The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Wednesday, April 15, 2015
(in North America and South America)

Thursday, April 16, 2015
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express calculations and answers as exact numbers such as $\pi + 1$ and $\sqrt{2}$, etc., rather than as $4.14\ldots$ or $1.41\ldots$, except where otherwise indicated.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:

1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked (light bulb), place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked (pencil), provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

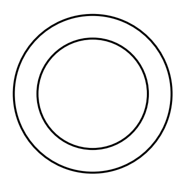
1. (a) What is value of $\dfrac{10^2 - 9^2}{10 + 9}$?

   (b) If $\dfrac{x+1}{x+4} = 4$, what is the value of $3x + 8$?

   (c) If $f(x) = 2x - 1$, determine the value of $(f(3))^2 + 2(f(3)) + 1$.

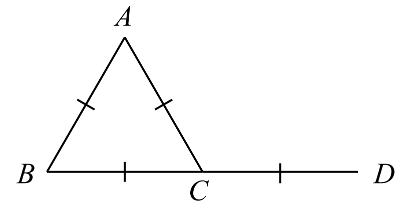
2. (a) If $\sqrt{a} + \sqrt{a} = 20$, what is the value of a?

   (b) Two circles have the same centre. The radius of the smaller circle is 1. The area of the region between the circles is equal to the area of the smaller circle. What is the radius of the larger circle?



   (c) There were 30 students in Dr. Brown's class. The average mark of the students in the class was 80. After two students dropped the class, the average mark of the remaining students was 82. Determine the average mark of the two students who dropped the class.

3. (a) In the diagram, $BD = 4$ and point C is the midpoint of BD. If point A is placed so that $\triangle ABC$ is equilateral, what is the length of AD?

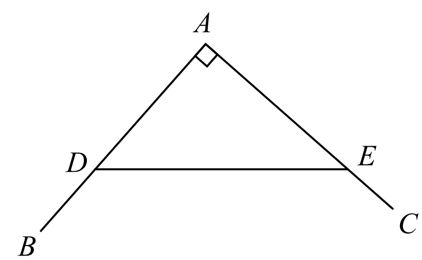


(b) $\triangle MNP$ has vertices $M(1,4)$, $N(5,3)$, and $P(5,c)$. Determine the sum of the two values of c for which the area of $\triangle MNP$ is 14.

4. (a) What are the x-intercepts and the y-intercept of the graph with equation $y = (x-1)(x-2)(x-3) - (x-2)(x-3)(x-4)$?

(b) The graphs of the equations $y = x^3 - x^2 + 3x - 4$ and $y = ax^2 - x - 4$ intersect at exactly two points. Determine all possible values of a.

5. (a) In the diagram, $\angle CAB = 90^\circ$. Point D is on AB and point E is on AC so that $AB = AC = DE$, $DB = 9$, and $EC = 8$. Determine the length of DE.

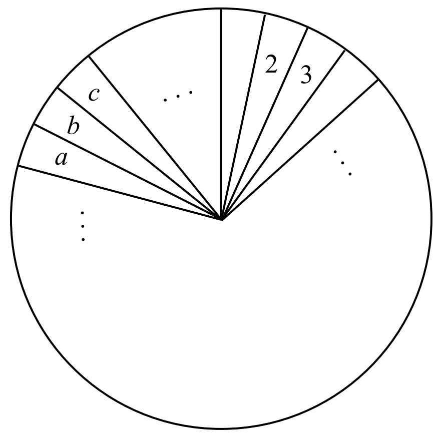


(b) Ellie has two lists, each consisting of 6 consecutive positive integers. The smallest integer in the first list is a, the smallest integer in the second list is b, and $a < b$. She makes a third list which consists of the 36 integers formed by multiplying each number from the first list with each number from the second list. (This third list may include some repeated numbers.) If

- the integer 49 appears in the third list,
- there is no number in the third list that is a multiple of 64, and
- there is at least one number in the third list that is larger than 75,

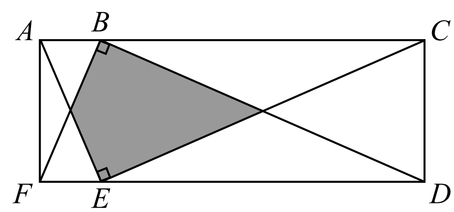
determine all possible pairs (a, b).

6. (a) A circular disc is divided into 36 sectors. A number is written in each sector. When three consecutive sectors contain a, b and c in that order, then $b = ac$. If the number 2 is placed in one of the sectors and the number 3 is placed in one of the adjacent sectors, as shown, what is the sum of the 36 numbers on the disc?



(b) Determine all values of x for which $0 < \dfrac{x^2 - 11}{x + 1} < 7$.

7. (a) In the diagram, $ACDF$ is a rectangle with $AC = 200$ and $CD = 50$. Also, $\triangle FBD$ and $\triangle AEC$ are congruent triangles which are right-angled at B and E, respectively. What is the area of the shaded region?

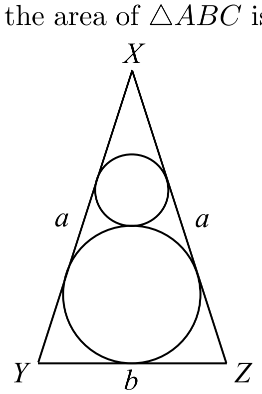


(b) The numbers $a_1, a_2, a_3, \ldots$ form an arithmetic sequence with $a_1 \neq a_2$. The three numbers a_1, a_2, a_6 form a geometric sequence in that order. Determine all possible positive integers k for which the three numbers a_1, a_4, a_k also form a geometric sequence in that order.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9$ are the first four terms of an arithmetic sequence.
A *geometric sequence* is a sequence in which each term after the first is obtained from the previous term by multiplying it by a non-zero constant. For example, 3, 6, 12 is a geometric sequence with three terms.)

8. (a) For some positive integers k, the parabola with equation $y = \dfrac{x^2}{k} - 5$ intersects the circle with equation $x^2 + y^2 = 25$ at exactly three distinct points A, B and C. Determine all such positive integers k for which the area of $\triangle ABC$ is an integer.

(b) In the diagram, $\triangle XYZ$ is isosceles with $XY = XZ = a$ and $YZ = b$ where $b < 2a$. A larger circle of radius R is inscribed in the triangle (that is, the circle is drawn so that it touches all three sides of the triangle). A smaller circle of radius r is drawn so that it touches XY, XZ and the larger circle. Determine an expression for $\dfrac{R}{r}$ in terms of a and b.



9. Consider the following system of equations in which all logarithms have base 10:

$$\begin{aligned}(\log x)(\log y) - 3\log 5y - \log 8x &= a\\ (\log y)(\log z) - 4\log 5y - \log 16z &= b\\ (\log z)(\log x) - 4\log 8x - 3\log 625z &= c\end{aligned}$$

(a) If $a = -4$, $b = 4$, and $c = -18$, solve the system of equations.

(b) Determine all triples (a, b, c) of real numbers for which the system of equations has an infinite number of solutions (x, y, z).

10. For each positive integer $n \geq 1$, let C_n be the set containing the n smallest positive integers; that is, $C_n = \{1, 2, \ldots, n-1, n\}$. For example, $C_4 = \{1, 2, 3, 4\}$. We call a set, F, of subsets of C_n a *Furoni family of* C_n if no element of F is a subset of another element of F.

(a) Consider $A = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Note that A is a Furoni family of C_4. Determine the two Furoni families of C_4 that contain all of the elements of A and to which no other subsets of C_4 can be added to form a new (larger) Furoni family.

(b) Suppose that n is a positive integer and that F is a Furoni family of C_n. For each non-negative integer k, define a_k to be the number of elements of F that contain exactly k integers. Prove that

$$\frac{a_0}{\binom{n}{0}} + \frac{a_1}{\binom{n}{1}} + \frac{a_2}{\binom{n}{2}} + \cdots + \frac{a_{n-1}}{\binom{n}{n-1}} + \frac{a_n}{\binom{n}{n}} \leq 1$$

(The sum on the left side includes $n + 1$ terms.)

(Note: If n is a positive integer and k is an integer with $0 \leq k \leq n$, then $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the number of subsets of C_n that contain exactly k integers, where $0! = 1$ and, if m is a positive integer, $m!$ represents the product of the integers from 1 to m, inclusive.)

(c) For each positive integer n, determine, with proof, the number of elements in the largest Furoni family of C_n (that is, the number of elements in the Furoni family that contains the maximum possible number of subsets of C_n).

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For students...

Thank you for writing the 2015 Euclid Contest! Each year, more than 200 000 students from more than 60 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2015 Canadian Senior Mathematics Contest, which will be written in November 2015.

Visit our website cemc.uwaterloo.ca to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

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- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results