**The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING**
cemc.uwaterloo.ca

2017 Gauss Contests

(Grades 7 and 8)

Wednesday, May 10, 2017
(in North America and South America)

Thursday, May 11, 2017
(outside of North America and South America)

Solutions

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Grade 7

1. Evaluating, $(2 + 4 + 6) - (1 + 3 + 5) = 12 - 9 = 3$.

   ANSWER: (B)

2. Reading from the tallest bar on the graph, approximately 50 students play soccer.
   Since this is larger than the number of students who play any of the other sports, then soccer is played by the most students.

   ANSWER: (C)

3. Michael has $280 in $20 bills and so the number of $20 bills that he has is $280 \div 20 = 14$.

   ANSWER: (C)

4. There are only two different products of two positive integers whose result is 14.
   These are 2×7 and 1×14.
   Since the two integers must be between 1 and 10, then the product must be 2×7.
   The sum of these two integers is $2 + 7 = 9$.

   ANSWER: (D)

5. Written as a fraction, three thousandths is equal to $\frac{3}{1000}$.
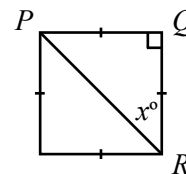   As a decimal, three thousandths is equal to $3 \div 1000 = 0.003$.

   ANSWER: (E)

6. *Solution 1*
   Since the given figure is a square, then $PQ = QR$ and $\angle PQR = 90°$.
   Since $PQ = QR$, $\triangle PQR$ is isosceles and so $\angle QPR = \angle QRP = x°$.
   The three angles in any triangle add to $180°$ and since $\angle PQR = 90°$,
   then $\angle QPR + \angle QRP = 180° - 90° = 90°$.
   Since $\angle QPR = \angle QRP$, then $\angle QRP = 90° \div 2 = 45°$, and so $x = 45$.
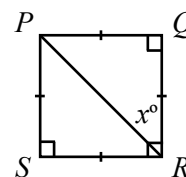
   Solution 2
   Diagonal PR divides square $PQRS$ into two identical triangles: $\triangle PQR$ and $\triangle PSR$.
   Since these triangles are identical, $\angle PRS = \angle PRQ = x°$.
   Since $PQRS$ is a square, then $\angle QRS = 90°$.
   That is, $\angle PRS + \angle PRQ = 90°$ or $x° + x° = 90°$ or $2x = 90$ and so $x = 45$.

   ANSWER: (B)

7. *Solution 1*
   Written as a mixed fraction, $\frac{35}{4} = 8\frac{3}{4}$.
   Since $\frac{3}{4}$ is closer to 1 than it is to 0, then $8\frac{3}{4}$ is closer to 9 than it is to 8.
   The integer closest in value to $\frac{35}{4}$ is 9.

   Solution 2
   Written as a decimal $\frac{35}{4} = 35 \div 4 = 8.75$.
   Since 0.75 is closer to 1 than it is to 0, then 8.75 is closer to 9 than it is to 8.
   The integer closest in value to $\frac{35}{4}$ is 9.

   ANSWER: (C)

8. When $n = 101$:

$$\begin{aligned}
3n &= 3 \times 101 = 303 \\
n + 2 &= 101 + 2 = 103 \\
n - 12 &= 101 - 12 = 89 \\
2n - 2 &= 2 \times 101 - 2 = 202 - 2 = 200 \\
3n + 2 &= 3 \times 101 + 2 = 303 + 2 = 305.
\end{aligned}$$

Of the expressions given, $2n - 2$ is the only expression which has an even value when $n = 101$. (In fact, the value of $2n - 2$ is an even integer for every integer n. Can you see why?)

ANSWER: (D)

9. The mean (average) of three integers whose sum is 153 is $\frac{153}{3} = 51$.
The mean of three consecutive integers equals the middle of the three integers.
That is, 51 is the middle integer of three consecutive integers and so the largest of these integers is 52.
(We may check that $50 + 51 + 52 = 153$.)

ANSWER: (A)

10. Each of the 4 smaller triangles is equilateral and thus has sides of equal length.
Each of these smaller triangles has a perimeter of 9 cm and so has sides of length $\frac{9}{3} = 3$ cm.
In $\triangle PQR$, side PQ is made up of two such sides of length 3 cm and thus $PQ = 2 \times 3 = 6$ cm.
Since $\triangle PQR$ is equilateral, then $PR = QR = PQ = 6$ cm.
Therefore, the perimeter of $\triangle PQR$ is $3 \times 6 = 18$ cm.

ANSWER: (E)

11. The denominators of the two fractions are 7 and 63.
Since $7 \times 9 = 63$, then we must also multiply the numerator 3 by 9 so that the fractions are equivalent.
That is, $\dfrac{3}{7} = \dfrac{3 \times 9}{7 \times 9} = \dfrac{27}{63}$.
Therefore, the number that goes into the $\square$ so that the statement true is 27.

ANSWER: (A)

12. If puzzles are bought individually for \$10 each, then 6 puzzles will cost $\$10 \times 6 = \$60$.
Since the cost for a box of 6 puzzles is \$50, it is less expensive to buy puzzles by the box than it is to buy them individually.
Buying 4 boxes of 6 puzzles gives the customer $4 \times 6 = 24$ puzzles and the cost is $4 \times \$50 = \$200$.
Buying one additional puzzle for \$10 gives the customer 25 puzzles at a minimum cost of \$210.

ANSWER: (A)

13. A translation moves (slides) an object some distance without altering it in any other way.
That is, the object is not rotated, reflected, and its exact size and shape are maintained.
Of the triangles given, the triangle labelled D is the only triangle whose orientation is identical to that of the shaded triangle.
Thus, D is the triangle which can be obtained when the shaded triangle is translated.

ANSWER: (D)

14. Since the time in Toronto, ON is 1:00 p.m. when the time in Gander, NL is 2:30 p.m., then the time in Gander is 1 hour and 30 minutes ahead of the time in Toronto.
A flight that departs from Toronto at 3:00 p.m. and takes 2 hours and 50 minutes will land in Gander at 5:50 p.m. Toronto time.
When the time in Toronto is 5:50 p.m., the time in Gander is 1 hour and 30 minutes ahead which is 7:20 p.m.

ANSWER: (A)

15. Henry was slower than Faiz and thus finished the race behind Faiz.
    Ryan was faster than Henry and Faiz and thus finished the race ahead of both of them.
    From fastest to slowest, these three runners finished in the order Ryan, Faiz and then Henry.
    Toma was faster than Ryan but slower than Omar.
    Therefore, from fastest to slowest, the runners finished in the order Omar, Toma, Ryan, Faiz, and Henry.
    The student who finished fourth was Faiz.

    ANSWER: (A)

16. The positive divisors of 20 are: $1, 2, 4, 5, 10, 20$.
    Of the 20 numbers on the spinner, 6 of the numbers are divisors of 20.
    It is equally likely that the spinner lands on any of the 20 numbers.
    Therefore, the probability that the spinner lands on a number that is a divisor of 20 is $\frac{6}{20}$.

    ANSWER: (E)

17. *Solution 1*
    Since 78 is 2 less than 80 and 82 is 2 greater than 80, the mean of 78 and 82 is 80.
    Since the mean of all four integers is 80, then the mean of 83 and x must also equal 80.
    The integer 83 is 3 greater than 80, and so x must be 3 less than 80.
    That is, $x = 80 - 3 = 77$.
    (We may check that the mean of $78, 83, 82$, and 77 is indeed 80.)

    Solution 2
    Since the mean of the four integers is 80, then the sum of the four integers is $4 \times 80 = 320$.
    Since the sum of 78, 83 and 82 is 243, then $x = 320 - 243 = 77$.
    Therefore, x is 77 which is 3 less than the mean 80.

    ANSWER: (D)

18. A discount of 20% on a book priced at $100 is a $0.20 \times \$100 = \$20$ discount.
    Thus for option (A), Sara's discounted price is $\$100 - \$20 = \$80$.
    A discount of 10% on a book priced at $100 is a $0.10 \times \$100 = \$10$ discount, giving a discounted price of $\$100 - \$10 = \$90$.
    A second discount of 10% on the new price of $90 is a $0.10 \times \$90 = \$9$ discount.
    Thus for option (B), Sara's discounted price is $\$90 - \$9 = \$81$.
    A discount of 15% on a book priced at $100 is a $0.15 \times \$100 = \$15$ discount, giving a discounted price of $\$100 - \$15 = \$85$.
    A further discount of 5% on the new price of $85 is a $0.05 \times \$85 = \$4.25$ discount.
    Thus for option (C), Sara's discounted price is $\$85 - \$4.25 = \$80.75$.
    A discount of 5% on a book priced at $100 is a $0.05 \times \$100 = \$5$ discount, giving a discounted price of $\$100 - \$5 = \$95$.
    A further discount of 15% on the new price of $95 is a $0.15 \times \$95 = \$14.25$ discount.
    Thus for option (D), Sara's discounted price is $\$95 - \$14.25 = \$80.75$.
    Therefore, the four options do not give the same price and option (A) gives Sara the best discounted price.

    ANSWER: (A)

19. In the diagram, rectangles $WQRZ$ and $PXYS$ are the two sheets
    of 11 cm× 8 cm paper.
    The overlapping square $PQRS$ has sides of length 8 cm.
    That is, $WQ = ZR = PX = SY = 11$ cm and
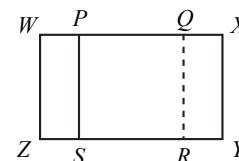 $WZ = QR = PS = XY = PQ = SR = 8$ cm.
    In rectangle $WQRZ$, $WQ = WP + PQ = 11$ cm and so
    $WP = 11 - PQ = 11 - 8 = 3$ cm.
    In rectangle $WXYZ$, $WX = WP + PX = 3 + 11 = 14$ cm.
    Since $WX = 14$ cm and $XY = 8$ cm, the area of $WXYZ$ is
    $14 \times 8 = 112$ cm^2.

    ANSWER: (B)

20. Points P, Q, R, S, T divide the bottom edge of the park into six segments of equal length, each
    of which has length $600 \div 6 = 100$ m.
    If Betty and Ann had met for the first time at point Q, then Betty would have walked a total
    distance of $600 + 400 + 4 \times 100 = 1400$ m and Ann would have walked a total distance of
    $400 + 2 \times 100 = 600$ m.
    When they meet, the time that Betty has been walking is equal to the time that Ann has been
    walking and so the ratio of Betty's speed to Ann's speed is equal to the ratio of the distance
    that Betty has walked to the distance that Ann has walked.
    That is, if they had met for the first time at point Q, the ratio of their speed's would be
    $1400 : 600$ or $14 : 6$ or $7 : 3$.
    Similarly, if Betty and Ann had met for the first time at point R, then Betty would have walked
    a total distance of $600 + 400 + 3 \times 100 = 1300$ m and Ann would have walked a total distance
    of $400 + 3 \times 100 = 700$ m.
    In this case, the ratio of their speed's would be $1300 : 700$ or $13 : 7$.
    When Betty and Ann actually meet for the first time, they are between Q and R.
    Thus Betty has walked less distance than she would have had they met at Q and more distance
    than she would have had they met at R.
    That is, the ratio of Betty's speed to Ann's speed must be less than $7 : 3$ and greater than
    $13 : 7$.
    We must determine which of the five given answers is a ratio that is less than $7 : 3$ and greater
    than $13 : 7$.
    One way to do this is to convert each ratio into a mixed fraction.
    That is, we must determine which of the five answers is less than $7 : 3 = \frac{7}{3} = 2\frac{1}{3}$ and greater
    than $13 : 7 = \frac{13}{7} = 1\frac{6}{7}$.
    Converting the answers, we get $\frac{5}{3} = 1\frac{2}{3}, \frac{9}{4} = 2\frac{1}{4}, \frac{11}{6} = 1\frac{5}{6}, \frac{12}{5} = 2\frac{2}{5}$, and $\frac{17}{7} = 2\frac{3}{7}$.
    Of the five given answers, the only fraction that is less than $2\frac{1}{3}$ and greater than $1\frac{6}{7}$ is $2\frac{1}{4}$.
    If Betty and Ann meet for the first time between Q and R, then the ratio of Betty's speed to
    Ann's speed could be $9 : 4$.

    ANSWER: (B)

21. *Solution 1*
    The first and tenth rectangles each contribute an equal amount to the perimeter.
    They each contribute two vertical sides (each of length 2), one full side of length 4 (the top side
    for the first rectangle and the bottom side for the tenth rectangle), and one half of the length
    of the opposite side.
    That is, the first and tenth rectangles each contribute $2 + 2 + 4 + 2 = 10$ to the perimeter.
    Rectangles two through nine each contribute an equal amount to the perimeter.

They each contribute two vertical sides (each of length 2), one half of a side of length 4, and one half of the length of the opposite side (which also has length 4).

That is, rectangles two through nine each contribute $2 + 2 + 2 + 2 = 8$ to the perimeter.

Therefore, the total perimeter of the given figure is $(2 \times 10) + (8 \times 8) = 20 + 64 = 84$.

Solution 2

One method for determining the perimeter of the given figure is to consider vertical lengths and horizontal lengths.

Each of the ten rectangles has two vertical sides (a left side and a right side) which contribute to the perimeter.

These 20 sides each have length 2, and thus contribute $20 \times 2 = 40$ to the perimeter of the figure.

Since these are the only vertical lengths contributing to the perimeter, we now determine the sum of the horizontal lengths.

There are two types of horizontal lengths which contribute to the perimeter: the bottom side of a rectangle, and the top side of a rectangle.

The bottom side of each of the first nine rectangles contributes one half of its length to the perimeter.

That is, the bottom sides of the first nine rectangles contribute $\frac{1}{2} \times 4 \times 9 = 18$ to the perimeter.

The entire bottom side of the tenth rectangle is included in the perimeter and thus contributes a length of 4.

Similarly, the top sides of the second rectangle through to the tenth rectangle contribute one half of their length to the perimeter.

That is, the top sides of rectangles two through ten contribute $\frac{1}{2} \times 4 \times 9 = 18$ to the perimeter.

The entire top side of the first rectangle is included in the perimeter and thus contributes a length of 4.

In total, the horizontal lengths included in the perimeter sum to $18 + 4 + 18 + 4 = 44$.

Since there are no additional lengths which contribute to the perimeter of the given figure, the total perimeter is $40 + 44 = 84$.

Solution 3

Before they were positioned to form the given figure, each of the ten rectangles had a perimeter of $2 \times (2 + 4) = 12$.

When the figure was formed, some length of each of the ten rectangles' perimeter was "lost" (and thus is not included) in the perimeter of the given figure.

These lengths that were lost occur where the rectangles touch one another.

There are nine such locations where two rectangles touch one another (between the first and second rectangle, between the second and third rectangle, and so on).

In these locations, each of the two rectangles has one half of a side of length 4 which is not included in the perimeter of the given figure.

That is, the portion of the total perimeter of the ten rectangles that is not included in the perimeter of the figure is $9 \times (2 + 2) = 36$.

Since the total perimeter of the ten rectangles before they were positioned into the given figure is $10 \times 12 = 120$, then the perimeter of the given figure is $120 - 36 = 84$.

ANSWER: (D)

22. The units digit of the product $1ABCDE \times 3$ is 1, and so the units digit of $E \times 3$ must equal 1. Therefore, the only possible value of E is 7.

Substituting $E = 7$, we get

$$\begin{array}{r} 1\,A\,B\,C\,D\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,C\,D\,7\,1 \end{array}$$

Since $7 \times 3 = 21$, 2 is carried to the tens column.
Thus, the units digit of $D \times 3 + 2$ is 7, and so the units digit of $D \times 3$ is 5.
Therefore, the only possible value of D is 5.
Substituting $D = 5$, we get

$$\begin{array}{r} 1\,A\,B\,C\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,C\,5\,7\,1 \end{array}$$

Since $5 \times 3 = 15$, 1 is carried to the hundreds column.
Thus, the units digit of $C \times 3 + 1$ is 5, and so the units digit of $C \times 3$ is 4.
Therefore, the only possible value of C is 8.
Substituting $C = 8$, we get

$$\begin{array}{r} 1\,A\,B\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,B\,8\,5\,7\,1 \end{array}$$

Since $8 \times 3 = 24$, 2 is carried to the thousands column.
Thus, the units digit of $B \times 3 + 2$ is 8, and so the units digit of $B \times 3$ is 6.
Therefore, the only possible value of B is 2.
Substituting $B = 2$, we get

$$\begin{array}{r} 1\,A\,2\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline A\,2\,8\,5\,7\,1 \end{array}$$

Since $2 \times 3 = 6$, there is no carry to the ten thousands column.
Thus, the units digit of $A \times 3$ is 2.
Therefore, the only possible value of A is 4.
Substituting $A = 4$, we get

$$\begin{array}{r} 1\,4\,2\,8\,5\,7 \\ \times \qquad\quad 3 \\ \hline 4\,2\,8\,5\,7\,1 \end{array}$$

Checking, we see that the product is correct and so $A+B+C+D+E = 4+2+8+5+7 = 26$.

ANSWER: (B)

23. Given 8 dimes (10 ¢ coins) and 3 quarters (25 ¢ coins), we list the different amounts of money (in cents) that can be created in the table below.
    When an amount of money already appears in the table, it has been stroked out.

Number of Dimes

25 ¢ \ 10 ¢	0	1	2	3	4	5	6	7	8
0	0	10	20	30	40	50	60	70	80
1	25	35	45	55	65	75	85	95	105
2	~~50~~	~~60~~	~~70~~	~~80~~	90	100	110	120	130
3	~~75~~	~~85~~	~~95~~	~~105~~	115	125	135	145	155

(left vertical label: **Number of Quarters**)

We may ignore the first entry in the table, 0, since we are required to use at least one of the 11 coins.
We are left with 27 different amounts of money that can be created using one or more of the 8 dimes and 3 quarters.

ANSWER: (A)

24. We begin by constructing rectangle $ABCD$ around the given quadrilateral $PQRS$, as shown.
    The vertical sides AB and DC pass through points Q and S, respectively.
    The horizontal sides AD and BC pass through points P and R, respectively.
    We determine the area of $PQRS$ by subtracting the areas of the four right-angled triangles, AQP, QBR, CSR, and SDP, from the area of $ABCD$.
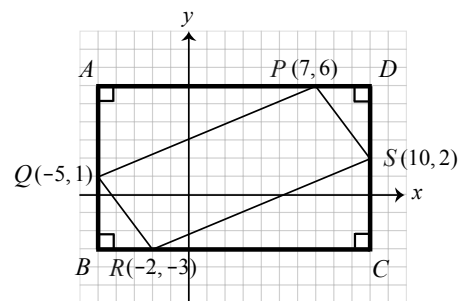
    To determine the horizontal side lengths of the right-angled triangles we count units along the x-axis, or we subtract the x-coordinates of two vertices.
    For example, since AB is vertical and passes through $Q(-5, 1)$, the x-coordinates of A and B are equal to that of Q, which is -5.
    Thus, the length of AP is determined by subtracting the x-coodinate of A from the x-coordinate of P, which is 7.
    Therefore the length of AP is $7 - (-5) = 12$.
    Similarly, the length of BR is $-2 - (-5) = 3$.
    Since DC is vertical and passes through $S(10, 2)$, the x-coordinates of D and C are equal to that of S, which is 10.
    Thus, the length of PD is $10 - 7 = 3$, and the length of RC is $10 - (-2) = 12$.
    To determine the vertical side lengths of the right-angled triangles we may count units along the y-axis, or we may subtract the y-coordinates of two vertices.
    For example, since AD is horizontal and passes through $P(7, 6)$, the y-coordinates of A and D are equal to that of P, which is 6.
    Thus, the length of AQ is determined by subtracting the y-coodinate of Q (which is 1) from the y-coordinate of A.
    Therefore the length of AQ is $6 - 1 = 5$.
    Similarly, the length of DS is $6 - 2 = 4$.

Since BC is horizontal and passes through $R(-2,-3)$, the y-coordinates of B and C are equal to that of R, which is -3. Thus, the length of QB is $1-(-3)=4$, and the length of SC is $2-(-3)=5$.

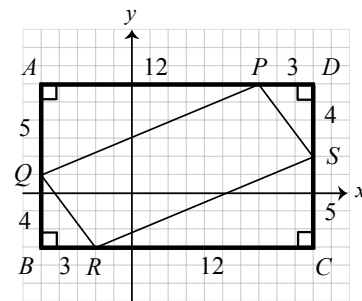The area of $\triangle AQP$ is $\frac{1}{2}\times AQ\times AP=\frac{1}{2}\times 5\times 12=30$.

The area of $\triangle CSR$ is also 30.

The area of $\triangle QBR$ is $\frac{1}{2}\times QB\times BR=\frac{1}{2}\times 4\times 3=6$.

The area of $\triangle SDP$ is also 6.

Since $AB=AQ+QB=5+4=9$ and $BC=BR+RC=3+12=15$, the area of $ABCD$ is $9\times 15=135$.

Finally, the area of $PQRS$ is $135-30\times 2-6\times 2=135-60-12=63$.

ANSWER: (B)

25. *Solution 1*

The sum of the positive integers from 1 to n is given by the expression $\dfrac{n(n+1)}{2}$.

For example when $n=6$, the sum $1+2+3+4+5+6$ can be determined by adding these integers to get 21, or by using the expression $\dfrac{6(6+1)}{2}=\dfrac{42}{2}=21$.

Using this expression, the sum of the positive integers from 1 to 2017, or $1+2+3+4+\cdots+2016+2017$ is $\dfrac{2017(2018)}{2}=\dfrac{4\,070\,306}{2}=2\,035\,153$.

To determine the sum of the integers which Ashley has not underlined, we must subtract from $2\,035\,153$ any of the 2017 integers which is a multiple of 2, or a multiple of 3, or a multiple of 5, while taking care not to subtract any number more than once.

First, we find the sum of all of the 2017 numbers which are a multiple of 2.

This sum contains 1008 integers and is equal to $2+4+6+8+\cdots+2014+2016$.

Since each number in this sum is a multiple of 2, then this sum is equal to twice the sum $1+2+3+4+\cdots+1007+1008$, since $2\times 1=2$, $2\times 2=4$, $2\times 3=6$, and so on.

That is, $2+4+6+8+\cdots+2014+2016=2(1+2+3+4+\cdots+1007+1008)$.

Using the formula above, the sum of the first 1008 positive integers is equal to $\dfrac{1008(1009)}{2}=\dfrac{1\,017\,072}{2}=508\,536$, and so $2+4+6+8+\cdots+2014+2016=2\times 508\,536=1\,017\,072$.

We may similarly determine the sum of all of the 2017 numbers which are a multiple of 3.

This sum is equal to $3+6+9+12+\cdots+2013+2016$ and contains 672 integers (since $3\times 672=2016$).

Since each of these numbers is a multiple of 3, $3+6+9+12+\cdots+2013+2016$ is equal to $3(1+2+3+4+\cdots+671+672)=3\times\dfrac{672(673)}{2}=3\times\dfrac{452\,256}{2}=3\times 226\,128=678\,384$.

The sum of all of the 2017 numbers which are a multiple of 5 is equal to $5+10+15+20+\cdots+2010+2015=5(1+2+3+4+\cdots+402+403)=5\times\dfrac{403(404)}{2}$ or $5\times 81\,406$ which is equal to $407\,030$.

We summarize this work in the table below.

Description	Sum	Result
All integers from 1 to 2017	$1+2+3+4+\cdots+2016+2017$	$2\,035\,153$
Integers that are a multiple of 2	$2+4+6+8+\cdots+2014+2016$	$1\,017\,072$
Integers that are a multiple of 3	$3+6+9+12+\cdots+2013+2016$	$678\,384$
Integers that are a multiple of 5	$5+10+15+20+\cdots+2010+2015$	$407\,030$

If we now subtract the sum of any of the 2017 integers which is a multiple of 2, or a multiple of 3, or a multiple of 5 from the sum of all 2017 integers, is the result our required sum?

The answer is no. Why?

There is overlap between the list of numbers that are a multiple of 2 and those that are a multiple of 3, and those that are a multiple of 5.

For example, any number that is a multiple of both 2 and 3 (and thus a multiple of 6) has been included in both lists and therefore has been counted twice in our work above.

We must add back into our sum those numbers that are a multiple of 6 (multiple of both 2 and 3), those that are a multiple of 10 (multiple of both 2 and 5), and those that are a multiple of 15 (multiple of both 3 and 5).

The sum of all of the 2017 numbers which are a multiple of 6 is equal to
$6 + 12 + 18 + 24 + \cdots + 2010 + 2016 = 6(1 + 2 + 3 + 4 + \cdots + 335 + 336)$, which is equal to
$6\left(\dfrac{336(337)}{2}\right) = 6 \times 56\,616 = 339\,696.$

The sum of all of the 2017 numbers which are a multiple of 10 is equal to
$10 + 20 + 30 + 40 + \cdots + 2000 + 2010 = 10(1 + 2 + 3 + 4 + \cdots + 200 + 201)$, which is equal to
$10\left(\dfrac{201(202)}{2}\right) = 10 \times 20\,301 = 203\,010.$

The sum of all of the 2017 numbers which are a multiple of 15 is equal to
$15 + 30 + 45 + 60 + \cdots + 1995 + 2010 = 15(1 + 2 + 3 + 4 + \cdots + 133 + 134)$, which is equal to
$15\left(\dfrac{134(135)}{2}\right) = 15 \times 9045 = 135\,675.$

We again summarize this work in the table below.

Description	Sum	Result
All integers from 1 to 2017	$1 + 2 + 3 + 4 + \cdots + 2016 + 2017$	2 035 153
Integers that are a multiple of 2	$2 + 4 + 6 + 8 + \cdots + 2014 + 2016$	1 017 072
Integers that are a multiple of 3	$3 + 6 + 9 + 12 + \cdots + 2013 + 2016$	678 384
Integers that are a multiple of 5	$5 + 10 + 15 + 20 + \cdots + 2010 + 2015$	407 030
Integers that are a multiple of 6	$6 + 12 + 18 + 24 + \cdots + 2010 + 2016$	339 696
Integers that are a multiple of 10	$10 + 20 + 30 + 40 + \cdots + 2000 + 2010$	203 010
Integers that are a multiple of 15	$15 + 30 + 45 + 60 + \cdots + 1995 + 2010$	135 675

If we take the sum of all 2017 integers, subtract those that are a multiple of 2, and those that are a multiple of 3, and those that are a multiple of 5, and then add those numbers that were subtracted twice (the multiples of 6, the multiples of 10, and the multiples of 15), then we get:

$$2\,035\,153 - 1\,017\,072 - 678\,384 - 407\,030 + 339\,696 + 203\,010 + 135\,675 = 611\,048$$

Is this the required sum?

The answer is still no, but we are close!

Consider any of the 2017 integers that is a multiple of 2, 3 and 5 (that is, a multiple of $2 \times 3 \times 5 = 30$).

Each number that is a multiple of 30 would have been underlined by Ashley, and therefore should not be included in our sum.

Each multiple of 30 was subtracted from the sum three times (once for each of the multiples of 2, 3 and 5), but then added back into our sum three times (once for each of the mutiples of 6, 10 and 15).

Thus, any of the 2017 integers that is a multiple of 30 must still be subtracted from 611 048 to achieve our required sum.

The sum of all of the 2017 numbers which are a multiple of 30 is equal to $30 + 60 + 90 + 120 + \cdots + 1980 + 2010 = 30(1 + 2 + 3 + 4 + \cdots + 66 + 67)$, which is equal to $30\left(\dfrac{67(68)}{2}\right) = 30 \times 2278 = 68\,340.$

Finally, the sum of the 2017 integers which Ashley has not underlined is $611\,048 - 68\,340 = 542\,708.$

Solution 2

We begin by considering the integers from 1 to 60.

When Ashley underlines the integers divisible by 2 and by 5, this will eliminate all of the integers ending in 0, 2, 4, 5, 6, and 8.

This leaves $1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 49, 51, 53, 57, 59.$

Of these, the integers $3, 9, 21, 27, 33, 39, 51, 57$ are divisible by 3.

Therefore, of the first 60 integers, only the integers

$$1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59$$

will not be underlined.

Among these 16 integers, we notice that the second set of 8 integers consists of the first 8 integers with 30 added to each.

This pattern continues, so that a corresponding set of 8 out of each block of 30 integers will not be underlined.

Noting that 2010 is the largest multiple of 30 less than 2017, this means that Ashley needs to add the integers

1	7	11	13	17	19	23	29
31	37	41	43	47	49	53	59
61	67	71	73	77	79	83	89
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1981	1987	1991	1993	1997	1999	2003	2009
2011	2017						

Let S equal the sum of these integers.

Before proceeding, we justify briefly why the pattern continues:

Every positive integer is a multiple of 30, or 1 more than a multiple of 30, or 2 more than a multiple of 30, and so on, up to 29 more than a multiple of 30. Algebraically, this is saying that every positive integer can be written in one of the forms

$$30k, 30k + 1, 30k + 2, 30k + 3, \ldots, 30k + 27, 30k + 28, 30k + 29$$

depending on its remainder when divided by 30.

Every integer with an even remainder when divided by 30 is even, since 30 is also even.

Similarly, every integer with a remainder divisible by 3 or 5 when divided by 30 is divisible by 3 or 5, respectively.

This leaves us with the forms

$$30k + 1, 30k + 7, 30k + 11, 30k + 13, 30k + 17, 30k + 19, 30k + 23, 30k + 29.$$

No integer having one of these forms will be underlined, since, for example, $30k + 11$ is one more than a multiple of 2 and 5 (namely, $30k + 10$) and is 2 more than a multiple of 3 (namely, $30k + 9$) so is not divisible by 2, 3 or 5.

The sum of the 8 integers in the first row of the table above is 120.

Since each of the integers in the second row of the table is 30 greater than the corresponding integer in the first row, then the sum of the numbers in the second row of the table is $120 + 8 \times 30$.

Similarly, the sum of the integers in the third row is $120 + 8 \times 60$, and so on.

We note that $2010 = 67 \times 30$ and $1980 = 66 \times 30$, so there are 67 complete rows in the table. Therefore,

$$
\begin{aligned}
S &= 120 + (120 + 8 \times 30) + (120 + 8 \times 60) + \cdots + (120 + 8 \times 1980) + (2011 + 2017) \\
&= 120 \times 67 + 8 \times (30 + 60 + \cdots + 1980) + 4028 \\
&= 8040 + 8 \times 30 \times (1 + 2 + \cdots + 65 + 66) + 4028 \\
&= 12\,068 + 240 \times (33 \times 67) \\
&= 12\,068 + 530\,640 \\
&= 542\,708
\end{aligned}
$$

Here, we have used the fact that the integers from 1 to 66 can be grouped into 33 pairs each of which adds to 67, as shown here:

$$1 + 2 + \cdots + 65 + 66 = (1 + 66) + (2 + 65) + \cdots + (33 + 34) = 67 + 67 + \cdots + 67 = 33 \times 67$$

ANSWER: (A)

Grade 8

1. Michael has \$280 in \$20 bills and so the number of \$20 bills that he has is $280 \div 20 = 14$.

   ANSWER: (C)

2. Evaluating, we get $4^2 - 2^3 = 16 - 8 = 8$.

   ANSWER: (A)

3. Exactly 1 of the 5 equal sections contains the number 4.
   Therefore, the probability that the spinner lands on 4 is $\frac{1}{5}$.

   ANSWER: (E)

4. The number of grade 8 students on the chess team is $160 \times 10\% = 160 \times \frac{10}{100} = 160 \times 0.10 = 16$.

   ANSWER: (B)

5. Since $22 = 2 \times 11$, then $44 \times 22 = 44 \times 2 \times 11$.
   Since $44 \times 2 = 88$, then $44 \times 2 \times 11 = 88 \times 11$.
   Therefore, $44 \times 22 = 88 \times 11$.

   ANSWER: (B)

6. In terms of x, the sum of the three side lengths of the triangle is $x + x + 1 + x - 1 = 3x$.
   Since the perimeter is 21, then $3x = 21$ and so $x = 7$.

   ANSWER: (B)

7. Reading from the graph, 20 students chose pink and 25 students chose blue.
   The ratio of the number of students who chose pink to the number of students who chose blue
   is $20 : 25$.
   After simplifying this ratio (dividing each number by 5), $20 : 25$ is equal to $4 : 5$.

   ANSWER: (A)

8. *Solution 1*
   To get the original number, we reverse the steps.
   That is, we add 6 and then divide by 3.
   Therefore, the original number is $(15 + 6) \div 3 = 21 \div 3 = 7$.

   Solution 2
   If the original number is x, then when it is tripled the result is $3x$.
   When this result is decreased by 6, we get $3x - 6$.
   Solving $3x - 6 = 15$, we get $3x = 15 + 6$ or $3x = 21$ and so $x = 7$.

   ANSWER: (D)

9. Tian travels 500 m every 625 steps and so she travels $500 \div 625 = 0.8$ m with each step.
   If Tian walks 10 000 steps at this same rate, she will walk a distance of $0.8 \times 10\,000 = 8000$ m.
   Since there are 1000 m in each kilometre, Tian will walk $8000 \div 1000 = 8$ km.

   ANSWER: (D)

10. Line segments PQ and RS intersect at T and so $\angle PTR$ and $\angle STQ$ are opposite angles and
    therefore $\angle STQ = \angle PTR = 88°$.
    Since $TS = TQ$, then $\triangle STQ$ is an isosceles triangle and so $\angle TSQ = \angle TQS = x°$.
    The three interior angles of any triangle add to $180°$.
    Thus, $88° + x° + x° = 180°$, and so $2x = 180 - 88$ or $2x = 92$ which gives $x = 46$.

    ANSWER: (B)

11. The volume of a rectangular prism is determined by multiplying the area of its base by its height.

    The area of the base for the given prism is $4 \times 5 = 20$ cm^2 and its height is x cm.

    Since the prism's volume is 60 cm^3, then $20x = 60$ and so $x = 3$.

    ANSWER: (D)

12. Since $\angle ACB = 90°$, then $\triangle ACB$ is a right-angled triangle.

    By the Pythagorean Theorem, $AB^2 = AC^2 + CB^2 = 8^2 + 15^2 = 64 + 225 = 289$ m^2.

    Since $AB > 0$, then $AB = \sqrt{289} = 17$ m and so Cindy walks a distance of 17 m.

    Walking from A to C to B, David walks a total distance of $8 + 15 = 23$ m.

    Thus, David walks $23 - 17 = 6$ m farther than Cindy.

    ANSWER: (D)

13. Each term of the sum $10 + 20 + 30 + \cdots + 990 + 1000$ is 10 times larger than its corresponding term in the sum $1 + 2 + 3 + \cdots + 99 + 100$, and so the required sum is 10 times larger than the given sum.

    Since $1 + 2 + 3 + \cdots + 99 + 100 = 5050$, then $10 + 20 + 30 + \cdots + 990 + 1000 = 5050 \times 10 = 50\,500$.

    ANSWER: (C)

14. If three of the students receive the smallest total number of pens possible, then the remaining student will receive the largest number of pens possible.

    The smallest number of pens that a student can receive is 1, since each student receives at least 1 pen.

    Since each student receives a different number of pens, the second smallest number of pens that a student can receive is 2 and the third smallest number of pens that a student can receive is 3.

    The smallest total number of pens that three students can receive is $1 + 2 + 3 = 6$.

    Therefore, the largest number of pens that a student can receive is $20 - 6 = 14$.

    ANSWER: (C)

15. The even integers between 1 and 103 are $2 = 2 \times 1, 4 = 2 \times 2, 6 = 2 \times 3, 8 = 2 \times 4$, and so on up to and including $102 = 2 \times 51$.

    Since there are 51 even integers in the list $2, 4, 6, \ldots, 100, 102$, then there are 51 even integers between 1 and 103.

    Next, we want to find a number N such that there are 51 odd integers between 4 and N.

    We notice that our lower bound, 4, is 3 greater than our original lower bound of 1.

    By increasing each of the 51 even integers from above by 3, we create the first 51 odd integers which are greater than 4.

    These odd integers are $2 \times 1 + 3 = 5, 2 \times 2 + 3 = 7, 2 \times 3 + 3 = 9, 2 \times 4 + 3 = 11$, and so on up to and including $2 \times 51 + 3 = 105$.

    Since there are 51 odd integers in the list $5, 7, 9, \ldots, 103, 105$, then there are 51 odd integers between 4 and 106.

    That is, the number of even integers between 1 and 103 is the same as the number of odd integers between 4 and 106.

    ANSWER: (E)

16. Label points S, T, U, V, W, as shown in Figure 1.
    Each shaded triangle is equilateral, $\triangle PQR$ is equilateral, and so
    $\angle VSU = \angle VTW = \angle SPT = 60°$.
    Therefore, $\angle PSV = 180° - \angle VSU = 120°$ (since PSU is a straight
    angle), and similarly, $\angle PTV = 120°$.
    In quadrilateral $PSVT$, $\angle SVT = 360° - \angle PSV - \angle SPT - \angle PTV$
    or $\angle SVT = 360° - 120° - 60° - 120° = 60°$.
    Therefore, $PSVT$ is a parallelogram, and since $SV = TV = 2$, then
    $PS = PT = 2$ (opposite sides of a parallelogram are equal in length).
    Join S to T, as shown in Figure 2.
   Since $SV = TV$, then $\angle VST = \angle VTS = \frac{1}{2}(180° - 60°) = 60°$.
    That is, $\triangle SVT$ is equilateral with side length 2, and is therefore
    congruent to each of the shaded triangles.
    Similarly, $\angle SPT = 60°$, $PS = PT = 2$, and so $\triangle PST$ is congruent
    to each of the shaded triangles.
    We may also join U to X and W to Y (as in Figure 3), and similarly
    show that $\triangle UQX, \triangle UXV, \triangle WRY$, and $\triangle WYV$ are also congruent
    to the shaded triangles.
    Thus, $\triangle PQR$ can be divided into 9 congruent triangles.
    Since 3 of these 9 triangles are shaded, the fraction of the area of
    $\triangle PQR$ that is shaded is $\frac{3}{9}$ or $\frac{1}{3}$.

Figure 1

Figure 2

Figure 3
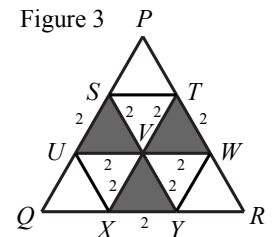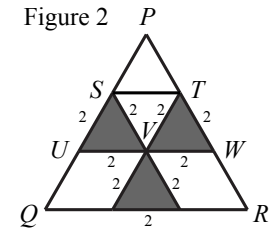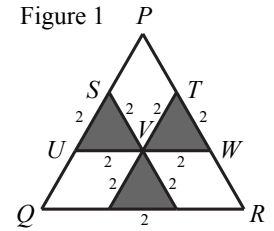
ANSWER: (B)

17. The range of the players' heights is equal to the difference between the height of the tallest
    player and the height of the shortest player.
    Since the tallest player, Meghan, has a height of 188 cm, and the range of the players' heights
    is 33 cm, then the shortest player, Avery, has a height of $188 - 33 = 155$ cm.
    Thus, answer (D) is a statement which provides enough information to determine Avery's
    height, and so must be the only one of the five statements which is enough to determine
    Avery's height.
    (Can you give a reason why each of the other four answers does not provide enough information
    to determine Avery's height?)

ANSWER: (D)

18. When Brodie and Ryan are driving directly towards each other at constant speeds of 50 km/h
    and 40 km/h respectively, then the distance between them is decreasing at a rate of
    $50 + 40 = 90$ km/h.
    If Brodie and Ryan are 120 km apart and the distance between them is decreasing at 90 km/h,
    then they will meet after $\dfrac{120}{90}$ h or $\dfrac{4}{3}$ h or $1\frac{1}{3}$ h.
    Since $\frac{1}{3}$ of an hour is $\frac{1}{3} \times 60 = 20$ minutes, then it will take Brodie and Ryan 1 h 20 min to
    meet.

ANSWER: (E)

19. The mean age of three of the friends is 12 years and 3 months which is equal to
    $12 \times 12 + 3 = 144 + 3 = 147$ months.
    Since the mean equals the sum of the ages divided by 3, then the sum of the ages of these three
    friends is $3 \times 147 = 441$ months.
    The mean age of the remaining four friends is 13 years and 5 months or
    $12 \times 13 + 5 = 156 + 5 = 161$ months.

Thus, the sum of the ages of these four friends is $4 \times 161 = 644$ months.

The sum of the ages of all seven friends is $441 + 644 = 1085$ months, and so the mean age of all seven friends is $\frac{1085}{7} = 155$ months.

ANSWER: (E)

20. The units digit of the product $1ABCDE \times 3$ is 1, and so the units digit of $E \times 3$ must equal 1. Therefore, the only possible value of E is 7.

Substituting $E = 7$, we get

$$
\begin{array}{r}
1\,ABCD\,7 \\
\times \qquad\qquad 3 \\
\hline
ABCD\,7\,1
\end{array}
$$

Since $7 \times 3 = 21$, 2 is carried to the tens column.

Thus, the units digit of $D \times 3 + 2$ is 7, and so the units digit of $D \times 3$ is 5.

Therefore, the only possible value of D is 5.

Substituting $D = 5$, we get

$$
\begin{array}{r}
1\,ABC\,5\,7 \\
\times \qquad\qquad 3 \\
\hline
ABC\,5\,7\,1
\end{array}
$$

Since $5 \times 3 = 15$, 1 is carried to the hundreds column.

Thus, the units digit of $C \times 3 + 1$ is 5, and so the units digit of $C \times 3$ is 4.

Therefore, the only possible value of C is 8.

Substituting $C = 8$, we get

$$
\begin{array}{r}
1\,AB\,8\,5\,7 \\
\times \qquad\qquad 3 \\
\hline
AB\,8\,5\,7\,1
\end{array}
$$

Since $8 \times 3 = 24$, 2 is carried to the thousands column.

Thus, the units digit of $B \times 3 + 2$ is 8, and so the units digit of $B \times 3$ is 6.

Therefore, the only possible value of B is 2.

Substituting $B = 2$, we get

$$
\begin{array}{r}
1\,A\,2\,8\,5\,7 \\
\times \qquad\qquad 3 \\
\hline
A\,2\,8\,5\,7\,1
\end{array}
$$

Since $2 \times 3 = 6$, there is no carry to the ten thousands column.

Thus, the units digit of $A \times 3$ is 2.

Therefore, the only possible value of A is 4.

Substituting $A = 4$, we get

$$
\begin{array}{r}
1\,4\,2\,8\,5\,7 \\
\times \qquad\qquad 3 \\
\hline
4\,2\,8\,5\,7\,1
\end{array}
$$

Checking, we see that the product is correct and so $A + B + C + D + E = 4 + 2 + 8 + 5 + 7 = 26$.

ANSWER: (B)

21. On the bottom die, the two visible faces are showing 2 dots and 4 dots.

Since the number of dots on opposite faces of this die add to 7, then there are 5 dots on the face opposite the face having 2 dots, and 3 dots on the face opposite the face having 4 dots.

Therefore, the top face of this bottom die (which is a face that is hidden between the dice) has either 1 dot on it or it has 6 dots on it.

On the second die from the bottom, the sum of the number of dots on the top and bottom faces (the faces hidden between the dice) is 7 since the number of dots on opposite faces add to 7. (We do not need to know which faces these are, though we could determine that they must have 1 and 6 dots on them.)

Similarly, on the third die from the bottom, the sum of the number of dots on the top and bottom faces (the faces hidden between the dice) is also 7.

Finally, the top face of the top die shows 3 dots, and so the bottom face of this die (which is a face that is hidden between the dice) contains 4 dots.

Therefore, the sum of the number of dots hidden between the dice is either $4 + 7 + 7 + 1 = 19$ or $4 + 7 + 7 + 6 = 24$.

Of these two possible answers, 24 is the only answer which appears among the five given answers.

ANSWER: (C)

22. To give $Y^X - W^V$ the greatest possible value, we make Y^X as large as possible while making W^V as small as possible.
To make Y^X as large as possible, we make Y and X as large as possible.
Thus, we must assign Y and X the two largest values, and assign W and V the two smallest values.
Let Y and X equal 4 and 5 in some order.
Since $4^5 = 1024$ and $5^4 = 625$, we let $Y = 4$ and $X = 5$ so that Y^X is as large as possible.
Similarly, we assign W and V the smallest possible numbers, 2 and 3.
Since $2^3 = 8$ and $3^2 = 9$, we let $W = 2$ and $V = 3$ so that W^V is as small as possible.
Thus, the greatest possible value of $Y^X - W^V$ is equal to $4^5 - 2^3 = 1024 - 8 = 1016$, which gives $X + V = 5 + 3 = 8$.

ANSWER: (D)

23. We introduce the letter M to represent a game that Mike has won, and the letter A to represent a game that Alain has won.
We construct a tree diagram to show all possible outcomes.
The M at the far left of the tree represents the fact that Mike won the first game.
The second "column" shows the two possible outcomes for the second game - Mike could win (M), or Alain could win (A).
The third, fourth and fifth columns show the possible outcomes of games 3, 4 and 5, respectively.
A branch of the tree is linked by arrows and each branch gives the outcomes of the games which lead to one of the players becoming the champion.
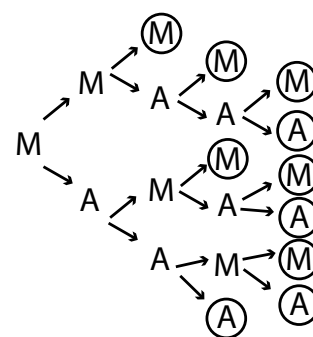

Once Mike has won 3 games, or Alain has won 3 games, the branch ends and the outcome of that final game is circled.
Since the first player to win 3 games becomes the champion, one way that Mike could become the champion is to win the second and third games (since he has already won the first game).
We represent this possibility as MMM, as shown along the top branch of the tree diagram.
In this MMM possibility, the final M is circled in the diagram, meaning that Mike has become the champion.
Since we are asked to determine the probability that Mike becomes the champion, we search

for all branches through the tree diagram which contain 3 Ms (paths ending with a circled M).
The tree diagram shows six such branches: MMM, MMAM, MMAAM, MAMM, MAMAM, and MAAMM.

All other branches end with a circled A, meaning that Alain has won 3 games and becomes the champion.

Since each player is equally likely to win a game, then Mike wins a game with probability $\frac{1}{2}$, and Alain wins a game with probability $\frac{1}{2}$.

Of the six ways that Mike can win (listed above), only one of these ends after 3 games (MMM).
The probability that Mike wins in exactly 3 games is equal to the probability that Mike wins the second game, which is $\frac{1}{2}$, multiplied by the probability that Mike wins the third game, which is also $\frac{1}{2}$.

That is, the probability that Mike becomes the champion by winning the first 3 games is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Of the six ways that Mike can win, two of these end after 4 games (MMAM, MAMM).
The probability that Mike wins games two and four but Alain wins game three (MMAM) is equal to the probability that Mike wins the second game, which is $\frac{1}{2}$, multiplied by the probability that Alain wins the third game, which is also $\frac{1}{2}$, multiplied by probability that Mike wins the fourth game, $\frac{1}{2}$.

In this case, Mike becomes the champion with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.
Similary, the probability that Mike becomes the champion by winning games three and four, but loses game two, is also $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$.

Finally, we determine the probabilty that Mike becomes the champion by winning in exactly 5 games (there are three possibilities: MMAAM, MAMAM and MAAMM).
Each of these three possibilities happens with probability $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$.

Therefore, if Mike has won the first game, then the probability that he becomes champion is $\frac{1}{4} + 2 \times \frac{1}{8} + 3 \times \frac{1}{16} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} = \frac{4+4+3}{16} = \frac{11}{16}$.

ANSWER: (C)

24. Let the area of the shaded region that lies outside of both semi-circles be X.

Let the area of the shaded region that lies inside of both semi-circles be Y.

The sum of the areas of both semi-circles counts the shaded area Y twice (since the area of overlap of the semi-circles is Y).

Therefore, if we subtract Y from the sum of the areas of both semi-circles, and add X, we get the area of the quarter-circle ABC.
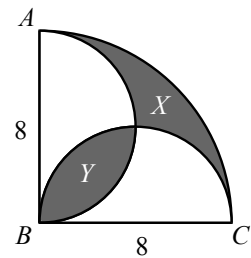
That is, the area of quarter-circle ABC is equal to

(the area of the semi-circle drawn on AB)+(the area of the semi-circle drawn on BC)$-Y+X$.

The area of quarter-circle ABC is $\frac{1}{4}\pi(8)^2 = 16\pi$.

The area of the semi-circle drawn on AB is $\frac{1}{2}\pi(4)^2 = 8\pi$.

The area of the semi-circle drawn on BC is also 8π.

Thus, $16\pi = 8\pi + 8\pi - Y + X$ or $16\pi = 16\pi - Y + X$, and so $Y = X$.

We build square $DBEF$ so that D is 4 vertical units from B, and
E is 4 horizontal units to the right of B. Next, we will show that F
lies on both semi-circles.
Since ABC is a quarter of a circle, then $\angle ABC = 90°$.
Beginning at point B, we move up vertically 4 units to point D, and
then move right 4 units in a direction perpendicular to AB.
After these two moves, we arrive at the point labelled F.
Since the diameter of the semi-circle drawn on AB has length 8, then
the radius of this semi-circle is 4.
Therefore, D is the centre of this semi-circle (since $DB = 4$), and F
lies on this semi-circle (since $DF = 4$).
Beginning again at point B, we move right 4 units to point E, and then move up vertically 4
units in a direction perpendicular to BC.
Since these are the same two moves we made previously (up 4 and right 4), then we must again
arrive at F.
Since the diameter of the semi-circle drawn on BC has length 8, then the radius of this semi-
circle is 4.
Therefore, E is the centre of this semi-circle (since $EB = 4$), and F lies on this semi-circle
(since $EF = 4$).
The two semi-circles intersect at exactly one point (other than point B).
Since we have shown that F lies on both semi-circles, then F must be this point of intersection
of the two semi-circles.
Therefore, $DBEF$ is a square with side length 4, and F is the point of intersection of the two
semi-circles.

Finally, we find the value of Y.
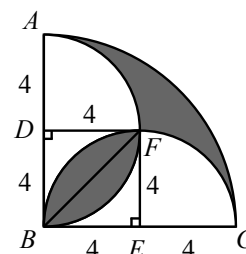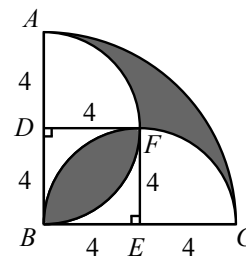First we construct BF, the diagonal of square $DBEF$.
By symmetry, BF divides the shaded area Y into two equal areas.
Each of these equal areas, $\dfrac{Y}{2}$, is equal to the area of $\triangle BEF$ sub-
tracted from the area of the quarter-circle BEF.
That is, $\dfrac{Y}{2} = \frac{1}{4}\pi(4)^2 - \frac{1}{2}(4)(4)$, and so $\dfrac{Y}{2} = 4\pi - 8$, or $Y = 8\pi - 16$.
The area of the shaded region is $X + Y = 2Y = 16\pi - 32$.
Of the answers given, $16\pi - 32$ is closest to 18.3.

ANSWER: (D)

25. *Solution 1*
Let the number of black plates, gold plates, and red plates be b, g and r, respectively (b, g and r
are whole numbers).
Brady is stacking 600 plates, and so $b + g + r = 600$, where b is a multiple of 2, g is a multiple
of 3, and r is a multiple of 6.
Rewrite this equation as $g = 600 - b - r$ and consider the right side of the equation.
Since 600 is a multiple of 2, and b is a multiple of 2, and r is a multiple of 2 (any multiple of 6
is a multiple of 2), then $600 - b - r$ is a multiple of 2 (the difference between even numbers is
even).
Since the right side of the equation is a multiple of 2, then the left side, g, must also be a
multiple of 2.
We are given that g is a multiple of 3, and since g is also a multiple of 2, then g must be an
even multiple of 3 or a multiple of 6.

Similarly, rewriting the equation as $b = 600 - g - r$ and considering the right side of the equation: 600 is a multiple of 6, and g is a multiple of 6, and r is a multiple of 6, so then $600 - g - r$ is a multiple of 6 (the difference between multiples of 6 is a multiple of 6).

Since the right side of the equation is a multiple of 6, then the left side, b, must also be a multiple of 6.

That is, each of b, g and r is a multiple of 6, and so we let $b = 6B$, $g = 6G$, and $r = 6R$ where B, G and R are whole numbers.

So then the equation $b + g + r = 600$ becomes $6B + 6G + 6R = 600$, which is equivalent to $B + G + R = 100$ after dividing by 6.

Each solution to the equation $B + G + R = 100$ corresponds to a way that Brady could stack the 600 plates, and every possible way that Brady could stack the 600 plates corresponds to a solution to the equation $B + G + R = 100$.

For example, $B = 30, G = 50, R = 20$ corresponds to $b = 6 \times 30 = 180, g = 6 \times 50 = 300$, $r = 6 \times 20 = 120$, which corresponds to Brady stacking 180 black plates, below 300 gold plates, which are below 120 red plates.

Since $B + G + R = 100$, then each of B, G and R has a maximum possible value of 100.

If $B = 100$, then $G = R = 0$.

If $B = 99$, $G + R = 100 - B = 1$.

Thus, $G = 0$ and $R = 1$ or $G = 1$ and $R = 0$.

That is, once we assign values for B and G, then there is no choice for R since it is determined by the equation $R = 100 - B - G$.

Thus, to determine the number of solutions to the equation $B + G + R = 100$, we must determine the number of possible pairs (B, G) which lead to a solution.

For example, above we showed that the three pairs $(100, 0)$, $(99, 0)$, and $(99, 1)$ each correspond to a solution to the equation.

Continuing in this way, we determine all possible pairs (B, G) (which give R) that satisfy $B + G + R = 100$.

Value of B	Value of G	Number of plates: b, g, r
100	0	600,0,0
99	0	594,0,6
99	1	594,6,0
98	0	588,0,12
98	1	588,6,6
98	2	588,12,0
97	0	582,0,18
97	1	582,6,12
97	2	582,12,6
97	3	582,18,0
$\vdots$	$\vdots$	$\vdots$
$100 - n$	0	$6(100 - n), 0, 6n$
$100 - n$	1	$6(100 - n), 6, 6(n - 1)$
$100 - n$	2	$6(100 - n), 12, 6(n - 2)$
$\vdots$	$\vdots$	$\vdots$
$100 - n$	n	$6(100 - n), 6n, 0$

We see from the table that if the value of B is $100 - n$ for some whole number $n \leq 100$, then G can equal any whole number from 0 to n and so there are $n + 1$ possible choices for G.

That is, when $B = 100$, there is 1 choice for G, when $B = 99$, there are 2 choices for G, when

$B = 98$, there are 3 choices for G, and so on.

Each additional decrease of 1 in B gives 1 additional choice for G until we arrive at $B = 0$ ($n = 100$), which gives $n + 1 = 100 + 1 = 101$ possible choices for G $(0, 1, 2, 3, \ldots, 100)$.

Therefore, the total number of solutions to $B + G + R = 100$ is given by the sum $1 + 2 + 3 + \cdots + 99 + 100 + 101$.

Using the fact that the sum of the first m positive integers $1 + 2 + 3 + \cdots + m$ is equal to $\dfrac{m(m + 1)}{2}$, we get $1 + 2 + 3 + \cdots + 99 + 100 + 101 = \dfrac{101(102)}{2} = 5151$.

Since each of these solutions corresponds to a way that Brady could stack the plates, there are 5151 ways that Brady could stack the plates under the given conditions.

Solution 2

In a given way of stacking the plates, let b be the number of groups of 2 black plates, g be the number of groups of 3 gold plates, and r be the number of groups of 6 red plates.

Then there are $2b$ black plates, $3g$ gold plates, and $6r$ red plates.

Since the total number of plates in a stack is 600, then $2b + 3g + 6r = 600$.

We note that the numbers of black, gold and red plates completely determines the stack (we cannot rearrange the plates in any way), and so the number of ways of stacking the plates is the same as the number of ways of solving the equation $2b + 3g + 6r = 600$ where b, g, r are integers that are greater than or equal to 0.

Since r is at least 0 and $6r$ is at most 600, then the possible values for r are $0, 1, 2, 3, \ldots, 98, 99, 100$.

When $r = 0$, we obtain $2b + 3g = 600$.

Since g is at least 0 and $3g$ is at most 600, then g is at most 200.

Since $2b$ and 600 are even, then $3g$ is even, so g is even.

Therefore, the possible values for g are $0, 2, 4, \ldots, 196, 198, 200$.

Since $200 = 100 \times 2$, then there are 101 possible values for g.

When $g = 0$, we get $2b = 600$ and so $b = 300$.

When $g = 2$, we get $2b = 600 - 3 \times 2 = 594$ and so $b = 297$.

Each time we increase g by 2, the number of gold plates increases by 6, so the number of black plates must decrease by 6, and so b decreases by 3.

Thus, as we continue to increase g by 2s from 2 to 200, the values of b will decrease by 3s from 297 to 0.

In other words, every even value for g does give an integer value for b.

Therefore, when $r = 0$, there are 101 solutions to the equation.

When $r = 1$, we obtain $2b + 3g = 600 - 6 \times 1 = 594$.

Again, g is at least 0, is even, and is at most $594 \div 3 = 198$.

Therefore, the possible values of g are $0, 2, 4, \ldots, 194, 196, 198$.

Again, each value of g gives a corresponding integer value of b.

Therefore, when $r = 1$, there are 100 solutions to the equation.

Consider the case of an unknown value of r, which gives $2b + 3g = 600 - 6r$.

Again, g is at least 0 and is even.

Also, the maximum possible value of g is $\dfrac{600 - 6r}{3} = 200 - 2r$.

This means that there are $(100 - r) + 1 = 101 - r$ possible values for g. (Can you see why?)

Again, each value of g gives a corresponding integer value of b.

Therefore, for a general r between 0 and 100, inclusive, there are $101 - r$ solutions to the equation.

We make a table to summarize the possibilities:

r	g	b	# of solutions
0	$0, 2, 4, \ldots, 196, 198, 200$	$300, 297, 294, \ldots, 6, 3, 0$	101
1	$0, 2, 4, \ldots, 194, 196, 198$	$297, 294, 291 \ldots, 6, 3, 0$	100
2	$0, 2, 4, \ldots, 192, 194, 196$	$294, 291, 288 \ldots, 6, 3, 0$	99
$\vdots$	$\vdots$	$\vdots$	$\vdots$
98	$0, 2, 4$	$6, 3, 0$	3
99	$0, 2$	$3, 0$	2
100	0	0	1

Therefore, the total number of ways of stacking the plates is

$$101 + 100 + 99 + \cdots + 3 + 2 + 1$$

We note that the integers from 1 to 100 can be grouped into 50 pairs each of which has a sum of 101 $(1 + 100, 2 + 99, 3 + 98, \ldots, 50 + 51)$.

Therefore, the number of ways that Brady could stack the plates is $101 + 50 \times 101 = 5151$.

ANSWER: (E)