The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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# *2018 Cayley Contest*

**(Grade 10)**

**Tuesday, February 27, 2018**
**(in North America and South America)**

**Wednesday, February 28, 2018**
**(outside of North America and South America)**

*Solutions*

1. Since $3 \times n = 6 \times 2$, then $3n = 12$ or $n = \frac{12}{3} = 4$.

   Answer: (E)

2. The $4 \times 5$ grid contains 20 squares that are $1 \times 1$.
   For half of these to be shaded, 10 must be shaded.
   Since 3 are already shaded, then $10 - 3 = 7$ more need to be shaded.

   Answer: (C)

3. Since the number line between 0 and 2 is divided into 8 equal parts, then the portions between 0 and 1 and between 1 and 2 are each divided into 4 equal parts.
   In other words, the divisions on the number line mark off quarters, and so $S = 1 + 0.25 = 1.25$.

   Answer: (D)

4. Since $9 = 3 \times 3$, then $9^4 = (3 \times 3)^4 = 3^4 \times 3^4 = 3^8$.
   Alternatively, we can note that

   $$9^4 = 9 \times 9 \times 9 \times 9 = (3 \times 3) \times (3 \times 3) \times (3 \times 3) \times (3 \times 3) = 3^8$$

   Answer: (D)

5. The entire central angle in a circle measures $360^\circ$.
   In the diagram, the central angle which measures $120^\circ$ represents $\frac{120^\circ}{360^\circ} = \frac{1}{3}$ of the entire central angle.
   Therefore, the area of the sector is $\frac{1}{3}$ of the area of the entire circle, or $\frac{1}{3} \times 9\pi = 3\pi$.

   Answer: (B)

6. For any value of $x$, we have $x^2 + 2x - x(x+1) = x^2 + 2x - x^2 - x = x$.
   When $x = 2018$, the value of this expression is thus 2018.

   Answer: (B)

7. We want to calculate the percentage increase from 24 to 48.
   The percentage increase from 24 to 48 is equal to $\dfrac{48 - 24}{24} \times 100\% = 1 \times 100\% = 100\%$.
   Alternatively, since 48 is twice 24, then 48 represents an increase of 100% over 24.

   Answer: (D)

8. A line segment joining two points is parallel to the $x$-axis exactly when the $y$-coordinates of the two points are equal.
   Here, this means that $2k + 1 = 4k - 5$ and so $6 = 2k$ or $k = 3$.
   (We can check that when $k = 3$, the coordinates of the points are $(3, 7)$ and $(8, 7)$.)

   Answer: (B)

9. Since $5, a, b$ have an average of 33, then $\dfrac{5 + a + b}{3} = 33$.
   Multiplying by 3, we obtain $5 + a + b = 3 \times 33 = 99$, which means that $a + b = 94$.
   The average of $a$ and $b$ is thus equal to $\dfrac{a+b}{2} = \dfrac{94}{2} = 47$.

   Answer: (E)

10. Of the given uniform numbers,

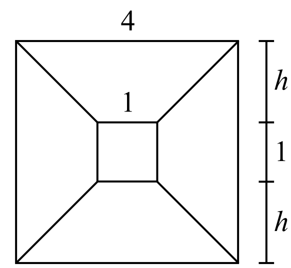
    - 11 and 13 are prime numbers
    - 16 is a perfect square
    - 12, 14 and 16 are even

    Since Karl's and Liu's numbers were prime numbers, then their numbers were 11 and 13 in some order.
    Since Glenda's number was a perfect square, then her number was 16.
    Since Helga's and Julia's numbers were even, then their numbers were 12 and 14 in some order. (The number 16 is already taken.)
    Thus, Ioana's number is the remaining number, which is 15.

    Answer: (D)

11. *Solution 1*
    The large square has side length 4, and so has area $4^2 = 16$.
    The small square has side length 1, and so has area $1^2 = 1$.
    The combined area of the four identical trapezoids is the difference in these areas, or $16 - 1 = 15$.
    Since the 4 trapezoids are identical, then they have equal areas, each equal to $\frac{15}{4}$.

    *Solution 2*
    Suppose that the height of each of the four trapezoids is $h$.
    Since the side length of the outer square is 4, then $h + 1 + h = 4$ and so $h = \frac{3}{2}$.
    Each of the four trapezoids has parallel bases of lengths 1 and 4, and height $\frac{3}{2}$.
    Therefore, the area of each is $\frac{1}{2}(1+4)(\frac{3}{2}) = \frac{5}{2}\left(\frac{3}{2}\right) = \frac{15}{4}$.

    Answer: (D)

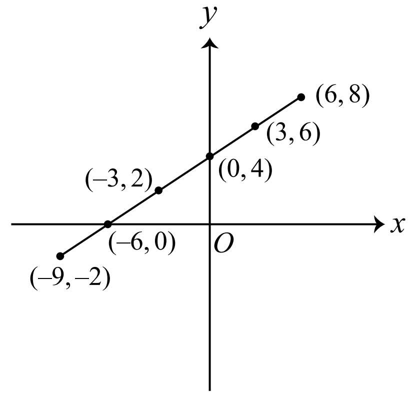
12. We are told that 1 Zed is equal in value to 16 Exes.
    We are also told that 2 Exes are equal in value to 29 Wyes.
    Since 16 Exes is 8 groups of 2 Exes, then 16 Exes are equal in value to $8 \times 29 = 232$ Wyes.
    Thus, 1 Zed is equal in value to 232 Wyes.

    Answer: (C)

13. The problem is equivalent to determining all values of $x$ for which $x + 1$ is a divisor of 3.
    The divisors of 3 are $3, -3, 1, -1$.
    If $x + 1 = 3, -3, 1, -1$, then $x = 2, -4, 0, -2$, respectively. There are 4 such values.

    Answer: (A)

14. *Solution 1*




The line segment with endpoints $(-9,-2)$ and $(6,8)$ has slope $\dfrac{8-(-2)}{6-(-9)} = \dfrac{10}{15} = \dfrac{2}{3}$.
This means that starting at $(-9,-2)$ and moving "up 2 and right 3" (corresponding to the rise and run of 2 and 3) repeatedly will give other points on the line that have coordinates which are both integers.
These points are $(-9,-2),(-6,0),(-3,2),(0,4),(3,6),(6,8)$.
So far, this gives 6 points on the line with integer coordinates.
Are there any other such points?
If there were such a point between $(-9,-2)$ and $(6,8)$, its $y$-coordinate would have to be equal to one of $-1,1,3,5,7$, the other integer possibilities between $-2$ and 8.
Consider the point on this line segment with $y$-coordinate 7.
Since this point has $y$-coordinate halfway between 6 and 8, then this point must be the midpoint of $(3,6)$ and $(6,8)$, which means that its $x$-coordinate is $\frac{1}{2}(3+6) = 4.5$, which is not an integer.
In a similar way, the points on the line segment with $y$-coordinates $-1,1,3,5$ do not have integer $x$-coordinates.
Therefore, the 6 points listed before are the only points on this line segment with integer coordinates.

*Solution 2*

The line segment with endpoints $(-9,-2)$ and $(6,8)$ has slope $\dfrac{8-(-2)}{6-(-9)} = \dfrac{10}{15} = \dfrac{2}{3}$.
Since the line passes through $(6,8)$, its equation can be written as $y-8 = \frac{2}{3}(x-6)$ or $y = \frac{2}{3}x+4$.
Suppose that a point $(x,y)$ lies along with line between $(-9,-2)$ and $(6,8)$ and has both $x$ and $y$ integers.
Since $y$ is an integer and $\frac{2}{3}x = y-4$, then $\frac{2}{3}x$ is an integer.
This means that $x$ must be a multiple of 3.
Since $x$ is between $-9$ and 6, inclusive, then the possible values for $x$ are $-9,-6,-3,0,3,6$.
This leads to the points listed in Solution 1, and justifies why there are no additional points.
Therefore, there are 6 such points.

Answer: (E)

15. Since $\triangle PQS$ is equilateral, then $\angle QPS = 60^\circ$.
Since $\angle RPQ$, $\angle RPS$ and $\angle QPS$ completely surround point $P$, then the sum of their measures is $360^\circ$.
Since $\angle RPQ = \angle RPS$, this means that $2\angle RPQ + \angle QPS = 360^\circ$ or $2\angle RPQ = 360^\circ - 60^\circ$, which means that $\angle RPQ = \angle RPS = 150^\circ$.
Since $PR = PQ$, then in isosceles $\triangle PQR$, we have

$$\angle PRQ = \angle PQR = \tfrac{1}{2}(180^\circ - \angle RPQ) = \tfrac{1}{2}(180^\circ - 150^\circ) = 15^\circ$$

Similarly, $\angle PRS = \angle PSR = 15^\circ$.
This means that $\angle QRS = \angle PRQ + \angle PRS = 15^\circ + 15^\circ = 30^\circ$.

Answer: (A)

16. Elisabeth climbs a total of 5 rungs by climbing either 1 or 2 rungs at a time.
Since there are only 5 rungs, then she cannot climb 2 rungs at a time more than 2 times.
Therefore, she must climb 2 rungs either 0, 1 or 2 times.

If she climbs 2 rungs 0 times, then each step consists of 1 rung and so she climbs $1, 1, 1, 1, 1$ to get to the top.

If she climbs 2 rungs 1 time, then she climbs $2, 1, 1, 1$, since the remaining 3 rungs must be made up of steps of 1 rung.
But she can climb these numbers of rungs in several different orders.
Since she takes four steps, she can climb 2 rungs as any of her 1st, 2nd, 3rd, or 4th step.
Putting this another way, she can climb $2, 1, 1, 1$ or $1, 2, 1, 1$ or $1, 1, 2, 1$ or $1, 1, 1, 2$.
There are 4 possibilities in this case.

If she climbs 2 rungs 2 times, then she climbs $2, 2, 1$.
Again, she can climb these numbers of rungs in several different orders.
Since she takes three steps, she can climb 1 rung as any of her 1st, 2nd or 3rd step.
Putting this another way, she can climb $1, 2, 2$ or $2, 1, 2$ or $2, 2, 1$.
There are 3 possibilities in this case.

In total, there are $1 + 4 + 3 = 8$ ways in which she can climb.

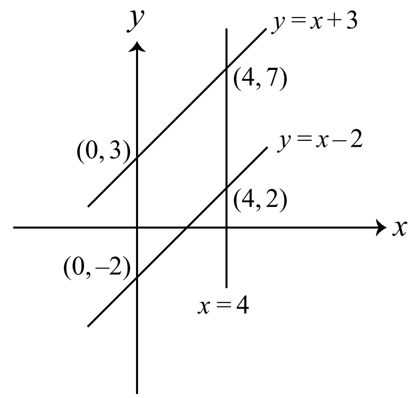
ANSWER: (E)

17. Since $\dfrac{x-y}{x+y} = 5$, then $x - y = 5(x + y)$.
This means that $x - y = 5x + 5y$ and so $0 = 4x + 6y$ or $2x + 3y = 0$.
Therefore, $\dfrac{2x+3y}{3x-2y} = \dfrac{0}{3x-2y} = 0$.
(A specific example of $x$ and $y$ that works is $x = 3$ and $y = -2$.
This gives $\dfrac{x-y}{x+y} = \dfrac{3-(-2)}{3+(-2)} = \dfrac{5}{1} = 5$ and $\dfrac{2x+3y}{3x-2y} = \dfrac{2(3)+3(-2)}{3(3)-2(-2)} = \dfrac{0}{13} = 0$.)
We should also note that if $\dfrac{x-y}{x+y} = 5$, then $2x+3y = 0$ (from above) and $x$ and $y$ cannot both be 0 (for $\dfrac{x-y}{x+y}$ to be well-defined).
Since $2x + 3y = 0$, then $x = -\frac{3}{2}y$ which means that it is not possible for only one of $x$ and $y$ to be 0, which means that neither is 0.
Also, the denominator of our desired expression $\dfrac{2x+3y}{3x-2y}$ is $3x - 2y = 3(-\frac{3}{2}y)) - 2y = -\frac{13}{2}y$ which is not 0, since $y \neq 0$.
Therefore, if $\dfrac{x-y}{x+y} = 5$, then $\dfrac{2x+3y}{3x-2y} = 0$.

ANSWER: (B)

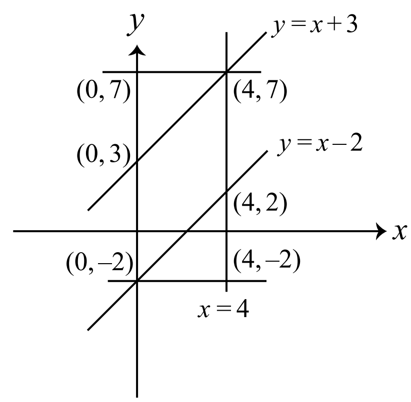
18. *Solution 1*

The lines with equations $x = 0$ and $x = 4$ are parallel vertical lines.
The lines with equations $y = x - 2$ and $y = x + 3$ are parallel lines with slope 1.
Since the quadrilateral has two sets of parallel sides, it is a parallelogram.
Thus, its area equals the length of its base times its height.
We consider the vertical side along the $y$-axis as its base.
Since the two sides of slope 1 have $y$-intercepts $-2$ and 3, then the length of the vertical base is $3 - (-2) = 5$.
Since the parallel vertical sides lie along the lines with equations $x = 0$ and $x = 4$, then these sides are a distance of 4 apart, which means that the height of the parallelogram is 4.
Therefore, the area of the quadrilateral is $5 \times 4 = 20$.




*Solution 2*

The lines with equations $x = 0$ and $x = 4$ are parallel vertical lines.
The lines with equations $y = x - 2$ and $y = x + 3$ are parallel lines with slope 1.
The lines with equations $y = x - 2$ and $y = x + 3$ intersect the line with equation $x = 0$ at their $y$-intercepts, namely $-2$ and 3, respectivley.
The lines with equations $y = x - 2$ and $y = x + 3$ intersect the line with equation $x = 4$ at $(4, 2)$ and $(4, 7)$, respectively (the point on each line with $x$-coordinate 4).
We draw horizontal lines through $(0, -2)$, intersecting $x = 4$ at $(4, -2)$, and through $(4, 7)$, intersecting $x = 0$ at $(0, 7)$.




The area of the quadrilateral equals the area of the large rectangle with vertices $(0, 7)$, $(4, 7)$, $(4, -2)$, $(0, -2)$ minus the combined area of the two triangles.
This rectangle has side lengths $4 - 0 = 4$ and $7 - (-2) = 9$ and so has area $4 \times 9 = 36$.
The two triangles are right-angled and have bases of length $4 - 0 = 4$ and heights of length $7 - 3 = 4$ and $2 - (-2) = 4$. This means that they can be combined to form a square with side length 4, and so have combined area $4^2 = 16$.
Therefore, the area of the quadrilateral is $36 - 16 = 20$.

Answer: (E)

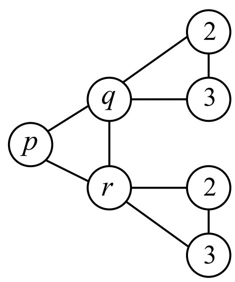
19. Suppose that $L$ is the area of the large circle, $S$ is the area of the small circle, and $A$ is the area of the overlapped region.
Since the area of the overlapped region is $\frac{3}{5}$ of the area of the small circle, then $A = \frac{3}{5}S$.
Since the area of the overlapped region is $\frac{6}{25}$ of the area of the large circle, then $A = \frac{6}{25}L$.
Therefore, $\frac{3}{5}S = \frac{6}{25}L$.
Multiplying both sides by 25 to clear denominators, we obtain $15S = 6L$.
Dividing both sides by 3, we obtain $5S = 2L$. Therefore, $\dfrac{5S}{L} = 2$ or $\dfrac{S}{L} = \dfrac{2}{5}$.
This means that the ratio of the area of the small circle to the area of the large circle is $2 : 5$.

Answer: (D)

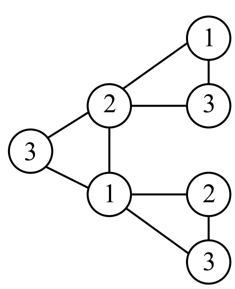
20. When the product of the three integers is calculated, either the product is a power of 2 or it is not a power of 2.
If $p$ is the probability that the product is a power of 2 and $q$ is the probability that the product is not a power of 2, then $p + q = 1$.
Therefore, we can calculate $q$ by calculating $p$ and noting that $q = 1 - p$.
For the product of the three integers to be a power of 2, it can have no prime factors other than 2. In particular, this means that each of the three integers must be a power of 2.
In each of the three sets, there are 3 powers of 2 (namely, 2, 4 and 8) and 2 integers that are not a power of 2 (namely, 6 and 10).
This means that the probability of choosing a power of 2 at random from each of the sets is $\frac{3}{5}$.
Since Abigail, Bill and Charlie choose their numbers independently, then the probability that each chooses a power of 2 is $\left(\frac{3}{5}\right)^3 = \frac{27}{125}$.
In other words, $p = \frac{27}{125}$ and so $q = 1 - p = 1 - \frac{27}{125} = \frac{98}{125}$.

Answer: (C)

21. Since each of $s, t, u, v$ is equal to one of 1, 2 or 3, and $s$ and $t$ are different and $u$ and $v$ are different, then their sum cannot be any larger than $2 + 3 + 2 + 3 = 10$.
This can only happen if $s$ and $t$ are 2 and 3 in some order, and if $u$ and $v$ are 2 and 3 in some order.




But $q, s, t$ are $1, 2, 3$ in some order and $r, u, v$ are $1, 2, 3$ in some order.
So if $s$ and $t$ are 2 and 3, then $q = 1$. Similarly, if $u$ and $v$ are 2 and 3, then $r = 1$.
But $p, q, r$ are $1, 2, 3$ in some order, so we cannot have $q = r = 1$.
Therefore, we cannot have $s + t + u + v = 10$.
The next largest possible value of $s + t + u + v$ would be 9.
We can construct the diagram with this value of $s + t + u + v$ by letting $s = 1$, $t = 3$, $u = 2$, $v = 3$, and proceeding as follows:



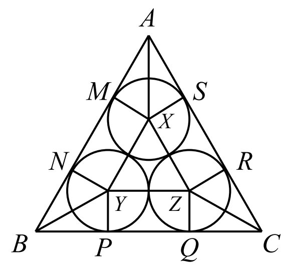

Therefore, the maximum possible value of $s + t + u + v$ is 9.

Answer: (B)

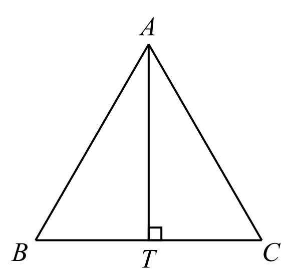
22. For the given expression to be equal to an integer, each prime factor of the denominator must divide out of the product in the numerator.
In other words, each prime number that is a factor of the denominator must occur as a factor at least as many times in the numerator as in the denominator.
We note that $25 = 5^2$ and so $25^y = 5^{2y}$.
Also, $36 = 6^2 = 2^2 3^2$ and so $36^x = (2^2 3^2)^x = 2^{2x} 3^{2x}$.
Therefore, the given expression is equal to $\dfrac{30!}{2^{2x}3^{2x}5^{2y}}$.
We count the number of times that each of 5, 3 and 2 is a factor of the numerator.
The product equal to 30! includes six factors which are multiples of 5, namely $5, 10, 15, 20, 25, 30$.
Each of these factors includes 1 factor of 5, except for 25 which includes 2 factors.
Therefore, the numerator includes 7 factors of 5.
For the numerator to include at least as many factors of 5 as the denominator, we must have $7 \geq 2y$.
Since $y$ is an integer, then $y \leq 3$.
The product equal to 30! includes 10 factors which are multiples of 3, namely $3, 6, 9, 12, 15, 18, 21, 24, 27, 30$.
Seven of these include exactly one factor of 3, namely $3, 6, 12, 15, 21, 24, 30$.
Two of these include exactly two factors of 3, namely $9, 18$.
One of these includes exactly three factors of 3, namely 27.
Therefore, the numerator includes $7(1) + 2(2) + 1(3) = 14$ factors of 3.
For the numerator to include at least as many factors of 3 as the denominator, we must have $14 \geq 2x$.
Since $x$ is an integer, then $x \leq 7$.
If $x \leq 7$, then the denominator includes at most 14 factors of 2. Since the product equal to 30! includes 15 even numbers, then the numerator includes at least 15 factors of 2, and so includes more factors of 2 than the denominator. This means that the number of 2s does not limit the value of $x$.
Since $x \leq 7$ and $y \leq 3$, then $x + y \leq 7 + 3 = 10$.
We note that if $x = 7$ and $y = 3$, then the given expression is an integer so this maximum can be achieved.

Answer: (A)

23. To determine the volume of the prism, we calculate the area of its base and the height of the prism.
First, we calculate the area of its base.
Any cross-section of the prism parallel to its base has the same shape, so we take a cross-section 1 unit above the base.
Since each of the spheres has radius 1, this triangular cross-section will pass through the centre of each of the spheres and the points of tangency between the spheres and the rectangular faces.
Let the vertices of the triangular cross-section be $A$, $B$ and $C$, the centres of the spheres be $X$, $Y$ and $Z$, and the points of tangency of the spheres (circles) to the faces of the prism (sides of the triangle) be $M$, $N$, $P$, $Q$, $R$, and $S$, as shown:

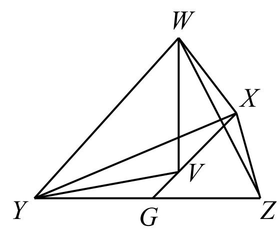



Join $X$ to $M$, $S$, $Y$, $Z$, $A$, join $Y$ to $N$, $P$, $Z$, $B$, and $Z$ to $Q$, $R$, $C$.
We determine the length of $BC$. A similar calculation will determine the lengths of $AB$ and $AC$. By symmetry, $AB = AC = BC$.
Note that $YP$ and $ZQ$ are perpendicular to $BC$ since radii are perpendicular to tangents.
Also, $YP = ZQ = 1$ since the radius of each sphere (and hence of each circle) is 1.
Since $ZYPQ$ has right angles at $P$ and $Q$ and has $YP = ZQ = 1$, then $ZYPQ$ is a rectangle.
Therefore, $YZ = PQ$.
Since $YZ$ passes through the point at which the circles touch, then $YZ$ equals the sum of the radii of the circles, or 2.
Thus, $PQ = YZ = 2$.
Since $AB = BC = CA$, then $\triangle ABC$ is equilateral and so $\angle ABC = 60^\circ$.
Now $YB$ bisects $\angle ABC$, by symmetry.
This means that $\angle YBP = 30^\circ$, which means that $\triangle BYP$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Since $YP = 1$, then $BP = \sqrt{3}$, using the ratios of side lengths in a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Similarly, $QC = \sqrt{3}$.
Therefore, $BC = BP + PQ + QC = \sqrt{3} + 2 + \sqrt{3} = 2 + 2\sqrt{3}$.
This means that $AB = BC = CA = 2 + 2\sqrt{3}$.
To calculate the area of $\triangle ABC$, we drop a perpendicular from $A$ to $T$ on $BC$.




Since $\angle ABC = 60^\circ$, then $\triangle ABT$ is a $30^\circ$-$60^\circ$-$90^\circ$ triangle.
Since $AB = 2 + 2\sqrt{3}$, then $AT = \frac{\sqrt{3}}{2}AB = \frac{\sqrt{3}}{2}(2 + 2\sqrt{3})$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(BC)(AT) = \frac{1}{2}(2 + 2\sqrt{3})\left(\frac{\sqrt{3}}{2}(2 + 2\sqrt{3})\right) = \frac{\sqrt{3}}{4}(2 + 2\sqrt{3})^2$.

This means that the area of the base of the prism (which is $\triangle ABC$) is $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2$.
Now we calculate the height of the prism.
Let the centre of the top sphere be $W$.
The vertical distance from $W$ to the top face of the prism equals the radius of the sphere, which is 1.
Similarly, the vertical distance from the bottom face to the plane through $X$, $Y$ and $Z$ is 1.
To finish calculating the height of the prism, we need to determine the vertical distance between the cross-section through $X$, $Y$, $Z$ and the point $W$.
The total height of the prism equals 2 plus this height.
Since the four spheres touch, then the distance between any pair of centres is the sum of the radii, which is 2.
Therefore, $WX = XY = YZ = WZ = WY = XZ = 2$. (This means that $WXYZ$ is a tetrahedron with equal edge lengths.)
We need to calculate the height of this tetrahedron.
Join $W$ to the centre, $V$, of $\triangle XYZ$.
By symmetry, $W$ is directly above $V$.




Let $G$ be the midpoint of $YZ$. This means that $YG = GZ = 1$.
Join $V$ to $Y$ and $G$.
Since $V$ is the centre of $\triangle XYZ$, then $VG$ is perpendicular to $YZ$ at $G$.
Since $\triangle XYZ$ is equilateral, then $\angle VYG = \frac{1}{2}\angle XYZ = 30^\circ$. This is because $V$, the centre of equilateral $\triangle XYZ$, lies on the angle bisectors of each of the angles.
$\triangle YVG$ is another $30^\circ$-$60^\circ$-$90^\circ$ triangle, and so $YV = \frac{2}{\sqrt{3}}YG = \frac{2}{\sqrt{3}}$.
Finally, $\triangle WYV$ is right-angled at $V$.
Thus, $WV = \sqrt{WY^2 - YV^2} = \sqrt{2^2 - \frac{4}{3}} = \sqrt{\frac{8}{3}}$.
This means that the height of the prism is $1 + 1 + \sqrt{\frac{8}{3}}$.
The volume of the prism is equal to the area of its base times its height, which is equal to $\frac{\sqrt{3}}{4}(2+2\sqrt{3})^2 \cdot \left(2 + \sqrt{\frac{8}{3}}\right)$ which is approximately 46.97.
Of the given answers, this is closest to 47.00.

Answer: (E)

24. Since there must be at least 2 gold socks ($G$) between any 2 black socks ($B$), then we start by placing the $n$ black socks with exactly 2 gold socks in between each pair:

$$BGGBGGBGGB \cdots BGGB$$

Since there are $n$ black socks, then there are $n-1$ "gaps" between them and so $2(n-1)$ or $2n-2$ gold socks have been used.
This means that there are 2 gold socks left to place. There are $n+1$ locations in which these socks can be placed: either before the first black sock, after the last black sock, or in one of the $n-1$ gaps.
These 2 socks can either be placed together in one location or separately in two locations.
If the 2 socks are placed together, they can be placed in any one of the $n+1$ locations, and so there are $n+1$ ways to do this. (The placement of the gold socks within these locations does not matter as the gold socks are all identical.)
The other possibility is that the 2 gold socks are placed separately in two of these $n+1$ spots. There are $n+1$ possible locations for the first of these socks. For each of these, there are then $n$ possible locations for the second of these socks (any other than the location of the first sock). Since these two socks are identical, we have double-counted the total number of possibilities, and so there are $\frac{1}{2}(n+1)n$ ways for these two gold socks to be placed.
In total, there are $(n+1) + \frac{1}{2}(n+1)n = (n+1)(1+\frac{1}{2}n) = \frac{1}{2}(n+1)(n+2)$ ways in which the socks can be arranged.
We want to determine the smallest positive integer $n$ for which this total is greater than 1 000 000.
This is equivalent to determining the smallest positive integer $n$ for which $(n+1)(n+2)$ is greater than 2 000 000.
We note that $(n+1)(n+2)$ increases as $n$ increases, since each of $n+1$ and $n+2$ is positive and increasing, and so their product is increasing.
When $n = 1412$, we have $(n+1)(n+2) = 1\,997\,982$.
When $n = 1413$, we have $(n+1)(n+2) = 2\,000\,810$.
Since $(n+1)(n+2)$ is increasing, then $n = 1413$ is the smallest positive integer for which $(n+1)(n+2)$ is greater than 2 000 000, and so it is the smallest positive integer for which there are more than 1 000 000 arrangements of the socks.
The sum of the digits of $n = 1413$ is $1+4+1+3 = 9$.

Answer: (A)

25. Since the terms in each such sequence can be grouped to get both positive and negative sums, then there must be terms that are positive and there must be terms that are negative.
Since the 15 terms have at most two different values and there are terms that are positive and terms that are negative, then the terms have exactly two different values, one positive and one negative.
We will call these values $x$ and $y$. We know that both are integers and we will add the condition that $x > 0$ and $y < 0$.
Consider one of these sequences and label the terms

$$a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13}, a_{14}, a_{15}$$

Since the sum of six consecutive terms is always positive, then $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 > 0$.
Since the sum of eleven consecutive terms is always negative, then $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$.
Since $a_1+a_2+a_3+a_4+a_5+a_6 > 0$ and $(a_1+a_2+a_3+a_4+a_5+a_6)+(a_7+a_8+a_9+a_{10}+a_{11}) < 0$, then $a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$.
The six term condition also tells us that $a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} > 0$.
Since $a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$ and $a_6 + (a_7 + a_8 + a_9 + a_{10} + a_{11}) > 0$, then $a_6 > 0$ and so $a_6 = x$.
The six term condition also tells us that $a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} > 0$.
Since $a_7 + a_8 + a_9 + a_{10} + a_{11} < 0$, then $a_{12} > 0$ which gives $a_{12} = x$.
We can repeat this argument by shifting all of the terms one further along in the sequence.
Starting with $a_2+a_3+a_4+a_5+a_6+a_7 > 0$ and $a_2+a_3+a_4+a_5+a_6+a_7+a_8+a_9+a_{10}+a_{11}+a_{12} < 0$ and using the same arguments as above will give $a_7 = a_{13} = x$.
By continuing to shift one more term further along twice more, we obtain $a_8 = a_{14} = x$ and $a_9 = a_{15} = x$.
So far this gives the sequence

$$a_1, a_2, a_3, a_4, a_5, x, x, x, x, a_{10}, a_{11}, x, x, x, x$$

We can continue to repeat this argument by starting at the right-hand end of the sequence.
Starting with $a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} > 0$ and $a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} < 0$, will allow us to conclude that $a_{10} = x$ and $a_4 = x$.
Shifting back to the left and repeating this argument gives $a_1 = a_2 = a_3 = a_4 = a_{10} = x$.
This means that the sequence has the form

$$x, x, x, x, a_5, x, x, x, x, x, a_{11}, x, x, x, x$$

At least one of $a_5$ and $a_{11}$ must equal $y$, otherwise all of the terms in the sequence would be positive.
In fact, it must be the case that both of these terms equal $y$.
To see this, suppose that $a_5 = y$ but $a_{11} = x$.
In this case, the sum of the first 6 terms in the sequence is $5x + y$ which is positive (that is, $5x + y > 0$).
Also, the sum of the first 11 terms in the sequence is $10x + y$ which is negative (that is, $10x + y < 0$).
But $x > 0$ and so $10x + y = 5x + (5x + y) > 0$, which contradicts $10x + y < 0$.
We would obtain the same result if we considered the possibility that $a_5 = x$ and $a_{11} = y$.
Therefore, $a_5 = a_{11} = y$ and so the sequence must be

$$x, x, x, x, y, x, x, x, x, x, y, x, x, x, x$$

In this case, each group of 6 consecutive terms includes exactly 5 $x$'s and so the sum of each group of 6 consecutive terms is $5x + y$.
We are told that $5x + y > 0$.
Also, the sum of each group of 11 consecutive terms is $9x + 2y$.
We are told that $9x + 2y < 0$.
We have now changed the original problem into an equivalent problem: count the number of pairs $(x, y)$ of integers with $x > 0$ and $y < 0$ and $5x + y > 0$ and $9x + 2y < 0$ and either $x$ is between 1 and 16, inclusive, or $y$ is between $-16$ and $-1$, inclusive.
Suppose that $1 \leq x \leq 16$.
From $5x + y > 0$ and $9x + 2y < 0$, we obtain $-5x < y < -4.5x$.
We can now make a chart that enumerates the values of $x$ from 1 to 16, the corresponding bounds on $y$, and the resulting possible values of $y$:

| $x$ | $-5x$ | $-4.5x$ | Possible $y$ |
|---|---|---|---|
| 1 | $-5$ | $-4.5$ | None |
| 2 | $-10$ | $-9$ | None |
| 3 | $-15$ | $-13.5$ | $-14$ |
| 4 | $-20$ | $-18$ | $-19$ |
| 5 | $-25$ | $-22.5$ | $-24, -23$ |
| 6 | $-30$ | $-27$ | $-29, -28$ |
| 7 | $-35$ | $-31.5$ | $-34, -33, -32$ |
| 8 | $-40$ | $-36$ | $-39, -38, -37$ |
| 9 | $-45$ | $-40.5$ | $-44, -43, -42, -41$ |
| 10 | $-50$ | $-45$ | $-49, -48, -47, -46$ |
| 11 | $-55$ | $-49.5$ | $-54, -53, -52, -51, -50$ |
| 12 | $-60$ | $-54$ | $-59, -58, -57, -56, -55$ |
| 13 | $-65$ | $-58.5$ | $-64, -63, -62, -61, -60, -59$ |
| 14 | $-70$ | $-63$ | $-69, -68, -67, -66, -65, -64$ |
| 15 | $-75$ | $-67.5$ | $-74, -73, -72, -71, -70, -69, -68$ |
| 16 | $-80$ | $-72$ | $-79, -78, -77, -76, -75, -74, -73$ |

Therefore, when $1 \leq x \leq 16$, there are $2(1 + 2 + 3 + 4 + 5 + 6 + 7) = 56$ pairs $(x, y)$ and so there are 56 sequences.
For example, if $x = 5$ and $y = -24$, we get the sequence

$$5, 5, 5, 5, -24, 5, 5, 5, 5, 5, -24, 5, 5, 5, 5$$

which satisfies the five given conditions.

Are there any additional sequences with $-16 \leq y \leq -1$?
Since $5x + y > 0$, then $x > -0.2y$. Since $y \geq -16$, then $x < 0.2(16) = 3.2$.
Since $9x + 2y < 0$, then $x < -\frac{2}{9}y$. Since $y \leq -1$, then $x > \frac{2}{9}(1) = \frac{2}{9}$.
Since $\frac{2}{9} < x < 3.2$, then any sequence with $-16 \leq y \leq -1$ also has $1 \leq x \leq 16$, and so we have counted this already.
This means that $N$, the total number of such sequences, equals 56.

Answer: (E)