The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2017 Canadian Team Mathematics Contest

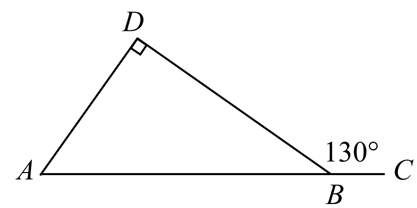
Team Problems

IMPORTANT NOTES:

- Calculators are not permitted.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

1. In the diagram, A, B and C form a straight angle. If $\triangle ADB$ is right-angled at D and $\angle DBC = 130°$, what is the measure of $\angle DAB$?



2. Evaluate $\left[\left(\dfrac{2017+2017}{2017}\right)^{-1} + \left(\dfrac{2018}{2018+2018}\right)^{-1}\right]^{-1}$.

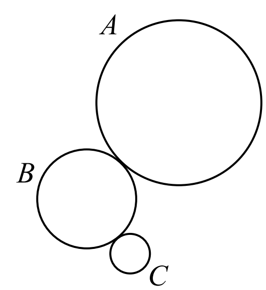
3. Bethany is told to create an expression from $2 \square 0 \square 1 \square 7$ by putting a $+$ in one box, a $-$ in another, and a $\times$ in the remaining box. There are 6 ways in which she can do this. She calculates the value of each expression and obtains a maximum value of M and a minimum value of m. What is $M - m$?

4. If n is the largest positive integer with $n^2 < 2018$ and m is the smallest positive integer with $2018 < m^2$, what is $m^2 - n^2$?

5. If N is a positive integer with $\sqrt{12} + \sqrt{108} = \sqrt{N}$, determine the value of N.

6. The ratio of the width to the height of a rectangular screen is $3 : 2$. If the length of a diagonal of the screen is 65 cm, what is the area of the screen, in cm^2?

7. In the diagram, wheel B touches wheel A and wheel C. Wheel A has radius 35 cm, wheel B has radius 20 cm, and wheel C has radius 8 cm. If wheel A rotates, it causes wheel B to rotate without slipping, which causes wheel C to rotate without slipping. When wheel A rotates through an angle of $72°$, what is the measure of the angle through which wheel C rotates?



8. Martina has a cylinder with radius 10 cm and height 70 cm that is closed at the bottom. Martina puts some solid spheres of radius 10 cm inside the cylinder and then closes the top end of the cylinder. If she puts the largest possible number of spheres in the cylinder, what is the volume, in cm^3, of the cylinder that is not taken up by the spheres?

9. Determine the value of the expression

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + 10 + 11 - 12 + \cdots + 94 + 95 - 96 + 97 + 98 - 99\ .$$

(The expression consists of 99 terms. The operations alternate between two additions and one subtraction.)

10. Two vertical chords are drawn in a circle, dividing the circle into 3 distinct regions. Two horizontal chords are added in such a way that there are now 9 regions in the circle. A fifth chord is added that does not lie on top of one of the previous four chords. The maximum possible number of resulting regions is M and the minimum possible number of resulting regions is m. What is $M^2 + m^2$?

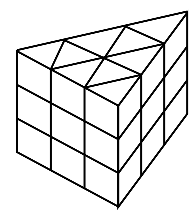
11. The product of the roots of the quadratic equation $2x^2 + px - p + 4 = 0$ is 9. What is the sum of the roots of this equation?

12. The four positive integers a, b, c, d satisfy $a < b < c < d$. When the sums of the six pairs of these integers are calculated, the six answers are all different and the four smallest sums are 6, 8, 12, and 21. What is the value of d?

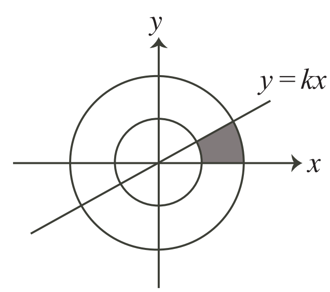
13. Determine all real values of x for which $16^x - \frac{5}{2}(2^{2x+1}) + 4 = 0$.

14. If $f(x)$ is a linear function with $f(k) = 4$, $f(f(k)) = 7$, and $f(f(f(k))) = 19$, what is the value of k?

15. A solid triangular prism is made up of 27 identical smaller solid triangular prisms, as shown. The length of every edge of each of the smaller prisms is 1. If the entire outer surface of the larger prism is painted, what fraction of the total surface area of all the smaller prisms is painted?



16. Two circles are drawn, as shown. Each is centered at the origin and the radii are 1 and 2. Determine the value of k so that the shaded region above the x-axis, below the line $y = kx$ and between the two circles has an area of 2.



17. Camp Koeller offers exactly three water activities: canoeing, swimming and fishing. None of the campers is able to do all three of the activities. In total, 15 of the campers go canoeing, 22 go swimming, 12 go fishing, and 9 do not take part in any of these activities. Determine the smallest possible number of campers at Camp Koeller.

18. Determine the area inside the circle with centre $\left(1, -\frac{\sqrt{3}}{2}\right)$ and radius 1 that lies inside the first quadrant.

19. When the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is divided by each of $x - 4$, $x - 3$, $x + 3$, and $x + 4$, the remainder in each case is 102. Determine all values of x for which $f(x) = 246$.

20. Determine the number of pairs of real numbers, (x, y), with $0 \le x \le \frac{\pi}{8}$ and $0 \le y \le \frac{\pi}{8}$ and $\cos^6(1000x) - \sin^6(1000y) = 1$.

21. Starting at midnight, Serge writes down the time after 1 minute, then 2 minutes after that, then 3 minutes after that, and so on. For example, the first four times that he writes are 12:01 a.m., 12:03 a.m., 12:06 a.m., and 12:10 a.m. He continues this process for 24 hours. What times does Serge write down that are exactly on the hour? (For example, 3:00 a.m. and 8:00 p.m. are exactly on the hour, while 11:57 p.m. is not.)

22. Figure 0 consists of a square with side length 18. For each integer $n \ge 0$, Figure $n+1$ consists of Figure n with the addition of two new squares constructed on each of the squares that were added in Figure n. The side length of the squares added in Figure $n+1$ is $\frac{2}{3}$ of the side length of the smallest square(s) in Figure n. Define A_n to be the area of Figure n for each integer $n \ge 0$. What is the smallest positive integer M with the property that $A_n < M$ for all integers $n \ge 0$?

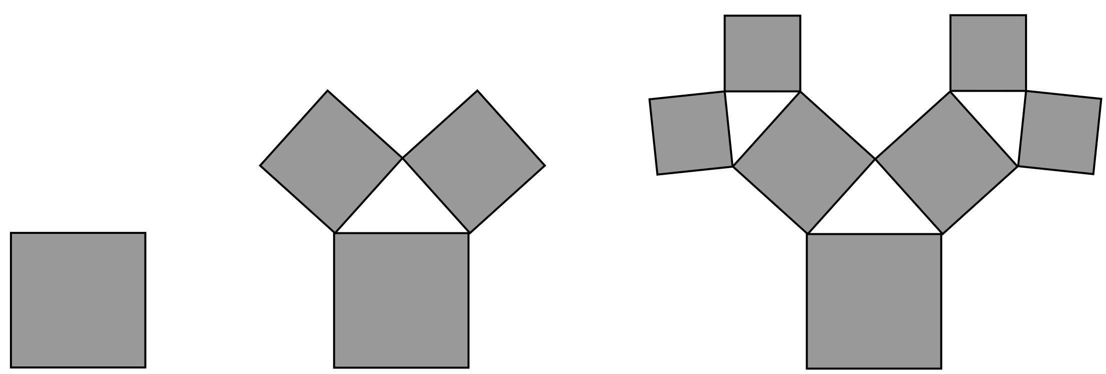
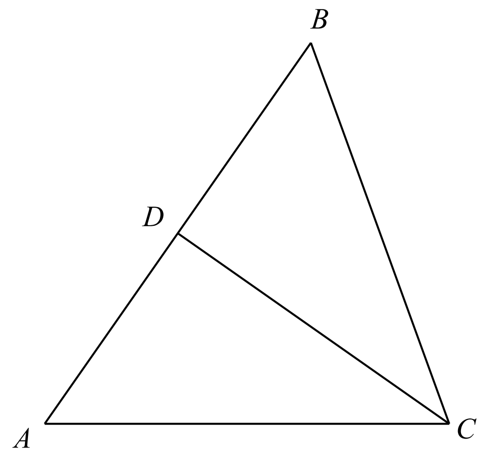


Figure 0 Figure 1 Figure 2

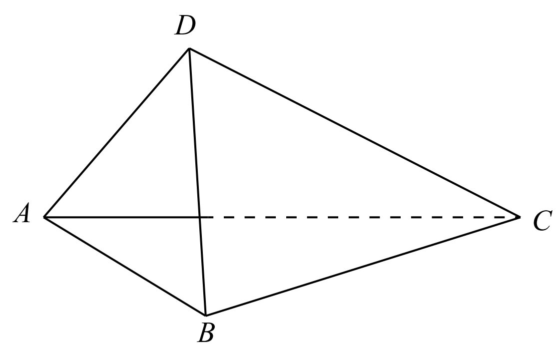
23. Brad answers 10 math problems, one after another. He answers the first problem correctly and answers the second problem incorrectly. For each of the remaining 8 problems, the probability that he answers the problem correctly equals the ratio of the number of problems that he has already answered correctly to the total number of problems that he has already answered. What is the probability that he answers exactly 5 out of the 10 problems correctly?

24. The lines with equations $x + y = 3$ and $2x - y = 0$ meet at point A. The lines with equations $x + y = 3$ and $3x - ty = 4$ meet at point B. The lines with equations $2x - y = 0$ and $3x - ty = 4$ meet at point C. Determine all values of t for which $AB = AC$.

25. A large piece of canvas is in the shape of an isosceles triangle with $AC = BC = 5$ m and $AB = 6$ m, as shown. Point D is the midpoint of AB.



The canvas is folded along median CD to create two faces of a tent ($\triangle ADC$ and $\triangle BDC$). The third face of the tent ($\triangle ABD$) is made from a separate piece of canvas. The bottom of the tent ($\triangle ABC$) is open to the ground.



What is the height of the tent (the distance from D to the base $\triangle ABC$) when the volume of the tent is as large as possible?