The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2016
Canadian Team Mathematics Contest

April 2016

Solutions

Individual Problems

1. Evaluating, $2^0 + 20^0 + 201^0 + 2016^0 = 1 + 1 + 1 + 1 = 4$.

   Answer: 4

2. Since 20 minutes is $\frac{1}{3}$ of an hour and Zeljko travelled at 30 km/h for $\frac{1}{3}$ of an hour, then Zeljko travelled $30 \cdot \frac{1}{3} = 10$ km during this portion of the trip.
   Since 30 minutes is $\frac{1}{2}$ of an hour and Zeljko travelled at 20 km/h for $\frac{1}{2}$ of an hour, then Zeljko travelled $20 \cdot \frac{1}{2} = 10$ km during this portion of the trip.
   In total, Zeljko travelled $10 + 10 = 20$ km.

   Answer: 20

3. Using the definition,

$$2 \ominus (2 \ominus 5) = 2 \ominus (2^5 - 5^2) = 2 \ominus (32 - 25) = 2 \ominus 7 = 2^7 - 7^2 = 128 - 49 = 79$$

   Answer: 79

4. We call the numbers tossed on the two dice A and B.
   There are 6 possible values for each of A and B, and each such value is equally likely.
   Therefore, there are $6 \cdot 6 = 36$ equally likely possible pairs of outcomes.
   If $A = 1$, then for $A + B < 10$, B can equal any of $1, 2, 3, 4, 5, 6$.
   If $A = 2$, then for $A + B < 10$, B can equal any of $1, 2, 3, 4, 5, 6$.
   If $A = 3$, then for $A + B < 10$, B can equal any of $1, 2, 3, 4, 5, 6$.
   If $A = 4$, then for $A + B < 10$, B can equal any of $1, 2, 3, 4, 5$.
   If $A = 5$, then for $A + B < 10$, B can equal any of $1, 2, 3, 4$.
   If $A = 6$, then for $A + B < 10$, B can equal any of $1, 2, 3$.
   Thus, $6+6+6+5+4+3 = 30$ pairs of outcomes satisfy the requirement and so the probability that $A + B$ is less than 10 is $\frac{30}{36}$ or $\frac{5}{6}$.

   Answer: $\frac{5}{6}$

5. An integer is divisible by 15 exactly when it is divisble by both 5 and 3.
   For a positive integer to be divisible by 5, its units (ones) digit must be 5 or 0.
   For a positive integer to be a palindrome, its first and last digits must be the same.
   Since a positive integer cannot have a first digit of 0, then the first and last digits of a palindrome that is divsible by 5 must be 5.
   Therefore, a five-digit palindrome that is divisible by 15 (and hence by 5) must be of the form $5aba5$. (Note that the thousands and tens digits must be the same.)
   For $5aba5$ to be as large as possible, we start with $a = b = 9$.
   This gives 59995 which is not divisible by 15. (To test this, we can use a calculator or use a test for divisibility by 3, since 59995 is already divisible by 5.)
   The next largest five-digit palindrome of the form $5aba5$ is 59895, which is divisible by 15. (If we reduced the value of a rather than the value of b, the resulting palindrome would be smaller than this one.)
   Therefore, the largest five-digit palindrome that is divisible by 15 is 59895.

   Answer: 59895

6. Since this arithmetic sequence of integers includes both even and odd integers, then the common difference in the sequence must be odd. (If it were even, then all of the terms in the sequence would be even or all would be odd.)
Suppose that this common difference is d.
Note that 555 is one of the terms in the sequence.
If $d = 1$, then either or both of 554 and 556 would be in the sequence. Since no term other than 555 can contain repeated digits, then $d \neq 1$.
If $d = 3$, then either or both of 552 and 558 would be in the sequence. Since no term other than 555 can contain repeated digits, then $d \neq 3$.
If $d = 5$, then 550 cannot be in the sequence, so 555 would have to be the smallest term in the sequence. This would mean that 560 and 565 were both in the sequence. Since no term other than 555 can contain repeated digits, then $d \neq 5$.
We can also rule out the possibility that $d \geq 9$:

> Since there are 14 terms in the sequence and the common difference is d, then the largest term is $13d$ larger than the smallest term.
> Since all terms are between 500 and 599, inclusive, then the maximum possible difference between two terms is 99.
> This means that $13d \leq 99$. Since d is an odd integer, then cannot be 9 or greater.

Since d is odd, d cannot be 1, 3 or 5, and $d \leq 7$, then it must be the case that $d = 7$.
Can we actually construct a sequence with the desired properties?
Starting with 555 and adding and subtracting 7s, we get the sequence

$$506, 513, 520, 527, 534, 541, 548, 555, 562, 569, 576, 583, 590, 597$$

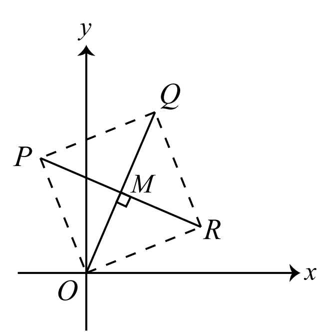
This sequence has 14 terms in the desired range, including 7 even and 7 odd terms, and no term other than 555 has repeated digits.
(Note that this sequence cannot be extended in either direction and stay in the desired range.)
Therefore, the smallest of the 14 house numbers must be 506.

ANSWER: 506

7. Since $Q(a+1, 4a+1)$ lies on the line with equation $y = ax + 3$, then

$$\begin{aligned} 4a + 1 &= a(a+1) + 3 \\ 0 &= a^2 - 3a + 2 \\ 0 &= (a-1)(a-2) \end{aligned}$$

Since $a > 1$, then we must have $a = 2$.
Therefore, Q has coordinates $(3, 9)$.
We want to find the coordinates of points P and R so that, together with $O(0,0)$ and $Q(3,9)$, the quadrilateral $OPQR$ is a square.
Since OQ is a diagonal of the square, then the midpoint, M, of OQ is the centre of the square.



Since O has coordinates $(0,0)$ and Q has coordinates $(3,9)$, then M has coordinates $(\frac{3}{2},\frac{9}{2})$.
Note that PM and MR are perpendicular to OM, since the diagonals of a square are perpendicular.
Since the slope of OM is $\dfrac{\frac{9}{2}-0}{\frac{3}{2}-0}=3$, then the slope of PM and MR is $-\frac{1}{3}$.
The equation of the line through M with slope $-\frac{1}{3}$ is $y-\frac{9}{2}=-\frac{1}{3}(x-\frac{3}{2})$ or $y=-\frac{1}{3}x+5$.
An arbitrary point on this line can be written in the form $A(3t,-t+5)$ for some real number t.
The points P and R will be the two points on this line for which OP and PQ are perpendicular and OR and RQ are perpendicular.
Thus, given an arbitrary point A on the line, we want to find all t for which OA and AQ are perpendicular (that is, for which the slopes of OA and AQ have a product of -1).
The slope of OA is $\dfrac{-t+5}{3t}$. The slope of AQ is $\dfrac{(-t+5)-9}{3t-3}$.
We have the following equivalent equations:

$$\begin{aligned}
\frac{-t+5}{3t}\cdot\frac{-t-4}{3t-3}&=-1\\
(-t+5)(-t-4)&=-3t(3t-3)\\
t^2-t-20&=-9t^2+9t\\
10t^2-10t-20&=0\\
t^2-t-2&=0\\
(t-2)(t+1)&=0
\end{aligned}$$

Thus, $t=-1$ or $t=2$.
Substituting into $(3t,-t+5)$, we obtain $P(-3,6)$ and $R(6,3)$.
We can check that the points $O(0,0)$, $P(-3,6)$, $Q(3,9)$, $R(6,3)$ give $OP=PQ=QR=RO$ and that consecutive sides are perpendicular, so $OQPR$ is a square.
Thus, the coordinates of the points P and R are $(-3,6)$ and $(6,3)$.

ANSWER: $(-3,6),(6,3)$

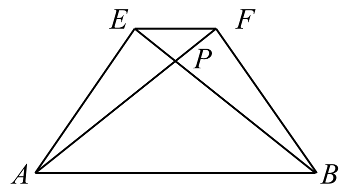
8. Suppose that Claudine has N candies in total.
When Claudine divides all of her candies equally among 7 friends, she has 4 candies left over.
This means that N is 4 more than a multiple of 7, or $N=7a+4$ for some non-negative integer a.
When Claudine divides all of her candies equally among 11 friends, she has 1 candy left over.
This means that N is 1 more than a multiple of 11, or $N=11b+1$ for some non-negative integer b.
Note then that $N+10=7a+14=7(a+2)$ and $N+10=11b+11=11(b+1)$.
In other words, $N+10$ is a multiple of 7 and a multiple of 11.
This means that $N+10$ is a multiple of 77, or N is 10 less than a multiple of 77.
Since Claude has p packages containing 19 candies each, then Claudine has $19p$ candies.
Putting these pieces together, we want to find the smallest non-negative integer p for which $19p$ is 10 less than a multiple of 77.
To do this efficiently, we list, in increasing order, the positive integers that are 10 less than a multiple of 77 until we reach one that is a multiple of 19:

$$67, 144, 221, 298, 375, 452, 529, 606, 683, 760$$

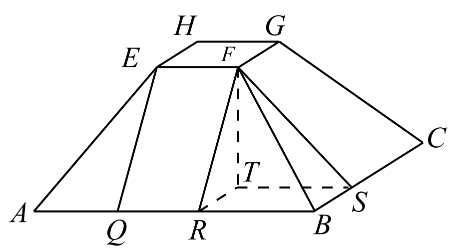
Since 760 is the first number in this list that is a multiple of 19, then 760 is the smallest multiple of 19 that is 4 more than a multiple of 7 and 1 more than a multiple of 11.
This means that the smallest possible value of p is $\frac{760}{19}$ or 40.

ANSWER: 40

9. By symmetry, the area of the shaded region on each of the four trapezoidal faces will be the same. Therefore, the total shaded area will be 4 times the shaded area on trapezoid $EFBA$.
Let h be the height of trapezoid $EFBA$. In other words, h is the distance between the parallel sides EF and AB.
Let P be the point of intersection of EB and FA.



Since $EFGH$ is a square with side length 1, then $EF = 1$. Similarly, $AB = 7$.
Therefore, the area of trapezoid $EFBA$ is $\frac{1}{2}(1+7)h = 4h$.
The shaded area on trapezoid $EFBA$ equals the area of $EFBA$ minus the area of $\triangle APB$.
Since EF and AB are parallel, then $\angle FEP = \angle PBA$.
This means that $\triangle FPE$ is similar to $\triangle APB$.
Since $AB : EF = 7 : 1$, then the ratio of the height of $\triangle APB$ to the height of $\triangle FPE$ must also be $7 : 1$.
Since the sum of the heights of these triangles is the height of the trapezoid, then the height of $\triangle APB$ is $\frac{7}{8}h$.
Thus, the area of $\triangle APB$ is $\frac{1}{2}AB \cdot \frac{7}{8}h = \frac{49}{16}h$, because $AB = 7$.
Therefore, the shaded area on trapezoid $EFBA$ is $4h - \frac{49}{16}h = \frac{15}{16}h$, and so the total shaded area on the truncated pyramid is $4 \cdot \frac{15}{16}h = \frac{15}{4}h$.
To complete our solution, we need to calculate h.
To do this, we begin by dropping perpendiculars from E and F to points Q and R on AB, from F to point S on BC, and from F to point T in square $ABCD$.



Since EF is parallel to QR and $\angle EQR = \angle FRQ = 90^\circ$, then $EFRQ$ is a rectangle.
Thus, $QR = 1$.
Since $EQ = FR = h$ and $EA = FB$ and each of $\triangle EQA$ and $\triangle FRB$ is right-angled, then these triangles are congruent, and so $AQ = RB$.
Since $AB = 7$ and $QR = 1$, then $AQ = RB = 3$. Similarly, $BS = 3$.
Now $RBST$ must be a square of side length 3, since EF lies directly above ST extended and FG lies directly above RT extended.
Finally, we have $TR = 3$, $FT = 4$, $FR = h$, and $\angle FTR = 90^\circ$.
By the Pythagorean Theorem, $h^2 = 3^2 + 4^2 = 25$, and so $h = 5$, since $h > 0$.
This means that the total shaded area is $\frac{15}{4}h = \frac{75}{4}$.

Answer: $\frac{75}{4}$

10. The inequality $x^4 + 8x^3y + 16x^2y^2 + 16 \leq 8x^2 + 32xy$ is equivalent to

$$x^4 + 16x^2y^2 + 16 + 8x^3y - 8x^2 - 32xy \leq 0$$

We note that

$$\begin{aligned}(x^2 + 4xy - 4)^2 &= (x^2)^2 + (4xy)^2 + (-4)^2 + 2x^2(4xy) + 2x^2(-4) + 2(4xy)(-4)\\ &= x^4 + 16x^2y^2 + 16 + 8x^3y - 8x^2 - 32xy\end{aligned}$$

Therefore, the original inequality is equivalent to $(x^2 + 4xy - 4)^2 \leq 0$.
Since the square of any real number is at least 0, then this inequality is equivalent to $(x^2 + 4xy - 4)^2 = 0$ or $x^2 + 4xy - 4 = 0$.

Similarly, since $(y^2 - 8xy + 5)^2 = y^4 + 64x^2y^2 + 25 + 10y^2 - 16xy^3 - 80xy$, then the inequality $y^4 + 64x^2y^2 + 10y^2 + 25 \leq 16xy^3 + 80xy$ is equivalent to $(y^2 - 8xy + 5)^2 \leq 0$, which is equivalent to $y^2 - 8xy + 5 = 0$.

Thus, the original system of inequalities is equivalent to the system of equations

$$\begin{aligned}x^2 + 4xy - 4 &= 0\\ y^2 - 8xy + 5 &= 0\end{aligned}$$

Adding 5 times the first of these equations to 4 times the second equation we obtain

$$5x^2 + 20xy - 20 + 4y^2 - 32xy + 20 = 0$$

or $5x^2 - 12xy + 4y^2 = 0$.
Factoring, we obtain $(5x - 2y)(x - 2y) = 0$, which gives $x = \frac{2}{5}y$ or $x = 2y$.

If $x = \frac{2}{5}y$, then $x^2 + 4xy - 4 = 0$ becomes $\frac{4}{25}y^2 + \frac{8}{5}y^2 - 4 = 0$ or $\frac{4}{25}y^2 + \frac{40}{25}y^2 = 4$ which gives $\frac{44}{25}y^2 = 4$ or $y^2 = \frac{100}{44} = \frac{25}{11}$, and so $y = \pm\frac{5}{\sqrt{11}}$.
Since $x = \frac{2}{5}y$, then $x = \pm\frac{2}{\sqrt{11}}$.
In this case, we get the pairs $(x, y) = (\frac{2}{\sqrt{11}}, \frac{5}{\sqrt{11}}), (-\frac{2}{\sqrt{11}}, -\frac{5}{\sqrt{11}})$.

If $x = 2y$, then $x^2 + 4xy - 4 = 0$ becomes $4y^2 + 8y^2 - 4 = 0$ which gives $12y^2 = 4$ or $y^2 = \frac{1}{3}$, and so $y = \pm\frac{1}{\sqrt{3}}$.
Since $x = 2y$, then $x = \pm\frac{2}{\sqrt{3}}$.
In this case, we get the pairs $(x, y) = (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Therefore, the pairs that satisfy the original system of inequalities are

$$(x, y) = (\tfrac{2}{\sqrt{11}}, \tfrac{5}{\sqrt{11}}), (-\tfrac{2}{\sqrt{11}}, -\tfrac{5}{\sqrt{11}}), (\tfrac{2}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}), (-\tfrac{2}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}})$$

Answer: $(\frac{2}{\sqrt{11}}, \frac{5}{\sqrt{11}}), (-\frac{2}{\sqrt{11}}, -\frac{5}{\sqrt{11}}), (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}), (-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$

Team Problems

1. Evaluating, $2 + 0 - 1 \times 6 = 2 + 0 - 6 = -4$.

   Answer: -4

2. Since the average of 5 numbers is 7, then the sum of the 5 numbers is $5 \times 7 = 35$.
   Therefore, $3 + 5 + 6 + 8 + x = 35$ or $22 + x = 35$ and so $x = 13$.

   Answer: 13

3. Since $\lfloor 1.6 \rfloor = 1$, then $\lfloor -2.3 + \lfloor 1.6 \rfloor \rfloor = \lfloor -2.3 + 1 \rfloor = \lfloor -1.3 \rfloor = -2$.

   Answer: -2

4. Starting with ABC, the magician

   - swaps the first and second cups to obtain BAC, then
   - swaps the second and third cups to obtain BCA, then
   - swaps the first and third cups to obtain ACB.

   The net effect of these 3 moves is to swap the second and third cups.
   When the magician goes through this sequence of moves 9 times, the net effect starting from ABC is that he swaps the second and third cups 9 times.
   Since 9 is odd, then the final order is ACB.

   Answer: ACB

5. Since $(1, 2)$ lies on the parabola with equation $y = ax^2 + bx + c$, then the coordinates of the point satisfy the equation of the parabola.
   Thus, $2 = a(1^2) + b(1) + c$ or $a + b + c = 2$.

   Answer: 2

6. Since $2^{11} = 2048$ and $2^8 = 256$, then $2^{11} - 2^8 = 2048 - 256 = 1792$.
   Therefore, $m = 11$ and $n = 8$, which gives $m^2 + n^2 = 11^2 + 8^2 = 121 + 64 = 185$.

   Alternatively, we can factor the left side of the equation $2^m - 2^n = 1792$ to obtain $2^n(2^{m-n} - 1) = 1792$.
   Since $m > n$, then $2^{m-n} - 1$ is an integer.
   Now, $1792 = 2^8 \cdot 7$.
   Since $2^n(2^{m-n} - 1) = 2^8 \cdot 7$, then $2^n = 2^8$ (which gives $n = 8$) and $2^{m-n} - 1 = 7$ (which gives $m - n = 3$ and so $m = 11$).

   Answer: 185

7. The number of two-digit integers in the range 10 to 99 inclusive is $99 - 10 + 1 = 90$.
   We count the number of two-digit positive integers whose tens digit is a multiple of the units (ones) digit.
   If the units digit is 0, there are no possible tens digits.
   If the units digit is 1, the tens digit can be any digit from 1 to 9. This gives 9 such numbers.
   If the units digit is 2, the tens digit can be $2, 4, 6, 8$. This gives 4 such numbers.
   If the units digit is 3, the tens digit can be $3, 6, 9$. This gives 3 such numbers.
   If the units digit is 4, the tens digit can be $4, 8$. This gives 2 such numbers.
   If the units digit is any of $5, 6, 7, 8, 9$, the only possibility is that the tens digit equals the units digit. This gives 5 such numbers.
   In total, there are $9 + 4 + 3 + 2 + 5 = 23$ two-digit positive integers whose tens digit is a multiple of the units digit, and so the probability that a randomly chosen two-digit integer has this property is $\frac{23}{90}$.

   Answer: $\frac{23}{90}$

8. The area of pentagon $ADCBZ$ equals the area of rectangle $ABCD$ minus the area of $\triangle AZB$.
Since the area of rectangle $ABCD$ is 2016, then $AB \cdot CB = 2016$.
The area of $\triangle AZB$ equals $\frac{1}{2}(AB)(ZH)$.
Since $ZH : CB = 4 : 7$, then $ZH = \frac{4}{7}CB$.
Thus, the area of $\triangle AZB$ is $\frac{1}{2}(AB)(\frac{4}{7}CB) = \frac{2}{7}(AB)(CB) = \frac{2}{7}(2016) = 576$.
Therefore, the area of pentagon $ADCBZ$ is $2016 - 576 = 1440$.

Answer: 1440

9. The integer with digits AAA is equal to $100A + 10A + A = 111A$.
Similarly, the integer with digits AAB is equal to $110A + B$, the integer with digits ABB is equal to $100A + 11B$, and the integer with digits BBB is equal to $111B$.
From the given addition, we obtain $111A + (110A + B) + (100A + 11B) + 111B = 1503$ or $321A + 123B = 1503$.
Dividing both sides by 3, we obtain $107A + 41B = 501$.
Rearranging gives $41B = 501 - 107A$.
Since A and B are positive integers, then $107A < 501$.
Since $107 \cdot 5 = 535$, then the possible values of A are $1, 2, 3, 4$.
If $A = 1$, then $501 - 107A = 394$ which is not a multiple of 41.
If $A = 2$, then $501 - 107A = 287$ which equals 7×41.
(Neither $A = 3$ nor $A = 4$ makes $501 - 107A$ a multiple of 41.)
Therefore, $A = 2$ and $B = 7$.
(Checking, we see that $222 + 227 + 277 + 777 = 1503$.)
Thus, $A^3 + B^2 = 2^3 + 7^2 = 8 + 49 = 57$.

Answer: 57

10. Suppose that, on the way from Appsley to Bancroft, Clara rides x km downhill, y km on level road, and z km uphill.
Then on the reverse trip, she travels x km uphill, y km on level road, and z km downhill.
Clara rides downhill at 24 km/h, on level road at 16 km/h, and uphill at 12 km/h.
Since the original trip takes 2 hours, then $\dfrac{x}{24} + \dfrac{y}{16} + \dfrac{z}{12} = 2$.
Since the reverse trip takes 2 hours and 15 minutes (or 2.25 hours), then $\dfrac{x}{12} + \dfrac{y}{16} + \dfrac{z}{24} = 2.25$.
Adding these two equations, we obtain

$$\left(\frac{1}{24} + \frac{1}{12}\right)x + \left(\frac{1}{16} + \frac{1}{16}\right)y + \left(\frac{1}{12} + \frac{1}{24}\right)z = 4.25$$

Since $\dfrac{1}{24} + \dfrac{1}{12} = \dfrac{1}{24} + \dfrac{2}{24} = \dfrac{3}{24} = \dfrac{1}{8}$, then the last equation is equivalent to

$$\frac{x}{8} + \frac{y}{8} + \frac{z}{8} = 4.25$$

Multiplying both sides by 8, we obtain $x + y + z = 34$.
The distance from Appsley to Bancroft is $(x + y + z)$ km, or 34 km.

Answer: 34

11. We write out the first few terms of the sequence using the given rule:

$$4, \quad 5, \quad \frac{5+1}{4} = \frac{3}{2}, \quad \frac{3/2+1}{5} = \frac{1}{2}, \quad \frac{1/2+1}{3/2} = 1, \quad \frac{1+1}{1/2} = 4, \quad \frac{4+1}{1} = 5, \quad \frac{5+1}{4} = \frac{3}{2}$$

Since the 6th and 7th terms are equal to the 1st and 2nd terms, respectively, then the sequence repeats with period 5. (This is because two consecutive terms determine the next term, so identical pairs of consecutive terms give identical next terms.)
Since $1234 = 246(5) + 4$, then the 1234th term is the 4th term after a cycle of 5 and so equals the 4th term, which is $\frac{1}{2}$.

ANSWER: $\frac{1}{2}$

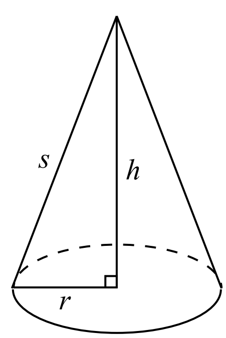
12. Since Austin chooses 2 to start and Joshua chooses 3 next, we can list all of the possible orders of choices of numbers:

Order of Numbers	Joshua's 1st Rd Score	Austin's 1st Rd Score	Joshua's 2nd Rd Score	Austin's 2nd Rd Score	Joshua's Total	Austin's Total
2, 3, 1, 4, 5	6	3	4	20	10	23
2, 3, 1, 5, 4	6	3	5	20	11	23
2, 3, 4, 1, 5	6	12	4	5	10	17
2, 3, 4, 5, 1	6	12	20	5	26	17
2, 3, 5, 1, 4	6	15	5	4	11	19
2, 3, 5, 4, 1	6	15	20	4	26	19

Since Austin's total is larger than Joshua's total in 4 of the 6 games, then the probability that Austin wins is $\frac{4}{6}$ or $\frac{2}{3}$.

ANSWER: $\frac{2}{3}$

13. Suppose that the cone has radius r, height h, and slant height s.



Since the lateral surface area of the cone is 80π and the total surface area is 144π, then the area of the base of the cone is $144\pi - 80\pi = 64\pi$.
Thus, $\pi r^2 = 64\pi$, which gives $r^2 = 64$ or $r = 8$ (since $r > 0$).
Since the lateral surface area is 80π and $r = 8$, then $\pi rs = 80\pi$ or $8s = 80$ which gives $s = 10$.
Now, by the Pythagorean Theorem, $s^2 = r^2 + h^2$ and so $h^2 = s^2 - r^2 = 10^2 - 8^2 = 36$.
Since $h > 0$, then $h = 6$.
Thus, the volume of the cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi 8^2(6) = 128\pi$.
Suppose that a sphere of equal volume has radius R. Then $\frac{4}{3}\pi R^3 = 128\pi$.
Thus, $R^3 = \frac{3}{4} \cdot 128 = 96$, and so $R = \sqrt[3]{96}$.

ANSWER: $\sqrt[3]{96}$

14. Using logarithm rules,

$$\begin{aligned}
&\log_3(1-\tfrac{1}{15})+\log_3(1-\tfrac{1}{14})+\log_3(1-\tfrac{1}{13})+\cdots+\log_3(1-\tfrac{1}{8})+\log_3(1-\tfrac{1}{7})+\log_3(1-\tfrac{1}{6})\\
&\quad=\log_3(\tfrac{14}{15})+\log_3(\tfrac{13}{14})+\log_3(\tfrac{12}{13})+\log_3(\tfrac{11}{12})+\log_3(\tfrac{10}{11})+\log_3(\tfrac{9}{10})\\
&\qquad\quad+\log_3(\tfrac{8}{9})+\log_3(\tfrac{7}{8})+\log_3(\tfrac{6}{7})+\log_3(\tfrac{5}{6})\\
&\quad=\log_3(\tfrac{14}{15}\cdot\tfrac{13}{14}\cdot\tfrac{12}{13}\cdot\tfrac{11}{12}\cdot\tfrac{10}{11}\cdot\tfrac{9}{10}\cdot\tfrac{8}{9}\cdot\tfrac{7}{8}\cdot\tfrac{6}{7}\cdot\tfrac{5}{6})\\
&\quad=\log_3(\tfrac{5}{15})\\
&\quad=\log_3(\tfrac{1}{3})\\
&\quad=-1
\end{aligned}$$

ANSWER: -1

15. We want to determine the number of four-digit positive integers of the form $4xyz$ that are divisible by 45.
A positive integer is divisible by 45 exactly when it is divisible by both 5 and 9.
For an integer to be divisible by 5, its units digit must be 0 or 5.
Thus, such numbers are of the form $4xy0$ or $4xy5$.
For an integer to be divisible by 9, the sum of its digits must be divisible by 9.

Case 1: $4xy0$
The sum of the digits of $4xy0$ is $4+x+y$.
Since x and y are between 0 and 9, inclusive, then $0 \leq x+y \leq 18$.
For $4+x+y$ to be divisible by 9, we need $x+y=5$ or $x+y=14$.
If $x+y=5$, then $(x,y)=(0,5),(1,4),(2,3),(3,2),(4,1),(5,0)$.
If $x+y=14$, then $(x,y)=(5,9),(6,8),(7,7),(8,6),(9,5)$.
There are 11 possible four-digit numbers in this case.

Case 1: $4xy5$
The sum of the digits of $4xy5$ is $9+x+y$.
Since x and y are between 0 and 9, inclusive, then $0 \leq x+y \leq 18$.
For $9+x+y$ to be divisible by 9, we need $x+y=0$ or $x+y=9$ or $x+y=18$.
If $x+y=0$, then $(x,y)=(0,0)$.
If $x+y=9$, then $(x,y)=(0,9),(1,8),(2,7),(3,6),(4,5)$ and their reverses.
If $x+y=18$, then $(x,y)=(9,9)$.
There are 12 possible four-digit numbers in this case.

Overall, there are $11+12=23$ possible combinations.

ANSWER: 23

16. Since $A_1A_2A_3\cdots A_n$ is a regular n-gon, then $\angle A_1A_2A_3=\angle A_2A_3A_4=\angle A_3A_4A_5=180^\circ-\dfrac{360^\circ}{n}$.
(This comes from the fact that the sum of the interior angles in a regular n-gon is $180^\circ(n-2)$.)
Therefore, $\angle PA_2A_3=\angle PA_4A_3=180^\circ-\left(180^\circ-\dfrac{360^\circ}{n}\right)=\dfrac{360^\circ}{n}$.
Also, the reflex angle at A_3 equals $360^\circ-\left(180^\circ-\dfrac{360^\circ}{n}\right)=180^\circ+\dfrac{360^\circ}{n}$.
Since the sum of the angles in quadrilateral $PA_2A_3A_4$ equals 360°, then

$$120^\circ+2\left(\frac{360^\circ}{n}\right)+180^\circ+\frac{360^\circ}{n}=360^\circ$$

or $\dfrac{1080^\circ}{n}=60^\circ$ and so $n=18$.

ANSWER: 18

17. There are $\binom{21}{3} = \dfrac{21(20)(19)}{3!} = 7(10)(19)$ ways of choosing 3 of the 21 marbles from the bag.
We count the number of ways in which 3 marbles of the same colour can be chosen.
There are 0 ways of choosing 3 magenta or 3 puce marbles, since there are fewer than 3 marbles of each of these colours.
There is 1 way of choosing 3 cyan marbles since there are exactly 3 cyan marbles in total.
Since there are 4 ecru marbles, there are $\binom{4}{3} = 4$ ways of choosing 3 ecru marbles.
Since there are 5 aquamarine marbles, there are $\binom{5}{3} = \dfrac{5(4)(3)}{3(2)(1)} = 10$ ways of choosing 3 aquamarine marbles.
Since there are 6 lavender marbles, there are $\binom{6}{3} = \dfrac{6(5)(4)}{3(2)(1)} = 20$ ways of choosing 3 lavender marbles.
Thus, there are $1 + 4 + 10 + 20 = 35$ ways of choosing 3 marbles of the same colour.
Since there are $7(10)(19)$ ways of choosing 3 marbles, then the probability that all three are the same colour is $\dfrac{35}{7(10)(19)} = \dfrac{7(5)}{7(10)(19)} = \dfrac{1}{2(19)} = \dfrac{1}{38}$.
For this probability to equal $\dfrac{1}{k}$, we must have $k = 38$.

ANSWER: 38

18. We begin by comparing pairs of the given expressions:

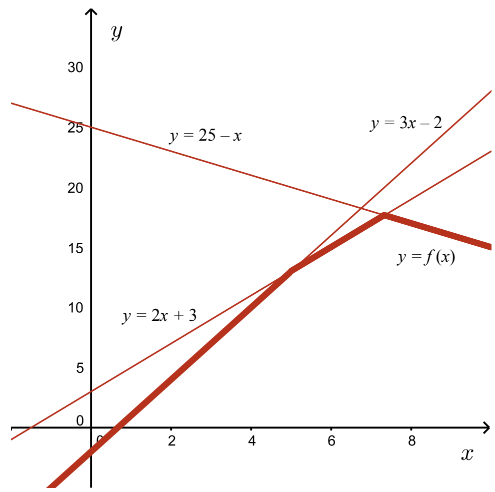
- $2x + 3 \le 3x - 2$ exactly when $5 \le x$
- $2x + 3 \le 25 - x$ exactly when $3x \le 22$ or $x \le \frac{22}{3}$
- $25 - x \le 3x - 2$ exactly when $27 \le 4x$ or $\frac{27}{4} \le x$

Therefore,

- when $x \le 5$, we have $3x - 2 \le 2x + 3$ and when $x \ge 5$, we have $2x + 3 \le 3x - 2$,
- when $x \le \frac{27}{4}$, we have $3x - 2 \le 25 - x$ and when $x \ge \frac{27}{4}$, we have $25 - x \le 3x - 2$, and
- when $x \le \frac{22}{3}$, we have $2x + 3 \le 25 - x$ and when $x \ge \frac{22}{3}$, we have $25 - x \le 2x + 3$.

Noting that $5 < \frac{27}{4} < \frac{22}{3}$, we combine these relationships to get that

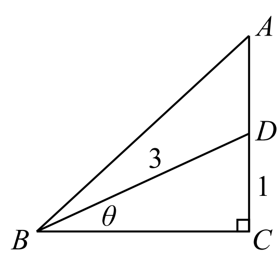
- when $x \le 5$, we have $3x - 2 \le 2x + 3$ and $3x - 2 \le 25 - x$ and $2x + 3 \le 25 - x$ and so the smallest of these is $3x - 2$, which means that $f(x) = 3x - 2$,
- when $5 \le x \le \frac{27}{4}$, we have $2x + 3 \le 3x - 2$ and $3x - 2 \le 25 - x$ and $2x + 3 \le 25 - x$ and so the smallest of these is $2x + 3$, which means that $f(x) = 2x + 3$,
- when $\frac{27}{4} \le x \le \frac{22}{3}$, we have $2x + 3 \le 3x - 2$ and $25 - x \le 3x - 2$ and $2x + 3 \le 25 - x$ and so the smallest of these is $2x + 3$, which means that $f(x) = 2x + 3$, and
- when $x \ge \frac{22}{3}$, we have $2x + 3 \le 3x - 2$ and $25 - x \le 3x - 2$ and $25 - x \le 2x + 3$ and so the smallest of these expressions is $25 - x$, which means that $f(x) = 25 - x$.



This tells us that $f(x) = 3x - 2$ when $x \le 5$, and $f(x) = 2x + 3$ when $5 \le x \le \frac{22}{3}$, and $f(x) = 25 - x$ when $x \ge \frac{22}{3}$.
Since $f(x)$ has positive slope when $x \le 5$, then the maximum value of $f(x)$ when $x \le 5$ occurs when $x = 5$. We have $f(5) = 3(5) - 2 = 13$.
Since $f(x)$ has positive slope when $5 \le x \le \frac{22}{3}$, then the maximum value of $f(x)$ when $5 \le x \le \frac{22}{3}$ occurs when $x = \frac{22}{3}$. We have $f(\frac{22}{3}) = 2 \cdot \frac{22}{3} + 3 = \frac{53}{3}$.
Since $f(x)$ has negative slope when $x \ge \frac{22}{3}$, then the maximum value of $f(x)$ when $x \ge \frac{22}{3}$ occurs when $x = \frac{22}{3}$. We already know that $f(\frac{22}{3}) = \frac{53}{3}$.
Comparing the maximum values of $f(x)$ over the three intervals, we can conclude that the maximum value of $f(x)$ is $\frac{53}{3}$, since $\frac{53}{3} > 13$.

Answer: $\frac{53}{3}$

19. Suppose that $\angle DBC = \theta$. Then $\angle ABC = 2\theta$.



Now $AD = AC - DC = AC - 1$.
Also, $\tan(\angle ABC) = \tan 2\theta = \dfrac{AC}{BC}$, so $AC = BC \tan 2\theta$.
In $\triangle DBC$, by the Pythagorean Theorem, we have

$$BC = \sqrt{BD^2 - DC^2} = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$$

Therefore, $AD = 2\sqrt{2}\tan 2\theta - 1$.
But $\tan\theta = \tan(\angle DBC) = \dfrac{DC}{BC} = \dfrac{1}{2\sqrt{2}}$.
This means that $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta} = \dfrac{1/\sqrt{2}}{1 - (1/8)} = \dfrac{1/\sqrt{2}}{7/8} = \dfrac{8}{7\sqrt{2}}$.
Finally, we obtain $AD = 2\sqrt{2}\tan 2\theta - 1 = \dfrac{16\sqrt{2}}{7\sqrt{2}} - 1 = \dfrac{16}{7} - 1 = \dfrac{9}{7}$.

Answer: $\dfrac{9}{7}$

20. *Solution 1*

Let T be the total number of animals.
Suppose that, in the first line, there are C cows and H horses.
Suppose that, in the second line, there are w cows and x horses that are standing opposite the cows in the first line, and that there are y cows and z horses that are standing opposite the horses in the first line.
Since there are 75 horses, then $H + x + z = 75$.
Since there are 10 more cows opposite cows than horses opposite horses, then $w = z + 10$.
The total number of animals is $T = C + H + w + x + y + z$.
But $y + z = H$ and $C = w + x$ so $T = (w + x) + H + w + x + H = 2(H + w + x)$.
Now $w = z + 10$ so $T = 2(H + z + 10 + x) = 20 + 2(H + x + z) = 20 + 2(75) = 170$, and so the total number of animals is 170.

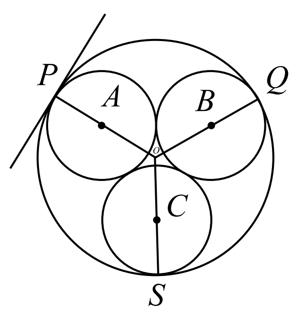
Solution 2

Let h be the number of pairs consisting of a horse opposite a horse.
Let c be the number of pairs consisting of a cow opposite a cow.
Let x be the number of pairs consisting of a horse opposite a cow.
Since the total number of horses is 75, then $75 = 2h + x$. (Each "horse-horse" pair includes two horses.)
Also, the given information tells us that $c = h + 10$.
The total number of animals is thus

$$2h + 2c + 2x = 2h + 2(h + 10) + 2(75 - 2h) = 2h + 2h + 20 + 150 - 4h = 170$$

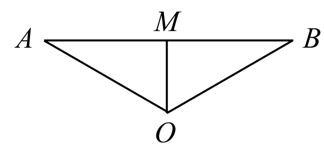
ANSWER: 170

21. Suppose that the three small circles have centres A, B and C.
For the larger circle to be as small as possible, these three small circles must just touch the larger circle (which has centre O) at P, Q and S, respectively.
Note that $AP = BQ = CS = 1$.



Let r be the radius of the larger circle. Then $OP = OQ = OS = r$.
Since the larger circle is tangent to each of the smaller circles at P, Q and S, this means that the circles share a common tangent at each of these points.
Consider the point P. Since AP is perpendicular to the tangent to the smaller circle at P and OP is perpendicular to the tangent to the larger circle at P and this tangent is common, then AP and OP must be parallel. Since AP and OP both pass through P, then AP lies along OP.
Since $AP = 1$ and $OP = r$, then $OA = r - 1$.
In a similar way, we can find that $OB = OC = r - 1$.
Now since the circles with centres A and B just touch, then the distance AB equals the sum of the radii of the circles, or $AB = 2$.

Similarly, $AC = BC = 2$.
Consider $\triangle AOB$. We have $OA = OB = r - 1$ and $AB = 2$.
By symmetry, $\angle AOB = \angle BOC = \angle COA = \frac{1}{3}(360°) = 120°$.
Let M be the midpoint of AB. Thus, $AM = MB = 1$.



Since $\triangle AOB$ is isosceles, then OM is perpendicular to AB and bisects $\angle AOB$.
This means that $\triangle AOM$ is a 30°-60°-90° triangle.
Therefore, $\dfrac{AO}{AM} = \dfrac{2}{\sqrt{3}}$ and so $\dfrac{r-1}{1} = \dfrac{2}{\sqrt{3}}$ or $r = 1 + \dfrac{2}{\sqrt{3}}$.
Thus, the radius of the smallest circle enclosing the other three circles is $1 + \dfrac{2}{\sqrt{3}}$

ANSWER: $1 + \dfrac{2}{\sqrt{3}}$

22. If we impose no restrictions initially, each of the 7 patients can be assigned to one of 3 doctors, which means that there are 3^7 ways to assign the patients to doctors.
To determine the number of ways in which the patients can be assigned to doctors so that each doctor is assigned at least one patient, we subtract from 3^7 the number of ways in which at least one doctor has 0 patients.
This can happen if all of the patients are assigned to one doctor. There are 3 ways to do this (all patients assigned to Huey, all patients assigned to Dewey, or all patients assigned to Louie).
This can also happen if all of the patients are assigned to two doctors and each of these two doctors are assigned at least one patient.
If each of the 7 patients are assigned to Huey and Dewey, for example, then there are 2 choices for each patient, giving 2^7 ways minus the 2 ways in which all of the patients are assigned to Huey and in which all of the patients are assgined to Dewey. This gives $2^7 - 2$ ways of assigning the patients in this way.
Similarly, there are $2^7 - 2$ ways of assigning the patients to Huey and Louie, and $2^7 - 2$ ways of assigning the patients to Dewey and Louise.
Overall, this means that there are $3^7 - 3 - 3(2^7 - 2) = 2187 - 3 - 3(126) = 1806$ ways of assigning the patients to the doctors.

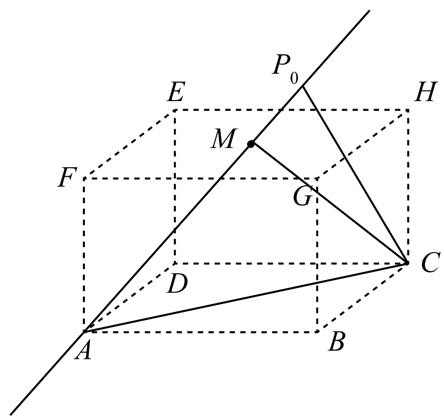
ANSWER: 1806

23. Suppose that $S = 1 + \dfrac{3}{a} + \dfrac{5}{a^2} + \dfrac{7}{a^3} + \cdots$.
Then $aS = a + 3 + \dfrac{5}{a} + \dfrac{7}{a^2} + \dfrac{9}{a^3} + \cdots$.
Therefore $aS - S = a + 2 + \dfrac{2}{a} + \dfrac{2}{a^2} + \dfrac{2}{a^3} + \cdots$.
Since $a > 1$, then $2 + \dfrac{2}{a} + \dfrac{2}{a^2} + \dfrac{2}{a^3} + \cdots$ is an infinite geometric series with common ratio $\dfrac{1}{a} < 1$.
Therefore, $2 + \dfrac{2}{a} + \dfrac{2}{a^2} + \dfrac{2}{a^3} + \cdots = \dfrac{2}{1 - \dfrac{1}{a}} = \dfrac{2a}{a-1}$.

Substituting, we obtain $aS - S = a + \dfrac{2a}{a-1}$ and so $(a-1)S = \dfrac{(a^2-a)+2a}{a-1}$ which gives $(a-1)S = \dfrac{a^2+a}{a-1}$ or $S = \dfrac{a^2+a}{(a-1)^2}$.

ANSWER: $\dfrac{a^2+a}{(a-1)^2}$

24. Let P_0 be the point on the line through AM that minimizes the distance from C.
Then CP_0 is perpendicular to AM.
(Note that any other point P on this line would form $\triangle CP_0P$ with right-angle at P_0, making CP the hypotenuse of the triangle and so the longest side. In particular, any other point P on the line gives $CP > CP_0$.)
Consider $\triangle AMC$.



The area of $\triangle AMC$ equals $\frac{1}{2}AC \cdot 2c = AC \cdot c$. This is because M lies directly above AC, which is a diagonal of the base of the prism, and so the height of $\triangle AMC$ equals the height of the prism, which is $2c$.
Also, the area of $\triangle AMC$ equals $\frac{1}{2}AM \cdot h$, where h is the perpendicular distance from C to AM. (Here, we are thinking of AM as a base of the triangle.)
But CP_0 is the corresponding height, so $h = CP_0$.
In other words, $\frac{1}{2}AM \cdot CP_0 = AC \cdot c$, and so $CP_0 = \dfrac{2AC \cdot c}{AM}$.
So we need to determine the length of AC and the length of AM.
AC is the hypotenuse of right-angled $\triangle ABC$.
Since $AB = 2a$ and $BC = AD = 2b$, then

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{(2a)^2 + (2b)^2} = \sqrt{4a^2 + 4b^2} = 2\sqrt{a^2+b^2}$$

AM is the hypotenuse of right-angled $\triangle AFM$.
Since $AF = 2c$ and $FM = \frac{1}{2}FH = \frac{1}{2}AC = \sqrt{a^2+b^2}$, then

$$AM = \sqrt{AF^2 + FM^2} = \sqrt{4c^2 + a^2 + b^2}$$

Therefore,

$$CP_0 = \frac{2(2\sqrt{a^2+b^2}) \cdot c}{\sqrt{4c^2+a^2+b^2}} = \frac{4c\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+4c^2}}$$

ANSWER: $\dfrac{4c\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+4c^2}}$

25. If m is a positive integer that ends in 9999, then $m+1$ must end in 0000.
This means that $m+1$ is a multiple of 10 000; that is, $m+1 = 10000k = 10^4k$ for some positive integer k.
Since m does not end in five or more 9s, then $m+1$ does not end in 00000 and so the units digit of k is not 0. In other words, k is not divisible by 10.

We determine the first several terms in each sequence to see if we can find a pattern.
By definition, $t_1 = 1$ and $t_2 = m$.
Next, $s_2 = t_1 + t_2 = m + 1 = 10^4k$.
This gives $t_3 = 3s_2 = 3 \cdot 10^4k$.
Next, $s_3 = t_1 + t_2 + t_3 = s_2 + t_3 = 10^4k + 3 \cdot 10^4k = 4 \cdot 10^4k$.
This gives $t_4 = 4s_3 = 4 \cdot 4 \cdot 10^4k$.
Next, $s_4 = s_3 + t_4 = 4 \cdot 10^4k + 4 \cdot (4 \cdot 10^4k) = 5 \cdot 4 \cdot 10^4k$.
This gives $t_5 = 5s_4 = 5 \cdot 5 \cdot 4 \cdot 10^4k$.
Next, $s_5 = s_4 + t_5 = 5 \cdot 4 \cdot 10^4k + 5 \cdot (5 \cdot 4 \cdot 10^4k) = 6 \cdot 5 \cdot 4 \cdot 10^4k$.
This gives $t_6 = 6 \cdot 6 \cdot 5 \cdot 4 \cdot 10^4k$.
Based on this apparent pattern, we guess that, for $n \geq 5$, we have $t_n = n \cdot n \cdot (n-1) \cdot \cdots \cdot 5 \cdot 4 \cdot 10^4k = \dfrac{n \cdot n! \cdot 10^4k}{3!}$ and $s_{n-1} = n \cdot (n-1) \cdot \cdots \cdot 5 \cdot 4 \cdot 10^4k = \dfrac{n! \cdot 10^k}{3!}$.
We will proceed based on this guess, which we will prove at the end of this solution.

Following this guess, $t_{30} = \dfrac{30 \cdot 30! \cdot 10^4k}{3!} = 5 \cdot 30! \cdot 10^4k$.

We want to find all positive integers N for which $5 \cdot 30! \cdot 10^4k = N!$ for some positive integer k that is not divisible by 10.
We note that 30! must be a divisor of $N!$ and so $N \geq 30$.
We also know that $N!$ contains at least five more factors of 5 than 30! since $N! = 30! \cdot 5 \cdot 5^4 \cdot 2^4k$.
This means that N must be enough larger than 30 so that $N!$ includes at least five more factors of 5.
We note that $35! = 35(34)(33)(32)(31)30!$, which contains one more factor of 5 than 30! and that 35! is the smallest factorial with this property.
Similarly, 40! contains one more factor of 5 than 35!.
Also, 45! contains one more factor of 5 than 40! and is the smallest factorial that includes three more factors of 5 than 30!.
Since 5^2 is a divisor of 50, then 50! contains two more factors of 5 than 45! and is the smallest factorial that thus includes five more factors of 5 than 30!. We conclude that $N \geq 50$.
We note that 50!, 51!, 52!, 53!, and 54! each contain the same number of factors of 5, and that 55! (and thus every $N!$ with $N \geq 55$) contains at least six more factors of 5 than 30!.
Now since $N \geq 50$, then 50! is a divisor of $N! = 30! \cdot 5^5 \cdot 2^4k$.
Rearranging, we get $N(N-1)(N-2)\cdots(33)(32)(31) = 5^5 \cdot 2^4k$ or $N(N-1)(N-2)\cdots(33)(2)(31) = 5^5k$.
Since the left side is even, then k is even.
Since k is even and k is not divisible by 10, then k is not divisible by 5. (That is, the right side contains exactly five factors of 5.)
This means that N cannot be 55 or greater, as in this case the number of factors of 5 (at least six on the left side) would not balance.
Therefore, the possible values of N are $50, 51, 52, 53, 54$.

To complete our solution, we prove that $t_n = \dfrac{n \cdot n! \cdot 10^4 k}{3!}$ and $s_{n-1} = \dfrac{n! \cdot 10^4 k}{3!}$ when $n \geq 5$.

We already know that $t_5 = 5 \cdot 5 \cdot 4 \cdot 10^4 k = \dfrac{5 \cdot 5! \cdot 10^4 k}{3!}$ and that $s_4 = 5 \cdot 4 \cdot 10^4 k = \dfrac{5! \cdot 10^4 k}{3!}$.

We prove that the desired formulas are true for all $n \geq 5$ by using a technique called *mathematical induction*.

We have proven that these formulas are correct for $n = 5$.

If we can prove that whenever the forms are correct for $n = j$, then they must be correct for $n = j + 1$, then their correctness for $n = 5$ will give their correctness for $n = 6$ which will give their correctness for $n = 7$ and so on, giving their correctness for all $n \geq 5$.

So suppose that $t_j = \dfrac{j \cdot j! \cdot 10^4 k}{3!}$ and that $s_{j-1} = \dfrac{j! \cdot 10^4 k}{3!}$.

Then $s_j = s_{j-1} + t_j = \dfrac{j! \cdot 10^4 k}{3!} + \dfrac{j \cdot j! \cdot 10^4 k}{3!} = \dfrac{j! \cdot 10^4 k}{3!}(1 + j) = \dfrac{(j+1)! \cdot 10^4 k}{3!}$, as expected.

This means that $t_{j+1} = (j+1)s_j = (j+1) \cdot \dfrac{(j+1)! \cdot 10^4 k}{3!} = \dfrac{(j+1) \cdot (j+1)! \cdot 10^4 k}{3!}$, as expected.

Therefore, if t_j and s_{j-1} have the correct forms, then t_{j+1} and s_j have the correct forms.

This proves that t_n and s_{n-1} have the correct forms for all $n \geq 5$, which completes our solution.

ANSWER: 50,51,52,53,54

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $10 - 2 \times 3 = 10 - 6 = 4$.

   (b) The area of a triangle with base of length $2t$ and height of length $3t+1$ is $\frac{1}{2}(2t)(3t+1)$ or $t(3t+1)$.
   Since the answer to (a) is 4, then $t = 4$, and so $t(3t+1) = 4(13) = 52$.

   (c) Since $AB = BC$, then $\angle BCA = \angle BAC = t^\circ$.
   Therefore, $\angle ABC = 180^\circ - \angle BCA - \angle BAC = 180^\circ - 2t^\circ$.
   Since the answer to (b) is 52, then $t = 52$, and so

$$\angle ABC = 180^\circ - 2 \cdot 52^\circ = 180^\circ - 104^\circ = 76^\circ$$

Answer: $4, 52, 76^\circ$

1. (a) Since $x : 6 = 15 : 10$, then $\dfrac{x}{6} = \dfrac{15}{10}$ which gives $x = \dfrac{6 \cdot 15}{10} = 9$.

   (b) If $\dfrac{3(x+5)}{4} = t + \dfrac{3-3x}{2}$, then $3(x+5) = 4t + 2(3-3x)$ or $3x + 15 = 4t + 6 - 6x$, which gives $9x = 4t - 9$ or $x = \frac{4}{9}t - 1$.
   Since the answer to (a) is 9, then $t = 9$ and so $x = \frac{4}{9}t - 1 = 4 - 1 = 3$.

   (c) We start with the given equation and complete the square:

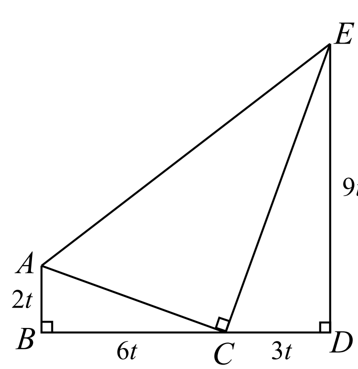
$$y = 3x^2+6\sqrt{m}x+36 = 3(x^2+2\sqrt{m}x+12) = 3((x+\sqrt{m})^2-m+12) = 3(x+\sqrt{m})^2+(36-3m)$$

   Therefore, the coordinates of the vertex of this parabola are $(-\sqrt{m}, 36 - 3m)$.
   Since the y-coordinate of the vertex is t, then $36 - 3m = t$ or $3m = 36 - t$ and so $m = 12 - \frac{1}{3}t$.
   Since the answer to (b) is 3, then $t = 3$, and so $m = 12 - \frac{1}{3}t = 12 - 1 = 11$.

Answer: $9, 3, 11$

2. (a) To find the x-intercept of the line with equation $20x + 16y - 40 = 0$, we set $y = 0$ and get $20x - 40 = 0$ or $x = 2$.
   To find the y-intercept of the line with equation $20x + 16y - 64 = 0$, we set $x = 0$ and get $16y - 64 = 0$ or $y = 4$.
   The sum of the intercepts is $2 + 4 = 6$.

   (b) Since $BC : CD = 2 : 1$, then $BC = 2CD$.
   Since $BD = 9t$, then $BC + CD = 9t$ or $CD + 2CD = 9t$ and so $3CD = 9t$ or $CD = 3t$, which gives $BC = 6t$.



The area of $\triangle ACE$, which is denoted k, equals the area of trapezoid $ABDE$ minus the areas of $\triangle ABC$ and $\triangle CDE$.
The area of trapezoid $ABDE$ is $\frac{1}{2}(AB + DE)(BD)$ or $\frac{1}{2}(2t + 9t)(9t)$ which equals $\frac{99}{2}t^2$.
The area of $\triangle ABC$ is $\frac{1}{2}(AB)(BC)$ or $\frac{1}{2}(2t)(6t)$ which equals $6t^2$.
The area of $\triangle CDE$ is $\frac{1}{2}(CD)(DE)$ or $\frac{1}{2}(3t)(9t)$ which equals $\frac{27}{2}t^2$.
Therefore, $k = \frac{99}{2}t^2 - 6t^2 - \frac{27}{2}t^2 = 30t^2$.
Since the answer to (a) is 6, then $t = 6$, and so $\frac{1}{36}k = \frac{30}{36}t^2 = 30$.

(c) The volume of a cylinder with radius $\sqrt{2}$ and height a is $\pi(\sqrt{2})^2 a = 2\pi a$.
The volume of a cylinder with radius $\sqrt{5}$ and height b is $\pi(\sqrt{5})^2 b = 5\pi b$.
From the given information, $2\pi a + 5\pi b = 10\pi t$ or $2a + 5b = 10t$.
Since $a > 0$ and $b > 0$, then $2a < 10t$ or $a < 5t$. Also, $5b < 10t$ or $b < 2t$.
Rearranging $2a + 5b = 10t$ gives $2a = 10t - 5b$ or $a = 5t - \frac{5}{2}b$.
If we assume that t is a positive integer, then for a to be a positive integer, it must be the case that b is even.
Since $b < 2t$, then the possible even values of b less than $2t$ are $2, 4, 6, \ldots, 2t - 4, 2t - 2$.
There are $t - 1$ such values.
Under the assumption that t is a positive integer, these are exactly the values of b that give positive integer values for a.
In particular, we obtain the pairs $(a, b) = (5t-5, 2), (5t-10, 4), \ldots, (10, 2t-4), (5, 2t-2)$, of which there are $t - 1$.
Since the answer to (b) is 30, then $t = 30$ (which is a positive integer), and so $t - 1 = 29$.
That is, there are 29 pairs of positive integers (a, b) that satisfy the requirements.

Answer: $6, 30, 29$

3. (a) Since $9^3 = 729$ and $10^3 = 1000$, then the largest positive integer a with $a^3 < 999$ is $a = 9$.
Since $2^5 = 32$ and $3^5 = 243$, then the smallest positive integer b with $b^5 > 99$ is $b = 3$.
Therefore, $a - b = 6$.

(b) Suppose that Oscar saw N birds in total.
From the given information, he saw $\frac{3}{5}N$ sparrows and $\frac{1}{4}N$ finches.
Therefore, he saw $N - \frac{3}{5}N - \frac{1}{4}N = N - \frac{12}{20}N - \frac{5}{20}N = \frac{3}{20}N$ cardinals.
Note that Oscar saw $\frac{3}{5}N$ sparrows and $\frac{3}{20}N$ cardinals. Since $\dfrac{3/5}{3/20} = 4$, then he saw four times as many sparrows as cardinals.
Since he saw $10t$ cardinals, then he saw $4 \times 10t = 40t$ sparrows.
Since the answer to (a) is 6, then $t = 6$, and so Oscar saw $40t = 240$ sparrows.

(c) Suppose that there are f seats in the first row.
Since every row after the first contains 4 more seats than the previous, then the remaining 19 rows contain $f + 4, f + 8, f + 12, \ldots, f + 72, f + 76$ seats.
The numbers of seats in the 20 rows form an arithmetic sequence with 20 terms, first term equal to f, and last term equal to $f + 76$.
The total number of seats is equal to the sum of this series, which equals $\frac{20}{2}(f + (f + 76))$ or $10(2f + 76)$.
Since we are told that there are $10t$ seats in total, then $10t = 10(2f + 76)$ or $t = 2f + 76$.
Therefore, $2f = t - 76$ or $f = \frac{1}{2}t - 38$.
Since the answer to (b) is 240, then $f = \frac{1}{2}t - 38 = 120 - 38 = 82$.
In other words, there are 82 seats in the first row.

Answer: $6, 240, 82$