The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2015 Fermat Contest

(Grade 11)

Tuesday, February 24, 2015
(in North America and South America)

Wednesday, February 25, 2015
(outside of North America and South America)

Solutions

1. The average of the five given numbers is $\frac{8+9+10+11+12}{5} = \frac{50}{5} = 10$.
   Alternatively, since there are an odd number of consecutive integers, the average is the middle number, which is 10.

   Answer: (E)

2. Evaluating, $\frac{2 \times 3 + 4}{2+3} = \frac{6+4}{5} = \frac{10}{5} = 2$.

   Answer: (A)

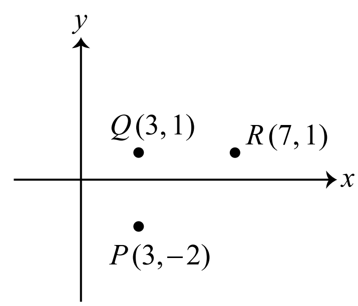
3. Suppose that the distance between consecutive points on the path is d.
   Then, walking from P to U, Emily walks a total distance of $5d$.
   Walking back from U to P, she also walks a distance of $5d$, for a total distance walked of $10d$.
   Since 70% of 10 is 7, then she has completed 70% of her walk after walking a distance of $7d$ (that is, after having walked 7 segments).
   She has walked a distance of $7d$ after walking a distance of $2d$ back from U, which gets her to S.

   Answer: (D)

4. Evaluating, $(x-3)^2 = (-3-3)^2 = (-6)^2 = 36$.

   Answer: (B)

5. Based on the diagram shown, the order of the vertices of the rectangle must be $PQRS$. (In any other configuration, either $\angle QRP$ or $\angle QPR$ would be an angle in the rectangle, which is not possible.)

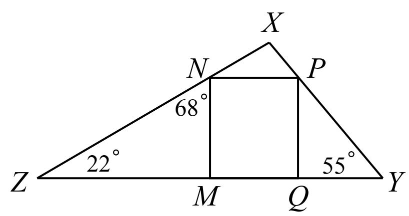
   

   Since P and Q have the same x-coordinate, then side PQ of the rectangle is vertical.
   This means that side SR must also be vertical, and so the x-coordinate of S is the same as the x-coordinate of R, which is 7.
   Since Q and R have the same y-coordinate, then side QR of the rectangle is horizontal.
   This means that side PS must also be horizontal, and so the y-coordinate of S is the same as the y-coordinate of P, which is -2.
   Thus, the coordinates of S are $(7, -2)$.

   Answer: (B)

6. Since $MNPQ$ is a rectangle, then $\angle NMQ$ is a right angle and thus so is $\angle NMZ$.
   The sum of the angles in $\triangle NMZ$ is 180°, so

$$\angle NZM = 180^\circ - \angle ZNM - \angle ZMN = 180^\circ - 68^\circ - 90^\circ = 22^\circ$$



   Since the sum of the angles in $\triangle ZXY$ is also 180°, then

$$\angle YXZ = 180^\circ - \angle XZY - \angle XYZ = 180^\circ - 22^\circ - 55^\circ = 103^\circ$$

   Answer: (E)

7. Violet starts with one-half of the money that she needed to buy the necklace.
After her sister gives her money, she has three-quarters of the amount that she needs.
This means that her sister gave her $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ of the total amount that she needs.
Since she now has three-quarters of the amount that she needs, then she still needs one-quarter of the total cost.
In other words, her father will give her the same amount that her sister gave her, or \$30.

Answer: (D)

8. Since $15^2 = 225$ and $15 = 3 \cdot 5$, then $225 = 15^2 = (3 \cdot 5)^2 = 3^2 \cdot 5^2$.
Therefore, $x = 2$ and $y = 2$, so $x + y = 4$.

Answer: (B)

9. *Solution 1*
The two teams include a total of $25 + 19 = 44$ players.
There are exactly 36 students who are at least one team.
Thus, there are $44 - 36 = 8$ students who are counted twice.
Therefore, there are 8 students who play both baseball and hockey.

Solution 2
Suppose that there are x students who play both baseball and hockey.
Since there are 25 students who play baseball, then $25-x$ of these play baseball and not hockey.
Since there are 19 students who play hockey, then $19-x$ of these play hockey and not baseball.
Since 36 students play either baseball or hockey or both, then

$$(25 - x) + (19 - x) + x = 36$$

(The left side is the sum of the numbers of those who play baseball and not hockey, those who play hockey and not baseball, and those who play both.)
Therefore, $44 - x = 36$ and so $x = 44 - 36 = 8$.
Thus, 8 students play both baseball and hockey.

Answer: (B)

10. Since Bruce drove 200 km at a speed of 50 km/h, this took him $\frac{200}{50} = 4$ hours.
Anca drove the same 200 km at a speed of 60 km/h with a stop somewhere along the way.
Since Anca drove 200 km at a speed of 60 km/h, the time that the driving portion of her trip took was $\frac{200}{60} = 3\frac{1}{3}$ hours.
The length of Anca's stop is the difference in driving times, or $4 - 3\frac{1}{3} = \frac{2}{3}$ hours.
Since $\frac{2}{3}$ hours equals 40 minutes, then Anca stops for 40 minutes.

Answer: (A)

11. For each of the three digits of such a positive integer, there are three choices of digit (7, 8 or 9).
Therefore, there are $3 \cdot 3 \cdot 3 = 27$ possible integers that use no digits other than 7, 8 or 9.
(We note that there are 9 such integers beginning with each of 7, 8 and 9.
The 9 such integers beginning with 7 are 777, 778, 779, 787, 788, 789, 797, 798, 799.)

Answer: (E)

12. Since $\cos 60^\circ = \frac{1}{2}$ and $\cos 45^\circ = \frac{1}{\sqrt{2}}$, then the given equation $\cos 60^\circ = \cos 45^\circ \cos\theta$ becomes $\frac{1}{2} = \frac{1}{\sqrt{2}}\cos\theta$.
Therefore, $\cos\theta = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.
Since $0^\circ \leq \theta \leq 90^\circ$, then $\theta = 45^\circ$.

ANSWER: (D)

13. We make a table of the total amount of money that each of Steve and Wayne have at the end of each year. After the year 2000, each entry in Steve's column is found by doubling the previous entry and each entry in Wayne's column is found by dividing the previous entry by 2. We stop when the entry in Steve's column is larger than that in Wayne's column:

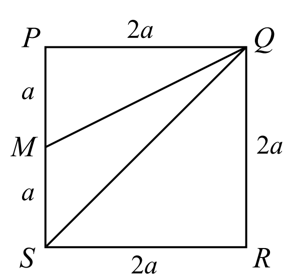
Year	Steve	Wayne
2000	\$100	\$10 000
2001	\$200	\$5000
2002	\$400	\$2500
2003	\$800	\$1250
2004	\$1600	\$625

Therefore, 2004 is the first time at which Steve has more money than Wayne at the end of the year.

ANSWER: (C)

14. *Solution 1*
Since $PQRS$ is a square, then its diagonal SQ cuts it into two equal areas.
Therefore, the ratio of the area of $\triangle PQS$ to the area of square $PQRS$ is $1 : 2$.
$\triangle PQS$ can be viewed as having base PS and height PQ.
$\triangle MQS$ can be viewed as having base MS and height PQ. (This is because PQ is perpendicular to the line containing MS.)
Since $MS = \frac{1}{2}PS$, then the area of $\triangle MQS$ is one-half of the area of $\triangle PQS$.
Since the ratio of the area of $\triangle PQS$ to the area of square $PQRS$ is $1 : 2$, then the ratio of the area of $\triangle QMS$ to the area of square $PQRS$ is $1 : 4$.

Solution 2
Suppose that the side length of square $PQRS$ is $2a$.
Then the area of square $PQRS$ is $(2a)^2 = 4a^2$.
Since M is the midpoint of side PS, then $PM = MS = a$.
Then $\triangle QMS$ can be seen as having base MS and height PQ. (This is because PQ is perpendicular to the line containing MS.)
Since $MS = a$ and $PQ = 2a$, then the area of $\triangle QMS$ is $\frac{1}{2}(MS)(PQ) = \frac{1}{2}a(2a) = a^2$.
Therefore, the ratio of the area of $\triangle QMS$ to the area of square $PQRS$ is $a^2 : 4a^2$ which equals $1 : 4$.



ANSWER: (B)

15. *Solution 1*

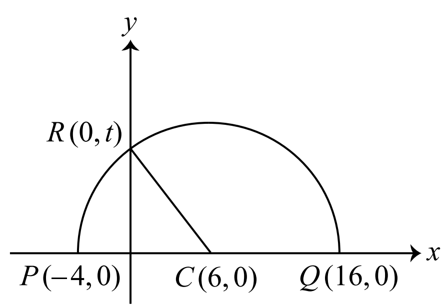
Zoltan answered 45 questions.
If all of his answers had been correct, his score would have been $45(4) = 180$ points.
Since his score was 135 points, then he lost $180 - 135 = 45$ points.
For each incorrect answer, he lost 5 points compared to a correct answer, since a correct answer adds 4 points and an incorrect answer subtracts 1 point.
Therefore, Zoltan had $45 \div 5 = 9$ incorrect answers.
(We can check that 36 correct, 9 incorrect, and 5 unanswered gives a score of $4(36)-1(9)+0(5)$ or 135 points.)

Solution 2

Suppose that Zoltan answered x questions incorrectly.
Since he answered 45 questions in total, then he answered $45 - x$ questions correctly.
Since the test included 50 questions and he answered 45, then he did not answer 5 questions.
Using the marking scheme, his score was $4(45 - x) - 1(x) + 0(5)$.
We are told that his score was 135 points.
Hence, $4(45 - x) - 1(x) + 0(5) = 135$ and so $180 - 4x - x = 135$.
Thus, $5x = 45$ or $x = 9$.
Therefore, Zoltan answered 9 questions incorrectly.
(We can check that 36 correct, 9 incorrect, and 5 unanswered gives a score of $4(36)-1(9)+0(5)$ or 135 points.)

ANSWER: (A)

16. Since $P(-4, 0)$ and $Q(16, 0)$ are endpoints of a diameter of the semi-circle, then the length of the diameter is $16 - (-4) = 20$.
Since a diameter of the semi-circle has length 20, then the radius of the semi-circle is $\frac{1}{2}(20) = 10$.
Also, the centre C is the midpoint of diameter PQ and so has coordinates $\left(\frac{1}{2}(-4+16), \frac{1}{2}(0+0)\right)$ or $(6, 0)$.



Now the distance between $C(6, 0)$ and $R(0, t)$ is 10, since CR is a radius.
Therefore,

$$\begin{aligned}\sqrt{(6-0)^2 + (0-t)^2} &= 10 \\ 36 + t^2 &= 100 \\ t^2 &= 64\end{aligned}$$

(Alternatively, we could have noted that if O is the origin, then $\triangle ROC$ is right-angled with $RO = t$, $RC = 10$ and $OC = 6$ and then used the Pythagorean Theorem to obtain $t^2 + 6^2 = 10^2$, which gives $t^2 = 64$.)
Since $t > 0$, then $t = 8$.

ANSWER: (C)

17. Since $\frac{a+b}{a-b}=3$, then $a+b=3(a-b)$ or $a+b=3a-3b$.
Thus, $4b=2a$ and so $2b=a$ or $2=\frac{a}{b}$.
(Note that $b\neq 0$, since otherwise the original equation would become $\frac{a}{a}=3$, which is not true.)

ANSWER: (D)

18. The equation $x^2+2kx+7k-10=0$ has two equal real roots precisely when the discriminant of this quadratic equation equals 0.
The discriminant, Δ, equals

$$\Delta=(2k)^2-4(1)(7k-10)=4k^2-28k+40$$

For the discriminant to equal 0, we have $4k^2-28k+40=0$ or $k^2-7k+10=0$ or $(k-2)(k-5)=0$.
Thus, $k=2$ or $k=5$.
We check that each of these values gives an equation with the desired property.
When $k=2$, the equation is $x^2+4x+4=0$ which is equivalent to $(x+2)^2=0$ and so only has one solution for x.
When $k=5$, the equation is $x^2+10x+25=0$ which is equivalent to $(x+5)^2=0$ and so only has one solution for x.
The sum of these values of k is $2+5=7$.

ANSWER: (E)

19. Suppose the slope of the three parallel lines is m.
The equation of a line with slope m and y-intercept 2 is $y=mx+2$.
To find the x-intercept in terms of m, we set $y=0$ and solve for x.
Doing this, we obtain $mx+2=0$ or $x=-\frac{2}{m}$.
Similarly, the line with slope m and y-intercept 3 has x-intercept $-\frac{3}{m}$.
Also, the line with slope m and y-intercept 4 has x-intercept $-\frac{4}{m}$.
Since the sum of the x-intercepts of these lines is 36, then $\left(-\frac{2}{m}\right)+\left(-\frac{3}{m}\right)+\left(-\frac{4}{m}\right)=36$.
Multiplying both sides by m, we obtain $-2-3-4=36m$ and so $36m=-9$ or $m=-\frac{1}{4}$.

ANSWER: (E)

20. First, we factor $a^{2014}+a^{2015}$ as $a^{2014}(1+a)$.
If $a=5$ or $a=10$, then the factor a^{2014} is a multiple of 5, so the original expression is divisible by 5.
If $a=4$ or $a=9$, then the factor $(1+a)$ is a multiple of 5, so the original expression is divisible by 5.
If $a=1,2,3,6,7,8$, then neither a^{2014} nor $(1+a)$ is a multiple of 5.
Since neither factor is a multiple of 5, which is a prime number, then the product $a^{2014}(1+a)$ is not divisible by 5.
Therefore, there are four integers a in the range $1\leq a\leq 10$ for which $a^{2014}+a^{2015}$ is divisible by 5.

ANSWER: (C)

21. If Amina wins, she can win on her first turn, on her second turn, or on her third turn.
If she wins on her first turn, then she went first and tossed tails.
This occurs with probability $\frac{1}{2}$.
If she wins on her second turn, then she tossed heads, then Bert tossed heads, then Amina tossed tails. This gives the sequence HHT. The probability of this sequence of tosses occurring is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$. (Note that there is only one possible sequence of Ts and Hs for which Amina wins on her second turn, and the probability of a specific toss on any turn is $\frac{1}{2}$.)
Similarly, if Amina wins on her third turn, then the sequence of tosses that must have occurred is HHHHT, which has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.
Therefore, the probability that Amina wins is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} = \frac{16+4+1}{32} = \frac{21}{32}$.

Answer: (A)

22. Since a, b and c form an arithmetic sequence in this order, then $a = b - d$ and $c = b + d$ for some real number d.
We note that $d \neq 0$, since otherwise we would have $a = b = c$ and then $abc = 17\,955$ would tell us that $b^3 = 17\,955$ or $b = \sqrt[3]{17\,955}$, which is not an integer.
Writing the terms of the geometric sequence in terms of b and d, we have

$$3a+b = 3(b-d)+b = 4b-3d \qquad 3b+c = 3b+(b+d) = 4b+d \qquad 3c+a = 3(b+d)+(b-d) = 4b+2d$$

Since $3a + b$, $3b + c$ and $3c + a$ form a geometric sequence in this order, then

$$\begin{aligned}
\frac{3b+c}{3a+b} &= \frac{3c+a}{3b+c} \\
(3b+c)^2 &= (3a+b)(3c+a) \\
(4b+d)^2 &= (4b-3d)(4b+2d) \\
16b^2 + 8bd + d^2 &= 16b^2 - 4bd - 6d^2 \\
12bd &= -7d^2 \\
12b &= -7d \qquad (\text{since } d \neq 0) \\
d &= -\tfrac{12}{7}b
\end{aligned}$$

Therefore, $a = b - d = b - (-\frac{12}{7}b) = \frac{19}{7}b$ and $c = b + d = b + (-\frac{12}{7}b) = -\frac{5}{7}b$.
Since $abc = 17\,955$, then $(\frac{19}{7}b)(b)(-\frac{5}{7}b) = 17\,955$ or $-\frac{95}{49}b^3 = 17\,955$ or $b^3 = -9261$ and so $b = -21$.
Thus, $a = \frac{19}{7}b = \frac{19}{7}(-21) = -57$ and $c = -\frac{5}{7}b = -\frac{5}{7}(-21) = 15$.
We can check that $a = -57$, $b = -21$ and $c = 15$ have a product of $17\,955$, that $-57, -21, 15$ is indeed an arithmetic sequence (with common difference 36), and that $3a + b = -192$, $3b + c = -48$, and $3c + a = -12$ form a geometric sequence (with common ratio $\frac{1}{4}$).
Therefore, $a + b + c = (-57) + (-21) + 15 = -63$.

Answer: (A)

23. Starting from the given equation, we obtain the equivalent equations

$$\begin{aligned} 5x^2 - 4xy + 2x + y^2 &= 624 \\ 5x^2 - 4xy + 2x + y^2 + 1 &= 625 \\ 4x^2 - 4xy + y^2 + x^2 + 2x + 1 &= 625 \\ (2x - y)^2 + (x + 1)^2 &= 625 \end{aligned}$$

Note that $625 = 25^2$.
Since x and y are both integers, then the left side of the given equation is the sum of two perfect squares. Since any perfect square is non-negative, then each of these perfect squares is at most $625 = 25^2$.
The perfect squares from 0^2 to 25^2 are:

$$0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144,$$

$$169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625$$

The pairs of perfect squares from this list that have a sum of 625 are

$$625 = 625 + 0 = 576 + 49 = 400 + 225$$

(We can verify this relatively quickly by checking the difference between 625 and each of the perfect squares in the list from 324 to 625 to see if this difference is itself a perfect square. We do not need to check anything less than 324, since 625 is odd and so one of the two squares adding to 625 must be larger than the other and hence larger than half of 625.)
Therefore, $(2x - y)^2$ and $(x + 1)^2$ equal 25^2 and 0^2 in some order, or 24^2 and 7^2 in some order, or 20^2 and 15^2 in some order.
Thus, $2x - y$ and $x + 1$ equal ± 25 and 0 in some order, or ± 24 and ± 7 in some order, or ± 20 and ± 15 in some order.
Since $x \geq 0$, then $x + 1 \geq 1$, so we need to consider the possibilities that $x + 1 = 25, 24, 7, 20, 15$:

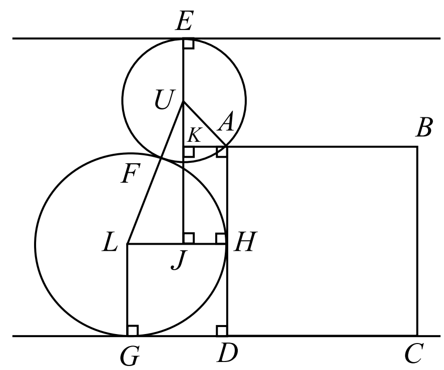
- If $x + 1 = 25$, then $x = 24$. If $2x - y = 0$ and $x = 24$, then $y = 48$.
- If $x + 1 = 24$, then $x = 23$. If $2x - y = 7$ and $x = 23$, then $y = 39$; if $2x - y = -7$ and $x = 23$, then $y = 53$.
- If $x + 1 = 7$, then $x = 6$. If $2x - y = 24$ and $x = 6$, then $y = -12$; if $2x - y = -24$ and $x = 6$, then $y = 36$.
- If $x + 1 = 20$, then $x = 19$. If $2x - y = 15$ and $x = 19$, then $y = 23$; if $2x - y = -15$ and $x = 19$, then $y = 53$.
- If $x + 1 = 15$, then $x = 14$. If $2x - y = 20$ and $x = 14$, then $y = 8$; if $2x - y = -20$ and $x = 14$, then $y = 48$.

From this list, the pairs of non-negative integers (x, y) that satisfy the condition $0 \leq x \leq y$ are $(x, y) = (24, 48), (23, 39), (23, 53), (6, 36), (19, 23), (19, 53), (14, 48)$.
There are 7 such pairs. (We can check by direct substitution that each pair satisfies the original equation.)

Answer: (E)

24. Let r be the radius of the lower circle.
We label the square as $ABCD$, the centre of the upper circle as U, and the centre of the lower circle as L. We call E the point at which the upper circle touches the top line, G the point at which the lower circle touches the bottom line, F the point at which the two circles touch, and H the point at which the lower circle touches the square.
We join EU, UF, FL, LG, LH, and UA.
We need to use two facts about circles:

- When the centre of a circle is joined to a point of tangency, the resulting line segment is perpendicular to the tangent line. Thus, UE is perpendicular to the top line, LG is perpendicular to the bottom line, and LH is perpendicular to AD.
- When two circles are tangent, the line segment joining their centres passes through the point of tangency between the circles. Thus, UFL is a straight line segment.



Extend EU downwards to meet LH at J. Since EU and AD are perpendicular to the parallel lines and LH is perpendicular to AD, then EJ is perpendicular to LH.
Extend BA to meet EJ at K. Using a similar argument, AK is perpendicular to EJ.
We focus on $\triangle UJL$, which is right-angled at J.
Since the radius of the upper circle is 65, then we know that $EU = UA = UF = 65$.
Since the radius of the lower circle is r, then we know that $FL = LH = LG = r$.
Thus, $UL = UF + FL = 65 + r = r + 65$.
Since the top and bottom lines are parallel, EJ and LG are perpendicular to these lines, and LJ is parallel to these lines, then $EU + UJ + LG$ equals the distance between the lines.
Thus, $65 + UJ + r = 400$ and so $UJ = 335 - r$.
Next, we note that $LJ = LH - JH = r - JH = r - AK$, since $AKJH$ is a rectangle.
Now $\triangle UKA$ is right-angled and has $UA = 65$.
Also, $UK = EK - EU = EK - 65$.
But EK equals the difference between the distance between the lines and the side length of the square, or $400 - 279 = 121$.
Thus, $UK = 121 - 65 = 56$.
Using the Pythagorean Theorem in $\triangle UKA$, we obtain $AK^2 = UA^2 - UK^2 = 65^2 - 56^2$ and so $AK^2 = 1089 = 33^2$.
Since $AK > 0$, then $AK = 33$.
Thus, $LJ = r - 33$.
Finally, using the Pythagorean Theorem in $\triangle UJL$, we obtain

$$\begin{aligned} UJ^2 + LJ^2 &= UL^2 \\ (335 - r)^2 + (r - 33)^2 &= (r + 65)^2 \\ r^2 - 670r + 335^2 + r^2 - 66r + 33^2 &= r^2 + 130r + 65^2 \\ r^2 - 866r + 109089 &= 0 \end{aligned}$$

By the quadratic formula, $r = \dfrac{866 \pm \sqrt{866^2 - 4(1)(109089)}}{2} = \dfrac{866 \pm 560}{2} = 153$ or 713.
Since r must be less than the distance between the two lines, which is 400, then $r = 153$. Of the given answers, this is closest to 153.

Answer: (C)

25. This solution is written in as mathematically complete a way as possible. Of course, while doing a multiple choice contest, those who attempt and/or complete this problem would be unlikely to work through all of these details.

Step 1: Using information about decimal equivalents to rewrite fractions
Consider a real number x whose decimal equivalent is of the form $0.g_1g_2\dots g_p\overline{r_1r_2\dots r_q}$ for some integers $p \geq 0$ and $q > 0$ and digits $g_1, g_2, \dots, g_p, r_1, r_2, \dots, r_q$.
(Note that if $p = 0$, then $0.g_1g_2\dots g_p\overline{r_1r_2\dots r_q} = 0.\overline{r_1r_2\dots r_q}$.)
Then $x = \dfrac{c}{10^p(10^q - 1)}$ for some positive integer c.
We demonstrate this in a specific example and leave the full algebraic derivation to the end of this solution. If $x = 0.12\overline{745}$, then

$$\begin{aligned}
x &= 0.12\overline{745} \\
100x &= 12.\overline{745} \\
10^2x - 12 &= 0.\overline{745} \\
1000(10^2x - 12) &= 745.\overline{745} \\
10^3(10^2x - 12) - 745 &= 0.\overline{745} \\
10^3(10^2x - 12) - 745 &= 10^2x - 12 \\
10^3(10^2x - 12) - (10^2x - 12) &= 745 \\
(10^3 - 1)(10^2x - 12) &= 745 \\
10^2x - 12 &= \frac{745}{10^3 - 1} \\
10^2x &= 12 + \frac{745}{10^3 - 1} \\
x &= \frac{12}{10^2} + \frac{745}{10^2(10^3 - 1)} \\
x &= \frac{(10^3 - 1)12 + 745}{10^2(10^3 - 1)}
\end{aligned}$$

Suppose that a fraction $\dfrac{m}{n}$ has m and n positive integers with $m < n$. Then $0 < \dfrac{m}{n} < 1$.
Suppose now that a fraction $\dfrac{m}{n}$ of positive integers with $0 < \dfrac{m}{n} < 1$ has the property that there is a sequence of consecutive digits of length 6 in its decimal equivalent that repeats consecutively and indefinitely. That is, suppose that $\dfrac{m}{n} = 0.g_1g_2\dots g_p\overline{r_1r_2\dots r_6}$ for some integer $p \geq 0$ and digits $g_1, g_2, \dots, g_p, r_1, r_2, r_3, r_4, r_5, r_6$.
From above, $\dfrac{m}{n} = \dfrac{c}{10^p \cdot 999999}$ for some positive integer c. (Note that $10^6 - 1 = 999999$.)

Step 2: Using further conditions given to analyze n
Continuing from above, $cn = 10^p \cdot 999999m$.
Since we are also told that $\dfrac{m}{n}$ is in lowest terms, then m and n have no common divisors larger than 1, and so n must be a divisor of $10^p \cdot 999999$.
Note that $10^p \cdot 999999 = 2^p \cdot 5^p \cdot 999 \cdot 1001 = 2^p \cdot 5^p \cdot (3^3 \cdot 37) \cdot (11 \cdot 91) = 2^p \cdot 5^p \cdot 3^3 \cdot 37 \cdot 11 \cdot 7 \cdot 13$.
Since n is a divisor of $2^p \cdot 5^p \cdot 3^3 \cdot 37 \cdot 11 \cdot 7 \cdot 13$, then n cannot contain any prime factors other than $2, 3, 5, 7, 11, 13, 37$.
Since n is not divisible by the square of any positive integer, then it cannot be divisible by the

square of any prime number.
Thus, n must be a divisor of $2 \cdot 5 \cdot 3 \cdot 37 \cdot 11 \cdot 7 \cdot 13 = 1111110$.

Step 3: Consolidating current progress
We now know that any fraction $\frac{m}{n}$ satisfying the properties

- m and n are positive integers with $m < n$,
- $\frac{m}{n}$ is in lowest terms,
- n is not divisible by the square of any integer larger than 1, and
- the decimal equivalent of $\frac{m}{n}$ includes a sequence of consecutive digits of length 6 that repeats consecutively and indefinitely,

can be written in the form $\frac{m}{n} = \frac{s}{1111110}$ for some positive integer s with $1 \leq s \leq 1111109$.
(We note that we have not yet determined whether the *shortest* sequence of consecutive digits that repeats consecutively and indefinitely has length 6.)

Step 4: Every fraction $\frac{s}{1111110}$ *with* $1 \leq s \leq 1111109$ *can be written as a fraction satisfying these four bullets*
Each $\frac{s}{1111110}$ is between 0 and 1, can be written in lowest terms and has denominator not divisible by the square of any positive integer larger than 1, so any equivalent fraction in lower (or lowest terms) shares this property as factors of the denominator will only be removed in reducing, not added.
Furthermore,

$$\frac{s}{1111110} = \frac{1}{10} \cdot \frac{9s}{999999} = \frac{1}{10}\left(y + \frac{z}{999999}\right)$$

for some non-negative integers y and z with $0 \leq y < 10$ and $0 \leq z < 999998$. (y and z are the quotient and remainder, respectively, when $9s$ is divided by 999999.)
Writing $z = r_1r_2r_3r_4r_5r_6$ for some digits $r_1, r_2, r_3, r_4, r_5, r_6$ (some or all possibly 0), then

$$\frac{s}{1111110} = \frac{1}{10}\left(y + \frac{r_1r_2r_3r_4r_5r_6}{999999}\right) = \frac{1}{10}\left(y.\overline{r_1r_2r_3r_4r_5r_6}\right) = 0.y\overline{r_1r_2r_3r_4r_5r_6}$$

so every $\frac{s}{1111110}$ can be written as a decimal with a repeating sequence of length 6.
Also, each $\frac{s}{1111110}$ is different and so will produce a different $\frac{m}{n}$.
Therefore, the number of such fractions $\frac{s}{1111110}$ (which is 1111109) will equal the number of fractions $\frac{m}{n}$ that satisfy the four bullets above.
Notice that we have not yet checked to see if the sequence of digits of length 6 is the *shortest* such sequence.

Step 5: Considering shorter possible lengths
Since 6 is to be the length of the shortest sequence of repeating digits after the decimal point, then there can be no sequence of repeating digits of length 1, 2, 3, 4, or 5.
Using a similar approach to the first derivation above, we see that $\frac{m}{n}$ thus cannot be written

in any of the forms $\dfrac{c}{10^p \cdot 9}$ or $\dfrac{c}{10^p \cdot 99}$ or $\dfrac{c}{10^p \cdot 999}$ or $\dfrac{c}{10^p \cdot 9999}$ or $\dfrac{c}{10^p \cdot 99999}$.

Using a similar approach to the analysis of prime factors above, we see that $\dfrac{m}{n}$ cannot be written in any of the forms $\dfrac{t}{2 \cdot 5 \cdot 3} = \dfrac{t}{30}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 11} = \dfrac{t}{330}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 37} = \dfrac{t}{1110}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 11 \cdot 101} = \dfrac{t}{33330}$ or $\dfrac{t}{2 \cdot 5 \cdot 3 \cdot 41 \cdot 271} = \dfrac{t}{333330}$.

It is possible that an $\dfrac{m}{n}$ with properties as above including a repeating sequence of length 6 in its decimal equivalent can also be written with a repeating sequence of length 1, 2 or 3.

This is because, for example,

$$0.\overline{r_1} = 0.\overline{r_1 r_1 r_1 r_1 r_1 r_1} \qquad 0.\overline{r_1 r_2} = 0.\overline{r_1 r_2 r_1 r_2 r_1 r_2} \qquad 0.\overline{r_1 r_2 r_3} = 0.\overline{r_1 r_2 r_3 r_1 r_2 r_3}$$

which all have sequences of length 6 that repeat.

It is not possible for an $\dfrac{m}{n}$ with a repeating sequence of length 6 in its decimal equivalent to also be written with a repeating sequence of length 4 or 5, without a repeating sequence of length 1 or 2.

This is because if, for example, there is a repeating sequence of length 4, then

$$\frac{s}{1111110} = \frac{s}{2 \cdot 5 \cdot 3 \cdot 37 \cdot 11 \cdot 7 \cdot 13} = \frac{t}{2 \cdot 5 \cdot 3 \cdot 11 \cdot 101}$$

for some positive integer t and so $2 \cdot 5 \cdot 3 \cdot 11 \cdot 101 \cdot s = 2 \cdot 5 \cdot 3 \cdot 37 \cdot 11 \cdot 7 \cdot 13 \cdot t$ and so $101 \cdot s = 37 \cdot 7 \cdot 13 \cdot t$ which tells us that 101 is divisor of t and so $\dfrac{s}{1111110} = \dfrac{t'}{2 \cdot 5 \cdot 3 \cdot 11} = \dfrac{t'}{330}$ for some positive integer t'.

Therefore, the decimal equivalent to $\dfrac{s}{1111110} = \dfrac{t'}{330}$ has a sequence of length 2 that repeats, so any fraction with a decimal equivalent that has a repeating sequence of length 4 will be dealt with among those with sequences of length 1, 2 or 3.

In a similar way, we can rule out decimal equivalents with sequences of length 5 that repeat.

Therefore, we need to (carefully) remove fractions from our count that have decimal equivalents with sequences of lengths 1, 2 or 3 that repeat.

Step 6: Considering overlap

We have 1111109 fractions $\dfrac{m}{n}$ to consider, each of which satisfies the four bullets above.

Since the decimal equivalent of $\dfrac{m}{n}$ cannot have sequences of lengths 1, 2 or 3 that repeat, then $\dfrac{m}{n}$ cannot be written in any of the forms $\dfrac{u}{1110}$ or $\dfrac{v}{330}$ or $\dfrac{w}{30}$ for positive integers u, v and w with $u < 1110$, $v < 330$, and $w < 30$.

Let U be the set of the 1009 fractions of the form $\dfrac{u}{1110}$, V be the set of the 329 fractions of the form $\dfrac{v}{330}$, and W be the set of the 29 fractions of the form $\dfrac{w}{30}$.

Any fraction of the form $\dfrac{w}{30}$ is also of the form $\dfrac{u}{1110}$ (since $\dfrac{w}{30} = \dfrac{37w}{1110}$) and is also of the form $\dfrac{v}{330}$ (since $\dfrac{w}{30} = \dfrac{11w}{330}$).

Therefore, every fraction in W is also in U and in V. In set notation $W \subseteq U$ and $W \subseteq V$.

Furthermore, any fraction that is in both U and V is also in W:

Suppose that a fraction can be written in both forms $\frac{u}{1110}$ and $\frac{v}{330}$.
Then $\frac{u}{1110} = \frac{v}{330}$ or $\frac{u}{37} = \frac{v}{11}$ and so $11u = 37v$.
Since $37v$ is a thus a multiple of 11 and 37 is not divisible by the prime number 11, then v is a multiple of 11.
This means that $\frac{v}{330} = \frac{11f}{330} = \frac{f}{30}$ for some positive integer f, and so is in W.

In set notation, $U \cap V \subseteq W$.
Since $W \subseteq U$ and $W \subseteq V$ and $U \cap V \subseteq W$, then $U \cap V = W$; in other words, the set of fractions in each of U and V is precisely the set of fractions W.

Step 7: Final counting
We start with 1111109 fractions, as above, and want to remove all of the fractions in U, V and W.
Since each fraction in W is in U and V, it is enough to remove those U and V only.
The total number of fractions in U and V (that is, in $U \cup V$) equals the number of fractions in U plus the number of fractions in V minus the number of fractions in their overlap (that is, in $U \cap V = W$). This is because any fraction in the overlap is "counted twice" when include all fractions in U and all fractions in V.
Therefore, we need to remove $1009 + 329 - 29$ fractions from the set of 1111109.
Therefore, F, the number of fractions having the desired properties, is

$$F = 1111109 - (1009 + 329 - 29) = 1109700$$

Since F has 7 digits, then $G = F + 7 = 1109707$. The sum of the squares of the digits of G is $1^2 + 1^2 + 0^2 + 9^2 + 7^2 + 0^2 + 7^2 = 1 + 1 + 81 + 49 + 49 = 181$.

Step 8: General algebraic derivation from Step 1
Consider a real number x whose decimal equivalent is of the form $0.g_1g_2\ldots g_p\overline{r_1r_2\ldots r_q}$ for some integers $p \geq 0$ and $q > 0$ and digits $g_1, g_2, \ldots, g_p, r_1, r_2, \ldots, r_q$. Then

$$\begin{aligned}
x &= 0.g_1g_2\ldots g_p\overline{r_1r_2\ldots r_q} \\
10^p x &= g_1g_2\ldots g_p.\overline{r_1r_2\ldots r_q} \\
10^p x - g_1g_2\ldots g_p &= 0.\overline{r_1r_2\ldots r_q} \\
10^q(10^p x - g_1g_2\ldots g_p) &= r_1r_2\ldots r_q.\overline{r_1r_2\ldots r_q} \\
10^q(10^p x - g_1g_2\ldots g_p) - r_1r_2\ldots r_q &= 0.\overline{r_1r_2\ldots r_q} \\
10^q(10^p x - g_1g_2\ldots g_p) - r_1r_2\ldots r_q &= 10^p x - g_1g_2\ldots g_p \\
10^q(10^p x - g_1g_2\ldots g_p) - (10^p x - g_1g_2\ldots g_p) &= r_1r_2\ldots r_q \\
(10^q - 1)(10^p x - g_1g_2\ldots g_p) &= r_1r_2\ldots r_q \\
10^p x - g_1g_2\ldots g_p &= \frac{r_1r_2\ldots r_q}{10^q - 1} \\
10^p x &= g_1g_2\ldots g_p + \frac{r_1r_2\ldots r_q}{10^q - 1} \\
x &= \frac{g_1g_2\ldots g_p}{10^p} + \frac{r_1r_2\ldots r_q}{10^p(10^q - 1)} \\
x &= \frac{(10^q - 1)g_1g_2\ldots g_p + r_1r_2\ldots r_q}{10^p(10^q - 1)}
\end{aligned}$$

ANSWER: (E)