The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2016 Canadian Team Mathematics Contest

Team Problems

1. What is the value of $2 + 0 - 1 \times 6$?

2. The average (mean) of 3, 5, 6, 8, and x is 7. What is the value of x?

3. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 4.2 \rfloor = 4$ and $\lfloor -5.4 \rfloor = -6$. What is the value of $\lfloor -2.3 + \lfloor 1.6 \rfloor \rfloor$?

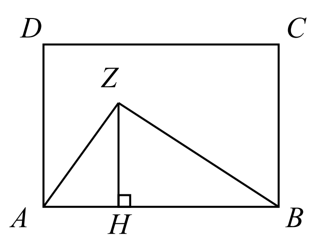
4. A street magician has three cups labelled, in order, A, B, C that he has upside down on his table. He has a sequence of moves that he uses to scramble the three cups: he swaps the first and second, then he swaps the second and third, then he swaps the first and third. If he goes through this sequence of three moves a total of nine times, in what order will the cups be?

5. A parabola has equation $y = ax^2 + bx + c$ and passes through the points $(-3, 50)$, $(-1, 20)$ and $(1, 2)$. Determine the value of $a + b + c$.

6. For some positive integers m and n, $2^m - 2^n = 1792$. Determine the value of $m^2 + n^2$.

7. A two-digit integer between 10 and 99, inclusive, is chosen at random. Each possible integer is equally likely to be chosen. What is the probability that its tens digit is a multiple of its units (ones) digit?

8. Rectangle $ABCD$ has area 2016. Point Z is inside the rectangle and point H is on AB so that ZH is perpendicular to AB. If $ZH : CB = 4 : 7$, what is the area of pentagon $ADCBZ$?



9. Suppose that A and B are digits with

$$\begin{array}{rccc} & A & A & A \\ & A & A & B \\ & A & B & B \\ + & B & B & B \\ \hline 1 & 5 & 0 & 3 \end{array}$$

What is the value of $A^3 + B^2$?

10. Clara takes 2 hours to ride her bicycle from Appsley to Bancroft. The reverse trip takes her 2 hours and 15 minutes. If she travels downhill at 24 km/h, on level road at 16 km/h and uphill at 12 km/h, what is the distance, in kilometres, between the two towns?

11. The first and second terms of a sequence are 4 and 5, respectively. Each term after the second is determined by increasing the previous term by one and dividing the result by the term before that. For example, the third term equals $\dfrac{5+1}{4}$. What is the 1234th term of this sequence?

12. Austin and Joshua play a game. Austin chooses a random number equal to 1, 2, 3, 4, or 5. Joshua then chooses randomly from the remaining four numbers. Joshua's first round score is equal to the product of his number and Austin's number. Austin then chooses randomly from the remaining three numbers, and his first round score is the product of his second number and Joshua's first number. The process is repeated until each of the five numbers has been chosen. The sum of each player's two scores is their final score and the player with the highest final score wins. If Austin chooses 2 to start and Joshua then chooses 3 (making Joshua's first round score 6), what is the probability that Austin will win?

13. A sphere and a cone have the same volume. The area of the lateral surface of the cone is 80π and the total surface area of the cone is 144π. Determine the radius of the sphere.

    (The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. The volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$. The surface area of a cone consists of two parts: its base and its lateral surface. The lateral surface area of a cone with radius r and slant height s is πrs.)

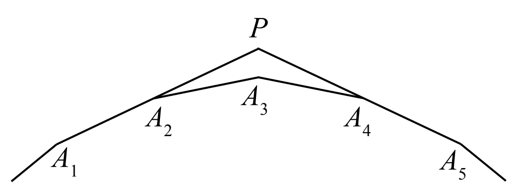
14. Determine the value of the following sum:

$$\log_3(1-\tfrac{1}{15}) + \log_3(1-\tfrac{1}{14}) + \log_3(1-\tfrac{1}{13}) + \cdots + \log_3(1-\tfrac{1}{8}) + \log_3(1-\tfrac{1}{7}) + \log_3(1-\tfrac{1}{6})$$

    (Note that the sum includes a total of 10 terms.)

15. A lock has a combination that is a four-digit positive integer. The first digit is 4 and the four-digit combination is divisible by 45. How many different possible combinations are there?

16. In a regular n-gon, $A_1A_2A_3\cdots A_n$, where $n > 6$, sides A_1A_2 and A_5A_4 are extended to meet at point P. If $\angle A_2PA_4 = 120^\circ$, determine the value of n.



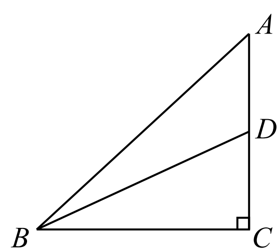
17. There are 21 marbles in a bag. The number of each colour of marble is shown in the following table:

Colour	Number
magenta	1
puce	2
cyan	3
ecru	4
aquamarine	5
lavender	6

For example, the bag contains 4 ecru marbles. Three marbles are randomly drawn from the bag without replacement. The probability that all three of these marbles are the same colour can be written as $\frac{1}{k}$. What is the value of k?

18. For each real number x, $f(x)$ is defined to be the minimum of the values of $2x + 3$, $3x - 2$ and $25 - x$. What is the maximum value of $f(x)$?

19. In the diagram, $\triangle ABC$ is right-angled at C. Point D is on AC so that $\angle ABC = 2\angle DBC$. If $DC = 1$ and $BD = 3$, determine the length of AD.



20. A group of cows and horses are randomly divided into two equal rows. (The animals are well-trained and stand very still.) Each animal in one row is directly opposite an animal in the other row. If 75 of the animals are horses and the number of cows opposite cows is 10 more than the number of horses opposite horses, determine the total number of animals in the group.

21. Three circles each with a radius of 1 are placed such that each circle touches the other two circles, but none of the circles overlap. What is the exact value of the radius of the smallest circle that will enclose all three circles?

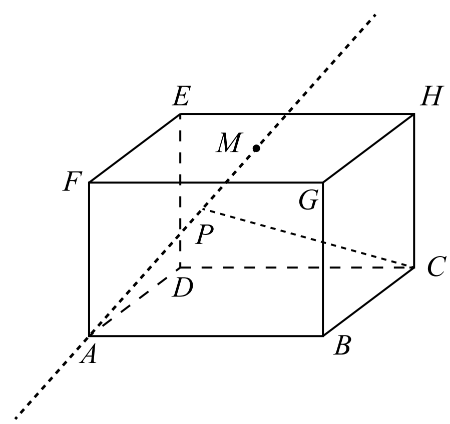
22. In a hospital, there are 7 patients (Doc, Grumpy, Happy, Sleepy, Bashful, Sneezy, and Dopey) who need to be assigned to 3 doctors (Huey, Dewey, and Louie). In how many ways can the patients be assigned to the doctors so that each patient is assigned to exactly one doctor and each doctor is assigned at least one patient?

23. Suppose that a is a positive integer with $a > 1$. Determine a closed form expression, in terms of a, equal to

$$1 + \frac{3}{a} + \frac{5}{a^2} + \frac{7}{a^3} + \cdots$$

(The infinite sum includes exactly the fractions of the form $\frac{2k-1}{a^{k-1}}$ for each positive integer k.)

24. In the diagram, rectangular prism $ABCDEFGH$ has $AB = 2a$, $AD = 2b$, and $AF = 2c$ for some $a, b, c > 0$. Point M is the centre of face $EFGH$ and P is a point on the infinite line passing through A and M. Determine the minimum possible length of line segment CP in terms of a, b, and c.



25. The sequences $t_1, t_2, t_3, \ldots$ and $s_1, s_2, s_3, \ldots$ are defined by

- $t_1 = 1$,
- $t_2 = m$ for some positive integer $m > 0$,
- $s_k = t_1 + t_2 + t_3 + \cdots + t_{k-1} + t_k$ for each $k \geq 1$, and
- $t_n = ns_{n-1}$ for each $n \geq 3$.

There exist positive integers m that end in in exactly four 9s and for which $t_{30} = N!$ for some positive integer N. Determine all of these corresponding values of N.

(If n is a positive integer, the symbol $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.)