The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2020 Canadian Team Mathematics Contest

Team Problems

IMPORTANT NOTES:

- Calculating devices are not permitted.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

1. What is the integer that is greater than $\sqrt{11}$ but less than $\sqrt{19}$?
2. What is the value of $\dfrac{3^5 - 3^4}{3^3}$?
3. The measures of the interior angles of a quadrilateral are in the ratio $1 : 2 : 3 : 4$. Determine the measure in degrees of the smallest interior angle.
4. Determine the integer equal to
$$2020 + 1919 + 1818 + 1717 + 1616 + 1515 + 1414 + 1313 + 1212 + 1111 + 1010\ .$$
5. What is the smallest eight-digit positive integer that has exactly four digits which are 4?
6. On Fridays, the price of a ticket to a museum is \$9. On one particular Saturday, there were 200 visitors to the museum, which was twice as many visitors as there were the day before. The total money collected from ticket sales on that particular Saturday was $\frac{4}{3}$ as much as the day before. The price of tickets on Saturdays is $\$k$. Determine the value of k.
7. What is the smallest four-digit positive integer that is divisible by both 5 and 9 and has only even digits?
8. The figure below was constructed by taking a semicircle with diameter 64 and replacing the diameter with four semicircles each having equal diameter. What is the perimeter of the figure?

9. How many times does the digit 0 appear in the integer equal to 20^{10}?
10. If $3y - 2x = 4$, determine the value of $\dfrac{16^{x+1}}{8^{2y-1}}$.

11. Let p_i be the i^{th} prime number; for example, $p_1 = 2, p_2 = 3$, and $p_3 = 5$. For each prime number, construct the point $Q_i(p_i, 0)$. Suppose A has coordinates $(0, 2)$. Determine the sum of the areas of the triangles $\triangle AQ_1Q_2, \triangle AQ_2Q_3, \triangle AQ_3Q_4, \triangle AQ_4Q_5, \triangle AQ_5Q_6$, and $\triangle AQ_6Q_7$.

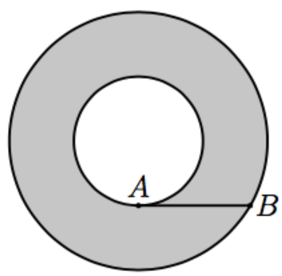
12. Éveriste listed all of the positive integers from 1 to 90. He then crossed out all of the multiples of 3 from the list. Of the remaining numbers, he then crossed out all of the multiples of 5. How many numbers were *not* crossed out?

13. The parabola with equation $y = -\frac{1}{4}x^2 + 5x - 21$ has its vertex at point A and crosses the x-axis at $B(b, 0)$ and $F(f, 0)$ where $b < f$. A second parabola with its vertex at B passes through A and crosses the y-axis at D. What are the coordinates of D?

14. Jeff caught 21 fish, each having a mass of at least 0.2 kg. He noticed that the average mass of the first three fish that he caught was the same as the average mass of all 21 fish. The total mass of the first three fish was 1.5 kg. What is the largest possible mass of any one fish that Jeff could have caught?

15. Suppose that a, b, c, and d are positive integers which are not necessarily distinct. If $a^2 + b^2 + c^2 + d^2 = 70$, what is the largest possible value of $a + b + c + d$?

16. In the diagram below, the two circles have the same centre. Point A is on the inner circle and point B is on the outer circle. Line segment AB has length 5 and is tangent to the inner circle at A. What is the area of the shaded region?



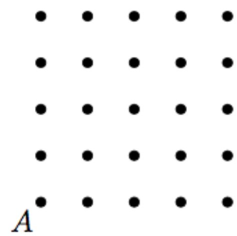
17. For each real number m, the function $f(x) = x^2 - 2mx + 8m + 4$ has a minimum value. What is the maximum of these minimum values?

18. A sequence $t_1, t_2, t_3, \ldots$ is defined by

$$t_n = \begin{cases} \dfrac{1}{7^n} & \text{when } n \text{ is odd} \\[2ex] \dfrac{2}{7^n} & \text{when } n \text{ is even} \end{cases}$$

for each positive integer n. Determine the sum of all of the terms in this sequence; that is, calculate $t_1 + t_2 + t_3 + \cdots$.

19. Pictured below is a rectangular array of dots with point the bottom left point labelled A. In how many ways can two points in the array be chosen so that they, together with point A, form a triangle with positive area?



20. Lyla and Isabelle run on a circular track both starting at point P. Lyla runs at a constant speed in the clockwise direction. Isabelle also runs in the clockwise direction at a constant speed 25% faster than Lyla. Lyla starts running first and Isabelle starts running when Lyla has completed one third of one lap. When Isabelle passes Lyla for the fifth time, how many times has Lyla returned to point P?

21. Suppose that $f(x) = 2\sin^2(\log x) + \cos(\log x^2) - 5$ for each $x > 0$. What is the value of $f(\pi)$?

22. How many triples (x, y, z) of positive integers satisfy both

    (i) $x + y + z$ is a multiple of 3, and

    (ii) $1 \leq x \leq 10$, $1 \leq y \leq 10$, and $1 \leq z \leq 10$?

23. Rectangle $ABCD$ has diagonal BD with endpoints $B(4, 2)$ and $D(12, 8)$. Diagonal AC lies on the line with equation $x + 2y - 18 = 0$. Determine the area of $ABCD$.

24. Determine the sum of the real numbers x for which $\dfrac{2x}{x^2 + 5x + 3} + \dfrac{3x}{x^2 + x + 3} = 1$.

25. A circle with centre O is inscribed in square $ABCD$ having side length 60. Point E is the midpoint of AD. Line segments AC and BE intersect the top half of the circle at points F and G respectively, and they intersect each other at point H. What is the total area of the shaded regions?