dropped off an equal number of red and blue shirts at Store 2.
   All 200 blue shirts were dropped off at Store 2, and thus 200 red shirts were also dropped off at Store 2. Ella began the day with 800 red shirts, dropped off 200 at Store 2, and so she dropped off $800 - 200 = 600$ red shirts at Store 1.

   (b) At Store 1, Ella dropped off 40% of $5x$ red shirts or $0.40 \times 5x = 2x$ red shirts.
   Ella dropped off the remaining $5x - 2x = 3x$ red shirts at Store 2, in addition to the $5x$ blue shirts. At Store 2, $\dfrac{5x}{3x + 5x} = \dfrac{5}{8}$ of the shirts dropped off were blue, which is $\frac{5}{8} \times 100\% = 62.5\%$ of the shirts.

   (c) Ella dropped off no blue shirts at Store 1, and so she dropped off all y blue shirts at Store 2.
   Since there were equal numbers of red, blue and green shirts dropped off at Store 2, then y green shirts (and y red shirts) were dropped off at Store 2.
   On Wednesday, Ella dropped off $3y$ red shirts, y blue shirts, and y green shirts, for a total

of $5y$ shirts.

Of all the shirts dropped off on Wednesday, $\frac{y}{5y} = \frac{1}{5}$ were green, which is $\frac{1}{5} \times 100\% = 20\%$ of the shirts.

3. (a) In Figure 2, each of the smaller pieces has the same length of crust.
Thus, we must determine the length of MN so that each of the smaller pieces has the same area.
The area of square $ABCD$ is $30 \times 30 = 900$, and so the area of each of the 3 smaller pieces is $\frac{900}{3} = 300$.
Since $ABCD$ is a square, then $AD = BC = 30$ and so $AM = 15$.
The smaller piece $AMNB$ is a trapezoid (AB and MN are perpendicular to AM and thus parallel to one another), and so $AMNB$ has area $\frac{AM}{2}(MN + AB)$ or $\frac{15}{2}(MN + 30)$.
Solving, we get $\frac{15}{2}(MN + 30) = 300$ or $MN + 30 = \frac{300\times 2}{15}$ or $MN + 30 = 40$, and so $MN = 10$.

Alternately, $\triangle BNC$ also has area 300 with base $BC = 30$, and so has height $h = \frac{300}{15} = 20$.
Thus, $MN = AB - h$ or $MN = 30 - 20 = 10$.

(b) In Figure 3, each of the 5 smaller pieces must have the same length of crust.
Since the slice of bread has crusts on 3 of its edges, the total length of crust is $3 \times 30 = 90$.
Thus the length of crust for each of the 5 smaller pieces is $\frac{90}{5} = 18$.
Since $\triangle TPQ$ is one of these smaller pieces, and its length of crust is PQ, then $PQ = 18$.

(c) The area of square $ABCD$ is $30 \times 30 = 900$, and so the area of each of the 5 smaller pieces is $\frac{900}{5} = 180$.
Consider a point W on BC so that MW is perpendicular to BC.
Since MW must pass through S and T, then TW is the altitude of $\triangle TPQ$.
The area of $\triangle TPQ$ is 180, and so $\frac{1}{2} \times PQ \times TW = 180$ or $\frac{1}{2} \times 18 \times TW = 180$, and so $TW = \frac{180}{9} = 20$.

Since MS and AU are both perpendicular to AM, then MS and AU are parallel to one another, and so $AMSU$ is a trapezoid.
Each of the 5 smaller pieces has the same length of crust, and so $AU = PQ = 18$.
The area of $AMSU$ is 180. Solving $\frac{AM}{2}(MS + AU) = 180$, we get $\frac{15}{2}(MS + 18) = 180$ or $MS + 18 = 24$, and so $MS = 6$.
Since $MW = AB = 30$, then $ST = 30 - MS - TW$ or $ST = 30 - 6 - 20$, and so $ST = 4$.

4. (a) Since each player has 3 distinct integers that they may spin, then the total number of possible outcomes is $3 \times 3 = 9$, and each outcome is equally probable.
If Alice spins a 5, she wins if Binh spins a 1, and loses if Binh spins 8 or 10.
If Alice spins a 9, she wins if Binh spins 1 or 8, and loses if Binh spins a 10.
If Alice spins an 11, she wins if Binh spins 1, 8 or 10.
We summarize these results in the table below, using an A to indicate that Alice wins and a B where Binh wins.

Alice's Spin

Binh's Spin

B \ A	5	9	11
1	A	A	A
8	B	A	A
10	B	B	A

We see that Alice wins 6 of the 9 possible outcomes, and thus the probability that Alice wins is $\frac{6}{9} = \frac{2}{3}$.

(b) Carole's spinner is {1,5,10}. Suppose that Darsh's spinner is $\{a, b, c\}$ with $a < b < c$ and where a, b, c are chosen from 2, 3, 4, 6, 7, 8, and 9.
Since each player has 3 distinct integers that they may spin, then the total number of possible outcomes is $3 \times 3 = 9$, and each outcome is equally probable.
Thus, Darsh's probability of winning is greater than Carole's probability of winning if the integer he spins is greater than Carole's for at least 5 of the 9 possible outcomes.

If Carole spins a 1, then Darsh wins since each of his integers must be greater than 1.
When Carole spins a 1, there are 3 winning outcomes for Darsh since he can spin a, b or c (and each is greater than 1).
If Carole spins a 10, then Darsh loses since each of his integers must be less than 10.
There are similarly 3 outcomes when Carole spins a 10, except each of these is a losing outcome for Darsh.
To this point, Darsh wins 3 of the possible outcomes if Carole spins a 1 or a 10.
This tells us that the probability of Darsh winning is determined by comparing a, b, c to Carole's spin of a 5.
Specifically, if at least two of Darsh's integers are greater than 5, then Darsh's probability of winning is greater than Carole's probability of winning (and the opposite is true if at least two of Darsh's integers are less than 5).
Why is this true? When Carole spins a 1, Darsh wins all 3 possible outcomes. When Carole spins a 10, Darsh wins 0 possible outcomes. When Carole spins a 5, Darsh wins at least 2 possible outcomes exactly when at least two of his integers are greater than 5, and thus Darsh wins at least $3 + 0 + 2 = 5$ of the 9 possible outcomes (and Carole wins 4 or fewer).
To determine the number of different spinners for which at least two of Darsh's integers are greater than 5, we consider the following two cases.

Case 1: All three of Darsh's integers are greater than 5.

In this case, Darsh wins 6 of the 9 possible outcomes, as shown. The integers from which Darsh may choose are 6, 7, 8, and 9, and so there are 4 possible spinners: {6,7,8}, {6,7,9}, {6,8,9}, and {7,8,9}.

Darsh's Spin (columns) / **Carole's Spin** (rows)

C \ D	a	b	c
1	D	D	D
5	D	D	D
10	C	C	C

Case 2: Two of Darsh's integers are greater than 5 and one of his integers is less than 5.

In this case, Darsh wins 5 of the 9 possible outcomes, as shown. There are 6 possible pairs of integers that are greater than 5. These are: {6,7}, {6,8}, {6,9}, {7,8}, {7,9}, and {8,9}. For each of these 6 pairs, there are 3 possible choices for the integer that is less than 5 (namely 2, 3 and 4). Thus, there are $6 \times 3 = 18$ possible spinners in this case.

Darsh's Spin (columns) / **Carole's Spin** (rows)

C \ D	a	b	c
1	D	D	D
5	C	D	D
10	C	C	C

Therefore, Darsh can make $4+18 = 22$ different spinners so that his probability of winning is greater than Carole's probability of winning.

(c) We begin by determining the value of p, the probability that Fynn beats Erin.
We summarize these results in the table below, using an F to indicate that Fynn wins and an E where Erin wins.

Fynn's Spin (columns) / **Erin's Spin** (rows)

E \ F	2	10	18
5	E	F	F
8	E	F	F
15	E	E	F

We see that Fynn wins 5 of the 9 possible outcomes, and thus $p = \frac{5}{9}$.
In the question, we are given that $p = q = r$ and thus $q = r = \frac{5}{9}$, or the probability that Erin beats Gina is $\frac{5}{9}$ and the probability that Gina beats Fynn is also $\frac{5}{9}$.
This means that Erin wins exactly 5 of the 9 possible outcomes when playing against Gina, and similarly, Gina wins exactly 5 of the 9 possible outcomes when playing against Fynn.
We begin by considering the game in which Erin plays Gina.
Erin's spinner is {5,8,15}. Gina's spinner is $\{x, y, z\}$ with $x < y < z$ and where x, y, z are chosen from

$$1, 3, 4, 6, 7, 9, 11, 12, 13, 14, 16, 17, 19, 20$$

Each of Erin's 3 integers (5, 8, 15) wins 0, 1, 2, or 3 possible outcomes.
For example, if Erin spins 15 and $15 > z$, then Erin wins all 3 possible outcomes (15 beats

each of x, y and z).
Conversely, if Erin spins 15 and $15 < x$, then Erin wins 0 possible outcomes.
We note that if a player's spinner is $\{p, q, r\}$ with $p < q < r$, then the number of winning outcomes when spinning r must be greater than or equal to the number of winning outcomes when spinning q, which must be greater than or equal to the number of winning outcomes when spinning p. Can you explain why this is?

There are three different ways that Erin can win exactly 5 of the 9 outcomes against Gina. These are:

- Erin's largest integer wins exactly 3 outcomes, her second largest integer wins exactly 2 outcomes, and her smallest integer wins 0 outcomes, or
- Erin's largest integer wins exactly 3 outcomes, and her remaining two integers each win exactly 1 outcome, or
- Erin's largest two integers each win exactly 2 outcomes, and her smallest integer wins exactly 1 outcome.

We will refer to the first of these bullets as the 3/2/0 result, the second bullet as the 3/1/1 result, and the final bullet as the 2/2/1 result.
We note that in each case, the sum of the number of winning outcomes is 5, and further, these are the only possible ways for exactly 5 winning outcomes to occur.
If the result in the game between Erin and Gina is 3/2/0, then Erin wins all 3 outcomes when spinning 15, wins 2 outcomes when spinning 8, and wins 0 outcomes when spinning 5. Recalling that Gina's spinner is $\{x, y, z\}$ with $x < y < z$, this means that if the result is 3/2/0, then the two spinners' six integers are ordered as follows,

$$5 < x < y < 8 < z < 15$$

That is, since 15 is greater than each of x, y, and z, then Erin wins 3 outcomes when spinning 15. Since 8 is greater than x and y, then Erin wins 2 outcomes when spinning 8, and since 5 is less than x, y and z, then Erin wins 0 outcomes when spinning 5.
Thus, if Gina makes her spinner with $x = 6$, $y = 7$ and $z = 9$, 11, 12, 13, or 14, then the probability that Erin beats Gina is $\frac{5}{9}$ since Erin will win exactly 5 of the 9 possible outcomes.

If the result is 3/1/1, then the two spinners' six integers are ordered as follows,

$$x < 5 < 8 < y < z < 15$$

In this case, a spin of 15 wins 3 times and spins of 5 and 8 each win exactly once.
Thus, if Gina makes her spinner with $x = 1, 3$ or 4, and y and z (with $y < z$) are chosen from 9, 11, 12, 13, 14, then the probability that Erin beats Gina is also $\frac{5}{9}$.
Finally, if Erin beats Gina with a 2/2/1 result, then the two spinners' six integers are ordered as follows,

$$x < 5 < y < 8 < 15 < z$$

and so $x = 1, 3$ or 4, $y = 6$ or 7, and $z = 16, 17, 19$, or 20.

We summarize these results in the table below.

Erin $\{5, 8, 15\}$ beats Gina $\{x, y, z\}$

Case	Result	Integer Ordering	Possible x	Possible y	Possible z
1	3/2/0	$5 < x < y < 8 < z < 15$	6	7	$9, 11, 12, 13, 14$
2	3/1/1	$x < 5 < 8 < y < z < 15$	$1, 3, 4$	$9, 11, 12, 13, 14$	$9, 11, 12, 13, 14$
3	2/2/1	$x < 5 < y < 8 < 15 < z$	$1, 3, 4$	$6, 7$	$16, 17, 19, 20$

(Note that in Case 2, the values of y are only possible provided that $y < z$.)
We also require Gina's spinner to be made so that the probability that Gina beats Fynn is $\frac{5}{9}$.
In the game between Gina and Fynn, we use the same process and notation as was used in the game between Erin and Gina.
In addition, we see from the table above that x, y, z must be chosen so that

$$x = 1, 3, 4, \text{or } 6, \text{ and } y = 6, 7, 9, 11, 12, 13, \text{or } 14, \text{ and } z = 9, 11, 12, 13, 14, 16, 17, 19, \text{or } 20$$

again noting that we require $x < y < z$.
In the table below, we exclude all other possible values of x, y, z not listed above.

Gina $\{x, y, z\}$ beats Fynn $\{2, 10, 18\}$

Case	Result	Integer Ordering	Possible x	Possible y	Possible z
4	3/2/0	$x < 2 < 10 < y < 18 < z$	1	11, 12, 13, 14	19, 20
5	3/1/1	$2 < x < y < 10 < 18 < z$	3, 4, 6	6, 7, 9	19, 20
6	2/2/1	$2 < x < 10 < y < z < 18$	3, 4, 6	11, 12, 13, 14	11, 12, 13, 14, 16, 17

(Note that in Cases 5 and 6, the values of x and y are only possible if $x < y < z$.)
To make a spinner for which Erin beats Gina and Gina beats Fynn, each with probability $\frac{5}{9}$, then we must determine values for x, y, z that satisfy at least one of the Cases 1, 2 or 3 while simultaneously satisfying at least one of the Cases 4, 5 or 6.

We begin by determining if there are values for x, y, z that satisfy Case 1 while simultaneously satisfying at least one of the Cases 4, 5 or 6.
To satisfy Case 1, $y = 7$ and $z = 9, 11, 12, 13,$ or 14, however, each of the Cases 4, 5 and 6 do not overlap the restrictions on y and z, and so there are no spinners for which Erin beats Gina with the 3/2/0 result.

Next, we determine if there are values for x, y, z that satisfy Case 2 while simultaneously satisfying at least one of the Cases 4, 5 or 6.
To satisfy Case 2, $z = 9, 11, 12, 13,$ or 14, however, each of the Cases 4 and 5 do not overlap the restrictions on z, and so we are left to consider Case 2 and Case 6.
If $x = 3$ or 4, and $y = 11, 12, 13,$ or 14, and $z = 11, 12, 13,$ or 14, then the restrictions on both Case 2 and 6 are satisfied.
Recalling that $y < z$, there are 6 pairs (y, z) satisfying these restrictions: $(y, z) = (11, 12), (11, 13), (11, 14), (12, 13), (12, 14), (13, 14)$.
For each of these 6 pairs, there are 2 possible choices for x (x can equal 3 or 4), and thus there are $6 \times 2 = 12$ spinners that Gina can make.

Finally, we determine if there are values for x, y, z that satisfy Case 3 while simultaneously satisfying at least one of the Cases 4, 5 or 6.
To satisfy Case 3, $y = 6$ or 7, however, each of the Cases 4 and 6 do not overlap the restrictions on y, and so we are left to consider Case 3 and Case 5.
If $x = 3$ or 4, and $y = 6$ or 7, and $z = 19$ or 20, then the restrictions on both Case 3 and Case 5 are satisfied.
Since there are 2 choices for each of x, y and z, then there are $2 \times 2 \times 2 = 8$ spinners that Gina can make (we note that each of the 8 spinners has $x < y < z$).

Recognizing that the $12 + 8 = 20$ spinners are different from one another, Gina can make 20 different spinners so that $p = q = r = \frac{5}{9}$.