The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015
Canadian Team Mathematics Contest

April 2015

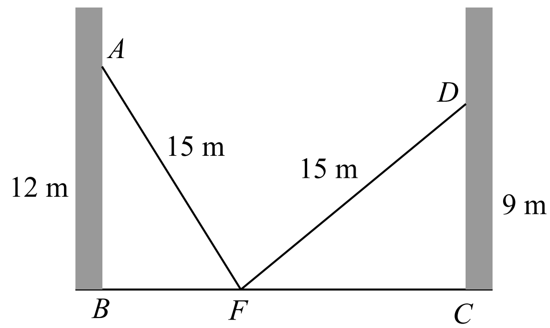
Solutions

Individual Problems

1. The inequality $5 + 3n > 300$ is equivalent to $3n > 295$, which is equivalent to $n > \frac{295}{3}$.
   Since $\frac{295}{3} = 98\frac{1}{3}$, then this is equivalent to $n > 98\frac{1}{3}$.
   Therefore, the smallest positive integer n for which $5 + 3n > 300$ is $n = 99$.

   Answer: 99

2. We label the diagram as shown.



   Since $AF = DF = 15$ m, then by the Pythagorean Theorem,

   $$BF = \sqrt{15^2 - 12^2} = \sqrt{81} = 9 \text{ m} \qquad FC = \sqrt{15^2 - 9^2} = \sqrt{144} = 12 \text{ m}$$

   Therefore, $BC = BF + FC = 9 + 12 = 21$ m, and so the walls are 21 m apart.

   Answer: 21

3. Let x be the number of friends who like to both ski and snowboard.
   Then $11 - x$ of the friends like to ski but not do not like to snowboard, and $13 - x$ of the friends like to snowboard but do not like to ski.
   Since 3 of the 20 friends do not like to ski or snowboard, then 17 like to either ski or snowboard or both.
   Thus, $(11 - x) + (13 - x) + x = 17$ and so $x = 7$.
   Therefore, 7 of the friends like to both ski and snowboard.

   Answer: 7

4. Since $(x, y) = (2, 5)$ is the solution of the system of equations, then $(x, y) = (2, 5)$ satisfies both equations.
   Since $(x, y) = (2, 5)$ satisfies $ax + 2y = 16$, then $2a + 10 = 16$ or $2a = 6$ and so $a = 3$.
   Since $(x, y) = (2, 5)$ satisfies $3x - y = c$, then $6 - 5 = c$ or $c = 1$.
   Therefore, $\frac{a}{c} = 3$.

   Answer: 3

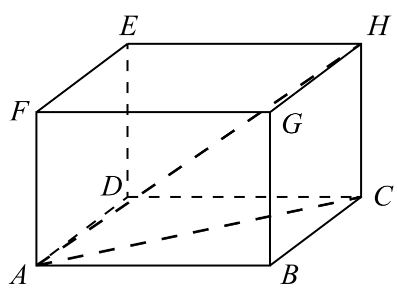
5. We note that $45 = 3^2 \cdot 5$ and so $45k = 3^2 \cdot 5 \cdot k$.
   For $45k$ to be a perfect square, each prime factor has to occur an even number of times.
   Therefore, k must be divisible by 5.
   We try the smallest two-digit possibilities for k that are divisible by 5, namely $k = 10, 15, 20, \ldots$.
   If $k = 10$, then $45k = 450$, which is not a perfect square.
   If $k = 15$, then $45k = 675$, which is not a perfect square.
   If $k = 20$, then $45k = 900 = 30^2$, which is a perfect square.
   Therefore, the smallest two-digit positive integer k for which $45k$ is a perfect square is $k = 20$.

   Answer: 20

6. Suppose that the distance that Clara has to travel is d km and that the time from when she starts to the scheduled meeting time is T hours.
When she travels at an average speed of 20 km/h, she arrives half an hour early.
Thus, she travels for $T - \frac{1}{2}$ hours and so $\dfrac{d}{20} = T - \dfrac{1}{2}$.
When she travels at an average speed of 12 km/h, she arrives half an hour late.
Thus, she travels for $T + \frac{1}{2}$ hours and so $\dfrac{d}{12} = T + \dfrac{1}{2}$.
Adding these two equations, we obtain $\dfrac{d}{20} + \dfrac{d}{12} = 2T$ or $\dfrac{3d}{60} + \dfrac{5d}{60} = 2T$.
Thus, $\dfrac{8d}{60} = 2T$ or $\dfrac{d}{15} = T$.
This means that if Clara travels at an average speed of 15 km/h, then it will take her exactly T hours to travel the d km, and so she will arrive at the scheduled time.
ANSWER: 15

7. Since the sum of any four consecutive terms is 17, then $8 + b + c + d = 17$ or $b + c + d = 9$.
Also, $b + c + d + e = 17$ and so $9 + e = 17$ or $e = 8$.
Similarly, $e + f + g + 2 = 17$ and $d + e + f + g = 17$ tell us that $d = 2$.
But $c + d + e + f = 17$ and $e = 8$ and $d = 2$, which gives $c + f = 17 - 8 - 2 = 7$.
ANSWER: 7

8. Suppose that the prism has $AB = x$, $AD = y$, and $AF = z$.
Since the sum of all of the edge lengths is 24, then $4x + 4y + 4z = 24$ or $x + y + z = 6$.
(The prism has 4 edges of each length.)
Since the surface area is 11, then $2xy + 2xz + 2yz = 11$.
(The prism has 2 faces that are x by y, 2 faces that are x by z, and two faces that are y by z.)
By the Pythagorean Theorem, $AH^2 = AC^2 + CH^2$.



Again, by the Pythagorean Theorem, $AC^2 = AB^2 + BC^2$, and so $AH^2 = AB^2 + BC^2 + CH^2$.
But $AB = x$, $BC = AD = y$, and $CH = AF = z$.
Therefore, $AH^2 = x^2 + y^2 + z^2$.
Finally,

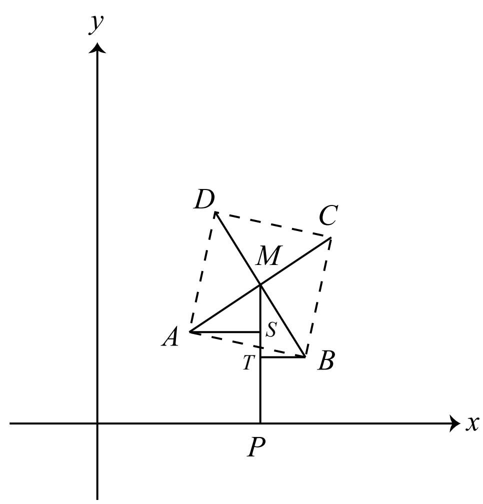
$$(x+y+z)^2 = (x + (y+z))^2 = x^2 + 2x(y+z) + (y+z)^2 = x^2 + 2xy + 2xz + y^2 + 2yz + z^2$$

and so $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$, which gives

$$AH^2 = (x+y+z)^2 - (2xy + 2xz + 2yz) = 6^2 - 11 = 25$$

Since $AH > 0$, then $AH = \sqrt{25} = 5$.
ANSWER: 5

9. Since the parabola opens upwards and has only one x-intercept, then its equation is of the form $y = a(x-r)^2$ for some real numbers $a, r > 0$.
Since the point $A(1,4)$ is on the parabola, then $(x,y) = (1,4)$ satisfies the equation of the parabola, which gives $4 = a(1-r)^2$.
We will obtain a second equation for a and r by determining the coordinates of B.
Since $ABCD$ is a square, its diagonals AC and BD cross at their common midpoint, M, and are perpendicular at M.
Since A has coordinates $(1,4)$ and C has coordinates $\left(\frac{39}{4}, \frac{37}{4}\right)$, then the coordinates of M are $\left(\frac{1}{2}\left(1+\frac{39}{4}\right), \frac{1}{2}\left(4+\frac{37}{4}\right)\right) = \left(\frac{43}{8}, \frac{53}{8}\right)$.
To get from A to M, we go $\frac{35}{8}$ units right and $\frac{21}{8}$ units up.
Thus, to get from M to B, we go $\frac{21}{8}$ units right and $\frac{35}{8}$ units down.
This means that the coordinates of B are $\left(\frac{43}{8}+\frac{21}{8}, \frac{53}{8}-\frac{35}{8}\right) = \left(8, \frac{9}{4}\right)$.
To see why this "over and up" approach is correct, draw a vertical line from M to P on the x-axis, and draw horizontal lines from A and B to points S and T, respectively, on MP.



Since AS and BT are perpendicular to MP, then $\triangle ASM$ and $\triangle MTB$ are right-angled.
Note that $AM = MB$, since $ABCD$ is a square and M is its centre.
Since AM and MB are perpendicular, then $\angle AMS + \angle TMB = 90^\circ$.
But $\angle SAM + \angle AMS = 90^\circ$, so $\angle SAM = \angle TMB$.
This means that $\triangle ASM$ and $\triangle MTB$ have equal corresponding angles.
Since $AM = MB$, these triangles are congruent, and so $AS = MT = \frac{35}{8}$ and $TB = SM = \frac{21}{8}$.
Thus, the coordinates of B are indeed $\left(8, \frac{9}{4}\right)$.

Since B is also on the parabola, then $\frac{9}{4} = a(8-r)^2$.

Dividing this equation by the first equation $4 = a(1-r)^2$, we obtain $\dfrac{9}{16} = \left(\dfrac{8-r}{1-r}\right)^2$.

Since $1 < r < 8$ (the vertex of the parabola is between A and B), then $\dfrac{8-r}{1-r} < 0$ and so $\dfrac{8-r}{1-r} = -\dfrac{3}{4}$.
Thus, $32 - 4r = 3r - 3$ and so $7r = 35$ or $r = 5$.
Since $4 = a(1-r)^2$, then $4 = a(1-5)^2$ and so $a = \frac{4}{16} = \frac{1}{4}$.
Therefore, the equation of the parabola is $y = \frac{1}{4}(x-5)^2$.

Answer: $y = \frac{1}{4}(x-5)^2$

10. First, we note that $257 = 256 + 1 = 2^8 + 1$.
Next, we note that $2^{2008} + 1$ is divisible by $2^8 + 1$.
This is because since

$$1 - 2^8 + 2^{16} - 2^{24} + \cdots - 2^{1992} + 2^{2000} = 1 + (-2^8) + (-2^8)^2 + (-2^8)^3 + \cdot + (-2^8)^{249} + (-2^8)^{250}$$

and the right side is a geometric sequence with first term 1, common ratio -2^8, and 251 terms, giving

$$1 - 2^8 + 2^{16} - 2^{24} + \cdots - 2^{1992} + 2^{2000} = \frac{1(1 - (-2^8)^{251})}{1 - (-2^8)} = \frac{2^{2008} + 1}{2^8 + 1}$$

Since the left side is an integer, then $2^{2008} + 1$ is divisible by $2^8 + 1 = 257$.
Since $2^{2008} + 1$ is divisible by 257, then any multiple of $2^{2008} + 1$ is divisible by 257.
In particular, $2^7(2^{2008} + 1) = 2^{2015} + 128$ is divisible by 257.
Since multiples of 257 differ by 257, then the largest multiple of 257 less than $2^{2015} + 128$ is $2^{2015} - 129$ which is smaller than 2^{2015}.
The next multiples of 257 larger than $2^{2015} + 128$ are $2^{2015} + 385$, $2^{2015} + 642$, $2^{2015} + 899$, and $2^{2015} + 1156$.
Those between 2^{2015} and $2^{2015} + 1000$ are $2^{2015} + 128$, $2^{2015} + 385$, $2^{2015} + 642$, $2^{2015} + 899$, which give values of N of 128, 385, 642, and 899.
The sum of these values is $128 + 385 + 642 + 899 = 2054$.

ANSWER: 2054

Team Problems

1. Re-arranging the terms and grouping into pairs,

   $$1+3+5+7+9+11+13+15+17+19 = (1+19)+(3+17)+(5+15)+(7+13)+(9+11) = 5\cdot 20 = 100$$

   Answer: 100

2. When the edge lengths of 10 are increased by 10%, the new edge lengths are 11.
   Therefore, the volume of the resulting cube is $11^3 = 1331$.

   Answer: 1331

3. Suppose that the cylinder has radius r.
   Since the circumference of its circular faces is 10π, then $2\pi r = 10\pi$ or $r = 5$.
   Therefore, the volume of the cylinder is $\pi r^2 h = \pi \cdot 5^2 \cdot 4 = 100\pi$.

   Answer: 100π

4. Every prime number other than 2 is odd.
   Since a, b and c are all prime and $a + b + c = 22$ which is even, it cannot be the case that all of a, b and c are odd (otherwise $a + b + c$ would be odd).
   Thus, at least one of a, b and c is even.
   Since $1 < a < b < c$, then it must be the case that $a = 2$ and b and c are odd primes.
   Since $a = 2$, then $b + c = 20$.
   Since b and c are primes with $b < c$, then $b = 3$ and $c = 17$, or $b = 7$ and $c = 13$.
   Therefore, there are two triples (a, b, c) of primes numbers that satisfy the requirements.

   Answer: 2

5. Since $\angle TBD = 110^\circ$, then $\angle DBC = 180^\circ - \angle TBD = 180^\circ - 110^\circ = 70^\circ$.
   Since $\angle BCN = 126^\circ$, then $\angle DCB = 180^\circ - \angle BCN = 180^\circ - 126^\circ = 54^\circ$.
   Since the sum of the angles in $\triangle DBC$ is 180°, then $\angle BDC = 180^\circ - 70^\circ - 54^\circ = 56^\circ$.
   Now $\angle BDC$ is an exterior angle of $\triangle ADC$.
   Thus, $\angle BDC = \angle DAC + \angle DCA$.
   But $\triangle ADC$ is isosceles with $\angle DAC = \angle DCA$.
   Thus, $\angle BDC = 2\angle DAC$.
   Since $\angle BDC = 56^\circ$, then $\angle DAC = \frac{1}{2}(56^\circ) = 28^\circ$.
   Therefore, $\angle PAC = 180^\circ - \angle DAC = 180^\circ - 28^\circ = 152^\circ$.

   Answer: 152°

6. Note that $234 = 9 \cdot 26 = 2 \cdot 3^2 \cdot 13$.
   Thus, $k \cdot 234 = k \cdot 2 \cdot 3^2 \cdot 13$.
   This product is divisible by $12 = 2^2 \cdot 3$ if and only if k contributes another factor of 2 (that is, if and only if k is even).
   Since k is a one-digit positive integer, then $k = 2, 4, 6, 8$.
   Therefore, there are 4 one-digit positive integers k for which $k \cdot 234$ is divisible by 12.

   Answer: 4

7. Since $A(5,-8)$, $B(9,-30)$, and $C(n,n)$ lie on the same straight line, then the slopes of AB and AC are equal.
Thus,

$$\begin{aligned}\frac{(-30)-(-8)}{9-5} &= \frac{n-(-8)}{n-5}\\ \frac{-22}{4} &= \frac{n+8}{n-5}\\ -22n+110 &= 4n+32\\ 78 &= 26n\\ n &= 3\end{aligned}$$

Therefore, $n = 3$.

ANSWER: 3

8. To find the largest possible three-digit positive integer with no repeated digits, we put the largest possible digit in each position, starting with the hundreds digit (9), then the tens digit (8), and finally the ones (units) digit (7).
Similarly, to find the smallest possible three-digit positive integer with no repeated digits, we put the smallest possible digit in each position, starting with the hundreds digit (1 because the leading digit cannot be 0), then the tens digit (0), and finally the ones (units) digit (2).
The difference between the numbers 987 and 102 is 885.

ANSWER: 885

9. Since $0 < x < y$, then $2x + 3y > 2x + 3x = 5x$.
Since $2x + 3y = 80$ and $2x + 3y > 5x$, then $80 > 5x$ or $16 > x$.
Since $2x + 3y = 80$, then $2x = 80 - 3y$. Since $2x$ is even, then $80 - 3y$ is even. Since 80 is even, then $3y$ is even, and so y must be even.
Set $y = 2Y$ for some positive integer Y.
Then $2x + 3y = 80$ becomes $2x + 6Y = 80$ or $x + 3Y = 40$ or $x = 40 - 3Y$.
Since $x < 16$, then $3Y > 24$ or $Y > 8$.
Since $x > 0$, then $3Y < 40$ or $Y < 13\frac{1}{3}$.
When $Y = 9$ (that is, $y = 18$), $x = 13$.
When $Y = 10$ (that is, $y = 20$), $x = 10$.
When $Y = 11$ (that is, $y = 22$), $x = 7$.
When $Y = 12$ (that is, $y = 24$), $x = 4$.
When $Y = 13$ (that is, $y = 26$), $x = 1$.
Therefore, there are 5 pairs that satisfy the given conditions: $(x, y) = (13, 18), (10, 20), (7, 22), (4, 24), (1, 26)$.

ANSWER: 5

10. Since $\angle BAC = 30^\circ$ and $\angle BCA = 90^\circ$, then $\triangle ABC$ is a 30°-60°-90° triangle.
Thus, $AB = 2BC$.
But $BC + AB = 10$ m (the height of the pole).
Therefore, $3BC = 10$ m or $BC = \frac{10}{3}$ m.

ANSWER: $\frac{10}{3}$ m

11. Since $a = 2^3$ and $b = 3^2$, then $a - b = 8 - 9 = -1$.
Therefore,

$$\frac{(a-b)^{2015}+1^{2015}}{(a-b)^{2015}-1^{2015}} = \frac{(-1)^{2015}+1}{(-1)^{2015}-1} = \frac{-1+1}{-1-1} = 0$$

ANSWER: 0

12. For a positive integer to be a perfect square, its prime factors must occur in pairs.
Since only one number from the list 1, 2, 3, 4, 5, 6, 7, 8, 9 is divisible by 5 (namely 5) and only one number from the list is divisible by 7, then neither 5 nor 7 can be one of the three integers chosen.
Therefore, we need to choose three numbers from the list 1, 2, 3, 4, 6, 8, 9.
We note that $9 \cdot 8 \cdot 2 = 144 = 12^2$, which is a perfect square.
Are there perfect squares larger than 144 that can be made?
If the product of three integers is odd, then all three integers are odd. The only three integers in the list that are odd are 1, 3, 9. Their product is 27, which is not a perfect square.
Therefore, any perfect square larger than 144 that can be made must be even.
Further, note that the largest possible product of three numbers in the second list is $9 \times 8 \times 6$ or 432, and so any perfect square that can be made is less than 432.
The even perfect squares between 144 and 432 are $14^2 = 196$, $16^2 = 256$, $18^2 = 324$, and $20^2 = 400$.
We cannot make 196 or 400, since no number in the list is a multiple of 7 or 5.
We cannot obtain a product of $18^2 = 324 = 3^4 2^2$, since 324 has 4 factors of 3, which would require using 3, 6 and 9, whose product is too small.
We cannot obtain a product of $16^2 = 256 = 2^8$, since if we multiply all 7 numbers from the list, there are only 7 factors of 2 in total (from 2, 4, 6, 8).
Therefore, 144 is the largest perfect square which is the product of three different one-digit positive integers.

Answer: 144

13. Suppose that the distance from Point A to Point B is d m, that Mario's speed is w m/s, and that the speed of the moving sidewalk is v m/s.
From the given information, $\frac{d}{w} = 45$ and $\frac{d}{v} = 90$.
The amount of the time, in seconds, that it takes Mario to walk along the sidewalk when it is moving is $\frac{d}{v+w}$, since his resulting speed is the sum of his walking speed and the speed of the sidewalk.
Here,

$$\frac{d}{v+w} = \frac{1}{\frac{v+w}{d}} = \frac{1}{\frac{v}{d} + \frac{w}{d}} = \frac{1}{\frac{1}{90} + \frac{1}{45}} = \frac{1}{\frac{3}{90}} = 30$$

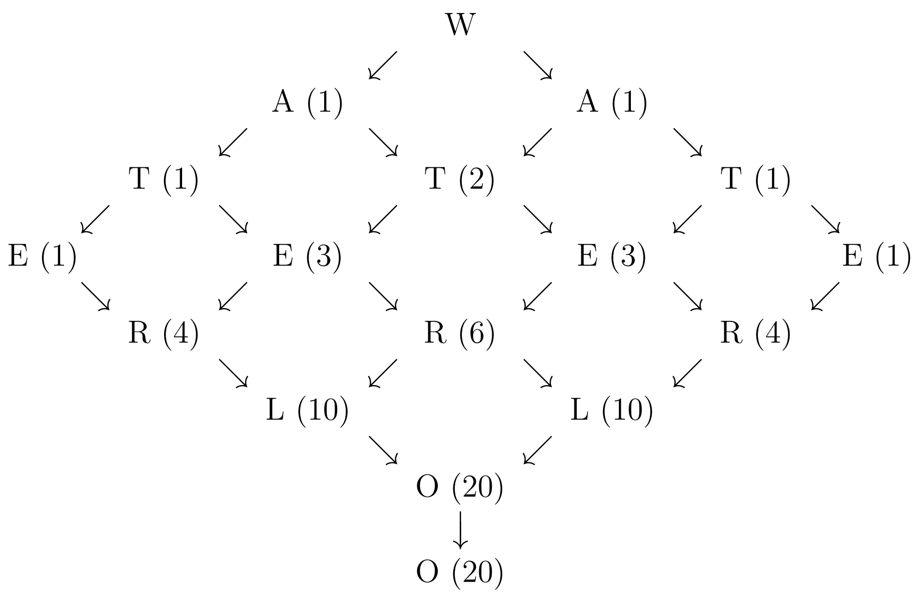
Therefore, it takes Mario 30 seconds to walk from Point A to Point B with the sidewalk moving.

Answer: 30 seconds

14. If a square has side length s, then the length of its diagonal is $\sqrt{2}s$.
Since we are told that the length of the diagonal is $s+1$, then $\sqrt{2}s = s+1$ or $\sqrt{2}s - s = 1$ or $(\sqrt{2}-1)s = 1$.
Thus, $s = \frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{\sqrt{2}+1}{2-1} = \sqrt{2}+1$.
The area of the square is $s^2 = (\sqrt{2}+1)^2 = 2 + 2\sqrt{2} + 1 = 3 + 2\sqrt{2}$.

Answer: $3 + 2\sqrt{2}$

15. In the diagram, every path that starts at the top and follows to the bottom spells the word "WATERLOO".
Therefore, we count the total number of paths from the top to the bottom.
There is one path leading from the W to each of the A's.
The number of paths leading to a given letter below the A's is the sum of the numbers of paths leading to the two letters above that point to the given letter.
This is because each path leading to a given letter must go through exactly one of the letters above pointing to the given letter.
We calculate the number of paths leading to each letter in the grid by proceeding downwards through the grid:



Therefore, there are 20 paths that spell "WATERLOO".

Answer: 20

16. Using exponent and trigonometric laws,

$$\begin{aligned}
4^{\sin^2 x} \cdot 2^{\cos^2 x} &= 2\sqrt[4]{8} \\
(2^2)^{\sin^2 x} \cdot 2^{\cos^2 x} &= 2\sqrt[4]{2^3} \\
2^{2\sin^2 x} \cdot 2^{\cos^2 x} &= 2^1 \cdot 2^{3/4} \\
2^{2\sin^2 x + \cos^2 x} &= 2^{7/4} \\
2\sin^2 x + \cos^2 x &= \tfrac{7}{4} \\
\sin^2 x + (\sin^2 x + \cos^2 x) &= \tfrac{7}{4} \\
\sin^2 x + 1 &= \tfrac{7}{4} \\
\sin^2 x &= \tfrac{3}{4}
\end{aligned}$$

Thus, $\sin x = \pm\frac{\sqrt{3}}{2}$.
The smallest positive angle for which one of these is true is $x = 60^\circ$ (or $x = \frac{1}{3}\pi$ in radians).

Answer: 60°

17. Using exponent laws,

$$\begin{aligned}
\log_{3n} 675\sqrt{3} &= \log_n 75 \\
(3n)^{\log_{3n} 675\sqrt{3}} &= (3n)^{\log_n 75} \\
675\sqrt{3} &= 3^{\log_n 75} \cdot n^{\log_n 75} \\
675\sqrt{3} &= 3^{\log_n 75} \cdot 75 \\
9\sqrt{3} &= 3^{\log_n 75} \\
3^2 \cdot 3^{1/2} &= 3^{\log_n 75} \\
3^{5/2} &= 3^{\log_n 75} \\
5/2 &= \log_n 75 \\
n^{5/2} &= 75 \\
n^5 &= 75^2
\end{aligned}$$

Therefore, $n^5 = 75^2 = 5625$.

Answer: 5625

18. For the quadratic equation $x^2 - dx + e = 0$, the sum of the roots is d and the product of the roots is e.
This is because if the roots are $x = f$ and $x = g$, then

$$x^2 - dx + e = (x - f)(x - g) = x^2 - fx - gx + fg = x^2 - (f + g)x + fg$$

Suppose that r and s are the roots of $x^2 - 5x + 2 = 0$.
Then $r + s = 5$ and $rs = 2$.
We are told that the roots of $x^2 + bx + c = 0$ are r^2 and s^2.
Then

$$c = r^2 \cdot s^2 = (rs)^2 = 2^2 = 4$$

and

$$-b = r^2 + s^2 = (r + s)^2 - 2rs = 5^2 - 2(2) = 21$$

which tells us that $b = -21$.
Therefore, $\dfrac{c}{b} = \dfrac{4}{-21} = -\dfrac{4}{21}$.
(We could also solve the problem by directly calculating the roots of $x^2 - 5x + 2 = 0$, squaring these roots, and creating a quadratic equation with these results as roots.)

Answer: $-\frac{4}{21}$

19. First, we note that $\dbinom{12}{3} = \dfrac{12!}{3! \cdot 9!} = \dfrac{12 \cdot 11 \cdot 10}{3 \cdot 2 \cdot 1} = 2 \cdot 11 \cdot 10 = 220$.
This means that there are 220 different combinations of 3 eggs that can be chosen from 12 eggs.
Zach randomly gives 3 eggs to each of the 4 children.
The probability that a specific one of these 4 children receives the specific set of all 3 special eggs is $\frac{1}{220}$, since there are 220 different combinations of 3 eggs that can be given and each set is equally likely to be chosen.
The probability that any one of these 4 children receives a specific set of 3 eggs is $4 \cdot \frac{1}{220} = \frac{1}{55}$ since the probability that each one receives the 3 special eggs is $\frac{1}{220}$ and these four events are disjoint.
Therefore, the probability that only one child receives an egg that contains a prize is $\frac{1}{55}$.

Answer: $\frac{1}{55}$

20. If $u \neq 0$, then

$$f(u) + f\left(\frac{1}{u}\right) = \frac{u^2}{1+u^2} + \frac{\frac{1}{u^2}}{1+\frac{1}{u^2}} = \frac{u^2}{1+u^2} + \frac{\frac{1}{u^2}}{\frac{u^2}{u^2}+\frac{1}{u^2}} = \frac{u^2}{1+u^2} + \frac{1}{u^2+1} = \frac{u^2+1}{u^2+1} = 1$$

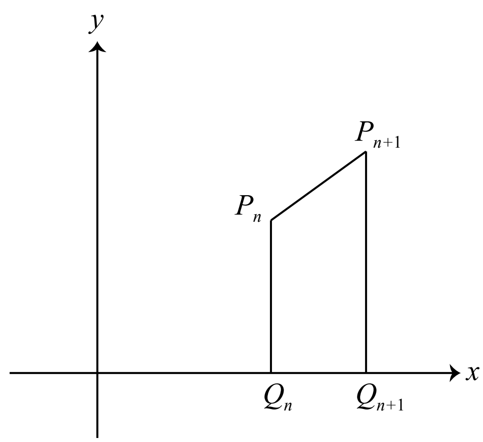
Therefore,

$$A + B = (f(1) + f(1)) + (f(2) + f(\tfrac{1}{2})) + (f(3) + f(\tfrac{1}{3})) + \cdots + (f(2015) + f(\tfrac{1}{2015}))$$

and the right side equals $2015 \cdot 1$.
Therefore, $A + B = 2015$.

ANSWER: 2015

21. We first note that the coordinates of P_{n+1} are $(((n+1)-1)^2, (n+1)((n+1)-1)) = (n^2, n(n+1))$ and that the coordinates of Q_{n+1} are $(((n+1)-1)^2, 0) = (n^2, 0)$.
Since P_n and Q_n have the same x-coordinate, then P_nQ_n is vertical.
Since P_{n+1} and Q_{n+1} have the same x-coordinate, then $P_{n+1}Q_{n+1}$ is vertical.
Since Q_n and Q_{n+1} have the same y-coordinate, then Q_nQ_{n+1} is horizontal.



Let A_n be the area of trapezoid $Q_nP_nP_{n+1}Q_{n+1}$.
Then

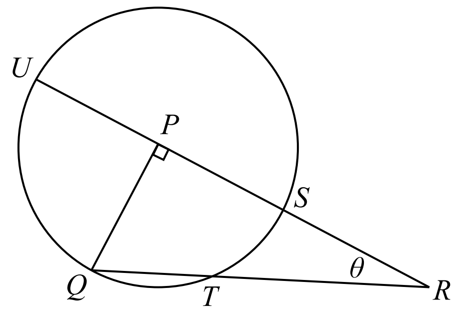
$$\begin{aligned} A_n &= \tfrac{1}{2}(P_nQ_n + P_{n+1}Q_{n+1})Q_nQ_{n+1} \\ &= \tfrac{1}{2}((n(n-1)-0) + (n(n+1)-0))(n^2 - (n-1)^2) \\ &= \tfrac{1}{2}(2n^2)(2n-1) \\ &= n^2(2n-1) \end{aligned}$$

We want to determine the number of integers n with $2 \leq n \leq 99$ for which A_n is a perfect square.
Since n^2 is a perfect square, then A_n is a perfect square if and only if $2n-1$ is a perfect square.
We note that $2n-1$ is odd and, since $2 \leq n \leq 99$, then $3 \leq 2n-1 \leq 197$.
Thus, the possible perfect square values of $2n-1$ are the odd perfect squares between 3 and 197, which are $3^2, 5^2, 7^2, 9^2, 11^2, 13^2$ (that is, $9, 25, 49, 81, 121, 169$).
Therefore, there are 6 integers n that satisfy the given conditions.

ANSWER: 6

22. *Solution 1*

Let $SR = x$ and the radius of the circle be r.
Extend RP to meet the circle again at U.
Then $PS = PQ = PU = r$, since each is a radius.



By the Pythagorean Theorem in $\triangle PQR$, we have $PQ^2 + PR^2 = QR^2$ or $r^2 + (r + x)^2 = 18^2$.
By the Secant-Secant Theorem, $RS \cdot RU = RT \cdot RQ$ or $x(x + 2r) = 10(18)$.
From the first equation, $r^2 + r^2 + 2xr + x^2 = 324$.
From the second equation, $x^2 + 2xr = 180$.
Substituting, we obtain $2r^2 + 180 = 324$ or $r^2 = 72$ and so $r = \sqrt{72} = 6\sqrt{2}$ since $r > 0$.
Since $\triangle PQR$ is right-angled, then $\theta = \angle PRQ$ is acute, which means that $\cos\theta > 0$.
Also,

$$\cos^2\theta = \left(\frac{PR}{RQ}\right)^2 = \frac{RQ^2 - PQ^2}{RQ^2} = \frac{324 - 72}{324} = \frac{252}{324} = \frac{7}{9}$$

Therefore, $\cos\theta = \sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{3}$.

Solution 2

Let $SR = x$ and the radius of the circle be r.
Join PT. Then $PQ = PT = r$.
Since $QT = 8$ and $TR = 10$, then $QR = 18$.
By the Pythagorean Theorem, $PR = \sqrt{QR^2 - PQ^2} = \sqrt{18^2 - r^2}$.
Therefore, $\cos\theta = \dfrac{PR}{QR} = \dfrac{\sqrt{18^2 - r^2}}{18}$.
By the cosine law in $\triangle PTR$,

$$\begin{aligned}
PT^2 &= PR^2 + TR^2 - 2(PR)(TR)\cos\theta \\
r^2 &= (18^2 - r^2) + 10^2 - 2\sqrt{18^2 - r^2}(10) \cdot \frac{\sqrt{18^2 - r^2}}{18} \\
r^2 &= 324 - r^2 + 100 - \tfrac{10}{9}(324 - r^2) \\
r^2 &= 424 - r^2 - 360 + \tfrac{10}{9}r^2 \\
\tfrac{8}{9}r^2 &= 64 \\
r^2 &= 72
\end{aligned}$$

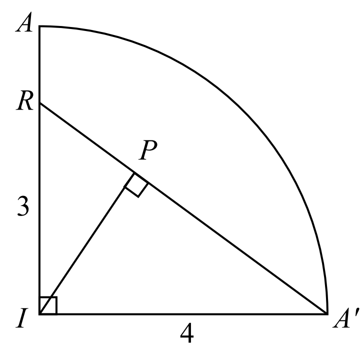
Therefore, $\cos\theta = \dfrac{\sqrt{324 - r^2}}{18} = \dfrac{\sqrt{252}}{18} = \dfrac{\sqrt{7}}{3}$.

Answer: $\frac{\sqrt{7}}{3}$

23. Suppose that the largest integer in S is L, the smallest integer in S is P, and the mean of the elements of S is m.
We are told that $m = \frac{2}{5}L$ and $m = \frac{7}{4}P$.
Thus, $\frac{7}{4}P = \frac{2}{5}L$ or $P = \frac{8}{35}L$.
Since P and L are positive integers, then L must be divisible by 35.
That is, $L = 35k$ for some positive integer k.
From this, $P = 8k$ and $m = 14k$.
Now, the sum of the elements of S is $nm = 14kn$.
But all of the elements of S are positive and they include at least $P = 8k$ and $L = 35k$, whose sum is $43k$.
Therefore, $14kn \geq 43k$, which means that $n \geq 4$.
If $n = 4$, then the sum of the elements in S is $56k$.
This means that there are two elements in addition to $8k$ and $35k$ and the sum of these two additional elements is $56k - 43k = 13k$.
But each of these elements must be at least $8k$ and so their sum is at least $16k$.
This is a contradiction, so there cannot be 4 elements in S.
If $n = 5$, then the sum of the elements in S is $70k$.
This means that there are three elements in addition to $8k$ and $35k$ and the sum of these three additional elements is $70k - 43k = 27k$.
If $n = 5$ and $k = 2$, we have $P = 16$ and $L = 70$ and we want the sum of the remaining three elements, which must be distinct integers larger than 16 and less than 70, to be 54.
This is possible with elements 17, 18, 19.
In this case, $S = \{16, 17, 18, 19, 70\}$. The sum of the elements of S is 140 and the average is 28.
Note that $28 = \frac{7}{4} \cdot 16 = \frac{2}{5} \cdot 70$.
Therefore, the minimum possible value of n is 5.

ANSWER: 5

24. We cut the lateral surface of the cone along line segment IA.
This forms a sector of a circle with radius $IA = 4$.
We label the second endpoint of the arc of this sector as A' (it previously connected with A).
Since the circumference of the base of the original cone is $2\pi(1) = 2\pi$, then the length of the arc of this sector is 2π.
Since a full circle with radius 4 has circumference 8π and length of the arc of this sector is 2π, then this sector is $\frac{1}{4}$ of a circle, so has central angle $\angle AIA' = 90^\circ$.
Point R still lies on IA with $IR = 3$.
The shortest path starting at R, travelling once around the cone, and ending at A is now the shortest path in this sector beginning at R and ending at A'. This is the straight line segment from R to A'.
Since $IR = 3$, $IA' = 4$, and $\angle RIA' = 90^\circ$, then $RA' = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$, by the Pythagorean Theorem.
The point P will be the point on RA' for which IP is perpendicular to RA'.



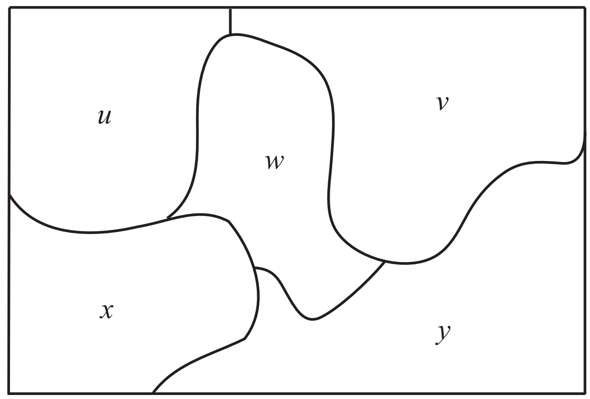
We can then calculate the area of $\triangle RIA'$ in two different ways

$$\text{Area} = \frac{1}{2} \cdot IR \cdot IA' = \frac{1}{2} \cdot RA' \cdot IP$$

to obtain $IR \cdot IA' = RA' \cdot IP$ or $3 \cdot 4 = 5 \cdot IP$ and so $IP = \frac{12}{5}$.
Therefore, the length of IP is $\frac{12}{5}$.
(We could also calculate the length IP from $\triangle RIA'$ using trigonometry or similar triangles.)

ANSWER: $\frac{12}{5}$

25. We label the regions with unknown integers u, v, w, x, y, as shown:



The given list of integers includes 4 odd integers $(1, 3, 5, 9)$ and 7 even integers $(2, 4, 6, 8, 10, 12, 14)$. We proceed by focusing on w and considering a number of cases:

(i) $\underline{w \text{ is even}}$
In this case, none of u, v, x, y can be even, by Rule 1.
Therefore, all four odd integers have to be used for u, v, x, y.
Note that 3 and 9 cannot be adjacent, by Rule 2.
Since 3, 5 and 9 are all used among u, v, x, y, then w cannot be 6, 10 or 12, by Rules 2 and 3.
Therefore, w can only be 2, 4, 8, 14.
There are 4 choices for w.
For each of these, there are 4 choices of placement for 3.
Once 3 is placed, the position of 9 is determined (it must be "opposite" 3).
Once 3 and 9 are placed, there are 2 choices of position for 5 and then 1 choice for 1.
In this case, there are $4 \times 4 \times 2 = 32$ possible labellings.

(ii) $\underline{w = 3 \text{ or } w = 9}$
Here, there are 2 choices for w.
The other possible value for w cannot be used among u, v, x, y, by Rule 2.
We cannot use 3 or 4 even numbers among u, v, x, y, otherwise two even numbers would be adjacement.
Therefore, at most 2 even numbers can be used.
But there are only 2 odd numbers left (1 and 5), so exactly two even numbers must be used and these cannot be adjacent. The even numbers used cannot include 6, 10, 12, by Rules 2 and 3 (each even number used will be adjacent to w (which is 3 or 9) and to 5).
Therefore, only 2, 4, 8, 14 can be used.
There are thus 2 choices for w, 4 choices of placement for 5, 1 choice of placement for 1 (opposite 5), $\binom{4}{2} = 6$ choices for the two even numbers, and 2 ways to place these even numbers.
In this case, there are $2 \times 4 \times 6 \times 2 = 96$ possible labellings.

(iii) $\underline{w = 5}$
Since 2 even numbers cannot be adjacent among u, v, x, y, then there must be 2 or 3 odd numbers among u, v, x, y.
Suppose that there are 3 odd numbers among u, v, x, y. (These must be 1, 3, 9.)
Note that 3 and 9 cannot be adjacent, so must be opposite.
There are 4 choices for placement of 3, which gives 1 placement for 9 (opposite 3), and

2 choices for 1 (in either open spot).
The remaining integer will be even and cannot be 6, 10 or 12, and so there are 4 choices for the remaining spot.
In this subcase, there are $4 \times 1 \times 2 \times 4 = 32$ possible labellings.

Suppose that there are now 2 odd numbers among u, v, x, y. These two cannot be adjacent (since the even numbers cannot be adjacent), so must be opposite. Therefore any of the $\binom{3}{2} = 3$ pairs from $\{1, 3, 9\}$ can be used.
There are 4 choices of placement for the smaller of these numbers and then the larger one is placed opposite.
The even numbers are chosen from among $\{2, 4, 8, 14\}$, as before. There are $\binom{4}{2} = 6$ choices for the pair of numbers and 2 ways of placing them.
In this subcase, there are thus $3 \times 4 \times 6 \times 2 = 144$ possible labellings.
In this case, there are thus $32 + 144 = 176$ possible labellings.

(iv) $\underline{w = 1}$
Since 2 even numbers cannot be adjacent among u, v, x, y, then there must be 2 or 3 odd numbers among u, v, x, y.
Suppose that there are 3 odd numbers among u, v, x, y.
Note that 3 and 9 cannot be adjacent, so must be opposite.
There are 4 choices for placement of 3, which gives 1 placement for 9 (opposite 3), and 2 choices for 5 (in either open spot).
The remaining integer will be even and cannot be 6 or 12, and so there are 5 choices for the remaining spot. (Note that the even number is now opposite 5 so could equal 10.)
In this subcase, there are $4 \times 1 \times 2 \times 5 = 40$ possible labellings.

Suppose that there are now 2 odd numbers among u, v, x, y. These two cannot be adjacent (since the even numbers cannot be adjacent), so must be opposite. Therefore any of the 3 pairs from $\{3, 5, 9\}$ can be used.
If 3 and 9 are used, there are 4 choices of placement for 3 and then 1 choice of placement for 9.
The even numbers are chosen from among $\{2, 4, 8, 10, 14\}$, as before. There are $\binom{5}{2} = 10$ choices for the pair of numbers and 2 ways of placing them.
In this subcase, there are thus $4 \times 10 \times 2 = 80$ possible labellings.

If 3 and 5 are used, there are 4 choices of placement for 3 and then 1 choice of placement for 5.
The even numbers are chosen from among $\{2, 4, 8, 14\}$, as before. There are $\binom{4}{2} = 6$ choices for the pair of numbers and 2 ways of placing them.
In this subcase, there are thus $4 \times 6 \times 2 = 48$ possible labellings.

If 9 and 5 are used, there are again 48 possible labellings.

In this case, there are thus $40 + 80 + 48 + 48 = 216$ possible labellings.

Therefore, in total there are $32 + 96 + 176 + 216 = 520$ possible labellings.

ANSWER: 520

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $2+0+1+5=8$.

   (b) The average of the numbers in the first list is $m = \dfrac{12+15+9+14+10}{5} = \dfrac{60}{5} = 12$.

   The average of the numbers in the second list is $n = \dfrac{24+t+8+12}{4} = \dfrac{44+t}{4} = 11+\dfrac{1}{4}t$.

   Therefore, $n-m = \left(11+\dfrac{1}{4}t\right) - 12 = \dfrac{1}{4}t - 1$.

   Since the answer to (a) is 8, then $t=8$, and so $n-m = 2-1 = 1$.

   (c) Since the two lines intersect at (a,b), then these coordinates satisfy the equation of each line.

   Therefore, $b=13$ and $b=3a+t$.

   Since $b=13$, then $13=3a+t$ or $3a=13-t$, and so $a=\frac{13}{3}-\frac{1}{3}t$.

   Since the answer to (b) is 1, then $t=1$, and so $a=\frac{13}{3}-\frac{1}{3}=4$.

   Answer: $8, 1, 4$

1. (a) Since $1024 = 2^{10}$ and $2^{k+4} = 1024$, then $k+4=10$ or $k=6$.

   (b) If $2t+2x-t-3x+4x+2t=30$, then $3t+3x=30$ or $t+x=10$ and so $x=10-t$.

   Since the answer to (a) is 6, then $t=6$ and so $x=10-t=4$.

   (c) We use the Pythagorean Theorem repeatedly and obtain

   $$DE^2 = CD^2 + CE^2 = CD^2 + (BC^2 + BE^2) = CD^2 + BC^2 + (AB^2 + AE^2)$$

   Since $AE=\sqrt{5}$, $AB=\sqrt{4}$, $BC=\sqrt{3}$, and $CD=\sqrt{t}$, then

   $$DE^2 = t+3+4+5 = t+12$$

   Since the answer to (b) is 4, then $t=4$, and so $DE^2=16$.

   Since $DE>0$, then $DE=\sqrt{16}=4$.

   Answer: $6, 4, 4$

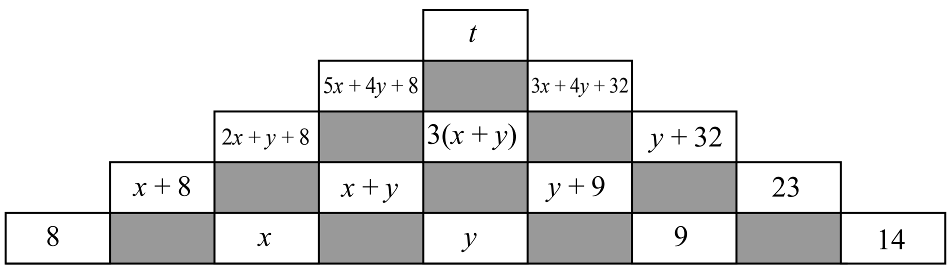
2. (a) Since A and B have the same y-coordinate, then AB is horizontal.
We treat AB as the base of $\triangle ABC$. It has length $5-(-1)=6$.
The height of $\triangle ABC$ is the vertical distance from $C(-4,-3)$ to the line through AB (which is $y=2$). This distance is $2-(-3)=5$.
Therefore, the area of $\triangle ABC$ is $\frac{1}{2}(6)(5)=15$.

(b) According to the given information, at last night's rehearsal, Canada's Totally Musical Choir (CTMC) spent $75-6-30-t=39-t$ minutes singing their pieces.
As a percentage of the full 75 minute rehearsal, this is

$$\frac{39-t}{75}\times 100\% = \left(\frac{4}{3}(39-t)\right)\% = \left(52-\frac{4}{3}t\right)\%$$

Since the answer to (a) is 15, then $t=15$ and so CTMC spent $\left(52-\frac{4}{3}t\right)\% = 32\%$ of their rehearsal singing their pieces, and so $N=32$.

(c) Using the rule that the sum of two entries in consecutive boxes in one row is the entry between these boxes in the row above, the entries in the second row from the bottom should be $x+8, x+y, y+9, 23$.
Thus, the first entry in the third row from the bottom should be $2x+y+8$ and the last entry in this row should be $y+32$.
Also, the two entries in the second row from the top should be $5x+4y+8$ and $3x+4y+32$.



Now the given entry $3(x+y)$ is the sum of the two entries below it.
Thus, $3x+3y=(x+y)+(y+9)$, which gives $2x+y=9$.
Also, the top entry t is the sum of the two entries below it.
Thus, $t=(5x+4y+8)+(3x+4y+32)$, which gives $8x+8y=t-40$ or $x+y=\frac{1}{8}t-5$.
Since $2x+y=9$ and $x+y=\frac{1}{8}t-5$, then $x=(2x+y)-(x+y)=9-\left(\frac{1}{8}t-5\right)=14-\frac{1}{8}t$.
Since $x+y=\frac{1}{8}t-5$ and $x=14-\frac{1}{8}t$, then $y=(x+y)-x=\frac{1}{4}t-19$.
Thus, $x-y=\left(14-\frac{1}{8}t\right)-\left(\frac{1}{4}t-19\right)=33-\frac{3}{8}t$.
Since the answer to (b) is 32, then $t=32$ and so $x-y=33-\frac{3}{8}t=21$.

Answer: 15, 32, 21

3. (a) A rectangular prism has six faces.
Since the prism has edge lengths 2, 3 and 4, then the prism has two faces with dimensions 2×3 (each has area 6), two faces with dimensions 2×4 (each has area 8), and two faces with dimensions 3×4 (each has area 12).
Therefore, the surface area of the prism is $2(6) + 2(8) + 2(12) = 52$.

(b) Since AB and CD are parallel, then $\angle FYZ = \angle AVY = 72^\circ$ (alternating angles).
Also, $\angle FZY = \angle DZG = x^\circ$ (opposite angles).
In $\triangle FYZ$, the sum of the angles is 180°, and so $\angle YFZ + \angle FYZ + \angle FZY = 180^\circ$.
Thus, $t^\circ + 72^\circ + x^\circ = 180^\circ$ or $x = 108 - t$.
Since the answer to (a) is 52, then $t = 52$ and so $x = 108 - t = 56$.

(c) We do some preliminary work assuming that the number t to be received is a positive integer. (This turns out to be the case; if it were not the case, we might have to change strategies.)
Since $1! = 1$, then $30t$ is divisible by $1!$ regardless of the value of t.
Since $2! = 2$, then $30t = 2(15t)$ is divisible by $2!$ regardless of the value of t.
Since $3! = 6$, then $30t = 6(15t)$ is divisible by $3!$ regardless of the value of t.
Next, consider $4! = 24$.
Since $\dfrac{30t}{4!} = \dfrac{30t}{24} = \dfrac{5t}{4}$, then $30t$ is divisible by $4!$ if and only if t is divisible by 4.
Next consider $5! = 120$.
Since $\dfrac{30t}{5!} = \dfrac{30t}{120} = \dfrac{t}{4}$, then $30t$ is divisible by $5!$ if and only if t is divisible by 4.
Next consider $6! = 720$.
Since $\dfrac{30t}{6!} = \dfrac{30t}{720} = \dfrac{t}{24}$, then $30t$ is divisible by $6!$ if and only if t is divisible by 24.
Since the answer to (b) is 56, then $t = 56$.
Since t is divisible by 4 but not by 24, then $30t$ is divisible by $1!$, $2!$, $3!$, $4!$, and $5!$, but not by $6!$.
Furthermore, $30t = 1680$ and if $n \geq 7$, then $n! \geq 7! = 5040$, so $30t$ is not divisible by $n!$ for $n \geq 7$ (because $n! > 30t$).
Therefore, there are 5 integers b for which $30t$ is divisible by $b!$.

Answer: $52, 56, 5$