The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2014
Canadian Team Mathematics Contest

April 2014

Solutions

Team Problems

1. David counts 100, 89, 78, 67, 56, 45, 34, 23, 12, 1, -10.
   The first number less than 0 that he counts is -10.

   Answer: -10

2. Evaluating, $1 + 2 + 3 + \cdots + 18 + 19 + 20 = \frac{1}{2}(20)(21) = 210$.

   Answer: 210

3. Using the Pythagorean Theorem in the given triangle, we obtain

   $$AC = \sqrt{AB^2 + BC^2} = \sqrt{24^2 + 10^2} = \sqrt{676} = 26$$

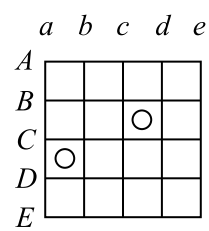
   since $AC > 0$.
   If AB and BC are each increased by 6, they become 30 and 16, so the hypotenuse of the new triangle is $\sqrt{30^2 + 16^2} = \sqrt{1156} = 34$.
   Therefore, AC increases by $34 - 26 = 8$.

   Answer: 8

4. In $\triangle AEF$, we have $\angle FAE = 180^\circ - \angle AEF - \angle AFE = 180^\circ - 90^\circ - 50^\circ = 40^\circ$.
   Since BFE is a straight line, then $\angle AFB = 180^\circ - \angle AFE = 180^\circ - 50^\circ = 130^\circ$.
   Now $\triangle AFB$ is isosceles with $AF = FB$.
   Thus, $\angle FAB = \angle FBA = \frac{1}{2}(180^\circ - \angle AFB) = \frac{1}{2}(180^\circ - 130^\circ) = 25^\circ$.
   Thus, $\angle BAE = \angle FAB + \angle FAE = 25^\circ + 40^\circ = 65^\circ$.
   Finally, $\angle ADC = \angle BAE$. (This is because a perpendicular dropped from C to a point P on AD would create $\triangle CPD$ which was congruent to $\triangle BEA$ – these triangles are each right-angled, have equal hypotenuses $BA = CD$ and have equal heights $BE = CP$, since BC and AD are parallel.)
   Therefore, $\angle ADC = 65^\circ$.

   Answer: 65°

5. We label the horizontal lines A, B, C, D, E from top to bottom and the vertical lines a, b, c, d, e from left to right.

   We determine the possibilities for the different sides of the rectangle using the fact that it must contain both circles.
   The left side of a rectangle containing both circles must be line a.
   The right side of the rectangle could be d or e.
   The top side of the rectangle could be A or B.
   The bottom side of the rectangle could be D or E.
   Each combination of these possibilities for the sides will give a rectangle containing both circles.
   This means that there are $1 \times 2 \times 2 \times 2 = 8$ possible rectangles that contain both circles.

   Answer: 8

6. The prime factorizations of the given integers are:

$$2\ ,\ 2^2\ ,\ 2\cdot 3\ ,\ 2^3\ ,\ 2\cdot 5\ ,\ 2^2\cdot 3\ ,\ 2\cdot 7\ ,\ 2^4\ ,\ 2\cdot 3^2\ ,\ 2^2\cdot 5$$

To find the least common multiple of these integers, we determine the prime numbers used in their prime factorizations (2, 3, 5, 7), determine the largest power of each prime used $(2^4, 3^2, 5^1, 7^1)$, and multiply these together to obtain $2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1 = 5040$.

Answer: 5040

7. If x and y are integers with $x > y$ and $x + y = 7$ and $xy = 12$, then $x = 4$ and $y = 3$, so $A = x - y = 1$. (We can check that there are no other solutions.)
If m and n are integers with $m > n$ and $m + n = 13$ and $m^2 + n^2 = 97$, then $m = 9$ and $n = 4$, so $B = m - n = 5$. (We can check that there are no other solutions.)
Therefore, $A - B = 1 - 5 = -4$.

Answer: -4

8. Robert was born in the year n^2 and will be 89 years old in the year $(n + 1)^2$.
Therefore, $(n + 1)^2 - n^2 = 89$ or $n^2 + 2n + 1 - n^2 = 89$ and so $2n + 1 = 89$ or $n = 44$.
Thus, $n^2 = 1936$, so Robert was born in the year 1936.

Answer: 1936

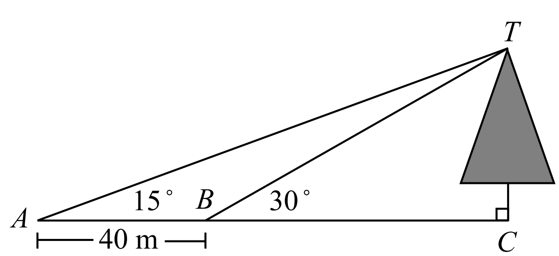
9. Simplifying,

$$\begin{aligned}
\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) &= 2014\\
\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{k+1}{k}\right)\left(\frac{k+2}{k+1}\right) &= 2014\\
\frac{3\cdot 4\cdot 5\cdot\cdots\cdot(k+1)\cdot(k+2)}{2\cdot 3\cdot 4\cdot\cdots\cdot k\cdot(k+1)} &= 2014\\
\frac{k+2}{2} &= 2014\\
k+2 &= 4028\\
k &= 4026
\end{aligned}$$

Therefore, $k = 4026$.

Answer: 4026

10. We label the diagram as shown and call the height of the tree h m.



Since $\triangle BCT$ is right-angled at C and $\angle TBC = 30°$, then $BC = \sqrt{3}TC = \sqrt{3}h$ m.
Since $\triangle ACT$ is right-angled at C, then $\tan(15°) = \dfrac{h}{40 + \sqrt{3}h}$.
This gives $40\tan(15°) + \sqrt{3}\tan(15°)h = h$ from which we obtain $h = \dfrac{40\tan(15°)}{1 - \sqrt{3}\tan(15°)} = 20$.
Therefore, the height of the tree is 20 m.

Answer: 20

11. To get from $P(6,-2)$ to $Q(-3,10)$, we go $6-(-3)=9$ units to the left and $10-(-2)=12$ units up.
One-third of this difference would be 3 units to the left and 4 units up, which means that the coordinates of R are $(a,b)=(6-3,-2+4)=(3,2)$.
Finally, $b-a=2-3=-1$.

Answer: -1

12. In any sequence of consecutive integers, the average, a, is either an integer (if there is an odd number of integers in the sequence) or halfway between two integers (if there is an even number of integers in the sequence).
Furthermore, the sum of this sequence is the product of the average a with the number of integers in the sequence, n.
We note that the minimum value for a sequence of 14 consecutive positive integers is $1+2+3+\cdots+13+14=\frac{1}{2}(14)(15)=105$ and any sequence of more than 14 consecutive positive integers has a larger sum.
In other words, $n\leq 14$.
We proceed by considering the equation $an=105$ and solving for a to obtain $a=\dfrac{105}{n}$.
For each n from 2 to 14, we calculate $\dfrac{105}{n}$ to see if it is an integer or is halfway between two integers:

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$\dfrac{105}{n}$	$52\frac{1}{2}$	35	$26\frac{1}{4}$	21	$17\frac{1}{2}$	15	$13\frac{1}{8}$	$11\frac{2}{3}$	$10\frac{1}{2}$	$9\frac{6}{11}$	$8\frac{3}{4}$	$8\frac{1}{13}$	$7\frac{1}{2}$

Each of the seven possibilities where $a=\dfrac{105}{n}$ gives a result that is an integer or halfway between two integers gives a sequence of positive integers whose sum is 105:

$52+53$ $34+35+36$ $19+20+21+22+23$ $15+16+17+18+19+20$ $12+13+14+15+16+17+18$

$6+7+8+9+10+11+12+13+14+15$ $1+2+3+4+5+6+7+8+9+10+11+12+13+14$

Therefore, there are 7 sequences of consecutive positive integers with a sum of 105.

Answer: 7

13. To get from A to B along edges only, the insect must travel along at least 3 edges. Thus, the insect travels at least 3 m, and so takes at least $\frac{3}{5}=0.6$ minutes.

To get from A to B along faces only, the insect must travel along at least 2 faces.
If we unfold the top and front faces to make a flat rectangle that is 1 m by 2 m, the insect can travel from A to B along the diagonal of this rectangle. The distance from A to B in this rectangle is $\sqrt{2^2+1^2}=\sqrt{5}$ m. Thus, the insect could travel $\sqrt{5}$ m, which would take $\frac{\sqrt{5}}{4}\approx 0.56$ minutes.

To get from A to B through the interior of the cube, the shortest possible path is the diagonal from A to B. Since the cube has edge length 1 m, then the length of the diagonal is $\sqrt{1^2+1^2+1^2}=\sqrt{3}$ m. Thus, the insect travels at least $\sqrt{3}$ m in this case, and so takes at least $\frac{\sqrt{3}}{3}\approx 0.58$ minutes.
Therefore, the shortest possible time that the insect can take is $\frac{\sqrt{5}}{4}$ minutes.

Answer: $\frac{\sqrt{5}}{4}$

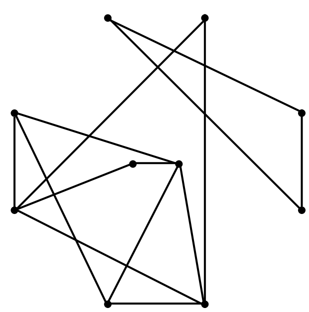
14. Suppose that the sequence has first term a and common difference d.
Then the 20 terms are $a, a+d, a+2d, \ldots, a+17d, a+18d, a+19d$.
The sum of an arithmetic sequence with first term a, common difference d, and n terms is $\frac{n}{2}(a+(a+(n-1)d)$, so the sum of this sequence is $\frac{20}{2}(a+(a+19d)) = 10(2a+19d)$.
Since the sum of the first three terms is 15, then $a+(a+d)+(a+2d) = 15$ or $3a+3d = 15$ and so $a+d = 5$.
Since the sum of the last three terms is 12, then $(a+17d)+(a+18d)+(a+19d) = 12$ or $3a+54d = 12$ and so $a+18d = 4$.
Therefore, the sum of all 20 terms is $10(2a+19d) = 10((a+d)+(a+18d)) = 10(5+4) = 90$.

Answer: 90

15. Since the parabola has equation $y = k^2 - x^2$, then its y-intercept is k^2, and so the height of rectangle $ABCD$ is k^2.
Since the parabola has equation $y = k^2 - x^2 = (k-x)(k+x)$, then its x-intercepts are $-k$ and k, so the width of rectangle is $k-(-k) = 2k$.
Since the perimeter is 48, then $2k^2 + 2(2k) = 48$ or $k^2 + 2k - 24 = 0$.
Factoring, we obtain $(k+6)(k-4) = 0$.
Since $k > 0$, then $k = 4$.

Answer: 4

16. Superimposing the top left diagram and the one to its right, we obtain the following graph with 10 vertices and 14 edges:



We can check the remaining eight diagrams to see that this diagram has all of the required edges.

Answer: 14

17. Since the grey paint has mass 12 kg and is 80% white paint, then the amount of white paint is $0.8 \times 12 = 9.6$ kg and so the amount of black paint is $12 - 9.6 = 2.4$ kg.
For the new mixture to be 90% white paint and 10% black paint, the mass of white paint should be 9 times the mass of black paint, or $9 \times 2.4 = 21.6$ kg.
In total, the new mixture has mass $2.4 + 21.6 = 24$ kg.

Answer: 24

18. Consider one of the n vertices of a regular polygon with n sides.
This vertex will be part of $(n-3)$ diagonals – one joining to every vertex other than itself and its two neighbours.
This means that there are $\frac{1}{2}n(n-3)$ diagonals in total, since each of n vertices is part of $n-3$ diagonals, and each diagonal is counted twice in this way.
For there to be 90 diagonals, then $\frac{1}{2}n(n-3) = 90$ or $n^2 - 3n - 180 = 0$.
Thus, $(n-15)(n+12) = 0$.
Since $n > 0$, then $n = 15$.

Answer: 15

19. We note that $18\,018 = 18 \times 1001 = 2 \times 3^2 \times 11 \times 91 = 2 \times 3^2 \times 7 \times 11 \times 13$.
We want to find all triples (a, b, c) of positive integers with $a + b + c = 100$ and $abc = 18\,018$.
First, we note that none of a, b, c can be a multiple of more than one of 7, 11 or 13:

> If one was a multiple of 11 and 13, it would be at least 143 which is too large.
> If one was a multiple of 7 and 13, it would be at least 91. It could not be a larger multiple of 91, so the remaining two integers would have a product of $2 \times 3^2 \times 11$ and a sum of 9, which is not possible.
> If one was a multiple of 7 and 11, it would be at least 77. It could not be a larger multiple of 77, so the remaining two integers would have a product of $2 \times 3^2 \times 13$ and a sum of 23, which is not possible.

Therefore, suppose that $a = 7k$, $b = 11m$, and $c = 13n$ for some positive integers k, m, n. We ignore the restriction that $a < b < c$ for the moment and then incorporate this later on.
Since $abc = 2 \times 3^2 \times 7 \times 11 \times 13$, then $(7k)(11m)(13n) = 2 \times 3^2 \times 7 \times 11 \times 13$ or $kmn = 18$.
So we check the possible combinations of k, m, n that have a product of 18 to see if the resulting sum $a + b + c$ equals 100. We note that none of k, m or n can equal 18 (otherwise the corresponding value of a, b or c would be larger than 100):

k	m	n	a	b	c	$a+b+c$
1	2	9	7	22	117	146
1	9	2	7	99	26	132
2	1	9	14	11	117	142
2	9	1	14	99	13	126
9	1	2	63	11	26	100
9	2	1	63	22	13	98
1	3	6	7	33	78	118
1	6	3	7	66	39	112
3	1	6	21	11	78	110
3	6	1	21	66	13	100
6	1	3	42	11	39	92
6	3	1	42	33	13	88
2	3	3	14	33	39	86
3	2	3	21	22	39	82
3	3	2	21	33	26	80

Therefore, there are two triples of positive integers with the required properties.
These are $(a, b, c) = (11, 26, 63)$ and $(a, b, c) = (13, 21, 66)$. (We have rearranged each triple in increasing order.)

ANSWER: 2

20. From the definition, $(x, 3) \diamond (x, y) = (x^2 - 3y, xy + 3x)$.
Since $(x, 3) \diamond (x, y) = (6, 0)$, then $x^2 - 3y = 6$ and $xy + 3x = 0$.
From the second equation $x(y + 3) = 0$ so $x = 0$ or $y = -3$.
If $x = 0$, then $x^2 - 3y = 6$ gives $0 - 3y = 6$ or $y = -2$.
If $y = -3$, then $x^2 - 3y = 6$ gives $x^2 + 9 = 6$ or $x^2 = -3$, which is not possible.
Therefore, $(x, y) = (0, -2)$.

ANSWER: $(0, -2)$

21. The numbers p, q and r must be three different numbers from the list 1, 2, 3, 4, 5, 6.
There are $\binom{6}{3} = 20$ ways of choosing 3 numbers from 6.
Once these numbers are chosen, there is only one way to assign them to p, q and r since $p < q < r$.
Each time a standard die is rolled, there are 6 equally likely results, so when the die is rolled 3 times, there are $6^3 = 216$ equally likely combinations that can occur.
Therefore, the probability of success (that is, of rolling three numbers in increasing order) is $\frac{20}{216}$ or $\frac{5}{54}$.

Answer: $\frac{5}{54}$

22. If a circle is tangent to both the x-axis and the y-axis, it must be entirely contained in one of the quadrants. Since we are looking for circles that pass through $(9, 2)$, they must be entirely in the first quadrant.
Consider a circle of radius r that is tangent to both the x-axis and the y-axis and is in the first quadrant.
Since the circle is tangent to the x-axis, its centre must be r units from the x-axis. Similarly, its centre must be r units from the y-axis, so the coordinates of the centre must be (r, r).
A circle with centre (r, r) and radius r has equation $(x - r)^2 + (y - r)^2 = r^2$.
For this circle to pass through $(9, 2)$, the pair $(9, 2)$ must satisfy the equation of the circle, so

$$\begin{aligned} (9-r)^2 + (2-r)^2 &= r^2 \\ r^2 - 18r + 81 + r^2 - 4r + 4 &= r^2 \\ r^2 - 22r + 85 &= 0 \\ (r-17)(r-5) &= 0 \end{aligned}$$

Therefore, $r = 5$ or $r = 17$, and so the largest possible radius is 17.

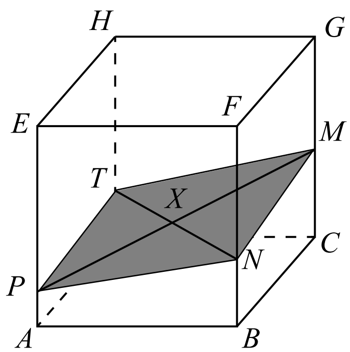
Answer: 17

23. Consider a positive integer n in the range $1000 \leq n \leq 9999$. Such an integer is of the form $abcd$ for some digits a, b, c, d.
Note that $n = 1000a + 100b + 10c + d$, since the thousands digit is a, the hundreds digit is b, the tens digit is c, and the units digit is d. Also, $s(n) = a + b + c + d$.
Therefore, we want to find all $n = abcd$ for which $\dfrac{1000a + 100b + 10c + d}{a + b + c + d} = 112$.
This is equivalent to $1000a+100b+10c+d = 112a+112b+112c+112d$ or $888a-111d = 12b+102c$ or $296a - 37d = 4b + 34c$ or $37(8a - d) = 4b + 34c$.
Since the LS is a multiple of 37, then the RS must also be a multiple of 37.
Note that the RS and is also even, and so is an even multiple of 37.
If $4b + 34c = 0(37) = 0$, then $b = c = 0$ and $8a - d = 0$. Since not all digits are zero, then we can have $a = 1$ and $d = 8$.
If $4b + 34c = 2(37) = 74$, there are no values of b and c in the appropriate range that work. (This is because $c = 0$ and $c = 1$ would give values of b larger than 9, $c = 2$ gives a value of b that is not an integer, and $c \geq 3$ give values of b that are negative.)
If $4b + 34c = 4(37) = 148$, then $b = 3$ and $c = 4$ are the only values that work. This gives $8a - d = 4$. For a and d to be digits, we must have $a = 1$ and $d = 4$.
If $4b + 34c = 6(37) = 222$, there are no values of b and c that work as in the case above.

If $4b + 34c = 8(37) = 296$, then $b = 6$ and $c = 8$ are the only values that work. This gives $8a - d = 8$. For a and d to be digits, we must have $a = 1$ and $d = 0$ or $a = 2$ and $d = 8$.
In total, there are four integers n that satisfy the required: 1008, 1344, 1680, and 2688.

ANSWER: 1008, 1344, 1680, 2688

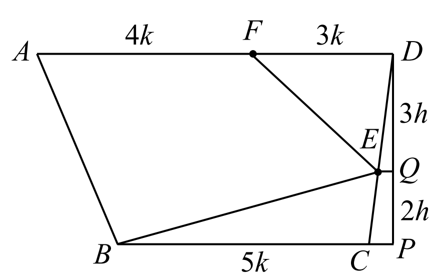
24. We draw the plane $MNPT$ and join MP and NT:



Since MP and NT both lie in the plane, then these line segments intersect.
Since MP joins GC to EA, it lies above the diagonal CA of the base.
Since NT joins FB to HD, it lies above the diagonal BD of the base.
Therefore, the point of intersection X of MP and NT lies above the point of intersection of CA and BD, which is the centre of the base.
Thus, X is the midpoint of MP and the midpoint of NT.
This means that the height of X will be the average of the heights of M and P above the base, and will be the average of the heights of N and T above the base. (This is because the height changes linearly along each line segment MP and NT.)
Therefore, $\frac{1}{2}(MC + PA) = \frac{1}{2}(NB + TD)$.
Thus, $MC + PA = NB + TD$ or $\frac{1}{2}a + \frac{1}{4}a = \frac{1}{3}a + DT$, and so $DT = \frac{6}{12}a + \frac{3}{12}a - \frac{4}{12}a = \frac{5}{12}a$.

ANSWER: $\frac{5}{12}a$

25. Since $BC : AD = 5 : 7$, then we let $BC = 5k$ and $AD = 7k$ for some real number $k > 0$.
Since $AD = 7k$ and F divides AD in the ratio $4 : 3$, then $AF = 4k$ and $FD = 3k$.
Drop a perpendicular from D to P on BC extended, and let the height of trapezoid $ABCD$ be $5h$, for some real number $h > 0$.
Draw a line through E parallel to AD and BC meeting DP at Q. Note that EQ is perpendicular to DP.
Now $\triangle DQE$ is similar to $\triangle DPC$ since each is right-angled, and the two triangles share a common angle at D.
Since $CE : ED = 2 : 3$, then $DE : DC = 3 : 5$.
By similarity, $DQ : DP = 3 : 5$; since $DP = 5h$, then $DQ = 3h$.
This means that $QP = DP - DQ = 2h$.



Trapezoid $ADBC$ has parallel sides $AD = 7k$ and $BC = 5k$, and height $5h$; thus, its area is $\frac{1}{2}(7k + 5k)(5h) = 30hk$.
$\triangle FDE$ has base $FD = 3k$ and height equal to the distance from FD to EQ, which is $3h$; thus, its area is $\frac{1}{2}(3k)(3h) = \frac{9}{2}hk$.
$\triangle BCE$ has base $BC = 5k$ and height equal to the distance from BC to EQ, which is $2h$; thus, its area is $\frac{1}{2}(5k)(2h) = 5hk$.
We are given that the area of quadrilateral $ABEF$ is 123.
Trapezoid $ABCD$ is made up of quadrilateral $ABEF$, $\triangle FDE$, and $\triangle BCE$.
Thus, the area of trapezoid $ABCD$ equals the sum of the areas of quadrilateral $ABEF$, $\triangle FDE$, and $\triangle BCE$.
This gives

$$\begin{aligned} 30hk &= 123 + \tfrac{9}{2}hk + 5hk \\ \tfrac{41}{2}hk &= 123 \\ hk &= 6 \end{aligned}$$

Since the area of trapezoid $ABCD$ equals $30hk$, then this area is $30(6)$ or 180.

ANSWER: 180

Individual Problems

1. Let x be the one-digit integer that is included 7 times.
Since x is at most 9, then $7x$ is at most 63.
If $x = 9$, then the other integer is $68 - 7x = 68 - 63 = 5$.
If $x \leq 8$, then $7x$ is at most 56, and so the other integer would have to be at least 12, which is not possible since it is a one-digit integer.
Thus, the other integer is 5.

ANSWER: 5

2. Since $120^2 = 14400$ and $130^2 = 16900$, then $15AB9$ is the square of an integer between 120 and 130.
Since $15AB9$ has a units digit of 9, then it is the square of an integer ending in 3 or 7.
Note that $123^2 = 15129$ and $127^2 = 16129$, then $A = 1$ and $B = 2$, so $A + B = 3$.

ANSWER: 3

3. Using the definition, $x = 4\heartsuit 2 = 4(2) - \dfrac{4}{2} = 6$ and $y = 2\heartsuit 2 = 2(2) - \dfrac{2}{2} = 3$.
Therfore, $x^y = 6^3 = 216$.

ANSWER: 216

4. The 8 numbers in the list have a sum of 236.
When these 8 numbers are grouped into 4 pairs with equal sums, this sum must be $\frac{236}{4} = 59$.
Therefore, 32 must be paired with $59 - 32 = 27$.
(Note that the pairings $36 + 23$ and $27 + 32$ and $42 + 17$ and $28 + 31$ satisfy the requirements.)

ANSWER: 27

5. Since $\angle FCB$ is an exterior angle for $\triangle ECD$, then $\angle FCB = \angle CED + \angle EDC = y^\circ + z^\circ$.
Since $\angle FBA$ is an exterior angle for $\triangle FBC$, then $\angle FBA = \angle BFC + \angle FCB = x^\circ + (y^\circ + z^\circ)$.
Since $\angle FBA = 124^\circ$, then $124^\circ = x^\circ + y^\circ + z^\circ$ or $x + y + z = 124$.

ANSWER: 124

6. Suppose that the numbers hidden behind the four stars, in order, are w, x, y, z.
Since the number on the first star is 280, then the product of the numbers behind the other three stars is 280, or $xyz = 280$.
Similarly, the numbers on the other three stars give the equations $wyz = 168$, $wxz = 105$, and $wxy = 120$.
Multiplying these four equations together, we obtain

$$(xyz)(wyz)(wxz)(wxy) = (280)(168)(105)(120)$$

or

$$w^3x^3y^3z^3 = (2^3 \cdot 5 \cdot 7)(2^3 \cdot 3 \cdot 7)(3 \cdot 5 \cdot 7)(2^3 \cdot 3 \cdot 5)$$

and so

$$(wxyz)^3 = 2^9 \cdot 3^3 \cdot 5^3 \cdot 7^3$$

Therefore, the product of the four hidden integers, or $wxyz$, equals $2^3 \cdot 3 \cdot 5 \cdot 7 = 840$.

ANSWER: 840

7. Since $ABOC$ is a quadrilateral, then the sum of its interior angles is 360°.
Thus, $\angle BOC = 360^\circ - \angle BAC - \angle ABO - \angle ACO = 360^\circ - 50^\circ - 90^\circ - 90^\circ = 130^\circ$.
Therefore, the shaded region is a sector of the circle with central angle 130°.
Since the radius of the circle is 6, then the area of the entire circle is $\pi 6^2 = 36\pi$, so the area of the shaded sector is $\frac{130^\circ}{360^\circ}(36\pi) = \frac{13}{36}(36\pi) = 13\pi$.

ANSWER: 13π

8. There are several possibilities for the groupings of students at the various positions:

   (i) 4 students tied
   (ii) 3 students tied plus 1 student separately
   (iii) 2 students tied plus 2 students tied
   (iv) 2 students tied plus 1 student separately plus 1 student separately
   (v) 1 student separately plus 1 student separately plus 1 student separately plus 1 student separately

In the following table, we summarize the information about each case, the number of ways of arranging the finish of the groups (a), the number of ways of selecting the students to be placed in the groups (b), and the total number of possible orders in this case (ab):

Case	a	b	ab
(i) 4	1	1	1
(ii) $3+1$	2	4	8
(iii) $2+2$	1	6	6
(iv) $2+1+1$	3	12	36
(v) $1+1+1+1$	1	24	24

We explain the calculations for two of the cases: (ii) and (iv).
In case (ii), there are 2 possible orderings for the groups of student (group of 3 followed by group of 1, or group of 1 followed by group of 3) and there are 4 ways to place the students in groups (we choose 1 of 4 to be the group of 1, and the remaining 3 are automatically placed). This gives $2 \times 4 = 8$ orderings overall.
In case (iv), there are 3 possible orderings for the groups (2 then 1 then 1, or 1 then 2 then 1, or 1 then 1 then 2) and 12 ways to place the student (there are 4 choices for the student in the first group of 1, for each of which there are 3 choices for the student in the next group of 1, and then the remaining students are placed in the group of 2). This gives $3 \times 12 = 36$ orderings overall.
In total, there are $1 + 8 + 6 + 36 + 24 = 75$ possible orders of finish.

ANSWER: 75

9. Suppose that the given sequence has first term $t_1 = a$ and common ratio r.
Thus, $t_2 = ar$.
Since $t_1 + t_2 = \dfrac{15}{2}$, then $a + ar = \dfrac{15}{2}$ or $a(1+r) = \dfrac{15}{2}$, from which we obtain $a = \dfrac{15}{2(1+r)}$.
(Note that r cannot equal -1.)
Since $(t_1)^2 + (t_2)^2 = \dfrac{153}{4}$, then $a^2 + a^2r^2 = \dfrac{153}{4}$, or $a^2(1+r^2) = \dfrac{153}{4}$.

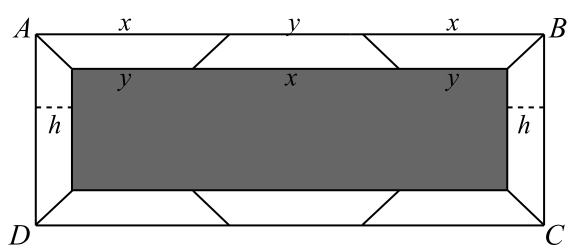
Substituting for a, we obtain

$$\begin{aligned}
\left(\frac{15}{2(1+r)}\right)^2(1+r^2) &= \frac{153}{4} \\
\frac{225(1+r^2)}{4(1+r)^2} &= \frac{153}{4} \\
\frac{25(1+r^2)}{(1+r)^2} &= 17 \\
25+25r^2 &= 17(1+2r+r^2) \\
8r^2-34r+8 &= 0 \\
4r^2-17r+4 &= 0 \\
(r-4)(4r-1) &= 0
\end{aligned}$$

Thus, $r=4$ or $r=\frac{1}{4}$.
Since $t_1 > t_2$, then $r<1$, so $r=\frac{1}{4}$.
Since $a=\frac{15}{2(1+r)}$, then $a=\frac{15}{2(\frac{5}{4})}=6$.
Finally, $t_5 = ar^4 = 6\left(\frac{1}{4}\right)^4 = \frac{6}{4^4} = \frac{6}{256} = \frac{3}{128}$.

Answer: $\frac{3}{128}$

10. Suppose that each of the identical trapezoids has parallel sides of length x and y (with $x>y$), height h, and area p, for some prime number p.
We may assume that $p>2$, since we eventually will want to make p as large as we can, subject to the other restrictions of the problem. Requiring $p>2$ ensures that p is odd.
Since the area of each trapezoid is p, then $\frac{1}{2}(x+y)h=p$.
Using the given diagram, the outer length of the frame is $x+y+x=2x+y$ and the inner length is $y+x+y=x+2y$.



The difference between the outer length and the inner length is twice the height of one of the trapezoids, so $(2x+y)-(x+2y)=2h$ or $x-y=2h$.
Since $\frac{1}{2}(x+y)h=p$, then $(x+y)(2h)=4p$ or $(x+y)(x-y)=4p$.
Since x and y are integers, then $x+y$ and $x-y$ are either both even or both odd. (We can see this by checking the various possibilities for x and y themselves being even or odd.)
Since the product of $x+y$ and $x-y$ is even, then each of them must be even.
Since $4p=2^2p$ and p is an odd prime number, then the only way of writing $4p$ as the product of two even integers is $2\cdot 2p$.
Since $x+y>x-y$, then we must have $x+y=2p$ and $x-y=2$.

Adding these gives $2x = 2p + 2$ or $x = p + 1$.
Substituting back gives $y = (x + y) - x = 2p - (p + 1) = p - 1$.
Now the shaded region has length $x + 2y$ (which equals $(p+1)+2(p-1)$, or $3p-1$) and width y (which equals $p - 1$).
Therefore, the area of the shaded region, in terms of p, is $(p - 1)(3p - 1)$.
We want to find the largest possible value less than 2000 that this expression can take, subject to the restriction that p is prime.
We note that

- when $p = 23$ we obtain $(p - 1)(3p - 1) = 1496$,
- when $p = 29$ (the next prime), we obtain $(p - 1)(3p - 1) = 2408$, and
- $(p - 1)(3p - 1)$ is increasing when $p > 2$.

Therefore, the maximum possible area less than 2000 for the shaded region is 1496.

ANSWER: 1496

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each Relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $2 \times 0 + 1 \times 4 = 0 + 4 = 4$.

   (b) The average of the first list is $m = \dfrac{13 + 16 + 10 + 15 + 11}{5} = \dfrac{65}{5} = 13$.
   The average of the second list is $n = \dfrac{16 + t + 3 + 13}{4} = \dfrac{32 + t}{4} = 8 + \dfrac{1}{4}t$.
   Therefore, $m - n = 13 - \left(8 + \dfrac{1}{4}t\right) = 5 - \dfrac{1}{4}t$.
   Since the answer to (a) is 4, then $t = 4$, and so $m - n = 5 - 1 = 4$.

   (c) Since the two lines intersect at (a, b), then these coordinates satisfy the equation of each line.
   Therefore, $b = 12$ and $b = 2a + t$.
   Since $b = 12$, then $12 = 2a + t$ or $2a = 12 - t$, and so $a = 6 - \frac{1}{2}t$.
   Since the answer to (b) is 4, then $t = 4$, and so $a = 6 - 2 = 4$.

ANSWER: $4, 4, 4$

1. (a) Evaluating, $\dfrac{1}{2}\left(\dfrac{1}{\frac{1}{9}} + \dfrac{1}{\frac{1}{6}} - \dfrac{1}{\frac{1}{5}}\right) = \dfrac{1}{2}(9 + 6 - 5) = \dfrac{1}{2}(10) = 5$.

   (b) Since $2 : m : t = m : 32 : x$, then $\dfrac{2}{m} = \dfrac{m}{32}$, which gives $m^2 = 64$ or $m = \pm 8$.
   If $m = 8$, the original ratio becomes $2 : 8 : t = 8 : 32 : x$.
   If $m = -8$, the original ratio becomes $2 : (-8) : t = (-8) : 32 : x$.
   Since the answer to (a) is 5, then $t = 5$.
   In the first case, $t = 5$ gives $2 : 8 : 5 = 8 : 32 : x$. Since each number on the RS is each 4 times the corresponding number on the LS, then $x = 4(5) = 20$.
   In the second case, $x = -20$, so x is not a positive integer.
   Therefore, $x = 20$.

   (c) Consider $\triangle CDE$ and its exterior angle, $\angle ACD$.
   We have $\angle ACD = \angle CDE + \angle CED$ or $(2x)^\circ = t^\circ + (2t)^\circ$ which gives $2x = 3t$.
   Now $\triangle BAC$ is isosceles with $AB = BC$, so $\angle BAC = \angle BCA = x^\circ$.
   Thus, $\angle ABC = 180^\circ - \angle BAC - \angle BCA = 180^\circ - (2x)^\circ$.
   Since $2x = 3t$, then $\angle ABC = 180^\circ - (3t)^\circ$.
   Since the answer to (b) is 20, then $t = 20$, and so $\angle BAC = 180^\circ - 60^\circ = 120^\circ$.

ANSWER: $5, 20, 120^\circ$

2. (a) Repeatedly dividing, $324 = 3 \times 108 = 3^2 \times 36 = 3^2 \times 3^2 \times 2^2 = 2^2 \times 3^4$.
Therefore, $a = 2$ and $b = 4$, so $2^b \times 3^a = 2^4 \times 3^2 = 16 \times 9 = 144$.

(b) Sarah eats $\frac{1}{3}t$ chocolates and Andrew eats $\frac{3}{8}t$ chocolates.
Since Cecily eats the rest of the chocolates, she eats $t - \frac{1}{3}t - \frac{3}{8}t = \frac{24-8-9}{24}t = \frac{7}{24}t$.
Therefore, Sarah eats $\frac{1}{3}t - \frac{7}{24}t = \frac{1}{24}t$ more chocolates than Cecily.
Since the answer to (a) is 144, then $t = 144$ and so Sarah eats 6 more chocolates than Cecily.

(c) Since the diameter of the circle has length t, then the radius of the circle is $\dfrac{t}{2}$ and so the area of the circle is $\pi\left(\dfrac{t}{2}\right)^2 = \dfrac{\pi t^2}{4}$.
Since $CD = 2AD$, then the area of rectangle $ABCD$ is $(AD)(CD) = 2AD^2$.
Since $ABCD$ is a rectangle, then $\angle ADC = 90^\circ$.
By the Pythagorean Theorem in $\triangle ADC$, we have $AD^2 + CD^2 = AC^2$.
Thus, $AD^2 + (2AD)^2 = t^2$ and so $5AD^2 = t^2$, or $2AD^2 = \dfrac{2t^2}{5}$. (Recall that this is the area of rectangle $ABCD$.)
The area of the shaded region is the area of the circle minus the area of the rectangle, which equals $\dfrac{\pi t^2}{4} - \dfrac{2t^2}{5}$.
Since the answer to (b) is 6, then $t = 6$ and so the shaded area is $\dfrac{\pi(36)}{4} - \dfrac{2(36)}{5} = 9\pi - \dfrac{72}{5}$.

ANSWER: $144, 6, 9\pi - \dfrac{72}{5}$

3. (a) Since $36 = 2^2 \times 3^2$ and $45 = 3^2 \times 5$, then the greatest common divisor of 36 and 45 is 9.
Since 9 is also a divisor of 495, then it is the greatest common divisor of the three integers.

(b) The length of the bottom segment of the figure is $3t$.
The sum of the lengths of the top segments must equal the length of the bottom segment, so $k + \frac{3}{2}k + t + \frac{1}{2}k = 3t$ or $3k = 2t$ and so $k = \frac{2}{3}t$.
Since the answer to (a) is 9, then $t = 9$ and so $k = \frac{2}{3}(9) = 6$.
(Note that we did not in fact use the given perimeter; this value of t does give a perimeter of 162.)

(c) Expanding,

$$(tx+3)^3 = (tx+3)(tx+3)^2 = (tx+3)(t^2x^2+6tx+9) = t^3x^3 + 9t^2x^2 + 27tx + 27$$

In terms of t, the coefficients are $a = t^3$, $b = 9t^2$, $c = 27t$, and $d = 27$.
Since the answer to (b) is 6, then $t = 6$, and so the coefficients are $a = 6^3 = 216$, $b = 9(36) = 324$, $c = 27(6) = 162$, and $d = 27$.
Therefore, the largest of the values of a, b, c, d is 324.

ANSWER: $9, 6, 324$