The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2018
Canadian Team Mathematics Contest

April 2018

Solutions

Individual Problems

1. Since $(a, 0)$ is on the line with equation $y = x + 8$, then $0 = a + 8$ or $a = -8$.

   ANSWER: -8

2. Simplifying,

$$\begin{aligned} x &= \left(1-\tfrac{1}{12}\right)\left(1-\tfrac{1}{11}\right)\left(1-\tfrac{1}{10}\right)\left(1-\tfrac{1}{9}\right)\left(1-\tfrac{1}{8}\right)\left(1-\tfrac{1}{7}\right)\left(1-\tfrac{1}{6}\right)\left(1-\tfrac{1}{5}\right)\left(1-\tfrac{1}{4}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{2}\right) \\ &= \left(\tfrac{11}{12}\right)\left(\tfrac{10}{11}\right)\left(\tfrac{9}{10}\right)\left(\tfrac{8}{9}\right)\left(\tfrac{7}{8}\right)\left(\tfrac{6}{7}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{2}\right) \\ &= \tfrac{11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{12\cdot11\cdot10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2} \\ &= \tfrac{1}{12} \qquad \text{(dividing out common factors)} \end{aligned}$$

   ANSWER: $\frac{1}{12}$

3. The following pairs of rectangles are not touching: AC, AE, CD.
   There are 3 such pairs.

   ANSWER: 3

4. Let the side length of the square be s.
   Since the diagonal of length 10 is the hypotenuse of a right-angled triangle with two sides of the square as legs, then $s^2 + s^2 = 10^2$ or $2s^2 = 100$, which gives $s^2 = 50$.
   Since the area of the square equals s^2, then the area is 50.

   ANSWER: 50

5. To make n as large as possible, we make each of the digits a, b, c as large as possible, starting with a.
   Since a is divisible by 2, its largest possible value is $a = 8$, so we try $a = 8$.
   Consider the two digit integer $8b$.
   This integer is a multiple of 3 exactly when $b = 1, 4, 7$.
   We note that 84 is divisible by 6, but 81 and 87 are not.
   To make n as large possible, we try $b = 7$, which makes $n = 87c$.
   For $n = 87c$ to be divisible by 5, it must be the case that $c = 0$ or $c = 5$.
   But $875 = 7 \cdot 125$ so 875 is divisible by 7.
   Therefore, for n to be divisible by 5 and not by 7, we choose $c = 0$.
   Thus, the largest integer n that satisfies the given conditions is $n = 870$.

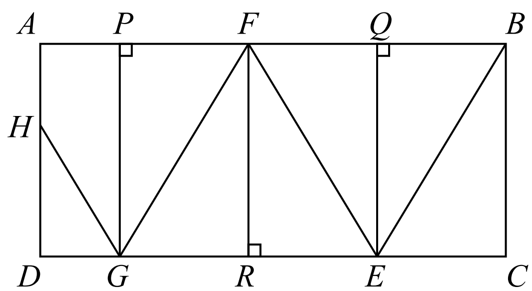
   ANSWER: 870

6. First, we note that $a^2 \geq 0$ for every real number a.
   Therefore, $(4x^2 - y^2)^2 \geq 0$ and $(7x + 3y - 39)^2 \geq 0$.
   Since $(4x^2 - y^2)^2 + (7x + 3y - 39)^2 = 0$, then it must be the case that $(4x^2 - y^2)^2 = 0$ and $(7x + 3y - 39)^2 = 0$.
   This means that $4x^2 - y^2 = 0$ and $7x + 3y - 39 = 0$.
   The equation $4x^2 - y^2 = 0$ is equivalent to $4x^2 = y^2$ and to $y = \pm 2x$.
   If $y = 2x$, then the equation $7x + 3y - 39 = 0$ becomes $7x + 6x = 39$ or $x = 3$.
   This gives $y = 2x = 6$.
   If $y = -2x$, then the equation $7x + 3y - 39 = 0$ becomes $7x - 6x = 39$ or $x = 39$.
   This gives $y = -2x = -78$.
   Therefore, the solutions to the original equation are $(x, y) = (3, 6), (39, -78)$.

   ANSWER: $(x, y) = (3, 6), (39, -78)$

7. In an arithmetic sequence with common difference d, the difference between any two terms must be divisible by d. This is because to get from any term in the sequence to any term later in the sequence, we add the common difference d some number of times.
In the given sequence, this means that $468 - 3 = 465$ is a multiple of d and $2018 - 468 = 1550$ is a multiple of d.
Thus, we want to determine the possible positive common divisors of 465 and 1550.
We note that $465 = 5 \cdot 93 = 3 \cdot 5 \cdot 31$ and that $1550 = 50 \cdot 31 = 2 \cdot 5^2 \cdot 31$.
Therefore, the positive common divisors of 465 and 1550 are $1, 5, 31, 155$. (These come from finding the common prime divisors.)
Since $d > 1$, the possible values of d are $5, 31, 155$. The sum of these is $5 + 31 + 155 = 191$.
ANSWER: 191

8. Let points P and Q be on AB so that GP and EQ are perpendicular to AB.
Let point R be on DC so that FR is perpendicular to DC.



Note that each of $\triangle BCE$, $\triangle EQB$, $\triangle EQF$, $\triangle FRE$, $\triangle FRG$, and $\triangle FPG$ is right-angled and has height equal to BC, which has length 10 m.
Since $\angle BEC = \angle FEG$, then the angles of $\triangle BCE$ and $\triangle FRE$ are equal. Since their heights are also equal, these triangles are congruent.
Since $FREQ$ and $BCEQ$ are rectangles (each has four right angles) and each is split by its diagonal into two congruent triangles, then $\triangle EQF$ and $\triangle EQB$ are also congruent to $\triangle BCE$.
Similarly, $\triangle FPG$ and $\triangle FRG$ are congruent to these triangles as well.
Let $CE = x$ m. Then $ER = RG = EC = x$ m.
Since $\triangle HDG$ is right-angled at D and $\angle HGD = \angle FGE$, then $\triangle HGD$ is similar to $\triangle FRG$.
Since $HD : FR = 6 : 10$, then $DG : RG = 6 : 10$ and so $DG = \frac{6}{10}RG = \frac{3}{5}x$ m.
Since $AB = 18$ m and $ABCD$ is a rectangle, then $DC = 18$ m.
But $DC = DG + RG + ER + EC = \frac{3}{5}x + 3x = \frac{18}{5}x$ m.
Thus, $\frac{18}{5}x = 18$ and so $x = 5$.
Since $x = 5$, then by the Pythagorean Theorem in $\triangle BCE$,

$$BE = \sqrt{BC^2 + EC^2} = \sqrt{(10 \text{ m})^2 + (5 \text{ m})^2} = \sqrt{125 \text{ m}^2} = 5\sqrt{5} \text{ m}$$

Now $FG = EF = BE = 5\sqrt{5}$ m and $GH = \frac{6}{10}FG = \frac{3}{5}(5\sqrt{5} \text{ m}) = 3\sqrt{5}$ m.
Therefore, the length of the path $BEFGH$ is $3(5\sqrt{5} \text{ m}) + (3\sqrt{5} \text{ m})$ or $18\sqrt{5}$ m.
ANSWER: $18\sqrt{5}$ m

9. The box contains $R+B$ balls when the first ball is drawn and $R+B-1$ balls when the second ball is drawn.
Therefore, there are $(R+B)(R+B-1)$ ways in which two balls can be drawn.
If two red balls are drawn, there are R balls that can be drawn first and $R-1$ balls that can be drawn second, and so there are $R(R-1)$ ways of doing this.
Since the probability of drawing two red balls is $\frac{2}{7}$, then $\frac{R(R-1)}{(R+B)(R+B-1)} = \frac{2}{7}$.
If only one of the balls is red, then the balls drawn are either red then blue or blue then red.
There are RB ways in the first case and BR ways in the second case, since there are R red balls and B blue balls in the box.
Since the probability of drawing exactly one red ball is $\frac{1}{2}$, then $\frac{2RB}{(R+B)(R+B-1)} = \frac{1}{2}$.
Dividing the first equation by the second, we obtain successively

$$\begin{aligned}
\frac{R(R-1)}{(R+B)(R+B-1)} \cdot \frac{(R+B)(R+B-1)}{2RB} &= \frac{2}{7} \cdot \frac{2}{1} \\
\frac{R-1}{2B} &= \frac{4}{7} \\
R &= \frac{8}{7}B + 1
\end{aligned}$$

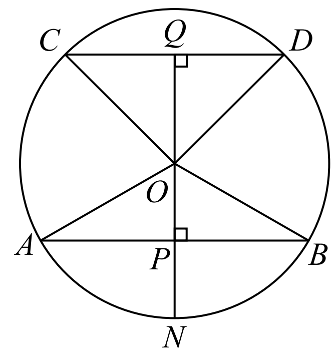
Substituting into the second equation, we obtain successively

$$\begin{aligned}
\frac{2\left(\frac{8}{7}B+1\right)B}{\left(\frac{8}{7}B+1+B\right)\left(\frac{8}{7}B+1+B-1\right)} &= \frac{1}{2} \\
\frac{2\left(\frac{8}{7}B+1\right)B}{\left(\frac{15}{7}B+1\right)\left(\frac{15}{7}B\right)} &= \frac{1}{2} \\
\frac{2(8B+7)}{15\left(\frac{15}{7}B+1\right)} &= \frac{1}{2} \qquad \text{(since } B \neq 0\text{)} \\
32B + 28 &= \tfrac{225}{7}B + 15 \\
13 &= \tfrac{1}{7}B \\
B &= 91
\end{aligned}$$

Since $R = \frac{8}{7}B + 1$, then $R = 105$ and so $(R, B) = (105, 91)$.
(We can verify that the given probabilities are correct with these starting numbers of red and blue balls.)

Answer: $(R, B) = (105, 91)$

10. Consider the front face of the tank, which is a circle of radius 10 m.
Suppose that when the water has depth 5 m, its surface is along horizontal line AB.
Suppose that when the water has depth $(10 + 5\sqrt{2})$ m, its surface is along horizontal line CD.
Let the area of the circle between the chords AB and CD be x m^2.
Since the tank is a cylinder which is lying on a flat surface, the volume of water added can be viewed as an irregular prism with base of area x m^2 and length 30 m.
Thus, the volume of water equals $30x$ m^3. Therefore, we need to calculate the value of x.



Let O be the centre of the circle, N be the point where the circle touches the ground, P the midpoint of AB, and Q the midpoint of CD.
Join O to A, B, C, and D. Also, join O to N and to Q.
Since Q is the midpoint of chord CD and O is the centre of the circle, then OQ is perpendicular to CD.
Since P is the midpoint of AB, then ON passes through P and is perpendicular to AB.
Since AB and CD are parallel and OP and OQ are perpendicular to these chords, then $QOPN$ is a straight line segment.

Since the radius of the circle is 10 m, then $OC = OD = OA = ON = OB = 10$ m.
Since AB is 5 m above the ground, then $NP = 5$ m.
Since $ON = 10$ m, then $OP = ON - NP = 5$ m.
Since CD is $(10 + 5\sqrt{2})$ m above the ground and $ON = 10$ m, then $QO = 5\sqrt{2}$ m.
Since $\triangle AOP$ is right-angled at P and $OP : OA = 1 : 2$, then $\triangle AOP$ is a 30°-60°-90° triangle with $\angle AOP = 60°$. Also, $AP = \sqrt{3}OP = 5\sqrt{3}$ m.
Since $\triangle CQO$ is right-angled at Q and $OC : OQ = 10 : 5\sqrt{2} = 2 : \sqrt{2} = \sqrt{2} : 1$, then $\triangle CQO$ is a 45°-45°-90° triangle with $\angle COQ = 45°$. Also, $CQ = OQ = 5\sqrt{2}$ m.

We are now ready to calculate the value of x.
The area between AB and CD is equal to the area of the circle minus the combined areas of the region under AB and the region above CD.
The area of the region under AB equals the area of sector AOB minus the area of $\triangle AOB$.
The area of the region above CD equals the area of sector COD minus the area of $\triangle COD$.
Since $\angle AOP = 60°$, then $\angle AOB = 2\angle AOP = 120°$.
Since $\angle COQ = 45°$, then $\angle COD = 2\angle COQ = 90°$.
Since the complete central angle of the circle is 360°, then sector AOB is $\frac{120°}{360°} = \frac{1}{3}$ of the whole circle and sector COD is $\frac{90°}{360°} = \frac{1}{4}$ of the whole circle.
Since the area of the entire circle is $\pi(10\text{ m})^2 = 100\pi\text{ m}^2$, then the area of sector AOB is $\frac{100}{3}\pi\text{ m}^2$ and the area of sector COD is $25\pi\text{ m}^2$.
Since P and Q are the midpoints of AB and CD, respectively, then $AB = 2AP = 10\sqrt{3}$ m and $CD = 2CQ = 10\sqrt{2}$ m.
Thus, the area of $\triangle AOB$ is $\frac{1}{2} \cdot AB \cdot OP = \frac{1}{2}(10\sqrt{3}\text{ m})(5\text{ m}) = 25\sqrt{3}\text{ m}^2$.
Also, the area of $\triangle COD$ is $\frac{1}{2} \cdot CD \cdot OQ = \frac{1}{2}(10\sqrt{2}\text{ m})(5\sqrt{2}\text{ m}) = 50\text{ m}^2$.
This means that the area of the region below AB is $\left(\frac{100}{3}\pi - 25\sqrt{3}\right)\text{ m}^2$ and the area of the region above CD is $(25\pi - 50)\text{ m}^2$.
Finally, this means that the volume of water added, in m^3, is

$$\begin{aligned}30x &= 30\left(100\pi - \left(\tfrac{100}{3}\pi - 25\sqrt{3}\right) - (25\pi - 50)\right)\\ &= 3000\pi - 1000\pi + 750\sqrt{3} - 750\pi + 1500\\ &= 1250\pi + 1500 + 750\sqrt{3}\end{aligned}$$

Therefore, $a\pi + b + c\sqrt{p} = 1250\pi + 1500 + 750\sqrt{3}$ and so $(a, b, c, p) = (1250, 1500, 750, 3)$.

ANSWER: $(1250, 1500, 750, 3)$

Team Problems

1. Evaluating,
$$\sqrt{1+2+3+4+5+6+7+8} = \sqrt{36} = 6$$
ANSWER: 6

2. Since the bucket is $\frac{2}{3}$ full and contains 9 L of maple syrup, then if it were $\frac{1}{3}$ full, it would contain $\frac{1}{2}(9 \text{ L}) = 4.5$ L.
Therefore, the capacity of the full bucket is $3 \cdot (4.5 \text{ L}) = 13.5$ L.
ANSWER: 13.5 L

3. Since the four integers are consecutive odd integers, then they differ by 2.
Let the four integers be $x-6, x-4, x-2, x$.
Since the sum of these integers is 200, then $(x-6)+(x-4)+(x-2)+x = 200$.
Simplifying and solving, we obtain $4x - 12 = 200$ and $4x = 212$ and $x = 53$.
Therefore, the largest of the four integers is 53.
ANSWER: 53

4. Since $80 = 20 \cdot 4$, then to make $80 = 20 \cdot 4$ thingamabobs, it takes $20 \cdot 11 = 220$ widgets.
Since $220 = 44 \cdot 5$, then to make $220 = 44 \cdot 5$ widgets, it takes $44 \cdot 18 = 792$ doodads.
Therefore, to make 80 thingamabobs, it takes 792 doodads.
ANSWER: 792

5. Since $BP_1 = P_1P_2 = P_2P_3 = P_3P_4 = P_4P_5 = P_5P_6 = P_6P_7 = P_7P_8$, then each of $\triangle BP_1P_2$, $\triangle P_1P_2P_3$, $\triangle P_2P_3P_4$, $\triangle P_3P_4P_5$, $\triangle P_4P_5P_6$, $\triangle P_5P_6P_7$, and $\triangle P_6P_7P_8$ is isosceles.
Since $\angle ABC = 5^\circ$, then $\angle BP_2P_1 = \angle ABC = 5^\circ$.
Next, $\angle P_2P_1P_3$ is an exterior angle for $\triangle BP_1P_2$.
Thus, $\angle P_2P_1P_3 = \angle P_1BP_2 + \angle P_1P_2B = 10^\circ$.
(To see this in another way, $\angle BP_1P_2 = 180^\circ - \angle P_1BP_2 - \angle P_1P_2B$ and
$$\angle P_2P_1P_3 = 180^\circ - \angle BP_1P_2 = 180^\circ - (180^\circ - \angle P_1BP_2 - \angle P_1P_2B) = \angle P_1BP_2 + \angle P_1P_2B$$
The first of these equations comes from the sum of the angles in the triangle and the second from supplementary angles.)
Continuing in this way,
$$\begin{aligned}
\angle P_2P_3P_1 &= \angle P_2P_1P_3 = 10^\circ \\
\angle P_3P_2P_4 &= \angle P_2P_3P_1 + \angle P_2BP_3 = 15^\circ \\
\angle P_3P_4P_2 &= \angle P_3P_2P_4 = 15^\circ \\
\angle P_4P_3P_5 &= \angle P_3P_4P_2 + \angle P_3BP_4 = 20^\circ \\
\angle P_4P_5P_3 &= \angle P_4P_3P_5 = 20^\circ \\
\angle P_5P_4P_6 &= \angle P_4P_5P_3 + \angle P_4BP_5 = 25^\circ \\
\angle P_5P_6P_4 &= \angle P_5P_4P_6 = 25^\circ \\
\angle P_6P_5P_7 &= \angle P_5P_6P_4 + \angle P_5BP_6 = 30^\circ \\
\angle P_6P_7P_5 &= \angle P_6P_5P_7 = 30^\circ \\
\angle P_7P_6P_8 &= \angle P_6P_7P_5 + \angle P_6BP_7 = 35^\circ \\
\angle P_7P_8P_6 &= \angle P_7P_6P_8 = 35^\circ \\
\angle AP_7P_8 &= \angle P_7P_8P_6 + \angle P_7BP_8 = 40^\circ
\end{aligned}$$
ANSWER: 40°

6. Suppose that the base of the pyramid has n sides.
The base will also have n vertices. Since the pyramid has one extra vertex (the apex), then it has $n+1$ vertices in total.
The pyramid has $n+1$ faces: the base plus n triangular faces formed by each edge of the base and the apex.
The pyramid has $2n$ edges: the n sides that form the base plus one edge joining each of the n vertices of the base to the apex.
From the given information, $2n + (n+1) = 1915$.
Thus, $3n = 1914$ and so $n = 638$.
Since the pyramid has $n+1$ faces, then it has 639 faces.

ANSWER: 639

7. Since $2^{11} = 2048$ and $2^5 = 32$, the eight values are

$$\begin{array}{cccc} 2^{11}+2^5+2=2082 & 2^{11}+2^5-2=2078 & 2^{11}-2^5+2=2018 & 2^{11}-2^5-2=2014 \\ -2^{11}+2^5+2=-2014 & -2^{11}+2^5-2=-2018 & -2^{11}-2^5+2=-2078 & -2^{11}-2^5-2=-2082 \end{array}$$

The third largest value is $2^{11} - 2^5 + 2 = 2018$.

ANSWER: $2^{11} - 2^5 + 2 = 2018$

8. For every real number a, $(-a)^3 = -a^3$ and so $(-a)^3 + a^3 = 0$.
Therefore,

$$(-n)^3 + (-n+1)^3 + \cdots + (n-2)^3 + (n-1)^3 + n^3 + (n+1)^3$$

which equals

$$((-n)^3 + n^3) + ((-n+1)^3 + (n-1)^3) + \cdots + ((-1)^3 + 1^3) + 0^3 + (n+1)^3$$

is equal to $(n+1)^3$.
Since $14^3 = 2744$ and $15^3 = 3375$ and n is an integer, then $(n+1)^3 < 3129$ exactly when $n+1 \leq 14$.
There are 13 positive integers n that satisfy this condition.

ANSWER: 13

9. Using the given definition, the following equations are equivalent:

$$\begin{aligned} (2\nabla x) - 8 &= x\nabla 6 \\ (2x - 4x) - 8 &= 6x - 6x^2 \\ 6x^2 - 8x - 8 &= 0 \\ 3x^2 - 4x - 4 &= 0 \end{aligned}$$

The sum of the values of x that satisfy the original equation equals the sum of the roots of this quadratic equation.
This sum equals $-\frac{-4}{3}$ or $\frac{4}{3}$.
(We could calculate the roots and add these, or use the fact that the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $-\dfrac{b}{a}$.)

ANSWER: $\frac{4}{3}$

10. Suppose Birgit's four numbers are a, b, c, d.
This means that the totals $a+b+c$, $a+b+d$, $a+c+d$, and $b+c+d$ are equal to 415, 442, 396, and 325, in some order.
If we add these totals together, we obtain

$$\begin{aligned}(a+b+c)+(a+b+d)+(a+c+d)+(b+c+d) &= 415+442+396+325 \\ 3a+3b+3c+3d &= 1578 \\ a+b+c+d &= 526\end{aligned}$$

since the order of addition does not matter.
Therefore, the sum of Luciano's numbers is 526.

Answer: 526

11. Let v_1 km/h be Krikor's constant speed on Monday.
Let v_2 km/h be Krikor's constant speed on Tuesday.
On Monday, Krikor drives for 30 minutes, which is $\frac{1}{2}$ hour.
Therefore, on Monday, Krikor drives $\frac{1}{2}v_1$ km.
On Tuesday, Krikor drives for 25 minutes, which is $\frac{5}{12}$ hour.
Therefore, on Tuesday, Krikor drives $\frac{5}{12}v_2$ km.
Since Krikor drives the same distance on both days, then $\frac{1}{2}v_1 = \frac{5}{12}v_2$ and so $v_2 = \frac{12}{5} \cdot \frac{1}{2}v_1 = \frac{6}{5}v_1$.
Since $v_2 = \frac{6}{5}v_1 = \frac{120}{100}v_1$, then v_2 is 20% larger than v_1.
That is, Krikor drives 20% faster on Tuesday than on Monday.

Answer: 20%

12. Using logarithm laws,

$$\begin{aligned}&\pi \log_{2018} \sqrt{2} + \sqrt{2}\log_{2018} \pi + \pi \log_{2018}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{2}\log_{2018}\left(\frac{1}{\pi}\right) \\ &\quad = \log_{2018}\left(\sqrt{2}^{\pi}\right) + \log_{2018}\left(\pi^{\sqrt{2}}\right) + \log_{2018}\left(\frac{1}{\sqrt{2}^{\pi}}\right) + \log_{2018}\left(\frac{1}{\pi^{\sqrt{2}}}\right) \\ &\quad = \log_{2018}\left(\frac{\sqrt{2}^{\pi} \cdot \pi^{\sqrt{2}}}{\sqrt{2}^{\pi} \cdot \pi^{\sqrt{2}}}\right) \\ &\quad = \log_{2018}(1) \\ &\quad = 0\end{aligned}$$

Answer: 0

13. We make a table that lists, for each possible value of k, the digits, the possible three-digit integers made by these digits, and $k+3$:

k	$k, k+1, k+2$	Possible integers	$k+3$
0	$0, 1, 2$	$102, 120, 201, 210$	3
1	$1, 2, 3$	$123, 132, 213, 231, 312, 321$	4
2	$2, 3, 4$	$234, 243, 324, 342, 423, 432$	5
3	$3, 4, 5$	$345, 354, 435, 453, 534, 543$	6
4	$4, 5, 6$	$456, 465, 546, 564, 645, 654$	7
5	$5, 6, 7$	$567, 576, 657, 675, 756, 765$	8
6	$6, 7, 8$	$678, 687, 768, 786, 867, 876$	9
7	$7, 8, 9$	$789, 798, 879, 897, 978, 987$	10

When $k=0$, the sum of the digits of each three-digit integer is 3, so each is divisible by 3.
When $k=1$, only two of the three-digit integers are even: 132 and 312. Each is divisible by 4.
When $k=2$, none of the three-digit integers end in 0 or 5 so none is divisible by 5.
When $k=3$, only two of the three-digit integers are even: 354 and 534. Each is divisible by 6.
When $k=4$, the integer 546 is divisible by 7. The rest are not. (One way to check this is by dividing each by 7.)
When $k=5$, only two of the three-digit integers are even: 576 and 756. Only 576 is divisible by 8.
When $k=6$, the sum of the digits of each of the three-digit integers is 21, which is not divisible by 9, so none of the integers is divisible by 9.
When $k=7$, none of the three-digit integers end in 0 so none is divisible by 10.
In total, there are $4+2+2+1+1=10$ three-digit integers that satisfy the required conditions.

ANSWER: 10

14. Suppose that d is the common difference in this arithmetic sequence.
Since $t_{2018}=100$ and t_{2021} is 3 terms further along in the sequence, then $t_{2021}=100+3d$.
Similarly, $t_{2036}=100+18d$ since it is 18 terms further along.
Since $t_{2018}=100$ and t_{2015} is 3 terms back in the sequence, then $t_{2015}=100-3d$.
Similarly, $t_{2000}=100-18d$ since it is 18 terms back.
Therefore,

$$\begin{aligned} t_{2000}+5t_{2015}+5t_{2021}+t_{2036} &= (100-18d)+5(100-3d)+5(100+3d)+(100+18d) \\ &= 1200-18d-15d+15d+18d \\ &= 1200 \end{aligned}$$

ANSWER: 1200

15. The area of the square wall with side length n metres is n^2 square metres.
The combined area of n circles each with radius 1 metre is $n \cdot \pi \cdot 1^2$ square metres or $n\pi$ square metres.
Given that Mathilde hits the wall at a random point, the probability that she hits a target is the ratio of the combined areas of the targets to the area of the wall, or $\dfrac{n\pi \text{ m}^2}{n^2 \text{ m}^2}$, which equals $\dfrac{\pi}{n}$.
For $\dfrac{\pi}{n} > \dfrac{1}{2}$, it must be the case that $n < 2\pi \approx 6.28$.
The largest value of n for which this is true is $n=6$.

ANSWER: $n=6$

16. First, we count the number of factors of 7 included in 200!.
Every multiple of 7 includes least 1 factor of 7.
The product 200! includes 28 multiples of 7 (since $28 \times 7 = 196$).
Counting one factor of 7 from each of the multiples of 7 (these are $7, 14, 21, \ldots, 182, 189, 196$), we see that 200! includes at least 28 factors of 7.
However, each multiple of $7^2 = 49$ includes a second factor of 7 (since $49 = 7^2, 98 = 7^2 \times 2$, etc.) which was not counted in the previous 28 factors.
The product 200! includes 4 multiples of 49, since $4 \times 49 = 196$, and thus there are at least 4 additional factors of 7 in 200!.
Since $7^3 > 200$, then 200! does not include any multiples of 7^3 and so we have counted all possible factors of 7.
Thus, 200! includes exactly $28+4=32$ factors of 7, and so $200! = 7^{32} \times t$ for some positive

integer t that is not divisible by 7.
Counting in a similar way, the product 90! includes 12 multiples of 7 and 1 multiple of 49, and thus includes 13 factors of 7.
Therefore, $90! = 7^{13} \times r$ for some positive integer r that is not divisible by 7.
Also, 30! includes 4 factors of 7, and thus $30! = 7^4 \times s$ for some positive integer s that is not divisible by 7.
Therefore, $\dfrac{200!}{90!30!} = \dfrac{7^{32} \times t}{(7^{13} \times r)(7^4 \times s)} = \dfrac{7^{32} \times t}{(7^{17} \times rs)} = \dfrac{7^{15} \times t}{rs}$.
Since we are given that $\dfrac{200!}{90!30!}$ is equal to a positive integer, then $\dfrac{7^{15} \times t}{rs}$ is a positive integer.
Since r and s contain no factors of 7 and $7^{15} \times t$ is divisible by rs, then it must be the case that t is divisible by rs.
In other words, we can re-write $\dfrac{200!}{90!30!} = \dfrac{7^{15} \times t}{rs}$ as $\dfrac{200!}{90!30!} = 7^{15} \times \dfrac{t}{rs}$ where $\dfrac{t}{rs}$ is an integer.
Since each of r, s and t does not include any factors of 7, then the integer $\dfrac{t}{rs}$ is not divisible by 7.
Therefore, the largest power of 7 which divides $\dfrac{200!}{90!30!}$ is 7^{15}, and so $n = 15$.

ANSWER: 15

17. Let $BD = h$.
Since $\angle BCA = 45^\circ$ and $\triangle BDC$ is right-angled at D, then $\angle CBD = 180^\circ - 90^\circ - 45^\circ = 45^\circ$.
This means that $\triangle BDC$ is isosceles with $CD = BD = h$.
Since $\angle BAC = 60^\circ$ and $\triangle BAD$ is right-angled at D, then $\triangle BAD$ is a 30°-60°-90° triangle.
Therefore, $BD : DA = \sqrt{3} : 1$.
Since $BD = h$, then $DA = \dfrac{h}{\sqrt{3}}$.
Thus, $CA = CD + DA = h + \dfrac{h}{\sqrt{3}} = h\left(1 + \dfrac{1}{\sqrt{3}}\right) = \dfrac{h(\sqrt{3}+1)}{\sqrt{3}}$.
We are told that the area of $\triangle ABC$ is $72 + 72\sqrt{3}$.
Since BD is perpendicular to CA, then the area of $\triangle ABC$ equals $\dfrac{1}{2} \cdot CA \cdot BD$.
Thus,

$$\begin{aligned}
\frac{1}{2} \cdot CA \cdot BD &= 72 + 72\sqrt{3} \\
\frac{1}{2} \cdot \frac{h(\sqrt{3}+1)}{\sqrt{3}} \cdot h &= 72(1+\sqrt{3}) \\
\frac{h^2}{2\sqrt{3}} &= 72 \\
h^2 &= 144\sqrt{3}
\end{aligned}$$

and so $BD = h = 12\sqrt[4]{3}$ since $h > 0$.

ANSWER: $12\sqrt[4]{3}$

18. Since each word is to be 7 letters long and there are two choices for each letter, there are $2^7 = 128$ such words.
We count the number of words that do contain three or more A's in a row and subtract this total from 128.

There is 1 word with exactly 7 A's in a row: AAAAAAA.

There are 2 words with exactly 6 A's in a row: AAAAAAB and BAAAAAA.

Consider the words with exactly 5 A's in a row.
If such a word begins with exactly 5 A's, then the 6th letter is B and so the word has the form AAAAABx where x is either A or B. There are 2 such words.
If such a word has the string of exactly 5 A's beginning in the second position, then it must be BAAAAAB since there cannot be an A either immediately before or immediately after the 5 A's.
If such a word ends with exactly 5 A's, then the 2nd letter is B and so the word has the form xBAAAAA where x is either A or B. There are 2 such words.
There are 5 words wth exactly 5 A's in a row.

Consider the words with exactly 4 A's in a row.
Using similar reasoning, such a word can be of one of the following forms: AAAABxx, BAAAABx, xBAAAAB, xxBAAAA.
Since there are two choices for each x, then there are $4 + 2 + 2 + 4 = 12$ words.

Consider the words with exactly 3 A's in a row.
Using similar reasoning, such a word can be of one of the following forms: AAABxxx, BAAABxx, xBAAABx, xxBAAAB, xxxBAAA.
Since there are two choices for each x, then there appear to be $8 + 4 + 4 + 4 + 8 = 28$ such words. However, the word AAABAAA is counted twice. (This the only word counted twice in any of these cases.) Therefore, there are 27 such words.

In total, there are $1 + 2 + 5 + 12 + 27 = 47$ words that contain three or more A's in a row, and so there are $128 - 47 = 81$ words that do not contain three or more A's in a row.

Answer: 81

19. Let $t = x^{1/5}$. Thus, $x^{2/5} = t^2$ and $x^{3/5} = t^3$.
Therefore, the following equations are equivalent:

$$\begin{aligned} x^{3/5} - 4 &= 32 - x^{2/5} \\ t^3 + t^2 - 36 &= 0 \\ (t-3)(t^2 + 4t + 12) &= 0 \end{aligned}$$

Thus, $t = 3$ or $t^2 + 4t + 12 = 0$.
The discriminant of this quadratic equation is $4^2 - 4(1)(12) < 0$, which means that there are no real values of t that are solutions. This in turn means that there are no corresponding real values of x.
This gives $x^{1/5} = t = 3$ and so $x = 3^5 = 243$ is the only solution.

Answer: $3^5 = 243$

20. Let $AC = x$.
Thus $BC = AC - 1 = x - 1$.
Since $AC = AB - 1$, then $AB = AC + 1 = x + 1$.
The perimeter of $\triangle ABC$ is $BC + AC + AB = (x - 1) + x + (x + 1) = 3x$.
By the cosine law in $\triangle ABC$,

$$\begin{aligned} BC^2 &= AB^2 + AC^2 - 2(AB)(AC)\cos(\angle BAC) \\ (x-1)^2 &= (x+1)^2 + x^2 - 2(x+1)x\left(\tfrac{3}{5}\right) \\ x^2 - 2x + 1 &= x^2 + 2x + 1 + x^2 - \tfrac{6}{5}(x^2 + x) \\ 0 &= x^2 + 4x - \tfrac{6}{5}(x^2 + x) \\ 0 &= 5x^2 + 20x - 6(x^2 + x) \\ x^2 &= 14x \end{aligned}$$

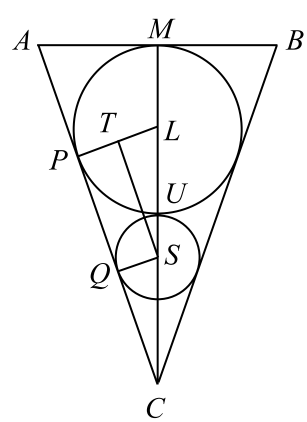
Since $x > 0$, then $x = 14$.
Therefore, the perimeter of $\triangle ABC$, which equals $3x$, is 42.

ANSWER: 42

21. Since $f(2x-3) - 2f(3x-10) + f(x-3) = 28 - 6x$ for all real numbers x, then when $x = 2$, we obtain $f(2(2)-3) - 2f(3(2)-10) + f(2-3) = 28 - 6(2)$ and so $f(1) - 2f(-4) + f(-1) = 16$.
Since f is an odd function, then $f(-1) = -f(1)$ or $f(1) + f(-1) = 0$.
Combining with $f(1) - 2f(-4) + f(-1) = 16$, we obtain $-2f(-4) = 16$ and so $-f(-4) = 8$.
Since f is an odd function, then $f(4) = -f(-4) = 8$.

ANSWER: 8

22. Let the radius of the small sphere be r and the radius of the large sphere be $2r$.
Draw a vertical cross-section through the centre of the top face of the cone and its bottom vertex.
By symmetry, this will pass through the centres of the spheres.
In the cross-section, the cone becomes a triangle and the spheres become circles.



We label the vertices of the triangle A, B, C.
We label the centres of the large circle and small circle L and S, respectively.
We label the point where the circles touch U.
We label the midpoint of AB (which represents the centre of the top face of the cone) as M.
Join L and S to the points of tangency of the circles to AC. We call these points P and Q.
Since LP and SQ are radii, then they are perpendicular to the tangent line AC at P and Q,

respectively.
Draw a perpendicular from S to T on LP.
The volume of the cone equals $\frac{1}{3}\pi \cdot AM^2 \cdot MC$. We determine the lengths of AM and MC in terms of r.
Since the radii of the small circle is r, then $QS = US = r$.
Since $TPQS$ has three right angles (at T, P and Q), then it has four right angles, and so is a rectangle.
Therefore, $PT = QS = r$.
Since the radius of the large circle is $2r$, then $PL = UL = ML = 2r$.
Therefore, $TL = PL - PT = 2r - r = r$.
Since MC passes through L and S, it also passes through U, the point of tangency of the two circles.
Therefore, $LS = UL + US = 2r + r = 3r$.
By the Pythagorean Theorem in $\triangle LTS$,

$$TS = \sqrt{LS^2 - TL^2} = \sqrt{(3r)^2 - r^2} = \sqrt{8r^2} = 2\sqrt{2}r$$

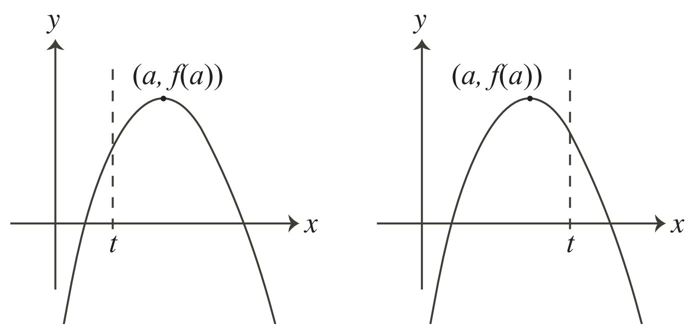
since $TS > 0$ and $r > 0$.
Consider $\triangle LTS$ and $\triangle SQC$.
Each is right-angled, $\angle TLS = \angle QSC$ (because LP and QS are parallel), and $TL = QS$.
Therefore, $\triangle LTS$ is congruent to $\triangle SQC$.
Thus, $SC = LS = 3r$ and $QC = TS = 2\sqrt{2}r$.
This tells us that $MC = ML + LS + SC = 2r + 3r + 3r = 8r$.
Also, $\triangle AMC$ is similar to $\triangle SQC$, since each is right-angled and they have a common angle at C.
Therefore, $\dfrac{AM}{MC} = \dfrac{QS}{QC}$ and so $AM = \dfrac{8r \cdot r}{2\sqrt{2}r} = 2\sqrt{2}r$.
This means that the volume of the original cone is $\frac{1}{3}\pi \cdot AM^2 \cdot MC = \frac{1}{3}\pi(2\sqrt{2}r)^2(8r) = \frac{64}{3}\pi r^3$.
The volume of the large sphere is $\frac{4}{3}\pi(2r)^3 = \frac{32}{3}\pi r^3$.
The volume of the small sphere is $\frac{4}{3}\pi r^3$.
The volume of the cone not occupied by the spheres is $\frac{64}{3}\pi r^3 - \frac{32}{3}\pi r^3 - \frac{4}{3}\pi r^3 = \frac{28}{3}\pi r^3$.
The fraction of the volume of the cone that this represents is $\dfrac{\frac{28}{3}\pi r^3}{\frac{64}{3}\pi r^3} = \frac{28}{64} = \frac{7}{16}$.

Answer: $\frac{7}{16}$

23. Let $f(x) = -x^2 + 2ax + a$.
Since $(x-a)^2 = x^2 - 2ax + a^2$, then

$$-x^2 + 2ax + a = -(x^2 - 2ax + a^2) + a^2 + a = -(x-a)^2 + a^2 + a$$

Therefore, $M(t)$ is the maximum value of $-(x-a)^2 + a^2 + a$ over all real numbers x with $x \leq t$.
Now the graph of $y = f(x) = -(x-a)^2 + a^2 + a$ is a parabola opening downwards with vertex at $(a, a^2 + a)$.
Since the parabola opens downwards, then the parabola reaches its maximum at the vertex $(a, a^2 + a)$.
Therefore, $f(x)$ is increasing when $x < a$ and decreasing when $x > a$.
This means that, when $t < a$, the maximum of the values of $f(x)$ with $x \leq t$ is $f(t)$ (because $f(x)$ increases until $f(t)$) and when $t \geq a$, the maximum of the values of $f(x)$ with $x \leq t$ is $f(a)$ (because the maximum value of $f(x)$ is to the left of t).



In other words,

$$M(t) = \begin{cases} f(t) & t < a \\ f(a) & t \geq a \end{cases}$$

Since $a - 1 < a$, then $M(a-1) = f(a-1)$.
Since $a + 2 > a$, then $M(a+2) = f(a)$.
Therefore,

$$\begin{aligned} M(a-1) + M(a+2) &= f(a-1) + f(a) \\ &= (-((a-1)-a)^2 + a^2 + a) + (-(a-a)^2 + a^2 + a) \\ &= -1 + a^2 + a - 0 + a^2 + a \\ &= 2a^2 + 2a - 1 \end{aligned}$$

ANSWER: $2a^2 + 2a - 1$

24. We find the points of intersection of $y = 2\cos 3x + 1$ and $y = -\cos 2x$ by equating values of y and obtaining the following equivalent equations:

$$\begin{aligned}
2\cos 3x + 1 &= -\cos 2x \\
2\cos(2x + x) + \cos 2x + 1 &= 0 \\
2(\cos 2x \cos x - \sin 2x \sin x) + \cos 2x + 1 &= 0 \\
2((2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x) + (2\cos^2 x - 1) + 1 &= 0 \\
2(2\cos^3 x - \cos x - 2\sin^2 x \cos x) + 2\cos^2 x &= 0 \\
4\cos^3 x - 2\cos x - 4(1 - \cos^2 x)\cos x + 2\cos^2 x &= 0 \\
4\cos^3 x - 2\cos x - 4\cos x + 4\cos^3 x + 2\cos^2 x &= 0 \\
8\cos^3 x + 2\cos^2 x - 6\cos x &= 0 \\
4\cos^3 x + \cos^2 x - 3\cos x &= 0 \\
\cos x(4\cos^2 x + \cos x - 3) &= 0 \\
\cos x(\cos x + 1)(4\cos x - 3) &= 0
\end{aligned}$$

Therefore, $\cos x = 0$ or $\cos x = -1$ or $\cos x = \frac{3}{4}$.
Two of these points of intersection, P and Q, have x-coordinates between $\frac{17\pi}{4} = 4\pi + \frac{\pi}{4}$ and $\frac{21\pi}{4} = 5\pi + \frac{\pi}{4}$.
Since $\cos 4\pi = 1$ and $\cos(4\pi + \frac{\pi}{4}) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{2\sqrt{2}}{4} < \frac{3}{4}$, then there is an angle θ between 4π and $\frac{17\pi}{4}$ with $\cos\theta = \frac{3}{4}$ and so there is not such an angle between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$.
Therefore, we look for angles x between $\frac{17\pi}{4}$ and $\frac{21\pi}{4}$ with $\cos x = 0$ and $\cos x = -1$.
Note that $\cos 5\pi = -1$ and that $\cos\frac{18\pi}{4} = \cos\frac{9\pi}{2} = \cos\frac{\pi}{2} = 0$.
Thus, the x-coordinates of P and Q are 5π and $\frac{9\pi}{2}$.
Suppose that P has x-coordinate 5π.
Since P lies on $y = -\cos 2x$, then its y-coordinate is $y = -\cos 10\pi = -1$.
Suppose that Q has x-coordinate $\frac{9\pi}{2}$.
Since Q lies on $y = -\cos 2x$, then its y-coordinate is $y = -\cos 9\pi = 1$.
Therefore, the coordinates of P are $(5\pi, -1)$ and the coordinates of Q are $(\frac{9\pi}{2}, 1)$.
The slope of the line through P and Q is $\dfrac{1 - (-1)}{\frac{9\pi}{2} - 5\pi} = \dfrac{2}{-\frac{\pi}{2}} = -\frac{4}{\pi}$.
The line passes through the point with coordinates $(5\pi, -1)$.
Thus, its equation is $y - (-1) = -\frac{4}{\pi}(x - 5\pi)$ or $y + 1 = -\frac{4}{\pi}x + 20$ or $y = -\frac{4}{\pi}x + 19$.
This line intersects the y-axis at $A(0, 19)$. Thus, $AO = 19$.
This line intersects the x-axis at point B with y-coordinate 0 and hence the x-coordinate of B is $x = 19 \cdot \frac{\pi}{4} = \frac{19\pi}{4}$. Thus, $BO = \frac{19\pi}{4}$.
Since $\triangle AOB$ is right-angled at the origin, then its area equals $\frac{1}{2}(AO)(BO) = \frac{1}{2}(19)(\frac{19\pi}{4}) = \frac{361\pi}{8}$.

ANSWER: $\frac{361\pi}{8}$

25. Let $f(2n)$ be the number of different ways to draw n non-intersecting line segments connecting pairs of points so that each of the $2n$ points is connected to exactly one other point.
Then $f(2) = 1$ (since there are only 2 points) and $f(6) = 5$ (from the given example).
Also, $f(4) = 2$. (Can you see why?)
We show that

$$f(2n) = f(2n-2)+f(2)f(2n-4)+f(4)f(2n-6)+f(6)f(2n-8)+\cdots+f(2n-4)f(2)+f(2n-2)$$

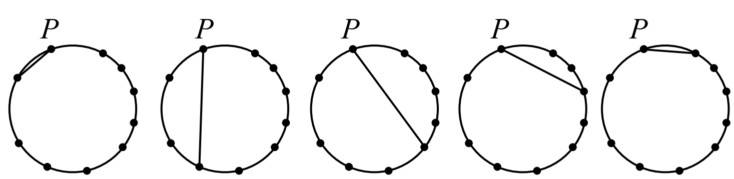
Here is a justification for this equation:

Pick one of the $2n$ points and call it P.
P could be connected to the 1st point counter-clockwise from P. This leaves $2n-2$ points on the circle. By definition, these can be connected in $f(2n-2)$ ways.
P cannot be connected to the 2nd point counter-clockwise, because this would leave an odd number of points on one side of this line segment. An odd number of points cannot be connected in pairs as required.
Similarly, P cannot be connected to any of the 4th, 6th, 8th, $(2n-2)$th points.
P can be connected to the 3rd point counter-clockwise, leaving 2 points on one side and $2n-4$ points on the other side. There are $f(2)$ ways to connect the 2 points and $f(2n-4)$ ways to connect the $2n-4$ points. Therefore, in this case there are $f(2)f(2n-4)$ ways to connect the points. We cannot connect a point on one side of the line to a point on the other side of the line because the line segments would cross, which is not allowed.
P can be connected to the 5th point counter-clockwise, leaving 4 points on one side and $2n-6$ points on the other side. There are $f(4)$ ways to connect the 4 points and $f(2n-6)$ ways to connect the $2n-6$ points. Therefore, in this case there are $f(4)f(2n-6)$ ways to connect the points.
Continuing in this way, P can be connected to every other point until we reach the last $((2n-1)$th) point which will leave $2n-2$ points on one side and none on the other. There are $f(2n-2)$ possibilities in this case.
Adding up all of the cases, we see that there are

$$f(2n) = f(2n-2)+f(2)f(2n-4)+f(4)f(2n-6)+f(6)f(2n-8)+\cdots+f(2n-4)f(2)+f(2n-2)$$

ways of connecting the points.

The figures below show the case of $2n = 10$.



We can use this formula to successively calculate $f(8), f(10), f(12), f(14), f(16)$:

$$\begin{aligned}
f(8) &= f(6) + f(2)f(4) + f(4)f(2) + f(6) \\
&= 5 + 1(2) + 2(1) + 5 \\
&= 14
\end{aligned}$$

$$\begin{aligned}
f(10) &= f(8) + f(2)f(6) + f(4)f(4) + f(6)f(2) + f(8) \\
&= 14 + 1(5) + 2(2) + 5(1) + 14 \\
&= 42
\end{aligned}$$

$$\begin{aligned}
f(12) &= f(10) + f(2)f(8) + f(4)f(6) + f(6)f(4) + f(8)f(2) + f(10) \\
&= 42 + 1(14) + 2(5) + 5(2) + 14(1) + 42 \\
&= 132
\end{aligned}$$

$$\begin{aligned}
f(14) &= f(12) + f(2)f(10) + f(4)f(8) + f(6)f(6) + f(8)f(4) + f(10)f(2) + f(12) \\
&= 132 + 1(42) + 2(14) + 5(5) + 14(2) + 42(1) + 132 \\
&= 429
\end{aligned}$$

$$\begin{aligned}
f(16) &= f(14) + f(2)f(12) + f(4)f(10) + f(6)f(8) + f(8)f(6) + f(10)f(4) + f(12)f(2) + f(14) \\
&= 429 + 1(132) + 2(42) + 5(14) + 14(5) + 42(2) + 132(1) + 429 \\
&= 1430
\end{aligned}$$

Therefore, there are 1430 ways to join the 16 points.
(The sequence $1, 2, 5, 12, 42, \ldots$ is a famous sequence called the *Catalan numbers*.)

ANSWER: 1430

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{9+3\times 3}{3} = \dfrac{9+9}{3} = \dfrac{18}{3} = 6$.

   (b) The area of a triangle with base $2t$ and height $3t+2$ is $\frac{1}{2}(2t)(3t+2)$ or $t(3t+2)$.
   Since the answer to (a) is 6, then $t = 6$, and so $t(3t+2) = 6(20) = 120$.

   (c) Since $AB = BC$, then $\angle BCA = \angle BAC$.
   Since $\angle ABC = t^\circ$, then $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = \frac{1}{2}(180^\circ - t^\circ) = 90^\circ - \frac{1}{2}t^\circ$.
   Since the answer to (b) is 120, then $t = 120$, and so

$$\angle BAC = 90^\circ - \tfrac{1}{2}(120^\circ) = 30^\circ$$

ANSWER: $6, 120, 30^\circ$

1. (a) We find the prime factorization of 390:

$$390 = 39 \cdot 10 = 3 \cdot 13 \cdot 2 \cdot 5$$

   Since 9450 is divisible by 10, then 2 and 5 are also prime factors of 9450.
   Since the sum of the digits of 9450 is $9+4+5+0 = 18$ which is a multiple of 3, then 9450 is also divisible by 3.
   (We can check that 9450 is not divisible by 13.)
   Therefore, the sum of the three common prime divisors of 390 and 9450 is $2+3+5 = 10$.

   (b) Simplifying,

$$\begin{aligned} n &= \frac{(4t^2 - 10t - 2) - 3(t^2 - t + 3) + (t^2 + 5t - 1)}{(t+7) + (t-13)} \\ &= \frac{4t^2 - 10t - 2 - 3t^2 + 3t - 9 + t^2 + 5t - 1}{2t - 6} \\ &= \frac{2t^2 - 2t - 12}{2t - 6} \\ &= \frac{t^2 - t - 6}{t - 3} \\ &= \frac{(t-3)(t+2)}{t-3} \\ &= t + 2 \end{aligned}$$

   assuming that $t \neq 3$.
   Since the answer to (a) is 10, then $t = 10$, and so $n = t + 2 = 12$.

   (c) We determine the average by calculating the sum of the 36 possible sums, and then dividing by 36.
   To determine the sum of the 36 possible sums, we determine the sum of the 36 values that appear on the top face of each of the two dice.
   Each of the 6 faces on the first dice is rolled in 6 of the 36 possibilities.
   Thus, these faces contribute $6(1+2+3+4+5+6) = 6(21) = 126$ to the sum of the 36 possible sums.
   Each of the 6 sides on the second dice is rolled in 6 of the 36 possibilities.

Thus, these faces contribute $6((t-10)+t+(t+10)+(t+20)+(t+30)+(t+40))$ or $6(6t+90)$ or $36t+540$ to the sum of the 36 possible sums.
Therefore, the sum of the 36 sums is $126+36t+540=36t+666$, and so the average of the 36 sums is $\dfrac{36t+666}{36}=t+\dfrac{111}{6}=t+\dfrac{37}{2}$.
Since the answer to (b) is 12, then $t=12$ and so the average of the 36 possible sums is $12+\dfrac{37}{2}=\dfrac{61}{2}=30.5$.

ANSWER: $10, 12, \dfrac{61}{2}$

2. (a) Expanding and simplifying,

$$2(x-3)^2-12=2(x^2-6x+9)-12=2x^2-12x+6$$

Thus, $a=2$, $b=-12$, and $c=6$.
This means that $10a-b-4c=10(2)-(-12)-4(6)=8$.

(b) The line through the points $(11,-7)$ and $(15,5)$ has slope $\dfrac{5-(-7)}{15-11}=\dfrac{12}{4}=3$.
Thus, a line perpendicular to this line has slope $-\dfrac{1}{3}$.
Therefore, the slope of the line through the points $(-4,t)$ and (k,k) has slope $-\dfrac{1}{3}$.
Thus, $\dfrac{k-t}{k-(-4)}=-\dfrac{1}{3}$.
We solve for k:

$$\begin{aligned}\frac{k-t}{k+4}&=-\frac{1}{3}\\ 3k-3t&=-k-4\\ 4k&=3t-4\\ k&=\frac{3}{4}t-1\end{aligned}$$

Since the answer to (a) is 8, then $t=8$ and so $k=\dfrac{3}{4}(8)-1=5$.

(c) The sum of the entries in the second row is

$$(4t-1)+(2t+12)+(t+16)+(3t+1)=10t+28$$

This means that the sum of the four entries in any row, column or diagonal will also be $10t+28$.
Looking at the fourth column, the top right entry is thus

$$10t+28-(3t+1)-(t+15)-(4t-5)=2t+17$$

Looking at the top row, the top left entry is thus

$$10t+28-(3t-2)-(4t-6)-(2t+17)=t+19$$

Looking at the southeast diagonal, the third entry is thus

$$10t+28-(t+19)-(2t+12)-(4t-5)=3t+2$$

Looking at the third row,

$$N = 10t + 28 - (4t - 2) - (3t + 2) - (t + 15) = 2t + 13$$

Since the answer to (b) is 5, then $t = 5$. Thus, $N = 23$.
We note that we can complete the grid, both in terms of t and using $t = 5$ as follows:

$t + 19$	$3t - 2$	$4t - 6$	$2t + 17$
$4t - 1$	$2t + 12$	$t + 16$	$3t + 1$
$2t + 13$	$4t - 2$	$3t + 2$	$t + 15$
$3t - 3$	$t + 20$	$2t + 16$	$4t - 5$

24	13	14	27
19	22	21	16
23	18	17	20
12	25	26	15

ANSWER: $8, 5, 23$

3. (a) Since (a, a^2) lies on the line with equation $y = 5x + a$, then $a^2 = 5a + a$ or $a^2 = 6a$. Since $a \neq 0$, then $a = 6$.

(b) The team scored a total of $10t \cdot 4 + 20 \cdot g = 40t + 20g$ points over their $4 + g$ games. Since their average number of points per game was 28, then

$$\begin{aligned} \frac{40t + 20g}{g + 4} &= 28 \\ 40t + 20g &= 28g + 112 \\ 40t - 112 &= 8g \\ g &= 5t - 14 \end{aligned}$$

Since the answer to (a) is 6, then $t = 6$ and so $g = 5(6) - 14 = 16$.

(c) Since $(x, y) = (a, b)$ satisfies the system of equations, then

$$\begin{aligned} a^2 + 4b &= t^2 \\ a^2 - b^2 &= 4 \end{aligned}$$

Subtracting the second equation from the first, we obtain successively

$$\begin{aligned} (a^2 + 4b) - (a^2 - b^2) &= t^2 - 4 \\ b^2 + 4b &= t^2 - 4 \\ b^2 + 4b + 4 &= t^2 \\ (b + 2)^2 &= t^2 \\ b + 2 &= \pm t \\ b &= -2 \pm t \end{aligned}$$

Since the answer to (b) is 16, then $t = 16$.
Therefore, $b = -2 + t = 14$ or $b = -2 - t = -18$.
Since $b > 0$, then $b = 14$.

ANSWER: $6, 16, 14$