The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2021 Pascal Contest

(Grade 9)

Tuesday, February 23, 2021
(in North America and South America)

Wednesday, February 24, 2021
(outside of North America and South America)

Solutions

1. Since Q is between P and R, then $PQ + QR = PR$.
   Since $PR = 12$ and $PQ = 3$, then $QR = PR - PQ = 12 - 3 = 9$.

   ANSWER: (D)

2. The fraction $\frac{4}{8}$ is equivalent to the fraction $\frac{1}{2}$.
   Therefore, the number 4 should be placed in the $\square$.

   ANSWER: (C)

3. Elena works for 4 hours and earns \$13.25 per hour.
   This means that she earns a total of $4 \times \$13.25 = \$53.00$.

   ANSWER: (E)

4. The perimeter of each of the squares of side length 1 is $4 \times 1 = 4$.
   The perimeters of the 7 squares in the diagram do not overlap, and so the perimeter of the entire figure is $7 \times 4 = 28$.

   ANSWER: (D)

5. Since there are 60 seconds in 1 minute, the number of seconds in 1.5 minutes is $1.5 \times 60 = 90$.
   Thus, Wesley's times were 63 seconds, 60 seconds, 90 seconds, 68 seconds, and 57 seconds.
   When these times in seconds are arranged in increasing order, we obtain 57, 60, 63, 68, 90.
   Thus, the median time is 63 seconds.

   ANSWER: (A)

6. The area of the original rectangle is $13 \times 10 = 130$.
   When the dimensions of the original rectangle are each increased by 2, we obtain a rectangle that is 15 by 12.
   The area of the new rectangle is $15 \times 12 = 180$, and so the area increased by $180 - 130 = 50$.

   ANSWER: (A)

7. *Solution 1*
   10% of 500 is $\frac{1}{10}$ of 500, which equals 50.
   Thus, 110% of 500 equals $500 + 50$, which equals 550.

   Solution 2
   110% of 500 is equal to $\frac{110}{100} \times 500 = 110 \times 5 = 550$.

   ANSWER: (E)

8. *Solution 1*
   We undo each of the operations in reverse order.
   The final result, 85, was obtained by multiplying a number by 5. This number was $85 \div 5 = 17$.
   The number 17 was obtained by decreasing n by 2. Thus, $n = 17 + 2 = 19$.

   Solution 2
   When n is decreased by 2, we obtain $n - 2$.
   When $n - 2$ is multiplied by 5, we obtain $5 \times (n - 2)$.
   From the given information, $5 \times (n - 2) = 85$ which means that $5n - 10 = 85$.
   From this, we obtain $5n = 95$ and so $n = \frac{95}{5} = 19$.

   ANSWER: (B)

9. Because 2 circles balance 1 triangle and 1 triangle balances 3 squares, then 2 circles balance 3 squares.
Because 2 circles balance 3 squares, then $2 + 2 = 4$ circles balance $3 + 3 = 6$ squares, which is choice (E).
(Can you argue that none of the other choices is equivalent to 6 squares?)

ANSWER: (E)

10. The integers that are multiples of both 5 and 7 are the integers that are multiples of 35.
The smallest multiple of 35 greater than 100 is $3 \times 35 = 105$. (The previous multiple of 35 is $2 \times 35 = 70$.)
Starting at 105 and counting by 35s, we obtain

$$105, 140, 175, 210, 245, 280, 315$$

The integers in this list that are between 100 and 300 and are not multiples of 10 (that is, whose units digit is not 0) are 105, 175, 245, of which there are 3.

ANSWER: (C)

11. Since $a \nabla b = a^b \times b^a$, then $2 \nabla 3 = 2^3 \times 3^2 = 8 \times 9 = 72$.

ANSWER: (B)

12. Since $\triangle PQR$ is right-angled at Q, then $\angle PRQ = 90^\circ - \angle QPR = 90^\circ - 54^\circ = 36^\circ$.
Since $\angle PRS = \angle QRS$, then $\angle QRS = \frac{1}{2}\angle PRQ = \frac{1}{2}(36^\circ) = 18^\circ$.
Since $\triangle RQS$ is right-angled at Q, then $\angle RSQ = 90^\circ - \angle QRS = 90^\circ - 18^\circ = 72^\circ$.

ANSWER: (E)

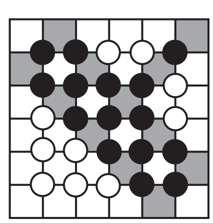
13. Since $m + 1 = \dfrac{n-2}{3}$, then $3(m + 1) = n - 2$.
This means that $3m + 3 = n - 2$ and so $3m - n = -2 - 3 = -5$.

ANSWER: (B)

14. Starting at 38, the robot moves 2 squares forward to 36, then rotates 90° clockwise to face 29 and then moves to 29.
Starting at 29, the robot moves 2 squares forward to 15, then rotates 90° clockwise to face 16 and then moves to 16.

ANSWER: (A)

15. There are 25 possible locations for the disc to be placed.
In the diagram below, each of these locations is marked with a small black disc if it is touching 2 shaded and unshaded squares (an equal number) and a small white disc if it is touching different numbers of shaded and unshaded squares.



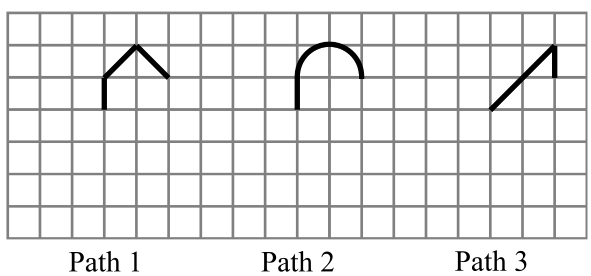
Therefore, there are 15 locations where the disc is touching an equal number of shaded and unshaded squares.
This means that the desired probability is $\frac{15}{25}$, which equals $\frac{3}{5}$.

ANSWER: (E)

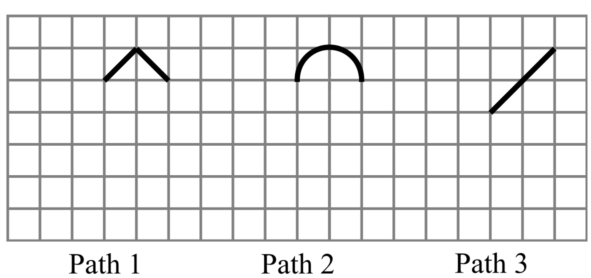
16. Perfect cubes have the property that the number of times that each prime factor occurs is a multiple of 3. This is because its prime factors can be separated into three identical groups; in this case, the product of each group is the cube root of the original number.
In particular, if $n^3 = 2^4 \times 3^2 \times 5^5 \times k$ where n is an integer, then the number of times that the prime factors 2, 3 and 5 occur in the integer n^3 must be multiples of 3.
Since n^3 already includes 4 factors of 2, then k must include at least 2 additional factors of 2, so that n^3 has a factor of 2^6. (k could also include more factors of 2, as long as the total number of factors of 2 is a multiple of 3.)
Since n^3 already includes 2 factors of 3, then k must include at least 1 additional factor of 3.
Since n^3 already includes 5 factors of 5, then k must include at least 1 additional factor of 5.
Therefore, k includes at least 2 factors of 2, at least 1 factor of 3, and at least 1 factor of 5.
This means that the smallest possible value of k is $2^2 \times 3 \times 5 = 60$. In principle, k could also include other prime factors, but to make k as small as possible, we do not need to consider this further.

ANSWER: (C)

17. To compare the lengths of these Paths, we begin by removing identical portions. In particular, we remove the horizontal segment of length 2, a vertical segment of length 1 from the left, and a vertical segment of length 4 from the right to obtain the following images:

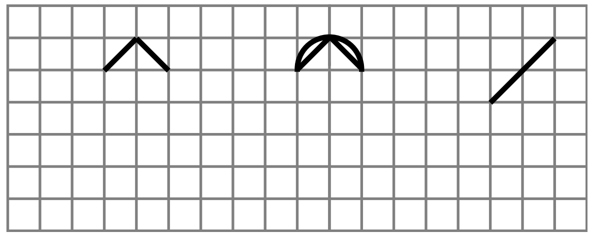


By removing the same lengths, we do not change the *relative* lengths of the Paths.
Each of the Paths still has a vertical segment of length 1, so we remove each of these segments, again maintaining the relative lengths of the Paths.



Each of Path 1 and Path 3 now consists of the diagonals of two of the grid squares. Thus, their original lengths were equal and so $x = z$.
This means that the final answer must equal (C) or (E), depending on whether $x = z$ is less than y or greater than y.

To answer this question, we re-draw the remaining segments of Path 1 under Path 2:



Path 1 Path 2 Path 3

Since a straight line path between two points is shorter than any other path between these two points, the length of the semi-circle is longer than the total length of the two straight line segments.
This means that $x = z$ and $z < y$.
(As an alternate approach, can you determine the length of each of the original Paths and compare these numerically?)

Answer: (C)

18. The length of time between 10:10 a.m. and 10:55 a.m. is 45 minutes.
The length of time between 10:55 a.m. and 11:58 a.m. is 1 hour and 3 minutes, or 63 minutes.
Since trains arrive at each of these times and we are told that trains arrive every x minutes, then both 45 and 63 must be multiples of x. (In other words, if we count forward repeatedly by x minutes starting at 10:10 a.m., we will eventually count 10:55 a.m. and then eventually count 11:58 a.m.)
Of the given choices (9, 7, 5, 10, 11), only 9 is a factor of each of 45 and 63.

Answer: (A)

19. *Solution 1*
We work backwards through the given information.
At the end, there is 1 candy remaining.
Since $\frac{5}{6}$ of the candies are removed on the fifth day, this 1 candy represents $\frac{1}{6}$ of the candies left at the end of the fourth day.
Thus, there were $6 \times 1 = 6$ candies left at the end of the fourth day.
Since $\frac{4}{5}$ of the candies are removed on the fourth day, these 6 candies represent $\frac{1}{5}$ of the candies left at the end of the third day.
Thus, there were $5 \times 6 = 30$ candies left at the end of the third day.
Since $\frac{3}{4}$ of the candies are removed on the third day, these 30 candies represent $\frac{1}{4}$ of the candies left at the end of the second day.
Thus, there were $4 \times 30 = 120$ candies left at the end of the second day.
Since $\frac{2}{3}$ of the candies are removed on the second day, these 120 candies represent $\frac{1}{3}$ of the candies left at the end of the first day.
Thus, there were $3 \times 120 = 360$ candies left at the end of the first day.
Since $\frac{1}{2}$ of the candies are removed on the first day, these 360 candies represent $\frac{1}{2}$ of the candies initially in the bag.
Thus, there were $2 \times 360 = 720$ in the bag at the beginning.

Solution 2
Suppose that there were x candies in the bag at the beginning.

On the first day, $\frac{1}{2}$ of the candies are eaten, which means that $1-\frac{1}{2}=\frac{1}{2}$ of the candies remain. Since there were x candies at the beginning of the first day, there are $\frac{1}{2}x$ candies at the end of the first day.
On the second day, $\frac{2}{3}$ of the remaining candies are eaten, which means that $1-\frac{2}{3}=\frac{1}{3}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{2}x$ candies at the beginning of the second day, there are $\frac{1}{3}\times\frac{1}{2}x=\frac{1}{6}x$ candies at the end of the second day.
On the third day, $\frac{3}{4}$ of the remaining candies are eaten, which means that $1-\frac{3}{4}=\frac{1}{4}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{6}x$ candies at the beginning of the third day, there are $\frac{1}{4}\times\frac{1}{6}x=\frac{1}{24}x$ candies at the end of the third day.
On the fourth day, $\frac{4}{5}$ of the remaining candies are eaten, which means that $1-\frac{4}{5}=\frac{1}{5}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{24}x$ candies at the beginning of the fourth day, there are $\frac{1}{5}\times\frac{1}{24}x=\frac{1}{120}x$ candies at the end of the fourth day.
On the fifth day, $\frac{5}{6}$ of the remaining candies are eaten, which means that $1-\frac{5}{6}=\frac{1}{6}$ of the candies from the beginning of the day remain at the end of the day.
Since there were $\frac{1}{120}x$ candies at the beginning of the fifth day, there are $\frac{1}{6}\times\frac{1}{120}x=\frac{1}{720}x$ candies at the end of the fifth day.
Since 1 candy remains, then $\frac{1}{720}x=1$ which gives $x=720$, and so there were 720 candies in the bag before the first day.

ANSWER: (B)

20. We make a chart of the possible integers, building their digits from left to right. In each case, we could determine the required divisibility by actually performing the division, or by using the following tests for divisibility:

- An integer is divisible by 3 when the sum of its digits is divisible by 3.
- An integer is divisible by 4 when the two-digit integer formed by its tens and units digits is divisible by 4.
- An integer is divisible by 5 when its units digit is 0 or 5.

$8R$	$8RS$	$N=8RST$
81	812	8120
		8125
	816	8160
		8165
84	840	8400
		8405
	844	8440
		8445
	848	8480
		8485
87	872	8720
		8725
	876	8760
		8765

In the first column, we note that the integers between 80 and 89 that are multiples of 3 are 81, 84 and 87. In the second column, we look for the mutiples of 4 between 810 and 819, between

840 and 849, and between 870 and 879. In the third column, we add units digits of 0 or 5.
This analysis shows that there are 14 possible values of N.

ANSWER: (E)

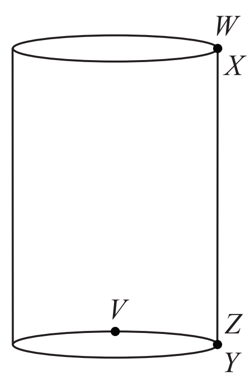
21. Since the average volume of three cubes is 700 cm^3, their total volume is 3×700 cm^3 or 2100 cm^3.
The volume of a cube with edge length s cm is s^3 cm^3.
Therefore, $3^3 + 12^3 + x^3 = 2100$ and so $27 + 1728 + x^3 = 2100$ or $x^3 = 345$.
Since $x^3 = 345$, then $x \approx 7.01$, which is closest to 7.

ANSWER: (E)

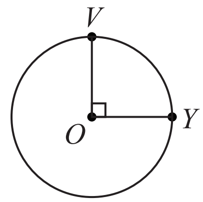
22. The height of each block is 2, 3 or 6.
Thus, the total height of the tower of four blocks is the sum of the four heights, each of which equals 2, 3 or 6.
If 4 blocks have height 6, the total height equals $4 \times 6 = 24$.
If 3 blocks have height 6, the fourth block has height 3 or 2.
Therefore, the possible heights are $3 \times 6 + 3 = 21$ and $3 \times 6 + 2 = 20$.
If 2 blocks have height 6, the third and fourth blocks have height 3 or 2.
Therefore, the possible heights are $2\times6+3+3 = 18$ and $2\times6+3+2 = 17$ and $2\times6+2+2 = 16$.
If 1 block has height 6, the second, third and fourth blocks have height 3 or 2.
Therefore, the possible heights are $6+3+3+3 = 15$ and $6+3+3+2 = 14$ and $6+3+2+2 = 13$ and $6 + 2 + 2 + 2 = 12$.
If no blocks have height 6, the possible heights are $3 + 3 + 3 + 3 = 12$ and $3 + 3 + 3 + 2 = 11$ and $3 + 3 + 2 + 2 = 10$ and $3 + 2 + 2 + 2 = 9$ and $2 + 2 + 2 + 2 = 8$.
The possible heights are thus $24, 21, 20, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8$.
There are 14 possible heights.

ANSWER: (B)

23. When the cylinder is created, W and X touch and Z and Y touch.



This means that WY is vertical and so is perpendicular to the plane of the circular base of the cylinder.
This means that $\triangle VYW$ is right-angled at Y.
By the Pythagorean Theorem, $WV^2 = WY^2 + VY^2$.
Note that WY equals the height of the rectangle, which is 3 (the length of WZ) and that VY is now measured *through* the cylinder, not along the line segment ZY.
Let O be the centre of the circular base of the cylinder.
In the original rectangle, $ZY = WX = 4$ and $ZV = 3$, which means that $VY = 1 = \frac{1}{4}ZY$.
This means that V is one-quarter of the way around the circumference of the circular base from Y back to Z.



As a result, $\angle YOV = 90^\circ$, since 90° is one-quarter of a complete circular angle.
Thus, $\triangle YOV$ is right-angled at O.
By the Pythagorean Theorem, $VY^2 = VO^2 + OY^2$.
Since YO and OV are radii of the circular base, then $VO = OY$ and so $YV^2 = 2VO^2$.
Since the circumference of the circular base is 4 (the original length of ZY), then if the radius of the base is r, we have $2\pi r = 4$ and so $r = \dfrac{4}{2\pi} = \dfrac{2}{\pi}$.
Since $VO = r$, then $YV^2 = 2VO^2 = 2\left(\dfrac{2}{\pi}\right)^2 = \dfrac{8}{\pi^2}$.
This means that

$$WV^2 = WY^2 + YV^2 = 9 + \frac{8}{\pi^2} = \frac{9\pi^2 + 8}{\pi^2} = \frac{8 + 9\pi^2}{\pi^2}$$

and so $WV = \sqrt{\dfrac{8 + 9\pi^2}{1 \cdot \pi^2}}$.
Since the coefficient of π^2 in the denominator is 1, it is not possible to "reduce" the values of a, b and c any further, and so $a = 8$, $b = 9$, and $c = 1$, which gives $a + b + c = 18$.

ANSWER: (C)

24. Starting with a list of $66 = 2 \times 33$ items, the items in the first 33 positions

$$1, 2, 3, \ldots, 31, 32, 33$$

are moved by an in-shuffle to the odd positions of the resulting list, namely to the positions

$$1, 3, 5, \ldots, 61, 63, 65$$

respectively. This means that an item in position x with $1 \leq x \leq 33$ is moved by an in-shuffle to position $2x - 1$.
We can see why this formula works by first moving the items in positions $1, 2, 3, \ldots, 31, 32, 33$ to the even positions $2, 4, 6, \ldots, 62, 64, 66$ (doubling the original position numbers) and then shifting each backwards one position to $1, 3, 5, \ldots, 61, 63, 65$.

Also, the items in the second 33 positions

$$34, 35, 36, \ldots, 64, 65, 66$$

are moved by an in-shuffle to the even positions of the resulting list, namely to the positions

$$2, 4, 6, \ldots, 62, 64, 66$$

respectively. This means that an item in position x with $34 \leq x \leq 66$ is moved by an in-shuffle to position $2(x - 33)$.
We can see why this formula works by first moving the items in positions $34, 35, 36, \ldots, 64, 65, 66$ backwards 33 positions to $1, 2, 3, \ldots, 31, 32, 33$ and then doubling their position numbers to obtain $2, 4, 6, \ldots, 62, 64, 66$.

In summary, the item in position x is moved by an in-shuffle to position

- $2x - 1$ if $1 \leq x \leq 33$
- $2(x - 33)$ if $34 \leq x \leq 66$

Therefore, the integer 47 is moved successively as follows:

List	Position
1	47
2	$2(47 - 33) = 28$
3	$2(28) - 1 = 55$
4	$2(55 - 33) = 44$
5	$2(44 - 33) = 22$
6	$2(22) - 1 = 43$
7	$2(43 - 33) = 20$
8	$2(20) - 1 = 39$
9	$2(39 - 33) = 12$
10	$2(12) - 1 = 23$
11	$2(23) - 1 = 45$
12	$2(45 - 33) = 24$
13	$2(24) - 1 = 47$

Because the integer 47 moves back to position 47 in list 13, this means that its positions continue in a cycle of length 12:

$$47, 28, 55, 44, 22, 43, 20, 39, 12, 23, 45, 24$$

This is because the position to which an integer moves is completely determined by its previous position and so the list will cycle once one position repeats.
We note that the integer 47 is thus in position 24 in every 12th list starting at the 12th list.
Since $12 \times 83 = 996$ and $12 \times 84 = 1008$, the cycle occurs a total of 83 complete times and so the integer 47 is in the 24th position in 83 lists. Even though an 84th cycle begins, it does not conclude and so 47 does not occur in the 24th position for an 84th time among the 1001 lists.

ANSWER: (C)

25. When Yann removes 4 of the n integers from his list, there are $n - 4$ integers left.
Suppose that the sum of the $n - 4$ integers left is T.
The average of these $n - 4$ integers is $89.5625 = 89.5 + 0.0625 = 89 + \frac{1}{2} + \frac{1}{16} = 89\frac{9}{16} = \frac{1433}{16}$.
Since the sum of the $n-4$ integers is T, then $\frac{T}{n-4} = \frac{1433}{16}$ which means that $16T = 1433(n-4)$.
Since 1433 and 16 have no common divisor larger than 1 (the positive divisors of 16 are 1, 2, 4, 8, 16, none of which other than 1 is a divisor of 1433), the value of $n - 4$ is a multiple of 16.

Since $100 < p < q < r < s$, the original list includes more than 100 numbers.
Since the original list includes consecutive integers starting at 1 and only 4 of more than 100 numbers are removed, it seems likely that the average of the original list and the average of the new list should be relatively similar.
Since the average of the new list is 89.5625 which is close to 90, it seems reasonable to say that the average of the original list is close to 90.
Since the original list is a list of consecutive positive integers starting at 1, this means that we would guess that the original list has roughly 180 integers in it.

In other words, n appears to be near 180.
We do know that $n-4$ is a multiple of 16. The closest multiples of 16 to 180 are 160, 176 and 192, which correspond to $n=164$, $n=180$, and $n=196$.

Suppose that $n=180$, which seems like the most likely possibility. We will show at the end of the solution that this is the only possible value of n.
The equation $\dfrac{T}{n-4}=89.5625$ gives $T=176\times 89.5625=15\,763$.
The sum of the n integers in the original list is

$$1+2+3+4+\cdots+(n-1)+n=\frac{1}{2}n(n+1)$$

When $n=180$, the sum of the integers $1,2,3,\ldots,178,179,180$ is $\frac{1}{2}(180)(181)=16\,290$.
Since the sum of the numbers in the original list is 16 290 and the sum once the four numbers are removed is 15 763, the sum of the four numbers removed is $16\,290-15\,763=527$.
In other words, $p+q+r+s=527$.

We now want to count the number of ways in which we can choose p, q, r, s with the conditions that $100<p<q<r<s\leq 180$ and $p+q+r+s=527$ with at least three of p, q, r, s consecutive.
The fourth of these integers is at least 101 and at most 180, which means that the sum of the three consecutive integers is at least $527-180=347$ and is at most $527-101=426$.
This means that the consecutive integers are at least $115,116,117$ (whose sum is 348) since $114+115+116=345$ which is too small and smaller integers will give sums that are smaller still.
If p, q, r equal 115, 116, 117, then $s=527-348=179$.
The consecutive integers are at most $141,142,143$ (whose sum is 426) since $142+143+144=429$ which is too large and larger integers will give sums that are larger still.
If p, q, r equal 141, 142, 143, then $s=527-426=101$.
When each of the three consecutive integers is increased by 1 and the sum is constant, the fourth integer is decreased by 3 to maintain this constant sum.
Using all of this, we obtain the following lists p, q, r, s:

$$115,116,117,179 \;\;;\;\; 116,117,118,176 \;\;;\;\; \ldots \;\;;\;\; 130,131,132,134$$

$$128,132,133,134 \;\;;\;\; 125,133,134,135 \;\;;\;\; \ldots \;\;;\;\; 101,141,142,143$$

Note that we cannot use 131, 132, 133, 131, since p, q, r, s must be distinct.
There are 26 lists of integers that can be removed (16 in the first set and 10 in the second set).
The corresponding values of s are:

$$179,176,173,170,167,164,161,158,155,152,149,146,143,140,137,134$$

$$134,135,136,137,138,139,140,141,142,143$$

There are 4 values of s that overlap between the two lists, and so there are $26-4=22$ possible values for s.

Why is $n=180$ the only possible value of n?
To see this, we use the fact that the average of the list of consecutive integers starting at a and ending at b equals the average of a and b, or $\dfrac{a+b}{2}$. (This is true because the integers in the

list have a constant difference and are thus evenly distributed, which means that the average of the first and last integers will equal the average of all of the integers in the list.)

The original list of integers is $1, 2, \ldots, n-1, n$ which has an average of $\dfrac{n+1}{2}$.

If the four largest integers are removed from the list, the new list is $1, 2, \ldots, n-5, n-4$, which has an average of $\dfrac{n-3}{2}$.

If the four smallest integers are removed from the list, the new list is $5, 6, \ldots, n-1, n$, which has an average of $\dfrac{n+5}{2}$.

When any four integers are removed, the sum of the remaining integers is greater than or equal to the sum of $1, 2, \ldots, n-5, n-4$ and less than or equal to the sum of $5, 6, \ldots, n-1, n$. Since the denominator in the average calculation remains the same, the average of any of the lists after four numbers are removed is at least $\dfrac{n-3}{2}$ and at most $\dfrac{n+5}{2}$.

This means that the actual average (which is 89.5625) is greater than or equal to $\dfrac{n-3}{2}$ and less than or equal to $\dfrac{n+5}{2}$.

Since $89.5625 \geq \dfrac{n-3}{2}$, then $n-3 \leq 179.125$ and so $n \leq 182.125$.

Since $89.5625 \leq \dfrac{n+5}{2}$, then $n+5 \geq 179.125$ and so $n \geq 174.125$.

Since n is an integer, then $175 \leq n \leq 182$ and so $171 \leq n-4 \leq 178$.

Since $n-4$ is a multiple of 16, then $n-4 = 176$ and so $n = 180$, as required.

Answer: 22

(The correct answer was missing from the original version of the problem.)