The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2022 Fermat Contest

(Grade 11)

Wednesday, February 23, 2022
(in North America and South America)

Thursday, February 24, 2022
(outside of North America and South America)

Solutions

1. Evaluating, $6 + (3 \times 6) - 12 = 6 + 18 - 12 = 12$.

   ANSWER: (C)

2. *Solution 1*
   Since the average of two numbers is 7, their sum is $2 \times 7 = 14$.
   Since one of the numbers is 5, the other is $14 - 5 = 9$.

   Solution 2
   The average of two numbers is 7 and one of the numbers is 5.
   Since 5 is 2 less than 7, the other number must be 2 more than 7, and so is 9.
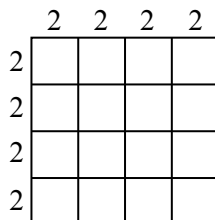
   ANSWER: (E)

3. From the day on which she walks 500 m to the day on which she walks 4500 m, Gauravi increases her distance by $(4500 \text{ m}) - (500 \text{ m}) = 4000$ m.

   Since Gauravi increases her distance by 500 m each day, then it takes $\dfrac{4000 \text{ m}}{500 \text{ m}} = 8$ days to increase from 500 m to 4500 m.

   Starting from Monday and counting forward by 8 days (which is 1 week and 1 day) gets to Tuesday, and so Gauravi walks exactly 4500 m on a Tuesday.

   ANSWER: (C)

4. By arranging 4 rows of 4 squares of side length 2, a square of side length 8 can be formed.

   Thus, $4 \cdot 4 = 16$ squares can be arranged in this way. Since these smaller squares completely cover the larger square, it is impossible to use more 2×2 squares, so 16 is the largest possible number.

   ANSWER: (C)

5. Since the list includes 15 integers, then an integer has a probability of $\frac{1}{3}$ of being selected if it occurs $\frac{1}{3} \cdot 15 = 5$ times in the list.
   The integer 5 occurs 5 times in the list and no other integer occurs 5 times, so $n = 5$.

   ANSWER: (E)

6. The given triangle can be considered to have base PQ (which is vertical along the line $x = 2$) and perpendicular height which runs from R horizontally to PQ.
   The line segment joining $P(2, 6)$ to $Q(2, 2)$ has length 4.
   Point $R(8, 5)$ is 6 units to the right of the line with equation $x = 2$.
   Thus, $\triangle PQR$ has area $\frac{1}{2} \cdot 4 \cdot 6 = 12$.

   ANSWER: (D)

7. Evaluating, $(1 + 2 + 3)(1 + \frac{1}{2} + \frac{1}{3}) = 6(1 + \frac{1}{2} + \frac{1}{3}) = 6 + \frac{6}{2} + \frac{6}{3} = 6 + 3 + 2 = 11$.

   ANSWER: (B)

8. Since $10x + y = 75$ and $10y + x = 57$, then

$$(10x + y) + (10y + x) = 75 + 57$$

   and so

$$11x + 11y = 132$$

   Dividing by 11, we get $x + y = 12$.
   (We could have noticed initially that $(x, y) = (7, 5)$ is a pair that satisfies the two equations, thence concluding that $x + y = 12$.)

   ANSWER: (A)

9. Since Pearl digs 4 holes in 7 days and $\frac{21}{7} = 3$, then in 21 days, Pearl digs $3 \cdot 4 = 12$ holes.
   Since Miguel digs 2 holes in 3 days and $\frac{21}{3} = 7$, then in 21 days, Miguel digs $7 \cdot 2 = 14$ holes.
   In total, they dig $12 + 14 = 26$ holes in 21 days.

   ANSWER: (D)

10. Manipulating the left side, $2^{11} \times 6^5 = 2^{11} \times (2 \times 3)^5 = 2^{11} \times 2^5 \times 3^5 = 2^{16} \times 3^5$.
    Since $4^x \times 3^y = 2^{16} \times 3^5$ and x and y are positive integers, then $y = 5$ (because 4^x has no factors of 3).
    This also means that $4^x = 2^{16}$.
    Since $4^x = (2^2)^x = 2^{2x}$, then $4^x = 2^{16}$ gives $2^{2x} = 2^{16}$ and so $2x = 16$ or $x = 8$.
    Therefore, $x + y = 8 + 5 = 13$.

    ANSWER: (D)

11. We use A, B, C, D, and E to represent Andy, Bev, Cao, Dhruv, and Elcim, respectively.
    We use the notation $D > B$ to represent the fact "Dhruv is older than Bev".
    The five sentences give $D > B$ and $B > E$ and $A > E$ and $B > A$ and $C > B$. These show us that Dhruv and Cao are older than Bev, and Elcim and Andy are younger than Bev.
    This means that two people are older than Bev and two people are younger than Bev, which means that Bev must be the third oldest.
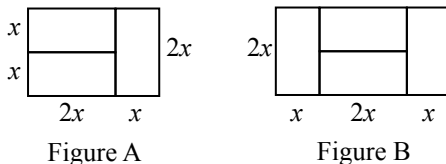
    ANSWER: (B)

12. Since d is an odd integer, then $d + d$ is even and $d \times d$ is odd.
    Since e is an even integer, then $e + e$ is even, which means that $(e + e) \times d$ is even.
    Also, $e + d$ is odd, which means that $d \times (e + d)$ is odd.
    Thus, 2 of the 4 expressions are equal to an odd integer.

    ANSWER: (C)

13. Suppose that each of the small rectangles has shorter sides of length x.

Then the height of the rectangle in Figure A is $2x$, which means that the longer side of each small rectangle has length $2x$.

Therefore, the rectangle in Figure A has height $2x$ and width $2x + x = 3x$, which means that its perimeter is $2(3x + 2x) = 10x$. Also, the rectangle in Figure B has height $2x$ and width $x + 2x + x = 4x$, so its perimeter is $2(4x + 2x) = 12x$.

Figure A   Figure B

Therefore, the ratio of the perimeter of Figure A to the perimeter of Figure B is $10x : 12x$ which is equal to $5 : 6$, since $x \neq 0$.

ANSWER: (E)

14. Zebadiah must remove at least 3 shirts.

If he removes 3 shirts, he might remove 2 red shirts and 1 blue shirt.

If he removes 4 shirts, he might remove 2 red shirts and 2 blue shirts.

Therefore, if he removes fewer than 5 shirts, it is not guaranteed that he removes either 3 of the same colour or 3 of different colours.

Suppose that he removes 5 shirts. If 3 are of the same colour, the requirements are satisfied. If no 3 of the 5 shirts are of the same colour, then at most 2 are of each colour (for example, 2 red, 2 blue and 1 green). This means that he must remove shirts of 3 colours, since if he only removed shirts of 2 colours, he would remove at most $2 + 2 = 4$ shirts.

In other words, if he removes 5 shirts, it is guaranteed that there are either 3 of the same colour or shirts of all 3 colours.

Thus, the minimum number is 5.

ANSWER: (D)

15. If a is odd, the output is $a + 3$, which is even because it is the sum of two odd integers.

If a is even, the output is $a + 5$, which is odd, because it is the sum of an even integer and an odd integer.

Starting with $a = 15$ and using the machine 2 times, we obtain $15 \to 15 + 3 = 18 \to 18 + 5 = 23$.

Starting with 23 and using the machine 2 times, we obtain $23 \to 23 + 3 = 26 \to 26 + 5 = 31$.

Starting with an odd integer and using the machine 2 times, the net result is adding 8 to the input, because the odd input generates a first output that is 3 larger (and so even) and a second output that is 5 larger than the first output.

This generates a net result that is $3 + 5$ larger than the input.

Therefore, using the machine 46 more times (that is, repeating the 2 steps a total of 23 more times), we add 8 a total of 23 more times to obtain the output $31 + 23 \cdot 8 = 215$.

To this point, the machine has been used 50 times.

Using the machine for the 51st time, $215 \to 215 + 3 = 218$ and so the final output is 218.

ANSWER: (B)

16. Since the remainder when 111 is divided by n is 6, then $111 - 6 = 105$ is a multiple of n and $n > 6$ (since, by definition, the remainder must be less than the divisor).

Since $105 = 3 \cdot 5 \cdot 7$, the positive divisors of 105 are $1, 3, 5, 7, 15, 21, 35, 105$.

Therefore, the possible values of n are 7, 15, 21, 35, 105, of which there are 5.

ANSWER: (A)

17. Suppose that the original can has radius r cm and height h cm.

    Since the surface area of the original can is 300 cm^2, then $2\pi r^2 + 2\pi rh = 300$.

    When the radius of the original can is doubled, its new radius is $2r$ cm, and so an expression for its surface area, in cm^2, is $2\pi(2r)^2 + 2\pi(2r)h$ which equals $8\pi r^2 + 4\pi rh$, and so $8\pi r^2 + 4\pi rh = 900$.

    When the height of the original can is doubled, its new height is $2h$ cm, and so an expression for its surface area, in cm^2, is $2\pi r^2 + 2\pi r(2h)$ which equals $2\pi r^2 + 4\pi rh$.

    Multiplying $2\pi r^2 + 2\pi rh = 300$ by 3, we obtain $6\pi r^2 + 6\pi rh = 900$.

    Since $8\pi r^2 + 4\pi rh = 900$, we obtain

    $$6\pi r^2 + 6\pi rh = 8\pi r^2 + 4\pi rh$$
    $$2\pi rh = 2\pi r^2$$
    $$\pi rh = \pi r^2$$

    Since $2\pi r^2 + 2\pi rh = 300$ and $\pi rh = \pi r^2$, then $2\pi r^2 + 2\pi r^2 = 300$ and so $4\pi r^2 = 300$ or $\pi r^2 = 75$.

    Since $\pi rh = \pi r^2 = 75$, then $2\pi r^2 + 4\pi rh = 6 \cdot 75 = 450$, and so the surface area of the cylinder with its height doubled is 450 cm^2.

    ANSWER: (A)

18. Let A be Aria's starting point, B be Bianca's starting point, and M be their meeting point.

    It takes Aria 42 minutes to walk from A to M and 28 minutes from M to B.

    (Note that 9:00 a.m. is 18 minutes after 8:42 a.m., and 9:10 a.m. is 10 minutes after 9:00 a.m..)

    Since Aria walks at a constant speed, then the ratio of the distance AM to the distance MB is equal to the ratio of times, or $42 : 28$, which is equivalent to $3 : 2$.

    Since it takes 42 minutes for Bianca to walk from B to M, the ratio of distances AM to MB is $3 : 2$, and Bianca walks at a constant speed, then it takes Bianca $\frac{3}{2} \times 42 = 63$ minutes to walk from M to A.

    Therefore, Bianca arrives at Aria's starting point at 9:45 a.m.

    (Note that 9:00 a.m. is 18 minutes after 8:42 a.m., and 9:45 a.m. is 45 minutes after 9:00 a.m..)

    ANSWER: (D)

19. Since $\triangle PQR$ is right-angled at R, then by the Pythagorean Theorem,

    $$PQ^2 = PR^2 + QR^2 = 12^2 + 16^2 = 144 + 256 = 400$$

    Since $PQ > 0$, then $PQ = 20$.

    Since M is the midpoint of PQ, then $MQ = \frac{1}{2}PQ = 10$.

    Now $\triangle NMQ$ is similar to $\triangle PRQ$, since each is right-angled and they share a common angle at Q.

    Therefore, $\dfrac{NQ}{PQ} = \dfrac{MQ}{RQ}$ and so $\dfrac{NQ}{20} = \dfrac{10}{16}$ which gives $NQ = \dfrac{20 \cdot 10}{16} = \dfrac{25}{2}$.

    Thus, $RN = RQ - NQ = 16 - \dfrac{25}{2} = \dfrac{32}{2} - \dfrac{25}{2} = \dfrac{7}{2}$.

    Since $\triangle PNR$ is right-angled at R, its area equals $\dfrac{1}{2} \cdot PR \cdot RN = \dfrac{1}{2} \cdot 12 \cdot \dfrac{7}{2} = 21$.

    ANSWER: (A)

20. We note that

$$t_1 = \frac{1}{1} - \frac{1}{3} = \frac{2}{3} \approx 0.67$$

$$t_1 + t_2 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{2}{3} + \frac{1}{4} = \frac{11}{12} \approx 0.92$$

$$t_1 + t_2 + t_3 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5}$$

$$= \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} = 1.05$$

$$t_1 + t_2 + t_3 + t_4 = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6}$$

$$= \frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6} \approx 1.13$$

This means that the sum of the first k terms is less than 1.499 for $k = 1, 2, 3, 4$.
When $k > 4$, we can extend the pattern that we saw for $k = 3$ and $k = 4$ to note that

$$t_1 + t_2 + t_3 + \ldots + t_{k-1} + t_k = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots$$

$$+ \left(\frac{1}{k-1} - \frac{1}{k+1}\right) + \left(\frac{1}{k} - \frac{1}{k+2}\right)$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} + \frac{1}{k} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \cdots - \frac{1}{k+1} - \frac{1}{k+2}$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots + \frac{1}{k-1} - \frac{1}{k-1} + \frac{1}{k} - \frac{1}{k} - \frac{1}{k+1} - \frac{1}{k+2}$$

$$= \frac{1}{1} + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}$$

$$= 1.500 - \frac{1}{k+1} - \frac{1}{k+2}$$

This means that the sum of the first k terms is less than 1.499 exactly when $\frac{1}{k+1} + \frac{1}{k+2}$ is greater than 0.001.

As k increases from 4, each of $\frac{1}{k+1}$ and $\frac{1}{k+2}$ decreases, which means that their sum decreases as well.

When $k = 1998$, $\frac{1}{k+1} + \frac{1}{k+2} = \frac{1}{1999} + \frac{1}{2000} > \frac{1}{2000} + \frac{1}{2000} = \frac{1}{1000} = 0.001$.

When $k = 1999$, $\frac{1}{k+1} + \frac{1}{k+2} = \frac{1}{2000} + \frac{1}{2001} < \frac{1}{2000} + \frac{1}{2000} = \frac{1}{1000} = 0.001$.

This means that $\frac{1}{k+1} + \frac{1}{k+2}$ is greater than 0.001 exactly when $k \leq 1998$ and is less than 0.001 when $k \geq 1999$.

In other words, the sum of the first k terms is less than 1.499 for $k = 1, 2, 3, 4$ as well as for $5 \leq k \leq 1998$, which is the same as saying that this is true for $1 \leq k \leq 1998$.
Therefore, $k = 1998$ is the largest positive integer for which the sum of the first k terms is less than 1.499.

ANSWER: (E)

21. The total mass of the six steel bars in the bags is at least $1 + 2 + 3 + 4 + 5 + 6 = 21$ kg and at most $10 + 11 + 12 + 13 + 14 + 15 = 75$ kg. This is because the masses of the 15 given bars are 1 kg, 2 kg, 3 kg, ..., 14 kg, and 15 kg.
Since the six bars are divided between three bags with the same total mass in each bag, then the total mass in each bag is at least $21 \div 3 = 7$ kg and at most $75 \div 3 = 25$ kg.
There are $25 - 7 + 1 = 19$ masses that are an integer number of kilograms in this range (7 kg, 8 kg, 9 kg, ..., 23 kg, 24 kg, 25 kg).
Each of 19 these masses is indeed possible. To see this, we note that

$$1 + 6 = 2 + 5 = 3 + 4 = 7 \qquad 1 + 7 = 2 + 6 = 3 + 5 = 8 \qquad 1 + 8 = 2 + 7 = 3 + 6 = 9$$

which shows that 7, 8 and 9 are possible values of M.
Continuing to increase the larger values to 15, 14, 13, we eventually obtain

$$1 + 15 = 2 + 14 = 3 + 13 = 16$$

and also that each value of M between 10 and 16, inclusive, will be a possible value of M.
Now, we increase the smaller values, starting from the last three pairs:

$$2+15 = 3+14 = 4+13 = 17 \qquad 3+15 = 4+14 = 5+13 = 18 \qquad \cdots \qquad 10+15 = 11+14 = 12+13 = 25$$

which shows that 17, 18, 19, 20, 21, 22, 23, 24, and 25 are also possible values of M.
This shows that every integer value of M with $7 \leq M \leq 25$ is possible.
In summary, there are 19 possible values of M.

ANSWER: 19

22. We enclose the given rectangle in a larger rectangle with horizontal and vertical sides so that the vertices of the smaller rectangle lie on the sides of the larger rectangle.
We will also remove the units from the problem and deal with dimensionless quantities.
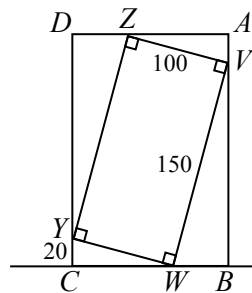
Since $VWYZ$ is a rectangle, then $YW = ZV = 100$ and $ZY = VW = 150$.
Since $\triangle YCW$ is right-angled at C, by the Pythagorean Theorem,

$$CW^2 = YW^2 - YC^2 = 100^2 - 20^2 = 10\,000 - 400 = 9600$$

Since $CW > 0$, then $CW = \sqrt{9600} = \sqrt{1600 \cdot 6} = \sqrt{1600} \cdot \sqrt{6} = 40\sqrt{6}$.

The height of Z above the horizontal line is equal to the length of DC, which equals $DY + YC$ which equals $DY + 20$.

Now $\triangle ZDY$ is right-angled at D and $\triangle YCW$ is right-angled at C.

Also, $\angle DYZ + \angle ZYW + \angle WYC = 180°$, which means that $\angle DYZ + \angle WYC = 90°$, since $\angle ZYW = 90°$.

Since $\angle CWY + \angle WYC = 90°$ as well (using the sum of the angles in $\triangle YCW$), we obtain $\angle DYZ = \angle CWY$, which tells us that $\triangle ZDY$ is similar to $\triangle YCW$.

Therefore, $\dfrac{DY}{ZY} = \dfrac{CW}{YW}$ and so $DY = \dfrac{ZY \cdot CW}{YW} = \dfrac{150 \cdot 40\sqrt{6}}{100} = 60\sqrt{6}$.

Finally, $DC = DY + 20 = 60\sqrt{6} + 20 \approx 166.97$. Rounded to the nearest integer, DC is 167.

Since the length of DC to the nearest integer is $100 + x$ and this must equal 167, then $x = 67$.

ANSWER: 67

23. Suppose that k is a fixed, but unknown, positive integer.

Suppose also that the lines with equations $9x + 4y = 600$ and $kx - 4y = 24$ intersect at the point with positive integer coordinates (x, y).

Since $9x + 4y = 600$ and $kx - 4y = 24$, adding these equations, we get $9x + kx = 624$ and so $(9 + k)x = 624$.

Since x and y are to be positive integers and $k > 0$, then $9 + k$ and x are a positive divisor pair of 624 with $9 + k > 9$.

Now $624 = 6 \cdot 104 = 6 \cdot 8 \cdot 13 = 2^4 3^1 13^1$, and so the positive divisors of 624 are

$$1, 2, 3, 4, 6, 8, 12, 13, 16, 24, 26, 39, 48, 52, 78, 104, 156, 208, 312, 624$$

We also want the value of y to be a positive integer.

Since the point (x, y) lies on the line with equation $9x + 4y = 600$, then $4y = 600 - 9x$ which gives $y = 150 - \frac{9}{4}x$, which is an integer exactly when x is a multiple of 4.

Therefore, we want x to be a positive divisor of 624 which is a multiple of 4.

Thus, the possible values of x are $4, 8, 12, 16, 24, 48, 52, 104, 156, 208, 312, 624$.

The corresponding values of $9 + k$ are $156, 78, 52, 39, 26, 13, 12, 6, 4, 3, 2, 1$.

Since $9 + k > 9$, we eliminate $6, 4, 3, 2, 1$ from this list.

Thus, the possible values of $9 + k$ are $156, 78, 52, 39, 26, 13, 12$.

The corresponding values of k are $147, 69, 43, 30, 17, 4, 3$.

These correspond to the following values of x: $4, 8, 12, 16, 24, 48, 52$.

Using $y = 150 - \frac{9}{4}x$, these give the following values of y: $141, 132, 123, 114, 96, 42, 33$. These are indeed all positive.

This means that there are 7 values of k with the required properties.

ANSWER: 07

24. Since $f(p) = 17$, then $ap^2 + bp + c = 17$.
Since $f(q) = 17$, then $aq^2 + bq + c = 17$.
Subtracting these two equations, we obtain $a(p^2 - q^2) + b(p - q) = 0$.
Since $p^2 - q^2 = (p - q)(p + q)$, this becomes $a(p - q)(p + q) + b(p - q) = 0$.
Since $p < q$, then $p - q \neq 0$, so we divide by $p - q$ to get $a(p + q) + b = 0$.
Since $f(p + q) = 47$, then $a(p + q)^2 + b(p + q) + c = 47$ and so $(p + q)(a(p + q) + b) + c = 47$.
Since $a(p + q) + b = 0$, then $(p + q)(0) + c = 47$ which tells us that $c = 47$.
Since $ap^2 + bp + c = 17$, then $ap^2 + bp = -30$ and so $p(ap + b) = -30$.
Similarly, $q(aq + b) = -30$.
Since p and q are prime numbers and a and b are integers, then p and q must be prime divisors of -30. We note that $30 = 2 \cdot 3 \cdot 5$ and also that p and q must be distinct.
Since $p < q$, then $p = 2$ and $q = 3$, or $p = 2$ and $q = 5$, or $p = 3$ and $q = 5$.

Alternatively, we could note that since $f(p) = f(q) = 17$, then $f(p) - 17 = f(q) - 17 = 0$.
Therefore, $f(x) - 17$ is a quadratic polynomial with roots p and q, which means that we can write $f(x) - 17 = a(x - p)(x - q)$, since the quadratic polynomial has leading coefficient a.
Since $f(p + q) = 47$, then $f(p + q) - 17 = a(p + q - p)(p + q - q)$ which gives $47 - 17 = aqp$ or $apq = 30$.
As above, $p = 2$ and $q = 3$, or $p = 2$ and $q = 5$, or $p = 3$ and $q = 5$.

If $p = 2$ and $q = 3$, the equations $p(ap + b) = -30$ becomes $2(2a + b) = -30$ (or $2a + b = -15$) and the equation $q(aq + b) = -30$ becomes $3(3a + b) = -30$ (or $3a + b = -10$).
Subtracting $2a + b = -15$ from $3a + b = -10$, we obtain $a = 5$ (note that $a > 0$) which gives $b = -15 - 2 \cdot 5 = -25$.
Therefore, $f(x) = 5x^2 - 25x + 47$.
Since $pq = 6$, then $f(pq) = 5(6^2) - 25(6) + 47 = 77$.

If $p = 2$ and $q = 5$, we get $2a + b = -15$ and $5a + b = -6$.
Subtracting the first of these from the second, we obtain $3a = 9$ which gives $a = 3$ (note that $a > 0$) and then $b = -15 - 2 \cdot 3 = -21$.
Therefore, $f(x) = 3x^2 - 21x + 47$.
Since $pq = 10$, then $f(pq) = 3(10^2) - 21(10) + 47 = 137$.

If $p = 3$ and $q = 5$, we get $3a + b = -10$ and $5a + b = -6$.
Subtracting the first of these from the second, we obtain $2a = 4$ which gives $a = 2$ (note that $a > 0$) and then $b = -10 - 3 \cdot 2 = -16$.
Therefore, $f(x) = 2x^2 - 16x + 47$.
Since $pq = 15$, then $f(pq) = 2(15^2) - 16(15) + 47 = 257$.

The sum of these values of $f(pq)$ is $77 + 137 + 257 = 471$.
The rightmost two digits of this integer are 71.

ANSWER: 71

25. Consider the grid as laid out in the problem:

a	b	c
d	5	e
f	g	h

We know that the sums of the integers along each row, along each column, and along the two main diagonals are all divisible by 5.

We start by removing all but the integers 5, a, c, f, and h from the grid.

a		c
	5	
f		h

There are 9 choices for each of a and c.

Since the sum of the entries on each diagonal is a multiple of 5, then $a + 5 + h$ is a multiple of 5, which is equivalent to saying that $a + h$ is a multiple of 5.

Note that each of a and h is between 1 and 9.

If $a = 1$, then $h = 4$ or $h = 9$. If $a = 6$, then $h = 4$ or $h = 9$.
If $a = 2$, then $h = 3$ or $h = 8$. If $a = 7$, then $h = 3$ or $h = 8$.
If $a = 3$, then $h = 2$ or $h = 7$. If $a = 8$, then $h = 2$ or $h = 7$.
If $a = 4$, then $h = 1$ or $h = 6$. If $a = 9$, then $h = 1$ or $h = 6$.
If $a = 5$, then $h = 5$.

We write $h = \overline{a}$ to show that h depends on a. (Note that h does not depend on c.) We will remember later that there might be 1 or 2 possible values for h, depending on the value of a.

Similarly, $c + 5 + f$ is a multiple of 5, or equivalently that $c + f$ is a multiple of 5 and this same combinations of possible values for c and f exist as for a and h.

We write $f = \overline{c}$. This gives us

a		c
	5	
$\overline{c}$		$\overline{a}$

Since $a + b + c$ is a multiple of 5, then $b + (a + c)$ is a multiple of 5. We write $b = \overline{a + c}$, since b depends on $a + c$.

This gives us

a	$\overline{a + c}$	c
	5	
$\overline{c}$		$\overline{a}$

Since a and c are each between 1 and 9, then $a + c$ is between 2 and 18. Recall that $b = \overline{a + c}$ is also between 1 and 9.

If $a + c$ is one of 2, 7, 12, and 17, the possible values for $b = \overline{a + c}$ are 3 and 8.
If $a + c$ is one of 3, 8, 13, and 18, the possible values for $b = \overline{a + c}$ are 2 and 7.
If $a + c$ is one of 4, 9, and 14, the possible values for $b = \overline{a + c}$ are 1 and 6.
If $a + c$ is one of 5, 10, and 15, then $b = \overline{a + c} = 5$.
If $a + c$ is one of 6, 11, and 16, the possible values for $b = \overline{a + c}$ are 4 and 9.

We can now start to consider a number of cases. Because we have seen above that the number of possibilities for some of the entries depend on whether or not a and c are 5, we look at (i) $a = c = 5$, (ii) $a = 5$ and $c \neq 5$, (iii) $c = 5$ and $a \neq 5$, and (iv) $a \neq 5$ and $c \neq 5$.

Case 1: $a = c = 5$
From above, there is only one choice for each of $\overline{a}$ and $\overline{c}$: each must equal 5.
Also, $a + c = 10$ and so $\overline{a + c}$ must also equal 5, giving the grid:

5	5	5
	5	
5		5

Similarly, each of the remaining cells can only be filled with 5, so there is only 1 way of completing the grid in this case.

Case 2: $a = 5$ and $c \neq 5$
Since $a = 5$, then $\overline{a} = 5$.
Also, since $a = 5$, then $a + c = 5 + c$ which means that $\overline{a + c}$ is the same as $\overline{c}$, giving the grid:

5	$\overline{c}$	c
	5	
$\overline{c}$		5

Here, there are 8 choices for c (everything but 5) and 2 choices for each occurrence of $\overline{c}$ (since c is not 5).
Furthermore, the possibilities for the 3 empty cells are determined by either the value of c or by the value of $\overline{c}$, neither of which can be a multiple of 5.
Thus, there are 2 possibilities for each of these 3 empty cells.
Combining this information, in this case, there are thus $1^2 \cdot 8 \cdot 2^2 \cdot 2^3 = 2^8$ grids.

Case 3: $c = 5$ and $a \neq 5$
If $c = 5$ and $a \neq 5$, there are also 2^8 grids.

Next, we consider the situation when $a \neq 5$ and $c \neq 5$.
We know here that there are two possible values for each of $\overline{a}$ and $\overline{c}$. However, the number of possible values for $\overline{a + c}$ depends on whether $a + c$ is a multiple of 5. Additionally, the number of possibilities for the 3 unlabelled cells also depend on the values of combinations of a, c, $\overline{a}$, and $\overline{c}$. This leads to three more cases in which $a \neq 5$ and $c \neq 5$.

Case 4: $a \neq 5$ and $c \neq 5$ and $a + c$ is a multiple of 5
There are 8 choices for a (everything but 5).
There are then 2 choices for c (either of the choices that makes $a + c$ a multiple of 5).
There are 2 choices for each of $\overline{a}$ and $\overline{c}$, since neither a nor c is 5.
Also, $\overline{a + c} = 5$ since $a + c$ is a multiple of 5.
This gives the grid:

a	5	c
	5	
$\overline{c}$		$\overline{a}$

The empty cell in the bottom row must be filled with a 5 to make the sum of the middle column a multiple of 5.

We now examine the first and third columns and see that neither $a + \bar{c}$ nor $c + \bar{a}$ can be a multiple of 5.

One way to justify this is to note that, since $a + c$ is a multiple of 5, the remainders when a and c are divided by 5 must add to 5.

This means that a and $\bar{c}$ have the same non-zero remainder when divided by 5, which in turn means that their sum is not divisible by 5.

Therefore, the remaining 2 empty cells each have 2 possible entries to make their column sums multiples of 5.

There are 8 choices for a, 2 choices for c, 2 cells which must be filled with 5, and 2 choices for each of the remaining 4 cells.

In this case, there are thus $8 \cdot 2 \cdot 1^2 \cdot 2^4 = 2^8$ grids.

Finally, we look at the grids where $a \neq 5$ and $c \neq 5$ and $a + c$ is not a multiple of 5, separating the situations where $a - c$ is a multiple of 5 and $a - c$ is not a multiple of 5.

Case 5: $a \neq 5$ and $c \neq 5$ and $a + c$ is a not multiple of 5 and $a - c$ is a multiple of 5

There are 8 choices for a.

There are then 2 choices for c: either c with the same remainder as a when divided by 5.

There are 2 choices for each of $\bar{a}$ and $\bar{c}$ and $\overline{a + c}$ since none of a, c and $a + c$ is a multiple of 5.

a	$\overline{a + c}$	c
	5	
$\bar{c}$		$\bar{a}$

Since $a - c$ is a multiple of 5, then $a + \bar{c}$ and $c + \bar{a}$ are both multiples of 5.

To see this, note that $\bar{c} = 5 - c$ or $\bar{c} = 10 - c$ or $\bar{c} = 15 - c$, and so $a + \bar{c}$ is equal to one of $5 + a - c$ or $10 + a - c$ or $15 + a - c$ which are all multiples of 5 since $a - c$ is.

This means that each of the empty side cells must be filled with 5.

Finally, there are 2 choices for the bottom entry (since $\overline{a + c}$ is not a multiple of 5).

In this case, there are $8 \cdot 2 \cdot 2^3 \cdot 1^2 \cdot 2 = 2^8$ grids.

Case 6: $a \neq 5$ and $c \neq 5$ and $a + c$ is a not multiple of 5 and $a - c$ is not a multiple of 5

There are 8 choices for a.

There are then 4 choices for c (not 5, not either choice that makes $a + c$ a multiple of 5, not either choice that makes $a - c$ a multiple of 5).

There 2 choices for each of $\bar{a}$, $\bar{c}$, and $\overline{a + c}$.

There are also 2 choices for each of the 3 remaining entries in the grid since the two entries in each of the first column, third column and third row do not add to a multiple of 5.

In this case, there are $8 \cdot 4 \cdot 2^3 \cdot 2^3 = 2^{11}$ grids.

Combining all of the cases, the number of possible ways to complete the grid is

$$N = 1 + 2^8 + 2^8 + 2^8 + 2^8 + 2^{11} = 1 + 4 \cdot 2^8 + 2^{11} = 3073$$

The rightmost two digits of N are 73.

ANSWER: 73