The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015 Pascal Contest

(Grade 9)

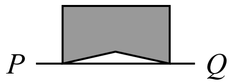
Tuesday, February 24, 2015
(in North America and South America)

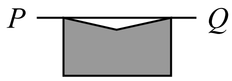
Wednesday, February 25, 2015
(outside of North America and South America)

Solutions

1. Evaluating, $\dfrac{20+15}{30-25} = \dfrac{35}{5} = 7$.

Answer: (D)

2. When the shaded figure in 

is reflected about the line segment PQ, the resulting figure is 

This is because the figure started above the line and so finished below the line, and the figure initially touched the line at two points, so finishes still touching the line at two points.

Answer: (A)

3. Since $8+6 = n+8$, then subtracting 8 from both sides, we obtain $6 = n$ and so n equals 6.
Alternatively, since the order in which we add two numbers does not change the sum, then $8+6 = 6+8$ and so $n = 6$.

Answer: (C)

4. Each of 0.07, -0.41, 0.35, and -0.9 is less than 0.7 (that is, each is to the left of 0.7 on the number line).
The number 0.8 is greater than 0.7.

Answer: (C)

5. Converting from fractions to decimals, $4 + \frac{3}{10} + \frac{9}{1000} = 4 + 0.3 + 0.009 = 4.309$.

Answer: (B)

6. Since the average of the three ages is 22, then the sum of the three ages is $3 \cdot 22 = 66$.
Since Andras' age is 23 and Frances' age is 24, then Gerta's age is $66 - 23 - 24 = 19$.

Answer: (A)

7. When $n = 7$, we have

$$9n = 63 \qquad n+8 = 15 \qquad n^2 = 49 \qquad n(n-2) = 7(5) = 35 \qquad 8n = 56$$

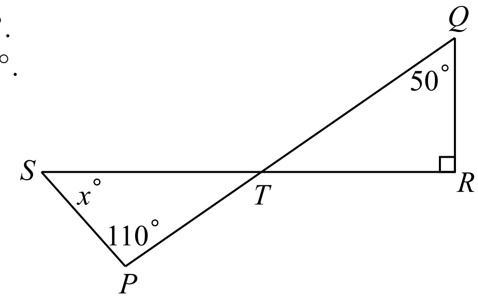
Therefore, $8n$ is even.
We note that for every integer n, the expression $8n$ is equal to an even integer, since 8 is even and the product of an even integer with any integer is even.
If n were even, then in fact all five choices would be even. If n is odd, only $8n$ is even.

Answer: (E)

8. After Jitka hiked 60% of the trail, $100\% - 60\% = 40\%$ of the trail was left.
From the given information, 40% of the length of the trail corresponds to 8 km.
This means that 10% of the trail corresponds to one-quarter of 8 km, or 2 km.
Since 10% of the trail has length 2 km, then the total length of the trail is $10 \cdot 2 = 20$ km.

Answer: (E)

9. Since the sum of the angles in a triangle is 180°, then $\angle QTR = 180^\circ - \angle TQR - \angle QRT = 180^\circ - 50^\circ - 90^\circ = 40^\circ$.
Since opposite angles are equal, then $\angle STP = \angle QTR = 40^\circ$.
Since the sum of the angles in $\triangle STP$ is 180°, then



$$\begin{aligned} \angle PST + \angle SPT + \angle STP &= 180^\circ \\ x^\circ + 110^\circ + 40^\circ &= 180^\circ \\ x + 150 &= 180 \\ x &= 30 \end{aligned}$$

Therefore, the value of x is 30.

ANSWER: (A)

10. Evaluating,

$$\sqrt{16 \times \sqrt{16}} = \sqrt{16 \times 4} = \sqrt{64} = 8$$

Since $8 = 2 \times 2 \times 2 = 2^3$, then $\sqrt{16 \times \sqrt{16}} = 2^3$.

ANSWER: (C)

11. *Solution 1*
The sequence of symbols includes 5 ♡'s and 2 ♠'s.
This means that, each time the sequence is written, there are $5 - 2 = 3$ more ♡'s written than ♠'s.
When the sequence is written 50 times, in total there are $50 \cdot 3 = 150$ more ♡'s written than ♠'s.

Solution 2
The sequence of symbols includes 5 ♡'s and 2 ♠'s.
When the sequence is written 50 times, there will be a total of $50 \cdot 5 = 250$ ♡'s written and a total of $50 \cdot 2 = 100$ ♠'s written.
This means that there are $250 - 100 = 150$ more ♡'s written than ♠'s.

ANSWER: (B)

12. Since 9 is a multiple of 3, then every positive integer that is a multiple of 9 is also a multiple of 3.
Therefore, we can simplify the problem to find the smallest positive integer that is a multiple of each of 5, 7 and 9.
The smallest positive integer that is a multiple of each of 7 and 9 is $7 \cdot 9 = 63$, since 7 and 9 have no common divisor larger than 1. (We could also list the positive multiples of 9 until we found the first one that is also a multiple of 7.)
Thus, the positive integers that are multiples of 7 and 9 are those which are multiples of 63.
We list the multiples of 63 until we find the first one that is divisible by 5 (that is, that ends in a 0 or in a 5):

$$63 \cdot 1 = 63 \qquad 63 \cdot 2 = 126 \qquad 63 \cdot 3 = 189 \qquad 63 \cdot 4 = 252 \qquad 63 \cdot 5 = 315$$

Therefore, the smallest positive integer that is a multiple of each of 3, 5, 7, and 9 is 315.

ANSWER: (D)

13. Each of the squares that has area 400 m^2 has side length $\sqrt{400} = 20$ m.
Anna's path walks along exactly 20 side lengths of squares, so has length $20 \cdot 20 = 400$ m.
Aaron's path walks along exactly 12 side lengths of squares, so has length $12 \cdot 20 = 240$ m.
Therefore, the total distance that they walk is $400 + 240 = 640$ m.

Answer: (E)

14. From the given definition,

$$4 \otimes 8 = \frac{4}{8} + \frac{8}{4} = \frac{1}{2} + 2 = \frac{5}{2}$$

Answer: (E)

15. We make a table of the total amount of money that each of Steve and Wayne have at the end of each year. After the year 2000, each entry in Steve's column is found by doubling the previous entry and each entry in Wayne's column is found by dividing the previous entry by 2. We stop when the entry in Steve's column is larger than that in Wayne's column:

Year	Steve	Wayne
2000	\$100	\$10 000
2001	\$200	\$5000
2002	\$400	\$2500
2003	\$800	\$1250
2004	\$1600	\$625

Therefore, the end of 2004 is the first time at which Steve has more money than Wayne at the end of the year.

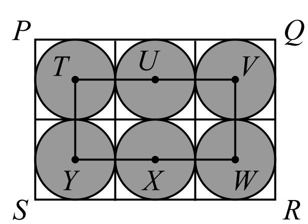
Answer: (C)

16. Since Bruce drove 200 km at a speed of 50 km/h, this took him $\frac{200}{50} = 4$ hours.
Anca drove the same 200 km at a speed of 60 km/h with a stop somewhere along the way.
Since Anca drove 200 km at a speed of 60 km/h, the time that the driving portion of her trip took was $\frac{200}{60} = 3\frac{1}{3}$ hours.
The length of Anca's stop is the difference in driving times, or $4 - 3\frac{1}{3} = \frac{2}{3}$ hours.
Since $\frac{2}{3}$ hours equals 40 minutes, then Anca stopped for 40 minutes.

Answer: (A)

17. Let r be the radius of each of the six circles.
Then $TY = TU = UV = YX = XW = VW = 2r$ and $PQ = SR = 6r$ and $PS = QR = 4r$:

> Since each circle has radius r, then each circle has diameter $2r$, and so can be enclosed in a square with side length $2r$ whose sides are parallel to the sides of rectangle $PQRS$.
> Each circle touches each of the four sides of its enclosing square.
> Because each of the circles touches one or two sides of rectangle $PQRS$ and each of the circles touches one or two of the other circles, then these six squares will fit together without overlapping to completely cover rectangle $PQRS$.



Therefore, $PQ = SR = 6r$ and $PS = QR = 4r$ since rectangle $PQRS$ is three squares wide and two squares tall.
Finally, since the centre of each circle is the centre of its square, then the distance between the centres of each pair of horizontally or vertically neighbouring squares is $2r$ (which is two times half the side length of one of the squares).

Therefore, the perimeter of rectangle $TVWY$ is

$$TV + TY + YW + VW = 2r + 4r + 4r + 2r = 12r$$

Since the perimeter of rectangle $TVWY$ is 60, then $12r = 60$ or $r = 5$.
Since $r = 5$, then in the larger rectangle, we have $PQ = SR = 30$ and $PS = QR = 20$.
Therefore, the area of rectangle $PQRS$ is $PQ \cdot PS = 30 \cdot 20 = 600$.

Answer: (A)

18. In a magic square, the numbers in each row, the numbers in each column, and numbers on each diagonal have the same sum.
Since the sum of the numbers in the first row equals the sum of the numbers in the first column, then $a + 13 + b = a + 19 + 12$ or $b = 19 + 12 - 13 = 18$.
Therefore, the sum of the numbers in any row, in any column, or along either diagonal equals the sum of the numbers in the third column, which is $18 + 11 + 16 = 45$.
Using the first column, $a + 19 + 12 = 45$ or $a = 14$.
Using the second row, $19 + c + 11 = 45$ or $c = 15$.
Thus, $a + b + c = 14 + 18 + 15 = 47$.

Answer: (C)

19. *Solution 1*
We work backwards from the last piece of information given.
Krystyna has 16 raisins left after giving one-half of her remaining raisins to Anna.
This means that she had $2 \cdot 16 = 32$ raisins immediately before giving raisins to Anna.
Immediately before giving raisins to Anna, she ate 4 raisins, which means that she had $32 + 4 = 36$ raisins immediately before eating 4 raisins.
Immediately before eating these raisins, she gave one-third of her raisins to Mike, which would have left her with two-thirds of her original amount.
Since two-thirds of her original amount equals 36 raisins, then one-third equals $\frac{36}{2} = 18$ raisins.
Thus, she gave 18 raisins to Mike and so started with $36 + 18 = 54$ raisins.

Solution 2
Suppose Krystyna starts with x raisins.
She gives $\frac{1}{3}x$ raisins to Mike, leaving her with $x - \frac{1}{3}x = \frac{2}{3}x$ raisins.
She then eats 4 raisins, leaving her with $\frac{2}{3}x - 4$ raisins.
Finally, she gives away one-half of what she has left to Anna, which means that she keeps one-half of what she has left, and so she keeps $\frac{1}{2}\left(\frac{2}{3}x - 4\right)$ raisins.
Simplifying this expression, we obtain $\frac{2}{6}x - \frac{4}{2} = \frac{1}{3}x - 2$ raisins.
Since she has 16 raisins left, then $\frac{1}{3}x - 2 = 16$ and so $\frac{1}{3}x = 18$ or $x = 54$.
Therefore, Krystyna began with 54 raisins.

Answer: (B)

20. Since $\$10 = 2 \cdot \$5$, then \$10 can be formed using 0, 1 or 2 \$5 bills and cannot be formed using more than 2 \$5 bills.

Using 2 \$5 bills, we obtain \$10 exactly. There is no choice in this case, and so this gives exactly 1 way to make \$10.

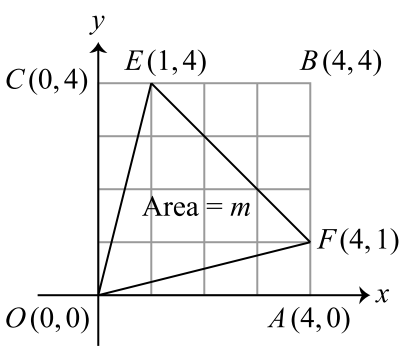
Using 1 \$5 bill, we need an additional \$5.
Since 5 is odd and any amount of dollars made up using \$2 coins will be even, we need an odd number of \$1 coins to make up the difference.
We can use 5 \$1 coins and 0 \$2 coins, or 3 \$1 coins and 1 \$2 coin, or 1 \$1 coin and 2 \$2 coins to obtain \$5.
This is 3 more ways.

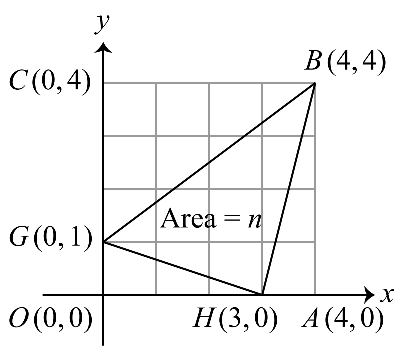
Using 0 \$5 bills, we need an additional \$10.
Since 10 is even and any amount of dollars made up using \$2 coins will be even, we need an even number of \$1 coins to make up the difference.
The numbers of \$1 coins and \$2 coins that we can use are 10 and 0, 8 and 1, 6 and 2, 4 and 3, 2 and 4, or 0 and 5.
This is 6 more ways.

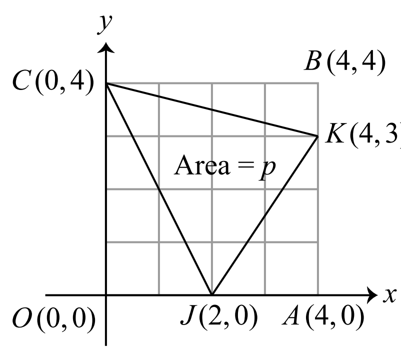
In total, there are $1 + 3 + 6 = 10$ ways in which André can make \$10.

Answer: (A)

21. In each diagram, we label the origin $(0,0)$ as O, the point $(4,0)$ as A, the point $(4,4)$ as B, and the point $(0,4)$ as C.
Thus, in each diagram, square $OABC$ is 4 by 4 and so has area 16.
In the first diagram, we label $(1,4)$ as E and $(4,1)$ as F.
In the second diagram, we label $(0,1)$ as G and $(3,0)$ as H.
In the third diagram, we label $(2,0)$ as J and $(4,3)$ as K.







In the first diagram, the area of $\triangle OEF$ equals the area of square $OABC$ minus the areas of $\triangle OCE$, $\triangle EBF$ and $\triangle FAO$.
Each of these three triangles is right-angled at a corner of the square.
Since $OC = 4$ and $CE = 1$, the area of $\triangle OCE$ is $\frac{1}{2}(4)(1) = 2$.
Since $EB = 3$ and $BF = 3$, the area of $\triangle EBF$ is $\frac{1}{2}(3)(3) = \frac{9}{2}$.
Since $FA = 1$ and $AO = 4$, the area of $\triangle FAO$ is $\frac{1}{2}(1)(4) = 2$.
Therefore, the area of $\triangle OEF$ equals $16 - 2 - \frac{9}{2} - 2 = \frac{15}{2}$, or $m = \frac{15}{2}$.

In the second diagram, the area of $\triangle GBH$ equals the area of square $OABC$ minus the areas of $\triangle GCB$, $\triangle BAH$ and $\triangle HOG$.

Each of these three triangles is right-angled at a corner of the square.
Since $GC = 3$ and $CB = 4$, the area of $\triangle GCB$ is $\frac{1}{2}(3)(4) = 6$.
Since $BA = 4$ and $AH = 1$, the area of $\triangle BAH$ is $\frac{1}{2}(4)(1) = 2$.
Since $HO = 3$ and $OG = 1$, the area of $\triangle HOG$ is $\frac{1}{2}(1)(3) = \frac{3}{2}$.
Therefore, the area of $\triangle HOG$ equals $16 - 6 - 2 - \frac{3}{2} = \frac{13}{2}$, or $n = \frac{13}{2}$.

In the third diagram, the area of $\triangle CKJ$ equals the area of square $OABC$ minus the areas of $\triangle CBK$, $\triangle KAJ$ and $\triangle JOC$.
Each of these three triangles is right-angled at a corner of the square.
Since $CB = 4$ and $BK = 1$, the area of $\triangle CBK$ is $\frac{1}{2}(4)(1) = 2$.
Since $KA = 3$ and $AJ = 2$, the area of $\triangle KAJ$ is $\frac{1}{2}(3)(2) = 3$.
Since $JO = 2$ and $OC = 4$, the area of $\triangle JOC$ is $\frac{1}{2}(2)(4) = 4$.
Therefore, the area of $\triangle CKJ$ equals $16 - 2 - 3 - 4 = 7$, or $p = 7$.

Since $m = \frac{15}{2} = 7\frac{1}{2}$, and $n = \frac{13}{2} = 6\frac{1}{2}$, and $p = 7$, then $n < p < m$.

Answer: (D)

22. *Solution 1*
Let $\$c$ be the cost per square metre of installing carpeting.
Then in each situation, the area of the room times the cost per square metre equals the total price.
From the top left entry in the table, $15 \cdot 10 \cdot \$c = \$397.50$.
From the top right entry in the table, $15 \cdot y \cdot \$c = \$675.75$.
From the bottom left entry in the table, $x \cdot 10 \cdot \$c = \$742.00$.
From the bottom right entry in the table, $x \cdot y \cdot \$c = \$z$.
Now,

$$z = x \cdot y \cdot c = x \cdot y \cdot c \cdot \frac{10 \cdot 15 \cdot c}{10 \cdot 15 \cdot c} = \frac{(x \cdot 10 \cdot c) \cdot (15 \cdot y \cdot c)}{15 \cdot 10 \cdot c} = \frac{(742.00) \cdot (675.75)}{397.50} = 1261.40$$

Therefore, $z = 1261.40$.

Solution 2
Let $\$c$ be the cost per square metre of installing carpeting.
Then in each situation, the area of the room times the cost per square metre equals the total price.
From the top left entry in the table, $15 \cdot 10 \cdot \$c = \$397.50$.
Thus, $c = \dfrac{397.50}{15 \cdot 10} = 2.65$.
From the top right entry in the table, $15 \cdot y \cdot \$c = \$675.75$.
Thus, $y = \dfrac{675.75}{15 \cdot 2.65} = 17$.
From the bottom left entry in the table, $x \cdot 10 \cdot \$c = \$742.00$.
Thus, $x = \dfrac{742.00}{10 \cdot 2.65} = 28$.
From the bottom right entry in the table, $x \cdot y \cdot \$c = \$z$.
Thus, $z = 28 \cdot 17 \cdot 2.65 = 1261.40$.
Therefore, $z = 1261.40$.

Answer: (E)

23. Since the left side of the given equation is a multiple of 6, then the right side, c^2, is also a multiple of 6.
Since c^2 is a multiple of 6, then c^2 is a multiple of 2 and a multiple of 3.
Since 2 and 3 are different prime numbers, then the positive integer c itself must be a multiple of 2 and a multiple of 3. This is because if c is not a multiple of 3, then c^2 cannot be a multiple of 3, and if c is not even, then c^2 cannot be even.
Therefore, c is a multiple of each of 2 and 3, and so is a multiple of 6.
Thus, there are five possible values for c in the given range: 6, 12, 18, 24, 30.
If $c = 6$, then $6ab = 36$ and so $ab = 6$.
Since $1 \leq a < b < 6$ (because $c = 6$), then $a = 2$ and $b = 3$.
If $c = 12$, then $6ab = 144$ and so $ab = 24$.
Since $1 \leq a < b < 12$, then $a = 3$ and $b = 8$ or $a = 4$ and $b = 6$.
(The divisor pairs of 24 are $24 = 1 \cdot 24 = 2 \cdot 12 = 3 \cdot 8 = 4 \cdot 6$. Only the pairs $24 = 3 \cdot 8 = 4 \cdot 6$ give solutions that obey the given restrictions, since in the other two pairs, the larger divisor does not satisfy the restriction of being less than 12.)
If $c = 18$, then $6ab = 324$ and so $ab = 54$.
Since $1 \leq a < b < 18$, then $a = 6$ and $b = 9$.
(The divisor pairs of 54 are $54 = 1 \cdot 54 = 2 \cdot 27 = 3 \cdot 18 = 6 \cdot 9$.)
If $c = 24$, then $6ab = 576$ and so $ab = 96$.
Since $1 \leq a < b < 24$, then $a = 6$ and $b = 16$ or $a = 8$ and $b = 12$.
(The divisor pairs of 96 are $96 = 1 \cdot 96 = 2 \cdot 48 = 3 \cdot 32 = 4 \cdot 24 = 6 \cdot 16 = 8 \cdot 12$.)
If $c = 30$, then $6ab = 900$ and so $ab = 150$.
Since $1 \leq a < b < 30$, then $a = 6$ and $b = 25$ or $a = 10$ and $b = 15$.
(The divisor pairs of 150 are $150 = 1 \cdot 150 = 2 \cdot 75 = 3 \cdot 50 = 5 \cdot 30 = 6 \cdot 25 = 10 \cdot 150$.)
Therefore, the triples (a, b, c) of positive integers that are solutions to the equation $6ab = c^2$ and that satisfy $a < b < c \leq 35$ are

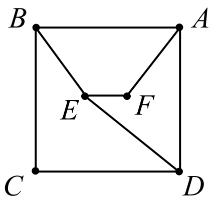
$$(a, b, c) = (2, 3, 6), (3, 8, 12), (4, 6, 12), (6, 9, 18), (6, 16, 24), (8, 12, 24), (6, 25, 30), (10, 15, 30)$$

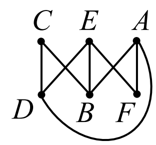
There are 8 such triplets.

Answer: (B)

24. We show that two of the five drawings can represent the given information.
In each drawing, we call each point a *vertex* (points are *vertices*), each line or curve joining two vertices an *edge*, and the number of edges meeting at each vertex the *degree* of the vertex.
We use the labels A, B, C, D, E, and F to represent Ali, Bob, Cai, Dee, Eve, and Fay, respectively.

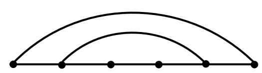
The second and fourth drawings can represent the data using the following labelling:





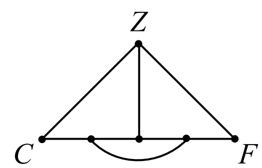
In each case, the 8 links $AB, BC, CD, DE, EF, FA, AD, BE$ are shown by edges and no additional edges are present.
Therefore, these drawings represent the given information.

The first drawing cannot represent the given information as it only includes 7 edges which cannot represent the 8 given links.

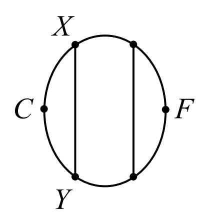


To analyze the third and fifth drawings, we note that there are exactly two suspects (Cai and Fay) who are part of only two links (Cai: BC and CD; Fay: EF and FA).

In the fifth drawing, there are two vertices of degree 2 (that is, at which exactly two edges meet). If this drawing is to represent the given information, these vertices must be C and F in some order. Because the diagram is symmetrical, we can label them as shown without loss of generality.
Consider the vertex labelled Z.
Since it is linked to C, it must be B or D.
But Z is also linked to F, and neither B nor D is linked to F.
Therefore, this drawing cannot represent the given information.



In the third drawing, there are two vertices of degree 2. If this drawing is to represent the given information, these vertices must be C and F in some order. Because the diagram is symmetrical, we can label them as shown without loss of generality.
Consider the vertices labelled X and Y.
Since they are linked to C, they must be B and D in some order.
But X and Y are joined by an edge, while B and D are not linked by the given information.
This means that this drawing cannot represent the given information.



Therefore, two of the five drawings can be labelled to represent the given data.

Answer: (B)

25. We begin by noting that an integer either appears in column V or it appears in one or more of the columns W, X, Y, Z:

    If an integer v appears in column V, then v cannot have appeared in an earlier row in the table. In particular, v cannot have appeared in any of columns W, X, Y, Z earlier in the table.
    Furthermore, v cannot appear again later in the table, since each entry in its row is larger than it and each entry in later rows is again larger (since the entries in V in those later rows will be larger).

    If an integer a appears in one or more of the columns W, X, Y, Z, then it will not appear in column V later in the table, since every entry in V is one that has not yet appeared in the table.
    Furthermore, a cannot have appeared in V earlier in the table since every entry in V before a appears in W, X, Y, or Z will be smaller than a.

    Therefore, an integer that appears in the table either appears in column V or it appears in one or more of the columns W, X, Y, Z.

    If an integer v has not appeared in the table, then it will eventually be the smallest positive integer that has not appeared in the table so far, and so v will appear in column V in the next row.

    We check to see if 2731 appears in column Z.
    Since $2731 = 7(390) + 1$, then 2731 appears in Z if 390 appears in V.
    Since 389 is not a multiple of 2, 3, 5 or 7 (we can check by dividing by each of these), then 390 cannot appear in W, X, Y, Z (because 390 cannot be written in the form $2n+1$, $3n+1$, $5n+1$, or $7n+1$ for some positive integer n).
    This means that 390 appears in V and so $2731 = 7(390) + 1$ appears in Z in this row.
    This eliminates answer (A).
    We note also that, since 2731 appears in Z, it cannot appear in V.

    Next, we show that 2731 appears in column Y.
    Since $2731 = 5(546) + 1$, then 2731 appears in Y if 546 appears in V.
    We now show that 546 does appear in V.
    Since 545 is a multiple of 5, but not of 2, 3 or 7, then either 546 appears in column V or in column Y, as explained above.
    We will show that 546 does not appear in Y.
    Since $546 = 5(109) + 1$, then 546 appears in Y if 109 appears in V.
    We will show that 109 does not appear in V.
    Since 108 is a multiple of 2 and 3, but not of 5 or 7, then 109 could appear in V, W or X.
    We will show that 109 appears in X.
    Since $109 = 3(36) + 1$, then 109 appears in X if 36 appears in V.
    We will show that 36 appears in V.
    Since 35 is a multiple of 5 and 7, but not of 2 or 3, then 36 could appear in V, Y or Z.
    If 36 appears in Z, then the V entry in this row would be 5.
    If 36 appears in Y, then the V entry in this row would be 7.
    Neither 5 nor 7 is in column V. (Each appears in the second row.)
    Therefore, 36 appears in V, which means that 109 appears in X.
    This means that 109 does not appear in V and so 546 does not appear in Y.
    Since 546 is in V or Y, then 546 is in V, which means that 2731 is in Y.
    This eliminates answers (C) and (E).

Lastly, we show that 2731 does not appear in column X.
Since 29 is not a multiple of 2, 3, 5 or 7, then 30 must appear in V.
Since 30 appears in V, then $151 = 5(30) + 1$ appears in Y.
Since 151 appears in Y, it does not appear in V.
Since 151 does not appear in V, then $303 = 2(151) + 1$ does not appear in W.
Since 303 does not appear in W, then 303 appears in V. (This is because 302 is not a multiple of 3, 5 or 7 and so 303 cannot appear in X, Y or Z.)
Since 303 appears in V, then $910 = 3(303) + 1$ appears in X.
Since 910 appears in X, then 910 does not appear in V.
Since 910 does not appear in V, then $2731 = 3(910) + 1$ does not appear in X.

Therefore, 2731 appears in W, Y and Z.

Answer: (D)