The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018 Galois Contest

Thursday, April 12, 2018
(in North America and South America)

Friday, April 13, 2018
(outside of North America and South America)

Solutions

1. (a) Simplifying, we get $\dfrac{12x^2}{3x} = 4x$ for $x \neq 0$.

   (b) Since $\dfrac{12x^2}{3x} = 4x$, the value of the expression $\dfrac{12x^2}{3x}$ is equal to the value of the simplified expression $4x$ for all values of $x \neq 0$.

   So when $x = 5$, the value of the expression $\dfrac{12x^2}{3x}$ is equal to $4(5) = 20$.

   (c) Simplifying, we get $\dfrac{8mn}{3m^2} = \dfrac{8n}{3m}$ for $m \neq 0$.

   The value of the expression $\dfrac{8mn}{3m^2}$ is equal to the value of the simplified expression $\dfrac{8n}{3m}$ for all values of $m \neq 0$.

   Substituting $n = 2m$ into $\dfrac{8n}{3m}$ and simplifying, we get $\dfrac{8(2m)}{3m} = \dfrac{16m}{3m} = \dfrac{16}{3}$.

   When $n = 2m$ and $m \neq 0$, the value of the expression $\dfrac{8mn}{3m^2}$ is $\dfrac{16}{3}$.

   (d) Simplifying, we get $\dfrac{8p^2q}{5pq^2} = \dfrac{8p}{5q}$ for $p \neq 0$, $q \neq 0$.

   When $q = 6$, we get $\dfrac{8p}{5q} = \dfrac{8p}{5(6)} = \dfrac{8p}{30} = \dfrac{4p}{15}$.

   That is, when $q = 6$ (and $p \neq 0$) the expression $\dfrac{8p^2q}{5pq^2}$ is equal to $\dfrac{4p}{15}$, and so the solution to $3 \leq \dfrac{8p^2q}{5pq^2} \leq 4$ is equivalent to the solution to $3 \leq \dfrac{4p}{15} \leq 4$.

   Solving $3 \leq \dfrac{4p}{15} \leq 4$, we get $45 \leq 4p \leq 60$ or $\dfrac{45}{4} \leq p \leq \dfrac{60}{4}$, and so $11.25 \leq p \leq 15$.

   Since p is a positive integer, then $p = 12, 13, 14, 15$.

   Note: In each of the solutions to (b), (c) and (d), we chose to simplify the expression before substituting. Changing the order to substitution followed by simplification would also allow us to solve these problems.

2. (a) In $\triangle ABC$, $\angle ABC = 90^\circ$.
   Using the Pythagorean Theorem, we get $AC^2 = AB^2 + BC^2$ or $AC^2 = 8^2 + 15^2$, and so $AC = \sqrt{64 + 225} = \sqrt{289} = 17$ (since $AC > 0$).

   (b) In Figure 2, EF is a diameter and so its length is twice the radius or 26.
   From the second fact, we know that $\angle EDF = 90^\circ$.
   Using the Pythagorean Theorem, we get $DF^2 = EF^2 - DE^2$ or $DF^2 = 26^2 - 24^2$, and so $DF = \sqrt{676 - 576} = \sqrt{100} = 10$ (since $DF > 0$).

   (c) Since SQ is a diameter, then $\angle SPQ = \angle SRQ = 90^\circ$.
   In $\triangle SPQ$, $SP = PQ$ which means that $\triangle SPQ$ is isosceles and so

   $\angle PQS = \angle PSQ = \dfrac{180^\circ - 90^\circ}{2} = 45^\circ$.

   Since $\angle RQP = 80^\circ$, then $\angle RQO = \angle RQP - \angle PQS = 80^\circ - 45^\circ = 35^\circ$.
   In $\triangle ROQ$, $OR = OQ$ (both are radii) and so $\angle QRO = \angle RQO = 35^\circ$ and $\angle ROQ = 180^\circ - 2 \times 35^\circ = 110^\circ$.
   In $\triangle SRQ$, we get $\angle RSQ = 180^\circ - \angle SRQ - \angle RQS = 180^\circ - 90^\circ - 35^\circ = 55^\circ$.

3. (a) A cylinder having radius r and height h has volume $\pi r^2 h$.
Cylinder A has radius 12 and height 25, and so its volume is $\pi(12)^2(25) = 3600\pi$.
Before Cylinder B is lowered into Cylinder A, the height of water in Cylinder A is 19, and so initially the volume of water in Cylinder A is $\pi(12)^2(19) = 2736\pi$.
The height of Cylinder B, 30, is greater than the height of Cylinder A, and so it is not possible for water to pour out of Cylinder A and into Cylinder B.
When Cylinder B is lowered to the bottom of Cylinder A, the portion of Cylinder B lying inside Cylinder A has radius 9 and height 25 (the height of Cylinder A).
Thus, the volume that Cylinder B occupies within Cylinder A is $\pi(9)^2(25) = 2025\pi$.
Since water cannot pour into Cylinder B, the space available for water within Cylinder A (and outside Cylinder B) is the difference between the volume of Cylinder A and the volume of Cylinder B lying inside Cylinder A, or $3600\pi - 2025\pi = 1575\pi$.
The volume of water in Cylinder A was initially 2736π and once Cylinder B is lowered to the bottom of Cylinder A, the space available for water in Cylinder A becomes 1575π.
Therefore, the volume of water that spills out of Cylinder A and onto the ground is $2736\pi - 1575\pi = 1161\pi$.

(b) As Cylinder B is lowered into Cylinder A, water spills out of Cylinder A and onto the ground when:

(i) the volume of water in Cylinder A exceeds the volume inside Cylinder A and outside Cylinder B, and

(ii) the top of Cylinder B lies above the top of Cylinder A.
(See Figure 1 given in the question.)

As Cylinder B is lowered into Cylinder A, water spills out of Cylinder A and into Cylinder B when:

(i) the top of Cylinder B lies below the top of Cylinder A, and

(ii) the volume of water in Cylinder A (and outside Cylinder B) exceeds the volume inside Cylinder A that lies below the top of Cylinder B and outside Cylinder B, and

(iii) Cylinder B is not full of water.
(See Figure 2 given in the question.)

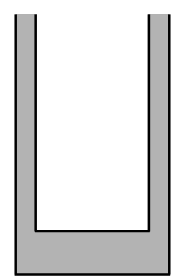
Figure 3

In Figure 3 shown, the top of Cylinder B has been lowered to the same level as the top of Cylinder A.
At this point, the volume of space inside Cylinder A and outside Cylinder B is $\pi(12)^2(25) - \pi(9)^2(20) = 3600\pi - 1620\pi = 1980\pi$.
The initial volume of water in Cylinder A was 2736π, and so at this point the volume of water that has spilled out of Cylinder A and onto the ground is $2736\pi - 1980\pi = 756\pi$.
(Since the top of Cylinder B is not below the top of Cylinder A, no water has spilled out of Cylinder A and into Cylinder B at this point.)
As Cylinder B is lowered below this level, water will spill out of Cylinder A and into Cylinder B. How much water will spill into Cylinder B?

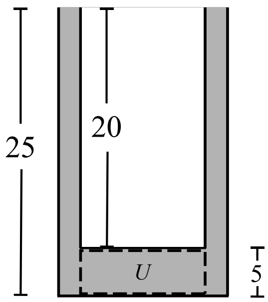

Figure 4

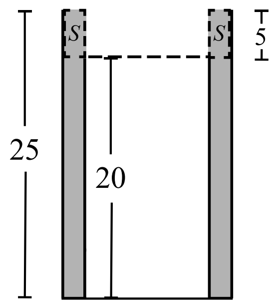

Figure 5

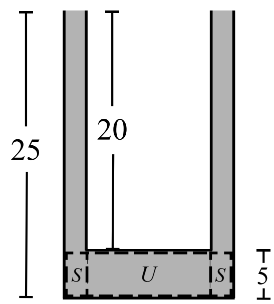

Figure 6

In Figure 4, the volume of water labelled U (lying directly underneath Cylinder B) will be displaced by Cylinder B when it is lowered to the bottom of Cylinder A.
This volume of water will spill into Cylinder B (since the top of Cylinder B will be below the top of Cylinder A).
The shape of the water labelled U is cylindrical, has radius equal to that of Cylinder B, 9, and has height $25 - 20 = 5$.
So the volume of the water labelled U is $\pi(9)^2(5) = 405\pi$.
In addition, the water labelled S in Figure 5 will also spill into Cylinder B when it is lowered to the bottom of Cylinder A.
The shape of the water labelled S is a cylindrical ring, inside Cylinder A and outside Cylinder B, having height $25 - 20 = 5$, and so has volume $\pi(12)^2(5) - \pi(9)^2(5) = 315\pi$.
The volume of water that spills from Cylinder A into Cylinder B is $405\pi + 315\pi = 720\pi$.
The depth, d, of water in Cylinder B when it is on the bottom of Cylinder A is given by $\pi(9)^2(d) = 720\pi$ and so $d = \dfrac{720\pi}{81\pi} = \dfrac{80}{9}$.
Note: We could have determined the volume of water that spills into Cylinder B by noticing that the volume labelled S (in Figure 5), is equal to the volume of water surrounding the water labelled U (see Figure 6).
Since both volumes have height 5, their combined volume is equal to that of a cylinder with radius 12 and height 5, and so $V = \pi(12)^2(5) = 720\pi$, as we previously determined.

(c) *Solution 1*

We begin by finding the range of values of h for which some water will spill out of Cylinder A when Cylinder B is lowered to the bottom of Cylinder A.
Consider lowering Cylinder B into Cylinder A until the water level reaches the top of Cylinder A, as shown in Figure 7 (we know this is possible for some values of h since it occured in part (a)).
Let y be the distance between the bottoms of the two cylinders, and so the distance between the top of Cylinder A and the bottom of Cylinder B is $25 - y$.

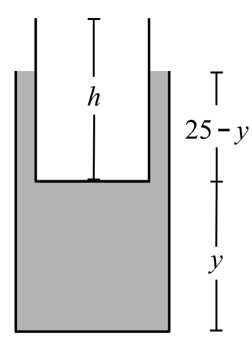

Figure 7

The volume of water, V_w, is equal to the volume of Cylinder A that lies below the bottom of Cylinder B, or $\pi(12)^2(y)$, added to the volume inside Cylinder A and outside Cylinder B between the top of Cylinder A and the bottom of Cylinder B, or $\pi(12)^2(25 - y) - \pi(9)^2(25 - y) = \pi(12^2 - 9^2)(25 - y)$.
That is, $V_w = \pi(12)^2(y) + \pi(12^2 - 9^2)(25 - y) = 144\pi y + 63\pi(25 - y) = 81\pi y + 1575\pi$.
From part (a), the initial volume of water is 2736π, and so we get $81\pi y + 1575\pi = 2736\pi$ or $81\pi y = 1161\pi$, and so $y = \frac{43}{3}$.
So if $h > 25 - y = 25 - \frac{43}{3} = \frac{32}{3}$, then water will spill out of Cylinder A onto the ground.
What if $h \leq \frac{32}{3}$?

When $h \leq \frac{32}{3}$, Cylinder B may be lowered so that its top is level with the top of Cylinder A without any water spilling out of Cylinder A onto the ground.
In this case when $h \leq \frac{32}{3}$, then $y \geq 25 - \frac{32}{3} = \frac{43}{3}$, and so $y > h$.
That is, when Cylinder B is lowered so that its top is level with the top of Cylinder A, the volume of water that lies directly below Cylinder B is greater that the volume of Cylinder B and so Cylinder B will be completely full of water when it is lowered to the bottom of Cylinder A.
In this question, we require that Cylinder B not be full and so $h > \frac{32}{3}$ and water will spill out of Cylinder A onto the ground before the top of Cylinder B is level with the top of Cylinder A.

Next, we will further restrict the range of values of h so that when Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full.
Consider lowering Cylinder B to the point where the tops of the two cylinders are level with one another (so then $h \leq 25$).
Some water has spilled out of Cylinder A.
When Cylinder B is lowered beyond this point (so then we require $h < 25$), water will spill from Cylinder A into Cylinder B (and not onto the ground).

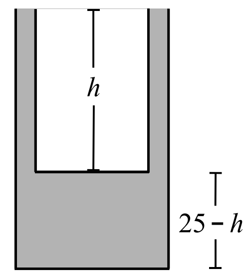

Figure 8

From the solution in part (b), recall that when Cylinder B is lowered to the bottom of Cylinder A, the volume of water that will spill from Cylinder A into Cylinder B is equal to the volume of water inside Cylinder A that lies below the bottom of Cylinder B (as in Figure 8).
This cylinder has radius 12 and height $25 - h$, and so has volume $\pi(12)^2(25 - h)$.
Assume that when this volume of water has spilled into Cylinder B, it fills Cylinder B to a depth of d.
Once Cylinder B is lowered to the bottom of Cylinder A, the volume of water in Cylinder B, $\pi(9)^2(d)$, must equal $\pi(12)^2(25 - h)$.
Solving for d, we get $81\pi d = 144\pi(25 - h)$ or $d = \dfrac{3600 - 144h}{81}$, and so $d = \dfrac{400 - 16h}{9}$.
The depth of water in Cylinder B must be less than the height of Cylinder B (Cylinder B cannot be full), so then $d < h$ or $\dfrac{400 - 16h}{9} < h$ or $400 - 16h < 9h$ or $400 < 25h$, and so $16 < h$.
As noted earlier, no water can spill into Cylinder B unless its height is less than that of Cylinder A, and so $h < 25$.
When Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full when $16 < h < 25$.

Solution 2
Let the volume of Cylinder A be V_A, the volume of Cylinder B be V_B, and the initial volume of water be V_W.
As we determined in Solution 1, $V_A = 3600\pi$, $V_B = 81\pi h$, and $V_W = 2736\pi$.
If $V_W + V_B > V_A$, then water spills out of the large cylinder onto the ground.
This gives $2736\pi + 81\pi h > 3600\pi$ or $81\pi h > 864\pi$, and so $h > \frac{32}{3}$.
If $\frac{32}{3} < h < 25$, water will spill onto the ground and then into B. (If $h \leq \frac{32}{3}$, B will actually be full of water when lowered into A, since no water spills out of A and the height of B is less than the initial height of water.)

Assume that $\frac{32}{3} < h < 25$.
Then the volume of water that spills out of Cylinder A onto the ground is

$$V_{\text{water on ground}} = V_W + V_B - V_A = 81\pi h - 864\pi.$$

When the tops of the two cylinders are at the same level (Figure 9), no water has spilled into Cylinder B, and so the volume of water in Cylinder A is the initial volume of water less the volume of water that has spilled out onto the ground.

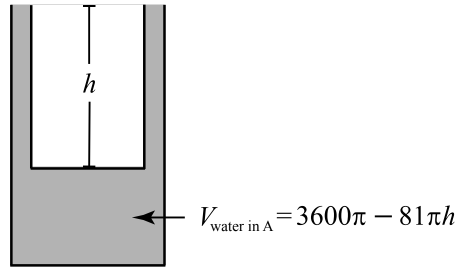

Figure 9

That is,

$$V_{\text{water in A}} = 2736\pi - V_{\text{water on ground}} = 2736\pi - (81\pi h - 864\pi) = 3600\pi - 81\pi h.$$

From this point on, all water stays in Cylinder B or in Cylinder A.
When Cylinder B is on the bottom of Cylinder A (Figure 10), the volume of water outside of Cylinder B (but inside Cylinder A), is the volume of Cylinder A that lies below the top of Cylinder B less the volume of Cylinder B.

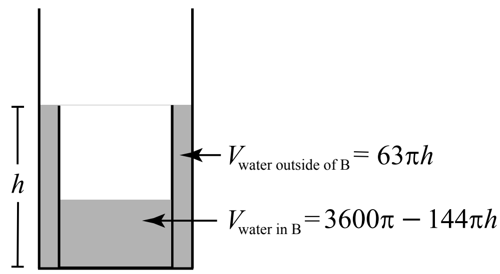

Figure 10

That is,

$$V_{\text{water outside of B}} = \pi(12^2)h - \pi(9^2)h = 63\pi h.$$

Further, the volume of water in Cylinder A, $3600\pi - 81\pi h$, must be equal to the volume of water outside of Cylinder B plus the volume of water inside of Cylinder B.
That is,

$$\begin{aligned} V_{\text{water in A}} &= V_{\text{water outside of B}} + V_{\text{water in B}} \\ V_{\text{water in B}} &= V_{\text{water in A}} - V_{\text{water outside of B}} \\ &= 3600\pi - 81\pi h - 63\pi h \\ &= 3600\pi - 144\pi h. \end{aligned}$$

The volume of water in Cylinder B must be less than the volume of Cylinder B, and so $3600\pi - 144\pi h < 81\pi h$ or $3600\pi < 225\pi h$, and thus $16 < h$.
When Cylinder B is on the bottom of Cylinder A, there is some water in Cylinder B but it is not full when $16 < h < 25$.

4. (a) As a sum of one of more consecutive positive integers, 45 can be written as

45, 22+23, 14+15+16, 7+8+9+10+11, 5+6+7+8+9+10, and 1+2+3+4+5+6+7+8+9,

and there are no other such lists.
The value of $C(45)$ is 6.

(b) The sum of the positive integers from 1 to n is given by the formula $\frac{1}{2}n(n+1)$.
The sum of the positive integers from 4 to n ($n \geq 4$) is equal to the sum of the positive

integers from 1 to n less the sum of the positive integers from 1 to 3, or $1+2+3=6$.
Therefore,

$$\begin{aligned} m &= 4+5+6+\cdots+n \\ &= \tfrac{1}{2}n(n+1)-6 \\ &= \tfrac{1}{2}(n(n+1)-12) \\ &= \tfrac{1}{2}(n^2+n-12) \\ &= \tfrac{1}{2}(n-3)(n+4). \end{aligned}$$

Since $m=\frac{1}{2}(n+a)(n+b)$ with $a<b$, then $a=-3$ and $b=4$.

(c) If $m=(a+1)+(a+2)+\cdots+n$, for integers $a\geq 0$ and $n\geq a+1$, then m is equal to the sum of the integers from 1 to n less the sum of the integers from 1 to a.
That is, $m=\frac{1}{2}n(n+1)-\frac{1}{2}a(a+1)$.
Simplifying, we get

$$\begin{aligned} m &= \tfrac{1}{2}n(n+1)-\tfrac{1}{2}a(a+1) \\ &= \tfrac{1}{2}(n^2+n-a^2-a) \\ &= \tfrac{1}{2}(n^2-a^2+n-a) \\ &= \tfrac{1}{2}((n-a)(n+a)+n-a) \\ &= \tfrac{1}{2}(n-a)(n+a+1) \end{aligned}$$

Each pair of integers (a,n) ($a\geq 0$ and $n\geq a+1$) for which $m=\frac{1}{2}(n-a)(n+a+1)$ gives a unique sum of one or more consecutive positive integers from $a+1$ to n whose sum is m.
In this question, we are asked to determine the number of such pairs (a,n) given that $m=2\times 3^4\times 5^6$.
Since $m=\frac{1}{2}(n-a)(n+a+1)$, then $2m=(n-a)(n+a+1)$.
That is, $2m$ can be expressed as the product of two positive integers $n+a+1$ and $n-a$.
The difference between these two integers is $(n+a+1)-(n-a)=2a+1$, which is an odd integer for all integers $a\geq 0$.
Since the difference between $n+a+1$ and $n-a$ is odd, then one of these integers must be even and the other odd (we say that they have different *parity*).
Thus, the problem of evaluating $C(m)$ appears to be equivalent to counting the number of factor pairs of $2m$ ($n+a+1$ and $n-a$) that have different parity.
At this point, we have shown that each pair of integers (a,n) ($a\geq 0$ and $n\geq a+1$) for which $m=\frac{1}{2}(n-a)(n+a+1)$ gives a factor pair with different parity.
We must now show that the converse is also true; that is, each factor pair with different parity gives a unique pair (a,n).
Suppose that $2m=d\cdot e$ for some positive odd integer d and positive even integer e.
We show that each pair d and e will give a pair of integers a and n.
If $d>e$, suppose that $d=n+a+1$ and $e=n-a$ (since $n+a+1>n-a$).
Adding the equations $n+a+1=d$ and $n-a=e$, we get $2n+1=d+e$ or $n=\frac{1}{2}(d+e-1)$.
Subtracting the two equations $n+a+1=d$ and $n-a=e$, we get $2a+1=d-e$ or $a=\frac{1}{2}(d-e-1)$.
Since d and e have different parity, then each of $d+e$ and $d-e$ is odd, and so each of $d+e-1$ and $d-e-1$ is even.
Therefore, each of $n=\frac{1}{2}(d+e-1)$ and $a=\frac{1}{2}(d-e-1)$ is an integer and $n>a$.
(If we assume that $d<e$, we can make a similar argument to show there exist corresponding integers a and n with $n>a$.)

That is, each factor pair (d, e) having different parity gives a unique pair (a, n) with $n > a$. This confirms that evaluating $C(m)$ is equivalent to counting the number of factor pairs of $2m$ that have different parity.

Before evaluating $C(2 \times 3^4 \times 5^6)$, we apply this to part (a) to confirm that $C(45) = 6$, and to demonstrate that for each odd factor of 2×45, there exists a corresponding unique list of consecutive positive integers whose sum is 45.

Since $m = 45 = 3^2 \times 5$, then $2m = 2 \times 3^2 \times 5$, and so the odd factors of $2 \times 3^2 \times 5$ must be of the form $3^i \times 5^j$ for integers $0 \leq i \leq 2$ and $0 \leq j \leq 1$ (that is, odd numbers have only odd divisors).
Since there are 3 choices for i (0,1,2), and 2 choices for j (0,1), there are $3 \times 2 = 6$ odd factors of $2 \times 3^2 \times 5$ (these are $1, 3, 5, 9, 15$, and 45).
Next, we demonstrate that each of these 6 odd factors gives a unique pair (a, n) for which:

(i) $(n - a)(n + a + 1) = 2 \times 45$, and

(ii) $n + a - 1$ and $n - a$ have different parity, and

(iii) $(a + 1) + (a + 2) + \cdots + n = 45$.

The odd factors $1, 3, 5, 9, 15, 45$ give the factor pairs $(1, 90), (3, 30), (5, 18), (9, 10), (15, 6)$, and $(45, 2)$ (we notice that the two numbers in each pair do indeed have different parity).
Next we note that since $a \geq 0$, then $n+a+1 > n-a$ and so for example, using the factor pair $(5, 18)$, we get $n - a = 5$ and $n + a + 1 = 18$.
Adding the two equations to solve this system of equations, we get $2n + 1 = 23$ and so $n = 11$ and $a = 6$.
This pair $(6, 11)$ gives the sum $7 + 8 + 9 + 10 + 11 = 45$.
We summarize the results using the other factor pairs in the table below.

Factor Pair	$a - n$	$a + n + 1$	n	a	$(a + 1) + (a + 2) + \cdots + n$
$(1, 90)$	1	90	45	44	45
$(3, 30)$	3	30	16	13	$14 + 15 + 16 = 45$
$(5, 18)$	5	18	11	6	$7 + 8 + 9 + 10 + 11 = 45$
$(9, 10)$	9	10	9	0	$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$
$(15, 6)$	6	15	10	4	$5 + 6 + 7 + 8 + 9 + 10 = 45$
$(45, 2)$	2	45	23	21	$22 + 23 = 45$

Comparing this table to our answer in part (a), we see that indeed each odd factor of 2×45 gives a unique list of consecutive positive integers whose sum is 45.

Finally, we turn our focus to evaluating $C(2 \times 3^4 \times 5^6)$, that is, counting the number of odd factors of $2^2 \times 3^4 \times 5^6$.
The odd factors of $2^2 \times 3^4 \times 5^6$ are of the form $3^i \times 5^j$ for integers $0 \leq i \leq 4$ and $0 \leq j \leq 6$.
Since there are 5 choices for i and 7 choices for j, there are $5 \times 7 = 35$ odd factors of $2^2 \times 3^4 \times 5^6$, and so $C(2 \times 3^4 \times 5^6) = 35$.

(d) We would like to determine the smallest positive integer k for which $C(k) = 215 = 5 \times 43$ (both 5 and 43 are prime numbers).
If $k = 2^a$ for some non-negative integer a, then $C(k) = 1$, and so k must have some odd prime factors $p_1, p_2, \ldots, p_n$. Can you see why?
That is, $k = 2^a \cdot p_1^{q_1} \cdot p_2^{q_2} \cdots p_n^{q_n}$ for distinct prime numbers $p_i, 1 \leq i \leq n$ and positive integers $q_j, 1 \leq j \leq n$.

From part (c), we know that $C(k) = (q_1 + 1)(q_2 + 1) \cdots (q_n + 1)$.
Since $C(k) = 5 \times 43 = (q_1 + 1)(q_2 + 1) \cdots (q_n + 1)$, then we let $n = 2$ and $q_1 + 1 = 5$ or $q_1 = 4$, and $q_2 + 1 = 43$ or $q_2 = 42$.
To minimize k, we let $a = 0$, and choose the smallest distinct odd primes $p_1 = 5$ and $p_2 = 3$ ($p_1 = 3$ and $p_2 = 5$ gives $k = 3^4 \times 5^{42}$, which is a much larger value for k).
The smallest positive integer k for which $C(k) = 215$ is $k = 5^4 \times 3^{42}$.