The **CENTRE for EDUCATION**
in MATHEMATICS and COMPUTING
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2014 Hypatia Contest

Wednesday, April 16, 2014
(in North America and South America)

Thursday, April 17, 2014
(outside of North America and South America)

Solutions

1. (a) Using the given definition, $8 \odot 7 = \sqrt{8 + 4(7)} = \sqrt{36} = 6$.

   (b) Since $16 \odot n = 10$, then $\sqrt{16 + 4n} = 10$ or $16 + 4n = 100$ (by squaring both sides) or $4n = 84$ and so $n = 21$.
   We check that indeed $16 \odot n = 16 \odot 21 = \sqrt{16 + 4(21)} = \sqrt{100} = 10$.

   (c) We first determine the value inside the brackets: $9 \odot 18 = \sqrt{9 + 4(18)} = \sqrt{81} = 9$.
   So then $(9 \odot 18) \odot 10 = 9 \odot 10 = \sqrt{9 + 4(10)} = \sqrt{49} = 7$.

   (d) Using the definition, $k \odot k = \sqrt{k + 4k} = \sqrt{5k}$.
   So we are asked to solve the equation $\sqrt{5k} = k$.
   Squaring both sides we get, $5k = k^2$ and so $k^2 - 5k = 0$ or $k(k - 5) = 0$, and so $k = 0$ or $k = 5$.
   Checking $k = 0$, we obtain $k \odot k = 0 \odot 0 = \sqrt{0 + 4(0)} = \sqrt{0} = 0 = k$, as required.
   Checking $k = 5$, we obtain $k \odot k = 5 \odot 5 = \sqrt{5 + 4(5)} = \sqrt{25} = 5 = k$, as required.
   Thus, the only possible solutions are $k = 0$ and $k = 5$.

2. (a) The song's position on week 1 ($w = 1$), is $P(1) = 3(1)^2 - 36(1) + 110 = 77$.

   (b) The song's position is given by the quadratic function $P = 3w^2 - 36w + 110$, the graph of which is a parabola opening upward.
   The minimum value of this parabola is achieved at its vertex.
   To find the coordinates of the vertex, we may complete the square.

$$\begin{aligned} P &= 3w^2 - 36w + 110 \\ &= 3(w^2 - 12w) + 110 \\ &= 3(w^2 - 12w + 36 - 36) + 110 \\ &= 3(w^2 - 12w + 36) - 108 + 110 \\ &= 3(w - 6)^2 + 2 \end{aligned}$$

   Therefore, the vertex of the parabola occurs at $w = 6$ and $P = 2$.

   (i) The best position that the song "Recursive Case" reaches is position #2.

   (ii) The song reaches its best position on week 6.

   (c) To determine the last week that "Recursive Case" appears on the Top 200 list, we want to find the largest w such that $P = 3w^2 - 36w + 110 \le 200$.
   Using the vertex form from part (b), we have $3(w - 6)^2 + 2 \le 200$ or $3(w - 6)^2 \le 198$ or $(w - 6)^2 \le 66$.
   To determine the largest positive integer w such that $(w - 6)^2 \le 66$, we want to find the largest square that is less than or equal to 66.
   Since $8^2 \le 66$ and $9^2 > 66$, then the largest w satisfies $w - 6 = 8$ and so $w = 14$.
   The last week that "Recursive Case" appears on the Top 200 list is week 14.
   To check this we note that,

$$P(14) = 3(14 - 6)^2 + 2 = 194 \le 200 \text{ but } P(15) = 3(15 - 6)^2 + 2 = 245 > 200.$$

3. (a) We will denote the area of a figure using vertical bars.
For example, $|\triangle BCE|$ is the area of $\triangle BCE$.
Since $ABCD$ has equal side lengths (it is a square) and
$EA = EB = EC = ED$, then the 4 triangular faces of
pyramid $ABCDE$ are congruent and so all have equal area.
The surface area of pyramid $ABCDE$ is equal to the sum of
the base area and the areas of the 4 triangular faces or
$|ABCD| + |\triangle EAB| + |\triangle EBC| + |\triangle ECD| + |\triangle EDA| = |ABCD| + 4|\triangle EAB|$.
Square $ABCD$ has side length 20 and so $|ABCD| = 20 \times 20 = 400$.
To determine $|\triangle EAB|$, we construct altitude EJ as shown.
$\triangle EAB$ is isosceles and so EJ bisects AB with $AJ = JB = 10$.
$\triangle EAJ$ is a right-angled triangle and so by the Pythagorean
Theorem, $EA^2 = AJ^2 + EJ^2$ or $18^2 = 10^2 + EJ^2$, so then
$EJ = \sqrt{224}$ or $EJ = 4\sqrt{14}$ (since $EJ > 0$).

The area of $\triangle EAB$ is $\frac{1}{2}(AB)(EJ) = \frac{1}{2}(20)(4\sqrt{14}) = 40\sqrt{14}$.
Thus the surface area of $ABCDE$ is $|ABCD| + 4|EAB| = 400 + 4(40\sqrt{14}) = 400 + 160\sqrt{14}$.

(b) As in part (b), J is positioned such that EJ is an altitude of
$\triangle EAB$ and so $EJ = 4\sqrt{14}$. Since EF is perpendicular to
the base of the pyramid, then EF is perpendicular to FJ, as
shown.
Further, F is the centre of the base $ABCD$ and J is
the midpoint of AB, so then FJ is parallel to CB and
$FJ = \frac{1}{2} \times CB = \frac{1}{2} \times 20 = 10$.
By the Pythagorean Theorem, $EJ^2 = EF^2 + FJ^2$ or $224 = EF^2 + 100$ and so
$EF = \sqrt{124} = 2\sqrt{31}$ (since $EF > 0$).
Therefore, the height EF of the pyramid $ABCDE$ is $2\sqrt{31}$.

(c) Points G and H are the midpoints of ED and EA, respectively, and so
$EG = GD = EH = HA = 9$.
Thus, GH is a midsegment of $\triangle EDA$ and so GH is parallel to DA and $GH = \frac{1}{2} \times DA = 10$.
(Note that this result follows from the fact that $\triangle EGH$ is similar to $\triangle EDA$. Can you
prove this?)
Since GH is parallel to DA and DA is parallel to CB, then GH is parallel to CB.
That is, quadrilateral $BCGH$ (whose area we are asked to find) is a trapezoid.
To determine the area of trapezoid $BCGH$, we need the lengths of the parallel sides
($GH = 10$ and $CB = 20$) and we need the perpendicular distance between these two
parallel sides.
We will proceed by showing that HT (in the diagram below) is such a perpendicular height
of the trapezoid and also by determining its length.
Join H to I, the midpoint of EB, so that HI is a midsegment
of $\triangle EAB$ with $HI = 10$.
Position P on the base of the pyramid such that HP is
perpendicular to the base.
Similarly, position M on the base such that IM is perpen-
dicular to the base.
Let MP extended intersect the edge BC at T and the edge
AD at K, as shown.
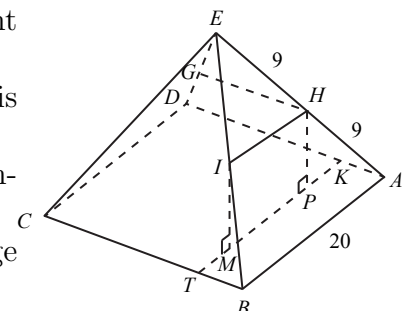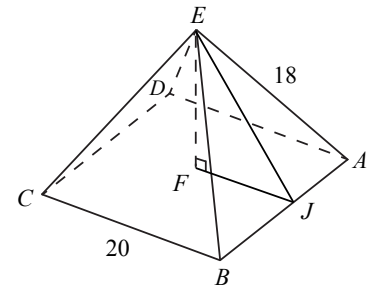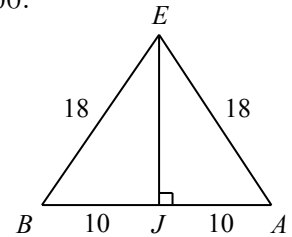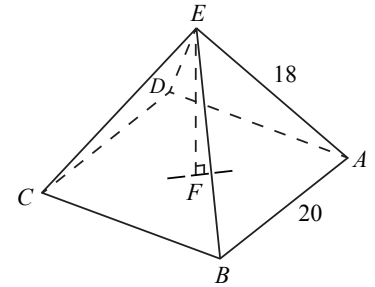
By symmetry, $HP = IM$ and so $HPMI$ is a rectangle with $PM = HI = 10$.

Further, since AB is parallel to HI and HI is parallel to KT (both are perpendicular to HP and IM), then AB is parallel to KT. So then $ABTK$ is a rectangle and $KT = AB = 20$.

Also by symmetry, PM is centred on line segment KT such that $KP = MT = \frac{20-10}{2} = 5$ ($EA = EB$ and E lies vertically above the centre of the square base).
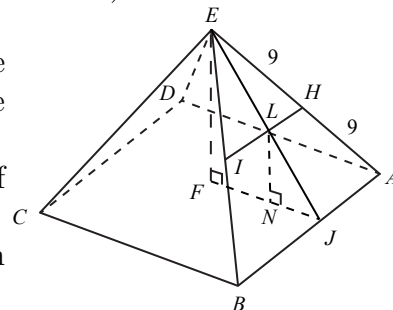
Therefore $PT = PM + MT = 10 + 5 = 15$.

Next, let the midpoint of HI be L and position N on the base of the pyramid such that LN is perpendicular to the base.

Since F is the centre of the square and J is the midpoint of edge AB, then FJ passes through N.

Since $\triangle EFJ$ is similar to $\triangle LNJ$ (by $AA \sim$), then $\frac{LN}{EF} = \frac{LJ}{EJ} = \frac{1}{2}$ (since HI is a midsegment of $\triangle EAB$).

Therefore, $LN = \frac{1}{2}(EF) = \frac{1}{2}(2\sqrt{31}) = \sqrt{31}$.

Since HI is parallel to the base of the pyramid, $ABCD$, then $HP = LN = \sqrt{31}$ (both are perpendicular to the base).
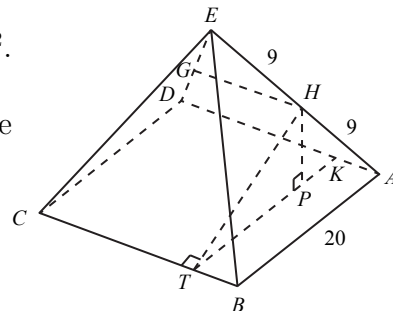
In right-angled $\triangle HPT$, $HT^2 = HP^2 + PT^2 = (\sqrt{31})^2 + 15^2$.

So $HT^2 = 256$ and $HT = 16$ (since $HT > 0$).

Since the plane containing $\triangle HPT$ is perpendicular to the base $ABCD$, then HT is perpendicular to BC.

That is, HT is the height of trapezoid $BCGH$.

Finally, $|BCGH| = \frac{HT}{2}(GH + CB) = \frac{16}{2}(10 + 20) = 240$.

4. (a) If $(4, y, z)$ is an APT, then $4^2 + y^2 = z^2 + 1$ or $z^2 - y^2 = 15$ and so $(z - y)(z + y) = 15$.

Since y and z are positive integers, then $(z + y)$ is a positive integer and thus $(z - y)$ is also a positive integer (since the product of the two factors is 15).

That is, $(z - y)$ and $(z + y)$ are the possible pairs of factors of 15, of which there are two: 1 and 15, and 3 and 5.

Since $z + y > z - y$, we have the following two systems of equations to solve:

$$z - y = 1 \qquad\qquad\qquad z - y = 3$$
$$z + y = 15 \qquad\qquad\qquad z + y = 5$$

Adding the first pair of equations, we get $2z = 16$ or $z = 8$ and so $y = 7$.

Adding the second pair of equations, we get $2z = 8$ or $z = 4$ and so $y = 1$.

Since $y > 1$, this second solution is not an APT.

The only APT with $x = 4$ is $(4, 7, 8)$.

(b) Let positive integers u, v, w be the lengths of the sides of $\triangle UVW$ (with side length u opposite vertex U, v opposite V, and w opposite W).

Without loss of generality, assume (u, v, w) forms an APT such that $u^2 + v^2 = w^2 + 1$ with $u > 1$ and $v > 1$.

The area of $\triangle UVW$ is given by $A = \frac{1}{2}uv \sin W$ $\quad(\star)$ (we will derive this formula at the end of the solution).

Assume that this area, A, is an integer.

In $\triangle UVW$, the cosine law gives $w^2 = u^2 + v^2 - 2uv \cos W$ or $\cos W = \dfrac{u^2 + v^2 - w^2}{2uv}$.

However, (u, v, w) is an APT and thus $u^2 + v^2 = w^2 + 1$ and so $u^2 + v^2 - w^2 = 1$.

Substituting, we get $\cos W = \dfrac{1}{2uv}$ and since $\sin^2 W = 1 - \cos^2 W$, then

$$\sin^2 W = 1 - \left(\dfrac{1}{2uv}\right)^2.$$

Squaring $(\star)$ and substituting for $\sin^2 W$, we get $A^2 = \frac{1}{4}u^2v^2\left(1 - \left(\dfrac{1}{2uv}\right)^2\right)$.

Simplifying, this last equation we get $A^2 = \frac{1}{4}u^2v^2\left(\dfrac{(2uv)^2 - 1}{(2uv)^2}\right)$ or $A^2 = \frac{1}{4}u^2v^2\left(\dfrac{(2uv)^2 - 1}{4u^2v^2}\right)$

or $16A^2 = (2uv)^2 - 1$ and so $4u^2v^2 - 16A^2 = 1$.

The left side of this equation has a common factor of 4 and so is an even integer for all integers u, v, A.

However, the right side of the equation is 1 (an odd integer) and so this equation has no solutions.

Our assumption that A is an integer is false and thus the area of any triangle whose sides lengths form an APT is not an integer.

Derivation of $A = \frac{1}{2}uv\sin W$

In $\triangle UVW$, construct the perpendicular from V meeting side UW (or side UW extended) at T.

In this case, $\triangle UVW$ has height VT and base UW.

In right-angled $\triangle VWT$, we have $\sin W = \dfrac{VT}{VW}$ and so $VT = u\sin W$.

(Note that if $\angle W$ is obtuse, then height VT lies outside the triangle and we have $\sin(180° - W) = \dfrac{VT}{VW}$ and so $VT = u\sin W$ since $\sin(180° - W) = \sin W$).

Thus, the area of $\triangle UVW$ is $\frac{1}{2}(UW)(VT) = \frac{1}{2}uv\sin W$.

(c) Since $(5t + p, bt + q, ct + r)$ is an APT, then $(5t + p)^2 + (bt + q)^2 = (ct + r)^2 + 1$.

Expanding, we get $25t^2 + 10pt + p^2 + b^2t^2 + 2bqt + q^2 = c^2t^2 + 2crt + r^2 + 1$, or $(25 + b^2)t^2 + (10p + 2bq)t + (p^2 + q^2) = c^2t^2 + 2crt + (r^2 + 1)$.

Each side of this equation is a quadratic polynomial in the variable t.

Since this equation is true for all positive integers t, then the corresponding coefficients on the left side and the right side are equal.

That is,

$$
\begin{align}
25 + b^2 &= c^2 \tag{1}\\
10p + 2bq &= 2cr \tag{2}\\
p^2 + q^2 &= r^2 + 1 \tag{3}
\end{align}
$$

Equation (1) is the Pythagorean relationship, and since b, c are positive integers then the Pythagorean triple $(5, 12, 13)$ satisfies (1) with $b = 12$ and $c = 13$.

(These are in fact the only values of b and c that work but we don't need to show this.)

Substituting b and c into (2) and simplifying we get $5p + 12q = 13r$ (4).

Squaring equation (4), we get $25p^2 + 120pq + 144q^2 = 169r^2$ (5).

From equation (3), we get $r^2 = p^2 + q^2 - 1$, and substituting this into (5) gives $25p^2 + 120pq + 144q^2 = 169(p^2 + q^2 - 1)$ or $144p^2 - 120pq + 25q^2 = 169$ or $(12p - 5q)^2 = 169$ and so $12p - 5q = \pm13$.

We search for positive integers p and q with $p \geq 100$ satisfying $12p - 5q = \pm13$ or $5q = 12p \pm 13$.

Since $12p$ is even for all integers p and 13 is odd, then $12p \pm 13$ is odd and so $5q$ must also be odd.

The units digit of $5q$ is either 0 or 5 for all positive integers q and since $5q$ is odd, then its units digit is 5.

The units digit of $5q - 13$ is 2 and the units digit of $5q + 13$ is 8, and so the units digit of $12p$ is either 2 or 8 (since $12p = 5q \pm 13$).

A value of $p \geq 100$ such that $12p$ has units digit 2 is $p = 101$.

Substituting, we get $12(101) = 5q - 13$ or $5q = 1225$ and so $q = 245$.

Substituting p and q into (4) gives $r = 265$.

A value of $p \geq 100$ such that $12p$ has units digit 8 is $p = 104$.

Substituting, we get $12(104) = 5q + 13$ or $5q = 1235$ and so $q = 247$.

Substituting p and q into (4) gives $r = 268$.

Therefore, two possible 5-tuples (b, c, p, q, r) satisfying the given conditions are $(12, 13, 101, 245, 265)$ and $(12, 13, 104, 247, 268)$.

(Note that there are an infinite number of possible solutions.)