The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Hypatia Contest

(Grade 11)

Wednesday, April 10, 2019
(in North America and South America)

Thursday, April 11, 2019
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: 75 minutes

Do not open this booklet until instructed to do so.

Number of questions: 4 **Each question is worth 10 marks**

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 2 or 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions must be supplied by your supervising teacher and inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

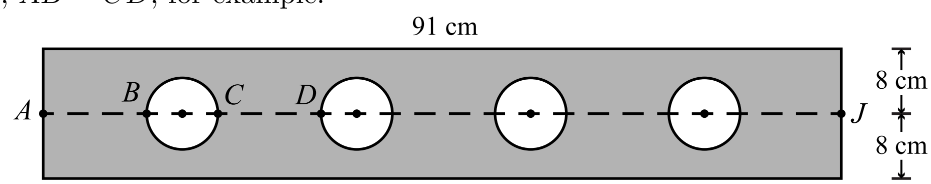
Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked, place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked, provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions, and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
7. No student may write more than one of the Fryer, Galois and Hypatia Contests in the same year.

1. A rectangular piece of metal measures 91 cm by 16 cm. Four identical circular discs are punched out of this piece of metal. The centres of the circular holes are on the midline of the rectangle, AJ, as shown. These four holes are equally spaced along the piece of metal. That is, $AB = CD$, for example.



(a) If the radius of each hole is 2 cm, what is the distance along the midline between adjacent holes (i.e. what is the length of CD)?

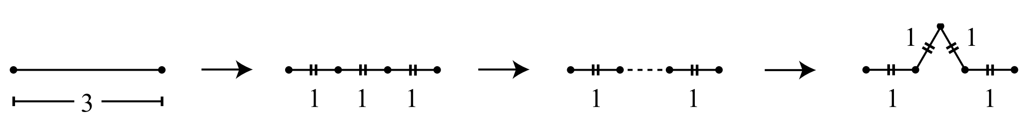
(b) If the distance along the midline between adjacent holes is equal to the radius of each hole, what is the radius of each hole?

(c) Show why the fact that holes must be circles means that the distance between adjacent holes cannot be 5 cm.

2. A *bump* can be added to any line segment through the following process:

- break the segment into three segments of equal length,
- remove the middle segment,
- add an equilateral triangular shaped *bump* with each side length equal to the removed segment.

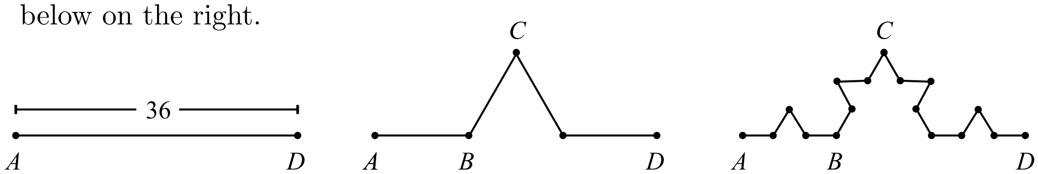
The series of diagrams below shows a *bump* being added to a line segment of length 3, transforming it into a path of length 4.



(a) A line segment has length 21. How long will the path be after a bump is added?

(b) A path with exactly one bump has length 240. How long was the original line segment?

(c) Lin starts with a line segment that has length 36 and adds a bump to it. She then adds bumps to each line segment of that path. The resulting figure is shown below on the right.



What is the total path length of the resulting figure?

(d) Ann starts with a line segment having length equal to some positive integer n and adds a bump to it resulting in Path 1. Ann then adds bumps to each line segment of Path 1 resulting in Path 2. She continues this process to create Path 3, Path 4, and finally Path 5. If the length of Path 5 is an integer, determine the smallest possible value of n.

3. The *arithmetic mean* of two positive real numbers x and y is half the sum of the two numbers, or $\frac{x+y}{2}$. The *geometric mean* of two positive real numbers x and y is the square root of the product of the two numbers, or $\sqrt{xy}$.

(a) What are the arithmetic and geometric means of 36 and 64?

(b) Determine a pair of positive real numbers whose arithmetic mean is 13 and geometric mean is 12.

(c) For two positive integers x and y, the arithmetic mean minus the geometric mean is equal to 1. Determine, with justification, all such pairs (x, y) where $x < y \leq 50$.

4. (a) Suppose that c is a real number. Solve the following system of equations for x and y in terms of c:

$$\begin{aligned} 3x + 4y &= 10 \\ 5x + 6y &= c \end{aligned}$$

(b) Determine all integers d for which the system of equations

$$\begin{aligned} x + 2y &= 3 \\ 4x + dy &= 6 \end{aligned}$$

has a solution (x, y), where x and y are integers.

(c) Determine a positive integer k for which there are exactly 8 integers n for which the system of equations

$$\begin{aligned} (9n+6)x - (3n+2)y &= 3n^2 + 6n + (3k+5) \\ (6n+4)x + (3n^2+2n)y &= -n^2 + (2k+2) \end{aligned}$$

has a solution (x, y), where x and y are integers.

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For students...

Thank you for writing the 2019 Hypatia Contest! Each year, more than 260 000 students from more than 80 countries register to write the CEMC's Contests.

Encourage your teacher to register you for the Canadian Intermediate Mathematics Contest or the Canadian Senior Mathematics Contest, which will be written in November 2019.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2019/2020 contests
- Register your students for the Canadian Senior and Intermediate Mathematics Contests which will be written in November
- Look at our free online courseware for senior high school students
- Learn about our face-to-face workshops and our web resources
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- Find your school's contest results