The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2014 Cayley Contest

(Grade 10)

Thursday, February 20, 2014
(in North America and South America)

Friday, February 21, 2014
(outside of North America and South America)

Solutions

1. We rearrange the given expression to obtain $2000 + 200 - 80 - 120$.
Since $200 - 80 - 120 = 0$, then $2000 + 200 - 80 - 120 = 2000$.
Alternatively, we could have evaluated each operation in order to obtain
$$2000 - 80 + 200 - 120 = 1920 + 200 - 120 = 2120 - 120 = 2000$$
Answer: (A)

2. Since $(2)(3)(4) = 6x$, then $6(4) = 6x$. Dividing both sides by 6, we obtain $x = 4$.
Answer: (E)

3. The unlabelled angle inside the triangle equals its vertically opposite angle, or 40°.
Since the sum of the angles in a triangle is 180°, then $40^\circ + 60^\circ + x^\circ = 180^\circ$ or $100 + x = 180$.
Thus, $x = 80$.
Answer: (C)

4. The line representing a temperature of 3° is the horizontal line passing halfway between 2° and 4° on the vertical axis.
There are two data points on this line: one at 2 p.m. and one at 9 p.m.
The required time is 9 p.m.
Answer: (A)

5. Since $2n + 5 = 16$, then $2n - 3 = (2n + 5) - 8 = 16 - 8 = 8$.
Alternatively, we could solve the equation $2n + 5 = 16$ to obtain $2n = 11$ or $n = \frac{11}{2}$.
From this, we see that $2n - 3 = 2(\frac{11}{2}) - 3 = 11 - 3 = 8$.
Answer: (A)

6. Since $3 = \frac{6}{2}$ and $\frac{5}{2} < \frac{6}{2}$, then $\frac{5}{2} < 3$.
Since $3 = \sqrt{9}$ and $\sqrt{9} < \sqrt{10}$, then $3 < \sqrt{10}$.
Thus, $\frac{5}{2} < 3 < \sqrt{10}$, and so the list of the three numbers in order from smallest to largest is $\frac{5}{2}, 3, \sqrt{10}$.
Answer: (B)

7. 20% of the number 100 is 20, so when 100 is increased by 20%, it becomes $100 + 20 = 120$.
50% of a number is half of that number, so 50% of 120 is 60.
Thus, when 120 is increased by 50%, it becomes $120 + 60 = 180$.
Therefore, Meg's final result is 180.
Answer: (E)

8. Since $\triangle PQR$ is right-angled at P, we can use the Pythagorean Theorem.
We obtain $PQ^2 + PR^2 = QR^2$ or $10^2 + PR^2 = 26^2$.
This gives $PR^2 = 26^2 - 10^2 = 676 - 100 = 576$ and so $PR = \sqrt{576} = 24$, since $PR > 0$.
Since $\triangle PQR$ is right-angled at P, its area equals $\frac{1}{2}(PR)(PQ) = \frac{1}{2}(24)(10) = 120$.
Answer: (B)

9. We use A, B, C, D, E to represent Amy, Bob, Carla, Dan, and Eric, respectively.
We use the greater than symbol ($>$) to represent "is taller than" and the less than symbol ($<$) to represent "is shorter than".
From the first bullet, $A > C$.
From the second bullet, $D < E$ and $D > B$ so $E > D > B$.
From the third bullet, $E < C$ or $C > E$.
Since $A > C$ and $C > E$ and $E > D > B$, then $A > C > E > D > B$, which means that Bob is the shortest.
Answer: (B)

10. *Solution 1*

We start from the `OUTPUT` and work back to the `INPUT`.

Since the `OUTPUT` 32 is obtained from adding 16 to the previous number, then the previous number is $32 - 16 = 16$.

$$\boxed{\texttt{INPUT}} \rightarrow \text{Subtract } 8 \rightarrow \boxed{} \rightarrow \text{Divide by } 2 \rightarrow \boxed{16} \rightarrow \text{Add } 16 \rightarrow \boxed{32}$$

Since 16 is obtained by dividing the previous number by 2, then the previous number is 2×16 or 32.

$$\boxed{\texttt{INPUT}} \rightarrow \text{Subtract } 8 \rightarrow \boxed{32} \rightarrow \text{Divide by } 2 \rightarrow \boxed{16} \rightarrow \text{Add } 16 \rightarrow \boxed{32}$$

Since 32 is obtained by subtracting 8 from the `INPUT`, then the `INPUT` must have been $32+8 = 40$.

$$\boxed{40} \rightarrow \text{Subtract } 8 \rightarrow \boxed{32} \rightarrow \text{Divide by } 2 \rightarrow \boxed{16} \rightarrow \text{Add } 16 \rightarrow \boxed{32}$$

Solution 2

Suppose that the `INPUT` is x.

Subtracting 8 gives $x - 8$.

Dividing this result by 2 gives $\frac{1}{2}(x - 8)$ or $\frac{1}{2}x - 4$.

Adding 16 to this result gives $(\frac{1}{2}x - 4) + 16 = \frac{1}{2}x + 12$, which is the `OUTPUT`.

$$\boxed{x} \rightarrow \text{Subtract } 8 \rightarrow \boxed{x-8} \rightarrow \text{Divide by } 2 \rightarrow \boxed{\tfrac{1}{2}x-4} \rightarrow \text{Add } 16 \rightarrow \boxed{\tfrac{1}{2}x+12}$$

If the `OUTPUT` is 32, then $\frac{1}{2}x + 12 = 32$ or $\frac{1}{2}x = 20$ and so $x = 40$.

Therefore, the `INPUT` must have been 40.

Answer: (D)

11. We consider the equation of the line shown in the form $y = mx + b$.

The slope, m, of the line shown is negative.

The y-intercept, b, of the line shown is positive.

Of the given choices only $y = -2x + 3$ has $m < 0$ and $b > 0$.

Therefore, a possible equation for the line is $y = -2x + 3$.

Answer: (E)

12. Since $x = 2y$, then $(x - y)(2x + y) = (2y - y)(2(2y) + y) = (y)(5y) = 5y^2$.

Answer: (A)

13. Erika assembling 9 calculators is the same as assembling three groups of 3 calculators.

Since Erika assembles 3 calculators in the same amount of time that Nick assembles 2 calculators, then he assembles three groups of 2 calculators (that is, 6 calculators) in this time.

Since Nick assembles 1 calculator in the same amount of time that Sam assembles 3 calculators, then Sam assembles 18 calculators while Nick assembles 6 calculators.

Thus, the three workers assemble $9+6+18 = 33$ calculators while Erika assembles 9 calculators.

Answer: (E)

14. Since 1 GB = 1024 MB, then Julia's 300 GB hard drive has $300 \times 1024 = 307\,200$ MB of storage space.

When Julia puts 300 000 MB of data on the empty hard drive, the amount of empty space remaining is $307\,200 - 300\,000 = 7200$ MB.

Answer: (C)

15. From the second row, $\triangle + \triangle + \triangle + \triangle = 24$ or $4\triangle = 24$, and so $\triangle = 6$.
From the first row, $\heartsuit + \triangle + \triangle + \heartsuit = 26$ or $2\heartsuit + 2\triangle = 26$.
Since $\triangle = 6$, then $2\heartsuit = 26 - 12 = 14$, and so $\heartsuit = 7$.
From the fourth row, $\square + \heartsuit + \square + \triangle = 33$.
Since $\triangle = 6$ and $\heartsuit = 7$, then $2\square + 7 + 6 = 33$, and so $2\square = 20$ or $\square = 10$.
Finally, from the third row, $\square + \blacklozenge + \heartsuit + \blacklozenge = 27$.
Since $\square = 10$ and $\heartsuit = 7$, then $2\blacklozenge = 27 - 10 - 7 = 10$.
Thus, $\blacklozenge = 5$.

Answer: (A)

16. The mean number of hamburgers eaten per student equals the total number of hamburgers eaten divided by the total number of students.
12 students each eat 0 hamburgers. This is a total of 0 hamburgers eaten.
14 students each eat 1 hamburger. This is a total of 14 hamburgers eaten.
8 students each eat 2 hamburgers. This is a total of 16 hamburgers eaten.
4 students each eat 3 hamburgers. This is a total of 12 hamburgers eaten.
2 students each eat 4 hamburgers. This is a total of 8 hamburgers eaten.
Thus, a total of $0 + 14 + 16 + 12 + 8 = 50$ hamburgers are eaten.
The total number of students is $12 + 14 + 8 + 4 + 2 = 40$.
Therefore, the mean number of hamburgers eaten is $\frac{50}{40} = 1.25$.

Answer: (C)

17. A circle with area 36π has radius 6, since the the area of a circle with radius r equals πr^2 and $\pi(6^2) = 36\pi$.
The circumference of a circle with radius 6 equals $2\pi(6) = 12\pi$.
Therefore, each quarter-circle contributes $\frac{1}{4}(12\pi) = 3\pi$ to the circumference.
The perimeter of the given figure consists of three quarter-circle sections and two radii from the circle.
Thus, its perimeter is $3(3\pi) + 2(6) = 9\pi + 12$.

Answer: (B)

18. Suppose that the number of 2¢ stamps that Sonita buys is x.
Then the number of 1¢ stamps that she buys is $10x$.
The total value of the 2¢ and 1¢ stamps that she buys is $2(x) + 1(10x) = 12x$¢.
Since she buys some 5¢ stamps as well and the total value of the stamps that she buys is 100¢, then the value of the 5¢ stamps that she buys is $(100 - 12x)$¢.
Thus, $100 - 12x$ must be a multiple of 5. Since 100 is a multiple of 5, then $12x$ must be a multiple of 5, and so x is a multiple of 5 (since 12 has no divisors larger than 1 in common with 5).
Note that $x > 0$ (since she buys some 2¢ stamps) and $x < 9$ (since $12x$ is less than 100). The only multiple of 5 between 0 and 9 is 5, so $x = 5$.
When $x = 5$, the value of 5¢ stamps is $100 - 12x = 100 - 12(5) = 40$¢ and so she buys $\frac{40}{5} = 8$ 5¢ stamps.
Finally, she buys 5 2¢ stamps, 50 1¢ stamps, and 8 5¢ stamps, for a total of $5 + 50 + 8 = 63$ stamps.
(Checking, these stamps are worth $5(2) + 50(1) + 8(5) = 10 + 50 + 40 = 100$¢ in total, as required.)

Answer: (D)

19. There are ten possible pairs of numbers that can be chosen: -3 and -1; -3 and 0; -3 and 2; -3 and 4; -1 and 0; -1 and 2; -1 and 4; 0 and 2; 0 and 4; 2 and 4. Each pair is equally likely to be chosen.
Pairs that include 0 (4 pairs) have a product of 0; pairs that do not include 0 (6 of them) do not have a product of 0.
Therefore, the probability that a randomly chosen pair has a product of 0 is $\frac{4}{10}$ or $\frac{2}{5}$.

Answer: (D)

20. The layer sum of $wxyz$ equals 2014.
This means that the sum of the integer with digits $wxyz$, the integer with digits xyz, the integer with digits yz, and the integer z is 2014.
Note that the integer with digits $wxyz$ equals $1000w + 100x + 10y + z$, the integer with digits xyz equals $100x + 10y + z$, and the integer with digits yz equals $10y + z$.
Therefore, we have

$$(1000w + 100x + 10y + z) + (100x + 10y + z) + (10y + z) + z = 2014$$

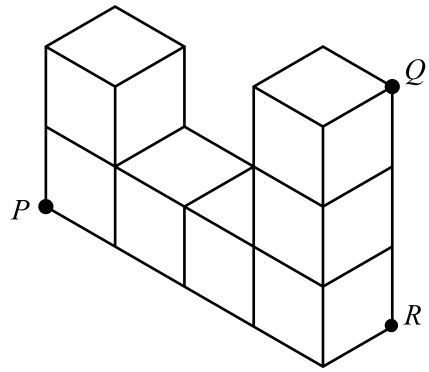
or

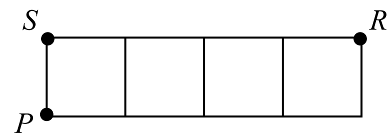
$$1000w + 200x + 30y + 4z = 2014 \quad (*)$$

Each of w, x, y, z is a single digit and $w \neq 0$.
Now w cannot be 3 or greater, or the left side of $(*)$ would be at least 3000, which is too large.
Thus, $w = 1$ or $w = 2$.
If $w = 2$, then $2000+200x+30y+4z = 2014$ and so $200x+30y+4z = 14$ or $100x+15y+2z = 7$.
This would mean that $x = y = 0$ (since otherwise the terms $100x + 15y$ would contribute more than 7), which gives $2z = 7$ which has no integer solutions. Thus, $w \neq 2$.
Therefore, $w = 1$.
This gives $1000+200x+30y+4z = 2014$ and so $200x+30y+4z = 1014$ or $100x+15y+2z = 507$.
Since $0 \leq y \leq 9$ and $0 \leq z \leq 9$, then $0 \leq 15y + 2z \leq 15(9) + 2(9) = 153$.
Since $100x$ is a multiple of 100 and $0 \leq 15y + 2z \leq 153$, then $100x = 400$ or $100x = 500$ so $15y + 2z = 507 - 400 = 107$ or $15y + 2z = 507 - 500 = 7$. From above, we saw that $15y + 2z$ cannot equal 7, so $15y + 2z = 107$, which means that $100x = 400$ or $x = 4$.
Thus, $15y + 2z = 107$.
Since $2z$ is even, then $15y$ must be odd to make $15y + 2z$ odd.
The odd multiples of 15 less than 107 are $15, 45, 75, 105$.
Since $0 \leq 2z \leq 18$, then we must have $15y = 105$ or $y = 7$. This gives $2z = 2$ or $z = 1$.
Therefore, the integer $wxyz$ is 1471. (Checking, $1471 + 471 + 71 + 1 = 2014$.)
Finally, $w + x + y + z = 1 + 4 + 7 + 1 = 13$.

Answer: (D)

21. Suppose that R is the point at the bottom of the solid directly under Q and S is the back left bottom corner of the solid (unseen in the problem's diagram).
Since QR is perpendicular to the bottom surface of the solid, then $\triangle PRQ$ is right-angled at R and so $PQ^2 = PR^2 + RQ^2$.
We note also that $\triangle PSR$ is right-angled at S, since the solid is made up of cubes.
Therefore, $PR^2 = PS^2 + SR^2$.





This tells us that $PQ^2 = PS^2 + SR^2 + RQ^2$.
Since the edge length of each cube in the diagram is 1, then $PS = 1$, $SR = 4$, and $RQ = 3$.
Therefore, $PQ^2 = 1^2 + 4^2 + 3^2 = 26$.
Since $PQ > 0$, then $PQ = \sqrt{26}$.

Answer: (B)

22. We write such a five-digit positive integer with digits $VWXYZ$.
We want to count the number of ways of assigning 1, 3, 5, 7, 9 to the digits V, W, X, Y, Z in such a way that the given properties are obeyed.
From the given conditions, $W > X$, $W > V$, $Y > X$, and $Y > Z$.
The digits 1 and 3 cannot be placed as W or Y, since W and Y are larger than both of their neighbouring digits, while 1 is smaller than all of the other digits and 3 is only larger than one of the other possible digits.
The digit 9 cannot be placed as V, X or Z since it is the largest possible digit and so cannot be smaller than W or Y. Thus, 9 is placed as W or as Y.
Therefore, the digits W and Y are 9 and either 5 or 7.

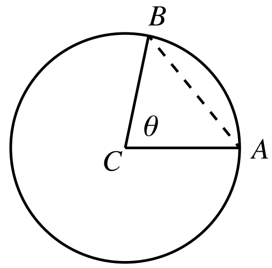
Suppose that $W = 9$ and $Y = 5$. The number is thus $V9X5Z$.
Neither X or Z can equal 7 since $7 > 5$, so $V = 7$. X and Z are then 1 and 3 or 3 and 1.
There are 2 possible integers in this case.
Similarly, if $Y = 9$ and $W = 5$, there are 2 possible integers.

Suppose that $W = 9$ and $Y = 7$. The number is thus $V9X7Z$.
The digits 1, 3, 5 can be placed in any of the remaining spots. There are 3 choices for the digit V. For each of these choices, there are 2 choices for X and then 1 choice for Z.
There are thus $3 \times 2 \times 1 = 6$ possible integers in this case.
Similarly, if $Y = 9$ and $W = 7$, there are 6 possible integers.

Overall, there are thus $2 + 2 + 6 + 6 = 16$ possible integers.

Answer: (C)

23. We call Clarise's spot C and Abe's spot A.
Consider a circle centred at C with radius 10 m. Since A is 10 m from C, then A is on this circle.
Bob starts at C and picks a direction to walk, with every direction being equally likely to be chosen. We model this by having Bob choose an angle θ between 0° and 360° and walk 10 m along a segment that makes this angle when measured counterclockwise from CA.
Bob ends at point B, which is also on the circle.



We need to determine the probability that $AB < AC$.
Since the circle is symmetric above and below the diameter implied by CA, we can assume that θ is between 0° and 180° as the probability will be the same below the diameter.
Consider $\triangle CAB$ and note that $CA = CB = 10$ m.
It will be true that $AB < AC$ whenever AB is the shortest side of $\triangle ABC$.
AB will be the shortest side of $\triangle ABC$ whenever it is opposite the smallest angle of $\triangle ABC$.
(In any triangle, the shortest side is opposite the smallest angle and the longest side is opposite the largest angle.)
Since $\triangle ABC$ is isosceles with $CA = CB$, then $\angle CAB = \angle CBA$.
We know that $\theta = \angle ACB$ is opposite AB and $\angle ACB + \angle CAB + \angle CBA = 180^\circ$.
Since $\angle CAB = \angle CBA$, then $\angle ACB + 2\angle CAB = 180^\circ$ or $\angle CAB = 90^\circ - \frac{1}{2}\angle ACB$.
If $\theta = \angle ACB$ is smaller than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be greater than 60°.
Similarly, if $\angle ACB$ is greater than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be smaller than 60°.
Therefore, AB is the shortest side of $\triangle ABC$ whenever θ is between 0° and 60°.
Since θ is uniformly chosen in the range 0° to 180° and $60^\circ = \frac{1}{3}(180^\circ)$, then the probability that θ is in the desired range is $\frac{1}{3}$.
Therefore, the probability that Bob is closer to Abe than Clarise is to Abe is $\frac{1}{3}$.
(Note that we can ignore the cases $\theta = 0^\circ$, $\theta = 60^\circ$ and $\theta = 180^\circ$ because these are only three specific cases out of an infinite number of possible values for θ.)

ANSWER: (B)

24. For each positive integer n, $S(n)$ is defined to be the smallest positive integer divisible by each of $1, 2, 3, \ldots, n$. In other words, $S(n)$ is the least common multiple (lcm) of $1, 2, 3, \ldots, n$.
To calculate the lcm of a set of numbers, we

- determine the prime factorization of each number in the set,
- determine the list of prime numbers that occur in these prime factorizations,
- determine the highest power of each prime number from this list that occurs in the prime factorizations, and
- multiply these highest powers together.

For example, to calculate $S(8)$, we determine the lcm of 1, 2, 3, 4, 5, 6, 7, 8.
The prime factorizations of the numbers 2, 3, 4, 5, 6, 7, 8 are $2, 3, 2^2, 5, 2 \cdot 3, 7, 2^3$.

The primes used in this list are 2, 3, 5, 7, with highest powers $2^3, 3^1, 5^1, 7^1$.
Therefore, $S(8) = 2^3 \cdot 3^1 \cdot 5^1 \cdot 7^1$.

Since $S(n)$ is the lcm of $1, 2, 3, \ldots, n$ and $S(n{+}4)$ is the lcm of $1, 2, 3, \ldots, n, n{+}1, n{+}2, n{+}3, n{+}4$, then $S(n) \neq S(n+4)$ if either (i) there are prime factors that occur in $n+1, n+2, n+3, n+4$ that don't occur in $1, 2, 3, \ldots, n$ or (ii) there is a higher power of a prime that occurs in the factorizations of one of $n+1, n+2, n+3, n+4$ that doesn't occur in any of $1, 2, 3, \ldots, n$.
For (i) to occur, consider a prime number p that is a divisor of one of $n+1, n+2, n+3, n+4$ and none of $1, 2, 3, \ldots, n$. This means that the smallest positive integer that has p as a divisor is one of the integers $n+1, n+2, n+3, n+4$, which in fact means that this integer equals p. (The smallest multiple of a prime p is $1 \cdot p$, or p itself.)
Thus, for (i) to occur, one of $n+1, n+2, n+3, n+4$ is a prime number.
For (ii) to occur, consider a prime power p^k (with $k > 1$) that is a divisor of one of $n+1$, $n+2, n+3, n+4$ and none of $1, 2, 3, \ldots, n$. Using a similar argument to condition (i), one of $n+1, n+2, n+3, n+4$ must equal that prime power p^k.
Therefore, $S(n) \neq S(n+4)$ whenever one of $n+1, n+2, n+3, n+4$ is a prime number or a prime power.
In other words, $S(n) = S(n+4)$ whenever none of $n+1, n+2, n+3, n+4$ is a prime number or a prime power.
Therefore, we want to determine the positive integers n with $1 \leq n \leq 100$ for which none of $n+1, n+2, n+3, n+4$ is a prime number or a prime power.
The prime numbers less than or equal to 104 are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103$$

(We go up to 104 since n can be as large as 100 so $n+4$ can be as large as 104.)
The prime powers (with exponent at least 2) less than or equal to 100 are

$$4, 8, 16, 32, 64, 9, 27, 81, 25, 49$$

There are 5 powers of 2, 3 powers of 3, 1 power of 5, and 1 power of 7 in this list. No primes larger than 7 have a power less than 100.
Therefore, we want to count the positive integers n with $1 \leq n \leq 100$ for which none of $n+1, n+2, n+3, n+4$ appear in the list

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37, 41, 43, 47, 49, 53, 59, 61, 64,$$

$$67, 71, 73, 79, 81, 83, 89, 97, 101, 103$$

For four consecutive integers not to occur in this list, we need a difference between adjacent numbers to be at least 5.
The values of n that satisfy this condition are $n = 32, 53, 54, 73, 74, 83, 84, 89, 90, 91, 92$.
(For example, 54 is a value of n that works since none of $55, 56, 57, 58$ appears in the list.)
Therefore, there are 11 values of n with $1 \leq n \leq 100$ for which $S(n) = S(n+4)$.

Answer: (C)

25. Suppose that P has coordinates $P(0, 2a)$ for some real number a.
Since P has y-coordinate greater than 0 and less than 100, then $0 < 2a < 100$ or $0 < a < 50$.
We determine an expression for the radius of the circle in terms of a and then determine how many values of a give an integer radius.
We determine the desired expression by first finding the coordinates of the centre, C, of the

circle in terms of a, and then calculating the distance from C to one of the points O, P, Q.
If a circle passes through the three vertices O, P and Q of a triangle, then its centre is the point of intersection of the perpendicular bisectors of the sides OP, OQ, and PQ of the triangle.
We determine the centre of the circle by finding the point of intersection of the perpendicular bisectors of OP and OQ. (We could use PQ instead, but this would be more complicated algebraically.)
Since O has coordinates $(0, 0)$ and P has coordinates $(0, 2a)$, then OP is vertical so its perpendicular bisector is horizontal.
The midpoint of OP is $(\frac{1}{2}(0 + 0), \frac{1}{2}(0 + 2a)) = (0, a)$.
Therefore, the perpendicular bisector of OP is the horizontal line through $(0, a)$, and so has equation $y = a$.
Since O has coordinates $(0, 0)$ and Q has coordinates $(4, 4)$, then OQ has slope $\frac{4-0}{4-0} = 1$.
Therefore, a line perpendicular to OQ has slope -1.
The midpoint of OQ is $(\frac{1}{2}(0 + 4), \frac{1}{2}(0 + 4)) = (2, 2)$.
Therefore, the perpendicular bisector of OQ has slope -1 and passes through $(2, 2)$, so has equation $y - 2 = (-1)(x - 2)$ or $y = -x + 4$.
The centre of the desired circle is thus the point of intersection of the lines with equations $y = a$ and $y = -x + 4$.
The y-coordinate of this point is a and the x-coordinate is obtained by solving $a = -x + 4$ and obtaining $x = 4 - a$.
Therefore, the coordinates of C are $(4 - a, a)$.
The radius, r, of the circle is the distance from C to any of the three points O, P and Q. It is easiest to find the distance from O to C, which is

$$r = \sqrt{(4 - a)^2 + a^2} = \sqrt{a^2 - 8a + 16 + a^2} = \sqrt{2a^2 - 8a + 16}$$

We rewrite this as

$$r = \sqrt{2(a^2 - 4a + 8)} = \sqrt{2(a^2 - 4a + 4 + 4)} = \sqrt{2((a - 2)^2 + 4)} = \sqrt{2(a - 2)^2 + 8}$$

Since $(a - 2)^2 \geq 0$ and $(a - 2)^2 = 0$ only when $a = 2$, then the minimum value of $2(a - 2)^2 + 8$ is 8 and this occurs when $a = 2$. Thus, $r \geq \sqrt{8}$.
The expression $\sqrt{2(a - 2)^2 + 8}$ is decreasing from $a = 0$ to $a = 2$ and then increasing from $a = 2$ to $a = 50$.
When $a = 0$, $r = \sqrt{2(a - 2)^2 + 8} = \sqrt{2(-2)^2 + 8} = 4$.
When $a = 2$, $r = \sqrt{2(a - 2)^2 + 8} = \sqrt{2(0)^2 + 8} = \sqrt{8} \approx 2.83$.
When $a = 50$, $r = \sqrt{2(a - 2)^2 + 8} = \sqrt{2(48)^2 + 8} = \sqrt{4616} \approx 67.94$.
Therefore, when $0 < a \leq 2$, we have $\sqrt{8} \leq r < 4$ and when $2 \leq a < 50$, we have $\sqrt{8} \leq r < \sqrt{4616}$.
The expression $r = \sqrt{2(a - 2)^2 + 8}$ will take every real number value in each of these ranges, because $b = 2(a - 2)^2 + 8$ represents the equation of a parabola which is a "smooth" curve.
Between $\sqrt{8} \approx 2.83$ and 4, there is one integer value (namely, 3) which is achieved by the expression. (We do not count 4 since it is an endpoint that is not included.)
Between $\sqrt{8} \approx 2.83$ and $\sqrt{4616} \approx 67.94$, there are 65 integer values (namely, 3 to 67, inclusive) which are achieved by the expression.
In total, there are $1 + 65 = 66$ integer values achieved by the expression in the allowable range for a, so there are 66 positions of P for which the radius is an integer.

ANSWER: (C)