



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Fermat Contest

(Grade 11)

Thursday, February 20, 2014
(in North America and South America)

Friday, February 21, 2014
(outside of North America and South America)

Solutions

1. Evaluating the numerator first, we obtain $\frac{15 - 3^2}{3} = \frac{15 - 9}{3} = \frac{6}{3} = 2$.
ANSWER: (A)

2. Since $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, $10^4 = 10\,000$, and $10^5 = 100\,000$, then 2014 is between 10^3 and 10^4 .
ANSWER: (D)

3. When $x = 2$, we have $(x + 2 - x)(2 - x - 2) = (2 + 2 - 2)(2 - 2 - 2) = (2)(-2) = -4$.
Alternatively, we could simplify $(x + 2 - x)(2 - x - 2)$ to obtain $(2)(-x)$ or $-2x$ and then substitute $x = 2$ to obtain a result of $-2(2)$ or -4 .
ANSWER: (E)

4. The positive integer divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.
The pairs of divisors that give a product of 24 are 24×1 , 12×2 , 8×3 , and 6×4 .
We want to find two positive integers x and y whose product is 24 and whose difference is 5.
Since $8 \times 3 = 24$ and $8 - 3 = 5$, then $x = 8$ and $y = 3$ are the required integers.
Here, $x + y = 8 + 3 = 11$.
ANSWER: (B)

5. Since square $WXYZ$ has area 9, then its side length is $\sqrt{9} = 3$.
Since W is the centre of the circle and X lies on the circumference of the circle, then WX is a radius of the circle, so the radius of the circle has length 3.
Therefore, the area of the circle is $\pi(3^2) = 9\pi$.
ANSWER: (C)

6. The percentage 50% is equivalent to the fraction $\frac{1}{2}$, while 75% is equivalent to $\frac{3}{4}$.
Since 50% of N is 16, then $\frac{1}{2}N = 16$ or $N = 32$.
Therefore, 75% of N is $\frac{3}{4}N$ or $\frac{3}{4}(32)$, which equals 24.
ANSWER: (D)

7. *Solution 1*

$\angle SRP$ is an exterior angle for $\triangle PQR$.

Therefore, $\angle SRP = \angle RPQ + \angle RQP$ or $(180 - x)^\circ = 30^\circ + 2x^\circ$.

Thus, $180 - x = 30 + 2x$ or $3x = 150$ and so $x = 50$.

Solution 2

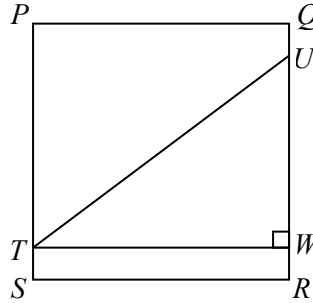
Since QRS is a straight line segment and $\angle SRP = (180 - x)^\circ$, then $\angle PRQ$ is the supplement of $\angle SRP$ so $\angle PRQ = x^\circ$.

Since the angles in a triangle add to 180° , then $\angle PRQ + \angle PQR + \angle RPQ = 180^\circ$, and so $x^\circ + 2x^\circ + 30^\circ = 180^\circ$.

From this, we obtain $3x = 150$ and so $x = 50$.
ANSWER: (E)

8. We use A, B, C, D, E to represent Amy, Bob, Carl, Dan, and Eric, respectively.
We use the greater than symbol ($>$) to represent “is taller than” and the less than symbol ($<$) to represent “is shorter than”.
From the first bullet, $A > C$.
From the second bullet, $D < E$ and $D > B$ so $E > D > B$.
From the third bullet, $E < C$ or $C > E$.
Since $A > C$ and $C > E$ and $E > D > B$, then $A > C > E > D > B$, which means that Bob is the shortest.
ANSWER: (B)

9. We draw a line through T to point W on QR so that TW is perpendicular to QR .



Since $TWRS$ has three right angles (at W , R and S), then it must be a rectangle.

Therefore, $WR = TS = 1$ and $TW = SR = 8$.

Since $QU = 1$, then $UW = QR - QU - WR = 8 - 1 - 1 = 6$.

Now, $\triangle TWU$ is right-angled at W .

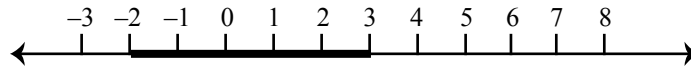
By the Pythagorean Theorem, we have $TU^2 = TW^2 + UW^2$.

Thus, $TU^2 = 8^2 + 6^2 = 64 + 36 = 100$.

Since $TU > 0$, then $TU = \sqrt{100} = 10$.

ANSWER: (C)

10. After the first rotation, the line segment lies between -2 and 3 .

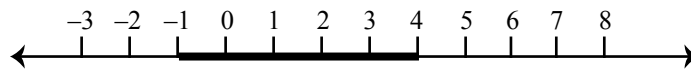


The line segment is now to be rotated about the point at 1.

Since the right endpoint of the segment is 2 units to the right of 1 before the rotation, then the left endpoint of the segment will be 2 units to the left of 1 after the rotation.

Thus, the left endpoint will be at -1 .

Since the line segment has length 5, then its right endpoint will be at $-1 + 5 = 4$.



Thus, the line segment lies between -1 and 4 .

ANSWER: (B)

11. Since $a = \frac{2}{3}b$, then $3a = 2b$. Since $b \neq 0$, then $a \neq 0$.

Thus, $\frac{9a + 8b}{6a} = \frac{9a + 4(2b)}{6a} = \frac{9a + 4(3a)}{6a} = \frac{21a}{6a} = \frac{7}{2}$ (since $a \neq 0$).

Alternatively, $\frac{9a + 8b}{6a} = \frac{3(3a) + 8b}{2(3a)} = \frac{3(2b) + 8b}{2(2b)} = \frac{14b}{4b} = \frac{7}{2}$ (since $b \neq 0$).

ANSWER: (A)

12. Since $100 = 10^2$, then $100^4 = (10^2)^4 = 10^8$.

Therefore, we must solve the equation $10^x \cdot 10^5 = 10^8$, which is equivalent to $10^{x+5} = 10^8$.

Thus, $x + 5 = 8$ or $x = 3$.

ANSWER: (E)

13. We note that the sum of the digits of 1000 is not 3. Every other positive integer in the given range has two or three digits.

For the sum of the digits of an integer to be 3, no digit can be greater than 3.

If a two-digit integer has sum of digits equal to 3, then its tens digit is 1, 2 or 3. The possible integers are 12, 21 and 30.

If a three-digit integer has sum of digits equal to 3, then its hundreds digit is 1, 2 or 3.

If the hundreds digit is 3, then the units and tens digits add to 0, so must be each 0. The integer must thus be 300.

If the hundreds digit is 2, then the units and tens digits add to 1, so must be 1 and 0 or 0 and 1. The possible integers are 210 and 201.

If the hundreds digit is 1, then the units and tens digits add to 2, so must be 2 and 0, or 1 and 1, or 0 and 2, giving possible integers 120, 111 and 102.

Overall, there are 9 such positive integers.

ANSWER: (D)

14. Let x be the number of days on which Pat worked.

On each of these days, he earned \$100 and had no food costs, so he earned a total of $100x$ dollars.

Since Pat worked for x of the 70 days, then he did not work on $70 - x$ days.

On each of these days, he earned no money and was charged \$20 for food, so was charged a total of $20(70 - x)$ dollars for food.

After 70 days, the money that he earned minus his food costs equalled \$5440.

Algebraically, we get $100x - 20(70 - x) = 5440$.

Thus, $100x - 1400 + 20x = 5440$ or $120x = 6840$, which gives $x = 57$.

Therefore, Pat worked on 57 of these 70 days.

(An alternative approach would be to test each of the five given choices to see how much money Pat earns after food costs are deducted.)

ANSWER: (D)

15. We make a chart that lists the possible results for the first spin down the left side, the possible results for the second spin across the top, and the product of the two results in the corresponding cells:

	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12
4	4	8	12	16

Since each spin is equally likely to stop on 1, 2, 3, or 4, then each of the 16 products shown in the chart is equally likely.

Since the product 4 appears three times in the table and this is more than any of the other numbers, then it is the product that is most likely to occur.

ANSWER: (B)

16. Since Jill never drove faster than 80 km/h over her 5 hour drive, then she could not have driven more than $5 \times 80 = 400$ km.

Since the initial odometer reading was 13 831 km, then the final odometer reading is no more than $13\,831 + 400 = 14\,231$ km.

Determining her greatest possible average speed can be done by first determining the greatest possible distance that she could have travelled, which can be done by determining the greatest

possible odometer reading.

Knowing that the final odometer reading was also a palindrome, we want to determine the greatest palindrome less than 14 231. This is 14 141. (To find this, we begin by trying to find palindromes that are at least 14 000. Such palindromes end with 41, so are of the form 14 x 41. The greatest such integer less than 14 231 is 14 141.)

Since Jill's greatest possible final odometer reading was 14 141, then she would have travelled $14\,141 - 13\,831 = 310$ km, and so her greatest possible average speed was $\frac{310}{5} = 62$ km/h.

ANSWER: (A)

17. Suppose that there are n employees at Sergio's store.

After his first average calculation, his n employees had sold an average of 75 items each, which means that a total of $75n$ items had been sold.

The next day, one employee sold 6 items, one sold 5, one sold 4, and the remaining $(n - 3)$ employees each sold 3 items.

After this day, the total number of items sold to date was $75n + (6 + 5 + 4 + (n - 3)3)$ or $75n + 15 + 3n - 9$ or $78n + 6$.

Since the new average number of items sold per employee was 78.3, then $\frac{78n + 6}{n} = 78.3$ or $78n + 6 = 78.3n$.

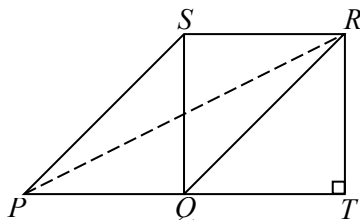
Therefore, $0.3n = 6$ or $n = 20$.

Thus, there are 20 employees in the store.

ANSWER: (C)

18. Suppose that the original square had side length x mm.

We extend PQ and draw a line through R perpendicular to PQ , meeting PQ extended at T .



$SRTQ$ is a square, since it has three right angles at S , Q , T (which makes it a rectangle) and since $SR = SQ$ (which makes the rectangle a square).

Now $RT = SQ = x$ mm and $PT = PQ + QT = 2x$ mm.

By the Pythagorean Theorem, $PR^2 = PT^2 + RT^2$ and so $90^2 = x^2 + (2x)^2$.

Therefore, $5x^2 = 8100$ or $x^2 = 1620$.

The area of the original square is x^2 mm², which equals 1620 mm².

ANSWER: (B)

19. Consider three three-digit numbers with digits RST , UVW and XYZ .

The integer with digits RST equals $100R + 10S + T$, the integer with digits UVW equals $100U + 10V + W$, and the integer with digits XYZ equals $100X + 10Y + Z$.

Therefore,

$$\begin{aligned} RST + UVW + XYZ &= 100R + 10S + T + 100U + 10V + W + 100X + 10Y + Z \\ &= 100(R + U + X) + 10(S + V + Y) + (T + W + Z) \end{aligned}$$

We note that each of $R, S, T, U, V, W, X, Y, Z$ can be any digit from 0 to 9, except that R, U and X cannot be 0.

Max wants to make $100(R + U + X) + 10(S + V + Y) + (T + W + Z)$ as large as possible. He does this by placing the largest digits (9, 8 and 7) as hundreds digits, the next largest digits (6, 5 and 4) as tens digits, and the next largest digits (3, 2 and 1) as units digits. We note that no digits can be repeated, and that the placement of the digits assigned to any of the place values among the three different three-digit numbers is irrelevant as it does not affect the actual sum. Max's sum is thus $100(9 + 8 + 7) + 10(6 + 5 + 4) + (3 + 2 + 1) = 2400 + 150 + 6 = 2556$.

Minnie wants to make $100(R + U + X) + 10(S + V + Y) + (T + W + Z)$ as small as possible. She does this by placing the smallest allowable digits (1, 2 and 3) as hundreds digits, the next smallest remaining digits (0, 4 and 5) as tens digits, and the next smallest digits (6, 7 and 8) as units digits.

Minnie's sum is thus $100(1 + 2 + 3) + 10(0 + 4 + 5) + (6 + 7 + 8) = 600 + 90 + 21 = 711$.

The difference between their sums is $2556 - 711 = 1845$.

ANSWER: (C)

20. Since $\triangle PQR$ has $PQ = QR = RP$, then $\triangle PQR$ is equilateral and all of its angles equal 60° . Since ST is parallel to QR , SV is parallel to PR , and TU is parallel to PQ , then all of the angles in $\triangle PST$, $\triangle SQV$ and $\triangle TUR$ equal 60° . In other words, each of these triangles is also equilateral.

Let $SQ = x$.

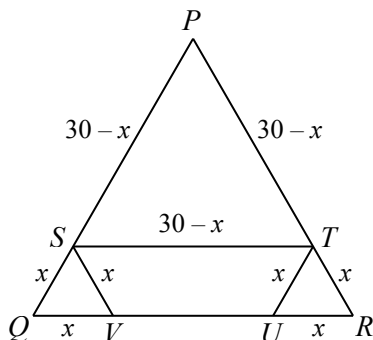
Since $\triangle SQV$ is equilateral, then $QV = VS = SQ = x$.

Since $PQ = 30$, then $PS = 30 - x$.

Since $\triangle PST$ is equilateral, then $ST = TP = PS = 30 - x$.

Since $PR = 30$, then $TR = 30 - (30 - x) = x$.

Since $\triangle TUR$ is equilateral, then $TU = UR = TR = x$.



Since $VS + ST + TU = 35$, then $x + (30 - x) + x = 35$ or $30 + x = 35$ and so $x = 5$.

Therefore, $VU = QR - QV - UR = 30 - x - x = 30 - 5 - 5 = 20$.

ANSWER: (D)

21. When 2 kg of the 10 kg of peanuts are removed, there are 8 kg of peanuts remaining.

Since 2 kg of raisins are added, then there are 2 kg of raisins in the bin.

The peanuts and raisins are thoroughly mixed.

Since 2 kg of this mixture is removed and this is one-fifth of the total mass, then one-fifth of the mass of peanuts (or $\frac{8}{5}$ kg) is removed and one-fifth of the mass of raisins (or $\frac{2}{5}$ kg) is removed.

This leaves $8 - \frac{8}{5} = \frac{32}{5}$ kg of peanuts, and $2 - \frac{2}{5} = \frac{8}{5}$ kg of raisins.

When 2 kg of raisins are added, the mass of raisins becomes $\frac{8}{5} + 2 = \frac{18}{5}$ kg.

There are $\frac{32}{5}$ kg of peanuts and $\frac{18}{5}$ kg of raisins in the bin.

Therefore, the ratio of the masses is $\frac{32}{5} : \frac{18}{5} = 32 : 18 = 16 : 9$.

ANSWER: (E)

22. As Jillian drives from J to G , suppose that she drives x km uphill, y km on flat ground, and z km downhill.

This means that when she drives from G to J , she will drive z km uphill, y km on flat ground, and x km downhill. This is because downhill portions become uphill portions on the return trip, while uphill portions become downhill portions on the return trip.

We are told that Jillian drives at 77 km/h on flat ground, 63 km/h uphill, and 99 km/h downhill. Since time equals distance divided by speed, then on her trip from J to G , her time driving uphill is $\frac{x}{63}$ hours, her time driving on flat ground is $\frac{y}{77}$ hours, and her time driving downhill is $\frac{z}{99}$ hours.

Since it takes her 3 hours and 40 minutes (which is $3\frac{2}{3}$ or $\frac{11}{3}$ hours), then

$$\frac{x}{63} + \frac{y}{77} + \frac{z}{99} = \frac{11}{3}$$

A similar analysis of the return trip gives

$$\frac{x}{99} + \frac{y}{77} + \frac{z}{63} = \frac{13}{3}$$

We are asked for the total distance from J to G , which equals $x + y + z$ km. Therefore, we need to determine $x + y + z$.

We add the two equations above and simplify to obtain

$$\begin{aligned} \frac{x}{63} + \frac{x}{99} + \frac{y}{77} + \frac{y}{77} + \frac{z}{99} + \frac{z}{63} &= \frac{24}{3} \\ x\left(\frac{1}{63} + \frac{1}{99}\right) + y\left(\frac{1}{77} + \frac{1}{77}\right) + z\left(\frac{1}{99} + \frac{1}{63}\right) &= 8 \\ x\left(\frac{1}{7 \cdot 9} + \frac{1}{9 \cdot 11}\right) + \frac{2}{77}y + z\left(\frac{1}{9 \cdot 11} + \frac{1}{7 \cdot 9}\right) &= 8 \\ x\left(\frac{11}{7 \cdot 9 \cdot 11} + \frac{7}{7 \cdot 9 \cdot 11}\right) + \frac{2}{77}y + z\left(\frac{7}{7 \cdot 9 \cdot 11} + \frac{11}{7 \cdot 9 \cdot 11}\right) &= 8 \\ x\left(\frac{18}{7 \cdot 9 \cdot 11}\right) + \frac{2}{77}y + z\left(\frac{18}{7 \cdot 9 \cdot 11}\right) &= 8 \\ x\left(\frac{2}{7 \cdot 11}\right) + \frac{2}{77}y + z\left(\frac{2}{7 \cdot 11}\right) &= 8 \\ \frac{2}{77}(x + y + z) &= 8 \end{aligned}$$

Thus, $x + y + z = \frac{77}{2} \cdot 8 = 77 \cdot 4 = 308$.

Finally, the distance from J to G is 308 km.

ANSWER: (C)

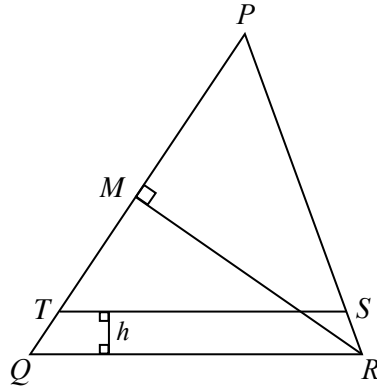
23. We label the other two vertices of the bottom section as S (on PR) and T (on PQ).

First, we calculate the area of $\triangle PQR$.

We do this by dropping a perpendicular from R to PQ .

Since $PR = QR$, then $\triangle PQR$ is isosceles and the perpendicular from R to PQ meets PQ at its midpoint M .

Thus, $PM = MQ = \frac{1}{2}(150) = 75$.



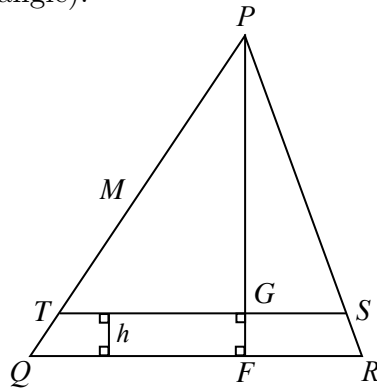
By the Pythagorean Theorem,

$$RM^2 = RQ^2 - MQ^2 = 125^2 - 75^2 = 15\,625 - 5\,625 = 10\,000$$

Since $RM > 0$, then $RM = \sqrt{10\,000} = 100$.

Therefore, the area of $\triangle PQR$ is $\frac{1}{2}(RM)(PQ) = \frac{1}{2}(100)(150) = 7500$.

Let F be the point on QR for which PF is perpendicular to QR and let G be the point where PF intersects TS (also at a 90° angle).



Then the area of $\triangle PQR$ is also equal to $\frac{1}{2}(QR)(PF)$.

Thus, $\frac{1}{2}(125)(PF) = 7500$ and so $PF = \frac{15\,000}{125} = 120$.

Since TS is parallel to QR , then $\angle PTS = \angle PQR$ and $\angle PST = \angle PRQ$.

This means that $\triangle PTS$ is similar to $\triangle PQR$.

Since each of the four sections of $\triangle PQR$ is one-quarter of the total area, then $\triangle PTS$ is three-quarters of the total area of $\triangle PQR$.

This means that the dimensions of $\triangle PTS$ are $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$ of those of $\triangle PQR$. (In general, if two similar triangles have areas in the ratio $k : 1$, then their corresponding sides are in the ratio $\sqrt{k} : 1$.)

Therefore, $PG = \frac{\sqrt{3}}{2}PF = \frac{\sqrt{3}}{2}(120) = 60\sqrt{3}$.

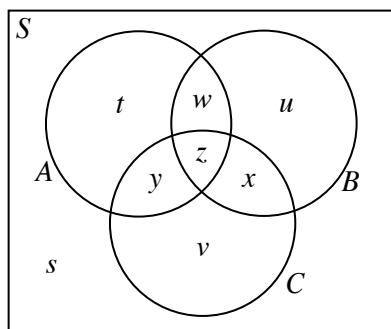
Finally, since TS is parallel to QR , then $h = GF = PF - PG = 120 - 60\sqrt{3} \approx 16.077$.

Of the given answers, the height is closest to 16.1.

ANSWER: (E)

24. If we have n balls to be placed in n boxes, one per box and without restriction, then there are $n! = n(n-1)(n-2)\cdots(3)(2)(1)$ ways to do this. (This is because there are n choices for the ball to be placed in box 1; for each of these, there are $n-1$ choices for the ball to be placed in box 2; for each of these pairs of choices, there are $n-2$ choices for the ball to be placed in box 3, and so on. In total, there are $n(n-1)(n-2)\cdots(3)(2)(1)$ ways to do this.)

We draw a Venn diagram where S represents all of the ways of placing the 8 balls in 8 boxes without restriction, circle A represents the ways in which the balls are placed with ball 1 going in box 1, circle B represents the ways with ball 2 going in box 2, and circle C represents the ways with ball 3 going in box 3.



Here, s represents the number of ways of putting the balls in boxes so that ball 1 is not in box 1 (s is outside circle A), ball 2 is not in box 2 (s is outside circle B), and ball 3 is not in box 3 (s is outside circle C). We want to calculate s .

The total number of ways in S is $8!$.

Circle A represents the ways when ball 1 is in box 1, and the other 7 balls are placed without restriction. There are $7!$ such ways.

Similarly, the number of ways inside each of circle B and circle C is $7!$.

In other words, $t + w + y + z = u + w + x + z = v + x + y + z = 7!$.

The overlap between circle A and B represents the ways with ball 1 in box 1 and ball 2 in box 2, with the other 6 balls placed without restriction. There are $6!$ such ways.

Similarly, there are $6!$ ways in the intersection of circles A and C , and circles B and C .

In other words, $w + z = y + z = x + z = 6!$.

Finally, the intersection of all three circles represents the ways in which ball 1 is in box 1, ball 2 is in box 2, ball 3 is in box 3, and the other 5 balls are placed without restriction. There are $5!$ such ways.

In other words, $z = 5!$.

Since $z = 5!$, then $w = x = y = 6! - 5!$.

Futhermore, $t = u = v = 7! - 2(6! - 5!) - 5! = 7! - 2(6!) + 5!$.

Finally,

$$\begin{aligned} s &= 8! - (t + u + v + w + x + y + z) \\ &= 8! - 3(7! - 2(6!) + 5!) - 3(6! - 5!) - 5! \\ &= 8! - 3(7!) + 6(6!) - 3(5!) - 3(6!) + 3(5!) - 5! \\ &= 8! - 3(7!) + 3(6!) - 5! \end{aligned}$$

Now, $5! = 5(4)(3)(2)(1) = 120$ so $6! = 6(5!) = 6(120) = 720$ and $7! = 7(6!) = 7(720) = 5040$ and $8! = 8(7!) = 8(5040) = 40320$.

Therefore, $s = 40320 - 3(5040) + 3(720) - 120 = 27240$.

Thus, the number of ways of putting the balls in the boxes with the given restrictions is 27 240.

We note that an alternative way of performing the algebraic steps is as follows:

$$\begin{aligned}
 s &= 8! - (t + u + v + w + x + y + z) \\
 &= 8! - (t + w + y + z) - (u + w + x + z) - (v + x + y + z) + w + x + y + 2z \\
 &= 8! - (t + w + y + z) - (u + w + x + z) - (v + x + y + z) + (w + z) + (x + z) + (y + z) - z \\
 &= 8! - 7! - 7! - 7! + 6! + 6! + 6! - 5! \\
 &= 8! - 3(7!) + 3(6!) - 5!
 \end{aligned}$$

which leads to the same final answer.

ANSWER: (A)

25. We first examine the conditions on the length and slope of PQ , then simplify the condition that $r + s + t + u = 27$, then finally incorporate the fact that points P and Q lie on the given parabola.

Since the slope of PQ is positive, then one of the points is “up and to the right” from the other. Without loss of generality, we assume that Q is “up and to the right” from P .

Thus, $u > s$ and $t > r$.

Since the slope of PQ equals $\frac{12}{5}$, then $\frac{u-s}{t-r} = \frac{12}{5}$, which means that $u-s = 12k$ and $t-r = 5k$ for some real number $k > 0$.

Since $PQ = 13$, then $(u-s)^2 + (t-r)^2 = 13^2$ or $(12k)^2 + (5k)^2 = 169$.

Thus, $144k^2 + 25k^2 = 169$ and so $169k^2 = 169$ or $k^2 = 1$, which gives $k = 1$ (since $k > 0$).

Thus, $u-s = 12$ (or $u = s + 12$) and $t-r = 5$ (or $t = r + 5$).

Thus, P has coordinates (r, s) and Q has coordinates $(r + 5, s + 12)$.

(This eliminates two of the variables from the problem.)

We eventually need to use the condition that $r + s + t + u = 27$.

Since $t = r + 5$ and $u = s + 12$, this is equivalent to $r + s + r + 5 + s + 12 = 27$ or $2r + 2s = 10$, which is equivalent to $r + s = 5$.

We will use this simplified condition shortly.

Since both $P(r, s)$ and $Q(r + 5, s + 12)$ are on the parabola with equation $y = x^2 - \frac{1}{5}mx + \frac{1}{5}n$, we obtain the following two equations:

$$\begin{aligned}
 s &= r^2 - \frac{1}{5}mr + \frac{1}{5}n \\
 s + 12 &= (r + 5)^2 - \frac{1}{5}m(r + 5) + \frac{1}{5}n
 \end{aligned}$$

Here, m and n are treated as known constants and r and s are variables for which we solve.

We want to determine the number of pairs (m, n) of positive integers with $n \leq 1000$ for which a solution (r, s) to this equation satisfies $r + s = 5$.

We solve this system of equations for r and s .

Expanding the second equation, we obtain

$$s + 12 = r^2 + 10r + 25 - \frac{1}{5}mr - m + \frac{1}{5}n$$

Subtracting the first equation, we obtain $12 = 10r + 25 - m$ and so $r = \frac{1}{10}(m - 13)$.

Substituting into the first equation, we obtain

$$\begin{aligned}
 s &= \left(\frac{1}{10}(m - 13)\right)^2 - \frac{1}{5}m\left(\frac{1}{10}(m - 13)\right) + \frac{1}{5}n \\
 &= \frac{1}{100}(m^2 - 26m + 169) - \frac{1}{50}(m^2 - 13m) + \frac{1}{5}n \\
 &= \frac{1}{100}(m^2 - 26m + 169 - 2(m^2 - 13m) + 20n) \\
 &= \frac{1}{100}(-m^2 + 169 + 20n)
 \end{aligned}$$

Note that each pair (m, n) gives a unique solution (r, s) .

We now need to determine the number of pairs (m, n) of positive integers with $n \leq 1000$ that produce $r = \frac{1}{10}(m - 13)$ and $s = \frac{1}{100}(-m^2 + 169 + 20n)$ satisfying the equation $r + s = 5$.

Substituting, we obtain

$$\begin{aligned} r + s &= 5 \\ \frac{1}{10}(m - 13) + \frac{1}{100}(-m^2 + 169 + 20n) &= 5 \\ 10m - 130 - m^2 + 169 + 20n &= 500 \\ 20n &= m^2 - 10m + 461 \\ 20n &= (m - 5)^2 + 436 \quad (\text{completing the square}) \end{aligned}$$

So we are left to determine the number of pairs (m, n) of positive integers with $n \leq 1000$ that satisfy the equation $20n = (m - 5)^2 + 436$.

Since $20n$ and 436 are both even integers, then $(m - 5)^2$ is an even integer, which means that $m - 5$ is an even integer and thus m is odd. (If $m - 5$ were odd, then $(m - 5)^2$ would also be odd.)

Therefore, we let $m = 2M - 1$ for some positive integer M .

Substituting, we obtain $20n = (2M - 6)^2 + 436$ or $20n = 4(M - 3)^2 + 436$, which is equivalent to $5n = (M - 3)^2 + 109$.

Therefore, we need to determine the number of pairs (M, n) of positive integers with $n \leq 1000$ that satisfy the equation $5n = (M - 3)^2 + 109$.

This is the same as determining the number of positive integer values of M for which the right side is a multiple of 5 that is at most 5000, because each such value of M will give a corresponding positive integer value for n that is at most 1000.

When $M = 1$, the right side equals 113, which is not a multiple of 5.

When $M = 2$, the right side equals 110, which is a multiple of 5. (This gives $n = 22$.)

When $M \geq 3$, we consider the units (ones) digits of $M - 3$, $(M - 3)^2$ and $(M - 3)^2 + 109$:

$M - 3$	0	1	2	3	4	5	6	7	8	9
$(M - 3)^2$	0	1	4	9	6	5	6	9	4	1
$(M - 3)^2 + 109$	9	0	3	8	5	4	5	8	3	0

Therefore, any positive integer M for which $M - 3$ ends in a 1, 4, 6, or 9 produces a right side divisible by 5. This is because in each of these cases the expression $(M - 3)^2 + 109$ has a units digit of 0 or 5 and hence is divisible by 5.

For $(M - 3)^2 + 109 \leq 5000$, we need $(M - 3)^2 \leq 4891$.

Since $\sqrt{4891} \approx 69.94$ and $M - 3$ is a positive integer, then $M - 3 \leq 69$.

Therefore, the values of M with the desired property are $M = 2$ and every positive integer M with $0 \leq M - 3 \leq 69$ for which $M - 3$ has a units digit of 1, 4, 6, or 9.

There are 28 positive integers M in the second list (four each with $M - 3$ between 0 and 9, 10 and 19, 20 and 29, 30 and 39, 40 and 49, 50 and 59, and 60 and 69). Thus, there are 29 such integers M overall.

Finally, this means that there are 29 pairs (m, n) which have the desired property.

ANSWER: (D)