The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2018 Canadian Team Mathematics Contest

Team Problems

IMPORTANT NOTES:

- Calculating devices are not permitted.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

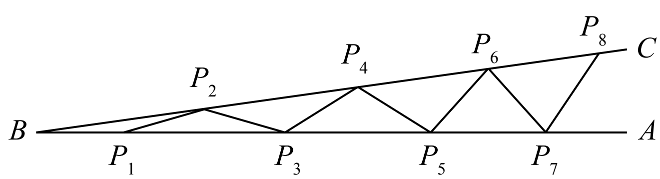
1. What is the value of $\sqrt{1+2+3+4+5+6+7+8}$?

2. A bucket that is $\frac{2}{3}$ full contains 9 L of maple syrup. What is the capacity of the bucket, in litres?

3. The sum of four consecutive odd integers is 200. What is the largest of these four integers?

4. It takes 18 doodads to make 5 widgets. It takes 11 widgets to make 4 thingamabobs. How many doodads does it take to make 80 thingamabobs?

5. In the diagram, points P_1, P_3, P_5, P_7 are on BA and points P_2, P_4, P_6, P_8 are on BC such that $BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6 = P_6P_7 = P_7P_8$. If $\angle ABC = 5°$, what is the measure of $\angle AP_7P_8$?



6. A *polygonal pyramid* is a three-dimensional solid. Its base is a regular polygon. Each of the vertices of the polygonal base is connected to a single point, called the *apex*. The sum of the number of edges and the number of vertices of a particular polygonal pyramid is 1915. How many faces does this pyramid have?

7. There are four ways to evaluate the expression "$\pm 2 \pm 5$":

$$2+5=7 \qquad 2-5=-3 \qquad -2+5=3 \qquad -2-5=-7$$

There are eight ways to evaluate the expression "$\pm 2^{11} \pm 2^5 \pm 2$". When these eight values are listed in decreasing order, what is the third value in the list?

8. For how many positive integers n is the sum

$$(-n)^3 + (-n+1)^3 + \cdots + (n-2)^3 + (n-1)^3 + n^3 + (n+1)^3$$

less than 3129?

9. For real numbers a and b, we define $a\nabla b = ab - ba^2$. For example, $5\nabla 4 = 5(4) - 4(5^2) = -80$. Determine the sum of the values of x for which $(2\nabla x) - 8 = x\nabla 6$.

10. Birgit has a list of four numbers. Luciano adds these numbers together, three at a time, and gets the sums 415, 442, 396, and 325. What is the sum of Birgit's four numbers?

11. On Monday, Krikor left his house at 8:00 a.m., drove at a constant speed, and arrived at work at 8:30 a.m. On Tuesday, he left his house at 8:05 a.m., drove at a constant speed, and arrived at work at 8:30 a.m. By what percentage did he increase his speed from Monday to Tuesday?

12. What is the value of $\pi \log_{2018} \sqrt{2} + \sqrt{2} \log_{2018} \pi + \pi \log_{2018} \left(\dfrac{1}{\sqrt{2}}\right) + \sqrt{2} \log_{2018} \left(\dfrac{1}{\pi}\right)$?

13. An Eilitnip number is a three-digit positive integer with the properties that, for some integer k with $0 \leq k \leq 7$:

   - its digits are k, $k+1$ and $k+2$ in some order, and
   - it is divisible by $k+3$.

   Determine the number of Eilitnip numbers.

14. An arithmetic sequence has 2036 terms labelled $t_1, t_2, t_3, \ldots, t_{2035}, t_{2036}$. Its 2018th term is $t_{2018} = 100$. Determine the value of $t_{2000} + 5t_{2015} + 5t_{2021} + t_{2036}$.

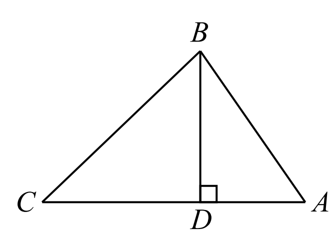
   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant, called the common difference. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence with common difference 2.)

15. A square wall has side length n metres. Guillaume paints n non-overlapping circular targets on the wall, each with radius 1 metre. Mathilde is going to throw a dart at the wall. Her aim is good enough to hit the wall at a single point, but poor enough that the dart will hit a random point on the wall. What is the largest possible value of n so that the probability that Mathilde's dart hits a target is at least $\frac{1}{2}$?

16. Determine the largest positive integer n for which 7^n is a divisor of the integer $\dfrac{200!}{90!30!}$.

   (Note: If n is a positive integer, the symbol $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.)

17. In the diagram, D is on side AC of $\triangle ABC$ so that BD is perpendicular to AC. Also, $\angle BAC = 60°$ and $\angle BCA = 45°$. If the area of $\triangle ABC$ is $72 + 72\sqrt{3}$, what is the length of BD?



18. BBAAABA and AAAAAAA are examples of "words" that are seven letters long where each letter is either A or B. How many seven-letter words in which each letter is either A or B do not contain three or more A's in a row?

19. Determine all real numbers x for which $x^{3/5} - 4 = 32 - x^{2/5}$.

20. In $\triangle ABC$, $BC = AC - 1$ and $AC = AB - 1$. If $\cos(\angle BAC) = \frac{3}{5}$, determine the perimeter of $\triangle ABC$.

21. The function f has the following properties:

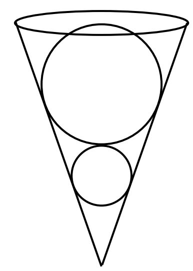
    (i) its domain is all real numbers,

    (ii) it is an odd function (that is, $f(-x) = -f(x)$ for every real number x), and

    (iii) $f(2x-3) - 2f(3x-10) + f(x-3) = 28 - 6x$ for every real number x.

    Determine the value of $f(4)$.

22. A right circular cone contains two spheres, as shown. The radius of the larger sphere is 2 times the radius of the smaller sphere. Each sphere is tangent to the other sphere and to the lateral surface of the cone. The larger sphere is tangent to the cone's circular base. Determine the fraction of the cone's volume that is *not* occupied by the two spheres.



23. Let a be a fixed real number. Define $M(t)$ to be the maximum value of $-x^2 + 2ax + a$ over all real numbers x with $x \leq t$. Determine a polynomial expression in terms of a that is equal to $M(a-1) + M(a+2)$ for every real number a.

24. The graphs $y = 2\cos 3x + 1$ and $y = -\cos 2x$ intersect at many points. Two of these points, P and Q, have x-coordinates between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$. The line through P and Q intersects the x-axis at B and the y-axis at A. If O is the origin, what is the area of $\triangle BOA$?

25. There are 16 distinct points on a circle. Determine the number of different ways to draw 8 non-intersecting line segments connecting pairs of points so that each of the 16 points is connected to exactly one other point. (For example, when the number of points is 6, there are 5 different ways to connect them, as shown.)